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Instantons on the six-sphere and twistors

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We consider the six-sphere $S^6 = G_2/\mathrm{SU}(3)$ and its twistor space $\mathcal{Z} = G_2/\mathrm{U}(2)$ associated with the SU(3)-structure on S^6. It is shown that a Hermitian Yang-Mills connection (instanton) on a smooth vector bundle over S^6 is equivalent to a flat partial connection on a vector bundle over the twistor space $\mathcal{Z}$. The relation with Tian's tangent instantons on $\mathbb{R}^7$ and their twistor description are briefly discussed.
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I. INTRODUCTION AND SUMMARY

The twistor description of solutions to chiral zero-rest-mass field equations on the six-dimensional space $\mathbb{C}^6$ or its real forms with various signatures (see, e.g., Ref. 1) was generalized recently to Abelian[2,3] and non-Abelian[4] holomorphic principal 2-bundles over the twistor space $Q'_6 \subset \mathbb{C}P^7 \setminus \mathbb{C}P^3$, corresponding to solutions of the 3-form self-duality equations on $\mathbb{C}^6$. The twistor approach was also extended to maximally supersymmetric Yang-Mills theory on $\mathbb{C}^6$.[5] Twistor methods have also been applied in the study of scattering amplitudes in this theory (see, e.g., Ref. 6).

The goal of our paper is to describe instantons in gauge theory on Euclidean six-dimensional space (i.e., the bosonic sector of maximally supersymmetric Yang-Mills theory) by using twistor methods. Recall that instantons in four dimensions are nonperturbative gauge-field configurations solving conformally invariant first-order anti-self-duality equations, which imply the full Yang-Mills equations.[7] The twistor approach allows one to describe instanton solutions and their moduli space very efficiently.[8–10] We will apply it to study gauge instantons on the six-dimensional sphere S^6, which is a natural compactification of $\mathbb{R}^6$. Our considerations are based on papers studying twistor spaces associated with higher-dimensional manifolds[11–19] as well as on papers considering instanton equations in dimensions higher than four.[20–29] Here, we consider instanton equations only on S^6. However, our results can be generalized to any nearly Kähler manifold in six and higher dimensions as well as to some other manifolds with G-structure.

We recall some definitions to clarify our purposes. Let X be a Riemannian manifold of dimension $2n$. We define the *metric* twistor space of X as the bundle $\mathrm{Tw}(X) \to X$ of almost Hermitian structures on X (i.e., almost complex structures compatible with the metric g on X and its orientation) associated with the principal bundle $P(X, \mathrm{SO}(2n))$ of orthonormal frames of X, i.e.,

$$\mathrm{Tw}(X) := P(X, \mathrm{SO}(2n)) \times_{\mathrm{SO}(2n)} \mathrm{SO}(2n)/\mathrm{U}(n)\ . \qquad (1.1)$$

It is well known that $\mathrm{Tw}(X)$ can be endowed with an almost complex structure $\mathcal{J}$, which is integrable if and only if the Weyl tensor of X vanishes identically when $n > 2$.[11] In the case $n = 2$, the Weyl tensor has to be anti-self-dual.[9,30] However, if the manifold X has a G-structure (which is

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not necessarily integrable) then one can often find a subbundle $\mathcal{Z}$ of Tw(X) associated with the G-structure bundle $P(X, G)$ for $G \subset$SO($2n$), such that an induced almost complex structure (also called $\mathcal{J}$) on $\mathcal{Z}$ is integrable. Many examples were considered in the literature.[11–13,15–17]

The six-sphere S^6 provides an interesting example. Considered as the round sphere

$$S^6 \;=\; \text{Spin}(7)/\text{Spin}(6)\ , \tag{1.2}$$

its metric twistor space is

$$\text{Tw}(S^6) \;=\; \text{Spin}(7)/\text{U}(3) \;\xrightarrow{\mathbb{C}P^3}\; S^6, \tag{1.3}$$

(see, e.g., Refs. 14 and 17), which may be recognized as a six-dimensional quadric[38] Q_6 in $\mathbb{C}P^7$. Alternatively, one may consider the six-sphere as a nearly Kähler homogeneous space with SU(3)-structure, namely

$$S^6 \;=\; G_2/\text{SU}(3)\ . \tag{1.4}$$

Then,

$$\mathcal{Z} \;=\; G_2/\text{U}(2) \;\xrightarrow{\mathbb{C}P^2}\; S^6 \tag{1.5}$$

is a complex subbundle of Tw(S^6).[11–13,16] Note that $\mathcal{Z}$ can be identified with a five-dimensional quadric $Q_5 \subset \mathbb{C}P^6$, and obviously $Q_5 \subset Q_6 = \text{Tw}(S^6)$. The twistor space (1.5) is a bundle of almost complex structures J on S^6, which are parametrized by the complex projective space $\mathbb{C}P^2$ at each point of S^6.

There was an attempt,[31] not quite successful, to obtain instanton-type configurations on S^6 from holomorphic bundles over Tw(S^6). However, natural instanton equations on S^6 (as well as on $\mathbb{R}^6$) are the Donaldson-Uhlenbeck-Yau (DUY) equations,[21,39] which are SU(3) invariant but not invariant under the SO(6) transformations on the round six-sphere.

The DUY equations are well defined on $S^6 = G_2/\text{SU}(3)$, and their solutions are natural connections $\mathcal{A}$ on pseudo-holomorphic vector bundles $E \to S^6$.[32] We will show that such bundles $(E, \mathcal{A})$ are pulled back to complex vector bundles $(\widetilde{E}, \widetilde{\mathcal{A}})$ over the complex twistor space $\mathcal{Z} = G_2/\text{U}(2)$ with flat partial connection $\widetilde{\mathcal{A}}$. For the definition and discussion of such connections, see, e.g., Refs. 19, 33, and 34. The bundle $\widetilde{E} \to \mathcal{Z}$ is not holomorphic. We would like to emphasize two outcomes of our study of instantons on S^6:

(i) the reduced twistor space $\mathcal{Z} \hookrightarrow \text{Tw}(X)$ of X may be more suitable for describing solutions of field equations on manifolds X with G-structure than the metric twistor space Tw(X),

(ii) the twistor description of gauge instantons in dimensions higher than four may lead to non-holomorphic bundles over the reduced twistor space $\mathcal{Z}$ even if $\mathcal{Z}$ is a complex manifold.

II. NEARLY KÄHLER STRUCTURE ON S^6

A. Almost complex structure

Let us consider the principal fibre bundle

$$G_2 \;\longrightarrow\; G_2/\text{SU}(3) \;=\; S^6 \tag{2.1}$$

with the Lie group SU(3) as the structure group. Let $\{e^a\}$ with $a = 1, \ldots, 6$ be a (local) coframe on S^6 compatible with the SU(3)-structure and $\{e^i\}$ with $i = 7, \ldots, 14$ be the components of an $su(3)$-valued connection on the bundle (2.1). Using e^a, one can introduce an almost complex structure J on S^6 such that

$$J\,\theta^\alpha = \mathrm{i}\,\theta^\alpha,\ \alpha = 1, 2, 3\ , \quad \text{for} \quad \theta^1 := e^1 + \mathrm{i}\,e^2,\ \theta^2 := e^3 + \mathrm{i}\,e^4,\ \theta^3 := e^5 + \mathrm{i}\,e^6, \tag{2.2}$$

as well as define forms

$$\omega := \tfrac{\mathrm{i}}{2}\big(\theta^1 \wedge \theta^{\bar 1} + \theta^2 \wedge \theta^{\bar 2} + \theta^3 \wedge \theta^{\bar 3}\big) \qquad \text{and} \qquad \Omega := \theta^1 \wedge \theta^2 \wedge \theta^3\ . \tag{2.3}$$

B. Flat connection on S^6

It is convenient to work with the matrices

$$\theta := \varkappa\left(\theta^1\,\theta^2\,\theta^3\right)\,, \quad \bar\theta := \varkappa\left(\theta^{\bar 1}\,\theta^{\bar 2}\,\theta^{\bar 3}\right)\,, \tag{2.4}$$

$$B := \sigma\begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix} \quad \text{and} \quad \bar B := \sigma\begin{pmatrix} 0 & \theta^{\bar 3} & -\theta^{\bar 2} \\ -\theta^{\bar 3} & 0 & \theta^{\bar 1} \\ \theta^{\bar 2} & -\theta^{\bar 1} & 0 \end{pmatrix} \tag{2.5}$$

with $\varkappa = \sqrt{\frac{2}{3}}$ and $\sigma = \sqrt{\frac{1}{3}}$. Using (2.4) and (2.5), one can introduce a flat Lie G_2-valued connection $\mathcal{A}_0$[35] on the trivial bundle $G_2 \times S^6 \to S^6$ as

$$\mathcal{A}_0 = \begin{pmatrix} \bar\Gamma & -\theta^\dagger & B \\ \theta & 0 & \bar\theta \\ \bar B & -\bar\theta^\dagger & \Gamma \end{pmatrix} \quad \text{with} \quad \Gamma = e^i I_i \quad \text{and} \quad \bar\Gamma = e^i \bar I_i = -e^i I_i^\top\,, \tag{2.6}$$

where $I_i = -I_i^\dagger$ are 3×3 matrix generators of the group SU(3) and $\Gamma = e^i I_i$ is the canonical connection in the bundle (2.1).

C. Maurer-Cartan equations on S^6

The flatness of the connection (2.6) (see Ref. 35) means that there exists a local G_2-valued function L, which is a coset representative of $G_2/\mathrm{SU}(3)$ such that $\mathcal{A}_0 = L^{-1}\mathrm{d}\,L$. Note that L is a local section of the bundle (2.1). For the curvature $\mathcal{F}_0 = \mathrm{d}\mathcal{A}_0 + \mathcal{A}_0 \wedge \mathcal{A}_0$, we find

$$\mathcal{F}_0 = \begin{pmatrix} \bar R - \theta^\dagger\wedge\theta + B\wedge\bar B & -(\mathrm{d}\theta^\dagger + \bar\Gamma\wedge\theta^\dagger + B\wedge\bar\theta^\dagger) & \mathrm{d}B + \bar\Gamma\wedge B - \theta^\dagger\wedge\bar\theta + B\wedge\Gamma \\ \mathrm{d}\theta + \theta\wedge\bar\Gamma + \bar\theta\wedge\bar B & -(\theta\wedge\theta^\dagger + \bar\theta\wedge\bar\theta^\dagger) & \mathrm{d}\bar\theta + \bar\theta\wedge\Gamma + \theta\wedge B \\ \mathrm{d}\bar B + \bar B\wedge\bar\Gamma - \bar\theta^\dagger\wedge\theta + \Gamma\wedge\bar B & -(\mathrm{d}\bar\theta^\dagger + \Gamma\wedge\bar\theta^\dagger + \bar B\wedge\theta^\dagger) & R - \bar\theta^\dagger\wedge\bar\theta + \bar B\wedge B \end{pmatrix}. \tag{2.7}$$

From $\mathcal{F}_0 = 0$, it follows that

$$\mathrm{d}\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \Gamma \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \frac{2}{\sqrt{3}}\begin{pmatrix} \theta^{\bar 2}\wedge\theta^{\bar 3} \\ \theta^{\bar 3}\wedge\theta^{\bar 1} \\ \theta^{\bar 1}\wedge\theta^{\bar 2} \end{pmatrix} \quad \Rightarrow \quad \mathrm{d}\theta^\alpha + \Gamma^\alpha_\beta\wedge\theta^\beta = T^\alpha\,, \tag{2.8}$$

where $\Gamma = (\Gamma^\alpha_\beta) = (\Gamma^i I^\alpha_{i\beta})$ is the canonical connection on the tangent bundle TS^6 associated to the bundle (2.1), and where $T^\alpha = \frac{1}{2}\,T^\alpha_{\bar\beta\bar\gamma}\theta^{\bar\beta}\wedge\theta^{\bar\gamma}$ is the intrinsic torsion of Γ (see, e.g., Ref. 36). Equation (2.8) and its complex conjugate constitute the Maurer-Cartan equations on the sphere S^6.

The curvature $R = \mathrm{d}\Gamma + \Gamma\wedge\Gamma$ of the connection Γ is read off (2.7) by equating to zero its lower right (or upper left) block,

$$R = \bar\theta^\dagger\wedge\bar\theta - \bar B\wedge B = \frac{1}{3}\begin{pmatrix} 2\theta^{1\bar 1} - \theta^{2\bar 2} - \theta^{3\bar 3} & 3\theta^{1\bar 2} & 3\theta^{1\bar 3} \\ 3\theta^{2\bar 1} & -\theta^{1\bar 1} + 2\theta^{2\bar 2} - \theta^{3\bar 3} & 3\theta^{2\bar 3} \\ 3\theta^{3\bar 1} & 3\theta^{3\bar 2} & -\theta^{1\bar 1} - \theta^{2\bar 2} + 2\theta^{3\bar 3} \end{pmatrix}, \tag{2.9}$$

where $\theta^{1\bar 1} = \theta^1\wedge\theta^{\bar 1}$ etc. Also, from (2.8) we see that the almost complex structure (2.2) on S^6 is not integrable due to the torsion T^α, which is a (0,2)-form with respect to J. It is easy to show that

$$\mathrm{d}\omega = 3\rho\,\mathrm{Im}\Omega \quad \text{and} \quad \mathrm{d}\Omega = 2\rho\,\omega\wedge\omega\,, \tag{2.10}$$

where $\rho \in \mathbb{R}$ is proportional to the inverse radius of S^6. The pair (ω, Ω) of forms subject to (2.10) turns S^6 into a nearly Kähler manifold (see, e.g.,Refs. 16, 32, and 36). It comes with a non-integrable SU(3)-structure.

D. Hermitian Yang-Mills equations

Consider an oriented $2n$-dimensional Riemannian manifold X^{2n} with an almost complex structure J and a complex vector bundle E over X^{2n} with a connection $\mathcal{A}$. According to Bryant,[32] a connection $\mathcal{A}$ on E defines a pseudo-holomorphic structure if it has curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of type (1,1) with respect to J, i.e., if $\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}$.

One can endow the bundle E with a Hermitian metric and choose $\mathcal{A}$ to be compatible with the Hermitian structure on E. If, in addition, ω is an almost Hermitian structure on (X^{2n}, J) and $c_1(E) = 0$,[40] then the equations

$$\mathcal{F}^{0,2} = -(\mathcal{F}^{2,0})^\dagger = 0 \qquad \text{and} \qquad \omega \lrcorner \mathcal{F} := \omega^{ab}\mathcal{F}_{ab} = 0 \tag{2.11}$$

are called the *Hermitian Yang-Mills equations*. The notation $\omega\lrcorner$ exploits the underlying Riemannian metric $g = \delta_{ab}e^a e^b$. In the case of an integrable almost complex structure J on X^{2n}, these equations were introduced by Donaldson and Uhlenbeck and Yau.[21]

We notice that the canonical connection Γ on the tangent bundle of $S^6 = G_2/\mathrm{SU}(3)$ satisfies the DUY Equation (2.11). In other words, its curvature obeys $R^{2,0} = 0 = R^{0,2}$ and $\omega \lrcorner R = 0$ with ω given in (2.3). This is easily seen from the explicit form (2.9) of the curvature R.

III. TWISTOR SPACES OF THE SIX-SPHERE

A. Twistor spaces of S^6

We mentioned in the Introduction that one can associate with S^6 two different twistor spaces, both with an integrable almost complex structure. The larger one, $\mathrm{Tw}(S^6) = \mathrm{Spin}(7)/\mathrm{U}(3)$, belongs to the round sphere $S^6 = \mathrm{Spin}(7)/\mathrm{SU}(4)$ having the full Lorentz symmetry $\mathrm{SO}(6) \cong \mathrm{SU}(4)/\mathbb{Z}_2$ on the tangent spaces and the Levi-Civita connection. The smaller twistor space, $\mathcal{Z} = G_2/\mathrm{U}(2)$, is associated with the nearly Kähler coset space $S^6 = G_2/\mathrm{SU}(3)$ having the canonical connection Γ with a torsion given in (2.8). The space $\mathcal{Z}$ is a complex submanifold of $\mathrm{Tw}(S^6)$. Note that the Hermitian Yang-Mills Equation (2.11) on S^6 is SU(3) invariant but not invariant under the full orthogonal group SO(6). This shows that the reduced twistor space $\mathcal{Z}$ is more suitable than $\mathrm{Tw}(S^6)$ for a description of instantons on S^6.

B. Coset representation of $\mathbb{C}P^2$

Let us consider the projection

$$\pi: \quad \mathcal{Z} \longrightarrow S^6 = G_2/\mathrm{SU}(3) \tag{3.1}$$

with fibres

$$\mathbb{C}P^2 = \mathrm{SU}(3)/\mathrm{U}(2)\ . \tag{3.2}$$

Let $J_{\mathbb{C}P^2}$ be a complex structure on $\mathbb{C}P^2$, $\{y^\alpha\}$ homogeneous coordinates on $\mathbb{C}P^2$ and

$$\lambda^1 = \frac{y^1}{y^3} \quad \text{and} \quad \lambda^2 = \frac{y^2}{y^3} \tag{3.3}$$

be local complex coordinates on the patch $\mathcal{U}_3 = \{y^3 \neq 0\} \subset \mathbb{C}P^2$.

One can choose as a coset representation of $\mathbb{C}P^2$ the matrix

$$V = \frac{1}{\gamma}\begin{pmatrix} W & \Lambda \\ -\Lambda^\dagger & 1 \end{pmatrix} := \frac{1}{\gamma}\begin{pmatrix} W_{11} & W_{12} & \lambda^1 \\ W_{21} & W_{22} & \lambda^2 \\ -\bar\lambda^{\bar 1} & -\bar\lambda^{\bar 2} & 1 \end{pmatrix} \in \mathrm{SU}(3)\ , \tag{3.4}$$

where

$$\gamma^2 := 1 + \Lambda^\dagger\Lambda = 1 + \lambda^1\bar\lambda^{\bar 1} + \lambda^2\bar\lambda^{\bar 2} \qquad \text{and} \qquad W = W^\dagger = \gamma\cdot\mathbb{1}_2 - \frac{1}{\gamma+1}\Lambda\Lambda^\dagger\ . \tag{3.5}$$

It is a local section of the bundle SU(3) $\to \mathbb{C}P^2 =$ SU(3)/U(2). From (3.4) and (3.5), it is easy to see that

$$W\Lambda = \Lambda \quad \text{and} \quad W^2 = \gamma^2 - \Lambda\Lambda^\dagger \qquad \Leftrightarrow \qquad V^\dagger V = \mathbb{1}_3 = VV^\dagger\ . \tag{3.6}$$

C. Flat connection on $\mathcal{Z}$

Using the group element (3.4) to parametrize the typical $\mathbb{C}P^2$-fibre in (3.1), we introduce a flat connection $\hat{\mathcal{A}}_0$ on the trivial bundle $G_2 \times \mathcal{Z} \to \mathcal{Z}$ as

$$\hat{\mathcal{A}}_0 \;=\; \hat{V}^\dagger \mathcal{A}_0 \hat{V} + \hat{V}^\dagger \mathrm{d}\hat{V} \qquad \text{with} \quad \hat{V} = \begin{pmatrix} \bar{V} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & V \end{pmatrix} \in G_2\ , \quad V \in \mathrm{SU}(3)\ , \tag{3.7}$$

where $\mathcal{A}_0$ is given in (2.6). One gets

$$\hat{\mathcal{A}}_0 \;=\; \begin{pmatrix} \bar{\hat{\Gamma}} & -\hat{\theta}^\dagger & \hat{B} \\ \hat{\theta} & 0 & \bar{\hat{\theta}} \\ \bar{\hat{B}} & -\bar{\hat{\theta}}^\dagger & \hat{\Gamma} \end{pmatrix} = \begin{pmatrix} \bar{V}^\dagger\bar{\Gamma}\bar{V} + \bar{V}^\dagger \mathrm{d}\bar{V} & -\bar{V}^\dagger\theta^\dagger & \bar{V}^\dagger B V \\ \theta\bar{V} & 0 & \bar{\theta}V \\ V^\dagger\bar{B}\bar{V} & -V^\dagger\bar{\theta}^\dagger & V^\dagger\Gamma V + V^\dagger \mathrm{d}V \end{pmatrix}, \tag{3.8}$$

and for the curvature $\hat{\mathcal{F}}_0 = \mathrm{d}\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_0 \wedge \hat{\mathcal{A}}_0$, we obtain

$$\hat{\mathcal{F}}_0 \;=\; \begin{pmatrix} \bar{\hat{R}} - \hat{\theta}^\dagger\wedge\hat{\theta} + \hat{B}\wedge\bar{\hat{B}} & -(\mathrm{d}\hat{\theta}^\dagger + \bar{\hat{\Gamma}}\wedge\hat{\theta}^\dagger + \hat{B}\wedge\bar{\hat{\theta}}^\dagger) & \mathrm{d}\hat{B} + \bar{\hat{\Gamma}}\wedge\hat{B} + \hat{B}\wedge\hat{\Gamma} - \hat{\theta}^\dagger\wedge\bar{\hat{\theta}} \\ \mathrm{d}\hat{\theta} + \hat{\theta}\wedge\bar{\hat{\Gamma}} + \bar{\hat{\theta}}\wedge\bar{\hat{B}} & -(\hat{\theta}\wedge\hat{\theta}^\dagger + \bar{\hat{\theta}}\wedge\bar{\hat{\theta}}^\dagger) & \mathrm{d}\bar{\hat{\theta}} + \bar{\hat{\theta}}\wedge\hat{\Gamma} + \hat{\theta}\wedge\hat{B} \\ \mathrm{d}\bar{\hat{B}} + \hat{\Gamma}\wedge\bar{\hat{B}} + \bar{\hat{B}}\wedge\bar{\hat{\Gamma}} - \bar{\hat{\theta}}^\dagger\wedge\hat{\theta} & -(\mathrm{d}\bar{\hat{\theta}}^\dagger + \hat{\Gamma}\wedge\bar{\hat{\theta}}^\dagger + \bar{\hat{B}}\wedge\hat{\theta}^\dagger) & \hat{R} - \bar{\hat{\theta}}^\dagger\wedge\bar{\hat{\theta}} + \bar{\hat{B}}\wedge\hat{B} \end{pmatrix}. \tag{3.9}$$

D. Maurer-Cartan equations on $\mathcal{Z}$

From the flatness $\hat{\mathcal{F}}_0 = 0$ with (3.8) and (3.9), it follows that

$$\bar{\hat{\theta}}^\dagger = V^\dagger\bar{\theta}^\dagger \;\Rightarrow\; \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix} W_{11} & W_{12} & -\lambda^1 \\ W_{21} & W_{22} & -\lambda^2 \\ \bar{\lambda}^{\bar{1}} & \bar{\lambda}^{\bar{2}} & 1 \end{pmatrix}\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} \quad \text{and}$$

$$\hat{B} = \bar{V}^\dagger B V = \sigma\begin{pmatrix} 0 & \hat{\theta}^3 & -\hat{\theta}^2 \\ -\hat{\theta}^3 & 0 & \hat{\theta}^1 \\ \hat{\theta}^2 & -\hat{\theta}^1 & 0 \end{pmatrix}, \tag{3.10}$$

where $\hat{\theta}^\alpha$ are (1,0)-forms with respect to $\pi^* J \oplus J_{\mathbb{C}P^2}$. The latter is not integrable since

$$\mathrm{d}\begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} + \hat{\Gamma}\wedge\begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} \;=\; \frac{2}{\sqrt{3}}\begin{pmatrix} \hat{\theta}^{\bar{2}}\wedge\hat{\theta}^{\bar{3}} \\ \hat{\theta}^{\bar{3}}\wedge\hat{\theta}^{\bar{1}} \\ \hat{\theta}^{\bar{1}}\wedge\hat{\theta}^{\bar{2}} \end{pmatrix} \qquad \Leftrightarrow \qquad \mathrm{d}\hat{\theta}^\alpha + \hat{\Gamma}^\alpha_\beta\wedge\hat{\theta}^\beta \;=\; \hat{T}^\alpha\ , \tag{3.11}$$

and we see a non-vanishing torsion $\hat{T}^\alpha$ with (0,2)-components. Here, the connection on the tangent bundle $T\mathcal{Z}$ reads

$$\hat{\Gamma} \;=\; V^\dagger\Gamma V + V^\dagger \mathrm{d}V \;=\; \begin{pmatrix} C_{11}+b & C_{12} & \hat{\theta}^4 \\ C_{21} & C_{22}+b & \hat{\theta}^5 \\ -\hat{\theta}^{\bar{4}} & -\hat{\theta}^{\bar{5}} & -2b \end{pmatrix}, \tag{3.12}$$

where

$$\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b\cdot\mathbf{1}_2 & 0 \\ 0 & -2b \end{pmatrix} \;=\; \begin{pmatrix} C_{11}+b & C_{12} & 0 \\ C_{21} & C_{22}+b & 0 \\ 0 & 0 & -2b \end{pmatrix} \tag{3.13}$$

is the canonical $u(2)$-valued connection on the principal bundle $G_2 \to G_2/\mathrm{U}(2) = \mathcal{Z}$, and $\hat\theta^4$, $\hat\theta^5$ are (1,0)-forms on the $\mathbb{C}P^2$ fibres of the twistor bundle (3.1).

E. Curvature of the connection $\hat\Gamma$

Consider the curvature $\hat R = \mathrm{d}\hat\Gamma + \hat\Gamma \wedge \hat\Gamma$ of the connection $\hat\Gamma$ on $T\mathcal{Z}$ given by (3.12) and (3.4). One can easily calculate $\hat R$ from (3.9) and obtain

$$\hat R = \begin{pmatrix} F^C_{11} + \mathrm{d}b - \hat\theta^{4\bar 4} & F^C_{12} - \hat\theta^{4\bar 5} & \mathrm{d}\hat\theta^4 + (C_{11}+3b)\wedge\hat\theta^4 + C_{12}\wedge\hat\theta^5 \\ F^C_{21} - \hat\theta^{5\bar 4} & F^C_{22} + \mathrm{d}b - \hat\theta^{5\bar 5} & \mathrm{d}\hat\theta^5 + C_{21}\wedge\hat\theta^4 + (C_{22}+3b)\wedge\hat\theta^5 \\ -(\mathrm{d}\hat\theta^{\bar 4} + \hat\theta^{\bar 4}\wedge(C_{11}+3b) + \hat\theta^{\bar 5}\wedge C_{21}) & -(\mathrm{d}\hat\theta^{\bar 5} + \hat\theta^{\bar 4}\wedge C_{12} + \hat\theta^{\bar 5}\wedge(C_{22}+3b)) & -2\mathrm{d}b + \hat\theta^{4\bar 4} + \hat\theta^{5\bar 5} \end{pmatrix},$$

$$= \begin{pmatrix} \frac13(2\hat\theta^{1\bar 1} - \hat\theta^{2\bar 2} - \hat\theta^{3\bar 3}) & \hat\theta^{1\bar 2} & \hat\theta^{1\bar 3} \\ \hat\theta^{2\bar 1} & \frac13(-\hat\theta^{1\bar 1} + 2\hat\theta^{2\bar 2} - \hat\theta^{3\bar 3}) & \hat\theta^{2\bar 3} \\ \hat\theta^{3\bar 1} & \hat\theta^{3\bar 2} & \frac13(-\hat\theta^{1\bar 1} - \hat\theta^{2\bar 2} + 2\hat\theta^{3\bar 3}) \end{pmatrix}, \qquad (3.14)$$

where

$$F^C = \mathrm{d}C + C\wedge C = \begin{pmatrix} F^C_{11} & F^C_{12} \\ F^C_{21} & F^C_{22} \end{pmatrix} \in su(2)\,. \qquad (3.15)$$

The components of F^C and $\mathrm{d}b$ can be read off from (3.14). Equation (3.14) also tells us that

$$\mathrm{d}\begin{pmatrix} \hat\theta^4 \\ \hat\theta^5 \end{pmatrix} + (C + 3b\cdot\mathbb{1}_2)\wedge\begin{pmatrix} \hat\theta^4 \\ \hat\theta^5 \end{pmatrix} = \begin{pmatrix} \hat\theta^{1\bar 3} \\ \hat\theta^{2\bar 3} \end{pmatrix}. \qquad (3.16)$$

Together with (3.11), this can be considered as the Maurer-Cartan equations on $\mathcal{Z}$ for the forms $\hat\theta^A$, $A = 1, \ldots, 5$. Those are (1,0)-forms with respect to an almost complex structure $\mathcal{J}_- = \pi^*J \oplus J_{\mathbb{C}P^2}$ on $\mathcal{Z}$ defined via

$$\mathcal{J}_-\hat\theta^A = \mathrm{i}\,\hat\theta^A\,. \qquad (3.17)$$

The non-vanishing (0,2)-type components of the torsion $\hat T^A$ obstruct the integrability of $\mathcal{J}_-$.

F. Integrable almost complex structure on $\mathcal{Z}$

We may introduce a different almost complex structure $\mathcal{J}_+$ on $\mathcal{Z}$ with the property

$$\mathcal{J}_+\vartheta^A = \mathrm{i}\,\vartheta^A \quad \text{for} \quad \vartheta^1 := \hat\theta^1,\quad \vartheta^2 := \hat\theta^2,\quad \vartheta^3 := \hat\theta^{\bar 3},\quad \vartheta^4 := \hat\theta^4 \quad\text{and}\quad \vartheta^5 := \hat\theta^5 \qquad (3.18)$$

and denote $\overline{\vartheta^A} =: \vartheta^{\bar A}$. Then from (3.11) and (3.16), we obtain

$$\mathrm{d}\vartheta^A + \widetilde\Gamma^A_B \wedge \vartheta^B = \widetilde T^A\,,$$

where the connection $\widetilde\Gamma = (\widetilde\Gamma^A_B)$ and the torsion $\widetilde T{=}(\widetilde T^A)$ are given by

$$\widetilde\Gamma = \begin{pmatrix} C_{11}+b & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22}+b & 0 & 0 & 0 \\ 0 & 0 & 2b & 0 & 0 \\ 0 & 0 & 0 & C_{11}+3b & C_{12} \\ 0 & 0 & 0 & C_{21} & C_{22}+3b \end{pmatrix} \quad\text{and}\quad \widetilde T = \begin{pmatrix} \frac{2}{\sqrt3}\vartheta^{3\bar 2} - \vartheta^{4\bar 3} \\ -\frac{2}{\sqrt3}\vartheta^{3\bar 1} - \vartheta^{5\bar 3} \\ -\frac{2}{\sqrt3}\vartheta^{12} + \vartheta^{4\bar 1} + \vartheta^{5\bar 2} \\ \vartheta^{13} \\ \vartheta^{23} \end{pmatrix}. \qquad (3.19)$$

Note that $\widetilde\Gamma$ is the *canonical* $u(2)$-valued connection on the tangent bundle $T\mathcal{Z}$, and $\widetilde T$ is the torsion of $\widetilde\Gamma$. The torsion $\widetilde T^A$ in (3.19) has no (0,2)-components with respect to the almost complex structure $\mathcal{J}_+$. Therefore, $\mathcal{J}_+$ is integrable, i.e., $(\mathcal{Z}, \mathcal{J}_+)$ is a complex manifold.

IV. TWISTOR DESCRIPTION OF INSTANTON BUNDLES OVER S^6

A. Pulled-back curvature

Consider a complex vector bundle E over S^6 with a connection one-form $\mathcal{A}$ having curvature $\mathcal{F}$. Recall that $(E, \mathcal{A})$ is called an instanton bundle if $\mathcal{A}$ satisfies the Hermitian Yang-Mills (HYM or DUY) Equation (2.11), which on S^6 can be written in the form

$$\mathcal{F}^{0,2} = 0 \qquad \Leftrightarrow \qquad \Omega \wedge \mathcal{F} = 0\,, \tag{4.1}$$

$$\omega \lrcorner \mathcal{F} = 0 \qquad \Leftrightarrow \qquad \omega \wedge \omega \wedge \mathcal{F} = 0\,. \tag{4.2}$$

Here, (ω, Ω) given in (2.3) are forms defining on S^6 a nearly Kähler structure. Note that, in this case, (4.2) follows from (4.1) due to (2.10).

Consider the twistor fibration (3.1). Let $(\widetilde{E}, \widetilde{\mathcal{A}}) = (\pi^* E, \pi^* \mathcal{A})$ be the pulled-back instanton bundle over $\mathcal{Z}$ with curvature $\widetilde{\mathcal{F}} = \mathrm{d}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}}$. We have

$$\widetilde{\mathcal{F}} \;=\; \tfrac{1}{2}\,\widetilde{\mathcal{F}}_{\alpha\beta}\,\vartheta^{\alpha} \wedge \vartheta^{\beta} + \widetilde{\mathcal{F}}_{\alpha\bar{\beta}}\,\vartheta^{\alpha} \wedge \vartheta^{\bar{\beta}} + \tfrac{1}{2}\,\widetilde{\mathcal{F}}_{\bar{\alpha}\bar{\beta}}\,\vartheta^{\bar{\alpha}} \wedge \vartheta^{\bar{\beta}} \;=\; \pi^* \mathcal{F}\,. \tag{4.3}$$

Using the relation (3.10) between θ^{α} and $\hat{\theta}^{\alpha}$ as well as the definition (3.18) of ϑ^{A}, we obtain

$$\widetilde{\mathcal{F}}_{\bar{1}\bar{2}} \;=\; \tfrac{1}{\gamma}\left\{\mathcal{F}_{\bar{1}\bar{2}} + \lambda^1 \mathcal{F}_{\bar{2}\bar{3}} + \lambda^2 \mathcal{F}_{\bar{3}\bar{1}}\right\}. \tag{4.4}$$

Vanishing of $\widetilde{\mathcal{F}}_{\bar{1}\bar{2}}$ for *all* values of $(\lambda^1, \lambda^2) \in \mathbb{C}P^2$ is equivalent to the instanton Eqs. (4.1) and (4.2). In contrast, for $\widetilde{\mathcal{F}}_{\bar{1}\bar{3}}$ and $\widetilde{\mathcal{F}}_{\bar{2}\bar{3}}$, we obtain complicated expressions, which vanish for all λ^1, λ^2 only if all components of the curvature $\mathcal{F}$ vanish. This yields the trivial case of a flat connection on E. In homogeneous coordinates y^{α} on $\mathbb{C}P^2$, this condition can be written as

$$\widetilde{\mathcal{F}}_{\bar{1}\bar{2}} = 0 \qquad \Leftrightarrow \qquad y^{\alpha} \varepsilon_{\alpha\beta\gamma} \mathcal{F}^{\beta\gamma} = 0\,, \tag{4.5}$$

where the indices $\bar{\alpha}, \bar{\beta}, \ldots$ are raised with the metric $\delta^{\alpha\bar{\beta}}$.

B. Correspondence of bundles

Let us denote by L_A vector fields on $\mathcal{Z}$ of type (1,0) (with respect to the complex structure $\mathcal{J}_+$) and by $L_{\bar{A}}$ their complex conjugates, $A = 1, \ldots, 5$. Then we can introduce a rank-2 subbundle[41] $\mathcal{T}^{0,1}_{(2)}$ of $T^{0,1}\mathcal{Z}$ spanned by $L_{\bar{1}}$ and $L_{\bar{2}}$ as well as a rank-4 subbundle $\mathcal{T}^{0,1}_{(4)}$ of $T^{0,1}\mathcal{Z}$ with $\{L_{\bar{1}}, L_{\bar{2}}, L_{\bar{4}}, L_{\bar{5}}\}$ as a basis. Note that $\widetilde{\mathcal{F}}_{\bar{1}\bar{2}}$ is the curvature of a partial connection $\nabla_{\mathcal{T}^{0,1}_{(2)}}$ (see Ref. 33) along the distribution $\mathcal{T}^{0,1}_{(2)}$, and we can extend it to a partial connection $\nabla_{\mathcal{T}^{0,1}_{(4)}}$ along $\mathcal{T}^{0,1}_{(4)}$ by putting $\widetilde{\mathcal{A}}_{\bar{4}} = 0 = \widetilde{\mathcal{A}}_{\bar{5}}$. Neither $\mathcal{T}^{0,1}_{(2)}$, nor $\mathcal{T}^{0,1}_{(4)}$ is integrable as a subbundle of $T^{0,1}\mathcal{Z}$ and, therefore, we cannot consider $\pi^* E$ as a Cauchy-Riemann (CR) bundle. This follows from the explicit form of the torsion (3.19) of the U(2)-structure on $\mathcal{Z}$. It would be interesting to repeat our analysis for the other three known homogeneous nearly Kähler spaces and to check whether there exist special cases where the integrability obstructions vanish. However, this is beyond the scope of our paper, which deals with S^6 only.

In summary, we have the following picture:

$$\begin{array}{ccc} \pi^* E & \longrightarrow & \mathcal{Z} \\ & & \pi \big\downarrow \mathbb{C}P^2 \\ E & \longrightarrow & S^6 \end{array} \tag{4.6}$$

where (E, ∇) with $\nabla = \mathrm{d} + \mathcal{A}$ is a Hermitian Yang-Mills bundle with curvature $\mathcal{F} = \nabla^2$ satisfying (4.1) and (4.2). From the above discussion, we obtain the equivalence of two assertions:

(i) $(\pi^* E, \pi^* \nabla)$ has its curvature $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$ vanishing along the distribution $\mathcal{T}^{0,1}_{(4)} \subset T^{0,1}\mathcal{Z}$.

(ii) $(E, \mathcal{A})$ is a HYM (instanton) bundle over S^6.
The non-integrability of the distribution $\mathcal{T}^{0,1}_{(4)}$ means that the HYM equations on S^6 are not integrable, contrary to the self-dual Yang-Mills equations on S^4. Hence, constructing instanton configurations in six dimensions is a task more complicated than one might expect.

C. Relation with instantons on $\mathbb{R}^7$

Note that the cone $C(S^6)$ over S^6 with the metric

$$ds_7^2 \;=\; dr^2 + r^2 ds^2_{S^6} \qquad \text{with} \quad r \in \mathbb{R}_+ \tag{4.7}$$

is flat space, $C(S^6) = \mathbb{R}^7\backslash\{0\}$, for a proper normalization of the S^6 coframe $\{e^a\}$ such that $\rho = 1$ in (2.10). Employing the forms (ω, Ω) defining the nearly Kähler structure on S^6, we introduce on $\mathbb{R}^7$ the 3-form

$$\psi \;:=\; r^2\omega \wedge dr + r^3\,\mathrm{Im}\Omega\;. \tag{4.8}$$

It is not difficult to show that (up to index permutation) its only nonzero coefficients are

$$\psi_{\hat a\hat b\hat c} = 1 \qquad \text{for} \quad (\hat a\hat b\hat c) = (136), (426), (145), (235), (127), (347), (567) \tag{4.9}$$

in the basis $\{dx^{\hat a}\}$ with coordinates $x^{\hat a}$ on $\mathbb{R}^7$ such that $\delta_{\hat a\hat b}x^{\hat a}x^{\hat b} = r^2$. The above 3-form ψ defines a G_2-structure on $\mathbb{R}^7$, i.e., it is invariant under the $G_2 \subset \mathrm{SO}(7)$ action. Its components (4.9) are often called octonionic structure constants.

Consider now a complex vector bundle $\mathcal{E}$ over $\mathbb{R}^7$ with a connection $\mathcal{A}'$ and curvature $\mathcal{F}'$. Employing the Hodge operator $*$ in $\mathbb{R}^7$, we impose on $\mathcal{A}'$ the first-order differential equations

$$*\psi \wedge \mathcal{F}' = 0 \qquad \Leftrightarrow \qquad \psi_{\hat a\hat b\hat c}\mathcal{F}'^{\hat b\hat c} = 0, \tag{4.10}$$

which are called G_2-instanton equations.[25] Their solutions automatically satisfy the Yang-Mills equations on $\mathbb{R}^7$.

It was shown by Tian[26] that solutions $\mathcal{A}'$ of (4.10) obeying also

$$\partial_r \lrcorner\, \mathcal{A}' = 0 \qquad \text{and} \qquad \partial_r \lrcorner\, \mathcal{F}' = 0 \tag{4.11}$$

are equivalent to solutions $\mathcal{A}$ of the HYM Equations (4.1) and (4.2) on S^6. He calls such configurations *tangent instantons* on $\mathbb{R}^7$. Examples of such instanton solutions were discussed in Ref. 29 and 37.

A twistor description of solutions to (4.10) on any 7-dimensional Riemannian manifold X with G_2-holonomy was recently proposed by Verbitsky.[18] Namely, he introduced a so-called CR twistor space of X as the bundle $\pi\colon S^6X \to X$ of unit six-spheres in the tangent bundle TX. For $\mathbb{R}^7$, this space is a direct product manifold

$$\mathrm{Tw}(\mathbb{R}^7) \;=\; \mathbb{R}^7 \times S^6\;. \tag{4.12}$$

It was shown[18] that the complexified tangent bundle of Tw(X) has an integrable complex rank-3 subbundle $\mathcal{T}^{0,1}_{(3)}$ if X is a G_2-holonomy manifold. For a bundle $\mathcal{E}$ over X with a connection $\mathcal{A}'$, one can introduce the pulled-back bundle $(\pi^*\mathcal{E}, \pi^*\mathcal{A}')$ over Tw(X). It was proven that G_2-instanton bundles over X correspond to CR-bundles over Tw(X) with a flat partial (0,1)-connection $\bar\partial_{\pi^*\mathcal{E}}$ defined on the distribution $\mathcal{T}^{0,1}_{(3)}$.[18] In other words, the G_2-instanton equations on X are equivalent to the equations $\bar\partial^2_{\pi^*\mathcal{E}} = 0$ on Tw(X). This theorem obviously applies to the case of $X = \mathbb{R}^7$. Specializing then to tangent solutions (in the sense of (4.11)) to the G_2-instanton Equation (4.10) on $\mathbb{R}^7$ will yield solutions of the HYM equations on S^6. Thus, the twistor description of instantons on S^6 is related to the twistor description of G_2-instanton solutions on $\mathbb{R}^7$.

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[38] Instead of S^6, one can also consider $\mathbb{R}^6 = S^6 \setminus \{\infty\}$ with a metric twistor space $Q_6' = \mathrm{Tw}(\mathbb{R}^6) \subset Q_6$ as an intersection of the quadric Q_6 with $\mathbb{C}P^7 \setminus \mathbb{C}P^3$. This space Q_6' is also the twistor space for the complexified space-time $\mathbb{C}^6$ with a double fibration establishing the correspondence between subspaces in Q_6' and $\mathbb{C}^6$ (see, e.g., Refs. 2 and 3 and references therein). There is no such correspondence in the real framework where the twistor space is the bundle of almost complex structures on a manifold.
[39] In the mathematical literature, they are often called Hermitian Yang-Mills equations.
[40] From a bundle with curvature $\mathcal{F}$ of non-zero degree, we can obtain a zero-degree bundle E by considering $\widetilde{\mathcal{F}} = \mathcal{F} - \frac{1}{k}(\mathrm{tr}\mathcal{F}) \cdot \mathbb{1}_k$, where $k = \mathrm{rank}E$.
[41] Recall that $T^{0,1}\mathcal{Z}$ is an integrable subbundle of the complexification of $T\mathcal{Z}$.