



Superstring limit of Yang–Mills theories



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ABSTRACT

It was pointed out by Shifman and Yung that the critical superstring on $X^{10} = \mathbb{R}^4 \times Y^6$, where Y^6 is the resolved conifold, appears as an effective theory for a $U(2)$ Yang–Mills–Higgs system with four fundamental Higgs scalars defined on $\Sigma_2 \times \mathbb{R}^2$, where Σ_2 is a two-dimensional Lorentzian manifold. Their Yang–Mills model supports semilocal vortices on $\mathbb{R}^2 \subset \Sigma_2 \times \mathbb{R}^2$ with a moduli space X^{10} . When the moduli of slowly moving thin vortices depend on the coordinates of Σ_2 , the vortex strings can be identified with critical fundamental strings. We show that similar results can be obtained for the low-energy limit of pure Yang–Mills theory on $\Sigma_2 \times T_p^2$, where T_p^2 is a two-dimensional torus with a puncture p . The solitonic vortices of Shifman and Yung then get replaced by flat connections. Various ten-dimensional superstring target spaces can be obtained as moduli spaces of flat connections on T_p^2 , depending on the choice of the gauge group. The full Green–Schwarz sigma model requires extending the gauge group to a supergroup and augmenting the action with a topological term.

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1. Introduction

Recently, Koroteev, Shifman and Yung [1–3] have shown that $U(2)$ solitonic vortex strings in certain $\mathcal{N} = 2$ super-Yang–Mills theories have an effective infrared dynamics of a critical fundamental string on a ten-dimensional target space $X^{10} = \mathbb{R}^4 \times Y^6$, where Y^6 is the resolved conifold.¹ More precisely, $\mathcal{N} = 2$ supersymmetric $U(2)$ Yang–Mills–Higgs theory on $\Sigma_2 \times \mathbb{R}^2$, where Σ_2 is a two-dimensional Lorentzian manifold, with a Fayet–Iliopoulos term and four flavor hypermultiplets in the fundamental of $U(2)$ admits non-Abelian semilocal vortices on \mathbb{R}^2 whose (translational, orientational and size) moduli are parametrized by X^{10} . Allowing the vortex moduli to depend on the coordinates of Σ_2 yields a string sigma model with worldsheet Σ_2 and target X^{10} , which describes the effective vortex dynamics.

In [1–3] the $\mathcal{N} = 2$ super-Yang–Mills model with fundamental matter was chosen because it admits vortex solutions with a Ricci-flat ten-dimensional moduli space. Also the metric on Σ_2 was taken as an independent variable. These two assumptions differ

from earlier treatments [8,9], where $\mathcal{N} = 4$ super-Yang–Mills theory on $\Sigma_2 \times \tilde{\Sigma}_2$ in the infrared limit ($\tilde{\Sigma}_2$ shrinking to a point) was reduced to certain sigma models on Σ_2 whose target space is the moduli space \mathcal{M} of flat connections² on a Riemann surface $\tilde{\Sigma}_2$.

In pure Yang–Mills theory and its standard supersymmetric extensions one gets flat connections instead of vortices. This is just as well, as we will demonstrate for $\tilde{\Sigma}_2 = T_p^2$, a two-dimensional torus T^2 with a puncture p . This case is simpler than that of a circle S^1 or a disk H^2 considered earlier [11–13], and it deserves a separate study. Therefore, in this paper we investigate the infrared limit of pure Yang–Mills theory on $\Sigma_2 \times T_p^2$, and we describe further examples of string backgrounds which can be obtained in this framework.

The organization of this paper is as follows. In Section 2 we describe a four-manifold $M^4 = \Sigma_2 \times T_p^2$ with an ε -deformed metric and introduce the ε -dependent Yang–Mills action on M^4 with a gauge group G , where $\varepsilon \in [0, \infty)$. In Section 3 we perform the low-energy limit $\varepsilon \rightarrow 0$ under which the Yang–Mills theory reduces to a stringy sigma model. We explain in some detail how gauge-field moduli become coordinates on the sigma-model target space (cf. [14,9,15]). Its effective action and Virasoro-type constraints will be derived. In Section 4 we provide a number of

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¹ For fine survey articles on non-Abelian vortices, their moduli spaces and reductions to effective $d = 2$ sigma models see e.g. [4–7] and references therein.

² From twisted super-Yang–Mills theories one can also get the cotangent bundle $T^*\mathcal{M}$ as target space, see e.g. [9,10].

examples of the above-mentioned target spaces, including supercosets such as $\text{PSU}(2, 2|4)/\text{SO}(4, 1) \times \text{SO}(5)$ related with $\text{AdS}_5 \times S^5$. The Conclusions summarize our findings and point out possible generalizations and applications.

2. Yang–Mills theory

Lie (super)groups In our approach the Green–Schwarz superstring action can be obtained from Yang–Mills theory in four dimensions with a supergroups as structure group (cf. [11–13]). However, here we mainly restrict ourselves to deriving the bosonic part of superstring actions, similarly as in [1–3]. This will make the discussion simpler and clearer. Green–Schwarz actions for various target spaces and the corresponding Lie supergroups will be briefly discussed in Section 4.

For the Yang–Mills structure group we consider a Lie group G with a closed subgroup H . Then, for the Lie algebras $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{h} = \text{Lie } H$ we have³

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad (2.1)$$

where \mathfrak{m} is the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to a metric $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . For matrix (super)algebras, $\langle X, Y \rangle = (\text{S})\text{tr}(XY)$ is the ordinary trace or supertrace. For additive groups like \mathbb{R}^k , it denotes the ordinary metric on vector spaces.

Gauge fields We consider Yang–Mills theory on a direct product manifold

$$M^4 = \Sigma_2 \times T_p^2 \quad \text{with coordinates} \quad (x^\mu) = (x^a, x^i) \\ \text{for } a = 1, 2 \quad \text{and} \quad i = 3, 4, \quad (2.2)$$

where Σ_2 is a two-dimensional Lorentzian manifold with a metric tensor $g_{\Sigma_2} = (g_{ab})$, and $T_p^2 = T^2 \setminus \{p\}$ is a two-dimensional torus with a point p removed (the puncture) and a metric $g_{T^2} = (g_{ij})$. We will just write T^2 (omitting the puncture) since we do not consider other tori in this paper. Then the metric tensor on M^4 reads $(g_{\mu\nu}) = (g_{ab}, g_{ij})$ with $\mu, \nu = 1, \dots, 4$. Fixing momentarily the size of T^2 , $\det(g_{ij}) = 1$, the metric g_{T^2} still depends on the complex shape parameter τ . For simplicity we choose the square torus $\tau = i$, i.e. we identify $x^i \sim x^i + 1$ for both homology circles.

We consider a topologically trivial bundle over M^4 (the principal G -bundle P and an associated vector bundle $E \rightarrow M^4$) with a gauge potential $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ (a connection) taking values in \mathfrak{g} . The \mathfrak{g} -valued gauge field (the curvature) reads

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \quad (2.3)$$

On $M^4 = \Sigma_2 \times T^2$ we have the obvious splitting

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + g_{ij} dx^i dx^j, \quad (2.4)$$

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \mathcal{A}_{\Sigma_2} + \mathcal{A}_{T^2} = \mathcal{A}_a dx^a + \mathcal{A}_i dx^i, \quad (2.5)$$

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{ab} dx^a \wedge dx^b + \mathcal{F}_{ai} dx^a \wedge dx^i + \frac{1}{2} \mathcal{F}_{ij} dx^i \wedge dx^j. \quad (2.6)$$

We note that there are mixed components \mathcal{F}_{ai} in (2.6).

Let us now deform the metric (2.4) and introduce

$$ds_\varepsilon^2 = g_{\mu\nu}^\varepsilon dx^\mu dx^\nu = g_{ab} dx^a dx^b + \varepsilon^2 g_{ij} dx^i dx^j \\ \text{hence} \quad g_{ab}^\varepsilon = g_{ab} \quad \text{and} \quad g_{ij}^\varepsilon = \varepsilon^2 g_{ij}, \quad (2.7)$$

where $\varepsilon \in [0, \infty)$ is a dimensionless real parameter. Then $\det(g_{\mu\nu}^\varepsilon) = \varepsilon^4 \det(g_{ab})$ and

$$\mathcal{F}_\varepsilon^{ab} = g_\varepsilon^{ac} g_\varepsilon^{bd} \mathcal{F}_{cd} = \mathcal{F}^{ab}, \quad \mathcal{F}_\varepsilon^{ai} = g_\varepsilon^{ac} g_\varepsilon^{ij} \mathcal{F}_{cj} = \varepsilon^{-2} \mathcal{F}^{ai}, \\ \mathcal{F}_\varepsilon^{ij} = g_\varepsilon^{ik} g_\varepsilon^{jl} \mathcal{F}_{kl} = \varepsilon^{-4} \mathcal{F}^{ij}, \quad (2.8)$$

where the indices in $\mathcal{F}^{\mu\nu}$ are raised by the nondeformed metric tensor $g^{\mu\nu}$. One can introduce on T^2 adapted coordinates $y^i = \varepsilon x^i$ for which $y^i \sim y^i + \varepsilon$. In other words, the deformation reintroduces the size modulus of T^2 : for $\varepsilon^2 \rightarrow 0$ the torus shrinks to a point.⁴ This limit is equivalent to the low-energy limit of gauge theory on $\Sigma_2 \times T^2$ [8].

Yang–Mills action For the deformed metric (2.7) the Yang–Mills action functional is

$$S_\varepsilon = \int_{M^4} d^4x \sqrt{|\det g_{\Sigma_2}|} \left\{ \varepsilon^2 \langle \mathcal{F}_{ab}, \mathcal{F}^{ab} \rangle + 2 \langle \mathcal{F}_{ai}, \mathcal{F}^{ai} \rangle \right. \\ \left. + \varepsilon^{-2} \langle \mathcal{F}_{ij}, \mathcal{F}^{ij} \rangle \right\}. \quad (2.9)$$

For $\varepsilon^2 = 1$ one has the standard Yang–Mills Lagrangian on $M^4 = \Sigma_2 \times T^2$ with the nondeformed metric (2.4), and for $\varepsilon^2 \rightarrow 0$ it reduces to a stringy sigma-model action on Σ_2 as we will see in a moment.

We play with the metric on T^2 , but the metric on Σ_2 can be dynamical, i.e. the Yang–Mills model is coupled to (two-dimensional) gravity. Therefore, one can add to the Lagrangian in (2.9) the term

$$\sqrt{|\det g_{M^4}|} R_{M^4} = \varepsilon^2 \sqrt{|\det g_{\Sigma_2}|} R_{\Sigma_2}, \quad (2.10)$$

where R_{M^4} and R_{Σ_2} are the scalar curvatures of M^4 and of Σ_2 , respectively, with the metric (2.7). The term (2.10) does not contribute to the equations of motion since integration of (2.10) over M^4 gives a topological invariant of Σ_2 . This is not so if we couple (2.10) with the dilaton field Φ , but anyway the term (2.10) vanishes in the limit $\varepsilon^2 \rightarrow 0$ which we consider. For this reason we do not add (2.10) to the Yang–Mills Lagrangian in (2.9).

The Yang–Mills equations following from (2.9) are

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_i \mathcal{F}^{ib} = 0, \quad (2.11)$$

$$D_a \mathcal{F}^{aj} + \varepsilon^{-2} D_i \mathcal{F}^{ij} = 0, \quad (2.12)$$

where D_a, D_i are Yang–Mills covariant derivatives on the curved background $M^4 = \Sigma_2 \times T^2$. The Euler–Lagrange equations for g_{Σ_2} yield the constraint equations

$$T_{ab}^\varepsilon \equiv \varepsilon^2 (g^{cd} \langle \mathcal{F}_{ac}, \mathcal{F}_{bd} \rangle - \frac{1}{4} g_{ab} \langle \mathcal{F}_{cd}, \mathcal{F}^{cd} \rangle) + g^{ij} \langle \mathcal{F}_{ai}, \mathcal{F}_{bj} \rangle \\ - \frac{1}{2} g_{ab} \langle \mathcal{F}_{ci}, \mathcal{F}^{ci} \rangle - \frac{1}{4} \varepsilon^{-2} g_{ab} \langle \mathcal{F}_{ij}, \mathcal{F}^{ij} \rangle = 0 \quad (2.13)$$

for the Yang–Mills energy-momentum tensor $T_{\mu\nu}^\varepsilon$, i.e. its components along Σ_2 are vanishing. Its other components, T_{ij}^ε or T_{aj}^ε , are not constrained. Note that we might employ the invariance under diffeomorphisms on Σ_2 to locally fix its metric, e.g., to a flat metric in the conformal gauge. Nevertheless, (2.13) must be added as an external constraint.

³ This splitting will be used later in defining a boundary condition for gauge connections.

⁴ It is usually assumed that \mathcal{A}_μ and $\mathcal{F}_{\mu\nu}$ smoothly depend on ε^2 with a well-defined limit for $\varepsilon^2 \rightarrow 0$.

3. Low-energy effective action

Adiabatic limit As usual in the adiabatic approach (see e.g. [16, 17]), we assume that the connection \mathcal{A} for small ε^2 can be expanded in a Taylor series in ε^2 , i.e. $\mathcal{A} = \mathcal{A}^0 + \varepsilon^2 \mathcal{A}^1 + O(\varepsilon^4)$. In particular, $\mathcal{A}_{T^2} = \mathcal{A}_{T^2}^0 + \varepsilon^2 \mathcal{A}_{T^2}^1 + O(\varepsilon^4)$ and therefore

$$\mathcal{F}_{ij} = \mathcal{F}_{ij}^0 + \varepsilon^2 (D_i^0 \mathcal{A}_j^1 - D_j^0 \mathcal{A}_i^1) + O(\varepsilon^4), \quad (3.1)$$

where $D_i^0 = \partial_i + [\mathcal{A}_i^0, \cdot]$ and $\mathcal{F}_{ij}^0 = [D_i^0, D_j^0]$. From (2.9) one sees that the term $\varepsilon^{-2} \langle \mathcal{F}_{ij}^0, \mathcal{F}^{0ij} \rangle$ in the Yang–Mills action diverges when $\varepsilon^2 \rightarrow 0$. To avoid this one should impose the condition

$$\mathcal{F}_{ij}^0 = 0 \quad (3.2)$$

on the components of the Yang–Mills field along T^2 . We denote by \mathcal{M}_{T^2} the moduli space of solutions (flat connections) to the equations (3.2) on T^2 with a puncture at p . It is known (see e.g. [17, 18]) that terms of order ε^{2k} in \mathcal{A}_{T^2} for $k \geq 1$ are orthogonal to \mathcal{M}_{T^2} and yield some massive modes in the effective theory on Σ_2 . A consideration of these modes goes beyond the scope of this paper. In the limit $\varepsilon^2 \rightarrow 0$ we keep only \mathcal{A}^0 and \mathcal{F}^0 (zero-mode moduli approximation), and from now on we omit the index “0” in connection \mathcal{A}^0 and the curvature \mathcal{F}^0 .

In the adiabatic approximation (when $\varepsilon^2 \rightarrow 0$), the Yang–Mills action (2.9) becomes

$$S_0 = \int_{M^4} d^4x \sqrt{|\det g_{\Sigma_2}|} \langle \mathcal{F}_{ai}, \mathcal{F}^{ai} \rangle. \quad (3.3)$$

As equations of motion one gets

$$D_i \mathcal{F}^{ib} \equiv \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_i \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} g^{ij} \mathcal{F}_{aj} \right) + [\mathcal{A}_i, \mathcal{F}^{ib}] = 0, \quad (3.4)$$

$$D_a \mathcal{F}^{aj} \equiv \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|} g^{ab} g^{ij} \mathcal{F}_{ib} \right) + [\mathcal{A}_a, \mathcal{F}^{aj}] = 0. \quad (3.5)$$

The constraint equations (2.13) in the limit $\varepsilon^2 \rightarrow 0$ have the form

$$T_{ab}^0 \equiv g^{ij} \langle \mathcal{F}_{ai}, \mathcal{F}_{bj} \rangle - \frac{1}{2} g_{ab} \langle \mathcal{F}_{ci}, \mathcal{F}^{ci} \rangle = 0. \quad (3.6)$$

Flat connection on T^2 It is well known that on smooth tori T^2 (compact, without punctures) there are no irreducible flat connections which are constant and take values in the Cartan subalgebra of \mathfrak{g} (see e.g. [8]). This so-called “abelianization” theorem is widely used in the literature on Yang–Mills confinement on $\mathbb{R}^3 \times S^1$ and $\mathbb{R}^2 \times T^2$. However, this theorem is not valid on Riemann surfaces with punctures or fixed points (see e.g. [20–22]). In particular, on tori T^2 with a puncture one can find irreducible flat connections on G -bundles over T^2 [21], and the same is true for higher genus (see e.g. [21,22]).⁵

Flat connections, i.e. solutions of (3.2), on a torus T^2 with a puncture can be described as follows [21]. The puncture $p \in T^2$ can be considered as infinity similar to the north pole on the two-sphere S^2 , and one can introduce cylindrical coordinates (ϱ, θ)

on a small disk centered at p via $x^3 = \exp(-\varrho) \cos \theta$ and $x^4 = \exp(-\varrho) \sin \theta$. The group of gauge transformations is defined as the Banach Lie group

$$\mathcal{G}_{T^2} = \{ \text{smooth maps } g : T^2 \rightarrow G \}, \quad (3.7)$$

whose topology is described in [21,22].

On the flat connections \mathcal{A}_{T^2} we impose the boundary condition

$$\mathcal{A}_{T^2} = \mathcal{A}_\varrho d\varrho + \mathcal{A}_\theta d\theta \rightarrow \mathcal{A}_p = a d\theta \quad \text{for } \varrho \rightarrow \infty. \quad (3.8)$$

Here a is either an arbitrary element of \mathfrak{m} for the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ introduced in (2.1), or $a = g_p h_0 g_p^{-1}$, where $h_0 \in \mathfrak{h}$ is fixed and $g_p \in G/H$ is arbitrary. Then \mathcal{A}_p is parametrized by $g_0 = \exp(2\pi a) \in G/H$ for $a \in \mathfrak{m}$ or $g_p \in G/H$, where the case $H = \{\text{Id}\}$ is included. If we denote by \mathcal{N} the space of all such flat connections then their moduli space is

$$\mathcal{M} = \mathcal{N} / \mathcal{G}_{T^2} = G/H. \quad (3.9)$$

In other words, the gauge group (3.7) forms the fibers over points in \mathcal{M} for the bundle

$$\pi : \mathcal{N} \xrightarrow{\mathcal{G}_{T^2}} \mathcal{M} = G/H. \quad (3.10)$$

Note that, if G/H is an adjoint orbit, e.g. the Kähler coset space

$$G/H = U(N)/U(N_1) \times \dots \times U(N_k) \quad \text{with } N_1 + \dots + N_k = N, \quad (3.11)$$

then \mathcal{M} is the moduli space of irreducible flat connections on vector bundles with parabolic structure (see [20,21] for more details).

Moduli space We endow our moduli space \mathcal{M} of flat connections on a punctured T^2 with local coordinates (ϕ^α) , with $\alpha = 1, \dots, \dim(\mathcal{M})$. In the adiabatic approach, the moduli approximation assumes that ϕ^α depend on $x^a \in \Sigma_2$ [14,4–10] In this way, the moduli of flat connections on T^2 define a map

$$\begin{aligned} \phi : \Sigma_2 \rightarrow \mathcal{M} \quad \text{via } (x^a) \mapsto \{\phi^\alpha(x^a)\} \\ \text{so that } \mathcal{A}_{T^2} = \mathcal{A}_{T^2}(\phi^\alpha(x^a), x^i). \end{aligned} \quad (3.12)$$

Now our space \mathcal{N} of solutions to (3.2) depends on $x \in \Sigma_2$ as well as on elements g of the gauge group \mathcal{G}_{T^2} . In fact, for any fixed $x \in \Sigma_2$ and $\mathcal{G}_x = \mathcal{G}_{T^2}(x^a)$, the gauge group \mathcal{G} of the full theory on $M^4 = \Sigma_2 \times T^2$ coincides with \mathcal{G}_{T^2} . Said differently, for any fixed $x \in \Sigma_2$ we have a copy of the moduli space $\mathcal{M}_x = \mathcal{N}_x / \mathcal{G}_x \cong G/H$ of flat connections on T^2 .

The maps (3.12) are not arbitrary – they are constrained by the equations (3.4)–(3.6). Since \mathcal{A}_{T^2} is a flat connection on T^2 for any point in Σ_2 , the derivatives $\partial_a \mathcal{A}_i$ have to satisfy the linearized (around \mathcal{A}_i) flatness equations (3.2). In other words, $\partial_a \mathcal{A}_i$ belong to the tangent space $T_{\mathcal{A}} \mathcal{N}$ of the solution space \mathcal{N} . Using the projection (3.10), one can orthogonally decompose $\partial_a \mathcal{A}_i$ into two parts,

$$T_{\mathcal{A}} \mathcal{N} = \pi^* T_{\mathcal{A}} \mathcal{M} \oplus T_{\mathcal{A}} \mathcal{G} \quad \Rightarrow \quad (3.13)$$

$$\begin{aligned} \frac{\partial \mathcal{A}_i}{\partial \phi^\alpha} &= \xi_{\alpha i} + D_i \epsilon_\alpha \quad \Leftrightarrow \\ \partial_a \mathcal{A}_i &= \frac{\partial \phi^\alpha}{\partial x^a} \frac{\partial \mathcal{A}_i}{\partial \phi^\alpha} = (\partial_a \phi^\alpha) \xi_{\alpha i} + D_i \epsilon_\alpha, \end{aligned} \quad (3.14)$$

where

$$dx^i D_i \epsilon_\alpha \in T_{\mathcal{A}} \mathcal{G} \quad \text{and} \quad \epsilon_\alpha := (\partial_a \phi^\alpha) \epsilon_\alpha, \quad (3.15)$$

⁵ Irreducible flat connections on complex vector bundles over smooth Riemann surfaces define *stable* holomorphic bundles. For vector bundles over Riemann surfaces with punctures, stability is replaced with Seshadri’s notion of parabolic stability [20,21].

i.e. ϵ_α are \mathfrak{g} -valued gauge parameters from the viewpoint of gauge theory on T^2 , and $\xi_\alpha = \xi_{\alpha i} dx^i \in T_{\mathcal{A}}\mathcal{M}$ can be identified with vector fields on $\mathcal{M} = G/H$. Thus, ξ_α correspond to generators from the subspace \mathfrak{m} in the Lie-algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

The fields ϵ_a are determined by the gauge-fixing conditions

$$g^{ij} D_i \xi_{\alpha j} = 0 \quad \Rightarrow \quad g^{ij} D_i D_j \epsilon_a = g^{ij} D_i \partial_a \mathcal{A}_j. \quad (3.16)$$

Note that, due to (3.2), one can solve the first equation,

$$2\varepsilon^{ij} D_i D_j = \varepsilon^{ij} \mathcal{F}_{ij} = 0 \quad \Rightarrow \quad \xi_{\alpha j} = \varepsilon_j^k D_k \xi_\alpha, \quad (3.17)$$

with $\varepsilon_j^k := g^{kl} \varepsilon_{lj}$ and $\varepsilon^{ij} = g^{ik} \varepsilon_k^j$.

Effective action Recall that \mathcal{A}_i obey (3.2) and have the moduli space $\mathcal{M} = G/H$, which parametrizes the boundary values of the connection at the puncture $p \in T^2$. The case $H = \{\text{Id}\}$ of a group manifold $\mathcal{M} = G$ is included. On the other hand, the components \mathcal{A}_a are yet free. It is natural to identify them with ϵ_a [9,15],⁶

$$\mathcal{A}_a = \epsilon_a \quad \Rightarrow \quad \mathcal{F}_{ai} = \partial_a \mathcal{A}_i - D_i \mathcal{A}_a = (\partial_a \phi^\alpha) \xi_{\alpha i} \in T_{\mathcal{A}}\mathcal{M}. \quad (3.18)$$

Substituting (3.18) into (3.4), we see that (3.4) is resolved due to (3.16). Plugging (3.18) into the action (3.3), we get the effective sigma-model action

$$S_0 = \int_{\Sigma_2} dx^1 dx^2 \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_a \phi^\alpha \partial_b \phi^\beta G_{\alpha\beta}, \quad (3.19)$$

where

$$G_{\alpha\beta}(\phi, \tau) = \int_{T^2} dx^3 dx^4 g^{ij} (\xi_{\alpha i}, \xi_{\beta j}) \quad (3.20)$$

is a metric on the moduli space \mathcal{M} , and the argument τ reminds us of a dependence on the shape of T^2 . One can also show that the equations (3.5) are equivalent to the Euler–Lagrange equations for ϕ^α following from (3.19) (cf. [11]). Finally, substituting (3.18) into (3.6), we arrive at

$$(\delta_a^c \delta_b^d - \frac{1}{2} g_{ab} g^{cd}) \partial_c \phi^\alpha \partial_d \phi^\beta G_{\alpha\beta} = 0, \quad (3.21)$$

which can also be obtained from (3.19) by varying the metric g_{Σ_2} . These are the Virasoro-type constraint equations.

4. Examples

Here we briefly discuss examples of $d = 10$ manifolds considered in the string literature. The list is not complete and serves only illustrative purposes. Superstring theories in all these backgrounds can be obtained from Yang–Mills theory via the adiabatic limit $\varepsilon^2 \rightarrow 0$ as discussed in the previous section.

AdS₄ × CP³ The background

$$G/H = \text{AdS}_4 \times \mathbb{C}P^3 = \frac{\text{SO}(3, 2)}{\text{SO}(3, 1)} \times \frac{\text{SU}(4)}{\text{U}(3)} \quad (4.1)$$

is considered in the context of the AdS₄/CFT₃ correspondence relating the IIA string in the coset (4.1) with $\mathcal{N} = 6$ super-Chern–Simons theory in three dimensions. Here $\mathbb{C}P^3$ is the standard complex projective space fibered over S^4 with $\mathbb{C}P^1$ -fibers,

$$\mathbb{C}P^3 \xrightarrow{\mathbb{C}P^1} S^4. \quad (4.2)$$

It has an integrable almost complex structure \mathcal{J}_+ defining (1,0)-forms ω^a on $\mathbb{C}P^3$ ($a = 1, 2, 3$) via $\mathcal{J}_+ \omega^a = i\omega^a$.

AdS₄ × CP³_{qK} The background (4.1) is not suitable for the consideration of heterotic strings since the Kähler space $\mathbb{C}P^3$ has a U(3) holonomy. The situation is changed if one switches from the integrable almost complex structure \mathcal{J}_+ on $\mathbb{C}P^3$ to a non-integrable one \mathcal{J}_- , which defines a quasi-Kähler space $\mathbb{C}P^3_{qK}$ isomorphic to $\mathbb{C}P^3$ as a smooth manifold. The (1,0)-forms Θ^a with respect to \mathcal{J}_- , obeying $\mathcal{J}_- \Theta^a = i\Theta^a$, relate to the previous ones as follows [23],

$$\Theta^1 = \omega^{\bar{1}}, \quad \Theta^2 = \omega^{\bar{2}} \quad \text{and} \quad \Theta^3 = \omega^{\bar{3}}. \quad (4.3)$$

The manifold $\mathbb{C}P^3_{qK}$, defined by \mathcal{J}_- and the (1,0)-forms (4.3), has the structure group $\text{U}(2) \subset \text{SU}(3)$, and its almost complex structure \mathcal{J}_- is non-integrable due to torsion [23]. Let Λ be the radius of S^4 and R be the radius of $\mathbb{C}P^1$ from (4.2). For $\Lambda^2 = 2R^2$ the space $\mathbb{C}P^3_{qK}$ is nearly Kähler and the torsion is totally antisymmetric. Since the latter may then be identified with the H -field flux, such manifolds appear in heterotic string compactifications with fluxes (see e.g. [24–26] and references therein).

Resolved conifold The resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{C}P^1$ can be obtained in our approach by considering a moduli space of flat connections on the punctured T^2 of the form

$$G/H = \mathbb{R}^{3,1} \times \frac{\text{SU}(5)}{\text{U}(4)} = \mathbb{R}^{3,1} \times \mathbb{C}P^4 \quad (4.4)$$

and restricting to the non-singular *quintic threefold* in $\mathbb{C}P^4$ (the zeros of a homogeneous quintic polynomial in the homogeneous $\mathbb{C}P^4$ coordinates) [27]. The same trick can be employed in the approach of Shifman and Yung, since $\mathbb{C}P^N$ spaces are the standard moduli spaces of non-Abelian vortices (see reviews [4–7]).

*Space T*S³* One can always view the cotangent bundle T^*K of a Lie group K as a Lie group. To this end, one performs a left trivialization (admitted by the parallelizability of K) and endows the resulting trivial bundle $K \times (\text{Lie}K)^*$ with the semi-direct product $K \ltimes (\text{Lie}K)^*$ by using the coadjoint action of K on the space $(\text{Lie}K)^*$ dual to $\text{Lie}K$. In the case of $K = \text{SU}(2)$ we can identify $\mathfrak{su}(2)^*$ with $\mathfrak{su}(2)$ and consider the six-dimensional real group manifold $\text{SU}(2) \ltimes \mathfrak{su}(2)$, which is diffeomorphic to the deformed conifold T^*S^3 . Choosing a proper metric tensor $G_{\alpha\beta}$ on this space, one can obtain string theory on $\mathbb{R}^{3,1} \times T^*S^3$ as the low-energy limit of Yang–Mills theory.

Flat d = 10 superspace For obtaining the Green–Schwarz superstring action (of type I, IIA or IIB) one should employ supergroups \tilde{G} instead of Lie groups G which can be embedded in \tilde{G} as bosonic subgroups, $G \subset \tilde{G}$, and the infrared limit of the corresponding supergroup gauge theories. This was demonstrated for superstrings in [11,13] and for supermembranes in [12]. Those papers treated the moduli space of flat connections on the circle S^1 or on the disk H^2 with proper boundary conditions. Here instead we use the moduli space of super-Lie-algebra valued flat connections on the punctured T^2 . This moduli space is a finite-dimensional supercoset space

$$\tilde{\mathcal{M}} = \tilde{G}/H, \quad (4.5)$$

and the analysis is simpler than in [13] where moduli spaces were loop supercosets. However, the derivation of the low-energy limit is so similar that we will not repeat it here and describe only the final results.

⁶ In fact, $\epsilon_a d\phi^\alpha$ in (3.14) is a connection on a G -bundle over \mathcal{M} , and $\epsilon_a dx^a$ from (3.15) is the pull-back of the connection $\epsilon_a d\phi^\alpha$ from the G -bundle over \mathcal{M} to the G -bundle over Σ_2 [9]. Therefore, $\mathcal{A}_a dx^a$ and $\epsilon_a dx^a$ are connections on the same bundle over Σ_2 , and it is natural to identify them.

So, for superstrings moving in Minkowski space $\mathbb{R}^{9,1}$, one should extend the bosonic Lie group of translation $G = \mathbb{R}^{9,1}$ to the supergroup (cf. [28])

$$\tilde{G} = \frac{N = 2 \text{ SUSY}}{\text{SO}(9, 1)}, \quad (4.6)$$

which is a subgroup of the $N = 2$ super-Poincaré group in ten dimensions generated by translations and $N = 2$ supersymmetry transformations. Coordinates on \tilde{G} are $(X^\Delta) = (X^\alpha, \theta^{Ap})$, where X^α with $\alpha = 0, \dots, 9$ parametrize $\mathbb{R}^{9,1}$ and θ^{Ap} with $A = 1, \dots, 32$ and $p = 1, 2$ are the components of two Majorana–Weyl spinors θ^p . By considering Yang–Mills theory on $M^4 = \Sigma_2 \times T^2$ with \tilde{G} as the gauge group and taking the adiabatic $\varepsilon^2 \rightarrow 0$ limit in (2.9), we get a string moving in the moduli space \tilde{G} of flat connections on the punctured T^2 . Its action functional reads

$$S_0 = \int_{\Sigma_2} dx^1 dx^2 \sqrt{|\det g_{\Sigma_2}|} g^{ab} \eta_{\alpha\beta} \Pi_a^\alpha \Pi_b^\beta, \quad (4.7)$$

where $\eta = (\eta_{\alpha\beta})$ is the Minkowski metric on $\mathbb{R}^{9,1}$, and

$$\Pi_a^\Delta = (\Pi_a^\alpha, \Pi_a^{Ap})$$

with $\Pi_a^\alpha = \partial_a X^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^\alpha \partial_a \theta^q$ and $\Pi_a^{Ap} = \partial_a \theta^{Ap}$ (4.8)

are the components of one-forms $\Pi^\Delta = dx^a \Pi_a^\Delta$ on Σ_2 pulled back from one-forms dX^α and $d\theta^{Ap}$ on \tilde{G} . Finally, γ^α are γ -matrices in $\mathbb{R}^{9,1}$ and $\bar{\theta}^p := (\theta^p)^\top C$, where C is the charge conjugation matrix.

The action (4.7) is not yet the full Green–Schwarz action, which needs an additional Wess–Zumino-type term [29]. This term may also be obtained from supergroup gauge theory, by extending Σ_2 and T^2 to a Lorentzian 3-manifold Σ_3 with boundary $\partial\Sigma_3 = \Sigma_2$ (as in [28]) and to a Riemannian 3-manifold B^3 with boundary $\partial B^3 = T^2$ (as in [13] for H^2). Then on $M^6 = \Sigma_3 \times B^3$ one can formulate the topological Yang–Mills term

$$S_{WZ} = \int_{\Sigma_3 \times B^3} f_{\Gamma\Delta\Lambda} \hat{F}^\Gamma \wedge \hat{F}^\Delta \wedge \hat{F}^\Lambda \quad (4.9)$$

for a Lie \tilde{G} -valued gauge field \hat{F} on M^6 , where the structure constants $f_{\Gamma\Delta\Lambda}$ are given in [28]. By the same calculations as in [13] one finds that in the low-energy limit $\varepsilon^2 \rightarrow 0$ the action (4.9) reduces to a Wess–Zumino-type action functional [28,29] which should be added to (4.7) with a proper coefficient. Also, similarly to [13] one can show that the Kalb–Ramond B -field appears from the topological term $\eta_{\alpha\beta} \mathcal{F}^\alpha \wedge \mathcal{F}^\beta$, whose integral in the adiabatic limit $\varepsilon^2 \rightarrow 0$ becomes

$$\int_{M^4} d^4x \varepsilon^{ab} \varepsilon^{ij} \langle \mathcal{F}_{ai}, \mathcal{F}_{bj} \rangle = \int_{\Sigma_2} dx^1 dx^2 \varepsilon^{cd} B_{\alpha\beta} \partial_c X^\alpha \partial_d X^\beta, \quad (4.10)$$

where

$$B_{\alpha\beta} = \int_{T^2} dx^3 dx^4 \varepsilon^{ij} \langle \xi_{\alpha i}, \xi_{\beta j} \rangle \quad (4.11)$$

are components of a two-form $\mathbb{B} = (B_{\alpha\beta})$ on the moduli space (4.6).

$\text{AdS}_5 \times S^5$ The coset space

$$G/H = \text{AdS}_5 \times S^5 = \frac{\text{SO}(4, 2)}{\text{SO}(4, 1)} \times \frac{\text{SO}(6)}{\text{SO}(5)} \quad (4.12)$$

is important in the $\text{AdS}_5/\text{CFT}_4$ correspondence between type IIB strings on this coset space and $\mathcal{N} = 4$ super–Yang–Mills theory on

the boundary $\mathbb{R}^{3,1}$ of AdS_5 . The group $G = \text{SO}(4, 2) \times \text{SO}(6)$ can be embedded into the supergroup $\tilde{G} = \text{PSU}(2, 2|4)$, and the supercoset \tilde{G}/H with $H = \text{SO}(4, 1) \times \text{SO}(5)$ is used for describing the superstring action [30]. Considering gauge theory with the supergroup $\tilde{G} = \text{PSU}(2, 2|4)$ on $M^4 = \Sigma_2 \times T^2$, we get in the $\varepsilon^2 \rightarrow 0$ limit the moduli space \tilde{G}/H of flat connections on T^2 as the string target space. Both (4.7) and (4.9) will apply with a proper choice of $G_{\alpha\beta}$ and $f_{\Gamma\Delta\Lambda}$ on G/H and \tilde{G}/H , because in this limit the non-vanishing components of \mathcal{F} (and \hat{F}) are proportional to the pull-back

$$L^\Delta = (dX^M) L_M^\Delta \rightarrow \Pi^\Delta = (dx^a) \Pi_a^\Delta$$

where $\Pi_a^\Delta = (\partial_a X^M) L_M^\Delta$, (4.13)

and the index Δ runs over the coset parts of the generators of $\text{psu}(2, 2|4)$ [30]. The explicit form of the superstring action (both kinetic and WZ terms) in terms of Π_a^Δ can be found in [30]. Similarly one can derive the full type IIA string action on $\text{AdS}_4 \times \mathbb{C}P^3$ by considering supergroup gauge theory on $\Sigma_2 \times T^2$ with $\tilde{G} = \text{OSp}(2, 2|6)$ and $H = \text{SO}(3, 1) \times \text{U}(3)$. Note that in (4.7) one will have the metric $G_{\alpha\beta}$ on the coset G/H instead of $\eta_{\alpha\beta}$.

5. Conclusions

We have shown that the Yang–Mills action on the product of a two-dimensional Lorentzian manifold Σ_2 and a singly-punctured two-torus T_p^2 , augmented by a topological term, flows to the Green–Schwarz superstring action on the worldsheet Σ_2 in the infrared limit, when T_p^2 shrinks to a point. Upon choosing a supergroup \tilde{G} as the gauge group and picking a closed subgroup $H \subset \tilde{G}$, the string target space becomes the supercoset \tilde{G}/H as the moduli space of flat Yang–Mills connections on T_p^2 . We mainly focused on the bosonic part of the superstring action because we want to emphasize the fundamental possibility of receiving superstring sigma models in an infrared limit of corresponding suitable Yang–Mills theories. A lot of backgrounds, including $\text{PSU}(2, 2|4)/\text{SO}(4, 1) \times \text{SO}(5)$ and $\text{OSp}(2, 2|6)/\text{SO}(3, 1) \times \text{U}(3)$, may appear as moduli spaces of flat connections on T_p^2 . Various other backgrounds can be obtained by generalizing the T_p^2 factor to a Riemann surface $\tilde{\Sigma}_2$ with punctures or boundaries, whose moduli space of flat connections will depend on the geometry and boundary conditions. In the infrared limit of gauge theory on $\Sigma_2 \times \tilde{\Sigma}_2$, this moduli space becomes the target space of a string sigma model on Σ_2 , promising a fresh perspective on the string vacuum landscape. Clearly, the relation between Yang–Mills and string theories deserve further study.

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References

- [1] M. Shifman, A. Yung, Critical string from non-Abelian vortex in four dimensions, Phys. Lett. B 750 (2015) 416, arXiv:1502.00683 [hep-th].
- [2] P. Koroteev, M. Shifman, A. Yung, Studying critical string emerging from non-Abelian vortex in four dimensions, Phys. Lett. B 759 (2016) 154, arXiv:1605.01472 [hep-th].
- [3] P. Koroteev, M. Shifman, A. Yung, Non-Abelian vortex in four dimensions as a critical string on a conifold, Phys. Rev. D 94 (2016) 065002, arXiv:1605.08433 [hep-th].
- [4] D. Tong, TASI lectures on solitons: instantons, monopoles, vortices and kinks, arXiv:hep-th/0509216.
- [5] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Solitons in the Higgs phase: the moduli matrix approach, J. Phys. A 39 (2006) R315, arXiv:hep-th/0602170.

- [6] M. Shifman, A. Yung, Supersymmetric solitons and how they help us understand non-Abelian gauge theories, *Rev. Mod. Phys.* 79 (2007) 1139, arXiv:hep-th/0703267.
- [7] D. Tong, Quantum vortex strings: a review, *Ann. Phys.* 324 (2009) 30, arXiv:0809.5060 [hep-th].
- [8] J.A. Harvey, G.W. Moore, A. Strominger, Reducing S-duality to T-duality, *Phys. Rev. D* 52 (1995) 7161, arXiv:hep-th/9501022.
- [9] M. Bershadsky, A. Johansen, V. Sadov, C. Vafa, Topological reduction of 4d SYM to 2d sigma models, *Nucl. Phys. B* 448 (1995) 166, arXiv:hep-th/9501096.
- [10] E. Witten, More on gauge theory and geometric Langlands, arXiv:1506.04293 [hep-th].
- [11] A.D. Popov, String theories as the adiabatic limit of Yang–Mills theory, *Phys. Rev. D* 92 (2015) 045003, arXiv:1505.07733 [hep-th].
- [12] O. Lechtenfeld, A.D. Popov, Supermembrane limit of Yang–Mills theory, *J. Math. Phys.* 57 (2016) 023520, arXiv:1508.06325 [hep-th].
- [13] A.D. Popov, Superstring theories as low-energy limit of supergroup gauge theories, arXiv:1605.01582 [hep-th].
- [14] N.S. Manton, A remark on the scattering of BPS monopoles, *Phys. Lett. B* 110 (1982) 54.
- [15] E.J. Weinberg, P. Yi, Magnetic monopole dynamics, supersymmetry, and duality, *Phys. Rep.* 438 (2007) 65, arXiv:hep-th/0609055.
- [16] D. Stuart, Dynamics of Abelian Higgs vortices in the near-Bogomolny regime, *Commun. Math. Phys.* 159 (1994) 51;
D. Stuart, The geodesic approximation for the Yang–Mills–Higgs equations, *Commun. Math. Phys.* 166 (1994) 149.
- [17] A.G. Sergeev, Adiabatic limit in the Ginzburg–Landau and Seiberg–Witten equations, *Proc. Steklov Inst. Math.* 289 (2015) 227.
- [18] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, N. Sakai, Higher derivative corrections to non-Abelian vortex effective theory, *Prog. Theor. Phys.* 128 (2012) 67, arXiv:1204.0773 [hep-th].
- [19] M.F. Atiyah, R. Bott, The Yang–Mills equations over Riemann surfaces, *Philos. Trans. R. Soc. Lond. A* 308 (1982) 523.
- [20] V.B. Mehta, C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* 248 (1980) 205.
- [21] G.D. Daskalopoulos, R.A. Wentworth, Geometric quantization for the moduli space of vector bundles with parabolic structure, in: *Geometry, Topology and Physics*, Campinas, 1996, de Gruyter, Berlin, 1997, p. 119.
- [22] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps, *J. Differ. Geom.* 48 (1998) 445.
- [23] J. Eells, S. Salamon, Constructions twistorielles des applications harmoniques, *C. R. Acad. Sci. Paris* 296 (1983) 685;
S. Salamon, Harmonic and holomorphic maps, *Lect. Notes Math.* 1164 (1985) 161.
- [24] G. Lopes Cardoso, G. Curio, G. Dall’Agata, D. Lüst, P. Manousselis, G. Zoupanos, Non-Kähler string backgrounds and their five torsion classes, *Nucl. Phys. B* 652 (2003) 5, arXiv:hep-th/0211118;
P. Manousselis, N. Prezas, G. Zoupanos, Supersymmetric compactifications of heterotic strings with fluxes and condensates, *Nucl. Phys. B* 739 (2006) 85, arXiv:hep-th/0511122;
I. Benmachiche, J. Louis, D. Martinez-Pedraza, The effective action of the heterotic string compactified on manifolds with SU(3) structure, *Class. Quantum Gravity* 25 (2008) 135006, arXiv:0802.0410 [hep-th].
- [25] O. Lechtenfeld, C. Nölle, A.D. Popov, Heterotic compactifications on nearly Kähler manifolds, *J. High Energy Phys.* 09 (2010) 074, arXiv:1007.0236 [hep-th];
A. Chatzistavrakidis, O. Lechtenfeld, A.D. Popov, Nearly Kähler heterotic compactifications with fermion condensates, *J. High Energy Phys.* 04 (2012) 114, arXiv:1202.1278 [hep-th];
K.P. Gemmer, A.S. Haupt, O. Lechtenfeld, C. Nölle, A.D. Popov, Heterotic string plus five-brane systems with asymptotic AdS₃, *Adv. Theor. Math. Phys.* 17 (2013) 771, arXiv:1202.5046 [hep-th].
- [26] M. Klaput, A. Lukas, C. Matti, E.E. Svanes, Moduli stabilising in heterotic nearly Kähler compactifications, *J. High Energy Phys.* 01 (2013) 015, arXiv:1210.5933 [hep-th];
X. de la Ossa, M. Larfors, E.E. Svanes, Exploring SU(3) structure moduli spaces with integrable G₂ structures, *Adv. Theor. Math. Phys.* 19 (2015) 837, arXiv:1409.7539 [hep-th].
- [27] P. Candelas, X. de la Ossa, P.S. Green, L. Parkes, A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, *Nucl. Phys. B* 359 (1991) 21.
- [28] M. Henneaux, L. Mezincescu, A sigma model interpretation of Green–Schwarz covariant superstring action, *Phys. Lett. B* 152 (1985) 340.
- [29] M.B. Green, J.H. Schwarz, Covariant description of superstrings, *Phys. Lett. B* 136 (1984) 367.
- [30] R.R. Metsaev, A.A. Tseytlin, Type IIB superstring action in AdS₅ × S⁵ background, *Nucl. Phys. B* 533 (1998) 109, arXiv:hep-th/9805028.