§1. Introduction. By a remarkable result of Erdös and Selfridge [3] in 1975, the diophantine equation

\[ y^k = (x + 1)(x + 2) \cdots (x + m), \]  

with integers \( k \geq 2 \) and \( m \geq 2 \), has only the trivial solutions \( x = -j \ (j = 1, \ldots, m), \ y = 0 \). This put an end to the old question whether the product of consecutive positive integers could ever be a perfect power; for a brief account of its history see [7].

From the viewpoint of algebraic geometry (1) represents a so-called superelliptic curve, and it seems to be more natural to ask for rational solutions \((x; y)\) instead of integer solutions. Rational points on elliptic curves are well understood, but for general \( k \) and \( m \), their nice arithmetic properties fade away. It follows from Faltings’s proof [4] of Mordell’s conjecture that, for fixed \( k \geq 1, m \geq 1 \) and \( k + m > 6 \), equation (1) has at most finitely many rational solutions (cf. [7]).

It was shown by the second author [7] that, for \( k^2 \) and \( m^2 \), all rational points \((x; y)\) on the superelliptic curve (1) are the trivial ones with \( x = -j \ (j = 1, \ldots, m) \) and \( y = 0 \), except for the case \( k = m = 2 \) where we have exactly those satisfying

\[ x = \frac{2c_1^2 - c_2^2}{c_2 - c_1^2}, \quad y = \frac{c_1 c_2}{c_2 - c_1^2} \]

with coprime integers \( c_1 \neq \pm c_2 \). The second author also made the following

**Conjecture.** For \( k \geq 2 \) and \( m \geq 2 \), all rational points \((x; y)\) on the superelliptic curve (1) are the trivial ones with \( x = -j \ (j = 1, \ldots, m) \) and \( y = 0 \), except for the case \( k = m = 2 \) with exactly those satisfying

\[ x = \frac{2c_1^2 - c_2^2}{c_2 - c_1^2}, \quad y = \frac{c_1 c_2}{c_2 - c_1^2} \]

with coprime integers \( c_1 \neq \pm c_2 \).

The purpose of this article is to prove the conjecture for \( m = 5 \) and all \( k \geq 2 \).
THEOREM. Let \( k \geq 2 \). Then the only rational points on the superelliptic curve
\[
y^k = (x + 1)(x + 2)(x + 3)(x + 4)(x + 5)
\]
are \( (x; 0) \) with \(-x \in \{1, 2, 3, 4, 5\}\).

§2. Notation and preliminary results. For a positive integer \( k \), let
\[
k_* := \begin{cases} k, & \text{for } 5 \nmid k, \\ k/5, & \text{for } 5 \mid k. \end{cases}
\]

For arbitrary integers \( a \) and \( b \) we define the greatest common divisors
\[
G_1 = G_1(a, b) := (a, a^2 - 4b^2), \\
G_2 = G_2(a, b) := (a^2 - b^2, a^2 - 4b^2).
\]

For \( (a, b) = 1 \) they satisfy
\[
G_1(a, b) = (a, 4) \in \{1, 2, 4\}, \\
G_2(a, b) = (a^2 - b^2, 3) \in \{1, 3\}.
\]

Our proof of the theorem uses results on the solutions of several diophantine equations, all of which are based on the work of Wiles [9]. An integral solution \( (x; y; z) \) of the equation
\[
aX^k + bY^l = cZ^m,
\]
with given integers \( a, b, c \) and positive integers \( k, l, m \), is called primitive if \( \gcd(x, y, z) = 1 \), and is called trivial if \( xyz \in \{0, \pm 1\} \).

**Theorem A (Ribet).** Let \( p \geq 3 \) be a prime, and let \( 2 \leq \alpha < p \). Then
\[
X^p + Y^p = 2^\alpha Z^p
\]
has only trivial solutions.

**Proof.** This is part of Ribet’s Theorem 3 in [6].

**Theorem B (Ribet, Darmon & Merel).** Let \( p \geq 3 \) be a prime. Then
\[
X^p + Y^p = 2Z^p
\]
has only trivial solutions.

**Proof.** This is part 1 of the main theorem of Darmon and Merel in [1].
Theorem C (Poonen). The primitive solutions of
\[ X^5 + Y^5 = Z^2 \]
are all trivial.

Proof. This result can be found in [5].

Theorem D (Serre, Ribet). For a positive integer \( \alpha \) the equation
\[ X^\alpha + Y^\alpha = Z^5 \]
has only trivial solutions.

Proof. This is shown in the same fashion as Ribet’s Theorem 1 in [6], since
Serre’s Théorème 2 in [8] also works in the case \( L = 3 \) and \( p = 5 \).

§3. Proof of the main theorem. According to the possible values of \( G_1 \) and \( G_2 \) (cf. (3) and (4)), the proof of our theorem falls into four parts.

Proposition 1. Let \( k \geq 2, a, b = b_1^{**}, c \) be integers satisfying
\[ G_2(a, b) = (a, b) = 1 \]
and
\[ c^k = a(a^2 - b^2)(a^2 - 4b^2). \]
(5)

Then \( c = 0 \).

Proof. Since \( (a, b) = 1 \), we have \( (a, a^2 - b^2) = 1 \). Then (5) and
\[ G_2(a, b) = 1 \]
imply that
\[ a^2 - b^2 = c_1^k \quad \text{and} \quad (a^2 - 4b^2)a = c_2^k \]
(6)
for some coprime integers \( c_1, c_2 \). By (3) we have \( G_1|4 \), and so we obtain from
the second equation in (6) that
\[ a = G_12c_3^k \quad \text{and} \quad a^2 - 4b^2 = G_12c_4^k \]
(7)
for some odd, coprime integers \( c_3, c_4 \) satisfying \( (c_1, c_3c_4) = 1 \), where
\[ (s, t) = \begin{cases} (0, 0), & \text{for } G_1 = 1, \\ (0, jk - 2), & \text{for } G_1 = 2, \\ (0, 0), & \text{for } G_1 = 4, 4 | a, \\ (jk - 4, 0), & \text{for } G_1 = 4, 8 | a, \\ \end{cases} \]
for a suitable positive integer \( j \). Since \( b = b_1^{**} \), (6) and (7) yield
\[ G_1^2c_2^2c_3^2 - b_1^{2k**} = c_1^k \]
in three pairwise coprime terms. Since \( G_1|4 \) by (3), that is an identity of type
\[ 2^{2k}X^{2k} - Y^{2k**} = Z^k \]
(8)
with \( \alpha = s + \log_2 G_1 \).

First, assume that there is a prime \( p \not\in \{2, 5\} \) dividing \( k \), or \( 5^2 \mid k \). Then (8) is of type

\[ 2^{2\alpha} X^n - Y^n = Z^n \]

in coprime terms for a prime \( p \neq 2 \). By Theorem A together with Theorem B this has only trivial solutions (more precisely \( XYZ = 0 \)); hence \( c_3c_1 = 0 \) and thus \( c = 0 \). We are left with the case \( k = 2^r5^e \) with a non-negative integer \( r \) and \( e \in \{0, 1\} \). For \( e = 0 \) we have \( r \geq 1 \), since \( k \geq 2 \). Therefore (8) is of type

\[ 2^{2\alpha} X^4 - Y^4 = Z^2, \]

which Euler knew to be unsolvable except for \( XYZ = 0 \) (cf. [2], p. 626). So we only have to deal with the case \( e = 1 \), when (8) is of type

\[ 2^{2\alpha} X^{10} - Y^2 = Z^2. \]

For \( \alpha = 0 \), we have an equation of the form \( X^5 + Z^5 = Y^2 \) in coprime terms, which has only trivial solutions by Theorem C and thus implies that \( c = 0 \). For \( \alpha \geq 1 \), we have

\[ (2^{\alpha} X^5 - Y, 2^{\alpha} X^5 + Y) = (2^{\alpha} X^5 - Y, 2^{\alpha+1} X^5) = 1; \]

hence, by (9),

\[ 2^{\alpha} X^5 - Y = Z_1^5 \quad \text{and} \quad 2^{\alpha} X^5 + Y = Z_2^5. \]

We obtain \( 2^{\alpha+1} X^5 = Z_1^5 + Z_2^5 \) in coprime terms, which again has only trivial solutions by Theorems A and B. This completes the proof of Proposition 1.

\textbf{Proposition 2.} \ Let \( k \geq 2, a, b = b_1 > 0 \) \ and \( c \) \ be \ integers \ satisfying \( G_1(a, b) = (a, b) = 1, G_2(a, b) = 3 \) \ and \( (5) \). Then \( c = 0 \).

\textit{Proof.} \ Since \( G_1 = (a, 4) = 1 \) by (3), we know that \( 2d_1a \), and hence

\[ (a - 2b, a + 2b) = (a - 2b, 2a) = (a - 2b, a) = 1. \quad (10) \]

Moreover, \( (a, b) = 1 \) implies that

\[ (a - b, a - 2b) = (a + b, a + 2b) = 1. \quad (11) \]

By the fact that \( 3 = G_2(a, b) = ((a - b)(a + b), (a - 2b)(a + 2b)) \), we have either \( (a - b, a + 2b) = 3, (a + b, a - 2b) = 1 \), or \( (a - b, a + 2b) = 1, (a + b, a - 2b) = 3 \). By the transformation \( b \mapsto -b \) each of the two cases turns into the other. Therefore, we may assume without loss of generality that

\[ (a - b, a + 2b) = 3 \quad \text{and} \quad (a + b, a - 2b) = 1. \quad (12) \]

Consequently, \( (a - b, 3) = 3 \), and clearly \( (a, (a^2 - b^2)(a^2 - 4b^2)) = G_1 = 1 \). By
RATIONAL POINTS ON THE ERDŐS-SEFRIDGE CURVE

virtue of (5), (10), (11) and (12), we only have to distinguish two cases, namely,

\[
\begin{align*}
    a &= c_1^5, \\
    a - 2b &= c_2^5, \\
    a + 2b &= 3^{k-1}c_3^5, \\
    a^2 - b^2 &= 3c_4^k,
\end{align*}
\]

(13)

and

\[
\begin{align*}
    a &= c_1^k, \\
    a - 2b &= c_2^k, \\
    a + 2b &= 3c_3^k, \\
    a^2 - b^2 &= 3^{k-1}c_4^k
\end{align*}
\]

(14)

for some pairwise coprime integers \(c_1, c_2, c_3, c_4\) satisfying \(2 \nmid c_1c_2c_3\) and \(3 \nmid c_1c_2\).

Since \(d := (a + b, a - b) = (a + b, 2) \in \{1, 2\}\), we have

\[(a + b, a(a - b)(a^2 - 4b^2)) = d.\]

So, by (5), we have in coprime terms

\[a + b = c_5^\mp \quad \text{or} \quad a + b = 2c_5^k \quad \text{or} \quad a + b = 2^{k-1}c_5^k\]

for a suitable \(c_5\). Both (13) and (14) imply that \(a = c_1^k\), and we have \(b = b_1^k\) by definition of \(b\). So we obtain an equation of type

\[X^k + Y^{k*} = 2^d \cdot Z^k\]

(15)

in coprime terms and \(d \in \{0, 1, k - 1\}\). If \(k\) has an odd prime divisor \(p \neq 5\) or \(5^2 \mid k\), then (15) has only trivial solutions by Theorems A and B, which all lead to \(c = 0\). The same is true in case \(4 \mid k\), which was known to Euler (cf. [2], p. 626). So we are left with the three exponents \(k \in \{2, 5, 10\}\).

For \(k \in \{2, 10\}\) we obtain (with \(b = b_1^{k*}\)) an equation of type \(X^4 - Y^4 = 3Z^2\) in coprime terms from the last identity in (13) or (14), respectively. This equation can be shown to have only trivial solutions by Fermat's method of descent (cf. [2], p. 634), or with a simple congruence argument by careful use of the divisibility properties we have.

It remains to consider the unique exponent \(k = 5\), which leaves us with four cases according to the two subcases (13) and (14) as well as \(d = (a + b, a - b) \in \{1, 2\}\). Let us first examine (13) with \(d = 1\). By (12) we have \(3 \mid (a - b)\), and thus \(3 \nmid (a + b)\). It follows from (13) that

\[
\begin{align*}
    a &= c_1^5, \\
    a - 2b &= c_2^5, \\
    a + 2b &= 3^4c_3^5, \\
    a - b &= 3c_5^k, \\
    a + b &= c_7^5
\end{align*}
\]
with coprime integers $c_6, c_7$. Consequently,
\[ b = (a + 2b) - (a + b) = 3^4 c_3^5 - c_7^5 \]
and
\[ 3b = (a + b) - (a - 2b) = c_7^5 - c_2^5. \]

Comparison of these two implies that
\[ (3c_3)^5 - 3c_7^5 = c_7^5 - c_2^5, \]
so that we have an equation of type $X^5 + Y^5 = 4Z^5$, which has only trivial solutions by Theorem A.

In case (13) with $d = 2$ we get
\[ a - b = 3 \cdot 2c_6^5 \quad \text{and} \quad a + b = 2^4 c_7^5 \]
or
\[ a - b = 3 \cdot 2^4 c_6^5 \quad \text{and} \quad a + b = 2c_7^5. \]

which imply that
\[
\begin{align*}
  a &= c_1^5, \\
  a - 2b &= c_2^5, \\
  a + 2b &= 3^4 c_3^5, \\
  a - b &= 2 \cdot 3c_6^5, \\
  a + b &= 2^4 c_7^5,
\end{align*}
\]

or
\[
\begin{align*}
  a &= c_1^5, \\
  a - 2b &= c_2^5, \\
  a + 2b &= 3^4 c_3^5, \\
  a - b &= 2^4 \cdot 3c_6^5, \\
  a + b &= 2c_7^5,
\end{align*}
\]

From (16) follow
\[ b = (a + 2b) - (a + b) = 3^4 c_3^5 - 2^4 c_7^5 \]
and
\[ 3b = (a + b) - (a - 2b) = 2^4 c_7^5 - c_2^5. \]

which lead to
\[ (3c_3)^5 + c_2^5 = 2(2c_7)^5. \]
an equation having only trivial solutions by Theorem B. In the other situation (17) we similarly obtain

\[(3c_3)^5 + c_2^5 = 8c_7^5,\]

which has only trivial solutions by Theorem A.

In case (14) with \(d = 1\) then follow

\[
\begin{align*}
  a &= c_1^5, \\
  a - 2b &= c_2^5, \\
  a + 2b &= 3c_3^5, \\
  a - b &= 3^4c_6^5, \\
  a + b &= c_7^5,
\end{align*}
\]

which imply that

\[ h = (a - b) - (a - 2b) = 3^4c_6^5 - c_2^5 \]

and

\[ 3b = (a + b) - (a - 2b) = c_7^5 - c_2^5, \]

thus

\[(3c_6)^5 - c_7^5 = 2c_2^5,\]

having only trivial solutions by Theorem B.

Finally we have to consider (14) with \(d = 2\), so that

\[
\begin{align*}
  a &= c_1^5, \\
  a - 2b &= c_2^5, \\
  a + 2b &= 3c_3^5, \\
  a - b &= 2 \cdot 3^4c_6^5, \\
  a + b &= 2^4c_7^5,
\end{align*}
\]

or

\[
\begin{align*}
  a &= c_1^5, \\
  a - 2b &= c_2^5, \\
  a + 2b &= 3c_3^5, \\
  a - b &= 2^4 \cdot 3^4c_6^5, \\
  a + b &= 2c_7^5.
\end{align*}
\]

From (18) follows

\[ 3c_1^5 = 3a = 2(a + b) + (a - 2b) = 2^5c_7^5 + c_2^5, \]
which means that
\[(2c_1)^5 + c_2^5 = 3c_4^5.\]

This equation of type \(X^5 + Y^5 = 3Z^5\) has only trivial solutions by Theorem D.

From (19) we obtain
\[c_1^5 = a = 2(a - b) - (a - 2b) = 2^5 \cdot 3^4c_6^5 - c_2^5;
\]
hence
\[c_1^5 + c_2^5 = 3^4(2c_6)^5,
\]
and again there are only trivial solutions by Theorem D. This proves Proposition 2.

**Proposition 3.** Let \(k \geq 2, a, b = b_1^* > 0\) and \(c\) be integers satisfying
\((a, b) = 1, G_1(a, b) = 2, G_2(a, b) = 3\) and (5). Then \(c = 0\).

**Proof.** It follows from \(G_1 = (a, a^2 - 4b^2) = 2\) that \(2 \mid a\). Then (5) implies,
by virtue of \((a, a^2 - b^2) = 1\), and \(G_2 = (a^2 - b^2, a^2 - 4b^2) = 3\), that
\[a = 2c_1^k \quad \text{and} \quad (a^2 - b^2)(a^2 - 4b^2) = 2^{k-1}c_2^5 \tag{20}\]
for some coprime integers \(c_1\) and \(c_2\) with odd \(c_1\). Since \((a, b) = 1\) and \(2 \mid a\), we
know that
\[(a - b, a + b) = (a - b, 2) = 1. \tag{21}\]

By the condition \(G_2 = 3\), we may assume without loss of generality (as in the
proof of Proposition 2) that
\[(a - b, a + 2b) = 3 \quad \text{and} \quad (a + b, a - 2b) = 1. \tag{22}\]

Since \(2 \mid a\), and thus \(2 \nmid b\), we have \(a \pm 2b \equiv 0 \mod 4\), and therefore
\[(a - 2b, a + 2b) = (a - 2b, 4b) = (a - 2b, 4) = 4. \tag{23}\]

Now (20), (21), (22) and \((a + b, a + 2b) = 1\) imply that
\[a + b = c_3^k, \tag{24}\]
and, with (23) in addition, we necessarily have one of the following four
situations:
\[
\begin{align*}
  a - b & = 3c_4^k, \\
  a - 2b & = 2^2c_5^k, \\
  a + 2b & = 2^{k-3}3^{k-1}c_6^k, \\
\end{align*}
\tag{25}
\]
or
\[
\begin{aligned}
&\begin{aligned}
& a - b = 3c_4^k, \\
& a - 2b = 2^{k-3}c_5^k, \\
& a + 2b = 2^23^{k-1}c_6^k,
\end{aligned} \\
\text{(26)}
\end{aligned}
\]

or
\[
\begin{aligned}
&\begin{aligned}
& a - b = 3^{k-1}c_4^k, \\
& a - 2b = 2^2c_5^k, \\
& a + 2b = 2^{k-3}3c_6^k,
\end{aligned} \\
\text{(27)}
\end{aligned}
\]

or
\[
\begin{aligned}
&\begin{aligned}
& a - b = 3^{k-1}c_4^k, \\
& a - 2b = 2^{k-3}c_5^k, \\
& a + 2b = 2^23c_6^k,
\end{aligned} \\
\text{(28)}
\end{aligned}
\]

where \(c_3, c_4, c_5\) and \(c_6\) are pairwise coprime integers.

First of all, (24), (20) and \(b = b_1^{k^*}\) imply that
\[
2c_1^k + b_1^{k^*} = c_3^k
\]

in coprime terms. For even \(k\) this equation is of type \(2X^2 + Y^2 = Z^2\) in coprime terms, where \(2 \not| XYZ\) by definition of \(c_1, c_3\) and \(b_1\), since \(2 \not| a\) and \(2 \not| b\). This is contradictory, because \(2X^2 + Y^2 \equiv 3 \not\equiv 1 \equiv Z^2 \mod 4\). For odd \(k\), (29) is an equation of type \(X^p + Y^p = 2Z^p\) for some prime \(p > 2\), unless \(k = 5\), which is the only exponent left by Theorem B.

For \(k = 5\) we obtain in both cases (25) and (26)
\[
a^2 - 4b^2 = (a - 2b)(a + 2b) = (2 \cdot 3)^4(c_5c_6)^5.
\]

With (20) then follows
\[
c_1^{10} - b^2 = 2^23^4(c_5c_6)^5.
\]

Since \((c_1, b) = 1\), we have \((c_1^5 - b, c_1^5 + b) = 2\), and we conclude that
\[
c_1^5 - b = 2 \cdot 3^4c_7^5 \quad \text{and} \quad c_1^5 + b = 2c_8^5
\]

or
\[
c_1^5 - b = 2c_7^5 \quad \text{and} \quad c_1^5 + b = 2 \cdot 3^4c_8^5
\]

for some coprime integers \(c_7\) and \(c_8\). In both cases, addition leads to an equation of type \(X^5 - Y^5 = 3^4Z^5\), which has only trivial solutions by Theorem D.
In the cases (25) and (26), we obtain, for $k = 5$,

$$a^2 - 4b^2 = (a - 2b)(a + 2b) = 2^4 \cdot 3(c_5 c_6)^5.$$ 

With (20) there follows

$$c_1^{10} - b^2 = 2^2 \cdot 3(c_5 c_6)^5.$$ 

Again, $(c_1, b) = 1$ implies that $(c_1^5 - b, c_1^5 + b) = 2$, and we conclude that

$$c_1^5 - b = 2 \cdot 3c_7^5 \quad \text{and} \quad c_1^5 + b = 2c_8^5$$ 

or

$$c_1^5 - b = 2c_7^5 \quad \text{and} \quad c_1^5 + b = 2 \cdot 3c_8^5$$ 

for some coprime integers $c_7$ and $c_8$. In both cases, addition leads to an equation of type $X^5 - Y^5 = 3Z^5$, which has only trivial solutions by Theorem D.

**Proposition 4.** Let $k \geq 2$, $a, b = b^k_1 > 0$ and $c$ be integers satisfying $(a, b) = 1$, $G_1(a, b) = 4$, $G_2(a, b) = 3$ and (5). Then $c = 0$.

**Proof.** It follows from $G_1 = (a, a^2 - 4b^2) = 4$ that $4 \mid a$; hence $2 \nmid b$ and so $4 \mid (a^2 - 4b^2)$. Then (5) implies, by virtue of $(a, a^2 - b^2) = 1$, that

$$a = 4^{k-1}c_1^k \quad \text{and} \quad (a^2 - b^2)(a^2 - 4b^2) = 4c_2^k$$ 

for some coprime integers $c_1$ and $c_2$ with odd $c_2$. Since $(a, b) = 1$ and $2 \nmid a$, we know that

$$(a - b, a + b) = (a - b, 2) = 1. \quad (31)$$

By the condition $G_2 = 3$, we may assume without loss of generality (as in the proof of Proposition 2) that

$$(a - b, a + b, a - 2b) = 3 \quad \text{and} \quad (a + b, a - 2b) = 1. \quad (32)$$

Now (30), (31), (32), $(a + b, a + 2b) = (a - b, a - 2b) = 1$ and $(a - 2b, a + 2b) = (a - 2b, 2a) = 2$ imply that

$$a + b = c_3^k \quad \text{and} \quad a - 2b = 2c_4^k. \quad (33)$$

and we necessarily have one of the following two situations:

$$a - b = 3c_5^k \quad \text{and} \quad a + 2b = 2 \cdot 3^{k-1}c_6^k \quad (34)$$

or

$$a - b = 3^{k-1}c_5^k \quad \text{and} \quad a + 2b = 2 \cdot 3c_6^k \quad (35)$$

with some pairwise coprime integers $c_3$, $c_4$, $c_5$ and $c_6$ satisfying $2 \nmid c_3 c_4 c_5 c_6$ and $3 \nmid c_3 c_4$. 

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From (33), (30) and $b = b_1^k$, we conclude that
\[ 4^{k-1}c_1^k + b_1^k = c_3^k \]  
(36)
in coprime terms. For even $k$ we consequently have $X^2 + Y^2 = Z^2 \mod 3$ with
\( 3 \mid XYZ \) by the definition of $c_1$, $b_1$ and $c_2$ and $3 \mid (a - b)$ by (35), but this
congruence cannot hold. For odd $k$, (36) is an equation of type $X^n - Y^n = 2^{n-2}(2Z)^n$ for some prime $p > 2$, unless $k = 5$, which is
the only exponent left by Theorem A or Theorem B, respectively.

For $k = 5$, we obtain in case (34) with (33)
\[ 0 = 3 \cdot 2a - 2 \cdot 3a = 3 \cdot ((a - 2b) + (a + 2b)) - 2 \cdot ((a - 2b) + 2(a + b)) \]
\[ = 3 \cdot (2c_5^5 + 2 \cdot 3c_6^5) - 2 \cdot (2c_5^5 + 2c_5^5) = 2c_5^5 + 2(3c_6^5 - 4c_3^5), \]
i.e., an equation with only trivial solutions by Theorem B. In case (35) for
$k = 5$ we obtain with (33)
\[ 0 = 3 \cdot 2a - 2 \cdot 3a = 3 \cdot ((a + b) + (a - b)) - 2 \cdot ((a - 2b) + 2(a + b)) \]
\[ = 3 \cdot (c_5^5 + 3d_5^5) - 2 \cdot (2c_5^5 + 2c_5^5) = -c_5^5 + (3c_5^5 - 4c_4^5), \]
and we have only trivial solutions by Theorem A. So the proof of Proposition
4 is complete.

Proof of the Theorem. By the transformation $x \mapsto x - 3$ equation (2) turns
into
\[ y^k = x(x^2 - 1)(x^2 - 4). \]  
(37)
Since $x$ and $y$ are rational numbers, we have $x = a/b$ and $y = c/d$ for suitable
integers $a, b > 0, c$ and $d > 0$ satisfying $(a, b) = (c, d) = 1$. We obtain from (37)
\[ \frac{c^k}{d^k} = \frac{a(a^2 - b^2)(a^2 - 4b^2)}{b^5}. \]
By virtue of $(a, b) = (c, d) = 1$ this is equivalent with
\[ c^k = a(a^2 - b^2)(a^2 - 4b^2) \quad \text{and} \quad b^5 = d^k. \]  
(38)
The second identity implies that $b = b_1^k$ for some positive integer $b_1$. Since
$(a, b) = 1$ it follows from (3) and (4) that Propositions 1–4 cover all possible
values of $G_1$ and $G_2$. Hence $c = 0$, and consequently $y = 0$ in (2), which proves
the theorem.

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