The Power of the Filtration Technique for Modal Logics with Team Semantics

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Abstract

Modal Team Logic (MTL) extends Väänänen's Modal Dependence Logic (MDL) by Boolean negation. Its satisfiability problem is decidable, but the exact complexity is not yet understood very well. We investigate a model-theoretical approach and generalize the successful filtration technique to work in team semantics. We identify an "existential" fragment of MTL that enjoys the exponential model property and is therefore, like Propositional Team Logic (PTL), complete for the class AEXP(poly). Moreover, superexponential filtration lower bounds for different fragments of MTL are proven, up to the full logic having no filtration for any elementary size bound. As a corollary, superexponential gaps of succinctness between MTL fragments of equal expressive power are shown.

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1 Introduction

Team semantics, introduced by Hodges [12], enjoys striking success as a compositional semantics for logics with incomplete information, like independence-friendly logic by Hintikka and Sandu [11]. In this vein, Väänänen [17] introduced *dependence logic* as an extension of first-order logic. At the heart of this extension is the atomic formula of *dependence*, written $=(x, y)$, stating that in a set (*team*) of assignments to the variables x and y, the value of y is a function of x. Related concepts are the atoms of *independence* $x \perp y$, *inclusion* $x \subseteq y$ and more [6, 7]. A rich family of logics of dependence and independence has developed, and dependence logic and its variants have found a broad range of applications like database theory, quantum mechanics and statistics.

The concept of team semantics has been introduced into other logics as well, like quantified Boolean logic [9] and modal logic [16]. A noticeable feature of this semantics is the loss of Boolean negation. Re-adding it as a special connective, often written $\sim$, yields what is called *team logic* [15, 17]. Team Logic based on propositional or modal logic is comprehensive enough to even express atoms like dependence, independence and inclusion as composite formulas using $\sim$ [10]. Therefore it is desirable to classify its exact computational complexity.

We study *Modal Team Logic* ($\mathcal{MTL}$). Its model checking problem was classified as PSPACE-complete by Müller [15]. Moreover, a finite model property was shown by Kontinen et al. [13]. The complexity of its satisfiability problem is however not yet understood well.

In this article, we pursue a model-theoretic approach to that problem via the prominent *filtration* technique. Filtration turned out to be a powerful tool to prove the *exponential model property* (if a formula φ has a model, then it has a model of size $2^{c\cdot|\varphi|}$) for a wide

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range of logics. Examples are modal logics [3], dynamic and temporal logics [2], and even fragments of first-order logic [8].

In Section 2, we give the necessary foundational definitions. The filtration technique for classical modal logic is generalized to $\mathcal{MTL}$, and applied in Section 3. It is however also shown unable to fully handle team-wide modalities. In Section 4, we improve the ideas of filtration to account for such modalities, and prove the exponential model property for a non-trivial fragment of $\mathcal{MTL}$, called $\mathcal{MTL}_{\mathsf{mon}}$. Nevertheless, in the final Section 5, it is shown that for full $\mathcal{MTL}$, filtration can only yield models of non-elementary size.

2 Preliminaries

Let $[m]$ denote the set $\{ 1, \dots, m \}$ for any $m \in \mathbb{N}$. We fix a countable set $\mathcal{PS}$ of atomic propositions $p_1, p_2, \dots$; $\mathcal{ML}$ then denotes the classical mono-modal logic, generated by the grammar

$$\alpha ::= \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Diamond\alpha \mid \Box\alpha \mid p,$$

where $p \in \mathcal{PS}$. Furthermore, we employ the usual abbreviations and define $\alpha \to \beta := \neg\alpha \vee \beta$, $\top := \alpha \to \alpha$ and $\bot := \neg\top$.

The semantics of $\mathcal{ML}$ are the usual Kripke semantics, i.e., $\mathcal{ML}$-formulas are evaluated over *Kripke structures* $\mathcal{K} = (W, R, V)$, where (W, R) is a directed graph and $V : \mathcal{PS} \to \mathcal{P}(W)$ is the *valuation* function.

Modal Team Logic $\mathcal{MTL}$ extends modal logic according to the following grammar, where φ denotes an $\mathcal{MTL}$ formula and α is any $\mathcal{ML}$ formula:

$$\varphi ::= {\sim}\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \alpha.$$

$\mathcal{MTL}$ is evaluated on whole *teams* of multiple (and potentially zero) worlds in a Kripke structure. Consequently, it replaces the point-wise negation $\neg$ by the *team-wise negation* $\sim$.

While classical $\mathcal{ML}$ formulas are denoted by the letters $\alpha, \beta, \gamma, \dots$, we use $\varphi, \psi, \vartheta, \dots$ for $\mathcal{MTL}$ formulas.

In order to generalize modal operators to teams, we define the following terms. If $T \subseteq W$ is a team in a Kripke structure (W, R, V), then its *image* is defined as $R(T) := \bigcup_{w \in T} R(w)$, where $R(w) := \{ v \in W \mid Rwv \}$. Analogously to R-successors of worlds, a team S is a *successor team* of T if $S \subseteq R(T)$ and $T \subseteq R^{-1}(S)$; that is, if every $w \in T$ has a successor in S and every $w' \in S$ has a predecessor in T.

The semantics of $\mathcal{MTL}$ is then as follows, where $\mathcal{K} = (W, R, V)$ is a Kripke structure, $T \subseteq W$, $\varphi, \psi \in \mathcal{MTL}$ and $\alpha \in \mathcal{ML}$:

$$
\begin{array}{lll}
(\mathcal{K}, T) \vDash \alpha & \text{iff} & \forall w \in T : (\mathcal{K}, w) \vDash \alpha \text{ (in Kripke semantics)}, \\
(\mathcal{K}, T) \vDash {\sim}\varphi & \text{iff} & (\mathcal{K}, T) \nvDash \varphi \\
(\mathcal{K}, T) \vDash \varphi \wedge \psi & \text{iff} & (\mathcal{K}, T) \vDash \varphi \text{ and } (\mathcal{K}, T) \vdash \psi \\
(\mathcal{K}, T) \vDash \varphi \vee \psi & \text{iff} & \exists S, U \subseteq T \text{ such that } T = S \cup U,\ (\mathcal{K}, S) \vDash \varphi, \text{ and } (\mathcal{K}, U) \vDash \psi \\
(\mathcal{K}, T) \vDash \Diamond\varphi & \text{iff} & \exists S \subseteq W \text{ such that } S \text{ is a successor team of } T \text{ and } (\mathcal{K}, S) \vDash \varphi \\
(\mathcal{K}, T) \vDash \Box\varphi & \text{iff} & R(T) \vDash \varphi
\end{array}
$$

The most striking difference to classical logic is perhaps that disjunction does not denote a truth-functional operation, but instead an existential quantification of subteams of the current team. We define the *Boolean* or *truth-functional disjunction* $\varphi_1 \otimes \varphi_2 := {\sim}({\sim}\varphi_1 \wedge {\sim}\varphi_2)$. The operator $\varovee$ is also sometimes called *intuitionistic* or *classical disjunction* in the context of team logic.[1]

The above semantics being well-defined on $\mathcal{ML}$ formulas (that is, on $\sim$-free formulas) is due to the *flatness property* of $\mathcal{ML}$ [16]. For instance, if every point $w \in T$ satisfies either p_1 or p_2 or both, and consequently $T \vDash p_1 \vee p_2$, then T has a division into T_1 satisfying p_1 and T_2 satisfying p_2, and vice versa. In general, the standard $\mathcal{ML}$ operators $(\wedge, \vee, \neg, \Diamond, \Box)$ can be evaluated locally, i.e., point-wise. When however containing the Boolean negation $\sim$ on the level of teams, then it is not possible to break down the semantics to local evaluation and the above semantics applies. This is in contrast to $\neg$, which is always applied locally and cannot contain a nested $\sim$; neither $\neg$ or $\sim$ can express the respective other one.

To express that at least one world w in a team T satisfies a given $\mathcal{ML}$ formula α (dually to the first line of the above definition), then we write $\mathsf{E}\alpha$, which can be defined as $\sim\neg\alpha$.

For $\Phi \subseteq \mathcal{MTL}$, let $\bigcirc\,\Phi := \{\bigcirc\varphi \mid \varphi \in \Phi\}$ for $\bigcirc \in \{\neg, \sim, \Box, \Diamond\}$.

For satisfaction we use the standard notation. $(\mathcal{K}, T) \vDash \Phi$ means that $(\mathcal{K}, T)$ satisfies every $\varphi \in \Phi$ in the above $\mathcal{MTL}$ semantics. $(\mathcal{K}, T)$ is then a *model* of Φ (or simply of φ in case $\Phi = \{\varphi\}$). Similarly, for sets $\Gamma \subseteq \mathcal{ML}$, $(\mathcal{K}, w)$ is called a model of Γ if it satisfies Γ in the standard semantics of modal logic.

$\Phi \vDash \Psi$ means that Φ entails all formulas in Ψ, i.e., any model of Φ is a model of every formula in Ψ. We also simply write $\varphi \vDash \psi$ instead of $\{\varphi\} \vDash \{\psi\}$. Likewise, if two formulas φ, ψ have the same models, then they are *equivalent*, written $\varphi \equiv \psi$.

If the structure $\mathcal{K}$ is clear, then we simply write $T \vDash \varphi$ or $w \vDash \alpha$ instead of $(\mathcal{K}, T) \vDash \varphi$ and $(\mathcal{K}, w) \vDash \alpha$.

The modality-free fragment of $\mathcal{MTL}$ is called $\mathcal{PTL}$; its satisfiability and validity problems are $\mathsf{AEXPTIME}(\text{poly})$-complete [9], where the class $\mathsf{AEXPTIME}(\text{poly})$ contains all languages that are decided by an alternating Turing machine with exponential runtime bound and polynomial alternation bound (see also Chandra et al. [4]).

2.1 Morphism and Filtrations

▶ **Definition 1** (Modal Homomorphism)**.** Let $\mathcal{K} = (W, R, V)$ and $\mathcal{K}' = (W', R', V')$ be Kripke structures. A mapping $h\colon W \to W'$ is a *homomorphism*, in symbols $h\colon \mathcal{K} \to \mathcal{K}'$, if

1. for all $p \in \mathcal{PS}$, if $w \in V(p)$, then $h(w) \in V'(p)$
2. for all $w, v \in W$, if Rwv, then $R'f(w)f(v)$.

If h is additionally surjective, then $\mathcal{K}'$ is called the *morphic image* of $\mathcal{K}$ and denoted $h(\mathcal{K})$.

If $\approx$ is an equivalence relation on a set S, then for every $s \in S$, $[s]_\approx := \{\, s' \in S \mid s' \approx s \,\}$ denotes the *equivalence class* of s in S. The set of all equivalence classes in S is the *quotient* $S/_\approx := \{\, [s]_\approx \mid s \in S \,\}$, and the *index* of $\approx$ is defined as the cardinality $|S/_\approx|$. If $U \subseteq S$, then $[U]_\approx := \{\, [s]_\approx \mid s \in U \,\}$. We often will drop the index and write $[s]$ and $[U]$. If $\approx$ and $\approx'$ are equivalence relations on S such that $s \approx' s'$ implies $s \approx s'$, then $\approx'$ is a *refinement* of $\approx$. Given two equivalence relations $\approx_1, \approx_2$ on S, their intersection $\approx_1 \cap \approx_2$ is again an equivalence relation on S and a refinement of both $\approx_1$ and $\approx_2$, and $|S/_{\approx_1 \cap \approx_2}| \leq |S/_{\approx_1}| \cdot |S/_{\approx_2}|$.

[1] The original term used by Väänänen was in fact "Boolean disjunction" [16, 17]. The synonymous adjective "classical" apparently stems from the disjunction acting truth-functionally and hence just as known from classical (point-wisely evaluated) logic. It is however potentially confusing, since already a simple expression like $p \varovee q$ is neither a classical formula, nor semantically equivalent to one. The term "intuitionistic disjunction" stems from a work of Abramsky and Väänänen [1] on connections between linear logic and team logic, but is more commonly found in linear logic. Of all mentioned terms, "Boolean disjunction" is probably the least confusing.

▶ **Definition 2** (Filtration). Let $\mathcal{K} = (W, R, V)$ be a Kripke structure. Let $\approx$ be an equivalence relation on W. Then the Kripke structure (W', R', V') defined by

$$\begin{aligned} W' &:= W/_{\approx}, && (1)\\ R'[w][v] &\Leftrightarrow \exists w' \in [w], \exists v' \in [v] \text{ such that } Rwv, && (2)\\ [w] \in V'(p) &\Leftrightarrow \exists w' \in [w] \cap V(p), && (3) \end{aligned}$$

is a *filtration of* $\mathcal{K}$ *through* $\approx$, denoted $\mathcal{K}/_{\approx}$.[2]

Clearly every filtration of a structure is also a morphic image of it, via the mapping $w \mapsto [w]$.

Standard Modal Logic allows, for any given formula $\alpha \in \mathcal{ML}$ and model $\mathcal{K}$, filtration down to a model of α of size exponential in α. The approach is the following: For fixed Kripke structures and a subset $\Gamma \subseteq \mathcal{ML}$, we define an equivalence relation $\approx_\Gamma$ of "agreement", namely $w \approx_\Gamma w'$ if and only if $\forall \alpha \in \Gamma : (\mathcal{K}, w) \vDash \alpha \Leftrightarrow (\mathcal{K}, w') \vDash \alpha$.

The *subformulas* of (sets of) $\mathcal{MTL}$ formulas are defined inductively:

$$\begin{aligned} &\mathrm{SF}(p) &&:= \{p\} && \text{if } p \in \mathcal{PS},\\ &\mathrm{SF}(\triangle\varphi) &&:= \{\triangle\varphi\} \cup \mathrm{SF}(\varphi) && \text{if } \triangle \in \{\neg, \sim, \Box, \Diamond\},\\ &\mathrm{SF}(\varphi \circ \psi) &&:= \{\varphi \circ \psi\} \cup \mathrm{SF}(\varphi) \cup \mathrm{SF}(\psi) && \text{if } \circ \in \{\wedge, \vee\},\\ &\mathrm{SF}(\Phi) &&:= \bigcup_{\varphi\in\Phi} \mathrm{SF}(\varphi) && \text{if } \Phi \subseteq \mathcal{MTL}. \end{aligned}$$

▶ **Theorem 3** ([3]). *Let $\mathcal{K} = (W, R, V)$ be a Kripke structure, and let $\Gamma \subseteq \mathcal{ML}$ be closed under taking subformulas, i.e., $\mathrm{SF}(\Gamma) = \Gamma$. Let $\approx'$ be a refinement of $\approx_\Gamma$.*

Then $(\mathcal{K}, w) \vDash \alpha$ iff $(\mathcal{K}/_{\approx}, [w]_{\approx'}) \vDash \alpha$, for all formulas $\alpha \in \Gamma$ and worlds $w \in W$.

Proof. Proven identically to [3, Theorem 2.39].[3] ◀

▶ **Corollary 4** (Small Model Property of Modal Logic). *Every satisfiable formula $\alpha \in \mathcal{ML}$ has a model of size at most $2^{|\alpha|}$.*

3 Filtration in Team Semantics

An obvious generalization of Theorem 3 is to replace the quantification "every world w in $\mathcal{K}$" by "every team T in $\mathcal{K}$". It is plausible that a similar inductive proof works for teams. The next definition is a first attempt: It permits to prove certain filtration results for team semantics, but its limits are quickly reached, as this section later demonstrates.

▶ **Definition 5.** If $\approx$ is an equivalence relation on a Kripke structure $\mathcal{K} = (W, R, V)$, $T \subseteq W$ is a team, and $\Phi \subseteq \mathcal{MTL}$, then $\approx$ is

Φ-*invariant on* $(\mathcal{K}, T)$ if $\forall \varphi \in \Phi : (\mathcal{K}, T) \vDash \varphi \;\Leftrightarrow\; (\mathcal{K}/_{\approx}, [T]_{\approx}) \vDash \varphi$,

Φ-*invariant on* $\mathcal{K}$ if it is $(\mathcal{K}, T)$-invariant for all $T \subseteq W$.

strongly Φ-*invariant* on $\mathcal{K}$ (resp. $(\mathcal{K}, T)$) if every refinement $\approx'$ of $\approx$ is Φ-invariant on $\mathcal{K}$ (resp. $(\mathcal{K}, T)$).

[2] The definition of R' used here is also known of the *minimal filtration* (of R), but all results in this paper can be proven for filtrations with a larger number of edges.

[3] The proof considers only $\approx' = \approx_\Gamma$. Nevertheless, it completely goes through if the first characterizing property of a filtration, namely $W' := \{ [w] \mid w \in W \}$, is relaxed, as long as the mapping $[\cdot]\colon w \mapsto [w]$ respects (2) and (3) in Definition 2.

▶ **Proposition 6.** *Let $\Gamma \subseteq \mathcal{ML}$. Then on any structure $\mathcal{K}$, the corresponding equivalence relation $\approx_{\mathrm{SF}(\Gamma)}$ is strongly Γ-invariant on $\mathcal{K}$.*

Proof. Let $\mathcal{K} = (W, V, R)$, and let $\approx'$ be a refinement of $\approx_{\mathrm{SF}(\Gamma)}$. We have to show that $\approx'$ is α-invariant on $\mathcal{K}$ for all $\alpha \in \Gamma$. By Theorem 3, it holds $w \vDash \alpha \Leftrightarrow [w]_{\approx'} \vDash \alpha$ for all $w \in W$. The statement is then proven, since for all $T \subseteq W$,

$$\begin{aligned}
& T \vDash \alpha \\
\Leftrightarrow\ & \forall w \in T : w \vDash \alpha && \text{(since } \alpha \in \mathcal{ML}\text{)} \\
\Leftrightarrow\ & \forall w \in T : [w]_{\approx'} \vDash \alpha && \text{(by assumption)} \\
\Leftrightarrow\ & \forall [w]_{\approx'} \in [T]_{\approx'} : [w]_{\approx'} \vDash \alpha && \text{(by definition of } [T]_{\approx'}\text{)} \\
\Leftrightarrow\ & [T]_{\approx'} \vDash \alpha && \text{(since } \alpha \in \mathcal{ML}\text{).}
\end{aligned}$$

◀

The above result demonstrates that team semantics has filtration when restricted to "flat" formulas. However, not all $\mathcal{MTL}$ formulas feature the flatness property. We proceed with fragments of $\mathcal{MTL}$ which are strictly stronger than $\mathcal{ML}$, and show that they still inherit the property to admit filtration.

▶ **Definition 7** **($\mathcal{B}$- and $\mathcal{S}$-closures).** If $\Phi \subseteq \mathcal{MTL}$, then $\mathcal{B}(\Phi)$ denotes the closure of Φ under $\sim$ and $\wedge$. Moreover, $\mathcal{S}(\Phi)$ denotes the closure of Φ under $\sim, \wedge$ and $\vee$.

Clearly $\Phi \subseteq \mathcal{B}(\Phi) \subseteq \mathcal{S}(\Phi) \subseteq \mathcal{MTL}$. However, every $\mathcal{MTL}$ formula is already expressively equivalent to even a $\mathcal{B}(\mathcal{ML})$ formula [13], a property that the recent axiomatization of $\mathcal{MTL}$ [14] allows to prove for each formula syntactically.

▶ **Lemma 8.** *Let $\mathcal{K} = (W, R, V)$ be a structure and $\Phi \subseteq \mathcal{MTL}$. If $\approx$ is strongly Φ-invariant on $\mathcal{K}$, then $\approx$ is also strongly $\mathcal{S}(\Phi)$-invariant on $\mathcal{K}$.*

Proof. Let $\approx'$ be a refinement of $\approx$. The proof is by induction on $|\varphi|$, where $\varphi \in \mathcal{S}(\Phi)$. The inductive step for the truth-functional connectives on the level of teams, i.e., $\sim$ and $\wedge$, are clear. So assume $\varphi = \psi_1 \vee \psi_2$.

Suppose $(\mathcal{K}, T) \vDash \varphi$ via $S \cup U = T$, $S \vDash \psi_1$ and $U \vDash \psi_2$. Then $[T]_{\approx'} = [S]_{\approx'} \cup [U]_{\approx'}$. By induction hypothesis, $[T]_{\approx'} \vDash \psi_1 \vee \psi_2$.

Conversely, let $[T]_{\approx'} \vDash \varphi$ via teams $\tilde{S}, \tilde{U} \subseteq W/_{\approx'}$ such that $\tilde{S} \cup \tilde{U} = [T]_{\approx'}$, $\tilde{S} \vDash \psi_1$ and $\tilde{U} \vDash \psi_2$. There is not necessarily a unique choice of $S, U \subseteq T$ such that $[S]_{\approx'} = \tilde{S}$ and $[U]_{\approx'} = \tilde{U}$, so we choose corresponding S and U as large as possible to ensure T is covered by $S \cup U$. Namely, define $S := \left\{ w \in T \mid [w]_{\approx'} \in \tilde{S} \right\}$ and $U := \left\{ w \in T \mid [w]_{\approx'} \in \tilde{U} \right\}$. If now $w \in T$, then $[w]_{\approx'} \in [T]_{\approx'}$, so $[w]_{\approx'} \in \tilde{S}$ or $[w]_{\approx'} \in \tilde{U}$, and consequently $w \in S$ or $w \in U$. Therefore $T \subseteq S \cup U$. By definition $S, U \subseteq T$, so $T = S \cup U$.

To show $T \vDash \psi_1 \vee \psi_2$ applying the induction hypothesis, it remains to show that actually $[S]_{\approx'} = \tilde{S}$ resp. $[U]_{\approx'} = \tilde{U}$ holds. Suppose $[w]_{\approx'} \in [S]_{\approx'}$. (The proof is analogous for U). Then by definition of $[\cdot]_{\approx'}$ there exists $\hat{w} \in S$ such that $\hat{w} \approx' w$, again implying by definiton of S that $[\hat{w}]_{\approx'} = [w]_{\approx'} \in \tilde{S}$. Hence $[S]_{\approx'} \subseteq \tilde{S}$.

Let conversely $[w]_{\approx'} \in \tilde{S}$. Then $[w]_{\approx'} \cap T$ is non-empty, since otherwise $[w]_{\approx'} \notin [T]_{\approx'}$ by definition of $[\cdot]_{\approx'}$, contradicting $[w]_{\approx'} \in \tilde{S} \subseteq [T]_{\approx'}$. Hence there exists some $\hat{w} \in T$ such that $\hat{w} \approx' w$. Since $\hat{w} \in T$ and $[\hat{w}]_{\approx'} = [w]_{\approx'} \in \tilde{S}$, it follows $\hat{w} \in S$ by the definition of S, and hence $[\hat{w}]_{\approx'} = [w]_{\approx'} \in [S]_{\approx'}$ as desired. ◀

▶ **Theorem 9.** *For every Kripke structure $\mathcal{K}$ and every finite $\Phi \subseteq \mathcal{S}(\mathcal{ML})$ there is an equivalence relation of index at most $\prod_{\varphi \in \Phi} 2^{|\varphi|}$ that is strongly Φ-invariant on $\mathcal{K}$.*

Proof. Let $\Gamma := \mathrm{SF}(\Phi) \cap \mathcal{ML}$. By Proposition 6, $\approx_\Gamma$ is Γ-invariant. $|\Gamma| \leq \sum_{\varphi\in\Phi} |\varphi|$, so $\approx_\Gamma$ has index at most $\prod_{\varphi\in\Phi} 2^{|\varphi|}$. Since $\Phi \subseteq \mathcal{S}(\Gamma)$, the theorem follows from Lemma 8. ◀

▶ **Corollary 10.** *Every satisfiable $\varphi \in \mathcal{S}(\mathcal{ML})$ has a model of size at most $2^{|\varphi|}$.*

3.1 Lower bounds for filtrations with invariance

In the context of modal logic, (strong) invariance appears as a natural property of filtrations. It seems like a straightforward tool to generalize the usual filtration technique, which preserves the truth of certain formulas in all *points* of a model, to team semantics. But it is inappropriate when team-wide modalities come into play, as the following counter-example shows. It employs a $\mathcal{PTL}$ formula of Hannula et al. [10], namely

$$\max(\Phi) := \sim \bigvee_{p\in\Phi} (p \otimes \neg p)\,,$$

where $\Phi \subseteq \mathcal{PS}$ is a finite set of propositions. Clearly $\max(\Phi)$ has length $\mathcal{O}(|\Phi|)$. Intuitively, it is true in a team T if and only if all Boolean assignments to variables in Φ are "realized" in worlds of T, formally:

$$\begin{aligned}(W,R,V,T) \vDash \max(\Phi) \Leftrightarrow \quad & \{\, f_w : \Phi \to \{1,0\} \mid w \in W, f_w(p) = 1 \Leftrightarrow w \in V(p) \,\} \\ &= \{\, f \mid f : \Phi \to \{1,0\} \,\}\end{aligned}$$

It is equivalent to the formula

$$\bigwedge_{\Phi'\in\mathcal{P}(X)} \mathsf{E}\left(\bigwedge_{p\in\Phi'} p \wedge \bigwedge_{p\in\Phi\setminus\Phi'} \neg p\right).$$

In the counter-example, we more generally consider homomorphisms that feature similar invariance properties as equivalence relations.

▶ **Definition 11.** Let $h\colon \mathcal{K} \to \mathcal{K}'$ be a homomorphism between Kripke structures. If T is a team in $\mathcal{K}$, then $h(T) := \{\, h(w) \mid w \in T \,\}$.
h is called Φ-*invariant on* $(\mathcal{K}, T)$ if $(\mathcal{K}, T) \vDash \varphi \Leftrightarrow (\mathcal{K}', h(T)) \vDash \varphi$ for all $\varphi \in \Phi$.
h is called Φ-*invariant on* $\mathcal{K}$ if it is Φ-invariant on $(\mathcal{K}, T)$ for all teams T in $\mathcal{K}$.

In the following, we also simply write that a homomorphism or equivalence relation is φ-invariant (resp. φ-preserving) instead of $\{\varphi\}$-invariant (resp. $\{\varphi\}$-preserving).

▶ **Theorem 12.** *Suppose $\varphi = \Box \max(\Phi)$ for $\Phi \subseteq \mathcal{PS}$. Then there is a structure $\mathcal{K}$ such that the image of any homomorphism φ-invariant on $\mathcal{K}$ has size at least $2^{2^{|\Phi|}}$.*

Proof. Let $\Phi := \{p_1, \ldots, p_n\} \subseteq \mathcal{PS}$, and let $\varphi := \Box \max(\Phi)$. Construct the structure $\mathcal{K} = (W, R, V)$ as follows. Let $W := \mathcal{P}(\mathcal{P}(\Phi))$, consequently $|W| = 2^{2^n}$. Intuitively, every world $\mathcal{A} \in W$ corresponds to a (possibly empty) subset of Boolean assignments to $p_1, \ldots, p_n$, each assignment being represented by a subset of Φ.

For any set $\mathcal{A} = \{\Phi'\} \in W$ of exactly one assignment $\Phi' \subseteq \Phi$, let $\mathcal{A} \in V(p) \Leftrightarrow p \in \Phi'$, that is, worlds in the Kripke structure that are singletons mimic the propositional labeling represented by their unique member Φ'. In all other worlds, all propositions are true, i.e., if $|\mathcal{A}| \neq 1$, then $\mathcal{A} \in V(p)$ for all $p \in \Phi$. Finally, for all $\mathcal{A} \in W$ and $\Phi' \in \mathcal{A}$, add the edge from $\mathcal{A}$ to $\{\Phi'\}$.

Let now $h(\mathcal{K})$ be a morphic image of $\mathcal{K}$ with h being φ-invariant. By definition, h has to preserve the truth of all propositions $p \in \Phi$. However, it also has to preserve their *falsity* everywhere: Otherwise there is some $\subseteq$-minimal $\Phi' \subseteq \Phi$ such that $\{\Phi'\} \nvDash p$, but $h(\{\Phi'\}) \vDash p$. Then the assignment Φ' itself cannot occur in $h(\mathcal{K})$ anymore, in particular, $h(\mathcal{K})$ contains no world w such that $w \nvDash p$ iff $p \notin \Phi'$. Then $\sim\varphi$ holds in all teams of $h(\mathcal{K})$ despite $(\mathcal{K}, W) \vDash \varphi$, contradiction to the φ-invariance.

Consequently, h preserves truth and falsity of propositions $p \in \Phi$. Suppose that the image $h(\mathcal{K})$ has *less* than 2^{2^n} worlds. By cardinality constraints, h is not injective, i.e., $h(\mathcal{A}) = h(\mathcal{A}')$ for distinct $\mathcal{A}, \mathcal{A}' \in W$. W.l.o.g. there is an assignment $\Phi' \in \mathcal{A} \setminus \mathcal{A}'$. Consider now the team $T := \{\, \mathcal{A}', \mathcal{P}(\Phi) \setminus \Phi' \,\}$ where neither world has $\{\Phi'\}$ as successor. Hence $T \nvDash \Box \max(\Phi)$.

But by definition, the second world in T has $\{\Phi''\}$ as successor for any assignment $\Phi'' \subseteq \Phi$ except Φ'. Note that

$$h(T) = \{h(\mathcal{A}'), h(\mathcal{P}(\Phi) \setminus \{\Phi'\})\} = \{h(\mathcal{A}), h(\mathcal{P}(\Phi) \setminus \{\Phi'\})\}.$$

As h preserves edges, for every $\Phi'' \subseteq \Phi$ an element in $h(T)$ has $h(\{\Phi''\})$ as a successor: for $\Phi'' = \Phi'$ it is $h(\mathcal{A})$, and for $\Phi'' \neq \Phi'$ it is $h(\mathcal{P}(\Phi) \setminus \Phi')$. Furthermore, for distinct singletons $\{\Phi''\}, \{\Phi'''\} \in W$ always $h(\{\Phi''\}) \neq h(\{\Phi'''\})$, as otherwise at least one proposition would not be preserved. Hence $h(T) \vDash \Box \max(\Phi)$. But as $T \nvDash \Box \max(\Phi)$, h cannot be φ-invariant. ◀

▶ **Corollary 13.** *There are $\mathcal{MTL}$ formulas φ and structures $\mathcal{K}$ such that every φ-invariant equivalence relation on $\mathcal{K}$ has index $2^{2^{\Omega(|\varphi|)}}$.*

Moreover, the theorem exhibits a gap between $\mathcal{B}(\mathcal{ML})$ and $\mathcal{S}(\mathcal{ML})$ with respect to distributing modal operators. Recall that according to Theorem 9, $\mathcal{S}(\mathcal{ML})$ admits exponential filtration.

▶ **Corollary 14.** *There are formulas $\Box\psi$, where $\psi \in \mathcal{S}(\mathcal{ML})$, such that every $\mathcal{S}(\mathcal{ML})$ formula equivalent to $\Box\psi$ has length $2^{\Omega(|\psi|)}$.*

In contrast, if $\varphi \in \mathcal{B}(\mathcal{ML})$, then $\Box\varphi$ has an equivalent $\mathcal{B}(\mathcal{ML})$ formula already of length $\leq 2|\varphi|$. The translation is immediate, since $\Box{\sim}\psi \equiv {\sim}\Box\psi$ and $\Box(\psi \wedge \psi') \equiv \Box\psi \wedge \Box\psi'$. In other words, $\Box$ *is easy to distribute over* $\wedge$ *and* $\sim$, *but hard to distribute over* $\vee$.

▶ **Corollary 15.** *There are formulas $\psi \in \mathcal{S}(\mathcal{ML})$ such that every $\mathcal{B}(\mathcal{ML})$ formula equivalent to ψ has length $2^{\Omega(|\psi|)}$.*

On the other hand, $\Diamond$ is easy to distribute over $\vee$, as $\Diamond(\psi \vee \psi') \equiv \Diamond\psi \vee \Diamond\psi'$.

4 Weaker filtrations for monotone MTL

An exponential model property can be obtained for larger fragments of $\mathcal{MTL}$, provided the requirements of filtration are weakened properly. An obvious candidate is the *invariance* property. To find a small model of φ starting from a given model $(\mathcal{K}, T)$, it is unnecessary to have $\sim\varphi$ preserved as well; hence we replace invariance by asymmetric preservation.

Moreover, a filtration $\approx$ does not need to preserve a formula φ in *all* teams of a model $(\mathcal{K}, T)$ — having φ true in $[T]_\approx$ would be completely sufficient. For this reason, we do not define preservation on the whole structure $\mathcal{K}$, but only locally:

▶ **Definition 16.** If $\approx$ is an equivalence relation on a Kripke structure $\mathcal{K} = (W, R, V)$, $T \subseteq W$ is a team, and $\Phi \subseteq \mathcal{MTL}$, then $\approx$ is

Φ-preserving on $(\mathcal{K}, T)$ if $\forall \varphi \in \Phi : (\mathcal{K}, T) \vDash \varphi \;\Rightarrow\; (\mathcal{K}/_\approx, [T]_\approx) \vDash \varphi$,

strongly Φ-preserving if every refinement $\approx'$ of $\approx$ is Φ-preserving.

Of course Φ-preservation does not imply $\sim\Phi$-preservation. The property is however still closed under application of *monotone* connectives. In this context, we consider the $\mathcal{MTL}$ operators $\land, \lor, \Diamond$ and $\Box$ as monotone, and also add the Boolean disjunction $\varobslash$. Accordingly, we define the following fragment.

▶ **Definition 17.** The fragment $\mathcal{MTL}_{\mathsf{mon}}$ of $\mathcal{MTL}$ is defined as the closure of $\mathcal{S}(\mathcal{ML})$ under $\land, \varobslash, \lor, \Box$ and $\Diamond$.

▶ **Theorem 18.** *For every finite $\Phi \subseteq \mathcal{MTL}_{mon}$, every structure $\mathcal{K} = (W, R, V)$, and every team $T \subseteq W$, there is an equivalence relation of index at most $\prod_{\varphi \in \Phi} 2^{|\varphi|}$ that is strongly Φ-preserving on $(\mathcal{K}, T)$.*

Note that we still quantify over all teams T, but are allowed to choose a different filtration for each team. The order of these quantifications makes a crucial difference here, in particular it eliminates the vulnerability against the method of Theorem 12 for filtration lower bounds.

In the following auxiliary lemmas, let $\mathcal{K} = (W, V, R)$ be a Kripke structure and $\approx$ an equivalence relation on W, and accordingly $\mathcal{K}/_{\approx} = (W/_{\approx}, R', V')$ as in Definition 2.

First we prove that subformulas starting with $\Diamond$ are preserved.

▶ **Lemma 19.** *If S is a successor team of T, then $[S]_{\approx}$ is an successor team of $[T]_{\approx}$.*

Proof. Suppose that S is a successor team of T. We have to show that every $[w] \in [T]$ has an R'-successor in $[S]$ and that every $[v] \in [S]$ has an R'-predecessor in $[T]$. Let $[w] \in [T]$, then $w' \in T$ for some $w' \approx w$. w' has an R-successor $v \in S$, so $[v] \in [S]$. But $Rw'v$ implies $R'[w'][v]$, so $[w'] = [w]$ has an R'-successor in $[S]$.

Conversely, if $[v] \in [S]$, then $v' \in S$ for some $v' \approx v$. v' has an R-predecessor $w \in T$. However, Rwv' again implies $R'[w][v']$, so $R'[w][v]$ for some $[w] \in [T]$. ◀

Formulas starting with $\Box$ are similarly preserved in teams T, at least as long as the filtration does not cross the boundaries of the preimage team T:

▶ **Lemma 20.** *If S is the image of T, and $w \approx w'$ implies $w \in T \Leftrightarrow w' \in T$, then $[S]$ is the image of $[T]$.*

Proof. As in the previous lemma, every $[v] \in [S]$ has an R'-predecessor in $[T]$. It remains to prove that $[S]$ contains all R'-successors $[v]$ of all $[w] \in [T]$. Let $R'[w][v]$ for $[w] \in [T]$. There exist $w' \approx w$ and $v' \approx v$ such that $Rw'v'$. By assumption of the lemma, $w' \in T$, so its R-successor v' must be in S, and $[v'] = [v] \in [S]$. ◀

It is easy to verify that the converse of the above lemmas is not true. Moreover, we are now ready to prove the theorem.

Proof of Theorem 18. Let $\mathcal{K} = (W, R, V)$, $T \subseteq W$ and $\Phi \subseteq \mathcal{MTL}_{\mathsf{mon}}$ be as in Theorem 18. W.l.o.g. $(\mathcal{K}, T) \vDash \Phi$. For this reason, we simply show that $\varphi := \bigwedge_{\psi \in \Phi} \psi$ is strongly preserved.

By definition of $\mathcal{MTL}_{\mathsf{mon}}$, φ is a monotone combination (i.e., using only operators $\land, \varobslash, \lor, \Diamond, \Box$) of $\mathcal{S}(\mathcal{ML})$ formulas. We exploit the monotonicity and define a *witness set* $\mathcal{T}(\psi)$ for certain subformulas ψ of φ. In the proof it suffices to preserve these subformulas in their corresponding witness teams instead of the whole structure.

For a simpler proof, we assume w.l.o.g. that every subformula of φ occurs only once in φ.

$\mathcal{T}(\psi) \subseteq W$ is defined in top-down manner, for $\psi \notin \mathcal{S}(\mathcal{ML})$, such that $(\mathcal{K}, \mathcal{T}(\psi)) \vDash \psi$. Accordingly, $\mathcal{T}(\varphi) := T$. Whenever $\mathcal{T}(\Box\psi)$ is defined for $\Box\psi \in \mathrm{SF}(\varphi)$, set $\mathcal{T}(\psi) := R(\mathcal{T}(\Box\psi))$. Similarly, $\mathcal{T}(\Diamond\psi)$ must have a successor team S that satisfies ψ, so set $\mathcal{T}(\psi) := S$. Any team $\mathcal{T}(\psi \lor \psi')$, for $\psi \lor \psi' \in \mathrm{SF}(\varphi)$, likewise can be split into $S \vDash \psi$ and $U \vDash \psi'$, consequently then $\mathcal{T}(\psi) := S$ and $\mathcal{T}(\psi') := U$. If $\psi = \psi' \wedge \psi''$ or $\psi = \psi' \varovee \psi''$ is in $\mathrm{SF}(\varphi)$, then $\mathcal{T}(\psi')$ and/or $\mathcal{T}(\psi'')$ simply equals $\mathcal{T}(\psi)$.

Similarly as in Theorem 9, $\Gamma := \mathrm{SF}(\varphi) \cap \mathcal{ML}$. Define $\approx'$ as the coarsest refinement of $\approx_\Gamma$ that does not cross the boundaries of witness teams $\mathcal{T}(\psi)$, i.e., $w \approx' w'$ if and only if $w \approx_\Gamma w'$ and, for all $\psi \in \mathrm{SF}(\varphi)$, $w \in \mathcal{T}(\psi) \Leftrightarrow w' \in \mathcal{T}(\psi)$.

Lemma 19 and 20 now allow to prove $(\mathcal{K}/_{\approx''}, [\mathcal{T}(\psi)]_{\approx''}) \vDash \psi$ for any refinement $\approx''$ of $\approx'$, and any $\psi \in \mathrm{SF}(\varphi)$, by induction on $|\psi|$. The splitting case is proven as in Lemma 8. As $\approx'$ has index at most $2^{|\varphi|}$, this proves the theorem. ◀

▶ **Corollary 21.** *Every satisfiable formula $\varphi \in \mathcal{MTL}_{\mathsf{mon}}$ has a model of size at most $2^{|\varphi|}$.*

▶ **Proposition 22.** *The satisfiability problem of $\mathcal{MTL}_{\mathsf{mon}}$ is* $\mathsf{AEXPTIME}(\mathrm{poly})$*-complete.*

Proof. The hardness already holds for $\mathcal{PTL}$ [9], so we prove only the upper bound. The model checking problem for $\mathcal{MTL}$ is decidable by an alternating Turing machine that, given $(\mathcal{K}, T, \varphi)$, runs in time polynomial in $|\mathcal{K}| + |\varphi|$, and with alternations polynomial in $|\varphi|$ [10]. This allows to decide the satisfiability problem of $\mathcal{MTL}_{\mathsf{mon}}$ as follows. Given a formula φ, guess a Kripke structure $\mathcal{K}$ of size up to $2^{|\varphi|}$ and a team T in $\mathcal{K}$. Then execute the above model checking algorithm on $(\mathcal{K}, T, \varphi)$. By the preceding corollary, the algorithm decides $\mathcal{MTL}_{\mathsf{mon}}$ in exponential runtime and polynomially many alternations. ◀

5 Lower bounds for MTL with alternating modalities

The established small model property for $\mathcal{MTL}_{\mathsf{mon}}$ crucially depends on the existential quantification of successor teams and splittings that is inherent to this fragment. If Boolean negations $\sim$ are permitted in front of arbitrary $\Diamond$ and $\vee$ operators, then *alternations* between existential and universal quantification of modalities are introduced.

Indeed, we prove in this section that already $\mathcal{MTL}$ formulas of the form $\sim\Diamond\sim\Diamond\cdots\varphi$ are highly resistant to filtration for very simple φ, e.g., of $\mathcal{B}(\mathcal{ML})$. Note that every $\mathcal{MTL}$ formula is equivalent to a $\mathcal{B}(\mathcal{ML})$ formula of the form

$$\bigovee_{i=1}^{n} \left(\alpha_i \wedge \bigwedge_{j=1}^{m_i} \mathsf{E}\beta_{i,j} \right),$$

where $\alpha_i, \beta_{i,j} \in \mathcal{ML}$ [13, 14]. We refer to this form as *disjunctive normal form (DNF)*, as $\varovee$ and $\wedge$ are the Boolean disjunction and conjunction over teams. $\mathcal{ML}$ formulas are then considered "literals." There is only one positive literal per disjunct as $\mathcal{ML}$ is closed under $\wedge$.

In the following, the positive literals of DNFs are w.l.o.g. pairwise distinct. This allows a concise notation: For finite $\Gamma, \Delta \subseteq \mathcal{ML}$ and $\lambda : \Gamma \to \mathcal{P}(\Delta)$, we write $(\Gamma, \Delta, \lambda)$ instead of

$$\bigovee_{\alpha \in \Gamma} \left(\alpha \wedge \bigwedge_{\beta \in \lambda(\alpha)} \mathsf{E}\beta \right).$$

Here, Γ is the set of positive literals, Δ is the set of negative literals, and λ defines the clauses.

To establish filtration lower bounds, we require the disjuncts of a DNF to be "independent enough". We formalize this with two properties.

▶ **Definition 23.** A set $\Gamma \subseteq \mathcal{ML}$ is called *multiplicative* if it is satisfiable, and furthermore for all $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \Gamma$ it holds that, if $\Gamma_1 \cup \neg\Gamma_2$ and $\Gamma_1 \cup \neg\Gamma_3$ are satisfiable, then also $\Gamma_1 \cup \neg(\Gamma_2 \cup \Gamma_3)$ is satisfiable.

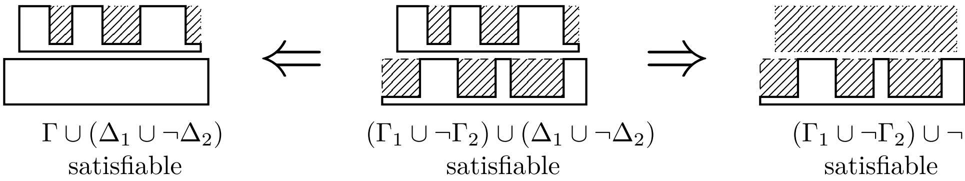

Figure 1 Hanoi property of sets $\Gamma, \Delta \subseteq \mathcal{ML}$.

Examples for multiplicative sets are $\mathcal{PS}$ and $\{ \Box\varphi \mid \varphi \in \mathcal{ML} \}$.

▶ **Definition 24.** Let $\Gamma, \Delta \subseteq \mathcal{ML}$. The tuple (Γ, Δ) has the *Hanoi property* if for all $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $\Delta_1, \Delta_2 \subseteq \Delta$:

1. $\Gamma \cup \neg\Delta$ is satisfiable,
2. whenever $\Gamma_1 \cup \neg\Gamma_2$ is satisfiable, then also $(\Gamma_1 \cup \neg\Gamma_2) \cup \neg\Delta$ is satisfiable,
3. whenever $\Delta_1 \cup \neg\Delta_2$ is satisfiable, then also $\Gamma \cup (\Delta_1 \cup \neg\Delta_2)$ is satisfiable.

For convenience, we say that a DNF $(\Gamma, \Delta, \lambda)$ has this property simply if (Γ, Δ) has it.

An example illustrating the Hanoi property[4] of a tuple (Γ, Δ) is depicted in Figure 1.

As a next step, we more carefully analyze the entailment relation $\vDash$ between the literals in each Γ and Δ. For the rest of the section, we make use of a number of definitions from order theory, like partial orders, lattices and filters. To refresh the foundations, the reader is referred to the very good textbook by Davey and Priestley [5].

An *antichain* is a set of pairwise unordered elements. The *width* $w(X)$ of a partially ordered finite set X is the cardinality of its largest antichain. If $S \subseteq S'$ is ordered by $\preceq$, then its *up-set* is $S^{\uparrow} := \{ s' \in S' \mid \exists s \in S : s \preceq s' \}$. Its *down-set* $S^{\downarrow}$ is defined analogously. $\mathcal{U}(X)$ resp. $\mathcal{L}(X)$ denotes the lattice of all upper resp. lower subsets of X. By convention, we assume the quasi-ordering $\vDash$ on $\mathcal{MTL}$, and the partial ordering $\subseteq$ on $\mathcal{P}(\mathcal{MTL})$.

Define the formula $\mathsf{NESub}\psi := \top \vee (\mathsf{E}\top \wedge \psi)$. It states that a non-empty subteam satisfies ψ. Furthermore, let $\mathsf{AS}\psi := {\sim}\mathsf{NESub}{\sim}\mathsf{NESub}\psi$.

▶ **Lemma 25.** *For any Kripke structure $\mathcal{K}$ and team T in $\mathcal{K}$, $(\mathcal{K}, T) \vDash \mathsf{AS}\psi$ if and only if $(\mathcal{K}, \{w\}) \vDash \psi$ for every singleton $\{w\} \subseteq T$.*

Proof. For the first direction, let $(\mathcal{K}, T) \vDash \mathsf{AS}\psi$ and $w \in T$. For the sake of contradiction suppose $(\mathcal{K}, \{w\})) \nvDash \psi$. Clearly the only non-empty subteam of $\{w\}$ is $\{w\}$ itself, so no non-empty subteam of $\{w\}$ satisfying ψ exists. For this reason, $(\mathcal{K}, \{w\}) \vDash {\sim}\mathsf{NESub}\psi$. Clearly $\{w\}$ is a non-empty subteam of T, consequently $(\mathcal{K}, T) \vDash \mathsf{NESub}{\sim}\mathsf{NESub}\psi$, contradicting $(\mathcal{K}, T) \vDash \mathsf{AS}\psi$.

For the other direction, suppose $(\mathcal{K}, \{w\}) \vDash \psi$ for every singleton $\{w\} \subseteq T$. Assume again for the sake of contradiction that $(\mathcal{K}, T) \nvDash \mathsf{AS}\psi$, i.e., $\mathsf{NESub}{\sim}\mathsf{NESub}\psi$ holds in T via some non-empty subteam $T' \subseteq T$ that satisfies ${\sim}\mathsf{NESub}\psi$. Since T' clearly contains at least one singleton that is also contained in T, it satisfies $\mathsf{NESub}\psi$ as well, contradiction. ◀

Using the above properties, we now identify a class of formulas in DNF in which the optimal filtration directly correlates with the size of an $\vDash$-antichain in the set of its positive literals.

[4] As in the *Towers of Hanoi* puzzle, the upper layer (true elements of Δ in Figure 1) should be empty unless the lower layer (true elements of Γ) is "full". The property then states that, starting at a model for any subsets of Γ and Δ, it is also possible to find models with the lower layer (Γ) changed to "full" or the upper one (Δ) changed to "empty".

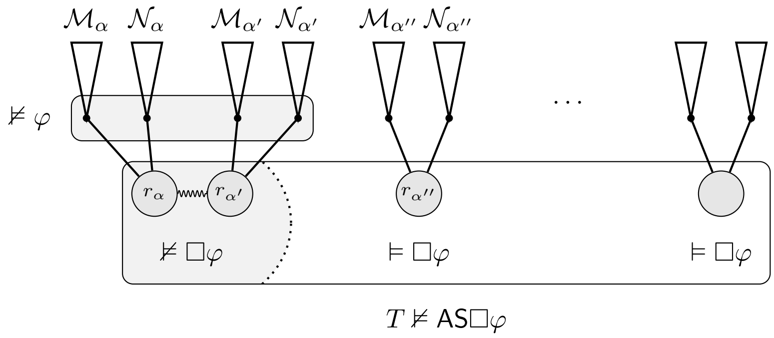

Figure 2 A filtration failing to preserve $\Box\varphi$ in the singleton $[r_\alpha] = [r_{\alpha'}]$.

▶ **Theorem 26.** *Let a DNF $\varphi = (\Gamma, \Delta, \lambda)$ satisfy the following properties:*
1. *φ has the Hanoi property and $\Gamma \cup \Delta$ is multiplicative,*
2. *Γ is partially ordered by $\vDash$,*
3. *$\lambda : \Gamma \to \Delta$, and $\alpha \vDash \alpha' \ \Leftrightarrow\ (\alpha' \wedge \lambda(\alpha')) \vDash (\alpha \wedge \lambda(\alpha))$.*

Then there exists a model $(\mathcal{K}, T)$ such that every equivalence relation $\approx$ that strongly preserves $\mathsf{AS}\Box\varphi$ on $(\mathcal{K}, T)$ has index at least $w(\Gamma)$.

Proof. Let $\Gamma^* \subseteq \Gamma$ be an antichain of cardinality $w(\Gamma)$. We construct $\mathcal{K}$ as follows. By multiplicativity and the Hanoi property, every $\alpha \in \Gamma$ is satisfiable by a model $\mathcal{M}_\alpha = (\mathcal{K}^{\mathcal{M}_\alpha}, w^{\mathcal{M}_\alpha})$ such that

$$\mathcal{M}_\alpha \vDash \{\alpha\} \cup \{\, \neg\alpha' \in \neg\Gamma \mid \alpha \nvDash \alpha' \,\} \cup \neg\Delta.$$

Likewise, let $\mathcal{N}_\alpha = (\mathcal{K}^{\mathcal{N}_\alpha}, w^{\mathcal{N}_\alpha})$ be a model such that

$$\mathcal{N}_\alpha \vDash \{\alpha \wedge \lambda(\alpha)\} \cup \left\{\neg(\alpha' \wedge \lambda(\alpha')) \mid \alpha' \in \Gamma \setminus \{\alpha\} \text{ and } \alpha \vDash \alpha'\right\}.$$

We show that $\mathcal{N}_\alpha$ exists. Due to multiplicativity, consider a single $\alpha' \neq \alpha$ such that $\alpha \vDash \alpha'$. Γ is partially ordered, so $\alpha' \nvDash \alpha$, and by **3.** $(\alpha \wedge \lambda(\alpha)) \nvDash (\alpha' \wedge \lambda(\alpha'))$. For this reason, $(\alpha \wedge \lambda(\alpha)) \wedge \neg(\alpha' \wedge \lambda(\alpha'))$ is satisfiable.

$\mathcal{K}$ is now the disjoint union of all $\mathcal{M}_\alpha, \mathcal{N}_\alpha$ and an additional world r_α for each $\alpha \in \Gamma^*$, which possesses edges to the roots of $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ (see Figure 2). Finally, let $T := \{\, r_\alpha \mid \alpha \in \Gamma^* \,\}$. It is straightforward that the image of each $\{r_\alpha\} \subseteq T$ satisfies the disjunct $\alpha \wedge \mathsf{E}\lambda(\alpha)$ of φ, so $T \vDash \mathsf{AS}\Box\varphi$.

Now suppose that $\approx$ is an equivalence relation with index less than $|\Gamma^*|$. Then $r_\alpha \approx r_{\alpha'}$ for some distinct $\alpha, \alpha' \in \Gamma^*$, i.e., the equivalence relation inevitably merges two worlds in T. Let $\approx'$ be the refinement of $\approx$ that merges nothing else. Suppose for the sake of contradiction that $\approx'$ preserves $\mathsf{AS}\Box\varphi$ on $(\mathcal{K}, T)$.

By definition of $\mathcal{M}_\alpha$, it holds $[r_\alpha]_{\approx'} \vDash \Diamond\neg\alpha''$, unless both $\alpha \vDash \alpha''$ and $\alpha' \vDash \alpha''$. Consequently, for $\{[r_\alpha]_{\approx'}\}$ to satisfy $\Box\varphi$, its image must necessarily satisfy a disjunct $\alpha'' \wedge \mathsf{E}\lambda(\alpha'')$ of φ with $\alpha \vDash \alpha''$ and $\alpha' \vDash \alpha''$.

Suppose such α'' exists. As Γ^* is an antichain, α, α' and α'' must be pairwise distinct. But then $(\alpha'' \wedge \lambda(\alpha''))$ is already false in the roots of $\mathcal{M}_\alpha$, $\mathcal{M}_{\alpha'}$, $\mathcal{N}_\alpha$, and $\mathcal{N}_{\alpha'}$. Since $\{[r_\alpha]_{\approx'}\} \vDash \Box{\sim}\alpha''$ or $\{[r_\alpha]_{\approx'}\} \vDash \Box\neg\lambda(\alpha'')$, α'' cannot exist. For this reason, ultimately $\{[r_\alpha]_{\approx'}\} \nvDash \Box\varphi$ holds and $\approx$ is not strongly $\mathsf{AS}\Box\varphi$-preserving. ◀

Of course we still have to ask whether suitable DNFs with large antichains actually exist. In Subsection 5.1, we introduce a candidate, namely the *naive expansion* of $\mathcal{MTL}$

formulas. In Subsection 5.2, we then investigate the naive expansion of formulas of the form $\underbrace{\sim\Diamond\sim\Diamond\cdots}_{n \text{ alternations}}\varphi$, and prove that its width grows faster than any elementary function in n.

5.1 Naive DNF expansions

Suppose that $\varphi = (\Gamma, \Delta, \lambda)$ is a DNF. If $\sim\varphi$ should hold in a team, then for each $\alpha \in \Gamma$, either $\mathsf{E}\neg\alpha$ holds, or $\neg\beta$ is true for some $\beta \in \lambda(\alpha)$. Depending on λ, there are many possible ways to falsify φ. We represent each one by a *choice function* $\mathfrak{f}_\varphi$. By convention, $\mathfrak{f}_\varphi(\alpha)$ being undefined shall mean that $\mathsf{E}\neg\alpha$ holds, and otherwise $\neg\mathfrak{f}_\varphi(\alpha)$ is true. The set of all choice functions with respect to φ is then:

$$F_\varphi = \{\, \mathfrak{f} : \Gamma' \to \Delta \mid \Gamma' \subseteq \Gamma \text{ and } \forall \alpha \in \Gamma' : \mathfrak{f}(\alpha) \in \lambda(\alpha) \,\}.$$

To now express $\sim\varphi$ as a DNF, every possible choice function $\mathfrak{f}$ must be considered. The DNF $(\Gamma, \Delta, \lambda)^\sim$, logically equivalent to $\sim(\Gamma, \Delta, \lambda)$, is defined as $(\Gamma', \Delta', \lambda')$ with

$$\Gamma' := \left\{ \bigwedge_{\alpha \in \mathsf{dom}\,\mathfrak{f}} \neg\mathfrak{f}(\alpha) \;\middle|\; \mathfrak{f} \in F_\varphi \right\}$$
$$\Delta' := \neg\Gamma,$$

and $\lambda'\left(\bigwedge_{\alpha \in \mathsf{dom}\,\mathfrak{f}} \neg\mathfrak{f}(\alpha)\right) := \neg(\Gamma \setminus \mathsf{dom}\,\mathfrak{f})$.

For $\Diamond$, recall that $\Box, \Diamond$ and $\vee$ all distribute over Boolean disjunction $\varovee$. But unlike the $\Box$ operator, $\Diamond$ and $\vee$ do not distribute over conjunction. Here, we merely have a "pseudo-distributive law" for $\Diamond$ (and a similar one for $\vee$),

$$\Diamond\left(\alpha \wedge \bigwedge_{i=1}^{n} \mathsf{E}\beta_i\right) \equiv \Diamond\alpha \wedge \bigwedge_{i=1}^{n} \mathsf{E}\Diamond(\alpha \wedge \beta_i),$$

where $\alpha, \beta \in \mathcal{ML}$ [14]. Then the DNF $(\Gamma, \Delta, \lambda)^\Diamond := (\Diamond\Gamma, \Delta', \lambda')$, where $\Delta' := \{\Diamond(\alpha \wedge \beta) \mid \alpha \in \Gamma, \beta \in \lambda(\alpha)\}$ and $\lambda'(\Diamond\alpha) := \{\Diamond(\alpha \wedge \beta) \mid \beta \in \lambda(\alpha)\}$, is equivalent to $\Diamond(\Gamma, \Delta, \lambda)$.

All in all, the combination of the above steps yields the following DNF $(\Gamma, \Delta, \lambda)^{\Diamond\sim} = (\Gamma', \Delta', \lambda')$ equivalent to $\sim\Diamond(\Gamma, \Delta, \lambda)$. We gather the "positive" literals of $\mathfrak{f} \in F_\varphi$ in the conjunction $\ulcorner\mathfrak{f}\urcorner := \bigwedge_{\alpha \in \mathsf{dom}\,\mathfrak{f}} \neg(\alpha \wedge \mathfrak{f}(\alpha))$, and simply define:

$$\Gamma' := \{\, \Box\ulcorner\mathfrak{f}\urcorner \mid \mathfrak{f} \in F_\varphi \,\}$$
$$\Delta' := \neg\Diamond\Gamma$$
$$\lambda'(\Box\ulcorner\mathfrak{f}\urcorner) := \neg\Diamond(\Gamma \setminus \mathsf{dom}\,\mathfrak{f}).$$

5.2 Existence of large width DNFs

In the naive expansion $(\Gamma, \Delta, \lambda)$ of a formula, many literals in Γ or Δ may happen to be redundant. To ensure that Γ contains a large antichain, and even is partially ordered under $\vDash$, we have to carefully "carve out" superfluous disjuncts. As a first step, we formalize a sufficient notion of redundancy of disjuncts in a DNF.

If f is a function and $X \subseteq \mathsf{dom}\, f$, let $f{\restriction}_X$ be the restriction of f to the domain X.

▶ **Definition 27.** Let $\varphi = (\Gamma, \Delta, \lambda)$ be a DNF. A *kernel* of φ is a DNF $\varphi' = (\Gamma', \Delta', \lambda')$ such that

1. $\Gamma' \subseteq \Gamma$, $\lambda' = \lambda{\restriction}_{\Gamma'}$, and $\Delta' = \bigcup_{\alpha \in \Gamma'} \lambda'(\alpha)$
2. for every $\alpha \in \Gamma$ there exists $\alpha' \in \Gamma'$ such that $\alpha \vDash \alpha'$ and $\lambda'(\alpha') \subseteq \lambda(\alpha)^{\uparrow}$.

The condition $\lambda'(\alpha') \subseteq \lambda(\alpha)^{\uparrow}$ can also be formulated as: for all $\beta' \in \lambda'(\alpha')$ it holds $\beta \vDash \beta'$ for some $\beta \in \lambda(\alpha)$.

▶ **Lemma 28.** *If φ' is a kernel of φ, then $\varphi \equiv \varphi'$.*

Proof. Since φ includes all disjuncts of φ', clearly $\varphi' \vDash \varphi$. Conversely, assume that a model $\mathcal{M}$ satisfies a disjunct $\alpha \wedge \mathsf{E}\beta_1 \wedge \cdots \wedge \mathsf{E}\beta_n$ of φ. Then there is a disjunct of φ' of the form $\alpha' \wedge \mathsf{E}\beta'_1, \wedge \cdots \wedge \mathsf{E}\beta'_m$, where $\alpha \vDash \alpha'$ and for every $i \in [m]$ there exists $j \in [n]$ such that $\beta_j \vDash \beta'_i$. Consequently, $\mathcal{M}$ satisfies $\alpha' \wedge \mathsf{E}\beta'_1, \wedge \cdots \wedge \mathsf{E}\beta'_m$ and therefore φ'. ◀

In the next lemma, we ensure that the Hanoi property, Definition 24, is preserved under application of $\sim\Diamond$.

▶ **Lemma 29.** *If a DNF $(\Gamma, \Delta, \lambda)$ has the Hanoi property, then also any kernel of $(\Gamma, \Delta, \lambda)^{\Diamond\sim}$ has it.*

Proof. Let $(\Gamma', \Delta', \lambda') := (\Gamma, \Delta, \lambda)^{\Diamond\sim}$. Proving the lemma for $(\Gamma', \Delta', \lambda')$ also proves it for every kernel.

Proof of condition 1. Since $\Gamma \cup \neg\Delta$ is satisfiable, clearly $\Diamond(\Gamma \cup \neg\Delta) \cup \Box(\Gamma \cup \neg\Delta)$ is satisfiable. The claim follows since $\Box\neg\Delta \vDash \Gamma'$ and $\Diamond\Gamma \equiv \neg\neg\Diamond\Gamma = \neg\Delta'$.

Proof of condition 2. Suppose $\Gamma'_1 \cup \neg\Gamma'_2$ is satisfiable for $\Gamma'_1, \Gamma'_2 \subseteq \Gamma'$. Note that $\Gamma' \cup \Delta'$ is multiplicative, since every formula in it is equivalent to an element of $\{\Box\varphi \mid \varphi \in \mathcal{ML}\}$ (cf. page 10). Since $\Gamma' \cup \neg\Delta'$ is satisfiable by condition **1.**, also $\Gamma'_1 \cup \neg\Gamma'_2 \cup \neg\Delta'$ is satisfiable.

Proof of condition 3. Let $\Delta'_1 \cup \neg\Delta'_2$ be satisfied by a model $\mathcal{M} = (\mathcal{K}, w)$. We augment $\mathcal{M}$ to additionally satisfy Γ'. Every formula in $\neg\Delta'_2$ is of the form $\neg\neg\Diamond\alpha$ for some $\alpha \in \Gamma$. The root w has a successor v_α for every such α such that $v_\alpha \vDash \{\alpha\} \cup \{\neg\alpha' \mid \neg\Diamond\alpha' \in \Delta'_1\}$. As condition **2.** holds for (Γ, Δ), we can for each such v_α add a new successor that is the root of a model of $\{\alpha\} \cup \{\neg\alpha' \mid \neg\Diamond\alpha' \in \Delta'_1\} \cup \neg\Delta$. If now all old successors of w are removed, the resulting structure satisfies $\Delta'_1 \cup \neg\Delta'_2 \cup \Gamma'$. ◀

In the rest of the section, we aim at applying Theorem 26 to a DNF $\varphi = (\Gamma, \Delta, \lambda)$ that is constructed to have a large width $w(\Gamma)$. The idea is to introduce alternations between positive and negative occurrences of modal operators, as stated at the beginning of this section. A lower bound of $w(\Gamma)$ is then proven by induction on the number of alternations. Unfortunately, the conditions **1.–3.** of Theorem 26 are not sufficient, since φ meeting them does not imply $\varphi^{\Diamond\sim}$ meeting them. Instead, we consider the following extended conditions which are used as an invariant during the induction step.

1. φ has the Hanoi property and $\Gamma \cup \Delta$ is multiplicative,
2. $\vDash$ is a partial ordering on both Γ and Δ,
3. **(a)** $\forall\alpha \in \Gamma : |\lambda(\alpha)| = 1$, and if $\lambda(\alpha) = \{\beta\}, \lambda(\alpha') = \{\beta'\}$, then $\alpha \vDash \alpha' \ \Leftrightarrow\ (\alpha' \wedge \beta') \vDash (\alpha \wedge \beta)$,
   (b) $\lambda : \Gamma \to \mathcal{U}(\Delta)$ is bijective and $\alpha \vDash \alpha' \ \Leftrightarrow\ \lambda(\alpha) \subseteq \lambda(\alpha')$,
4. **(a)** if $\alpha, \alpha' \in \Gamma$ are incomparable and $\beta \in \lambda(\alpha)$, then $\alpha \wedge \beta \nvDash \alpha'$,
   (b) if $\lambda(\alpha) \setminus \lambda(\alpha')$ contains an element incomparable to $\beta \in \Delta$, then $\alpha \wedge \beta \nvDash \alpha'$.

Moreover, if φ is a DNF with the properties **3.(a)** and **4.(a)**, then $\varphi^{\Diamond\sim}$ does not necessarily have a kernel which satisfies **3.(a)** and **4.(a)** again. Instead, the iterated expansion of a $\sim\Diamond$ prefix alternates between DNFs satisfying either **3.(a)–4.(a)** or **3.(b)–4.(b)**. This is made explicit in the following two technical lemmas. The proofs can be found in the appendix. Note that the lemmas work asymmetrically: the width $w(\Gamma)$ stays the same in the first lemma, but increases by a factorial in the second lemma.

▶ **Lemma 30.** *If a DNF $\varphi = (\Gamma, \Delta, \lambda)$ satisfies **1., 2., 3.(a)** and **4.(a)**, then $\varphi^{\Diamond\sim}$ has a kernel $(\Gamma', \Delta', \lambda')$ that satisfies **1., 2., 3.(b)** and **4.(b)** such that $w(\Delta') = w(\Gamma)$.*

▶ **Lemma 31.** *If a DNF $\varphi = (\Gamma, \Delta, \lambda)$ satisfies **1., 2., 3.(b)** and **4.(b)**, then $\varphi^{\Diamond\sim}$ has a kernel $(\Gamma', \Delta', \lambda')$ that satisfies **1., 2., 3.(a)** and **4.(a)** such that $w(\Gamma') \geq w(\Delta)!$.*

Applying the two lemmas in an inductive manner yields a width that grows roughly as fast as the iterated factorial. We write the n-times iterated factorial of n as:

$$n!^{(n)} := n\underbrace{!\ldots!}_{n \text{ times}}$$

The iteration count is essentially half the *negation depth* $\mathrm{nd}(\varphi)$, the maximal nesting depth of $\sim$ in φ. It is defined similarly as the *modal depth* $\mathrm{md}(\varphi)$, which is the maximal nesting depth of modalities [3].

It is not difficult to choose an initial DNF φ that satisfies **1.**, **2.**, **3.(a)** and **4.(a)** A simple example is $\varphi := \bigovee_{i=1}^{n}(p_i \wedge \mathsf{E}q_i)$, where $p_1, q_1, \ldots \in \mathcal{PS}$ are distinct. By applying Lemma 30 and 31 repeatedly, we obtain a DNF φ'. φ' satisfies **1.**, **2.**, **3.(a)** and **4.(a)**, and is by Lemma 28 equivalent to $(\sim\Diamond)^{2n}\varphi$, but its positive literal set has width at least $n!^{(n)}$.

Clearly an equivalence relation is φ-preserving if and only if it is φ'-preserving. A consequence of this fact and of Theorem 26 is the following main result of this section.

▶ **Theorem 32.** *There are $\mathcal{MTL}$ formulas $(\varphi_n)_{n\in\mathbb{N}}$ of length $\mathcal{O}(n)$, modal depth $2n + \mathcal{O}(1)$ and negation depth $2n + \mathcal{O}(1)$, and models $(\mathcal{K}, T)$ of φ_n, such that every equivalence relation that strongly preserves φ_n on $(\mathcal{K}, T)$ has index at least $n!^{(n)}$.*

Recall that $\mathcal{MTL}_{\mathsf{mon}}$ has, according to Theorem 18, strongly φ-preserving filtration with index exponential in $|\varphi|$. Therefore we conclude this section with a non-elementary succinctness gap between the equally expressive logics $\mathcal{MTL}$ and $\mathcal{MTL}_{\mathsf{mon}}$.

▶ **Corollary 33.** *There are $\mathcal{MTL}$ formulas $(\varphi_n)_{n\in\mathbb{N}}$ of length $\mathcal{O}(n)$, modal depth $2n + \mathcal{O}(1)$ and negation depth $2n + \mathcal{O}(1)$ for which any equivalent $\mathcal{MTL}_{\mathsf{mon}}$ formula has length at least $n!^{(n)}$.*

6 Conclusion

In this paper, the filtration technique was shown to be insufficient as a tool applied to modal logic considered under team semantics and supplied with Boolean negation.

On the one hand, filtration continues to work in an almost straightforward way for the extension $\mathcal{S}(\mathcal{ML})$ of $\mathcal{ML}$ – that is, the closure of $\mathcal{ML}$ under splitting $\vee$ and Boolean operators $\sim, \wedge$. Theorem 9, the exponential model property for $\mathcal{S}(\mathcal{ML})$, is not particularly surprising: for a given $\Gamma \subseteq \mathcal{ML}$, let Γ^* be the closure of Γ under $\neg$ and $\wedge$. Then every $\mathcal{S}(\Gamma)$ formula is equivalent to a $\mathcal{B}(\Gamma^*)$ formula [14].[5] As filtration upper bounds are clearly "closed under Boolean connectives", including all $\wedge$, $\neg$ and $\sim$, the strong upper bound in Section 3 naturally emerges.

[5] The translation from the first fragment to the latter plays a crucial role in the completeness of a recent axiomatic system for $\mathcal{MTL}$ [14].

On the other hand, the introduction of team-wide modalities prevents filtration very thoroughly and effectively, as Section 5 shows. The proof of the non-elementary filtration lower bound employs a formula $\mathsf{AS}\Box(\sim\Diamond)^{2n}\varphi$ where φ is a disjunctive normal form (using $\sim$ and $\wedge$) of only propositional variables. Besides $\Diamond$ operators alternating between positive and negative occurrences, for technical reasons that formula contains one additional $\Box$ and, contained in the definition of the AS operator, two instances of $\vee$.

The main conclusion that can be drawn from these results is the following. Filtration, being defined as a simple quotient construction merging individual worlds, preserves modal formulas that can be evaluated on points, but it cannot capture the interdependencies between different points in a Kripke structure that are addressed by the team-wide modalities.

Another subtle point is the distinction between an equivalence relation preserving the truth value of a formula φ on a pair $(\mathcal{K}, T)$, or even in all teams throughout the whole structure $\mathcal{K}$. Intuitively, the first variant corresponds to choosing a filtration after knowing the team T in which φ must be preserved, while the latter one corresponds to not knowing T beforehand. While $\mathcal{S}(\mathcal{ML})$ admits filtration in both senses, Section 3.1 shows that already the introduction of a single occurrence of a non-classical modality prohibits a filtration of exponential size unless T is fixed, i.e., known beforehand.

This more specialized approach is developed in Section 4 to a “weak filtration” for the fragment named $\mathcal{MTL}_{\mathsf{mon}}$. It has the exponential model property, at the cost of disallowing $\Diamond$ and $\Box$ to occur negatively. While this modal fragment appears rather artificial to allow filtration, it potentially permits a variant of the standard translation (see, e.g., [3]) to the existential fragment of a variant of first-order or second-order logic. It is well-known that the satisfiability problem for the analogous existential $\mathsf{SO}(\exists)$ fragment of second-order logic is not harder than for first-order logic FO.

The paper aims at classifying the actual complexity of Modal Team Logic $\mathcal{MTL}$, in the sense of both computational and model-theoretical complexity. From that perspective, the paper is clearly a negative result. Nevertheless, the used proof methods for the failure of filtration have useful side effects: namely they imply several succinctness lower bounds, even indicating a strict succinctness hierarchy of fragments inside $\mathcal{MTL}$. The detailed structure of that hierarchy is a potential target of future research, as well as closing the gap of known unconditional upper and lower bounds for model size.

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References

1 Samson Abramsky and Jouko Väänänen. From IF to BI. *Synthese*, 167(2):207–230, 2009. doi:10.1007/s11229-008-9415-6.

2 Ernest Allen Emerson. Temporal and modal logic. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science (Vol. B)*, pages 995–1072. MIT Press, Cambridge, MA, USA, 1990. URL: http://dl.acm.org/citation.cfm?id=114891.114907.

3 Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, New York, NY, USA, 2001.

4 Ashok K. Chandra, Dexter C. Kozen, and Larry J. Stockmeyer. Alternation. *Journal of the ACM*, 28(1):114–133, January 1981. doi:10.1145/322234.322243.

5 Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge university press, 2002.

6 Pietro Galliani. Inclusion and exclusion dependencies in team semantics – On some logics of imperfect information. *Annals of Pure and Applied Logic*, 163(1):68–84, January 2012. doi:10.1016/j.apal.2011.08.005.

7 Erich Grädel and Jouko Väänänen. Dependence and independence. *Studia Logica*, 101(2):399–410, 2013.

8 Erich Grädel, Phokion G. Kolaitis, and Moshe Y. Vardi. On the Decision Problem for Two-Variable First-Order Logic. *Bulletin of Symbolic Logic*, 3(01):53–69, March 1997. doi:10.2307/421196.

9 Miika Hannula, Juha Kontinen, Martin Lück, and Jonni Virtema. On Quantified Propositional Logics and the Exponential Time Hierarchy. In *Proceedings of the Seventh International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2016, Catania, Italy, 14-16 September 2016.*, pages 198–212, 2016. doi:10.4204/EPTCS.226.14.

10 Miika Hannula, Juha Kontinen, Jonni Virtema, and Heribert Vollmer. Complexity of Propositional Independence and Inclusion Logic. In Giuseppe F Italiano, Giovanni Pighizzini, and Donald T. Sannella, editors, *Mathematical Foundations of Computer Science 2015*, volume 9234, pages 269–280. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015. doi:10.1007/978-3-662-48057-1_21.

11 Jaakko Hintikka and Gabriel Sandu. Informational Independence as a Semantical Phenomenon. In *Studies in Logic and the Foundations of Mathematics*, volume 126, pages 571–589. Elsevier, 1989.

12 Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of IGPL*, 5(4):539–563, July 1997. doi:10.1093/jigpal/5.4.539.

13 Juha Kontinen, Julian-Steffen Müller, Henning Schnoor, and Heribert Vollmer. A Van Benthem Theorem for Modal Team Semantics. In *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, pages 277–291, 2015. doi:10.4230/LIPIcs.CSL.2015.277.

14 Martin Lück. Axiomatizations for Propositional and Modal Team Logic. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 33:1–33:18, Dagstuhl, Germany, 2016. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik. doi:10.4230/LIPIcs.CSL.2016.33.

15 Julian-Steffen Müller. *Satisfiability and Model Checking in team based logics*. PhD thesis, University of Hanover, 2014. URL: http://d-nb.info/1054741921.

16 Jouko Väänänen. Modal dependence logic. *New perspectives on games and interaction*, 4:237–254, 2008.

17 Jouko Väänänen. *Dependence logic: a new approach to independence friendly logic*. Number 70 in London Mathematical Society student texts. Cambridge University Press, Cambridge ; New York, 2007.

A Technical Appendix: Proofs for Section 5.2

A.1 Proofs of lemmas

▶ **Lemma 30.** *If a DNF $\varphi = (\Gamma, \Delta, \lambda)$ satisfies* **1.**, **2.**, **3.(a)** *and* **4.(a)**, *then $\varphi^{\Diamond\sim}$ has a kernel $(\Gamma', \Delta', \lambda')$ that satisfies* **1.**, **2.**, **3.(b)** *and* **4.(b)** *such that $w(\Delta') = w(\Gamma)$.*

Proof. We form the kernel $(\Gamma', \Delta', \lambda')$ of $(\Gamma'', \Delta'', \lambda'') := \varphi^{\Diamond\sim}$ using exactly the choice functions with upward closed domain, i.e., $F' := \{\, \mathfrak{f} \in F_\varphi \mid \mathsf{dom}\,\mathfrak{f} \in \mathcal{U}(\Gamma) \,\}$.

To meet the conditions of a kernel (cf. Definition 27), for every $\mathfrak{f} \in F_\varphi$ there has to exist $\mathfrak{f}' \in F'$ such that $\Box\ulcorner\mathfrak{f}\urcorner \vDash \Box\ulcorner\mathfrak{f}'\urcorner$ and $\lambda''(\Box\ulcorner\mathfrak{f}'\urcorner) \subseteq \lambda''(\Box\ulcorner\mathfrak{f}\urcorner)^{\uparrow}$. Obtain such $\mathfrak{f}'$ by restricting

$\mathfrak{f}$ to the domain $\Gamma \setminus (\Gamma \setminus \mathsf{dom}\,\mathfrak{f})^{\downarrow}$. Then $\mathsf{dom}\,\mathfrak{f}'$ is upward closed. Also, trivially $\ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner$. Furthermore,

$$\lambda''(\Box\ulcorner\mathfrak{f}'\urcorner) = \neg\Diamond\big((\Gamma \setminus \mathsf{dom}\,\mathfrak{f})^{\downarrow}\big) = (\neg\Diamond(\Gamma \setminus \mathsf{dom}\,\mathfrak{f}))^{\uparrow} = \lambda''(\Box\ulcorner\mathfrak{f}\urcorner)^{\uparrow}.$$

Consequently, $(\Gamma', \Delta', \lambda')$ with $\Gamma' := \{\Box\ulcorner\mathfrak{f}\urcorner \mid \mathfrak{f} \in F'\}$, and Δ', λ' defined accordingly, is a kernel. It is multiplicative and Lemma 29 applies, so condition **1.** is satisfied. Δ' is partially ordered and $w(\Delta') = w(\Gamma)$, since $\Delta' = \neg\Diamond\Gamma$. That Γ' is partially ordered follows from the proof of **3.(b)**, as $\subseteq$ is a partial order.

To prove the lemma, it remains to show **3.(b)** and **4.(b)**.

Proof of 3.(b) For every $\mathfrak{f} \in F'$, it is easy to verify that $\lambda'(\Box\ulcorner\mathfrak{f}\urcorner)$ is an upward closed subset of Δ', so $\lambda' : \Gamma' \to \mathcal{U}(\Delta')$. We prove only $\mathsf{dom}\,\mathfrak{f}' \subseteq \mathsf{dom}\,\mathfrak{f} \Leftrightarrow \ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner$ for all $\mathfrak{f}, \mathfrak{f}' \in F'$, since then

$$\begin{aligned}
& \Box\ulcorner\mathfrak{f}\urcorner \vDash \Box\ulcorner\mathfrak{f}'\urcorner \\
\Leftrightarrow\ & \quad \ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner \\
\Leftrightarrow\ & \quad \mathsf{dom}\,\mathfrak{f}' \subseteq \mathsf{dom}\,\mathfrak{f} \\
\Leftrightarrow\ & \lambda'(\Box\ulcorner\mathfrak{f}\urcorner) \subseteq \lambda'(\Box\ulcorner\mathfrak{f}'\urcorner) \qquad \text{(by definition of } \lambda'\text{)}.
\end{aligned}$$

Recall that $\lambda'(\alpha)$ is a singleton, and consequently $\mathfrak{f}(\alpha) = \mathfrak{f}'(\alpha)$ for every $\alpha \in \mathsf{dom}\,\mathfrak{f} \cap \mathsf{dom}\,\mathfrak{f}'$. For this reason, $\mathsf{dom}\,\mathfrak{f}' \subseteq \mathsf{dom}\,\mathfrak{f}$ implies $\ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner$. For the other direction, suppose $\mathsf{dom}\,\mathfrak{f}' \not\subseteq \mathsf{dom}\,\mathfrak{f}$, i.e., $\mathfrak{f}'(\alpha) = \beta$ for some $\alpha \in \mathsf{dom}\,\mathfrak{f}' \setminus \mathsf{dom}\,\mathfrak{f}$. We prove that

$$\delta := (\alpha \wedge \beta) \wedge \bigwedge_{\alpha' \in \mathsf{dom}\,\mathfrak{f}} \neg(\alpha' \wedge \mathfrak{f}(\alpha'))$$

and therefore $\neg\ulcorner\mathfrak{f}'\urcorner \wedge \ulcorner\mathfrak{f}\urcorner$ is satisfiable. Due to multiplicativity, we can simply ascertain the consistency of $(\alpha \wedge \beta)$ with all conjuncts $\neg(\alpha' \wedge \mathfrak{f}(\alpha'))$ of $\mathfrak{f}$. If $\alpha' \in \mathsf{dom}\,\mathfrak{f}$, then $\alpha' \nvDash \alpha$, as $\mathsf{dom}\,\mathfrak{f}$ is upward closed. Then **3.(a)** implies that $(\alpha \wedge \beta) \wedge \neg(\alpha' \wedge \mathfrak{f}(\alpha'))$ is satisfiable. λ' is injective: if $\lambda'(\Box\ulcorner\mathfrak{f}\urcorner) = \lambda'(\Box\ulcorner\mathfrak{f}'\urcorner)$, then $\mathsf{dom}\,\mathfrak{f}' = \mathsf{dom}\,\mathfrak{f}$, and consequently $\mathfrak{f}' = \mathfrak{f}$ by the same argument as stated above. By cardinality constraints, λ' is then surjective, as $|\Gamma'| = |\mathcal{U}(\Gamma)| = |\mathcal{L}(\Gamma)| = |\mathcal{U}(\Delta')|$. It follows that λ' is a bijection.

Proof of 4.(b) Suppose that $\lambda'(\Box\ulcorner\mathfrak{f}\urcorner) \setminus \lambda'(\Box\ulcorner\mathfrak{f}'\urcorner)$ contains an element $\neg\Diamond\alpha'$ incomparable to $\neg\Diamond\alpha \in \Delta'$. First note that α and α' must be incomparable as well, and furthermore, $\alpha' \in \mathsf{dom}\,\mathfrak{f}' \setminus \mathsf{dom}\,\mathfrak{f}$ by definition of λ'. Similar as before, we show that

$$\delta' := (\alpha' \wedge \mathfrak{f}'(\alpha')) \wedge \neg\alpha \wedge \bigwedge_{\alpha'' \in \mathsf{dom}\,\mathfrak{f}} \neg(\alpha'' \wedge \mathfrak{f}(\alpha''))$$

is satisfiable, implying $\ulcorner\mathfrak{f}\urcorner \wedge \neg\alpha \nvDash \ulcorner\mathfrak{f}'\urcorner$ and consequently **4.(b)**, as $\Box\ulcorner\mathfrak{f}\urcorner \wedge \Box\neg\alpha \nvDash \Box\ulcorner\mathfrak{f}'\urcorner$. Finally, the satisfiability of $(\alpha' \wedge \mathfrak{f}'(\alpha') \wedge \neg\alpha$ follows from **4.(a)**; while the satisfiability of $(\alpha' \wedge \mathfrak{f}'(\alpha')) \wedge \neg(\alpha'' \wedge \mathfrak{f}(\alpha''))$ follows again, since $\alpha' \notin \mathsf{dom}\,\mathfrak{f}$ implies $\alpha'' \nvDash \alpha'$, from **3.(a)** ◀

Next we prove the converse lemma, in which the width increases by a factorial. For the sake of clarity, several auxiliary claims again are proven in the appendix after the lemma itself.

▶ **Lemma 31.** *If a DNF $\varphi = (\Gamma, \Delta, \lambda)$ satisfies **1.**, **2.**, **3.(b)** and **4b.**, then $\varphi^{\Diamond\sim}$ has a kernel $(\Gamma', \Delta', \lambda')$ that satisfies **1.**, **2.**, **3a.** and **4a.** such that $w(\Gamma') \geq w(\Delta)!$.*

Proof. As in Lemma 30, we construct a kernel $(\Gamma', \Delta', \lambda')$ of $(\Gamma'', \Delta'', \lambda'') := \varphi^{\Diamond\sim}$. By the same argument as there, w.l.o.g. all $\mathfrak{f} \in F_\varphi$ have an upward closed domain.

We introduce a number of definitions.

Due to **3.(b)**, λ actually is an order isomorphism between Γ and the lattice $\mathcal{U}(\Delta)$ (cf. [5]). For any $\widehat{\Delta} \subseteq \Delta$, the lattice $\mathcal{U}(\Delta)$ furthermore contains a sublattice $L_{\widehat{\Delta}}$ of all upward closed subsets of Δ which are disjoint from $\widehat{\Delta}$. Let $\alpha_{\widehat{\Delta}} := \lambda^{-1}\left(\max L_{\widehat{\Delta}}\right)$.

Let $S_{\text{sub}}(\Delta)$ be the set of all permutations σ of subsets of Δ. If $\sigma = (\beta_1, \ldots, \beta_k) \in S_{\text{sub}}(\Delta)$, then σ_j is the prefix $(\beta_i)_{i\in[j]}$ of σ for all $0 \le j \le k$. The set underlying σ is $\Delta_\sigma := \{\beta_i\}_{i\in[k]}$, and we also write α_σ instead of α_{Δ_σ}.

A permutation $\sigma = (\beta_1, \ldots, \beta_k)$ is *lazy* if β_j is a minimal element in $\lambda(\alpha_{\sigma_{j-1}})$ (which is not necessarily unique) for all $j \in [k]$. It is a *linear extension* of Δ_σ if, for all $i, j \in [k]$, $\beta_i \vDash \beta_j$ implies $i \le j$.

▶ **Claim** (a). *$\sigma \in S_{\text{sub}}(\Delta)$ is lazy if and only if it is a linear extension of some $\widehat{\Delta} \in \mathcal{L}(\Delta)$.*

For every lazy permutation $\sigma = (\beta_1, \ldots, \beta_k) \in S_{\text{sub}}(\Delta)$, define the function $\mathfrak{f}_\sigma$ as follows. Let its domain $\mathsf{dom}\,\mathfrak{f}_\sigma$ be $\{\alpha_\emptyset\}^\downarrow \setminus \{\alpha_\sigma\}^\downarrow = \Gamma \setminus \{\alpha_\sigma\}^\downarrow$. For all $j \in [k]$ and $\alpha \in \{\alpha_{\sigma_{j-1}}\}^\downarrow \setminus \{\alpha_{\sigma_j}\}^\downarrow$, let $\mathfrak{f}(\alpha) = \beta_j$. Note that $\mathsf{dom}\,\mathfrak{f}_\sigma$ is an upward closed set. The range $\mathsf{ran}\,\mathfrak{f}_\sigma$ equals Δ_σ, which is a downward closed set by Claim (a).

Using these definitions, the restriction of F_φ to

$$F' := \{\, \mathfrak{f}_\sigma \mid \sigma \in S_{\text{sub}}(\Delta) \text{ is lazy} \,\}$$

permits to define the desired DNF $(\Gamma', \Delta', \lambda')$ using

$$\Gamma' := \{\, \Box\ulcorner \mathfrak{f}_\sigma \urcorner \mid \sigma \in S_{\text{sub}}(\Delta) \text{ is lazy} \,\},$$
$$\Delta' := \{\, \neg\Diamond\alpha_\sigma \mid \sigma \in S_{\text{sub}}(\Delta) \text{ is lazy} \,\},$$

and $\lambda'(\Box\ulcorner \mathfrak{f}_\sigma \urcorner) := \neg\Diamond\alpha_\sigma$.

▶ **Claim** (b). *$(\Gamma', \Delta', \lambda')$ is a kernel of $\varphi^{\Diamond\sim}$.*

It is straightforward that $(\Gamma', \Delta', \lambda')$ satisfies **1.** We proceed with the remaining properties stated in the lemma, using another auxiliary claim.

▶ **Claim** (c). *Let $\mathfrak{f}_\sigma, \mathfrak{f}_\tau \in F'$ be distinct.*

- *If σ is a prefix of τ, then $\ulcorner \mathfrak{f}_\tau \urcorner \vDash \ulcorner \mathfrak{f}_\sigma \urcorner$ and $\ulcorner \mathfrak{f}_\sigma \urcorner \nvDash \ulcorner \mathfrak{f}_\tau \urcorner$, and moreover $\ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma \vDash \ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau$ and $\ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma$.*
- *If σ is not a prefix of τ and vice versa, then $\ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma \nvDash \ulcorner \mathfrak{f}_\tau \urcorner$ and $\ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner \mathfrak{f}_\sigma \urcorner$.*

The kernel satisfies **2.**: Clearly $\Delta' \subseteq \neg\Diamond\Gamma$ is partially ordered, since Γ is. From the above claim it also follows that Γ' is partially ordered as well, since never both $\Box\ulcorner \mathfrak{f}_\sigma \urcorner \vDash \Box\ulcorner \mathfrak{f}_\tau \urcorner$ and $\Box\ulcorner \mathfrak{f}_\sigma \urcorner \vDash \Box\ulcorner \mathfrak{f}_\tau \urcorner$ hold simultaneously for $\sigma \neq \tau$. Moreover, in both cases **3.(a)** and **4.(a)** apply.

Finally, we determine $w(\Gamma')$. Let $\widehat{\Delta} \subseteq \Delta$ be an antichain with cardinality $n := w(\Delta)$. There are at least $n!$ distinct linear extensions of $(\widehat{\Delta})^\downarrow$, obtained by freely arranging the elements in $\widehat{\Delta}$, and by Claim (a) then at least $n!$ distinct lazy permutations of $(\widehat{\Delta})^\downarrow$. Since these permutations are of identical length, they are pairwise no prefix of each other. By Claim (c), then Γ' contains an antichain of size $n!$. ◀

A.2 Proofs of claims

▶ **Claim (a).** *$\sigma \in S_{\mathrm{sub}}(\Delta)$ is lazy if and only if is a linear extension of some $\widehat{\Delta} \in \mathcal{L}(\Delta)$.*

Proof. Let $\sigma = (\beta_1, \dots, \beta_k)$ be a linear extension of $\widehat{\Delta} \in \mathcal{L}(\Delta)$. We prove that β_j is a minimal element of $\lambda(\alpha_{\sigma_{j-1}})$ for all $j \in [k]$, so σ is lazy.

Observe that the elements $\alpha_{\sigma_0}, \alpha_{\sigma_1}, \dots, \alpha_{\sigma_k}$ form an descending chain in Γ whenever σ is lazy, that is, $\alpha_\sigma = \alpha_{\sigma_k} \vDash \alpha_{\sigma_{k-1}} \vDash \cdots \vDash \alpha_{\sigma_1} \vDash \alpha_{\sigma_0} = \alpha_\emptyset$.

First note that, for every $j \in [k]$, $\widehat{\Delta}_j := \{\beta_i \mid i \in [j]\}$ is downward closed as well. Conversely, the set $\Delta \setminus \widehat{\Delta}_{j-1}$ is disjoint from $\widehat{\Delta}_{j-1}$ and upward closed. β_j is an element of $\lambda(\alpha_{\sigma_{j-1}})$, since $\beta_j \in \Delta \setminus \widehat{\Delta}_{j-1}$ and $\lambda(\alpha_{\sigma_{j-1}})$ is the largest upper set disjoint from $\widehat{\Delta}_{j-1}$. β_j is indeed a minimal element: suppose that $\lambda(\alpha_{\sigma_{j-1}})$ contains an element below β_j. That element is in $\widehat{\Delta}$ by downward closure, so it equals some β_ℓ in σ. It holds $\ell < j$, since σ is a linear extension. Also, $\beta_\ell \notin \lambda(\alpha_{\sigma_\ell})$ by definition of α_{σ_ℓ}. But since $\lambda(\alpha_\emptyset) \supseteq \lambda(\alpha_{\sigma_1}) \supseteq \cdots \supseteq \lambda(\alpha_\sigma)$, β_ℓ is not in $\lambda(\alpha_{\sigma_{j-1}})$ either, contradiction. So σ is lazy.

For the other direction, assume that $\sigma = (\beta_1, \dots, \beta_k)$ is lazy. Define $\widehat{\Delta}_j$ as above. As a first step, we prove by induction on j that $\widehat{\Delta}_j$ is downward closed. As $j = 0$ is trivial, consider $j > 0$. Since every prefix of a lazy permutation is lazy, $\widehat{\Delta}_{j-1}$ is downward closed by induction hypothesis. Furthermore, β_j is a minimal element of $\lambda(\alpha_{\sigma_{j-1}})$ by assumption. $\lambda(\alpha_{\sigma_{j-1}})$ is the largest upward closed set disjoint to $\widehat{\Delta}_{j-1}$, so by downward closure of $\widehat{\Delta}_{j-1}$ it must equal $\Delta \setminus \widehat{\Delta}_{j-1}$. But then β_j is minimal in $\Delta \setminus \widehat{\Delta}_{j-1}$. As any element below β_j is in $\widehat{\Delta}_{j-1}$, the set $\widehat{\Delta}_{j-1} \cup \{\beta_j\} = \widehat{\Delta}_j$ is downward closed.

Finally, suppose σ is lazy, but not a linear extension of its underlying set $\widehat{\Delta}$. Then there are β_i, β_j such that $i < j$ and $\beta_j \vDash \beta_i$. It holds $\lambda(\alpha_{\sigma_i}) \supseteq \lambda(\alpha_{\sigma_j})$. But then $\beta_j \in \lambda(\alpha_{\sigma_i})$, which contradicts the fact that β_i is a minimal element of $\lambda(\alpha_{\sigma_i})$. ◀

▶ **Claim (b).** *$(\Gamma', \Delta', \lambda')$ is a kernel of $\varphi^{\Diamond\sim}$.*

Proof. Let $(\Gamma'', \Delta'', \lambda'') := \varphi^{\Diamond\sim}$. Say that $\mathfrak{f}$ *covers* $\mathfrak{f}'$ if $\ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner$ and $\lambda'(\Box\ulcorner\mathfrak{f}'\urcorner) \subseteq \lambda'(\Box\ulcorner\mathfrak{f}\urcorner)^{\uparrow}$. If $\sigma = (\beta_1, \dots, \beta_n)$ is a permutation not containing the element β_{n+1}, then (σ, β_{n+1}) denotes the permutation $(\beta_1, \dots, \beta_n, \beta_{n+1})$.

We prove that if $\mathfrak{f}_\sigma$ is the restriction of a function $\mathfrak{f}$, then either $\mathfrak{f}_\sigma = \mathfrak{f}$, or there exists $\beta \in \Delta$ such that $\mathfrak{f}_{(\sigma,\beta)}$ is the restriction of some $\mathfrak{f}' \in F$ covering $\mathfrak{f}$. Since covering is transitive, and since $\mathfrak{f}_{()}$ is the restriction of every function, by induction on $|\sigma|$ every $\mathfrak{f} \in F_\varphi$ is covered by some $\mathfrak{f}_\sigma \in F'$, proving the claim.

For the proof, assume $\mathfrak{f}_\sigma \neq \mathfrak{f}$, i.e., $\mathsf{dom}\,\mathfrak{f}_\sigma \subsetneq \mathsf{dom}\,\mathfrak{f}$. Then $\alpha_\sigma \in \mathsf{dom}\,\mathfrak{f}$, since $\mathsf{dom}\,\mathfrak{f} = \Gamma \setminus \{\alpha_\sigma\}^{\downarrow}$ and $\mathsf{dom}\,\mathfrak{f}$ is upward closed. Let β^* be a minimal element of $\lambda(\alpha_\sigma)$ below or equal to $\mathfrak{f}(\alpha_\sigma)$. Define

$$\mathfrak{f}'(\alpha) = \begin{cases} \beta^* \text{ if } \alpha \in \{\alpha_\sigma\}^{\downarrow} \setminus \{\alpha_{(\sigma,\beta')}\}^{\downarrow}, \\ \mathfrak{f}(\alpha) \text{ otherwise.} \end{cases}$$

Then $\mathfrak{f}_{(\sigma,\beta^*)}$ is a restriction of $\mathfrak{f}'$. That $\mathfrak{f}'$ covers $\mathfrak{f}$ can be seen as follows: Clearly $\lambda''(\Box\ulcorner\mathfrak{f}'\urcorner) \subseteq \lambda''(\Box\ulcorner\mathfrak{f}\urcorner)$, since $\mathsf{dom}\,\mathfrak{f}' \supseteq \mathsf{dom}\,\mathfrak{f}$. To show $\ulcorner\mathfrak{f}\urcorner \vDash \ulcorner\mathfrak{f}'\urcorner$, let $\neg(\alpha \wedge \beta)$ be a conjunct of $\ulcorner\mathfrak{f}'\urcorner$ not occurring in $\ulcorner\mathfrak{f}\urcorner$. Then $\alpha \in \{\alpha_\sigma\}^{\downarrow}$ and $\beta = \beta^*$, so $(\alpha \wedge \beta) \vDash (\alpha_\sigma \wedge \mathfrak{f}(\alpha_\sigma))$. It follows that $\ulcorner\mathfrak{f}\urcorner$ has a conjunct $\neg(\alpha_\sigma \wedge \mathfrak{f}(\alpha_\sigma))$ that implies $\neg(\alpha \wedge \beta)$. ◀

▶ **Claim (c).** *Let $\mathfrak{f}_\sigma, \mathfrak{f}_\tau \in F'$ be distinct.*

- *If σ is a prefix of τ, then $\ulcorner\mathfrak{f}_\tau\urcorner \vDash \ulcorner\mathfrak{f}_\sigma\urcorner$ and $\ulcorner\mathfrak{f}_\sigma\urcorner \nvDash \ulcorner\mathfrak{f}_\tau\urcorner$, and moreover $\ulcorner\mathfrak{f}_\sigma\urcorner \wedge \neg\alpha_\sigma \vDash \ulcorner\mathfrak{f}_\tau\urcorner \wedge \neg\alpha_\tau$ and $\ulcorner\mathfrak{f}_\tau\urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner\mathfrak{f}_\sigma\urcorner \wedge \neg\alpha_\sigma$.*
- *If σ is not a prefix of τ and vice versa, then $\ulcorner\mathfrak{f}_\sigma\urcorner \wedge \neg\alpha_\sigma \nvDash \ulcorner\mathfrak{f}_\tau\urcorner$ and $\ulcorner\mathfrak{f}_\tau\urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner\mathfrak{f}_\sigma\urcorner$.*

Proof of Claim (c).

(i) σ is a prefix of τ. Then $\mathsf{dom}\, \mathfrak{f}_\sigma \subseteq \mathsf{dom}\, \mathfrak{f}_\tau$ by the definition of $\mathfrak{f}_\sigma$ and $\mathfrak{f}_\tau$.
$\mathfrak{f}_\sigma$ and $\mathfrak{f}_\tau$ agree on $\mathsf{dom}\, \mathfrak{f}_\sigma$, so $\ulcorner \mathfrak{f}_\tau \urcorner \vDash \ulcorner \mathfrak{f}_\sigma \urcorner$. From $\mathsf{dom}\, \mathfrak{f}_\sigma = \Gamma \setminus \{\alpha_\sigma\}^{\downarrow}$ and $\mathsf{dom}\, \mathfrak{f}_\tau = \Gamma \setminus \{\alpha_\tau\}^{\downarrow}$ follows $\{\alpha_\sigma\}^{\downarrow} \supseteq \{\alpha_\tau\}^{\downarrow}$, so $\neg\alpha_\sigma \vDash \neg\alpha_\tau$. Also, from $\mathsf{dom}\, \mathfrak{f}_\tau \setminus \mathsf{dom}\, \mathfrak{f}_\sigma \subseteq \{\alpha_\sigma\}^{\downarrow}$ follows $\ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma \vDash \ulcorner \mathfrak{f}_\tau \urcorner$. As result, $\ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma \vDash \ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau$.
Since σ and τ are distinct, $\mathsf{dom}\, \mathfrak{f}_\tau \not\subseteq \mathsf{dom}\, \mathfrak{f}_\sigma$. Then $\mathsf{dom}\, \mathfrak{f}_\tau \cap \{\alpha_\sigma\}^{\downarrow}$ is non-empty, and by upward closure $\alpha_\sigma \in \mathsf{dom}\, \mathfrak{f}_\tau$. Since $\lambda(\alpha_\sigma) \cap \Delta_\sigma = \emptyset$ by definition of α_σ, but $\mathsf{ran}\, \mathfrak{f}_\sigma \subseteq \Delta_\sigma$, it follows that $\mathfrak{f}_\tau(\alpha_\sigma) \notin \mathsf{ran}\, \mathfrak{f}_\sigma$. As $\mathsf{ran}\, \mathfrak{f}_\sigma$ is downward closed, $\mathfrak{f}_\tau(\alpha_\sigma) \nvDash \beta$ for all $\beta \in \mathsf{ran}\, \mathfrak{f}_\sigma$. By the Hanoi property and multiplicativity, then

$$\delta := \alpha_\sigma \wedge \mathfrak{f}_\tau(\alpha_\sigma) \wedge \bigwedge_{\beta \in \mathsf{ran}\, \mathfrak{f}_\sigma} \neg\beta$$

is satisfiable, implying $\ulcorner \mathfrak{f}_\sigma \urcorner \nvDash \ulcorner \mathfrak{f}_\tau \urcorner$.
To also prove $\ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma$, we simply show that already $\neg\Delta \cup \{\neg\alpha_\tau, \alpha_\sigma\}$ is satisfiable. Since $\alpha_\tau \vDash \alpha_\sigma$, and Γ is partially ordered, $\alpha_\sigma \nvDash \alpha_\tau$. Then $\alpha_\sigma \wedge \neg\alpha_\tau$, and by the Hanoi property also $\neg\Delta \cup \{\neg\alpha_\tau, \alpha_\sigma\}$, is satisfiable.

(ii) σ is not a prefix of τ and vice versa.
We show $\ulcorner \mathfrak{f}_\sigma \urcorner \wedge \neg\alpha_\sigma \nvDash \ulcorner \mathfrak{f}_\tau \urcorner$; by symmetry reasons, then also $\ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\tau \nvDash \ulcorner \mathfrak{f}_\sigma \urcorner$.
For the proof, let $\sigma = (\beta_1, \ldots, \beta_k)$ and $\tau = (\beta'_1, \ldots, \beta'_m)$. There exists a minimal i, $1 \leq i \leq \min\{k, m\}$ such that $\beta_i \neq \beta'_i$. β_i and β'_i are both minimal in $\lambda(\alpha_{\sigma_{i-1}}) = \lambda(\alpha_{\tau_{i-1}})$, and consequently incomparable. Furthermore, $\beta_i \notin \lambda(\alpha_{\sigma_i})$ by definition of α_{σ_i}. Since now $\lambda(\alpha_{\tau_{i-1}}) \setminus \lambda(\alpha_{\sigma_i})$ contains an element β_i incomparable to β'_i, we can apply **4.(b)** and infer $\alpha_{\tau_{i-1}} \wedge \beta'_i \nvDash \alpha_{\sigma_i}$. Consider now the formula

$$\delta := (\alpha_{\tau_{i-1}} \wedge \beta'_i) \wedge \neg\alpha_{\sigma_i} \wedge \bigwedge_{\substack{\alpha' \in \mathsf{dom}\, \mathfrak{f}_\sigma \\ \alpha' \vDash \alpha_{\sigma_i}}} \neg\alpha' \wedge \bigwedge_{\substack{\alpha' \in \mathsf{dom}\, \mathfrak{f}_\sigma \\ \alpha' \nvDash \alpha_{\sigma_i}}} \neg\mathfrak{f}_\sigma(\alpha').$$

We argue that δ is satisfiable. As $\mathfrak{f}_\tau(\alpha_{\tau_{i-1}}) = \beta'_i$, the conjunction $(\alpha_{\tau_{i-1}} \wedge \beta'_i)$ entails $\neg\ulcorner \mathfrak{f}_\tau \urcorner$. Moreover, $\neg\alpha_{\sigma_i} \vDash \neg\alpha_\sigma$. Then δ itself implies $\neg\ulcorner \mathfrak{f}_\tau \urcorner \wedge \neg\alpha_\sigma \wedge \ulcorner \mathfrak{f}_\sigma \urcorner$, proving the claim.
The first large conjunction is consistent with $(\alpha_{\tau_{i-1}} \wedge \beta'_i)$ since $\alpha_{\tau_{i-1}} \wedge \beta'_i \nvDash \alpha'$ follows from $\alpha_{\tau_{i-1}} \wedge \beta'_i \nvDash \alpha_{\sigma_i}$ and $\alpha' \vDash \alpha_{\sigma_i}$.
For the second large conjunction, consider $\alpha' \in \mathsf{dom}\, \mathfrak{f}_\sigma$ such that $\alpha' \nvDash \alpha_{\sigma_i}$. $\mathfrak{f}_\sigma(\alpha')$ is then defined as some β_j in σ. By definition of $\mathfrak{f}_\sigma$, then $\alpha' \in \{\alpha_{\sigma_{j-1}}\}^{\downarrow} \setminus \{\alpha_{\sigma_j}\}^{\downarrow}$. But since $\alpha' \notin \{\alpha_{\sigma_i}\}^{\downarrow}$, and consequently $\alpha' \notin \{\alpha_{\sigma_i}\}^{\downarrow} \cup \{\alpha_{\sigma_{i+1}}\}^{\downarrow} \cup \cdots \cup \{\alpha_\sigma\}^{\downarrow}$, it holds $j - 1 < i$.
If $j = i$, then $\beta_j = \beta_i$, so β_j is incomparable to β'_i. Then $\beta'_i \wedge \neg\beta_j$, and due to the Hanoi property, $(\alpha_{\tau_{i-1}} \wedge \beta'_i) \wedge \neg\beta_j$ is satisfiable.
If $j < i$, then $\beta_j = \beta'_j$, so $\beta'_i \nvDash \beta'_j$, as τ is a linear extension. Again by the Hanoi property, $(\alpha_{\tau_{i-1}} \wedge \beta'_i) \wedge \neg\beta_j$ is then satisfiable. ◀