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Roger Bielawski*

Slices to sums of adjoint orbits, the Atiyah-Hitchin manifold, and Hilbert schemes of points

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Abstract: We show that the regular Slodowy slice to the sum of two semisimple adjoint orbits of $GL(n,\mathbb{C})$ is isomorphic to the deformation of the D_2-singularity if $n=2$, the Dancer deformation of the double cover of the Atiyah-Hitchin manifold if $n=3$, and to the Atiyah-Hitchin manifold itself if $n=4$. For higher n, such slices to the sum of two orbits, each having only two distinct eigenvalues, are either empty or biholomorphic to open subsets of the Hilbert scheme of points on one of the above surfaces. In particular, these open subsets of Hilbert schemes of points carry complete hyperkähler metrics. In the case of the double cover of the Atiyah-Hitchin manifold this metric turns out to be the natural L^2-metric on a hyperkähler submanifold of the monopole moduli space.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let (e,h,f) be an $\mathfrak{sl}(2,\mathbb{C})$-triple in $\mathfrak{g}^{\mathbb{C}}$. The affine subspace $S(f)=f+C(e)$ of $\mathfrak{g}^{\mathbb{C}}$, where $C(e)$ denotes the centraliser of e, is called the Slodowy slice to the nilpotent orbit of f [28]. It has the remarkable property of being transverse to any adjoint orbits it meets. Let M be a hyperkähler manifold with a tri-Hamiltonian action of G and let $\mu\colon M\to(\mathfrak{g}^{\mathbb{C}})^{\star}\simeq\mathfrak{g}^{\mathbb{C}}$ be the complex moment map for one of the complex structures. It has been shown in [4] (extending results of Kronheimer [23] on slices to nilpotent orbits) that, under mild assumptions on M, $\mu^{-1}(S(f))$ carries a natural hyperkähler structure. Moreover, as we show in the present paper, the hyperkähler metric on $\mu^{-1}(S(f))$ is complete if the original metric on M is complete.

A remarkable number of hyperkähler manifolds arise as such Slodowy slices to simpler hyperkähler manifolds:

- the moduli space of $SU(2)$-monopoles of charge k is the regular slice to $T^{\star}GL(k,\mathbb{C})$ for the $U(k)\times U(k)$-action [4] ("regular slice" means that the nilpotent orbit of f is regular);
- ALE gravitational instantons are subregular slices to regular semisimple adjoint orbits [4, 6] (this has been also rediscovered by Manolescu [24] in the case of A_{2m} ALE spaces and by Jackson [19] for other ALE spaces of types A and D);
- ALF gravitational instantons of type D_k, $k\geq 3$, are regular slices to regular semisimple adjoint orbits of $SL(k,\mathbb{C})$ for the action of $SU(2)\times SU(k-2)\subset SU(k)$ [11];
- the D_2 ALF gravitational instanton is the regular slice to the product of two semisimple adjoint orbits of $SL(2,\mathbb{C})$ [6].

As seen from these examples, slices to adjoint orbits provide a particularly rich source of hyperkähler manifolds. As far as we know, however, slices to products of two or more orbits have not been investigated, apart from the last example.

***Corresponding Author: Roger Bielawski:** Institut für Differentialgeometrie, Universität Hannover, Welfengarten 1, D-30167 Hannover

Since the complex moment map of an orbit $\mathcal{O}$ is the embedding $\mathcal{O} \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$, such a slice is the affine variety

$$\left\{ (A_1, \dots, A_k) \in \prod_{i=1}^{k} \mathcal{O}_i;\ A_1 + \dots + A_k \in S(f) \right\}.$$

It makes therefore sense to speak about a slice to the *sum* of orbits.

In the present paper we make the first step and consider regular Slodowy slices to the sum of two adjoint semisimple orbits of $GL(n, \mathbb{C})$, each orbit having only two eigenvalues with multiplicities k_i and l_i ($k_i + l_i = n$), $i = 1, 2$. They are nonempty only if $j = |k_1 - l_1| + |k_2 - l_2| \leq 2$, and in this case we obtain three series of complete hyperkähler manifolds depending on the value of $j = 0, 1, 2$. These three series turn out to be related to the first three ALF gravitational instantons of type D_k, i.e. the Atiyah-Hitchin manifold (D_0) if $j = 2$, Dancer's deformation of its double cover (D_1) if $j = 1$, and Hitchin's [16] D_2-manifold if $j = 0$. All three are affine surfaces given by respective equations

$$x^2 - zy^2 + y = 0,$$
$$x^2 - zy^2 - 1 + \alpha y = 0,$$
$$x^2 - zy^2 + x + \alpha y + \beta = 0.$$

We show that these surfaces and their hyperkähler metrics are realised as slices to the sum of two orbits in $GL(4, \mathbb{C})$ for the Atiyah-Hitchin manifold, in $GL(3, \mathbb{C})$ for its double cover, and in $GL(2, \mathbb{C})$ for the deformation of the D_2-singularity (in the last case, the complex structure, but not the metric, has been identified in [6]).

For higher n we identify the nonempty slices as open subsets of Hilbert schemes of points on these surfaces. These open subsets consist of those schemes Z of points for which the projection π onto the z-coordinate in the above equations induces an isomorphism between Z and its scheme-theoretic image $\pi(Z)$. We call this open subset the *Hilbert scheme of points transverse to* π. The construction of these goes back to Atiyah and Hitchin [1, Ch. 6], who realised spaces of based rational maps on $\mathbb{P}^1$ as such transverse Hilbert schemes. As observed by Atiyah and Hitchin, if we start with a hyperkähler 4-manifold, then we can apply the transverse Hilbert scheme construction fibrewise to its twistor space and obtain a new twistor space, which may lead to a new hyperkähler metric. This does not always work: we show (§6) that if we start with flat $\mathbb{C}^2$ and $\pi(x, y) = xy$, then the new twistor space has no twistor lines. It does however work in case of the D_0, D_1 and D_2 surfaces with $\pi(x, y, z) = z$ and we show that the hyperkähler metric on the transverse Hilbert scheme of m points obtained from the slice construction coincides with the one obtained from the fibrewise twistor construction. In the case of the double cover of the Atiyah-Hitchin manifold, we can identify this hyperkähler metric as the natural metric on certain hyperkähler submanifold of the moduli space of $SU(2)$-monopoles of charge $2m$.

Incidentally, the transverse Hilbert scheme of points on the ALE surfaces of type A_k ($k \geq 1$) and D_k ($k \geq 4$) have been identified as slices to single semisimple adjoint orbits by Seidel and Smith [27], Manolescu [24] and Jackson [19] (the identification is only as complex manifolds but I believe that going through the identification fibrewise on the twistor space will show that the hyperkähler metrics are those arising from the fibrewise Hilbert scheme construction). Hence, there arises the natural question whether the fibrewise transverse Hilbert scheme construction applied to twistor spaces of arbitrary ALF gravitational instantons also produces complete hyperkähler metrics. We show here that this is the case for type A_0, i.e. for the Taub-NUT metric (§6). Since the ALF gravitational instantons of type A_k, $k \geq 1$, arise as a Taub-NUT modification of the corresponding ALE spaces, we expect that a combination of the construction for the Taub-NUT metric and for the A_k ALE spaces will answer the question positively in this case. For ALF gravitational instantons of type D_k there is a whole unexplored world of possibilities among non-regular slices to sums of two orbits or regular slices to sums of three or more orbits and slices to orbits in other simple Lie groups.

The article is organised as follows. In the next section we recall the hyperkähler slice construction from [4]. In section 2 we discuss slices to sums of two orbits, each having only two eigenvalues, identify those pairs of orbits for which the regular slice is nonempty and define the three series of hyperkähler manifolds described above (depending on the value of $j \in \{0, 1, 2\}$). In sections 3 and 4 we identify the slices in the case of orbits in $GL(2, \mathbb{C})$, $GL(3, \mathbb{C})$, and $GL(4, \mathbb{C})$, i.e. the initial members of the three series. In sections 5 and 6 we discuss transverse Hilbert schemes of points and show that there exist complete hyperkähler metrics

on such Hilbert schemes of points for the Taub-NUT metric. In section 7 we show that nonempty slices to sums of two orbits (each orbit having only two eigenvalues) in $GL(n,\mathbb{C})$ with $n > 4$ are transverse Hilbert schemes of points on the D_0, D_1 or D_2 surface (depending on whether the value of j defined above is 2, 1 or 0). We also show there that the complete hyperkähler metrics on these, which arise from the hyperkähler slice construction, have twistor spaces obtained via the fibrewise transverse Hilbert scheme construction. Finally, in the appendix, we show that a hyperkähler slice to a complete hyperkähler manifold is again complete.

1 Slodowy slices and hyperkähler metrics

Let $\mathfrak{g}$ be a compact Lie algebra and $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ a homomorphism of Lie algebras. We extend ρ to a homomorphism of complexified Lie algebras and denote by (e,h,f) the resulting $\mathfrak{sl}(2,\mathbb{C})$-triple. The *Slodowy slice* [28] corresponding to ρ is the affine subspace $S(\rho) = f + C(e)$ of $\mathfrak{g}^{\mathbb{C}}$, where $C(e)$ denotes the centraliser of e. It has the remarkable property of being transverse to any adjoint orbits it meets.

As shown in [4], the manifold $S(\rho)\times G^{\mathbb{C}}$, where G is a compact Lie group with Lie algebra $\mathfrak{g}$, carries a natural hyperkähler metric. It is described (see Appendix) as the natural L^2-metric on the moduli space of $\mathfrak{g}$-valued solutions to Nahm's equations on the interval $(0,1]$ (rescaling the interval corresponds to rescaling the metric) with simple poles having residues determined by ρ at $t = 0$. Two solutions are identified if they differ by a gauge transformation which is identity at both $t = 0$ and $t = 1$. This hyperkähler metric on $S(\rho)\times G^{\mathbb{C}}$ admits a free tri-hamiltonian (in particular isometric) action of G (given by gauge transformations with arbitrary values at $t = 1$) and an isometric $SO(3)$-action rotating the complex structures. In particular, each complex structure of this hyperkähler manifold is that of $S(\rho)\times G^{\mathbb{C}}$. The completeness of the hyperkähler metric on $S(\rho)\times G^{\mathbb{C}}$ was not shown in [4] - an omission which we rectify in the appendix to this paper:

Theorem 1.1. *The natural hyperkähler metric on $S(\rho)\times G^{\mathbb{C}}$ is complete.*

Let now M be a hyperkähler manifold with a tri-Hamiltonian action of G. We can then form the hyperkähler quotient of $M\times S(\rho)\times G^{\mathbb{C}}$ by G which is always smooth and, in view of the above theorem, complete, if the metric on M is complete. Moreover, as shown in [4], the complex structure of this hyperkähler quotient is, under mild assumptions, easily identified:

Theorem 1.2. *[4] Let (M,g,I_1,I_2,I_3) be a hyperkähler manifold with a tri-hamiltonian action of a compact Lie group G. Let I be one of the complex structures and $\mu\colon M\to\mathfrak{g}^{\mathbb{C}}$ the corresponding I-holomorphic moment map (i.e. $\mu = \mu_2 + i\mu_3$ if $I = I_1$). Assume that the action of G extends to an I-holomorphic action of $G^{\mathbb{C}}$ and that M admits a global Kähler potential for the Kähler form $g(I\cdot,\cdot)$ which is bounded below on each $G^{\mathbb{C}}$-orbit. Then the hyperkähler quotient of $M\times S(\rho)\times G^{\mathbb{C}}$ by G is biholomorphic, with respect to the complex structure I, to $\mu^{-1}\big(S(\rho)\big)$.* □

We shall refer to the hyperkähler quotient of $M\times S(\rho)\times G^{\mathbb{C}}$ as a *hyperkähler slice to M*. In the current paper we shall be interested in the case $G = U(n)$, (e,h,f) a regular $\mathfrak{sl}(2,\mathbb{C})$-triple and M a product of flat space and adjoint $GL(n,\mathbb{C})$-orbits with their Kronheimer-Biquard-Kovalev (KBK) hyperkähler structure [9, 21–23].

In the case of slices to adjoint orbits, we need to be careful the identification of the complex-symplectic and hyperkähler quotients described in the above theorem, since the Kähler form of the KBK metric does not need to admit a global Kähler potential[1] (it may define a nonzero $H^{1,1}$-cohomology class).

The following proposition allows us to identify the complex structure of slices to arbitrary semisimple adjoint orbits.

1 KBK-metrics on a given complex adjoint orbit are parameterised by an element τ_1 of certain abelian subalgebra of $\mathfrak{g}$. Only the KBK-metric with $\tau_1 = 0$ admits a global Kähler potential.

Proposition 1.3. *Let H and G be two commuting closed subgroups of $Sp(n)$ with $G \cap H = \{1\}$ and let $\nu = (\nu_1, \nu_2, \nu_3) : \mathbb{H}^n \to \mathfrak{h} \otimes \mathbb{R}^3$ be the hyperkähler moment map for H. Let $M = \nu^{-1}(\tau)/H$ be a hyperkähler quotient. Assume that H acts freely on $Z_I = (\nu_2 + i\nu_3)^{-1}(\tau_2 + i\tau_3)$ and that the $H^{\mathbb{C}}$-orbits in Z_I are closed. Then the hyperkähler quotient of $M \times S(\rho) \times G^{\mathbb{C}}$ by G is biholomorphic, with respect to the complex structure I, to $\mu^{-1}\big(S(\rho)\big)$, where $\mu : M \to \mathfrak{g}^{\mathbb{C}}$ is the corresponding I-holomorphic moment map.*

Proof. The hyperkähler quotient of $M \times S(\rho) \times G^{\mathbb{C}}$ by G is isomorphic to the hyperkähler quotient of $V \times S(\rho) \times G^{\mathbb{C}}$ by $G \times H$. The statement is equivalent to saying that this second hyperkähler quotient is biholomorphic, with respect to the complex structure I, to the complex-symplectic quotient of $V \times S(\rho) \times G^{\mathbb{C}}$ by $G^{\mathbb{C}} \times H^{\mathbb{C}}$. The assumptions imply that all $G^{\mathbb{C}} \times H^{\mathbb{C}}$ orbits in the level set Z of the complex moment map are closed and that $G \times H$ acts freely there. Moreover V has a proper Kähler potential for the complex structure I. It follows, as in [4, §4], that Z has a global Kähler potential which is proper on each $G^{\mathbb{C}} \times H^{\mathbb{C}}$-orbit. The arguments in [13, §6.5.2] imply now that every $G^{\mathbb{C}} \times H^{\mathbb{C}}$-orbit in Z is stable and hence the hyperkähler and complex-symplectic quotients coincide.

□

Hyperkähler metrics on adjoint orbits of $GL(n, \mathbb{C})$ can be obtained, as shown by Nakajima [25] (see also [20]), as hyperkähler quotients of a quaternionic vector space by a product of unitary groups, and the assumption about the closedness of $H^{\mathbb{C}}$-orbits is fulfilled in the case of semisimple orbits. The fact that the Nakajima and the KBK hyperkähler stuctures coincide in this case is a consequence of the main result of [5].

Thus we conclude the above result holds for any product $\mathcal{O}_1 \times \cdots \times \mathcal{O}_k$ of semisimple adjoint orbits of $GL(n, \mathbb{C})$ (it also holds for nilpotent orbits, thanks to Theorem 1.2). This means that the hyperkähler slice to this product is biholomorphic to the affine variety

$$\left\{ (A_1, \ldots, A_k) \in \prod_{i=1}^{k} \mathcal{O}_i;\ A_1 + \cdots + A_k \in S(\rho) \right\}.$$

As mentioned above, we shall only consider the regular slice, i.e. the case where ρ defines an irreducible representation of $\mathfrak{su}(2)$. In this case, the Slodowy slice is isomorphic to the set $\mathcal{S}_n$ of companion matrices

$$\begin{pmatrix} 0 & \ldots & \ldots & 0 & s_n \\ 1 & \ddots & & 0 & s_{n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ddots & 0 & s_2 \\ 0 & \ldots & \ldots & 1 & s_1 \end{pmatrix}. \tag{1.1}$$

We shall denote the manifold $\mathcal{S}_n \times GL(n, \mathbb{C})$ with its natural hyperkähler metric by N_n. We shall need later a description of the twistor space of N_n, which has already been found in [7] (see the proof of Proposition 6.1 there). It is given by gluing two copies of $\mathbb{C} \times \mathcal{S}_n \times GL(n, \mathbb{C})$ with coordinates (ζ, S, g) and $(\tilde{\zeta}, \tilde{S}, \tilde{g})$ via

$$\tilde{\zeta} = \zeta^{-1}, \quad \tilde{S} = D(\zeta) S D(\zeta)^{-1}/\zeta^2, \quad \tilde{g} = g \exp(-S/\zeta) D(\zeta)^{-1}, \tag{1.2}$$

where $D(\zeta) = \operatorname{diag}\big(\zeta^{-n+1}, \zeta^{-n+3}, \ldots, \zeta^{n-1}\big)$.

2 Regular slices to sums of two orbits

We consider two semisimple adjoint orbits $\mathcal{O}_1, \mathcal{O}_2$ of $GL(n, \mathbb{C})$ both having only two different eigenvalues (i.e. orbits which are also complex symmetric spaces). After translating by multiples of the identity matrix we can assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ have eigenvalues $\mu_1, -\mu_1$ with multiplicities k_1, l_1 and $\mu_2, -\mu_2$ with multiplicities k_2, l_2, respectively. Thus $A^2 = \mu_i^2$ for $A \in \mathcal{O}_i$ and matrices A satisfying this equation fall into different orbits according to the value of $\operatorname{tr} A$.

Let us write $d_1 = k_1 - l_1$ and $d_2 = k_2 - l_2$, so that $\operatorname{tr} A = d_1\mu_1$, $\operatorname{tr} B = d_2\mu_2$. The numbers d_1 and d_2 have the same parity as n.

The regular Slodowy slice to the sum of these two orbits is the variety

$$\big(\mathcal{O}_1 + \mathcal{O}_2\big) \cap \mathcal{S}_n = \big\{(A, B) \in \mathcal{O}_1 \times \mathcal{O}_2;\ A + B \in \mathcal{S}_n\big\}, \tag{2.1}$$

where $\mathcal{S}_n$ is the set of companion matrices (1.1). As described in the previous section, this variety has a natural hyperkähler structure and it arises a complex-symplectic quotient by a free action of a Lie group and is therefore smooth. Moreover, were it nonempty, it would have dimension

$$(n^2 - k_1^2 - l_1^2) + (n^2 - k_2^2 - l_2^2) + n - n^2 = n - \frac{(d_1)^2}{2} - \frac{(d_2)^2}{2}. \tag{2.2}$$

However:

Proposition 2.1. *Suppose that* $|d_1| + |d_2| > 2$. *Then the variety* (2.1) *is empty.*

Proof. We can replace $\mathcal{S}_n$ by $f + C(e)$, i.e. by matrices with $s_{ij} = 0$ if $i > j + 1$, $s_{i+1,i} = 1$, and $s_{i,i+r}$ depending only on r. In particular $A + B \in f + C(e)$ if an only if $(A + aI) + (B + bI) \in f + C(e)$ for any $a, b \in \mathbb{C}$. Without loss of generality, we can assume that $k_i \geq l_i$, $i = 1, 2$. The matrices $A - \mu_1 I$ and $B - \mu_2 I$ have ranks $\leq l_1$, $\leq l_2$, respectively. Since their sum has rank $\geq n - 1$, the conclusion follows. □

Remark 2.2. From the point of view of hyperkähler geometry, the fact that (2.1) is empty in the case $\frac{(d_1)^2}{2} + \frac{(d_2)^2}{2} \leq n$ and $|d_1| + |d_2| > 2$ is interesting. It means that $N_n \times \mathcal{O}_1 \times \mathcal{O}_2$ is an example of a complete hyperkähler manifold with a tri-Hamiltonian action of $SU(n)$, such the image W of the hyperkähler moment map is open, but $0 \notin W$. I am not aware of any other examples of this kind.

The variety (2.1) can therefore be nonempty only if $|d_1| + |d_2| \leq 2$. Since changing μ_i to $-\mu_i$ is equivalent to replacing d_i with $-d_i$, we shall not lose generality in assuming that $d_i \geq 0$, $i = 1, 2$. Similarly, we shall not lose generality in assuming that $d_1 \geq d_2$. Furthermore, the d_i have the same parity as n, and hence there are only three possibilities:

- $n = 2m$ and $d_1 = d_2 = 0$. We shall denote the corresponding variety by $D_{2,m}(\mu_1, \mu_2)$.
- $n = 2m + 1$ and $d_1 = d_2 = 1$. The corresponding variety will be denoted by $D_{1,m}(\mu_1, \mu_2)$.
- $n = 2m + 2$ and $d_1 = 2$, $d_2 = 0$. The corresponding variety will be denoted by $D_{0,m}(\mu_1, \mu_2)$.

The reasons for this notation will become apparent. We shall also see that these varieties are nonempty for each $m \geq 1$, and therefore the dimension calculation (2.2) gives $2m$ as the dimension of each of them. Since the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ involved in the definition of $D_{2,m}(\mu_1, \mu_2)$ do not change if we change the sign of μ_1 or of μ_2, we conclude that $D_{2,m}(\mu_1, \mu_2)$ depends only on μ_1^2 and μ_2^2. Similarly $D_{0,m}(\mu_1, \mu_2) = D_{0,m}(\mu_1, -\mu_2)$. We shall see later that $D_{1,m}(\mu_1, \mu_2)$ depends only on $\mu_1 - \mu_2$ and $D_{0,m}(\mu_1, \mu_2)$ is independent (up to an affine isomorphism) of μ_1 and μ_2.

2.1 Characteristic polynomial of $A + B$

If (A, B) belong to the variety (2.1), then $S = A + B$ is a companion matrix (1.1). Recall that the characteristic polynomial $P(z) = \det(z - S)$ of a companion matrix is equal to $z^n - \sum_{i=1}^n s_i z^{n-i}$ and S is the matrix of the multiplication by z on $\mathbb{C}[z]/(P(z))$ in the basis $1, z, \ldots, z^{n-1}$.

We shall need the following algebraic result.

Proposition 2.3. *Let* A *and* B *be two* $n \times n$ *matrices satisfying* $A^2 = \mu_1^2$, $B^2 = \mu_2^2$, $\operatorname{tr} A = d_1\mu_1$, $\operatorname{tr} B = d_2\mu_2$, $\mu_1, \mu_2 \neq 0$, $d_1 \geq d_2 \geq 0$. *If* $S = A + B$ *is a regular element of* $\mathfrak{gl}(n, \mathbb{C})$, *then the characteristic polynomial of* S *has the form:*

$$\big(z - (\mu_1 + \mu_2)\big)^p \big(z - (\mu_1 - \mu_2)\big)^q \prod_{i=1}^{r} (z^2 - x_i), \tag{2.3}$$

where $p = \frac{d_1 + d_2}{2}$, $q = \frac{d_1 - d_2}{2}$, *and* $x_i \in \mathbb{C}$.

Proof. We observe that

$$\mu_1^2 = A^2 = (S-B)^2 = S^2 - SB - BS + \mu_2^2,$$

which we can write as

$$S(\frac{1}{2}S - B) + (\frac{1}{2}S - B)S = \mu_1^2 - \mu_2^2.$$

We abbreviate $Y = \frac{1}{2}S - B$ and $\tau = \mu_1^2 - \mu_2^2$, so that the equation we are concerned with is

$$SY + YS = \tau. \tag{2.4}$$

Lemma 2.4. *If $\lambda \neq 0$ is an eigenvalue of S and the corresponding eigenvector u satisfies $Yu \neq \frac{\tau}{2\lambda}u$, then $-\lambda$ is also an eigenvalue of S with the same algebraic multiplicity as λ.*

Proof. Suppose that $Su = \lambda u$. Then $SYu = -\lambda Yu + \tau u$ and it follows that if $\lambda \neq 0$, then

$$S\left(Yu - \frac{\tau}{2\lambda}u\right) = -\lambda\left(Yu - \frac{\tau}{2\lambda}u\right).$$

By assumption $-\lambda$ is also an eigenvalue of S with eigenvector $Yu - \frac{\tau}{2\lambda}u$.

Since S is regular, the geometric multiplicities of λ and $-\lambda$ are both equal to 1.

Let m be the algebraic multiplicity of λ, i.e. there is vector v such that $(S-\lambda)^{m-1}v \neq 0$ and $(S-\lambda)^m v = 0$. Denote by E_λ the kernel of $(S-\lambda)^m$, i.e. the subspace spanned by $v, (S-\lambda)v, \ldots, (S-\lambda)^{m-1}v$. The λ-eigenspace is generated by $u = (S-\lambda)^{m-1}v$ and the $-\lambda$-eigenspace by $Yu - \frac{\tau}{2\lambda}u$. Consider first the case $\tau = 0$. Then $Y(S-\lambda)^{m-1}v$ is an $-\lambda$-eigenvector and, since $Y(S-\lambda)^{m-1}v = (-S-\lambda)^{m-1}Yv$, this eigenvector belongs to $\mathrm{Im}(S+\lambda)^{m-1}$. It follows that the algebraic multiplicity of $-\lambda$ is at least as large as that of λ. By symmetry the two multiplicities are equal.

If $\tau \neq 0$, then using the fact that S^2 commutes with Y, we have for any $k \leq m$

$$\begin{aligned}(S+\lambda)^k Yv &= \sum \binom{k}{i} S^i \lambda^{k-i} Yv = \sum \binom{k}{i} Y(-S)^i \lambda^{k-i} v + \tau \sum_{i \equiv 1 \bmod 2} \binom{k}{i} \lambda^{k-i} v \\ &= Y(-S+\lambda)^k v + \tau \frac{(1+\lambda)^k - (1-\lambda)^k}{2} v.\end{aligned}$$

Setting $k = m$, we obtain, in particular, $v \in \mathrm{Im}(S+\lambda)^m$. The $-\lambda$-eigenvector $Y(S-\lambda)^{m-1}v - \frac{\tau}{2\lambda}(S-\lambda)^{m-1}v$ can be then written, thanks to the above formula, as

$$(-S-\lambda)^{m-1}Yv + (-1)^m \tau \frac{(1+\lambda)^{m-1} - (1-\lambda)^{m-1}}{2} v - \frac{\tau}{2\lambda}(S-\lambda)^{m-1}v,$$

and so it belongs $\mathrm{Im}(S+\lambda)^{m-1}$. Thus the algebraic multiplicities are equal also in this case. □

We now consider the remaining case $Yu = \frac{\tau}{2\lambda}u$.

Lemma 2.5. *If $\lambda \neq 0$ and u is a λ-eigenvector of S and $Yu = \frac{\tau}{2\lambda}u$, then u is a common eigenvector of A and B, and consequently $\lambda = \pm\mu_1 \pm \mu_2$.*

Proof. From assumption, u is a common eigenvector of $S = A + B$ and $Y = (A-B)/2$, i.e. u is a common eigenvector of A and B, with eigenvalues $\frac{1}{2}\left(\lambda + \frac{\tau}{\lambda}\right)$ and $\frac{1}{2}\left(\lambda - \frac{\tau}{\lambda}\right)$, respectively. These must be $\pm\mu_1$ and $\pm\mu_2$, so that such a λ must be equal to one of the four values $\pm\mu_1 \pm \mu_2$. □

Finally, for $\lambda = 0$ we have

Lemma 2.6. *Suppose that 0 is an eigenvalue of S. If $\tau \neq 0$ then its algebraic multiplicity is even. If $\tau = 0$, then A and B have a common eigenvector (with eigenvalues summing up to zero).*

Proof. Suppose that $\tau \neq 0$ and a nonzero v satisfies $S^m v = 0$ for an odd m. Then $S^m Yv = -YS^m v + \tau v = \tau v$, and so $S^m Yv \neq 0$ and $S^{m+1}Yv = 0$. Thus the algebraic multiplicity of $\lambda = 0$ is even. If $\tau = 0$ and $Su = 0$, then $SYu = 0$, and so $Yu = \rho u$ for some ρ, since the geometric multiplicity of each eigenvalue is 1. □

Taken together, these three lemmata imply that the characteristic polynomial of S has the form

$$\big(z-\delta_1(\mu_1+\mu_2)\big)^{m_1}\,\big(z-\delta_2(\mu_1-\mu_2)\big)^{m_2}\prod_{i=1}^{r}(z^2-x_i),$$

where $\delta_j=\pm1$. The trace of S is then $\delta_1 m_1(\mu_1+\mu_2)+\delta_2 m_2(\mu_1-\mu_2)$. On the other hand

$$\operatorname{tr} S=d_1\mu_1+d_2\mu_2=\frac{d_1+d_2}{2}(\mu_1+\mu_2)+\frac{d_1-d_2}{2}(\mu_1-\mu_2).$$

According to Proposition 2.1, we have only three possibilities (under the assumption $d_1\geq d_2$): $d_1=d_2=0$, $d_1=d_2=1$, or $d_1=2$, $d_2=0$. Comparing the two formulae for the trace proves the proposition. □

2.2 The varieties $D_{k,m}(\mu_1,\mu_2)$

We return to the case when (2.1) is nonempty, i.e. to the varieties $D_{k,m}(\mu_1,\mu_2)$, $k=0,1,2$. Their elements can be alternatively described as pairs of matrices (S,Y), where S is of the form (1.1), satisfying the equations

$$SY+YS=\mu_1^2-\mu_2^2,\quad (Y-S/2)^2=\mu_2^2, \tag{2.5}$$

together with $\operatorname{tr} S=d_1\mu_1+d_2\mu_2$, $\operatorname{tr} Y=\frac{1}{2}(d_1\mu_1-d_2\mu_2)$, where $(d_1,d_2)=(0,0)$ if $k=2$, $(d_1,d_2)=(1,1)$ if $k=1$, and $(d_1,d_2)=(2,0)$ if $k=0$.

Proposition 2.3 implies that the characteristic polynomial $P(z)$ of S is of the form

$$P(z)=(z-\mu_1-\mu_2)^{\epsilon_1}(z-\mu_1+\mu_2)^{\epsilon_2}Q(z),$$

where (ϵ_1,ϵ_2) is equal to $(0,0)$ if $k=2$, to $(1,0)$ if $k=1$, and to $(1,1)$ if $k=0$. The polynomial $Q(z)$ has degree $2m$ with coefficients of all odd powers of z equal to zero.

If $(\epsilon_1,\epsilon_2)\neq(0,0)$, we can simplify the form of S and Y:

Proposition 2.7. *(i) The affine variety $D_{1,m}(\mu_1,\mu_2)$ is canonically isomorphic to the variety of pairs of matrices (S,Y) satisfying equations* (2.5) *and having the following form:*

$$S=\begin{pmatrix}S_0 & 0\\ e & \mu_1+\mu_2\end{pmatrix},\quad Y=\begin{pmatrix}Y_0 & 0\\ v & (\mu_1-\mu_2)/2\end{pmatrix}, \tag{2.6}$$

where S_0 is the companion matrix (1.1) *to $Q(z)$ (in particular all s_{2i+1} are equal to zero) and $e=(0,\dots,0,1)$.*

(ii) The affine variety $D_{0,m}(\mu_1,\mu_2)$ is canonically isomorphic to the variety of pairs of matrices (S,Y) satisfying equations (2.5) *and having the following form:*

$$S=\begin{pmatrix}S_0 & 0 & 0\\ e & \mu_1+\mu_2 & 0\\ e & 0 & \mu_1-\mu_2\end{pmatrix},\quad Y=\begin{pmatrix}Y_0 & 0 & 0\\ v_1 & (\mu_1-\mu_2)/2 & 0\\ v_2 & 0 & (\mu_1+\mu_2)/2\end{pmatrix}, \tag{2.7}$$

where S_0 and e are as in case (i).

Proof. The affine isomorphism is given by the change of basis of $\mathbb{C}[z]/(P(z))$ from $1,z,\dots,z^{2m}$ to $1,\dots,z^{2m-1},Q(z)$ in case (i), and from $1,z,\dots,z^{2m+1}$ to $1,\dots,z^{2m-1},(2\mu_2)^{-1}(z-\mu_1+\mu_2)Q(z),-(2\mu_2)^{-1}(z-\mu_1-\mu_2)Q(z)$ in case (ii). Since S corresponds to multiplication by z on $\mathbb{C}[z]/(P(z))$, it has in both cases the stated form in the new basis. Furthermore, Lemmata 2.5, 2.6, and equations (2.5) imply that Y also has the stated form. □

Observe now that the blocks S_0 and Y_0 in the above proposition still satisfy equations (2.5) and $\operatorname{tr} S_0=\operatorname{tr} Y_0=0$. These blocks are therefore elements of the variety $D_{2,m}(\mu_1,\mu_2)$. Thus $(S,Y)\mapsto(S_0,Y_0)$ defines canonical

holomorphic maps from $D_{1,m}(\mu_1,\mu_2)$ or $D_{0,m}(\mu_1,\mu_2)$ to $D_{2,m}(\mu_1,\mu_2)$. Let us write ϕ for this holomorphic map from $D_{1,m}(\mu_1,\mu_2)$ to $D_{0,m}(\mu_1,\mu_2)$. In the case of $D_{0,m}$ we can associate to (S,Y) the minor matrices

$$S_1=\begin{pmatrix} S_0 & 0\\ e & \mu_1+\mu_2\end{pmatrix},\quad Y_1=\begin{pmatrix} Y_0 & 0\\ v_1 & (\mu_1-\mu_2)/2\end{pmatrix}, \tag{2.8}$$

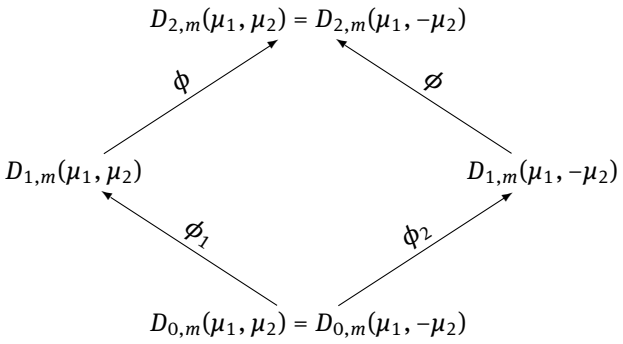
or

$$S_2=\begin{pmatrix} S_0 & 0\\ e & \mu_1-\mu_2\end{pmatrix},\quad Y_2=\begin{pmatrix} Y_0 & 0\\ v_1 & (\mu_1+\mu_2)/2\end{pmatrix}, \tag{2.9}$$

and so obtain canonical holomorphic maps

$$\phi_1\colon D_{0,m}(\mu_1,\mu_2)\to D_{1,m}(\mu_1,\mu_2),\quad \phi_2\colon D_{0,m}(\mu_1,\mu_2)\to D_{1,m}(\mu_1,-\mu_2). \tag{2.10}$$

We have a commutative diagram:



Remark 2.8. Observe that a pair (S,Y) of the form (2.7) satisfies equations (2.5) if and only if both pairs (S_1,Y_1) and (S_2,Y_2) satisfy these equations. Thus $D_{0,m}(\mu_1,\mu_2)$ is the fibred product $D_{1,m}(\mu_1,\mu_2)\times_{D_{2,m}(\mu_1,\mu_2)} D_{1,m}(\mu_1,-\mu_2)$.

3 The case $m=1$ - complex structures

We are going to identify the surfaces $D_{2,1}$, $D_{1,1}$ and $D_{0,1}$, i.e. we consider the variety (2.1) for $n=2,3,4$.

n=2. (cf. [6]) In this case $\operatorname{tr}A=0$ and $\operatorname{tr}B=0$, so that both S and Y are also traceless.

Write

$$S=\begin{pmatrix} 0 & x\\ 1 & 0\end{pmatrix},\quad Y=\begin{pmatrix} a & b\\ c & -a\end{pmatrix}.$$

The equation $SY+YS=\tau$ reduces to $b+cx=\tau$, and the equation $\mu_2^2=B^2=(Y-S/2)^2$ becomes then

$$a^2-\big(x(c+\frac{1}{2})-\tau\big)(c-\frac{1}{2})=\mu_2^2,$$

or

$$a^2-xc^2+\frac{1}{4}x+(\mu_1^2-\mu_2^2)c-\frac{1}{2}(\mu_1^2+\mu_2^2)=0, \tag{3.1}$$

which is a deformation of the D_2-singularity.

n=3. We can assume that $A\sim(\mu_1,\mu_1,-\mu_1)$ and $B\sim(\mu_2,\mu_2,-\mu_2)$. Proposition 2.7 implies that our variety is isomorphic to the variety of pairs (S,Y) satisfying (2.5) and of the form

$$S=\begin{pmatrix} 0 & x & 0\\ 1 & 0 & 0\\ 0 & 1 & \mu_1+\mu_2\end{pmatrix},\quad Y=\begin{pmatrix} a & b & 0\\ c & -a & 0\\ y & z & (\mu_1-\mu_2)/2\end{pmatrix}. \tag{3.2}$$

The equation $SY + YS = \tau$ gives again $b + cx = \tau$ and two more equations:

$$c + (\mu_1 + \mu_2)y + z = 0, \tag{3.3}$$

$$-a + (\mu_1 + \mu_2)z + yx + (\mu_1 - \mu_2)/2 = 0. \tag{3.4}$$

These allow us to express a, b and c as functions of y and z and substituting into $\mu_2^2 = B^2 = (Y - S/2)^2$ gives the coordinate ring of the variety (2.1), which is easily seen to be determined by a single equation

$$y^2x - z^2 + \frac{1}{4} + (\mu_1 - \mu_2)y = 0. \tag{3.5}$$

This is the Dancer deformation of the D_1-manifold [12].

n=4. This time $A \sim (\mu_1, \mu_1, \mu_1, -\mu_1)$ and $B \sim (\mu_2, \mu_2, -\mu_2, -\mu_2)$. Proposition 2.7 implies that our variety is isomorphic to the variety of pairs (S, Y) satisfying (2.5) and of the form

$$S = \begin{pmatrix} 0 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \mu_1 + \mu_2 & 0 \\ 0 & 1 & 0 & \mu_1 - \mu_2 \end{pmatrix},$$

$$Y = \begin{pmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ y & z & (\mu_1 - \mu_2)/2 & 0 \\ u & v & 0 & (\mu_1 + \mu_2)/2 \end{pmatrix}.$$

By Remark 2.8 (S, Y) satisfies (2.5) if and only if the 3×3 minor matrices obtained by removing the 3rd (resp. the 4th) row and the 3rd (resp. the 4th) column satisfy these equations, i.e. belong to $D_{1,1}(\mu_1, -\mu_2)$ (resp. $D_{1,1}(\mu_1, +\mu_2)$). The equation $SY + YS = \tau$ gives equations (3.3)-(3.4) and the following two equations:

$$c + (\mu_1 - \mu_2)u + v = 0, \tag{3.6}$$

$$-a + (\mu_1 - \mu_2)v + ux + (\mu_1 + \mu_2)/2 = 0. \tag{3.7}$$

The equation $(Y - \frac{1}{2}S)^2 = \mu_2^2$ for the two minor matrices is equivalent to equation (3.5), i.e.:

$$0 = y^2x - z^2 + \frac{1}{4} + (\mu_1 - \mu_2)y = \left(x - (\mu_1 - \mu_2)^2\right)y^2 + \left((\mu_1 - \mu_2)y + \frac{1}{2}\right)^2 - z^2, \tag{3.8}$$

$$0 = u^2x - v^2 + \frac{1}{4} + (\mu_1 + \mu_2)u = \left(x - (\mu_1 + \mu_2)^2\right)u^2 + \left((\mu_1 + \mu_2)u + \frac{1}{2}\right)^2 - v^2. \tag{3.9}$$

We can combine (3.6)-(3.7) with (3.3)-(3.4) to obtain two equations for u, v, y, z, which we row reduce to

$$z - v + (\mu_1 + \mu_2)y - (\mu_1 - \mu_2)u = 0, \tag{3.10}$$

$$2\mu_2 v + (x - (\mu_1 + \mu_2)^2)y + (-x + \mu_1^2 - \mu_2^2)u = \mu_2. \tag{3.11}$$

Thus both z and v are functions of y and u, and the solutions are:

$$z = \frac{1}{2\mu_2}\left((\mu_1 - \mu_2)^2 - x\right)(y - u) + (\mu_1 - \mu_2)y + \frac{1}{2},$$

$$v = \frac{1}{2\mu_2}\left((\mu_1 + \mu_2)^2 - x\right)(y - u) + (\mu_1 + \mu_2)u + \frac{1}{2},$$

Write $\alpha_\pm = \mu_1 \pm \mu_2$ and introduce a new variable $w = (y - u)/2\mu_2$. The last two equations become then $z = (\alpha_-^2 - x)w + \alpha_- y + \frac{1}{2}$ and $v = (\alpha_+^2 - x)w + \alpha_+ u + \frac{1}{2}$.

Substituting into (3.8)-(3.9) we obtain

$$\left(x-\alpha_-^2\right)\left((y+\alpha_- w)^2+w-xw^2\right)=0,$$

$$\left(x-\alpha_+^2\right)\left((u+\alpha_+ w)^2+w-xw^2\right)=0.$$

Since $\mu_1,\mu_2\neq 0$, it follows that $\alpha_-\neq\alpha_+$ and, consequently, we can eliminate $\left(x-\alpha_-^2\right)$ and $\left(x-\alpha_+^2\right)$ from these equations.

We easily check that

$$y+\alpha_- w=u+\alpha_+ w=\frac{(\mu_1+\mu_2)y-(\mu_1-\mu_2)u}{2\mu_2} \tag{3.12}$$

and, therefore, the defining ideal of $D_{0,1}(\mu_1,\mu_2)$ is generated by the single polynomial $(y+\alpha_- w)^2+w-xw^2$.

Making the final substitution $t=y+\alpha_- w=u+\alpha_+ w$ we obtain the equation of the Atiyah-Hitchin manifold (also known as the D_0-manifold):

$$t^2-xw^2+w=0. \tag{3.13}$$

Remark 3.1. If $n=2$ or $n=3$, then we can extend the above identifications of complex structures to the case when one or both of μ_1,μ_2 is equal to zero. Indeed, this means replacing a semisimple orbit with a minimal nilpotent orbit. Since the elements of the minimal nilpotent orbit still satisfy the quadratic equation $A^2=0$, all the considerations from this and from the previous section remain valid. Observe that for $n=3$ we get a smooth manifold for any value of μ_1,μ_2, since, if $A+B$ is a regular element of $\mathfrak{gl}(n,\mathbb{C})$, then neither A nor B can be the singular point 0 of the nilpotent variety.

If $n=4$, the above identification also extends to the case $\mu_1=0$ or $\mu_2=0$, i.e. when one or both orbits become nilpotent. In this case, however, the canonical form (2.7) is no longer valid: the lower-right 2×2-blocks of S and Y become nilpotent matrices. One needs to repeat the above computation separately for S, Y of this form.

4 The case $m=1$ - the metrics

As discussed in section 1 the manifolds $D_{k,m}(\mu_1,\mu_2)$ carry a natural hyperkähler structure arising from their construction as a moduli space of solutions to Nahm's equations. In the case of $k=1$ there exist of course well-known hyperkähler metrics on the complex surfaces found in the previous section: the ALF gravitational instantons [1, 16]. Since our description of these spaces is very different from previously known ones, we are going to show that the metrics on $D_{k,1}$, $k=2,1,0$, are the standard ones.

The hyperkähler metric on $D_{k,m}(\mu_1,\mu_2)$ is obtained as a hyperkähler quotient of the manifold N_n ($n=2m+2-k$), described in §1, and a pair of semisimple adjoint orbits with their Kronheimer-Biquard-Kovalev metrics. Both adjoint orbits admit a family of hyperkähler structures parameterised by real numbers: for any $r_1\in\mathbb{R}$ there is a hyperkähler structures on $\mathcal{O}_1$ such that the generic complex structure I_ζ ($\zeta\in\mathbb{P}^1$) is that the adjoint orbit with eigenvalues $\pm(\mu_1+2r_1\zeta-\overline{\mu}_1\zeta^2)$ with multiplicities k_1 and l_1, and analogously for $\mathcal{O}_2$.

The corresponding hyperkähler metric on $D_{k,m}(\mu_1,\mu_2)$ can be found by identifying real sections of the twistor space which is obtained as the fibrewise complex-symplectic quotient of the product of fibred product of the twistor space of N_n and the twistor spaces of the orbits $\mathcal{O}_1,\mathcal{O}_2$. The twistor space of N_n was described at the end of section 1. The sections of the twistor space of the orbits are simply $n\times n$ matrices with $\mathcal{O}(2)$-entries (belonging to the orbit of $(\mu_i+2r_i\zeta-\overline{\mu}_i\zeta^2)I_{k_i}\oplus-(\mu_1+2r_i\zeta-\overline{\mu}_i\zeta^2)I_{l_i}$ for each ζ, $i=1,2$).

If $X_1\in\mathcal{O}_1$, $X_2\in\mathcal{O}_2$ and $(S,g)\in N_n$, then the complex moment map equation is $X_1+X_2=gSg^{-1}$. In terms of our original equation $A+B=S$, $A+B=g^{-1}(X_1+X_2)g$, i.e. $Y=\frac{1}{2}g^{-1}(X_1-X_2)g$. It follows that

$$\tilde{S}=D(\zeta)(S/\zeta^2)D(\zeta)^{-1},\quad \tilde{Y}=D(\zeta)\exp(S/\zeta)(Y/\zeta^2)\exp(-S/\zeta)D(\zeta)^{-1}. \tag{4.1}$$

We shall now identify the twistor lines for our surfaces $D_{k,1}(\mu_1,\mu_2)$, $k=2,1,0$. We begin with the special case $D_{2,1}(\mu/2,\mu/2)=D_{2,1}(\mu/2,-\mu/2)$.

$D_{2,1}(\mu/2, \mu/2)$.

As long as $x \neq 0$, we can diagonalise S to $\mathrm{diag}(\lambda, -\lambda)$. The corresponding Y (which anticommutes with S) is of the form $\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$. The relationship between u, v and a, c in §3 is

$$a = \frac{u+v}{2}, \quad c = \frac{u-v}{2\lambda}.$$

To compute the transition functions in (1.2) for diagonal S, we compare the the new S, g to $S_{\mathrm{old}}, g_{\mathrm{old}}$ in (1.2): $S = hS_{\mathrm{old}}h^{-1}$, $g = g_{\mathrm{old}}h^{-1}$ and similarly $\tilde{S} = hD(\zeta)^{-1}\tilde{S}_{\mathrm{old}}D(\zeta)h^{-1}/\zeta^2$, $\tilde{g} = \tilde{g}_{\mathrm{old}}D(\zeta)h^{-1}$, where h is the diagonalising matrix. It follows that $\tilde{S} = S/\zeta^2$, $\tilde{g} = g\exp(-S/\zeta)$ and, consequently, the transition functions in (4.1) are $\tilde{S} = S/\zeta^2$, $\tilde{Y} = \exp(S/\zeta)(Y/\zeta^2)\exp(-S/\zeta)$. Therefore the transition functions for λ, u, v are $\tilde{\lambda} = \frac{\lambda}{\zeta^2}$, $\tilde{u} = e^{2\lambda/\zeta}\frac{u}{\zeta^2}$, $\tilde{v} = e^{-2\lambda/\zeta}\frac{v}{\zeta^2}$, and it follows that in terms of $\tilde{a}, \tilde{c}$ we have:

$$\tilde{a} = \frac{\tilde{u}+\tilde{v}}{2} = \frac{1}{\zeta^2}\left(a\cosh\frac{2\lambda}{\zeta} + c\lambda\sinh\frac{2\lambda}{\zeta}\right),$$

$$\tilde{c} = \frac{\tilde{u}-\tilde{v}}{2\tilde{\lambda}} = \frac{a}{\lambda}\sinh\frac{2\lambda}{\zeta} + c\cosh\frac{2\lambda}{\zeta}.$$

In particular observe that $a - \lambda c$ is a section of $L^2(2)$, where L^2 is a line bundle over $|\mathcal{O}(2)| \simeq T\mathbb{P}^1$ with transition function $\exp(-2\lambda/\zeta)$ from $\zeta \neq \infty$ to $\zeta \neq 0$. The equation (3.1) can be written as $(a+\lambda c)(a-\lambda c) = \frac{1}{4}(\mu^2 - x)$, so that twistor sections are given by a section $x(\zeta)$ of $\mathcal{O}(4)$ and a section $s(\zeta, \lambda)$ of $L^2(2)$ over the elliptic curve $\lambda^2 = x(\zeta)$ satisfying $s(\zeta, \lambda)s(\zeta, -\lambda) = \frac{1}{4}(\mu^2(\zeta) - x(\zeta))$ (satisfying the natural reality conditions). With a bit of extra care (cf. [8, p.308-309]) one can show that 2 zeros of $s(\zeta, \lambda)$ occur at the intersection of the elliptic curve with $\lambda = \mu(\zeta)$ and the two other zeros at the intersection of the the elliptic curve with $\lambda = -\mu(\zeta)$.

$D_{2,1}(\mu_1, \mu_2)$, $\mu_1 \neq \pm\mu_2$.

Following Hitchin [16], let us first rewrite (3.1). After multiplying it by x (this will not affect the determination of twistor lines), we can rewrite it as

$$xa^2 - (xc - \tau/2)^2 + \frac{1}{4}\left(x - (\mu_1 - \mu_2)^2\right)\left(x - (\mu_1 + \mu_2)^2\right) = 0,$$

where $\tau = \mu_1^2 - \mu_2^2$.

After introducing a new variable $w = xc - \tau/2$ we obtain the equation $w^2 - xa^2 = \frac{1}{4}\left(x - (\mu_1 - \mu_2)^2\right)\left(x - (\mu_1 + \mu_2)^2\right)$.

We now proceed as in the case $\mu_1 = \pm\mu_2$.

This time a point (S, Y) of our variety satisfies $SY + YS = \tau$ with $\tau \neq 0$. If $x \neq 0$, we can write $Y = \frac{1}{2\tau}S^{-1} + Y'$ with Y' anticommuting with S as before. Thus after diagonalising S, Y becomes

$$\begin{pmatrix} \frac{1}{2\lambda\tau} & u \\ v & \frac{-1}{2\lambda\tau} \end{pmatrix},$$

and, consequently,

$$a = \frac{u+v}{2}, \quad c = \frac{u-v}{2\lambda} + \frac{1}{2\lambda^2\tau},$$

so that

$$a = \frac{u+v}{2}, \quad w = \lambda\frac{u-v}{2}.$$

The same computation as above shows that the transition functions for a and w are

$$\tilde{a} = \frac{\tilde{u}+\tilde{v}}{2} = \frac{1}{\zeta^2}\left(a\cosh\frac{2\lambda}{\zeta} + w\lambda\sinh\frac{2\lambda}{\zeta}\right),$$

$$\tilde{w} = \tilde{\lambda}\frac{\tilde{u}-\tilde{v}}{2} = a\lambda\sinh\frac{2\lambda}{\zeta} + w\lambda^2\cosh\frac{2\lambda}{\zeta}.$$

Thus $w - \lambda a$ is a section $s(\zeta, \lambda)$ of $L^2(4)$ over the elliptic curve $\lambda^2 = x(\zeta)$ satisfying $s(\zeta, \lambda)s(\zeta, -\lambda) = \frac{1}{4}\left(x(\zeta) - (\mu_1(\zeta) - \mu_2(\zeta))^2\right)\left(x(\zeta) - (\mu_1(\zeta) + \mu_2(\zeta))^2\right)$ (again satisfying the natural reality conditions). Again, the precise location of the zeros of $s(\zeta, \lambda)$ can be determined.

This is Hitchin's description of the hyperkähler metric on the deformation of the D_2-singularity [16].

$D_{1,1}(\mu/2, \mu/2)$. We proceed similarly. In the basis in which $S = \mathrm{diag}(\lambda, -\lambda, \mu)$, Y is of the form

$$\begin{pmatrix} 0 & u & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The transition matrices between this basis and the one in which S and Y have the form (3.2) are easily computed using corresponding bases of $\mathbb{C}[z]/((z^2 - \lambda^2)(z - \mu))$. They are:

$$V = \begin{pmatrix} 1 & \lambda & 0 \\ 1 & -\lambda & 0 \\ 1 & \mu & \mu^2 - \lambda^2 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2\lambda} & -\frac{1}{2\lambda} & 0 \\ \frac{1}{2\lambda(\lambda-\mu)} & \frac{1}{2\lambda(\lambda+\mu)} & \frac{1}{\mu^2-\lambda^2} \end{pmatrix}.$$

We have:

$$\begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ y & z & 0 \end{pmatrix} = V^{-1} \begin{pmatrix} 0 & u & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V,$$

and consequently:

$$y = \frac{v}{2\lambda(\lambda + \mu)} + \frac{u}{2\lambda(\lambda - \mu)}, \quad z = \frac{v}{2(\lambda + \mu)} - \frac{u}{2(\lambda - \mu)}. \tag{4.2}$$

The transition functions for λ, u, v are the same as in the D_2-case and from an analogous calculation we obtain:

$$\tilde{y} = \zeta^2 \left(y \cosh \frac{2\lambda}{\zeta} - \frac{z}{\lambda} \sinh \frac{2\lambda}{\zeta} \right), \quad \tilde{z} = z \cosh \frac{2\lambda}{\zeta} - y\lambda \sinh \frac{2\lambda}{\zeta}.$$

In particular, observe that $z + \lambda y$ is a section of L^2. The equation of our D_1-manifold can be written as $(z + \lambda y)(z - \lambda y) = \frac{1}{4}$, so that twistor sections are given by a section $x(\zeta)$ of $\mathcal{O}(4)$ and a nonvanishing section $s(\zeta, \lambda)$ of L^2 over the elliptic curve $\lambda^2 = x(\zeta)$ satisfying $s(\zeta, \lambda)s(\zeta, -\lambda) = \frac{1}{4}$. This is the twistor description of the Atiyah-Hitchin metric on the D_1-manifold (it is easy to see that the reality conditions are the same; one can also check that the symplectic forms coincide, but already the above information determines the hypercomplex structure, hence the Levi-Civita connection, hence the metric up to a constant factor).

$D_{1,1}(\mu_1, \mu_2)$, $\mu_1 \neq \mu_2$. As in the D_2-case we multiply (3.5) by x and rewrite it as

$$\left(yx + (\mu_1 - \mu_2)/2\right)^2 - xz^2 = \frac{1}{4}(\mu_1 - \mu_2)^2. \tag{4.3}$$

With a new variable $q = yx + (\mu_1 - \mu_2)/2$ this becomes $q^2 - xz^2 = \frac{1}{4}(\mu_1 - \mu_2)^2$. Proceeding as in the D_2-case we recover Chalmers' description [10] of twistor lines for Dancer's deformation of the Atiyah-Hitchin metric.

$D_{0,1}(\mu_1, \mu_2)$. We first discuss the twistor description of the Atiyah-Hitchin metric. The Atiyah-Hitchin manifold arises as the quotient of the D_1-manifold $z^2 - y^2x = 1/4$ by the involution $(z, y) \mapsto (-z, -y)$. Setting $s = z^2$, $t = 4yz$, $w = -4y^2$, and substituting $s = (1 - wx)/4$ into $t^2 = -4ws$, we obtain $t^2 - xw^2 + w = 0$. Multiply this last equation by w and rewrite it as $(xw - 1/2)^2 - xt^2 = 1/4$, or $(xw - 1/2 + t\sqrt{x})(xw - 1/2 - t\sqrt{x}) = 1/4$. Now observe that

$$xw - 1/2 \pm t\sqrt{x} = -2(z \pm y\sqrt{x})^2.$$

The description of the twistor space of $D_{1,1}(\mu, \mu)$ implies that $z \pm y\sqrt{x}$ is a nonvanishing section section of $L^{\pm 2}$, so that $xw - 1/2 \pm t\sqrt{x}$ becomes a nonvanishing section of $L^{\pm 4}$. We shall therefore show that the hyperkähler metric on $D_{0,1}(\mu_1, \mu_2)$ is the Atiyah-Hitchin metric, if we can show that t and w arising via the calculation in §3 do make $xw - 1/2 \pm t\sqrt{x}$ a holomorphic section of $L^{\pm 4}$ (or any $L^{\pm c}$, $c > 0$, which corresponds to rescaling the metric).

According to §3, restricting the 4×4 matrices S, Y to appropriate 3×3 minor matrices produces two D_1-manifolds with equations

$$y^2x - z^2 + \frac{1}{4} + (\mu_1 - \mu_2)y = 0, \quad u^2x - v^2 + \frac{1}{4} + (\mu_1 + \mu_2)u = 0.$$

We can rewrite these as in (4.3), i.e.:

$$q^2 - xz^2 = \frac{1}{4}(\mu_1 - \mu_2)^2, \quad p^2 - xv^2 = \frac{1}{4}(\mu_1 + \mu_2)^2$$

where $q = yx + (\mu_1 - \mu_2)/2$ and $p = ux + (\mu_1 + \mu_2)/2$. It follows from the twistor description of $D_{1,1}(\mu_1, \mu_2)$ that $q \pm z\sqrt{x}$ and $p \pm v\sqrt{x}$ are sections of $L^{\pm 2}(2)$. Using the definition of t given in (3.12) and $w = (y - u)/2\mu_2$, together with (3.10), we obtain

$$\frac{q + z\sqrt{x} - p - v\sqrt{x}}{2\mu_2} = \frac{(q - p) + (z - v)\sqrt{x}}{2\mu_2} = xw - 1/2 - t\sqrt{x}$$

and

$$\frac{q - z\sqrt{x} - p + v\sqrt{x}}{2\mu_2} = \frac{(q - p) - (z - v)\sqrt{x}}{2\mu_2} = xw - 1/2 + t\sqrt{x}.$$

Thus both $xw - 1/2 + t\sqrt{x}$ and $xw - 1/2 - t\sqrt{x}$ are meromorphic sections of L^2 and L^{-2}, respectively, with poles possible only at zeros of $\mu_2(\zeta)$. Equation (3.13) and the calculation at the beginning of this subsection imply, however, that their product is constant (equal to $1/4$). Therefore $xw - 1/2 \pm t\sqrt{x}$ are holomorphic sections of $L^{\pm 2}$ and the metric on $D_{0,1}(\mu_1, \mu_2)$ is the Atiyah-Hitchin metric.

Remark 4.1. Due to general facts about isometries between real-analytic Riemannian manifolds, the natural maps $\phi\colon D_{1,m}(\mu_1, \mu_2) \to D_{2,m}(\mu_1, \mu_2)$ and $\phi_1\colon D_{0,m}(\mu_1, \mu_2) \to D_{1,m}(\mu_1, \mu_2)$, described in §2.2, cannot be (even local) isometries. On the other hand, it appears that the first of these maps is well-defined on each fibre of the twistor space and maps real sections to real sections. To see that this is not so, observe that computing v from (4.2) yields $v = (\lambda + \mu)(z + \lambda y)$ (and similarly $u = (\mu - \lambda)(z - \lambda y)$), so that the map ϕ sends a real section of the twistor space of $D_{1,1}(\mu/2, \mu/2)$ to a pair of sections $s_\pm$ of $L^{\pm 2}(2)$ over the elliptic curve $\lambda^2 = x(\zeta)$ such that all zeros of s_+ (resp. s_-) occur at the intersection points with $\lambda = -\mu(\zeta)$ (resp. $\lambda = \mu(\zeta)$). This real section does *not* arise from the hyperkähler structure of $D_{2,1}(\mu/2,\ mu/2)$ (see the remark at the end of the calculation of the metric for $D_{2,1}(\mu/2, \mu/2)$)).

The second of these maps, $\phi_1\colon D_{0,m}(\mu_1, \mu_2) \to D_{1,m}(\mu_1, \mu_2)$, is well-defined only if $\mu_2 \neq 0$, which means that is not well-defined on all fibres of the twistor space.

5 Hilbert schemes of points transverse to a projection

Let X be a complex manifold, C a 1-dimensional complex manifold, and $\pi\colon X \to C$ a surjective holomorphic map. Following Atiyah and Hitchin [1] we define an open subset $X^{[n]}_\pi$ of the Hilbert scheme $X^{[n]}$ of n points in X as consisting of these 0-dimensional subschemes Z of length n for which $\pi_*\mathcal{O}_Z$ is a cyclic $\mathcal{O}_C$ sheaf.

Equivalently, $\pi\colon Z \to \pi(Z)$ is an isomorphism onto its scheme-theoretic image.

Let $Z \in X^{[n]}_\pi$ and suppose that $\pi\colon Z \to \pi(Z)$ is an isomorphism, i.e. $\pi_*\mathcal{O}_Z = \mathcal{O}_{\pi(Z)}$.

If t_0 is a point in $\pi(Z)$ and t is a local coordinate on C, then a neighbourhood of t_0 is of the form $\mathbb{C}[t]/(t^m)$ for some $m \leq n$. Since $\pi\colon Z \to \pi(Z)$ is an isomorphism, there exists a morphism $\phi\colon \mathbb{C}[t]/(t^m) \to X$, the image of which is the corresponding open subset of Z. Such a morphism ϕ is an equivalence class of local smooth sections of π truncated up to order m. Globally, Z is an equivalence class of local smooth sections of π, defined in a neighbourhood of $\pi(Z)$, where the equivalence relation is as above. In other words, if s is a local section on $U \subset \mathbb{C}$, then the defining ideal of Z is $I_s + \pi^* I_{\pi(Z)}$, where I_s is the defining ideal of $s(U)$. We can formulate this as follows:

Proposition 5.1. *(cf. [1, Ch.6]) $X^{[n]}_\pi$ parameterises 0-dimensional subschemes Z of X of length n such that $I_Z = I_s + \pi^* I_T$, where $T \in C^{[n]} \simeq S^n C$, s is a local section of π defined in a neigbourhood U of T, and I_s is the defining ideal of $s(U)$.*

Remark 5.2. It follows from the construction that π induces a surjective holomorphic map $\pi^{[n]}\colon X_\pi^{[n]} \to C^{[n]} \simeq S^n(C)$.

We shall call $X_\pi^{[n]}$ *Hilbert scheme of n points transverse to* π or simply *transverse Hilbert scheme*.

Suppose now that $X \subset \mathbb{C}^k$ is an affine variety and $\pi\colon X \to \mathbb{C}$ is a restriction of a polynomial $P\colon \mathbb{C}^k \to \mathbb{C}$ to X. Without loss of generality we may assume that $P(w_1, \dots, w_k) = w_k$. Indeed, if this is not the case, then we can view X as an affine variety in $\mathbb{C}^{k+1}$ by adding the equation $w_{k+1} = P(w_1, \dots, w_k)$, so that π becomes the projection onto the last coordinate.

Let us therefore assume that $\pi(w_1, \dots, w_k) = w_k$ and write $z = w_k$.

The ideal of a $T \in \mathbb{C}^{[n]} \simeq S^n\mathbb{C}$ is generated by a monic polynomial $q(z) \in \mathbb{C}[z]$ of degree n.

If $Z \in X_\pi^{[n]}$ and $\pi(Z) = T$, then $q(z) \in I_Z$. If $q(z) = \prod_{i=1}^s (z - z_i)^{m_i}$, then a local section of π in a neighbourhood of z_i modulo $q(z)$ is of the form $(w_1^i(z), \dots, w_{k-1}^i(z), z)$, where $w_j^i(z)$ is polynomial of degree $\leq m_i - 1$. These local polynomials can be combined, using Lagrange interpolation, to give polynomials $w_1(z), \dots, w_{k-1}(z)$ of degree at most $n-1$ satisfying $w_j(z) = w_j^i(z) \mod (z - z_i)^{m_i}$.

It follows that the defining ideal of a point Z in $X_\pi^{[n]}$ is given by

$$\big(q(z), f(w_1(z), \dots, w_{k-1}(z), z)\big)_{f \in I},$$

where $q(z)$ is a monic polynomial of degree n, I is the defining ideal of X, and $w_1(z), \dots, w_{k-1}(z)$ are polynomials of degree at most $n-1$. Thus $X_\pi^{[n]}$ is an affine variety in $\mathbb{C}^{kn}$, with coordinates p_{ij}, q_j, $1 \leq i \leq k-1$, $0 \leq j \leq n-1$ defined by equations

$$f\left(\sum_{j=0}^{n-1} p_{1j} z^j, \dots, \sum_{j=0}^{n-1} p_{n-1,j} z^j, z\right) = 0 \mod \left(z^n - \sum_{j=0}^{n-1} q_j z^j\right) \tag{5.1}$$

for every $f \in I$.

Example 5.3. Let $X = \mathbb{C}^* \times \mathbb{C}$ and π the projection onto the second coordinate. We can view X as the affine variety $\{(x, y, z) \in \mathbb{C}^3; xy = 1\}$ with $\pi(x, y, z) = z$.

According to the above description $X_\pi^{[n]}$ is the variety of triples of polynomials $x(z)$, $y(z)$, $q(z)$ of degrees $n-1$, $n-1$ and n and q monic such that $x(z)y(z) = 1 \mod q(z)$. In other words $x(z)$ (or $y(z)$) does not vanish at any of the roots of $q(z)$ and, consequently, $X_\pi^{[n]}$ is isomorphic to the space of based rational maps of degree n (cf. [1]).

Example 5.4. Let X be the double cover of the Atiyah-Hitchin manifold, i.e. an affine surface in $\mathbb{C}^3$ defined by the equation $x^2 - zy^2 = 1$, $\pi(x, y, z) = z$. Again, $X_\pi^{[n]}$ is the variety of triples of polynomials $x(z)$, $y(z)$, $q(z)$ of degrees $n-1$, $n-1$ and n and q monic, such that $x(z)$, $y(z)$, z satisfy the defining equation modulo $q(z)$. Alternatively, consider the quadratic extension $z = u^2$, so that the defining equation becomes $(x+uy)(x-uy) = 1$. If $x(z)$ and $y(z)$ are polynomials of degree $n-1$, then $x(z) \pm uy(z) = x(u^2) \pm uy(u^2)$ and $q(z) = q(u^2)$. In other words, $q(u^2)$ is a polynomial of degree $2n$ with all coefficients of odd powers equal to 0 and $p(u) = x(u^2) + uy(u^2)$ is a polynomial of degree $2n-1$ satisfying $p(u)p(-u) = 1 \mod q(u^2)$. Thus $X_\pi^{[n]}$ is the space of degree $2n$ based rational maps of the form $p(u)/q(u^2)$ with p satisfying the above condition.

5.1 Hyperkähler metrics

As observed by Atiyah and Hitchin [1], the definition of $X_\pi^{[n]}$ is well-suited to twistorial constructions of new hyperkähler metrics (or hypercomplex structures) from old ones. Namely, let Z be the twistor space of a hypercomplex or hyperkähler manifold. In particular, Z comes equipped with a holomorphic submersion $p\colon Z \to \mathbb{P}^1$ and an antiholomorphic involution (real structure) σ covering the antipodal map. Suppose that Z also admits a holomorphic map π to the total space of the line bundle $\mathcal{O}(2r)$ which preserves the fibres over $\mathbb{P}^1$ and is compatible with the real structures, where the canonical real structure of $\mathcal{O}(2r)$ is

$$\left(\zeta, \eta\Big(\frac{\partial}{\partial\zeta}\Big)^{\otimes r}\right) \mapsto \left(-1/\bar\zeta, (-1)^r \frac{\bar\eta}{\bar\zeta^{2r}}\Big(\frac{\partial}{\partial\zeta}\Big)^{\otimes r}\right).$$

Assuming that π is surjective, we can apply the transverse Hilbert scheme construction fibrewise and obtain a new twistor space $Z_\pi^{[n]}$, which also fibres over $\mathbb{P}^1$ and has an induced real structure. Moreover, if $\dim Z = 2$ and Z had a fibrewise $\mathcal{O}(2)$-twisted symplectic form, then so does $Z_\pi^{[n]}$ [2].

From the description of $X_\pi^{[n]}$, given above, it is clear that a section s of $p\colon Z_\pi^{[n]} \to \mathbb{P}^1$ corresponds to a degree n curve C in Z such that $\pi_{|C}\colon C \to \pi(C)$ is an isomorphism, or, equivalently, a curve of degree n in the total space of the line bundle $\mathcal{O}(2r)$ together with a lift to Z. Moreover, the results of [6] imply that the normal bundle of $s(\mathbb{P}^1)$ is the sum of $\mathcal{O}(1)$-s if and only if the normal bundle N of C in Z satisfies the condition $H^0(C, N(-2)) = H^1(C, N(-2)) = 0$.

Example 5.5. Applying this construction to the twistor space of $X = \mathbb{C}^\star \times \mathbb{C}$ (cf. Example 5.3) produces the natural complete hyperkähler metric on the moduli space $\mathcal{M}_n$ of framed euclidean $SU(2)$-monopoles of charge n [1]. On the other hand applying the construction to the double cover of the Atiyah-Hitchin manifold as in Example 5.4 produces a totally geodesic submanifold of $\mathcal{M}_{2n}$, consisting of monopoles which are symmetric about the origin (in particular centred) and have total phase equal to 1. This follows easily by considering the effect of the involution $x \to -x$ in $\mathbb{R}^3$ on the twistor space of monopoles (cf. [17]).

6 Example: hyperkähler metrics on $\left(\mathbb{C}^2\right)_{\pi}^{[n]}$

We consider $X = \mathbb{C}^2$ and $\pi(x, y) = xy$. As described in the previous section, we view X as the variety

$$\{(x, y, z) \in \mathbb{C}^3;\ xy = z\},$$

and $\pi(x, y, z) = z$. The variety $\left(\mathbb{C}^2\right)_\pi^{[n]}$ is then a $2n$-dimensional affine variety in $\mathbb{C}^{3n}$ defined by equations

$$\left(\sum_{j=0}^{n-1} x_j z^j\right)\left(\sum_{j=0}^{n-1} y_j z^j\right) = z \mod \left(z^n - \sum_{j=0}^{n-1} q_j z^j\right). \tag{6.1}$$

For example, the defining equations of $\left(\mathbb{C}^2\right)_\pi^{[2]}$ are

$$x_0 y_0 + x_1 y_1 q_0 = 0,$$

$$x_1 y_0 + x_0 y_1 + x_1 y_1 q_1 = 1.$$

We recall Nakajima's description of the Hilbert scheme of n points in $\mathbb{C}^2$ [26]: $\left(\mathbb{C}^2\right)^{[n]}$ is isomorphic to the the quotient by $GL(n, \mathbb{C})$ of triples (B_1, B_2, i) consisting of commuting $n \times n$ matrices B_1, B_2 and an $i \in \mathbb{C}^n$ which satisfy the following stability condition: there is no proper subspace V of $\mathbb{C}^n$ containing i and invariant under B_1 and B_2. The defining ideal of the corresponding subscheme Z given by

$$I_Z = \{f \in \mathbb{C}[x, y];\ f(B_1, B_2) = 0\}.$$

The defining ideal of $\pi(Z)$ consists of functions $g \in \mathbb{C}[z]$ such that $\pi^\star g = 0$, i.e. $g(B_1 B_2) = 0$. We have an embedding $\mathcal{O}_{\pi(Z)} \hookrightarrow \pi_\star \mathcal{O}_Z$, $g \mapsto g(xy)$, which is an isomorphism exactly then, when $\dim \mathcal{O}_{\pi(Z)} = n$, which means that $B_1 B_2$ is a regular element of $\mathfrak{gl}(n, \mathbb{C})$.

We conclude therefore that $\left(\mathbb{C}^2\right)_\pi^{[n]}$ consists of $GL(n, \mathbb{C})$-orbits of triples (B_1, B_2, i) as above and such that $B_1 B_2$ is a regular element of $\mathfrak{gl}(n, \mathbb{C})$. Using again the isomorphism between Z and $\pi(Z)$ we also conclude that i is a cyclic vector for $B_1 B_2$. Every conjugacy class of regular matrices contains a unique companion matrix S, i.e. a matrix of the form (1.1).

If $B_1 B_2 = S$, then we can conjugate B_1 and B_2 by an element of the centraliser of S in order to make the vector i equal to $e_1 = (1, 0, \ldots, 0)^T$. The pair (S, e_1) has trivial stabiliser and we have thus shown:

Proposition 6.1. *The variety $\left(\mathbb{C}^2\right)_\pi^{[n]}$ is isomorphic to the variety of triples (B_1, B_2, S) of $n \times n$ matrices, such that S is of the form* (1.1)*, B_1,B_2 commute, and $B_1 B_2 = S$.* □

To recover the description of $\left(\mathbb{C}^2\right)^{[n]}_{\pi}$ given in (6.1), we observe that both B_1 and B_2 commute with the regular element of $\mathfrak{gl}(n,\mathbb{C})$ S, i.e. we can uniquely write:

$$B_1 = \sum_{i=0}^{n-1} x_i S^i, \quad B_2 = \sum_{i=0}^{n-1} y_i S^i,$$

and since $S^k = \sum_{i=0}^{n-1} q_i S^i$, the equation $B_1 B_2 = S$ is equivalent to (6.1).

$\mathbb{C}^2$ carries at least two complete hyperkähler metrics: the flat one and the Taub-NUT. We can therefore apply the fibrewise construction described in section 5.1 to both of them and try to obtain new hyperkähler metrics in higher dimensions.

The twistor space Z of the flat metric on $\mathbb{R}^4$ is the total space of $\mathcal{O}(1)\oplus\mathcal{O}(1)$ over $\mathbb{P}^1$. The map $\pi(x,y) = xy$ induces a projection to $\mathcal{O}(2)$ given by fibrewise multiplication. As discussed above, a section of $p\colon Z^{[k]}_{\pi} \to \mathbb{P}^1$ would correspond to a degree n curve in $|\mathcal{O}(2)|$ which can be lifted to $|\mathcal{O}(1)\oplus\mathcal{O}(1)| \simeq \mathbb{P}^3\backslash\mathbb{P}^1$. A degree n curve in $|\mathcal{O}(2)|$ has genus $(n-1)^2$ and its lift would be a degree n curve in $\mathbb{P}^3$ having such a genus. There are, however, no such curves owing to a classification result of Hartshorne [14]. Thus we conclude that $p\colon Z^{[n]}_{\pi} \to \mathbb{P}^1$ has no sections and we do not obtain new hyperkähler metrics.

6.1 Taub-NUT

The twistor space of the Taub-NUT metric is described for example in [3, pp. 393–395]. There is actually a family of such metrics depending on a positive real parameter c with the twistor space defined as

$$Z_c = \{(x,y) \in L^c(1)\oplus L^{-c}(1); xy = z\},$$

where L^c is a line bundle over $|\mathcal{O}(2)| \simeq T\mathbb{P}^1$ with transition function $\exp(-c\eta/\zeta)$ from $\zeta \neq \infty$ to $\zeta \neq 0$, and z is the tautological section of $\rho^*\mathcal{O}(2)$ over $|\mathcal{O}(2)|$, where $\rho\colon |\mathcal{O}(2)| \to \mathbb{P}^1$ is the projection.

It follows that a section of $p\colon (Z_c)^{[n]}_{\pi} \to \mathbb{P}^1$ corresponds to a degree n curve C in $|\mathcal{O}(2)|$ which can be lifted to Z_c, i.e. to a pair of sections s_1 of $L^c(1)_{|C}$ and s_2 of $L^{-c}(1)_{|C}$ such that $(s_1)+(s_2) = (z)$. It is a priori unclear that such sections exist for any n and, even if they do, that there is a σ-invariant family of them defining a complete hyperkähler metric. We shall now show that this is so by giving a construction of a complete hyperkähler metric on $\left(\mathbb{C}^2\right)^{[n]}_{\pi}$ as a hyperkähler quotient.

We observe namely that Proposition 6.1 is actually a description of $\left(\mathbb{C}^2\right)^{[n]}_{\pi}$ as a complex-symplectic quotient and it suggests what the hyperkähler quotient should be. We begin with the space V of $n\times n$ quaternionic matrices and two copies of N_n (defined in §1), but with metric rescaled by c (equivalently: we consider solutions to Nahm's equations on $(0,c]$ rather than on $(0,1]$). V has two tri-hamiltonian $U(n)$ actions and N_n also has a tri-Hamiltonian action of $U(n)$. The hyperkähler quotient M_n of $V\times N_n\times N_n$ by $U(n)\times U(n)$ is the hyperkähler slice to V as described in section 1. It is therefore a smooth and complete hyperkähler manifold. It remains to identify its complex structures. The complex structure of V is that of pairs B_1, B_2 of complex matrices with the two $\mathfrak{gl}(k,\mathbb{C})$-valued moment maps given by B_1B_2 and B_2B_1. The complex structure of N_n is that of pairs (S,g), S having the form (1.1) and $g \in GL(n,\mathbb{C})$, with the moment map gSg^{-1}. It follows from the results of [4] (recalled in §1) that M_n is biholomorphic to the variety of quadruples (B_1,B_2,S_1,S_2), with S_1, S_2 of the form (1.1) and satisfying $B_1B_2 = S_1$, $B_2B_1 = S_2$. Since the characteristic polynomials of B_1B_2 and B_2B_1 are the same, it follows that $S_1 = S_2$ and B_1, B_2 commute. Thus M_n as a complex manifold is isomorphic to $\left(\mathbb{C}^2\right)^{[n]}_{\pi}$.

We observe that in case $n = 1$, $V = \mathbb{H}$ and $N_1 = S^1\times\mathbb{R}^3$ so that the hyperkähler quotient M_1 is the Taub-NUT metric on $\mathbb{C}^2$. It is straightforward to check, given the description of the twistor space of N_n in §1, that the twistor space of M_n is isomorphic to $(Z_c)^{[n]}_{\pi}$.

Remark 6.2. Just as the Taub-NUT metric itself, the transverse Hilbert scheme of n points on it also admits a tri-Hamiltonian circle action. In the case $n = 2$ we can then perform the hyperkähler quotient by this circle and obtain again a hyperkähler 4-manifold. It is easy to compute from equations (7.4) that this is again a deformation of the D_2-singularity.

7 Hyperkähler metrics on $(D_k)^{[m]}_{\pi}$.

Recall the varieties $D_{k,m}(\mu_1,\mu_2)$ defined in section 2.2. These are regular slices to sums of two orbits and carry, therefore, natural complete hyperkähler metrics. In the case $m=1$ we have identified them as the D_0, D_1 and D_2 ALF gravitational instantons. We are now going to prove that $D_{k,m}(\mu_1,\mu_2)$ is the Hilbert scheme $\big(D_{k,1}(\mu_1,\mu_2)\big)^{[m]}_{\pi}$ of m points on $D_{k,1}(\mu_1,\mu_2)$ transverse to the projection onto the x-coordinate (in equations (3.1), (3.5) and (3.13)).

$D_{2,m}(\mu_1,\mu_2)$.

Let (S,Y) be an element of $D_{2,m}(\mu_1,\mu_2)$. The characteristic polynomial $P(z)$ of S is, thanks to Proposition 2.3, of the form $\prod_{i=1}^m(z^2-x_i)$. If the x_i are distinct, we can conjugate S to a block-diagonal form, with 2×2 blocks

$$S_i=\begin{pmatrix}0 & x_i\\ 1 & 0\end{pmatrix}. \tag{7.1}$$

Viewing S as multiplication by z on $\mathbb{C}[z]/(P(z))$, this corresponds to a change of basis from $1,\dots,z^{2m-1}$ to

$$f_1(z),zf_1(z),f_2(z),zf_2(z),\dots,f_m(z),zf_m(z),\quad f_i(z)=\frac{\prod_{j\neq i}(z^2-x_j)}{\prod_{j\neq i}(x_i-x_j)}. \tag{7.2}$$

In this basis Y becomes also block-diagonal with 2×2 blocks Y_i. Indeed, the equation $SY+YS=\tau$, implies that any 2×2 minor matrix Y^{ij} of Y of the form $\begin{pmatrix}y_{2i-1,2j-1} & y_{2i-1,2j}\\ y_{2i,2j-1} & y_{2i,2j}\end{pmatrix}$ satisfies the equation $S_iY^{ij}+Y^{ij}S_j=\delta_{ij}\tau$. Since $x_i\neq x_j$ for $i\neq j$, $Y^{ij}=0$ for $i\neq j$. The diagonal blocks (S_i,Y_i) belong to $D_2(\mu_1,\mu_2)$ and, consequently, the open dense subset $D_{2,m}(\mu_1,\mu_2)^o$ of $D_{2,m}(\mu_1,\mu_2)$, where the x_i are distinct is isomorphic to

$$\big\{(p_1,\dots,p_m)\in D_2(\mu_1,\mu_2);\ p_i=(a_i,c_i,x_i),\ \ \forall_{i\neq j}x_i\neq x_j\big\}/\Sigma_m. \tag{7.3}$$

Each Y_k, $k=1,\dots,m$, is of the form

$$Y_k=\begin{pmatrix}a_k & \tau-x_kc_k\\ c_k & -a_k\end{pmatrix},$$

for some complex numbers a_k and c_k. Let $a(x)$ and $c(x)$ be polynomials of degrees $\leq m-1$ such that $a(x_k)=a_k$, $c(x_k)=c_k$, $k=1,\dots,m$. Recalling the equation of $D_{2,1}(\mu_1,\mu_2)$ from the previous section, we conclude that the polynomials $a(x)$ and $c(x)$ satisfy the equation

$$a(x)^2-xc(x)^2+\frac14x+(\mu_1^2-\mu_2^2)c(x)-\frac12(\mu_1^2+\mu_2^2)=0\qquad \mathrm{mod}\ q(x) \tag{7.4}$$

where $q(x)=\prod(x-x_k)$ whenever the x_k are distinct. Let T be the variety defined by this equation and let T^o be its open subset where the x_i are distinct. Thus we have a biholomorphism $T^0\to D_{2,m}(\mu_1,\mu_2)^o$. To describe what happens if some x_i coalesce, we observe first that the change of basis matrix from $f_1,\dots,f_m,zf_1,\dots,zf_m$ to $1,\dots,z^{2m-1}$ is

$$\begin{pmatrix}V(x_1,\dots,x_m) & 0\\ 0 & V(x_1,\dots,x_m)\end{pmatrix}$$

where $V=V(x_1,\dots,x_m)$ is the Vandermonde matrix, i.e. $V_{ij}=x_i^{j-1}$. The inverse of the Vandermonde matrix has entries

$$\big(V^{-1}\big)_{ij}=(-1)^{m-i}\frac{e_{m-i}(x_1,\dots,\widehat{x_j},\dots,x_m)}{\prod_{k\neq j}(x_j-x_k)},$$

where e_l denotes the l-th elementary symmetric polynomial (with $e_0=1$). It follows that the Y corresponding to the $(Y_i)_{i=1,\dots,m}$ has entries given by

$$\sum_{k=1}^m(V^{-1})_{ik}Y_kV_{kj}. \tag{7.5}$$

We can compute these entries by observing that, if $u_1, \dots, u_m \in \mathbb{C}$, then the j-th column of $V^{-1}\,\mathrm{diag}(u_1, \dots, u_m)V$ is given by the coefficients of the polynomial $p_j(x)$ of degree $\leq m-1$ satisfying $p_j(x_k) = u_k x_k^{j-1}$. Thus the map $T^0 \to D_{2,m}(\mu_1, \mu_2)^o$ extends to a biholomorphism $T \to D_{2,m}(\mu_1, \mu_2)$ and conclude that:

Theorem 7.1. *The variety $D_{2,m}(\mu_1, \mu_2)$ is given by a monic polynomial $q(x)$ of degree m, and two polynomials $a(x)$, $c(x)$ of degrees at most $m-1$, such that (7.4) is satisfied.*

Comparing with (5.1) we conclude:

Corollary 7.2. *The variety $D_{2,m}(\mu_1, \mu_2)$ is isomorphic to the transverse Hilbert scheme $\big(D_{2,1}(\mu_1, \mu_2)\big)^{[m]}_{\pi}$ of m points on the deformation of the D_2-singularity defined by the equation*

$$a^2 - xc^2 + \frac{1}{4}x + (\mu_1^2 - \mu_2^2)c - \frac{1}{2}(\mu_1^2 + \mu_2^2) = 0,$$

with $\pi(a, c, x) = x$.

We can also conclude from the above proof that the hyperkähler metric on $D_{2,m}(\mu_1, \mu_2)$ is the one given by the fibrewise transverse Hilbert scheme construction, described in §5.1, applied to the twistor space of $D_{2,1}(\mu_1, \mu_2)$. Indeed, in the case $\mu_1 = \mu_2$, applying the above calculations fibrewise, shows that $q(x)$ defines a curve C of degree m in the total space of $\mathcal{O}(4)$ over $\mathbb{P}^1$ and $a(x) + c(x)\sqrt{x}$ defines a section $s(\zeta, x)$ of $L^2(2)$ over C such that $(s(\zeta, x)) + (s(\zeta, -x)) = (\mu^2(\zeta) - x$. Comparing with the description of the twistor space of $D_{2,1}(\mu, \mu)$, given in §4, proves the claim. Similarly, in the case $\mu_1 \neq \mu_2$, $xc(x) - \tau(x)/2 - a(x)\sqrt{x}$ defines an appropriate section of $L^2(4)$, and again, the cmparison with §4 shows that both constructions produce the same hyperkähler metric.

$D_{1,m}(\mu_1, \mu_2)$ and ***$D_{0,m}(\mu_1, \mu_2)$***. We proceed similarly. Let (S, Y) be an element of either of these varieties in the canonical form described in Proposition 2.7. The characteristic polynomial $Q(z)$ of S_0 still has the form $\prod_{i=1}^m (z^2 - x_i)$ and, if we assume that the x_i are distinct, we can pass from the basis of $\mathbb{C}[z]/(P(z))$ described in the proof of that Proposition to the basis consisting of polynomials

$$f_1(z), zf_1(z), f_2(z), zf_2(z), \dots, f_m(z), zf_m(z), \quad f_i(z) = \prod_{j\neq i}(z^2 - x_j), \tag{7.6}$$

plus $f_{2m+1} = Q(z)$ in the $D_{1,m}$-case (resp. $f_{2m+1} = (2\mu_2)^{-1}(z - \mu_1 + \mu_2)Q(z)$, $f_{2m+2} = -(2\mu_2)^{-1}(z - \mu_1 - \mu_2)Q(z)$ in the $D_{0,m}$-case). In this new basis S and Y are still of the form (2.6) or (2.7) with S_0 block-diagonal with blocks (7.1) and the covector e is equal to $(0, 1, 0, 1, \dots, 0, 1)$. The matrix Y_0 is then, owing to the argument given above for $D_{2,m}(\mu_1, \mu_2)$, also block-diagonal. It follows that the minor matrices consisting of a single block of S_0, the corresponding entries of the last row (or the last two rows in the $D_{0,m}$-case) and the corresponding entries of the last column (or, again, last two columns in the $D_{0,m}$-case), and the analogous minor matrix for Y are elements of $D_{1,1}(\mu_1, \mu_2)$ or $D_{0,1}(\mu_1, \mu_2)$. Now, analogously to the $D_{2,m}$-case, we change the basis back to the one given in the proof of Proposition 2.7 and conclude that $D_{1,m}(\mu_1, \mu_2)$ or $D_{0,m}(\mu_1, \mu_2)$ are described by the same affine equations which define $D_{1,1}(\mu_1, \mu_2)$ or $D_{0,1}(\mu_1, \mu_2)$ but this time in $\mathbb{C}[x]/(q(x)$, where $q(x) = \prod_{i=1}^m (x - x_i)$. Comparing with (5.1) and with example 5.4 yields:

Theorem 7.3. *The variety $D_{j,m}(\mu_1, \mu_2)$, $j = 0, 1$, is isomorphic to the transverse Hilbert scheme $\big(D_{j,1}(\mu_1, \mu_2)\big)^{[m]}_{\pi}$ of m points on the D_j-surface defined by equation (3.5) or (3.13) with π being the projection onto the x-coordinate. In particular, $D_{1,m}(\mu, \mu)$ is biholomorphic to the space of rational maps of degree $2m$ of the form $p(u)/q(u^2)$, $\deg p = 2m - 1$, $\deg q = m$, and satisfying $p(u)p(-u) = 1 \mod q(u^2)$.*

Once again, we can go through above calculations fibrewise on the twistor space of the D_1- or D_0-surface, and conclude that the hyperkähler metric obtained from the slice construction of [4] coincides with the one obtained from the fibrewise transverse Hilbert scheme construction described in §5.1. In particular, example 5.5 shows:

Corollary 7.4. *The hyperkähler manifold $D_{1,m}(\mu,\mu)$ is isometric to a totally geodesic submanifold of the moduli space of monopoles of charge $2m$, consisting of monopoles invariant under the involution $x \mapsto -x$ in $\mathbb{R}^3$ and with total phase equal to* 1.

A Completeness of hyperkähler slices

Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, and $\langle\ ,\ \rangle$ an invariant scalar product on $\mathfrak{g}$. Let $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ be a homomorphism of Lie algebras, and write α_i, $i = 1, 2, 3$ for the images of the standard generators of $\mathfrak{su}(2)$.

We consider quadruples of $\mathfrak{g}$-valued smooth functions $T_i(t)$ on $(0, 1]$ which satisfy the Nahm equations, i.e. $\dot{T}_1 + [T_0, T_1] = [T_2, T_3]$ and two further equations given by cyclic permutations of indices 1, 2, 3. The group of gauge transformations smooth on $(0, 1]$ acts on the set of solutions. The boundary conditions we impose at $t = 0$ are as follows:

(i) T_0 is smooth at $t = 0$,

(ii) there exists a smooth gauge transformation with $g(0) = 1$, such that the resulting T_1, T_2, T_3 have simple poles with residues α_i and $T_i(t) - \alpha_i/t$ are analytic in powers of $t^{1/2}$ (powers of t if $\operatorname{ad}\rho$ has only even weights).

The group $\mathcal{G}$ of smooth gauge transformations $g\colon [0, 1] \to G$, $g(0) = g(1) = 1$, acts on the set $\mathcal{Z}(\rho)$ of solutions having the above boundary conditions and the quotient is a finite-dimensional smooth manifold, which we denote by $N(\rho)$, and which is diffeomorphic to $S(\rho)\times G^{\mathbb{C}}$ (see §1 and [4]). The natural L^2-metric (with respect to $\langle\ ,\ \rangle$) on the infinite-dimensional manifold $\mathcal{Z}(\rho)$ is preserved by the group $\mathcal{G}$ and it induces a Riemannian metric on $N(\rho)$. This metric is hyperkähler and it is the one used in the hyperkähler slice construction of Theorem 1.2. There are two tri-hamiltonian group actions on $N(\rho)$: the group G acts by allowing gauge transformations with arbitrary values at $t = 1$; the group $H \subset G$, the Lie algebra of which is the centraliser of $\rho(\mathfrak{su}(2))$, acts by allowing gauge transformations $g(t)$ with $g(0) \in H$.

We are going to prove:

Theorem A.1. *The natural L^2-metric on $N(\rho)$ is complete.*

Proof. Our first goal is a suitable description of $N(\rho)$ as an infinite-dimensional hyperkähler quotient. We start with an affine space $\mathcal{M}(\rho)$ of quadruples $(X_0, S_1 + X_1, S_2 + X_2, S_3 + X_3)$ of $\mathfrak{g}$-valued functions on $(0, 1]$, where $S_i(t) = \frac{\alpha_i}{t}$ and $X_j \in L^2\big((0,1),\mathfrak{g}\big)$, $j = 0, 1, 2, 3$. It is a flat hyperkähler Hilbert manifold (modelled on $L^2\big((0,1),\mathfrak{g}\big)\otimes\mathbb{R}^4$) consisting of $\mathfrak{g}$-valued quadruples (T_0, T_1, T_2, T_3) with prescribed boundary behaviour. The relevant group $\mathcal{G}'$ of gauge transformation has the Lie algebra consisting of maps $\phi\colon [0, 1] \to \mathfrak{g}$ of class $W^{1,2}$ satisfying $\phi(0) = \phi(1) = 0$. The corresponding fundamental vector field $\tilde{\phi}$ is

$$\Big(-\dot{\phi} + [\phi, T_0], [\phi, T_1], [\phi, T_2], [\phi, T_3]\Big).$$

Since $\phi(t)$ belongs to $W^{1,2}(0, 1)$, it has the form $\int_0^t \kappa(\tau)d\tau$ for an L^2-function κ. The Hardy inequality implies then that $\frac{\phi}{t}$ is square-integrable, so that the vector field $\tilde{\phi}$ is indeed in $L^2\big((0,1),\mathfrak{g}\big)\otimes\mathbb{R}^4$. The action of $\mathcal{G}'$ is free, proper, and isometric. These conditions suffice to conclude that the hyperkähler quotient of $\mathcal{M}(\rho)$ by the Hilbert Lie group $\mathcal{G}'$ is a hyperkähler manifold. The zero-level set $\mathcal{Z}(\rho)'$ of the hyperkähler moment map consists of weak solutions to Nahm's equations.

We first need to show that the moduli space of weak solutions to Nahm's equations in $\mathcal{M}(\rho)$ modulo the group the Hilbert Lie group $\mathcal{G}'$ is isometric to $N(\rho)$. For this, it suffices to show that every $\mathcal{G}'$-orbit contains a smooth solution $(X_0, \dots, X_3)$, and since the X_0-component can be made equal to zero, using a $W^{1,2}$-transformation with $g(0) = 1$, it is enough to show that for any weak solution $(0, S_1 + X_1, S_2 + X_2, S_3 + X_3)$ to Nahm's equations in $\mathcal{M}_\rho$ the functions $X_i(t)$ have the prescribed regularity on $[0, 1]$.

We can rewrite the Nahm's equations as an equation on $X = (X_1, X_2, X_3)$ of the form

$$\frac{dX}{dt} = \frac{1}{t}A(X) + Q(X, X), \tag{A.1}$$

where

$$A(X) = \begin{pmatrix} 0 & -\operatorname{ad}\alpha_3 & \operatorname{ad}\alpha_2 \\ \operatorname{ad}\alpha_3 & 0 & -\operatorname{ad}\alpha_1 \\ -\operatorname{ad}\alpha_2 & \operatorname{ad}\alpha_1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

and Q is a bilinear operator with constant coefficients. Multiplying both sides by t and writing this equation as an equation for $\frac{d(tX)}{dt}$, we conclude that the L^1-norm of $\frac{d(tX)}{dt}$ is bounded. It follows that the C^0-norm of $tX(t)$ is bounded and equation (A.1) implies, by iteration, that $X(t)$ is smooth on $(0, 1]$. In order to prove the required regularity at $t = 0$, we observe that the smoothness on $(0, 1]$ suffices to conclude that the solution to Nahm's equations corresponds to a linear flow of 1-dimensional sheaves on $T\mathbb{P}^1$ with fixed scheme-theoretic support ("spectral curve"). This in turn implies as in [18, p. 624–625] that the entries of X are of the form $t^\lambda w(t)$, where λ is an eigenvalue of the operator A and w is analytic in t. Since the eigenvalues of A are multiples of $1/2$, it follows that if $\lambda < 0$ and $w(0) \neq 0$, then X is not in L^2 and we conclude that the X_i must have the regularity required in $N(\rho)$.

In order to prove completeness of $N(\rho)$, we need the following lemma.

Lemma A.2. *Any smooth curve $\gamma\colon [0, a) \to N(\rho)$ can be lifted to a horizontal curve in $\mathcal{Z}(\rho)'$.*

Proof. Since $N(\rho)$ can be alternatively described as the space of weak solutions to Nahm's equations modulo $W^{1,2}$ gauge transformations *or* the space of solutions satisfying the $N(\rho)$-regularity conditions modulo smooth gauge transformations, we can find a lift $T\colon [0, a) \to \mathcal{Z}(\rho)$ of γ. We seek a smooth map $g\colon [0, a) \to \mathcal{G}$ such that $(g(s).T(s))'$ is horizontal for each s. This means that $T'(s) + (g^{-1}(s)g'(s))\tilde{}$ is horizontal for each s, where $\tilde{\phi}$ denotes the fundamental vector field corresponding to a $\phi \in \operatorname{Lie}\mathcal{G}$. If we write $T(s) = (T_0(t, s), T_1(t, s), T_2(t, s), T_3(t, s))$ and $T'(s) = (t_0(t, s), t_1(t, s), t_2(t, s), t_3(t, s))$, then a ϕ such that $T'(s) + \tilde{\phi}$ is horizontal is a solution of the following linear differential equation

$$\ddot{\phi} - 2[\dot{\phi}, T_0] - [\phi, T_0] + \sum_{i=1}^{3}[T_i, [T_i, \phi]] = \dot{t}_0 + \sum_{i=0}^{3}[T_i, t_i],$$

with $\phi(0) = \phi(1) = 0$ and the dot denoting derivation with respect to t. The solution is a smooth map $\phi\colon [0, 1] \times [0, a) \to \mathfrak{g}$. We can solve for each t the linear equation $g^{-1}g' = \phi$ in the Lie group G and this produces the desired curve $g\colon [0, a) \to \mathcal{G}$. □

To finish the proof of completeness, let $m\colon [0, 1) \to N(\rho)$ be a smooth curve of finite length and let $\tilde{m}(t)$ be its horizontal lift to $Z(\rho)'$.

This lift also has finite length and therefore there is a sequence of $\tilde{m}(t_i)$, $t_i \to 1$, converging to a limit T^∞ in $\mathcal{M}(\rho)$. This limit is weak solution to Nahm's equations and hence $T^\infty \in \mathcal{Z}(\rho)'$. Since Riemannian submersions between Hilbert manifolds shorten distances, m_k converges in $N(\rho)$ to the equivalence class of T^∞. □

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