Upper limits on a stochastic gravitational-wave background using LIGO and Virgo interferometers at 600–1000 Hz
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I. INTRODUCTION

A major science goal of current and future generations of gravitational-wave detectors is the detection of a stochastic gravitational-wave background (SGWB)—a superposition of unresolvable gravitational-wave signals of astrophysical and/or cosmological origin. An astrophysical background is expected to be comprised of signals originating from astrophysical objects, for example, binary neutron stars [1], spinning neutron stars [2], magnetars [3], or core-collapse supernovae [4]. A cosmological background is expected to be generated by various physical processes in the early universe [5] and, as gravitational waves are so weakly interacting, to be essentially unattenuated since then. We expect that gravitational waves would decouple much earlier than other radiation, so a cosmological background would carry the earliest information accessible about the very early universe [6]. There are various production mechanisms from which we might expect cosmological gravitational waves including cosmic strings [7], amplification of vacuum fluctuations following inflation [8,9], pre-Big-Bang models [10,11], or the electroweak phase transition [12].

Whatever the production mechanism of a SGWB, the signal is usually described in terms of the dimensionless quantity,

\[ \Omega_{GW}(f) = \frac{\int p_{GW}}{\rho_c} \frac{df}{df}. \]

where \( d\rho_{GW} \) is the energy density of gravitational radiation contained in the frequency range \( f \) to \( f + df \) and \( \rho_c \) is the critical energy density of the universe [13]. As a SGWB signal is expected to be much smaller than current detector noise, and because we assume both the detector noise and the signal to be Gaussian random variables, it is not feasible to distinguish the two in a single interferometer. We must therefore search for the SGWB using two or more interferometers. The optimal method is to cross-correlate the strain data from a pair, or several pairs of detectors [13]. In recent years, several interferometric gravitational-wave detectors have been in operation in the USA and Europe. At the time that the data analyzed in this paper were taken, five interferometers were in operation. Two LIGO interferometers were located at the same site in Hanford, WA, one with 4 km arms and one with 2 km arms (referred to as H1 and H2, respectively). In addition, one LIGO 4 km interferometer, L1, was located in Livingston, LA [14]. The Virgo interferometer, V1, with 3 km arms was located near Pisa, Italy [15] and GEO600, with 600 m arms, was located near Hannover, Germany [16]. LIGO carried out its fifth science run, along with GEO600, between 5th November 2005 and 30th September 2007. They were joined from 18th May 2007 by Virgo, carrying out its first science run. In this paper we present a joint analysis of the data taken by the LIGO and Virgo detectors during these periods, in the frequency range 600–1000 Hz. This is the first search for a SGWB using data from both LIGO and Virgo interferometers, and the first using multiple baselines. Previous searches using the LIGO interferometers used just one baseline. The most sensitive direct limit obtained so far used the three LIGO interferometers, but as the two Hanford interferometers were collocated this involved just one baseline [17]. The most recent upper limit in frequency band studied in this paper was obtained using data from the LIGO-Livingston interferometer and the ALLEGRO bar detector, which were collocated for the duration of the analysis [18]. The addition of Virgo to the LIGO interferometers adds two further baselines, for which the frequency dependence of the sensitivity varies differently. The frequency range used in this paper was chosen because the addition of Virgo data was expected to most improve the sensitivity at these high frequencies. This is due in part to the relative orientation and separation of
the LIGO and Virgo interferometers, and in part to the fact that the Virgo sensitivity is closest to the LIGO sensitivity at these frequencies. The GEO600 interferometer was not included in this analysis as the strain sensitivity at these frequencies was insufficient to significantly improve the sensitivity of the search.

The structure of this paper is as follows. In Sec. II we describe the method used to analyze the data. In Sec. III we present the results of the analysis of data from the LIGO and Virgo interferometers. We describe validation of the results using software injections in Sec. IV. In Sec. V we compare our results to those of previous experiments and in Sec. VI we summarize our conclusions.

II. ANALYSIS METHOD

The output of an interferometer is assumed to be the sum of instrumental noise and a stochastic background signal,

$$s(t) = n(t) + h(t).$$

(2)

The gravitational-wave signal has a power spectrum, $S_{GW}(f)$, which is related to $\Omega_{GW}(f)$ by [19]

$$S_{GW}(f) = \frac{3H_0^2}{10\pi^2} \frac{\Omega_{GW}(f)}{f^3}. \tag{3}$$

Our signal model is a power law spectrum,

$$\Omega_{GW}(f) = \Omega_\alpha \left(\frac{f}{900 \text{ Hz}}\right)^\alpha, \tag{4}$$

where $\alpha$ is the spectral index, and $f_R$ a reference frequency, such that $\Omega_\alpha = \Omega_{GW}(f_R)$. For this analysis we create a filter using a model which corresponds to a white strain amplitude spectrum and choose a reference frequency of 900 Hz, such that

$$\Omega_{GW}(f) = \Omega_3 \left(\frac{f}{900 \text{ Hz}}\right)^3. \tag{5}$$

We choose this spectrum as it is expected that some astrophysical backgrounds will have a rising $\Omega_{GW}(f)$ spectrum in the frequency band we are investigating [2–4]. In fact, different models predict different values of the spectral index $\alpha$ in our frequency band, so we quote upper limits for several values.

For a pair of detectors, with interferometers labeled by $i$ and $j$, we calculate the cross-correlation statistic in the frequency domain

$$\hat{Y} = \int_{-\infty}^{+\infty} df Y(f)$$

$$\quad = \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' \delta_T(f - f') \hat{s}_i^*(f) \hat{s}_j(f') \hat{Q}_{ij}(f'), \tag{6}$$

where $\hat{s}_i(f)$ and $\hat{s}_j(f)$ are the Fourier transforms of the strain time-series of two interferometers, $\hat{Q}_{ij}(f)$ is a filter function, and $\delta_T$ is a finite-time approximation to the Dirac delta function, [13]

$$\delta_T(t) := \int_{-T/2}^{T/2} dt e^{-i2\pi ft} = \frac{\sin(\pi f T)}{\pi f}. \tag{7}$$

We assume the detector noise is Gaussian, stationary, uncorrelated between the two interferometers and much larger than the signal. Under these assumptions, the variance of the estimator $\hat{Y}$ is

$$\sigma^2_Y = \int_{0}^{+\infty} df \sigma^2_Y(f) = \frac{T}{2} \int_{0}^{+\infty} df P_i(f) P_j(f) \left| \hat{Q}_{ij}(f) \right|^2, \tag{8}$$

where $P_i(f)$ is the one-sided power spectral density of interferometer $i$ and $T$ is the integration time. By maximizing the expected signal-to-noise ratio (SNR) for a chosen model of $\Omega_{GW}(f)$, we find the optimal filter function,

$$\hat{Q}_{ij}(f) = \mathcal{N} \frac{\gamma_{ij}(f) f^{\alpha - 3}}{\int_{f_R}^{f} P_i(f) P_j(f)} \hat{Y}(f), \tag{9}$$

where $\gamma_{ij}(f)$ is the overlap reduction function (ORF) of the two interferometers and $\mathcal{N}$ is a normalization factor. We choose the normalization such that the cross-correlation statistic is an estimator of $\Omega_\alpha$, with expectation value $\langle \hat{Y} \rangle = \Omega_\alpha$. It follows that the normalization is

$$\mathcal{N} = \frac{f_R^{2\alpha}}{T} \left( \frac{10\pi^2}{3H_0^2} \right) \left[ \int_{0}^{+\infty} df \frac{\hat{Y}^2(f) \Omega_{GW}^2(f)}{P_i(f) P_j(f)} \right]^{-1}. \tag{10}$$

Using this filter function and normalization gives an optimal SNR of [13]

$$\text{SNR} = \frac{3H_0^2}{10\pi^2} \sqrt{2T} \left[ \int_{0}^{+\infty} df \frac{\gamma_{ij}^2(f) \Omega_{GW}^2(f)}{f^6 P_i(f) P_j(f)} \right]^{1/2}. \tag{11}$$

The ORF encodes the separation and orientations of the detectors and is defined as [13,20]

$$\gamma_{ij}(f) := \frac{5}{8\pi} \sum_A \int_{S^2} d\Omega e^{i2\pi f \hat{\Omega} \cdot \Delta \hat{x}} F^A_i(\hat{\Omega}) F^A_j(\hat{\Omega}), \tag{12}$$

where $\hat{\Omega}$ is a unit vector specifying a direction on the two-sphere, $\Delta \hat{x} = \hat{x}_i - \hat{x}_j$ is the separation of the two interferometers and

$$F^A_i(\hat{\Omega}) = e^{A} a_i(\hat{\Omega}) d_i^{ab}, \tag{13}$$

is the response of the $i$th detector to the $A = +, \times$ polarization, where $e^A_{ab}$ are the transverse traceless polarization tensors. The geometry of each interferometer is described by a response tensor,

$$d_i^{ab} = \frac{1}{2} (\hat{x}_a \delta_b^c - \hat{x}_b \delta_a^c), \tag{14}$$

which is constructed from the two unit vectors that point along the arms of the interferometer, $\hat{x}$ and $\hat{y}$ [20,21]. At zero frequency, the ORF is determined solely by the relative orientations of the two interferometers. The LIGO
interferometers are oriented in such a way as to maximize
the amplitude of the ORF at low frequency, while the
relative orientations of the LIGO-Virgo pairs are poor.
Thus at low frequency the amplitude of the ORF
between the Hanford and Livingston interferometers, \( \gamma_{HL}(f) \), is
larger than that of the overlap between Virgo and any of
the LIGO interferometers, \( \gamma_{HV}(f) \) or \( \gamma_{LV}(f) \) (note that the
“HL” and “HV” overlap reduction functions hold for
both H1 and H2 as they are collocated). However, at high
frequency the ORF behaves as a sinc function of the
frequency multiplied by the light-travel time between the
interferometers. As the LIGO interferometers are closer to
each other than to Virgo, their ORF \( \gamma_{HL}(f) \) oscillates less,
but decays more rapidly with frequency than the the ORFs
of the LIGO-Virgo pairs. Figure 1 shows the ORFs be-
 tween the LIGO interferometers, as well as the observing geometry, described by
\( \gamma_{ij}(f) \). For interferometers operating at design sensitivity,
this means that for frequencies above \( \sim 200 \) Hz the LIGO-
Virgo pairs make the dominant contribution to the sensi-
tivity [22]. During its first science run Virgo was closest to
design sensitivity at frequencies above several hundred Hz,
which informed our decision to use the 600–1000 Hz band.

The procedure by which we analyzed the data is as
follows. For each pair of interferometers, labeled by \( I \),
the coincident data were divided into segments, labeled by \( J \), of length \( T = 60 \) s. The data from each segment are
Hann windowed in order to minimize spectral leakage.
In order not to reduce the effective observation time, the
segments are therefore overlapped by 50%. For each
segment, the data from both interferometers were Fourier
transformed then coarse-grained to a resolution of 0.25 Hz.
The data from the adjacent segments were then used to
calculate power spectral densities (PSDs) with Welch’s
method. The Fourier transformed data and the PSDs were
used to calculate the estimator on \( \Omega_3 \), \( \tilde{Y}_J \), and its standard
deviation, \( \sigma_{ij} \). For each pair, the results from all segments
were optimally combined by performing a weighted aver-
age (with weights \( 1/\sigma_{ij}^2 \)), taking into account the correla-
tions that were introduced by the overlapping segments
[23]. The weighted average for each pair, \( \tilde{Y}_I \), has an
associated standard deviation \( \sigma_I \), also calculated by com-
bining the standard deviations from each segment (note
that \( \sigma_I \) is the equivalent of \( \sigma_Y \) [from Eq. (8)] for each pair,
\( I \), but we have dropped the \( Y \) subscript to simplify the
notation).

**A. Data quality**

Data quality cuts were made to eliminate data that was
too noisy or nonstationary, or that had correlated noise
between detectors. Time segments that were known to
contain large noise transients in one interferometer were
removed from the analysis. We also excluded times when
the digitizers were saturated, times with particularly high
noise, and times when the calibration was unreliable. This
also involved excluding the last thirty seconds before the
loss of lock in the interferometers, as they are known to
have an increase in noise in this period. Additionally, we
ensured that the data were approximately stationary over a
period of three minutes, as the PSD estimates, \( P_i(f) \), used
in calculating the optimal filter and standard deviation in
each segment are obtained from data in the immediately
adjacent segments. This was achieved by calculating a
measure of stationarity,

\[
\Delta \sigma_{ij} = \left| \sigma_{ij} - \sigma_{ij}^0 \right| / \sigma_{ij},
\]

for each segment, where \( \sigma_{ij}^0 \) was calculated [following
Eq. (8)] using the PSDs estimated from the adjacent seg-
ments, and \( \sigma_{ij}^0 \) was calculated using the PSDs estimated
using data from the segment itself. To ensure stationarity,
we set a threshold value, \( \zeta \), and all segments with values of \( \Delta \sigma_{1f} \) > \( \zeta \) are discarded. The threshold was tuned by analyzing the data with unphysical time offsets between the interferometers; a value of \( \zeta \) = 0.1 as this ensures that the remaining data are Gaussian.

In order to exclude correlations between the instruments caused by environmental factors we excluded certain frequencies from our analysis. The frequency bins to be removed were identified in two ways. Some correlations were identified, we also calculated the coherence, e.g. there are correlations at multiples of 60 Hz between the interferometers located in the USA due to the frequency of the power supply [19]. These were removed from the analysis, but in order to ensure that all coherent bins were identified, we also calculated the coherence,

\[
\Gamma(f) = \frac{|\tilde{x}_1(f)\tilde{x}_2(f)|^2}{P_1(f)P_2(f)},
\]

which is the ratio of the cross-spectrum to the product of the two power spectral densities, averaged over the whole run. This value was calculated first at a resolution of 0.1 Hz, then at 1 mHz to investigate in more detail the frequency distribution of the coherence. Several frequencies showed excess coherence; some had been identified a priori but two had not, so these were also removed from the analysis. The calculations of the power spectra and the cross correlation were carried out at a resolution of 0.25 Hz, so we removed the corresponding 0.25 Hz bin from our analysis. Excess coherence was defined as coherence exceeding a threshold of \( \Gamma(f) = 5 \times 10^{-3} \). This threshold was also chosen after analyzing the data with unphysical time offsets. The excluded bins for each interferometer can be seen in Table I.

### Table I. Table of the frequency bins excluded from the analysis for each interferometer.

<table>
<thead>
<tr>
<th>IFO</th>
<th>Notched frequencies (Hz)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>786.25</td>
<td>Harmonic of calibration line</td>
</tr>
<tr>
<td></td>
<td>961</td>
<td>Timing diagnostic line</td>
</tr>
<tr>
<td>H2</td>
<td>640</td>
<td>Excess noise</td>
</tr>
<tr>
<td></td>
<td>814.5</td>
<td>Harmonic of calibration line</td>
</tr>
<tr>
<td></td>
<td>961</td>
<td>Timing diagnostic line</td>
</tr>
<tr>
<td>L1</td>
<td>793.5</td>
<td>Harmonic of calibration line</td>
</tr>
<tr>
<td></td>
<td>961</td>
<td>Timing diagnostic line</td>
</tr>
<tr>
<td>V1</td>
<td>706</td>
<td>Harmonics of calibration lines</td>
</tr>
<tr>
<td></td>
<td>710</td>
<td></td>
</tr>
<tr>
<td></td>
<td>714</td>
<td></td>
</tr>
<tr>
<td></td>
<td>718</td>
<td></td>
</tr>
</tbody>
</table>

In order to be sure that the cross correlation is a measure of the gravitational-wave signal present in both detectors in a pair, we must be sure that the data collected in both detectors are truly coincident. Calibration studies were carried out to determine the timing offset, if any, between the detectors and to estimate the error on this offset. These studies are described in more detail in Ref. [24], but we summarize them here.

The output of each interferometer is recorded at a rate of 16384 Hz. Each data point has an associated time-stamp and we need to ensure that data taken with identical time-stamps are indeed coincident measurements of the strain, to within the calibration errors of the instruments. No offset between the instruments was identified, but several possible sources of timing error were investigated. First, approximations in our models of the interferometers can introduce phase errors. For the measurement of strain, we model the interferometers using the long-wavelength approximation (i.e. we assume that the wavelengths of the gravitational waves that we measure are much longer than the arm-lengths of the interferometers). We also make an approximation in the transfer function of the Fabry-Perot cavity; the exact function has several poles or singularities, but we use an approximation which includes only the lowest frequency pole [25]. The errors that these two approximations introduce largely cancel, with a residual error of \( \sim 2 \mu s \) or \( \sim 1^s \) at 1 kHz [24].

Second, there is some propagation time between strain manifesting in the detectors and the detector output being recorded in a frame file. This is well understood for all detectors and is accounted for (to within calibration errors) when the detector outputs are converted to strain. The time-stamp associated with each data point is therefore taken to be the GPS time at which the differential arm length occurs, to within calibration errors [24].

Third, the GPS time recorded at each site has some uncertainty. The timing precision of the GPS system is \( \sim 30 \) ns, which corresponds with the stated location accuracy of \( \sim 10 \) m. Each site necessarily uses its own GPS receiver, so the relative accuracy of these receivers has been checked, by taking a Virgo GPS receiver to a LIGO site and comparing the outputs. The relative accuracy was found to be better than 1 \( \mu s \). The receivers have also been checked against Network Time Protocol (NTP) and were found to have no offset [24]. The total error in GPS timing is far smaller than the instrumental phase calibration errors in the 600–1000 Hz frequency band (see Table II).

These investigations concluded that the timing offset between the instruments is zero for all pairs, with errors on these values that are smaller than the error in the phase calibration of each instrument. The phase calibration errors of the instruments are negligible in this analysis as their inclusion would produce a smaller than 1% change in the
results at this sensitivity, and therefore the relative timing error is negligible.

C. Combination of multiple pairs

We performed an analysis of all of the available data from LIGO’s fifth science run and Virgo’s first science run. However, we excluded the H1–H2 pair as the two instruments were built inside the same vacuum system, and so may have significant amounts of correlated noise. There is an ongoing investigation into identifying and removing these correlations [28], and for the present analysis, we consider only the five remaining pairs. As described above, the output of each pair yields an estimator, $\hat{Y}_I$, with a standard deviation, $\sigma_I$, where $I = 1 \ldots 5$ labels the detector pair.

Using the estimators $\hat{Y}_I$ and their associated error bars, $\sigma_I$, we construct a Bayesian posterior probability density function (PDF) on $\Omega_I$. Bayes theorem says that the posterior PDF of a set of unknown parameters, $\theta$, given a set of data, $D$, is given by

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)},$$

where $p(D|\theta)$ is the prior PDF on the unknown parameters—representing the state of knowledge before the experiment—$p(D|\theta)$ is the likelihood function and $p(D)$ is a normalization factor. In this case, the unknown parameters, $\theta$, are the value of $\Omega_3$ and the amplitude calibration factors of the instruments, which will be discussed below. The data set, $D$, is the set of five pairs of estimators, $\{\hat{Y}_I\}$, we obtain from the five pairs of interferometers.

In forming this posterior, we must consider the errors in the calibration of the strain data obtained by the interferometers. In the data from one interferometer, labeled by $i$, there may be an error on the calibration of both the amplitude and the phase, such that the value we measure is

$$\tilde{s}_i(f) = e^{\Lambda_{i1}} e^{i \phi_i} \hat{s}_i(f),$$

where $\hat{s}_i(f)$ is the “true” value that would be measured if the interferometer were perfectly calibrated. The phase calibration errors given in Table II are negligible, and the studies described in Sec. II B have shown that there is no significant relative timing error between the interferometers, so we can simply assume that $\phi_i = 0$. However, the amplitude calibration errors are not negligible, and the calibration factors take the values $\Lambda_i = 0 \pm \epsilon_{\Lambda,i}$, where $\epsilon_{\Lambda,i}$ are the fractional amplitude calibration errors of the instruments, which are quoted in Table II.

The calibration factors combine such that the estimator for a pair $I$ is

$$\hat{Y}_I = e^{\Lambda_{i1}} e^{\Lambda_{i2}} \hat{Y}_I,$$

where $\hat{Y}_I$ is the true value that would be measured with perfectly calibrated instruments and $\Lambda_{i1}$ and $\Lambda_{i2}$ are the calibration factors of the two instruments in pair $I$. The likelihood function for a single estimator is given by

$$p(\hat{Y}_I|\Omega, \sigma_I, \Lambda_I) = \frac{1}{\sigma_I \sqrt{2\pi}} \exp\left(-\frac{(\hat{Y}_I - e^{\Lambda_{i1}} \Omega)^2}{2\sigma_I^2}\right).$$

(21)

where we have used $\Lambda_I = \Lambda_{i1} + \Lambda_{i2}$. The joint likelihood function on all the data is the product over all pairs of Eq. (21)

$$p(\{\hat{Y}_I\}|\Omega_3, \{\sigma_I\}, \{\Lambda_I\}) = \prod_{i=1}^{n_{\text{IFO}}} p(\hat{Y}_I|\Omega_3, \sigma_I, \Lambda_I).$$

(22)

In order to form a posterior PDF, we define priors on the calibration factors of the individual interferometers, $\{\Lambda_i\}$. The calibration factors are assumed to be Gaussian distributed, with variance given by the square of the calibration errors quoted in Table II, such that

$$p(\{\Lambda_i\}|\epsilon_{\Lambda,i}) = \prod_{i=1}^{n_{\text{IFO}}} \frac{1}{\epsilon_{\Lambda,i} \sqrt{2\pi}} \exp\left(-\frac{\Lambda_i^2}{2\epsilon_{\Lambda,i}^2}\right).$$

(23)

We choose a flat prior on $\Omega_3$ because, although there has been an analysis in this band previously, it did not include data from the whole of the frequency band and an uninformative flat prior is conservative. We chose $\Omega_{\text{max}} = 10$, which is two orders of magnitude greater than the estimators and their standard deviations, such that the prior is essentially unconstrained.

We combine the prior and likelihood functions to give a posterior PDF

$$p(\Omega_3, \{\Lambda_i\}|\{\hat{Y}_I\}, \{\sigma_I\}, \{\epsilon_{\Lambda,i}\}) = p(\Omega_3)p(\{\Lambda_i\}|\epsilon_{\Lambda,i})p(\{\hat{Y}_I\}|\Omega_3, \{\sigma_I\}, \{\Lambda_i\}).$$

(25)

We marginalize this posterior analytically over all $\Lambda_i$ [36] to give us a posterior on $\Omega_3$ alone.
We find that summing the integrands from each pair: \[ 22 \]

A combined sensitivity integrand can also be found by

Under the assumption that the calibration factors

The optimal estimator, \( \hat{\tilde{Y}} \), is given by the combination of

It has a variance, \( \sigma \), given by

Under the assumption that the calibration factors \( \Lambda_i \) are all equal to zero, then the optimal way to combine the results from each pair is to perform a weighted average with weights \( 1/\sigma_i^2 \) (equivalently to combining results from multiple, uncorrelated, time segments) [22]

\[
\hat{\tilde{Y}} = \frac{\sum_i \tilde{Y}_i \sigma_i^{-2}}{\sum_i \sigma_i^{-2}}
\]

\[
\sigma^{-2} = \sum_i \sigma_i^{-2}.
\]

A combined sensitivity integrand can also be found by summing the integrands from each pair: [22]

\[
I(f) = \sum_i I_i(f)
\]

### III. RESULTS

We applied the analysis described in Sec. II to all of the available data from the LIGO and Virgo interferometers between November 2005 and September 2007. We obtained estimators of \( \Omega_3 \) from each of five pairs, which are listed in Table III along with their standard deviations. We also create the combined estimators and their standard deviations, using Eqs. (30) and (31), for the full network, and for the network including only the LIGO interferometers. We see that the addition of Virgo to the network reduces the size of the standard deviation by 23%.

Using the posterior PDF defined in Eq. (26) and the calibration errors in Table II we found a 95\% upper limit of \( \Omega_3 < 0.32 \), assuming the Hubble constant to be \( h_{100} = 0.71 \) [29] (see also [30]), while using only the LIGO instruments obtained an upper limit of \( \Omega_3 < 0.30 \). Both of the lower limits were zero. The posterior PDFs obtained by the search are shown in Fig. 2, while the sensitivity integrands, which show the contribution to the sensitivity of the search from each frequency bin, are shown in Fig. 3. The upper limit corresponds to a strain sensitivity of \( 8.5 \times 10^{-24} \text{ Hz}^{-1/2} \) using just the LIGO interferometers, or \( 8.7 \times 10^{-24} \text{ Hz}^{-1/2} \) using both LIGO and Virgo. The LIGO-only upper limit is, in fact, lower than the upper

![FIG. 2. Posterior PDFs on \( \Omega_3 \). The dashed line shows the posterior PDF obtained using just the LIGO detectors, the solid line shows the PDF obtained using LIGO and Virgo detectors. The filled areas show the 95\% probability intervals.](image)

<table>
<thead>
<tr>
<th>Network</th>
<th>Estimator ( \hat{\tilde{Y}}_I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1L1</td>
<td>0.11 ± 0.15</td>
</tr>
<tr>
<td>H1V1</td>
<td>0.55 ± 0.21</td>
</tr>
<tr>
<td>H2L1</td>
<td>-0.14 ± 0.25</td>
</tr>
<tr>
<td>H2V1</td>
<td>-0.51 ± 0.40</td>
</tr>
<tr>
<td>L1V1</td>
<td>0.18 ± 0.19</td>
</tr>
<tr>
<td>LIGO</td>
<td>0.05 ± 0.13</td>
</tr>
<tr>
<td>all</td>
<td>0.15 ± 0.10</td>
</tr>
</tbody>
</table>

# Table III. Table of values of \( \hat{\tilde{Y}}_I \), the estimator of \( \Omega_3 \), obtained by analyzing the data taken during LIGO’s fifth science run and Virgo’s first science run, over a frequency band of 600–1000 Hz, along with the standard deviation, \( \sigma_I \), of each result.
limit using the whole data set, even though the sensitivity of the combined LIGO-Virgo analysis is better. This is not surprising because the addition of Virgo also increases the value of the estimator. The estimator will usually lie somewhere between 0 and 2σ-in this case, the LIGO-only estimator was in the lower part of that range while the LIGO-Virgo estimator was not, but the two results are entirely consistent with each other. When we add Virgo, the likelihood excludes more of the parameter space below \( \Omega_3 = 0 \), but this is a region we already exclude by setting the priors. Monte Carlo simulations show that, in the absence of a signal, the probability of the combined LIGO-Virgo upper limit being at least this much larger than the LIGO-only upper limit is 4.3%. This probability is not so small as to indicate a non-null result and we therefore conclude that the LIGO-Virgo upper limit is larger due to statistical fluctuations.

We also used the same data to calculate the 95% probability intervals for gravitational-wave spectra with spectral indices ranging over \(-4 \leq \alpha \leq 4\), which correspond with different models of possible backgrounds in our frequency band. For example, a background of magnetar signals would be expected to have a spectral index of \( \alpha = 4 \) [3]. Figure 4 shows the values of these upper limits. Note that they were all calculated using a reference frequency of 900 Hz, and Hubble parameter \( h_{100} = 0.71 \).

### IV. VALIDATION OF RESULTS

In order to test our analysis pipeline, we created simulated signals and used software to add them to the data that had been taken during the first week of Virgo’s first science run (this week was then excluded from the full analysis). We generated frame files containing a simulated isotropic stochastic background, with \( \Omega_{\text{GW}}(f) \propto f^3 \). We were then able to scale this signal to several values of \( \Omega_3 \) and add it to the data taken from the instruments. We did not include H2 in this analysis, but used only H1, L1, and V1. Table IV shows the injected values of \( \Omega_3 \) and the recovered values and associated standard deviations, along with the SNR of the signal in the H1V1 pair. The recovered 95% probability intervals of the injections can be seen in Fig. 5. The intervals all contain the injected value of \( \Omega_3 \).

It should be noted that, in order to have detectable signals in this short amount of data, the larger injections are no longer in the small-signal limit. We usually make two assumptions based on this limit. The first is the approximation in Eq. (8), which only holds if the signal is much smaller than the noise, as we are ignoring terms that are first and second order in \( \Omega_{\text{GW}}(f) \) [13]. The second

<table>
<thead>
<tr>
<th>Injected ( \Omega_3 )</th>
<th>Estimator ( \hat{Y} )</th>
<th>95% probability interval</th>
<th>SNR in H1V1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.8 ± 1.3</td>
<td>(0.0, 4.1)</td>
<td>1.3</td>
</tr>
<tr>
<td>9.7</td>
<td>9.1 ± 1.5</td>
<td>(5.7, 12.8)</td>
<td>6.3</td>
</tr>
<tr>
<td>20.2</td>
<td>19.3 ± 1.8</td>
<td>(14.2, 24.8)</td>
<td>13.3</td>
</tr>
<tr>
<td>95.1</td>
<td>91.1 ± 3.7</td>
<td>(72.3, 110.6)</td>
<td>62.3</td>
</tr>
<tr>
<td>203.1</td>
<td>194.1 ± 6.2</td>
<td>(154.9, 234.3)</td>
<td>133.1</td>
</tr>
</tbody>
</table>
assumption enters into the calculation of the noise PSDs, \( P_i(f) \). We calculate these directly from the data, as in the small-signal limit we can assume that \( \langle |\tilde{s}_i(f)|^2 \rangle = \langle |\tilde{n}_i(f)|^2 \rangle \). The first assumption causes an over-estimation of the standard deviation, while the second causes our “optimal” filter to no longer be quite optimal. If we ignore these assumptions, we will underestimate the theoretical error bar, \( \sigma_\gamma \), and the width of the posterior PDFs. However, we still find 95% probability intervals that are consistent with the injected signals.

V. COMPARISON WITH OTHER RESULTS

The previous most sensitive direct upper limit in this frequency band was \( \Omega_{GW}(f) < 1.02 \), obtained by the joint analysis of data from the LIGO-Livingston interferometer and the ALLEGRO bar detector over a frequency band of \( 850 \text{ Hz} \leq f \leq 950 \text{ Hz} \) [18]. This result was obtained using a constant \( \Omega_{GW}(f) = \Omega_0 \), so should be compared with our upper limit for \( \alpha = 0 \). As can be seen in Fig. 4, our 95% upper limit for \( \alpha = 0 \) is \( \Omega_0 < 0.16 \) using all the available data, or \( \Omega_0 < 0.15 \) using just the LIGO interferometers, therefore our result has improved on the sensitivity of the LIGO-ALLEGRO result by a factor of \( \approx 7 \). The comparative strain sensitivity of the upper limits of the current search and the LIGO-ALLEGRO search can be seen in Fig. 6.

The previous most sensitive direct limit at any frequency was the analysis of data from the three LIGO detectors in the fifth science run [17]. The analysis was carried out using the same data as the analysis presented in this paper, but was restricted to the frequency band \( 40 \text{ Hz} \leq f \leq 500 \text{ Hz} \). This included the most sensitive frequency band of the three detectors. The 95% upper limit on \( \Omega_0 \) in this band was given as \( 6.9 \times 10^{-6} \), which is a factor of \( 2 \times 10^4 \) times smaller than our upper limit. They also found an upper limit on \( \Omega_3 \) of \( 7.1 \times 10^{-6} \). In order to compare that to our upper limit on \( \Omega_3 \), we must extend the spectrum to the frequency band analyzed in this paper. The \( 40 \text{ Hz} \leq f \leq 500 \text{ Hz} \) upper limit would correspond to an upper limit at \( 900 \text{ Hz} \) of \( \Omega_3 < 0.0052 \), which is a factor of \( \approx 60 \) smaller than the upper limit presented in this paper.

The search at lower frequencies is significantly more sensitive and we would expect that in the advanced detector era the combined analysis of LIGO and Virgo detectors at low frequencies will improve even further on the previously published upper limits.

We can also compare our results with indirect upper limits on the stochastic gravitational-wave background. In this band, the most stringent constraints come from Big Bang nucleosynthesis (BBN) and measurements of the cosmic microwave background (CMB). The BBN bound constrains the integrated energy density of gravitational waves over frequencies above \( 10^{-10} \text{ Hz} \), based on observations of different relative abundances of light nuclei today. The BBN upper limit is [6].
where $N_\nu$ is the effective number of neutrino species at the time of BBN. Recent constraints on $N_\nu$, obtained from CMB measurements, BBN modeling, and the observed abundances of light elements suggest that $3.5 \leq N_\nu \leq 4.4$ [31–34]. The CMB limit also constrains the integrated gravitational-wave energy density, and is obtained from the observed CMB and matter power spectra, as these would be altered if there were a higher gravitational-wave energy density at the time of decoupling. The CMB upper limit [35] is

$$\int \Omega_{GW}(f) d(\ln f) < 1.1 \times 10^{-5} (N_\nu - 3),$$

(33)

Our upper limit is not sensitive enough to improve on these indirect upper limits, however, these indirect bounds only apply to a background of cosmological origin, whereas the bound presented here applies to astrophysical signals as well.

VI. CONCLUSIONS

Data acquired by the LIGO and Virgo interferometers have been analyzed to search for a stochastic background of gravitational waves. This is the first time that data from LIGO and Virgo have been used jointly for such a search, and we have demonstrated that the addition of Virgo increases the sensitivity of the search significantly, reducing the error bar by 23% even though the length of time for which Virgo was taking data was approximately one fifth of the time of the LIGO run. The upper limit obtained with the LIGO interferometers only is the most sensitive direct result in this frequency band to date, improving on the previous best limit, set with the joint analysis of ALLEGRO and LIGO data, by a factor of $\approx 7$.

Adding Virgo improves the sensitivity across the frequency band, largely due to the addition of pairs which have different overlap reduction functions. This enables us to cover the frequency band more evenly, as well as effectively increasing the total observation time. We can see that the sensitivity of the search is much improved by adding Virgo by comparing the standard deviations in Table III.

However, in this case, the increased sensitivity did not lead to a decreased upper limit, as the joint estimator of $\Omega_3$ obtained by the full LIGO-Virgo search was higher than the estimator obtained by the LIGO-only analysis.

As part of this analysis, we have also developed a method of marginalizing over the error on the amplitude calibration of several interferometers. The methods used in this paper will be useful for future analyses of data from the network of interferometers, which we expect to grow, eventually including not only interferometers in North America and Europe, but also hopefully around the world.

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