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Hidden symmetries of deformed oscillators

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Abstract

We associate with each simple Lie algebra a system of second-order differential equations invariant under a non-compact real form of the corresponding Lie group. In the limit of a contraction to a Schrödinger algebra, these equations reduce to a system of ordinary harmonic oscillators. We provide two clarifying examples of such deformed oscillators: one system invariant under SO(2, 3) transformations, and another system featuring $G_{2(2)}$ symmetry. The construction of invariant actions requires adding semi-dynamical degrees of freedom; we illustrate the algorithm with the two examples mentioned.
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1. Introduction

It is widely believed that integrability of a mechanical system is related with a high degree of (usually hidden) symmetry. Identifying such symmetry for a given system may be very complicated, even in the simplest cases, like in harmonic oscillators. The inverse task – constructing a system possessing a given symmetry – seems to be more simple, since there are many ways to find its equations of motion. One of them is the method of nonlinear realizations [1,2], equipped

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with the inverse Higgs phenomenon [3]. For constructing a system of equations with a given symmetry, all one needs is the symmetry group together with the stability subgroup, which acts linearly on the mechanical coordinates.

Our recent paper [4] applies nonlinear realizations to the Schrödinger and ℓ-conformal Galilei algebra. These symmetries give rise to a system of ordinary harmonic oscillators and their higher-derivative (in time) extensions known as conformal Pais–Uhlenbeck oscillators [5,6]. Next to the one-dimensional conformal algebra $so(1,2) \sim su(1,1)$, there are only evident shift symmetries of the oscillators. However, when we deform the Schrödinger algebra in two space dimensions to $su(1,2)$, the corresponding oscillator is also deformed to a nonlinear one, but remains dynamically equivalent to the standard oscillator [7]. This suggests the existence of F-invariant nonlinearly deformed oscillator systems for every noncompact real Lie group F.

Crucial in our construction [4] of the deformed oscillators is the 5-grading of $su(1,2)$. Now, *any* finite-dimensional simple complex Lie algebra beyond sl_2 has at least one non-compact real form with a 5-graded decomposition [8,9]. A universal part of the 5-grading is the $su(1,1)$ subalgebra formed by the highest- and lowest-grade subspace together with the (grade-zero) grading operator L_0, so one-dimensional conformal symmetry is always present. In the present paper, we extend the procedure developed in [4] from $su(1,2)$ to a non-compact real form of any simple Lie algebra. It will provide a system of (generically nonlinear) second-order differential equations with the prescribed non-compact symmetry, which reduces to ordinary harmonic oscillators under the contraction to a Schrödinger algebra.

The existence of a corresponding invariant action is a more delicate matter, which we also investigate here. It is not guaranteed, because the equations of motion usually enjoy a larger symmetry than the action. In the following, we shall work out two explicit examples in detail, featuring $SO(2,3)$ and $G_{2(2)}$ symmetry, respectively. We shall see that the formulation of an action requires additional, semi-dynamical degrees of freedom which, however, do not affect the deformed oscillator equations. This provides an algorithm for the construction of invariant actions.

2. General construction

It is a well known fact [8,9] that every simple Lie algebra $\mathcal{F}$ (except for sl_2) admits 5-graded decompositions with respect to a suitable generator $L_0 \in \mathcal{F}$:

$$\mathcal{F} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{-\frac{1}{2}} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+\frac{1}{2}} \oplus \mathfrak{f}_{+1} \qquad \text{with} \qquad \left[\mathfrak{f}_i, \mathfrak{f}_j\right] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i,j \in \left\{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\right\} \tag{2.1}$$

($\mathfrak{f}_i = 0$ for $|i| > 1$ understood). There is an (up to automorphisms) unique 5-grading with one-dimensional spaces $\mathfrak{f}_{\pm 1}$. Choosing this one, we may write

$$\mathfrak{f}_{-1} = \mathbb{C}\, L_{-1}, \qquad \mathfrak{f}_{+1} = \mathbb{C}\, L_1 \qquad \text{and} \qquad \mathfrak{f}_0 = \mathcal{H} \oplus \mathbb{C}\, L_0, \tag{2.2}$$

where $\mathcal{H} \subset \mathcal{F}$ is a Lie subalgebra and L_0 commutes with $\mathcal{H}$. A basis for the spaces $\mathfrak{f}_{\pm\frac{1}{2}}$ (of some dimension d) is given by generators $G^A_{\pm\frac{1}{2}}$ with $A = 1, \dots, d$. They carry an irreducible representation of $\mathcal{H}$. In the following, we will deal with *real* Lie algebras and groups only, so some real form of $\mathcal{F}$ and $\mathcal{H}$ has to be picked. (We keep the same notation however.) Compatibility with the 5-grading requires this real form to be non-compact. Therefore, (L_{-1}, L_1, L_0) generate an $su(1,1)$ subalgebra of $\mathcal{F}$. Different real forms of $\mathcal{F}$ and $\mathcal{H}$ give rise to different non-compact quaternionic symmetric spaces W [8,9],

$$W = \frac{F}{H \times \mathrm{SU}(1,1)}, \tag{2.3}$$

where F, H and SU(1,1) are the (simply-connected) groups generated by $\mathcal{F}$, $\mathcal{H}$ and $su(1,1)$, respectively.

The main idea of our construction consists in enlarging the coset by slightly reducing the stability group from $H \times \mathrm{SU}(1,1)$ to $H \times \mathfrak{B}_{\mathrm{SU}(1,1)}$, where $\mathfrak{B}_{\mathrm{SU}(1,1)}$ denotes the positive Borel subgroup of SU(1,1), whose algebra $\mathfrak{b}_{su(1,1)}$ is generated by (L_0, L_1). In other words, we keep L_{-1} in the numerator and consider the coset

$$\mathcal{W} = \frac{F}{H \times \mathfrak{B}_{\mathrm{SU}(1,1)}}. \tag{2.4}$$

The elements of $\mathcal{W}$ can be parametrized as follows,

$$g = e^{t\left(L_{-1}+\omega^2 L_1\right)} e^{u(t)\cdot G_{-\frac{1}{2}}} e^{v(t)\cdot G_{\frac{1}{2}}}, \tag{2.5}$$

where we employed a $\cdot$ notation to suppress the summation over A. The parameter ω represents some freedom in the parametrization of $\mathcal{W}$. It yields the oscillation frequency of the deformed oscillators we are going to construct.

Defining the Cartan forms in the standard way (with a basis $\{h_s\}$ of $\mathcal{H}$),

$$g^{-1}dg = \omega_{-1}L_{-1} + \omega_0 L_0 + \omega_1 L_1 + \omega_{-\frac{1}{2}} \cdot G_{-\frac{1}{2}} + \omega_{\frac{1}{2}} \cdot G_{\frac{1}{2}} + \sum_s \omega_h^s h_s, \tag{2.6}$$

one can check that the constraints

$$\omega_{-\frac{1}{2}} = 0 \tag{2.7}$$

firstly are invariant under the whole group F, realized by left multiplication in the coset $\mathcal{W}$ (2.4), and secondly express the Goldstone fields $v(t)$ through the Goldstone fields $u(t)$ and their time derivatives in a covariant fashion (inverse Higgs phenomenon [3]). After imposing the constraints (2.7) we have a realization of the F transformations on the time t and the d coordinates $u_A(t)$.

Finally, one can impose the additional invariant constraints

$$\omega_{\frac{1}{2}} = 0, \tag{2.8}$$

which produces a system of second-order differential equations for the variables $u_A(t)$. These are the equations of motion. Hence, with every simple Lie algebra $\mathcal{F}$ one may associate a system of dynamical equations in d variables which is invariant under some non-compact real form of the group F.

Given the above structures, we can partially fix the commutator relations of $\mathcal{F}$:

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad \left[L_n, G_r^A\right] = \left(\tfrac{n}{2} - r\right) G_{n+r}^A,$$
$$m,n = -1,0,1, \; r = -\tfrac{1}{2}, \tfrac{1}{2}, \; A = 1,\ldots,d. \tag{2.9}$$

The $[G,G]$ commutators lands in $\mathcal{H} \oplus su(1,1)$. However, they can be made to vanish by a group contraction. To this end, one rescales the generators via $G^A_{\pm\frac{1}{2}} = \gamma^{-1}\widetilde{G}^A_{\pm\frac{1}{2}}$ with $\gamma \in \mathbb{R}_+$. The limit $\gamma \to 0$ preserves the relations (2.9) but lets all generators $\widetilde{G}^A_{\pm\frac{1}{2}}$ commute with one another. Thus, after the contraction we arrive at the algebra

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad \left[L_n, \widetilde{G}^A_r\right] = \left(\tfrac{n}{2} - r\right)\widetilde{G}^A_{n+r},$$
$$\left[\widetilde{G}^A_{\pm\frac12}, \widetilde{G}^B_{\pm\frac12}\right] = 0, \qquad \left[\widetilde{G}^A_{\pm\frac12}, \widetilde{G}^B_{\mp\frac12}\right] = 0. \tag{2.10}$$

This is the Schrödinger algebra in $d+1$ dimensions [4]. One may check that in this limit the equations (2.7) and (2.8) linearize to

$$\ddot{u}_A(t) + \omega^2 u_A(t) = 0 \qquad \text{for} \quad A = 1, \ldots, d. \tag{2.11}$$

Undoing the contraction, one may regard (2.7) and (2.8) as a deformation of (2.11). For this reason we refer to them as 'deformed oscillators'. The first example, for the algebra $\mathcal{F} = su(1,2)$ and $\mathcal{H} = u(1)$, was considered in [4] and [7].

Finally we note that the above construction yields only the equations of motion for the variables $u_A(t)$. The question of existence of a corresponding invariant action has to be answered independently. We will demonstrate below that a positive answer requires extending further the number of Goldstone fields.

In the following two sections we will consider two instructive examples in detail: $SO(2,3)$ and $G_{2(2)}$ invariant deformed oscillators.

3. $SO(2,3)$ invariant oscillator

The 10-dimensional $so(2,3)$ algebra admits a 5-graded structure with $d = 2$ and $\mathcal{H} = su(1,1)$. It can be visualized by writing the commutator relations as

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m}, \quad [M_a, M_b] = (a-b)M_{a+b},\\
&m, n = -1, 0, 1, \quad a, b = -1, 0, 1,\\
&\left[L_n, G_{r,A}\right] = \left(\tfrac{n}{2} - r\right) G_{n+r,A}, \quad \left[M_a, G_{r,A}\right] = \left(\tfrac{a}{2} - A\right) G_{r,a+A},\\
&r, s = -\tfrac12, \tfrac12, \quad A, B = -\tfrac12, \tfrac12,\\
&\left[G_{r,A}, G_{s,B}\right] = 2\left(A\,\delta_{A+B,0}\,L_{r+s} + r\,\delta_{r+s,0}\,M_{A+B}\right).
\end{aligned} \tag{3.1}$$

All generators may be taken to be antihermitian,

$$(L_n)^\dagger = -L_n, \qquad (M_a)^\dagger = -M_a, \qquad \left(G_{r,A}\right)^\dagger = -G_{r,A}. \tag{3.2}$$

Thus, we see that

$$\begin{aligned}
&\mathfrak{f}_{-\frac12} = \mathbb{R}\,G_{-\frac12,-\frac12} \oplus \mathbb{R}\,G_{-\frac12,+\frac12}, \quad \mathfrak{f}_{+\frac12} = \mathbb{R}\,G_{+\frac12,-\frac12} \oplus \mathbb{R}\,G_{+\frac12,+\frac12},\\
&\mathcal{H} = \mathbb{R}\,M_{-1} + \mathbb{R}\,M_0 + \mathbb{R}\,M_{+1}.
\end{aligned} \tag{3.3}$$

From the maximally non-compact four-dimensional quaternionic symmetric space $W = \mathrm{SO}(2,3)/\mathrm{SO}(2,2)$ we pass to the five-dimensional coset space

$$\mathcal{W} = \frac{\mathrm{SO}(2,3)}{\mathrm{SU}(1,1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}} \tag{3.4}$$

where the stability subgroup is generated by (L_0, L_1, M_a). The coset $\mathcal{W}$ is parametrized

$$\begin{aligned}
&g = e^{t(L_{-1}+\omega^2 L_1)}\, e^{u_1 G_{-\frac12,-\frac12} + u_2 G_{-\frac12,+\frac12}}\, e^{v_1 G_{+\frac12,-\frac12} + v_2 G_{+\frac12,+\frac12}},\\
&u^\star_{1,2} = u_{1,2}, \; v^\star_{1,2} = v_{1,2}, \quad g^\dagger = g^{-1}.
\end{aligned} \tag{3.5}$$

To find the equations of motion for the coordinates $u_1(t)$ and $u_2(t)$ we have to calculate the Cartan forms

$$g^{-1}dg = \sum_n \omega_{L_n} L_n + \sum_a \omega_{M_a} M_a + \sum_{r,A} \omega_{r,A} G_{r,A}. \tag{3.6}$$

Their explicit form reads (we will not need ω_{L_n})

$$\begin{aligned}
\omega_{-\frac12,-\frac12} &= du_1 - v_1 \left(dt + \tfrac12 (u_1\, du_2 - u_2\, du_1)\right),\\
\omega_{-\frac12,+\frac12} &= du_2 - v_2 \left(dt + \tfrac12 (u_1\, du_2 - u_2\, du_1)\right),\\
\omega_{+\frac12,-\frac12} &= dv_1 + \tfrac12 v_1 (v_2\, du_1 - v_1\, du_2) + \omega^2\, dt\, u_1 \left(1 + \tfrac12 (u_2 v_1 - u_1 v_2)\right),\\
\omega_{+\frac12,+\frac12} &= dv_2 + \tfrac12 v_2 (v_2\, du_1 - v_1\, du_2) + \omega^2\, dt\, u_2 \left(1 + \tfrac12 (u_2 v_1 - u_1 v_2)\right)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\omega_{M_{-1}} &= +\tfrac14 u_1 v_1^2\, du_2 - v_1 \left(1 + \tfrac14 u_2 v_1\right) du_1 + \tfrac12 dt \left(v_1^2 + \omega^2 u_1^2\right),\\
\omega_{M_{+1}} &= -\tfrac14 u_2 v_2^2\, du_1 - v_2 \left(1 - \tfrac14 u_1 v_2\right) du_2 + \tfrac12 dt \left(v_2^2 + \omega^2 u_2^2\right),\\
\omega_{M_0} &= -v_2 \left(1 + \tfrac12 u_2 v_1\right) du_1 - v_1 \left(1 - \tfrac12 u_1 v_2\right) du_2 + dt \left(v_1 v_2 + \omega^2 u_1 u_2\right).
\end{aligned} \tag{3.8}$$

In accordance with the general scheme outlined in Section 2, we firstly have to express v_1 and v_2 in terms of u_1 and u_2 by nullifying the forms $\omega_{-\frac12,-\frac12}$ and $\omega_{-\frac12,+\frac12}$. Doing so, we obtain

$$\omega_{-\frac12,\pm\frac12} = 0 \quad \Rightarrow \quad v_1 = \frac{\dot u_1}{1 + \frac12 (u_1 \dot u_2 - \dot u_1 u_2)} \quad \text{and} \quad v_2 = \frac{\dot u_2}{1 + \frac12 (u_1 \dot u_2 - \dot u_1 u_2)}. \tag{3.9}$$

Finally, to find the invariant equations of motion one has to nullify the forms $\omega_{+\frac12,-\frac12}$ and $\omega_{+\frac12,+\frac12}$ (with (3.9) taken into account). In this way we arrive at

$$\omega_{+\frac12,\pm\frac12} = 0 \quad \Rightarrow \quad \ddot u_1 + \omega^2 u_1 = 0 \quad \text{and} \quad \ddot u_2 + \omega^2 u_2 = 0. \tag{3.10}$$

Having expected two coupled nonlinear differential equations, we are surprised to have obtained just ordinary (decoupled) linear oscillator equations, even before taking the linearizing contraction limit to the Schrödingier algebra. We conclude that the equations of motion of the ordinary two-dimensional harmonic oscillator enjoy an SO(2,3) invariance!

It is instructive to present the SO(2,3) symmetry transformations of (3.10) explicitly. The SO(2,3) group is realized by left multiplication on the coset element (3.5),

$$g_0\, g = g'\, h \qquad \text{with} \qquad g_0 \in \mathrm{SO}(2,3) \qquad \text{and} \qquad h \in \mathrm{SU}(1,1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}. \tag{3.11}$$

Different elements g_0 effect different changes $g \mapsto g'$, which induce transformations of the time t and the coordinates (u_1, u_2, v_1, v_2). We display their infinitesimal versions (linear in the transformation parameters)[1]:

[1] The result of acting with $g_0 = e^{\varepsilon_1 G_{+\frac12,-\frac12} + \varepsilon_2 G_{+\frac12,+\frac12}}$ can be obtained from the commutator of (3.12) and (3.14).

$$g_0 = e^{a\,L_{-1}+b\,L_0+c\,L_{+1}} : \left\{ \begin{array}{l} \delta t \;\; = \frac{1+\cos(2\omega t)}{2}\,a + \frac{\sin(2\omega t)}{2\omega}\,b + \frac{1-\cos(2\omega t)}{2\omega^2}\,c \;\equiv\; f(t) \\ \delta u_1 = \frac{1}{2}\dot{f}\,u_1 \\ \delta u_2 = \frac{1}{2}\dot{f}\,u_2 \\ \delta v_1 = -\frac{1}{2}\dot{f}\,v_1 + \frac{1}{2}\ddot{f}\,u_1\left(1+\frac{1}{2}(u_2 v_1 - u_1 v_2)\right) \\ \delta v_2 = -\frac{1}{2}\dot{f}\,v_2 + \frac{1}{2}\ddot{f}\,u_2\left(1+\frac{1}{2}(u_2 v_1 - u_1 v_2)\right) \end{array} \right\}, \tag{3.12}$$

$$g_0 = e^{\alpha\,M_{-1}+\beta\,M_0+\gamma\,M_{+1}} : \left\{ \begin{array}{l} \delta t \;\; = 0 \\ \delta u_1 = \frac{1}{2}\beta\,u_1 - \alpha\,u_2 \\ \delta u_2 = -\frac{1}{2}\beta\,u_2 + \gamma\,u_1 \\ \delta v_1 = \frac{1}{2}\beta\,v_1 - \alpha\,v_2 \\ \delta v_2 = -\frac{1}{2}\beta\,v_2 + \gamma\,v_1 \end{array} \right\}, \tag{3.13}$$

$$g_0 = e^{\epsilon_1\,G_{-\frac{1}{2},-\frac{1}{2}}+\epsilon_2\,G_{-\frac{1}{2},+\frac{1}{2}}} : \left\{ \begin{array}{l} \delta t \;\; = \frac{1}{2}\cos(\omega t)\,(\epsilon_2\,u_1 - \epsilon_1\,u_2) \\ \delta u_1 = \cos(\omega t)\epsilon_1 - \frac{1}{2}\omega\sin(\omega t)\,u_1\;(\epsilon_2 u_1 - \epsilon_1 u_2) \\ \delta u_2 = \cos(\omega t)\epsilon_2 - \frac{1}{2}\omega\sin(\omega t)\,u_2\;(\epsilon_2 u_1 - \epsilon_1 u_2) \\ \delta v_1 = \frac{1}{4}\omega^2\cos(\omega t)u_1(2+u_2 v_1 - u_1 v_2)(\epsilon_1 u_2 - \epsilon_2 u_1) \\ \qquad -\,\omega\sin(\omega t)\epsilon_1(1+u_2 v_1 - u_1 v_2) \\ \delta v_2 = \frac{1}{4}\omega^2\cos(\omega t)u_2(2+u_2 v_1 - u_1 v_2)(\epsilon_1 u_2 - \epsilon_2 u_1) \\ \qquad -\,\omega\sin(\omega t)\epsilon_2(1+u_2 v_1 - u_1 v_2) \end{array} \right\}. \tag{3.14}$$

One may check that (3.9) as well as (3.10) are invariant under these transformations.

Can we invent an invariant action which yields the equations of motion (3.10)? The simplest candidate which produces (3.10) and is invariant under M_0 rotations (see (3.13)),

$$S_{\text{test}} = \int dt\,\left(\dot{u}_1\dot{u}_2 - \omega^2\,u_1 u_2\right) \tag{3.15}$$

is not invariant with respect to the other transformations in (3.12), (3.13) or (3.14). For instance, under an M_{-1} transformation (see again (3.13)) it changes by

$$\delta S_{\text{test}} = -\alpha \int dt\,\left(\dot{u}_2\dot{u}_2 - \omega^2\,u_2\,u_2\right) \neq 0. \tag{3.16}$$

In fact, with the given set of four coordinates u_A and v_A provided by the coset $\mathcal{W}$ via (3.5) it is impossible to construct an SO(2,3) invariant action. However, the variation of S_{test} suggests that we introduce additional coordinates to compensate for the variation (3.16). These new variables must experience constant shifts under the M_{-1} and M_1 transformations and carry the appropriate M_0 charge. Therefore, the idea is to further extend our coset space from five to seven dimensions,

$$\mathcal{W} = \frac{\mathrm{SO}(2,3)}{\mathrm{SU}(1,1)\times \mathfrak{B}_{\mathrm{SU}(1,1)}} \qquad \to \qquad \mathcal{W}_{\text{imp}} = \frac{\mathrm{SO}(2,3)}{\mathrm{U}(1)\times \mathfrak{B}_{\mathrm{SU}(1,1)}}, \tag{3.17}$$

where the U(1) factor is generated by M_0. The new Goldstone fields Λ_{-1} and Λ_{+1} associated with the generators M_{-1} and M_{+1}, respectively, come with determined transformation properties. Moreover, the Cartan form for the U(1) generator shifts by a time derivatives under any

SO(2,3) transformation (3.12), (3.13) or (3.14) and, therefore, may be considered for an invariant action.

To realize above mentioned procedure we have to perform the following steps.

- First, we must introduce the new coordinates $\Lambda_{\pm 1}$ by extending our coset element g (3.5) to
$$g_{\rm imp} = g\, e^{\Lambda_{-1}M_{-1}+\Lambda_{+1}M_{+1}}, \qquad \Lambda^{\star}_{\pm 1} = \Lambda_{\pm 1}. \tag{3.18}$$
- Second, one has to recalculate the Cartan forms. Let us denote their 'improved' version by Ω_{M_a} and $\Omega_{r,\alpha}$. Then, the simplest invariant action is
$$S = -\int \Omega_{M_0}. \tag{3.19}$$
- Third, one has to derive the 'improved' equations of motion for u_1 and u_2 from (3.19) and assert that they are unchanged, i.e. still coincide with (3.10).

The improved Cartan forms are defined through the coset element $g_{\rm imp}$ (3.18) via
$$g^{-1}_{\rm imp}\, d\, g_{\rm imp} = \sum_n \Omega_{L_n} L_n + \sum_a \Omega_{M_a} M_a + \sum_{r,A} \Omega_{r,A} G_{r,A} \tag{3.20}$$
and read[2]
$$\begin{aligned}
\Omega_{\pm\frac12,-\frac12} &= \frac{1}{\sqrt{1+\lambda_{-1}\lambda_{+1}}}\Big(\omega_{\pm\frac12,-\frac12} + \lambda_{-1}\,\omega_{\pm\frac12,+\frac12}\Big),\\
\Omega_{\pm\frac12,+\frac12} &= \frac{1}{\sqrt{1+\lambda_{-1}\lambda_{+1}}}\Big(\omega_{\pm\frac12,+\frac12} - \lambda_{+1}\,\omega_{\pm\frac12,-\frac12}\Big),\\
\Omega_{M_{-1}} &= \frac{1}{1+\lambda_{-1}\,\lambda_{+1}}\Big(d\lambda_{-1} + \omega_{M_{-1}} + \lambda_{-1}\omega_{M_0} + \lambda^2_{-1}\omega_{M_{+1}}\Big),\\
\Omega_{M_{+1}} &= \frac{1}{1+\lambda_{-1}\,\lambda_{+1}}\Big(d\lambda_{+1} + \omega_{M_{+1}} - \lambda_{+1}\omega_{M_0} + \lambda^2_{+1}\omega_{M_{-1}}\Big),\\
\Omega_{M_0} &= \frac{1}{1+\lambda_{-1}\lambda_{+1}}\big(2\lambda_{-1}\omega_{M_{+1}} - 2\lambda_{+1}\omega_{M_{-1}} + (1-\lambda_{-1}\lambda_{+1})\,\omega_{M_0}\\
&\quad + \lambda_{-1}d\lambda_{+1} - \lambda_{+1}d\lambda_{-1}\big).
\end{aligned} \tag{3.21}$$
Here, stereographically projected coordinates were employed for simplicity,
$$\lambda_{-1} = \frac{\tan\left(\sqrt{\Lambda_{-1}\Lambda_{+1}}\right)}{\sqrt{\Lambda_{-1}\Lambda_{+1}}}\Lambda_{-1} \qquad \text{and} \qquad \lambda_{+1} = \frac{\tan\left(\sqrt{\Lambda_{-1}\Lambda_{+1}}\right)}{\sqrt{\Lambda_{-1}\Lambda_{+1}}}\Lambda_{+1}. \tag{3.22}$$
The improved invariant constraints
$$\Omega_{-\frac12,\pm\frac12} = 0 \qquad \text{and} \qquad \Omega_{+\frac12,\pm\frac12} = 0 \tag{3.23}$$
imply the old constraints (3.9) and (3.10) and, therefore, indeed produce the previous equations of motion (3.10). For the new variables $\lambda_{\pm 1}$ one can get covariant equations of motion by imposing the extra constraints
$$\Omega_{M_{-1}} = \Omega_{M_{+1}} = 0, \tag{3.24}$$

[2] The forms $\Omega_{L_n} = \omega_{L_n}$ are unchanged. We do not need to know their explicit form.

which imply

$$\dot\lambda_{-1} = \frac{(\dot u_1 + \lambda_{-1}\,\dot u_2)^2}{2\left(1+\frac{1}{2}\,(u_1\dot u_2 - \dot u_1 u_2)\right)} - \frac{\omega^2}{2}\,(u_1 + \lambda_{-1}\,u_2)^2\,,$$
$$\dot\lambda_{+1} = \frac{(\dot u_2 - \lambda_{+1}\,\dot u_1)^2}{2\left(1+\frac{1}{2}\,(u_1\dot u_2 - \dot u_1 u_2)\right)} - \frac{\omega^2}{2}\,(u_2 - \lambda_{+1}\,u_1)^2\,. \qquad (3.25)$$

Like the oscillator equations (3.10), the above are invariant under the transformations (3.12), (3.13), (3.14) together with the corresponding transformations of λ_{-1} and λ_{+1} The latter take the generic form

$$\delta\lambda_{-1} = \mu_{-1} + \mu_0\lambda_{-1} + \mu_{+1}\lambda_{-1}^2 \qquad \text{and} \qquad \delta\lambda_{+1} = \mu_{+1} - \mu_0\lambda_{+1} + \mu_{-1}\lambda_{+1}^2, \qquad (3.26)$$

with the parameters μ given by

$$g_0 = e^{a\,L_{-1}+b\,L_0+c\,L_{+1}}\ :\ \mu_{-1} = \tfrac{1}{4}\dddot{f} u_1^2, \quad \mu_0 = \tfrac{1}{2}\dddot{f} u_1\,u_2, \quad \mu_{+1} = \tfrac{1}{4}\dddot{f} u_2^2,$$
$$g_0 = e^{\alpha\,M_{-1}+\beta\,M_0+\gamma\,M_{+1}}\ :\ \mu_{-1} = \alpha, \quad \mu_0 = \beta, \quad \mu_{+1} = \gamma,$$
$$g_0 = e^{\epsilon_1\,G_{-\frac{1}{2},-\frac{1}{2}}+\epsilon_2\,G_{-\frac{1}{2},+\frac{1}{2}}}\ :$$
$$\left\{\begin{array}{l} \mu_{-1} = \frac{1}{4}\omega^2\,\cos(\omega t)\,u_1^2\,(\epsilon_1 u_2 - \epsilon_2 u_1) - \omega\sin(\omega t)\,\epsilon_1\,u_1 \\ \mu_0 = \frac{1}{2}\omega^2\,\cos(\omega t)\,u_1\,u_2\,(\epsilon_1 u_2 - \epsilon_2 u_1) - \omega\sin(\omega t)\,(\epsilon_1 u_2 + \epsilon_2 u_1) \\ \mu_{+1} = \frac{1}{4}\omega^2\,\cos(\omega t)\,u_2^2\,(\epsilon_1 u_2 - \epsilon_2 u_1) - \omega\sin(\omega t)\,\epsilon_2\,u_2 \end{array}\right\}. \qquad (3.27)$$

Finally, the invariant action (3.19) acquires the form

$$S = \int dt \left[\frac{(1-\lambda_{-1}\,\lambda_{+1})\,\dot u_1\dot u_2 + \lambda_{-1}\dot u_2^2 - \lambda_{+1}\dot u_1^2}{(1+\lambda_{-1}\,\lambda_{+1})\left(1+\frac{1}{2}\,(u_1\dot u_2 - \dot u_1 u_2)\right)} + \frac{\dot\lambda_{-1}\lambda_{+1} - \lambda_{-1}\dot\lambda_{+1}}{1+\lambda_{-1}\,\lambda_{+1}} \right.$$
$$\left. -\omega^2\,\frac{(u_1+\lambda_{-1}u_2)\,(u_2-\lambda_{+1}u_1)}{1+\lambda_{-1}\,\lambda_{+1}} \right]. \qquad (3.28)$$

It is matter of quite lengthly calculations to check the invariance of this action with respect to the transformations (3.12), (3.13), (3.14) and (3.27). A somewhat less tedious task is to check that the equations of motion following from the action (3.28) coincide with the equations (3.10) and (3.25).

The action (3.28) describes an interaction of the coordinates u_1 and u_2 with isospinor variables λ_{-1} and λ_{+1}. Such kind of variables was firstly introduced within the supersymmetric Calogero model in [10]. Somewhat later, these isospin variables (a.k.a. spin variables) were re-introduced through an SU(2)-reduction procedure [11,12]. However, the action (3.28) has the following peculiarities, which distinguish it from a bosonic sector of some supersymmetric mechanics:

- We are dealing with the non-compact version of isospin variables, as they parametrize the coset $SU(1,1)/U(1)$. Moreover, this SU(1,1) is not an external automorphism group but belongs to the symmetry of our system.
- Despite the explicit interaction between isospin and ordinary variables in the action (3.28), the isospin variables decouple from u_1 and u_2 in the oscillator equations of motion (3.10). They serve only to provide the SO(2,3) invariance of the action.

For an expected relationship of the action (3.28) with those one constructed in [13,14], one has to turn to the Hamiltonian formalism. This will be done elsewhere.

4. $G_{2(2)}$ invariant oscillator

The 14-dimensional $g_{2(2)}$ algebra possesses a 5-grading with d=4 and again $\mathcal{H} = su(1,1)$. This is made manifest by its commutation relations,

$$
\begin{aligned}
&[L_n, L_m] = (n-m)\, L_{n+m}, \quad [M_a, M_b] = (a-b)\, M_{a+b},\\
&m, n = -1, 0, 1, \quad a, b = -1, 0, 1,\\
&\left[L_n, G_{r,A}\right] = \left(\tfrac{n}{2} - r\right) G_{n+r,A}, \quad \left[M_a, G_{r,A}\right] = \left(\tfrac{3a}{2} - A\right) G_{r,a+A},\\
&r, s = -\tfrac{1}{2}, \tfrac{1}{2}, \quad A, B = -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2},\\
&\left[G_{r,A}, G_{s,B}\right] = 3A\left(4A^2 - 5\right) \delta_{A+B,0}\, L_{r+s} + r\left(6A^2 - 8A\,B + 6B^2 - 9\right) \delta_{r+s,0}\, M_{A+B}.
\end{aligned}
\tag{4.1}
$$

As in the previous example (3.2), these generators are chosen to be anti-hermitian,

$$
(L_n)^\dagger = -L_n, \qquad (M_a)^\dagger = -M_a, \qquad \left(G_{r,A}\right)^\dagger = -G_{r,A}. \tag{4.2}
$$

Thus we have as basis elements

$$
G_{-\frac{1}{2},A} \in \mathfrak{f}_{-\frac{1}{2}}, \qquad G_{+\frac{1}{2},A} \in \mathfrak{f}_{+\frac{1}{2}} \qquad \text{and} \qquad M_a \in \mathcal{H} = su(1,1). \tag{4.3}
$$

We start from the eight-dimensional quaternionic symmetric space $W = G_{2(2)}/\mathrm{SO}(2,2)$ and enlarge it to the nine-dimensional coset

$$
\mathcal{W} = \frac{G_{2(2)}}{\mathrm{SU}(1,1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}} \tag{4.4}
$$

with the stability subgroup generated by (L_0, L_1, M_a) as before. It may be parameterized as

$$
\begin{aligned}
g = {} & e^{t(L_{-1}+\omega^2 L_1)} e^{u_1 G_{-\frac{1}{2},-\frac{3}{2}} + u_2 G_{-\frac{1}{2},-\frac{1}{2}} + u_3 G_{-\frac{1}{2},+\frac{1}{2}} + u_4 G_{-\frac{1}{2},+\frac{3}{2}}}\\
& \times e^{v_1 G_{+\frac{1}{2},-\frac{3}{2}} + v_2 G_{+\frac{1}{2},-\frac{1}{2}} + v_3 G_{+\frac{1}{2},+\frac{1}{2}} + v_4 G_{+\frac{1}{2},+\frac{3}{2}}}, \quad g^\dagger = g^{-1}.
\end{aligned}
\tag{4.5}
$$

The corresponding Cartan forms are rather complicated. To write them in a concise form we relabel the generators G and variables u and v in the spin-$\frac{3}{2}$ $\mathcal{H}$-representation with a symmetrized triple of spinor indices $\alpha, \beta, \gamma = 1, 2$:

$$
\begin{aligned}
&G_{\pm\frac{1}{2},-\frac{3}{2}} = 3G_{\pm\frac{1}{2},111}, \quad G_{\pm\frac{1}{2},-\frac{1}{2}} = 3G_{\pm\frac{1}{2},112}, \quad G_{\pm\frac{1}{2},+\frac{1}{2}} = 3G_{\pm\frac{1}{2},122},\\
&G_{\pm\frac{1}{2},+\frac{3}{2}} = 3G_{\pm\frac{1}{2},222},\\
&u_1 = \tfrac{1}{3}U^{111}, \quad u_2 = U^{112}, \quad u_3 = U^{122}, \quad u_4 = \tfrac{1}{3}U^{222},\\
&v_1 = \tfrac{1}{3}V^{111}, \quad v_2 = V^{112}, \quad v_3 = V^{122}, \quad v_4 = \tfrac{1}{3}V^{222},
\end{aligned}
\tag{4.6}
$$

such that (with spinor index triples completely symmetric)

$$
\begin{aligned}
&u_1 G_{-\frac{1}{2},-\frac{3}{2}} + u_2 G_{-\frac{1}{2},-\frac{1}{2}} + u_3 G_{-\frac{1}{2},+\frac{1}{2}} + u_4 G_{-\frac{1}{2},+\frac{3}{2}} = \sum_{\alpha\beta\gamma} U^{\alpha\beta\gamma} G_{-\frac{1}{2},\alpha\beta\gamma},\\
&v_1\, G_{+\frac{1}{2},-\frac{3}{2}} + v_2\, G_{+\frac{1}{2},-\frac{1}{2}} + v_3\, G_{+\frac{1}{2},+\frac{1}{2}} + v_4\, G_{+\frac{1}{2},+\frac{3}{2}} = \sum_{\alpha\beta\gamma} V^{\alpha\beta\gamma} G_{+\frac{1}{2},\alpha\beta\gamma}.
\end{aligned}
\tag{4.7}
$$

Clearly, $G_{\pm\frac{1}{2},\alpha\beta\gamma}$, $U^{\alpha\beta\gamma}$, and $V^{\alpha\beta\gamma}$ are real tensors totally symmetric in α, β, γ. Some of their multiple products will be abbreviated as follows,[3]

$$
\begin{aligned}
&(\mathcal{A}\mathcal{B})^{\alpha\beta} = \tfrac{1}{2}\sum \left(\mathcal{A}^{\alpha\gamma_1\gamma_2}\mathcal{B}_{\gamma_1\gamma_2}{}^{\beta} + \mathcal{A}^{\beta\gamma_1\gamma_2}\mathcal{B}_{\gamma_1\gamma_2}{}^{\alpha}\right),\\
&(\mathcal{A}\mathcal{B}) = \sum \mathcal{A}^{\alpha\beta\gamma}\mathcal{B}_{\alpha\beta\gamma},\\
&(\mathcal{A}\mathcal{B}\mathcal{C})^{\alpha\beta\gamma} = \tfrac{1}{3}\sum \left(\mathcal{A}^{\alpha\rho_1\rho_2}\mathcal{B}_{\rho_1\rho_2\rho_3}\mathcal{C}^{\rho_3\beta\gamma} + \mathcal{A}^{\beta\rho_1\rho_2}\mathcal{B}_{\rho_1\rho_2\rho_3}\mathcal{C}^{\rho_3\gamma\alpha} + \mathcal{A}^{\gamma\rho_1\rho_2}\mathcal{B}_{\rho_1\rho_2\rho_3}\mathcal{C}^{\rho_3\beta\alpha}\right),\\
&(\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}) = \sum \mathcal{A}^{\alpha\beta_1\beta_2}\mathcal{B}_{\beta_1\beta_2\gamma_1}\mathcal{C}^{\gamma_1\rho_1\rho_2}\mathcal{D}_{\rho_1\rho_2\alpha}.
\end{aligned}
\tag{4.8}
$$

Defining the Cartan forms

$$
g^{-1}dg = \sum_n \omega_{L_n} L_n + \sum_a \omega_{M_a} M_a + \sum_{\alpha\beta\gamma} \omega_u^{\alpha\beta\gamma} G_{-\frac{1}{2},\alpha\beta\gamma} + \sum_{\alpha\beta\gamma} \omega_v^{\alpha\beta\gamma} G_{+\frac{1}{2},\alpha\beta\gamma}, \tag{4.9}
$$

we arrive at

$$
\omega_u^{\alpha\beta\gamma} = dU^{\alpha\beta\gamma} + \omega^2 dt \left(U^3\right)^{\alpha\beta\gamma} - V^{\alpha\beta\gamma}\left[dt\left(1 - \tfrac{\omega^2}{2}\left(U^4\right)\right) + (UdU)\right], \tag{4.10}
$$

$$
\begin{aligned}
\omega_v^{\alpha\beta\gamma} = {} & dV^{\alpha\beta\gamma} + (V^3)^{\alpha\beta\gamma}\left[dt\left(1 - \tfrac{\omega^2}{2}\left(U^4\right)\right) + (UdU)\right]\\
& - 2(VdUV)^{\alpha\beta\gamma} - (VVdU)^{\alpha\beta\gamma}\\
& + \omega^2 dt\Big[U^{\alpha\beta\gamma} + 3(UUV)^{\alpha\beta\gamma} + 2(VU^3V)^{\alpha\beta\gamma} - (U^3VV)^{\alpha\beta\gamma}\Big].
\end{aligned}
\tag{4.11}
$$

In what follows we also need the forms ω_{M_a}

$$
\omega_{M_{-1}} = \tfrac{1}{2}\omega^{11}, \qquad \omega_{M_{+1}} = \tfrac{1}{2}\omega^{22}, \qquad \omega_{M_0} = \omega^{12}, \tag{4.12}
$$

where

$$
\begin{aligned}
\omega^{\alpha\beta} = {} & -4(VdU)^{\alpha\beta} + 2(VV)^{\alpha\beta}\left[dt\left(1 - \tfrac{\omega^2}{2}(U^4)\right) + (UdU)\right]\\
& + 2\omega^2 dt\Big[(UU)^{\alpha\beta} + (U^3V)^{\alpha\beta}\Big].
\end{aligned}
\tag{4.13}
$$

Now, imposing the conditions $\omega_u^{\alpha\beta\gamma} = 0$ we can express the coordinates $V^{\alpha\beta\gamma}$ in terms of $U^{\alpha\beta\gamma}$,

$$
\omega_u^{\alpha\beta\gamma} = 0 \qquad \Rightarrow \qquad V^{\alpha\beta\gamma} = \frac{\dot{U}^{\alpha\beta\gamma} + \omega^2\left(U^3\right)^{\alpha\beta\gamma}}{1 - \frac{\omega^2}{2}\left(U^4\right) + (U\dot{U})}. \tag{4.14}
$$

With these relations the forms ω_{M_a} given by (4.12) and (4.13) simplify to

$$
\omega_{M_{-1}} = \tfrac{1}{2}\tilde{\omega}^{11}, \qquad \omega_{M_{+1}} = \tfrac{1}{2}\tilde{\omega}^{22}, \qquad \omega_{M_0} = \tilde{\omega}^{12}, \tag{4.15}
$$

with

$$
\tilde{\omega}^{\alpha\beta} = -2dt\,\frac{(\dot{U}\dot{U})^{\alpha\beta} - \omega^2\left[\left(1 + (U\dot{U})\right)(UU)^{\alpha\beta} - (U^3\dot{U})^{\alpha\beta}\right]}{1 - \frac{\omega^2}{2}\left(U^4\right) + (U\dot{U})}. \tag{4.16}
$$

Finally, using the conditions $\omega_v^{\alpha\beta\gamma} = 0$ in (4.11) and the relations (4.14) we come to the covariant equations of motion (with $V = V(U)$ according to (4.14)):

[3] $su(1,1)$ indices are raised and lowered via $\mathcal{A}_\alpha = \epsilon_{\alpha\beta}\mathcal{A}^\beta$, $\mathcal{A}^\alpha = \epsilon^{\alpha\beta}\mathcal{A}_\beta$ with $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $\epsilon_{12} = \epsilon^{21} = 1$.

$$\dot V^{\alpha\beta\gamma} + (V^3)^{\alpha\beta\gamma}\Big[\Big(1-\tfrac{\omega^2}{2}\Big(U^4\Big)\Big)+(U\dot U)\Big] - 2(V\dot U V)^{\alpha\beta\gamma} - (VV\dot U)^{\alpha\beta\gamma}$$
$$+\,\omega^2\Big[U^{\alpha\beta\gamma}+3(UUV)^{\alpha\beta\gamma}+2(VU^3V)^{\alpha\beta\gamma}-(U^3VV)^{\alpha\beta\gamma}\Big]=0. \quad (4.17)$$

In the limit $\omega = 0$ these equations simplify to

$$\ddot U^{\alpha\beta\gamma} = 2\frac{(\dot U\dot U\dot U)^{\alpha\beta\gamma} - \dot U^{\alpha\beta\gamma}(\dot U\dot U\dot U\cdot U)}{1+(U\dot U)}$$
$$\text{with}\quad (\dot U\dot U\dot U\cdot U)\equiv\sum(\dot U\dot U\dot U)^{\alpha_1\alpha_2\alpha_3}U_{\alpha_1\alpha_2\alpha_3}, \quad (4.18)$$

and in the contraction limit $\gamma\to 0$ after the rescaling $G^A_{\pm\frac12}=\gamma^{-1}\widetilde G^A_{\pm\frac12}$ (see (2.10)) they linearize to

$$\ddot U^{\alpha\beta\gamma}+\omega^2\,U^{\alpha\beta\gamma}=0. \quad (4.19)$$

In full analogy with the SO(2,3) invariant oscillator considered in the previous section, in order to construct the invariant action one has to extend the coset to an eleven-dimensional one,

$$\mathcal W=\frac{G_{2(2)}}{\mathrm{SU}(1,1)\times\mathfrak B_{\mathrm{SU}(1,1)}}\qquad\to\qquad \mathcal W_{\mathrm{imp}}=\frac{G_{2(2)}}{\mathrm U(1)\times\mathfrak B_{\mathrm{SU}(1,1)}}, \quad (4.20)$$

with elements

$$g_{\mathrm{imp}}=g\,e^{\Lambda_{-1}M_{-1}+\Lambda_{+1}M_{+1}} \quad (4.21)$$

‘improving’ g of (4.5). Defining the improved Cartan forms

$$g_{\mathrm{imp}}^{-1}dg_{\mathrm{imp}}=\sum_{n=-1}^{+1}\Omega_{L_n}L_n+\sum_{a=-1}^{+1}\Omega_{M_a}M_a+\sum_{\alpha\beta\gamma}\Omega_u^{\alpha\beta\gamma}G_{-\frac12,\alpha\beta\gamma}+\sum_{\alpha\beta\gamma}\Omega_v^{\alpha\beta\gamma}G_{+\frac12,\alpha\beta\gamma} \quad (4.22)$$

with $\Omega_{L_n}=\omega_{L_n}$, one can see that $\Omega_u^{\alpha\beta\gamma}$ and $\Omega_v^{\alpha\beta\gamma}$ are linear combinations of the forms $\omega_u^{\alpha\beta\gamma}$ and $\omega_v^{\alpha\beta\gamma}$, because

$$\Omega_u^{\alpha\beta\gamma}G_{-\frac12,\alpha\beta\gamma}=e^{-\Lambda_{-1}M_{-1}-\Lambda_{+1}M_{+1}}\,\omega_u^{\alpha\beta\gamma}G_{-\frac12,\alpha\beta\gamma}\,e^{\Lambda_{-1}M_{-1}+\Lambda_{+1}M_{+1}},$$
$$\Omega_v^{\alpha\beta\gamma}G_{+\frac12,\alpha\beta\gamma}=e^{-\Lambda_{-1}M_{-1}-\Lambda_{+1}M_{+1}}\,\omega_v^{\alpha\beta\gamma}G_{+\frac12,\alpha\beta\gamma}\,e^{\Lambda_{-1}M_{-1}+\Lambda_{+1}M_{+1}}. \quad (4.23)$$

Therefore, the analogous constraints on $\Omega_u^{\alpha\beta\gamma}$ and $\Omega_v^{\alpha\beta\gamma}$ still imply the equations (4.14) and (4.17),

$$\Omega_u^{\alpha\beta\gamma}=\Omega_v^{\alpha\beta\gamma}=0\qquad\Rightarrow\qquad\omega_u^{\alpha\beta\gamma}=\omega_v^{\alpha\beta\gamma}=0. \quad (4.24)$$

The equations of motion for the additional variables $\lambda_{\pm1}$ related to $\Lambda_{\pm1}$ as in (3.22) follow from the invariant constraints

$$\Omega_{M_{-1}}=\frac{1}{1+\lambda_{-1}\lambda_{+1}}\Big(d\lambda_{-1}+\omega_{M_{-1}}+\lambda_{-1}\omega_{M_0}+\lambda_{-1}^2\omega_{M_{+1}}\Big)=0,$$
$$\Omega_{M_{+1}}=\frac{1}{1+\lambda_{-1}\lambda_{+1}}\Big(d\lambda_{+1}+\omega_{M_{+1}}-\lambda_{+1}\omega_{M_0}+\lambda_{+1}^2\omega_{M_{-1}}\Big)=0, \quad (4.25)$$

where the forms $\omega_{M_{-1}}$, ω_{M_0} and $\omega_{M_{+1}}$ were defined in (4.15) and (4.16).

Finally, the invariant action can be constructed from Ω_{M_0},

$$S = -\int \Omega_{M_0} = -\int \frac{1}{1+\lambda_{-1}\lambda_{+1}} \Big[\lambda_{-1}\tilde{\omega}^{22} - \lambda_{+1}\tilde{\omega}^{11} + (1-\lambda_{-1}\lambda_{+1})\,\tilde{\omega}^{12} + \lambda_{-1}d\lambda_{+1} - \lambda_{+1}d\lambda_{-1} \Big], \tag{4.26}$$

where the $\tilde{\omega}^{\alpha\beta}$ were given in (4.16).

A good way to verify that the equations of motion extremize the action (4.26) employs its first-order form

$$S = -\int \Omega_{M_0} = -\int \frac{1}{1+\lambda_{-1}\lambda_{+1}} \Big[\lambda_{-1}\omega^{22} - \lambda_{+1}\omega^{11} + (1-\lambda_{-1}\lambda_{+1})\,\omega^{12} + \lambda_{-1}d\lambda_{+1} - \lambda_{+1}d\lambda_{-1} \Big], \tag{4.27}$$

where the forms $\omega^{\alpha\beta}$ are given by the expressions (4.13). Then, varying this action over $V^{\alpha\beta\gamma}$ will yield (4.14), while the variations over $U^{\alpha\beta\gamma}$, λ_{-1} and λ_{+1} will reproduce $\omega_v^{\alpha\beta\gamma} = 0$ and (4.25), respectively.

The transformation properties of the time t and the coordinates $U^{\alpha\beta\gamma}$ and $\lambda_{\pm 1}$ under $G_{2(2)}$ are found from computing the $G_{2(2)}$ action on the improved coset elements (4.21) by left multiplication,

$$g_0\, g_{\text{imp}} = g'_{\text{imp}}\, h \qquad \text{with} \qquad g_0 \in G_{2(2)} \qquad \text{and} \qquad h \in \mathrm{U}(1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}. \tag{4.28}$$

Due to the commutator relations (4.1) it suffices to know the transformations generated by

$$g_0 = e^{\sum_{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma} G_{-\frac{1}{2},\alpha\beta\gamma}} \qquad \text{and} \qquad g_0 = e^{\sum_{\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma} G_{+\frac{1}{2},\alpha\beta\gamma}}. \tag{4.29}$$

The corresponding transformations can be written in the following concise way,

$$\delta t = \frac{(U^3\vartheta) + (U\,\theta)}{1 + \frac{\omega^2}{2}(U^4)}, \qquad \delta U^{\alpha\beta\gamma} = \theta^{\alpha\beta\gamma} + 2(U\vartheta U)^{\alpha\beta\gamma} + (\vartheta U U)^{\alpha\beta\gamma} - \omega^2 (U^3)^{\alpha\beta\gamma}\,\delta t,$$
$$\delta\lambda_{-1} = \Psi^{11} + 2\Psi^{12}\lambda_{-1} + \Psi^{22}\lambda_{-1}^2, \qquad \delta\lambda_{+1} = \Psi^{22} - 2\Psi^{12}\lambda_{+1} + \Psi^{11}\lambda_{+1}^2 \tag{4.30}$$

with

$$\Psi^{\alpha\beta} = (U\vartheta)^{\alpha\beta} - \omega^2 (UU)^{\alpha\beta}\,\delta t, \tag{4.31}$$

and the parameters $\theta^{\alpha\beta}$ and $\vartheta^{\alpha\beta}$ are related to those in (4.29) as

$$\theta^{\alpha\beta\gamma} = \begin{cases} \cos(mt)\,\epsilon^{\alpha\beta\gamma} \\ \frac{1}{m}\sin(mt)\,\epsilon^{\alpha\beta\gamma} \end{cases} \qquad \text{and} \qquad \vartheta^{\alpha\beta\gamma} = \begin{cases} -m\,\sin(mt)\,\epsilon^{\alpha\beta\gamma} \\ \cos(mt)\,\varepsilon^{\alpha\beta\gamma} \end{cases} \tag{4.32}$$

in the first and second instance of (4.29), respectively. A quite lengthy and tedious calculation confirms that the action (4.26) is indeed invariant under these transformations.

5. Conclusions

We proposed a procedure which associates with any simple Lie algebra a system of the second-order nonlinear differential equations which are invariant with respect to a non-compact real form of this symmetry. Two explicit examples considered in detail gave rise to a system of deformed oscillators invariant under SO(2,3) respective $G_{2(2)}$ transformations. For these cases, we also constructed invariant actions. These actions include additional, semi-dynamical variables which do not affect the equations of motion for the physical variables.

The five-graded decomposition of the Lie algebra, a key feature in our construction, coercively includes a one-dimensional conformal algebra $su(1,1)$. Therefore, all systems constructed in this fashion will possess conformal invariance. Due to our special choice of the stability subalgebra a dilaton is absent, and the conformal invariance is achieved without it. In a contraction limit, when the Lie algebra reduces to a Schrödinger algebra, the equations reduce to a system of ordinary harmonic oscillators.

The following further developments come to mind.

- Our choice of the coset parametrization (the ordering $\mathfrak{g}_{-1} \cdot \mathfrak{g}_{-\frac{1}{2}} \cdot \mathfrak{g}_{\frac{1}{2}}$) is rather special. Clearly, this is far from unique, and a reordering will give the equations a different appearance.
- The chosen coset parametrization is calculationally useful but provides an unusual form of the metric. It is desirable to bring the metric and connection to a more standard form through some reparametrization.
- Some Lie algebras possess other forms of grading (for example, there is a 7-graded basis for G_2). It will be interesting to learn how our equations change when the grading is altered.
- Our construction procedure for invariant actions works properly only in the presence of an $su(1,1)$ factor in the stability subalgebra. It should be clarified how to construct invariant actions when this is not so.
- A supersymmetric extension of the present approach may be of interest.
- Finally, a Hamiltonian description may illuminate the structure of conserved currents and help to relate our systems to others in the literature.

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