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Sasakian quiver gauge theories and instantons on the conifold

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Abstract

We consider Spin(4)-equivariant dimensional reduction of Yang–Mills theory on manifolds of the form $M^d \times T^{1,1}$, where M^d is a smooth manifold and $T^{1,1}$ is a five-dimensional Sasaki–Einstein manifold $\mathrm{Spin}(4)/\mathrm{U}(1)$. We obtain new quiver gauge theories on M^d extending those induced via reduction over the leaf spaces $\mathbb{C}P^1 \times \mathbb{C}P^1$ in $T^{1,1}$. We describe the Higgs branches of these quiver gauge theories as moduli spaces of Spin(4)-equivariant instantons on the conifold which is realized as the metric cone over $T^{1,1}$. We give an explicit construction of these moduli spaces as Kähler quotients.

1. Introduction

The idea of extra dimensions has become an important concept in physics, particularly in string theory wherein the compactification of these dimensions is a fundamental ingredient. In this approach one studies theories living on the product $M^d \times \mathbb{X}$ of a d-dimensional spacetime

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M^d and a Riemannian manifold $\mathbb{X}$. The latter space parameterizes the internal degrees of freedom and is usually chosen with reduced holonomy. While Calabi–Yau manifolds are particular examples, one faces an enormous number of possible geometric structures with each of them leading to a different effective theory on spacetime upon dimensional reduction.

Because of their intensive treatment in differential geometry and their symmetries, coset spaces $\mathbb{X} = G/H$ are typical candidates for the description of the internal degrees of freedom; dimensional reduction over these spaces is known as *coset space dimensional reduction* [1]. If one considers Yang–Mills theory on these spaces and imposes a G-equivariance condition on the pertinent bundles and connections, systematic restrictions follow and the effective field theories can be described as quiver gauge theories. The field content constitutes representations of certain quivers, which are oriented graphs whose arrow representatives can be interpreted as Higgs fields. A rigorous mathematical treatment can be found in [2], while brief reviews can be found in e.g. [3,4].

Typical coset spaces $\mathbb{X}$ that have been studied in the literature are homogeneous spaces carrying Kähler structures such as the complex projective line $\mathbb{C}P^1$ [5–9] and $\mathbb{C}P^1 \times \mathbb{C}P^1$ [10], or Kähler manifolds of the form $\mathrm{SU}(3)/H$ [11,12]. Since Sasakian geometry is the natural odd-dimensional counterpart of Kähler geometry, one may include five-dimensional Sasakian manifolds in this framework of quiver gauge theory [13]. In particular Sasaki–Einstein manifolds $\mathbb{X}$, whose metric cones $C(\mathbb{X})$ are Calabi–Yau threefolds, find applications in string theory where they provide explicit tests of AdS/CFT duality. In this setting the near horizon geometry of a stack of D3-branes is that of $AdS_5 \times \mathbb{X}$, and the supergravity D3-brane solution interpolates between $AdS_5 \times \mathbb{X}$ and $\mathbb{R}^{1,3} \times C(\mathbb{X})$. In the low-energy limit, the worldvolume theory on the D-branes thus gives rise to a superconformal quiver gauge theory in four dimensions which is the (naive) dimensional reduction of ten-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory over the cone $C(\mathbb{X})$.

As pointed out in [14], any complete homogeneous Sasaki–Einstein manifold of dimension five is a U(1)-bundle over either the complex projective plane $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$, which respectively realize the two most prominent examples: the five-sphere S^5 and the space $T^{1,1}$. In general, the notation $T^{p,q}$ [15,16] refers to a class of homogeneous spaces $\mathrm{Spin}(4)/\mathrm{U}(1) = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$, where the coprime integers p and q parameterize the embedding of U(1) and, equivalently, the Chern numbers $(p, q) \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{Z})$ of the circle bundle. The case $p = 1 = q$ is best known for the fact that its metric cone is the *conifold*, which has been intensively studied both in mathematics and string theory. Much attention has been paid to the dual $\mathcal{N} = 1$ superconformal quiver gauge theories [17,18] and to configurations of branes probing its conical singularity [19,20], as well as to deformations and (partial) resolutions thereof [21,22]. New classes of Sasaki–Einstein structures on $S^2 \times S^3$, denoted $Y^{p,q}$, have been constructed in [23] which contain the homogeneous space $T^{1,1} = Y^{1,0}$ as a special case [24]; these spaces have been studied in [13] in the context of their dual superconformal quiver gauge theories.

Manifolds with special geometry, such as those with reduced holonomy or G-structures, are of interest as backgrounds in string theory due to the benefits the additional geometric structures provide for the construction of explicit solutions. In this context it is shown in [25] that the existence of real Killing spinors, as is the case for Sasaki–Einstein manifolds, implies that a generalized instanton condition automatically leads to the Yang–Mills equations of gauge theory. Moreover, the generalized definition of an instanton from [25] includes the gaugino Killing spinor equation as one part of the BPS equations in heterotic string theory [26].

This article addresses the construction of new quiver gauge theories associated to the space $T^{1,1}$. We will obtain them by imposing $\mathrm{SU}(2) \times \mathrm{SU}(2)$-equivariance on the connections on vector bundles over this manifold, in the spirit of [2]. To implement the equivariance condition on the connection explicitly, we use an established framework [27] that is also applied for the investigation of instanton solutions e.g. in [28–30]. A similar study of the quiver gauge theories for the five-sphere S^5 and a class of its lens spaces has been performed in [31], extending the treatment of [32] which dealt uniformly with all Sasaki–Einstein three-manifolds; there these field theories were dubbed *Sasakian quiver gauge theories*.

This paper is organized as follows. In Section 2 we review the geometric properties of the space $T^{1,1}$ and provide the necessary basic tools for our ensuing calculations, in particular the choice of local coordinates and the structure equations. By imposing the Sasaki–Einstein condition, all pertinent parameters are fixed. The canonical connection, which is the starting point for the construction of instantons, is also introduced. Section 3 reviews the general construction of equivariant connections and determines the resulting quiver gauge theories for equivariant dimensional reduction over the coset space $T^{1,1}$. Since this space is a principal U(1)-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$, our descriptions follow closely those from [10]. Besides the general form of the quiver gauge theories, we consider some explicit examples and compare them with the quiver gauge theories obtained from dimensional reduction over the coset space $\mathbb{X} = \mathbb{C}P^1 \times \mathbb{C}P^1$ from [10]. We shall find that not only vertex loop modifications occur in the underlying quivers, as in [32,31], but also more general additional arrows, because the group $H = \mathrm{U}(1)$ is smaller than the maximal torus of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and consequently provides fewer restrictions. We further compute the curvatures of equivariant connections, and carry out the dimensional reduction of Yang–Mills theory to M^d. In order to understand the structure of the quiver gauge theory more clearly, we consider a special case in which the computations are significantly simplified due to a grading of the equivariant connections. In Section 4 we study quiver gauge theory on the metric cone $C(T^{1,1})$ over $T^{1,1}$ and impose the Hermitian Yang–Mills equations in order to obtain solutions of the generalized instanton equations. The moduli spaces of solutions to the resulting equations for spherically symmetric configurations in this framework have been analyzed in [33, 31] in terms of Kähler quotients and adjoint orbits, and we adapt this analysis to our setting. We also comment on the relation of this description of the Higgs branches of our quiver gauge theories to moduli spaces of BPS states of D-branes wrapping $C(T^{1,1})$. Finally, in Section 5 we close with some concluding remarks, while an appendix at the end of the paper contains some technical details involving connections and curvatures which are employed throughout the main text.

2. Geometry of the coset space $T^{1,1}$

2.1. Local geometry

In this section we shall review the geometry of the coset space $T^{1,1}$; this geometry is well-known both in the physics literature [15] and in mathematics literature on Sasakian geometry, see e.g. [14]. A description of the geometry of the five-dimensional Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2) = \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)$, which has the same structure as $T^{1,1}$ at the Lie algebra level, can be found e.g. in the classification [34].

We start by describing explicit local coordinates on $\mathrm{SU}(2) \simeq S^3$ and $\mathbb{C}P^1 \simeq S^2$, based on the defining representation of the Lie group SU(2) on $\mathbb{C}^2$ and the Maurer–Cartan form. Each element of SU(2) can be locally written as[1]

[1] This description is based on a treatment of the Hopf fibration $S^3 \to S^2$ and can be found e.g. in [35].

$$\underbrace{\frac{1}{(1+y_l\,\bar y_l)^{1/2}}\begin{pmatrix}1 & -\bar y_l\\ y_l & 1\end{pmatrix}}_{=\,g_l\,\in\,\mathbb{C}P^1\,\subset\,\mathrm{SU}(2)}\underbrace{\begin{pmatrix}\mathrm{e}^{\,\mathrm{i}\,\varphi_l} & 0\\ 0 & \mathrm{e}^{-\,\mathrm{i}\,\varphi_l}\end{pmatrix}}_{\in\,\mathrm{U}(1)}, \tag{2.1}$$

where y_l and $\bar y_l$ are stereographic coordinates on S^2, defined as in [10], and the index $l=1,2$ refers to the two copies of S^2 which are contained in $T^{1,1}$. The canonical flat connection A_l on the homogeneous space $\mathbb{C}P^1$ is given by the Maurer–Cartan form

$$\begin{aligned} A_l \;:=\; g_l^{-1}\,\mathrm{d}g_l \;&=\; \frac{1}{1+y_l\,\bar y_l}\begin{pmatrix}\frac12\,(\bar y_l\,\mathrm{d}y_l - y_l\,\mathrm{d}\bar y_l) & -\mathrm{d}\bar y_l\\ \mathrm{d}y_l & \frac12\,(y_l\,\mathrm{d}\bar y_l-\bar y_l\,\mathrm{d}y_l)\end{pmatrix}\\ &=:\;\begin{pmatrix}a_l & -\bar\beta_l\\ \beta_l & -a_l\end{pmatrix} \end{aligned} \tag{2.2}$$

which provides SU(2)-invariant 1-forms

$$a_l \;=\; -\bar a_l \;=\; \tfrac12\left(\bar y_l\,\beta_l - y_l\,\bar\beta_l\right)\,, \qquad \beta_l \;=\; \frac{\mathrm{d}y_l}{1+y_l\,\bar y_l} \tag{2.3}$$

with differentials

$$\mathrm{d}a_l \;=\; -\beta_l\wedge\bar\beta_l\,, \quad \mathrm{d}\beta_l \;=\; 2a_l\wedge\beta_l\,, \quad \mathrm{d}\bar\beta_l \;=\; -2a_l\wedge\bar\beta_l\,. \tag{2.4}$$

Since the geometry of $T^{1,1}$ involves the Hopf fibration, it has a close relation to quantities associated with magnetic monopoles as the appearance of the monopole forms (2.3) indicates.

To deal with two copies of SU(2), we can analogously to (2.1) start again from the defining representation and express an arbitrary element of SU(2) × SU(2) locally as

$$\underbrace{\begin{pmatrix}g_1 & \mathbf{0}_2\\ \mathbf{0}_2 & g_2\end{pmatrix}}_{\in\,\mathbb{C}P^1\times\mathbb{C}P^1}\underbrace{\mathrm{diag}\left(\mathrm{e}^{\,\mathrm{i}\,\varphi_1},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi_1},\,\mathrm{e}^{\,\mathrm{i}\,\varphi_2},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi_2}\right)}_{\in\,\mathrm{U}(1)\times\mathrm{U}(1)}. \tag{2.5}$$

To pass to the coset space $T^{p,q}$, we have to factor by the U(1) subgroup whose embedding is described by the coprime integers p and q, which sends $z\in\mathrm{U}(1)$ to $\mathrm{diag}(z^p, z^{-p}, z^{-q}, z^q)\in\mathrm{SU}(2)\times\mathrm{SU}(2)$. We will specialize to the case $p=q=1$, which means that the embedding of $H=\mathrm{U}(1)$ into $G=\mathrm{SU}(2)\times\mathrm{SU}(2)$ is such that H is generated by the difference of the two Cartan generators of G, i.e. $\mathfrak{h}=\langle I^3_{(1)}-I^3_{(2)}\rangle$.[2] Therefore we change U(1) coordinates to $\varphi=\frac12\,(\varphi_1+\varphi_2)$ and $\psi=\frac12\,(\varphi_1-\varphi_2)$, so that the U(1) × U(1) factor in (2.5) reads

$$\begin{aligned} &\mathrm{diag}\big(\mathrm{e}^{\,\mathrm{i}\,(\varphi+\psi)},\,\mathrm{e}^{-\,\mathrm{i}\,(\varphi+\psi)},\,\mathrm{e}^{\,\mathrm{i}\,(-\varphi+\psi)},\,\mathrm{e}^{\,\mathrm{i}\,(\varphi-\psi)}\big)\\ &\qquad = \mathrm{diag}\left(\mathrm{e}^{\,\mathrm{i}\,\varphi},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi},\,\mathrm{e}^{\,\mathrm{i}\,\varphi},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi}\right)\,\mathrm{diag}\big(\mathrm{e}^{\,\mathrm{i}\,\psi},\,\mathrm{e}^{-\,\mathrm{i}\,\psi},\,\mathrm{e}^{-\,\mathrm{i}\,\psi},\,\mathrm{e}^{\,\mathrm{i}\,\psi}\big)\,. \end{aligned} \tag{2.6}$$

By passing to the coset space $T^{1,1}$, the second term in (2.6) is divided out and one ends up with elements of the form

$$v \;=\; \begin{pmatrix}g_1 & \mathbf{0}_2\\ \mathbf{0}_2 & g_2\end{pmatrix}\,\mathrm{diag}\left(\mathrm{e}^{\,\mathrm{i}\,\varphi},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi},\,\mathrm{e}^{\,\mathrm{i}\,\varphi},\,\mathrm{e}^{-\,\mathrm{i}\,\varphi}\right)\,. \tag{2.7}$$

[2] Two different descriptions of $T^{1,1}$ occur in the literature: Some treatments (e.g. [15,17]) obtain the manifold $T^{1,1}$ from $S^3\times S^3$ by quotienting with the *sum* of the diagonal SU(2) generators, whereas others (e.g. [36]) quotient by the U(1) subgroup generated by the *difference*. Changing from one convention to the other simply inverts the complex structure on one of the two-spheres S^2 contained in $T^{1,1}$.

Hence the local description of $T^{1,1}$ is based on the quintuple of CR coordinates $(y_1, \bar{y}_1, y_2, \bar{y}_2, \varphi)$, and we derive a basis of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ left-invariant 1-forms by considering its canonical flat connection

$$A_0 := v^{-1}\,\mathrm{d}v = \begin{pmatrix} \mathrm{i}\,\mathrm{d}\varphi + a_1 & -\mathrm{e}^{-2\mathrm{i}\varphi}\,\bar{\beta}_1 & 0 & 0 \\ \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_1 & -(\mathrm{i}\,\mathrm{d}\varphi + a_1) & 0 & 0 \\ 0 & 0 & \mathrm{i}\,\mathrm{d}\varphi + a_2 & -\mathrm{e}^{-2\mathrm{i}\varphi}\,\bar{\beta}_2 \\ 0 & 0 & \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_2 & -(\mathrm{i}\,\mathrm{d}\varphi + a_2) \end{pmatrix} . \tag{2.8}$$

By introducing the definitions

$$\begin{aligned} &a := \tfrac{1}{2}(a_1 - a_2) , \quad \mathrm{i}\kappa\, e^5 := \mathrm{i}\,\mathrm{d}\varphi + \tfrac{1}{2}(a_1 + a_2) , \\ &\alpha_1\left(e^1 + \mathrm{i}\,e^2\right) := \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_1 , \quad \alpha_2\left(e^3 + \mathrm{i}\,e^4\right) := \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_2 , \end{aligned} \tag{2.9}$$

where α_1, α_2 and κ are real constants to be determined later from the Sasaki–Einstein condition, we obtain the expression

$$A_0 = \begin{pmatrix} \mathrm{i}\kappa\, e^5 + a & -\alpha_1\left(e^1 - \mathrm{i}\,e^2\right) & 0 & 0 \\ \alpha_1\left(e^1 + \mathrm{i}\,e^2\right) & -\mathrm{i}\kappa\, e^5 - a & 0 & 0 \\ 0 & 0 & \mathrm{i}\kappa\, e^5 - a & -\alpha_2\left(e^3 - \mathrm{i}\,e^4\right) \\ 0 & 0 & \alpha_2\left(e^3 + \mathrm{i}\,e^4\right) & -\mathrm{i}\kappa\, e^5 + a \end{pmatrix} . \tag{2.10}$$

2.2. Sasaki–Einstein geometry

Based on the choice of the basis 1-forms $e^1, \ldots, e^5$ on $T^{1,1}$ according to (2.9), the structure equations can be determined. The equation for e^5 follows directly from its definition and the differentials (2.4), while the other equations can be obtained from the flatness condition on the canonical connection, $\mathrm{d}A_0 = -A_0 \wedge A_0$. Since the forms a_l and consequently also a are purely imaginary, one ends up with the equations

$$\begin{aligned} &\mathrm{d}e^1 = 2\kappa\, e^{25} - 2\,\mathrm{i}\,e^2 \wedge a , \quad \mathrm{d}e^2 = -2\kappa\, e^{15} + 2\,\mathrm{i}\,e^1 \wedge a , \\ &\mathrm{d}e^3 = 2\kappa\, e^{45} + 2\,\mathrm{i}\,e^4 \wedge a , \quad \mathrm{d}e^4 = -2\kappa\, e^{35} - 2\,\mathrm{i}\,e^3 \wedge a , \\ &\mathrm{d}e^5 = \tfrac{1}{\kappa}\left(\alpha_1^2\, e^{12} + \alpha_2^2\, e^{34}\right) , \end{aligned} \tag{2.11}$$

where in general we write $e^{\mu_1\cdots\mu_k} := e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}$.

The remaining scaling factors in the definition of the 1-forms in (2.9) can be fixed by imposing the Sasaki–Einstein condition. Among the numerous definitions concerning contact geometry, we use the description given in [37]. Then a Sasaki–Einstein five-manifold is characterized by a special SU(2)-structure which can be defined by an orthonormal cobasis $\{e^\mu\}$ and forms

$$\eta = -e^5 , \quad \omega^1 = e^{23} + e^{14} , \quad \omega^2 = e^{31} + e^{24} , \quad \omega^3 = e^{12} + e^{34} \tag{2.12}$$

satisfying the equations

$$\mathrm{d}\eta = 2\omega^3 , \quad \mathrm{d}\omega^1 = -3\eta \wedge \omega^2 , \quad \mathrm{d}\omega^2 = 3\eta \wedge \omega^1 . \tag{2.13}$$

Here η is the contact 1-form which is a connection on the Sasakian fibration,[3] and ω^3 is the Kähler 2-form of its base. Calculating the differentials with the help of the structure equations (2.11) yields

$$\begin{aligned} \mathrm{d}\omega^1 &= 4\kappa\left(e^{135} - e^{245}\right) = 4\kappa\,\eta\wedge\omega^2 , \\ \mathrm{d}\omega^2 &= 4\kappa\left(e^{145} + e^{235}\right) = -4\kappa\,\eta\wedge\omega^1 , \\ \mathrm{d}\eta &= -\mathrm{d}e^5 = -\tfrac{1}{\kappa}\left(\alpha_1^2\,e^{12} + \alpha_2^2\,e^{34}\right) . \end{aligned} \tag{2.14}$$

Hence in order to fulfil the Sasaki–Einstein condition one has to impose

$$\kappa = -\tfrac{3}{4} , \qquad \alpha_1^2 = \alpha_2^2 = \tfrac{3}{2} . \tag{2.15}$$

With this choice of parameters the definition of the basic 1-forms is given by

$$e^1 + \mathrm{i}\,e^2 = \sqrt{\tfrac{2}{3}}\,\mathrm{e}^{2\,\mathrm{i}\,\varphi}\,\beta_1 , \quad e^3 + \mathrm{i}\,e^4 = \sqrt{\tfrac{2}{3}}\,\mathrm{e}^{2\,\mathrm{i}\,\varphi}\,\beta_2 , \quad e^5 = -\tfrac{4}{3}\,\mathrm{d}\varphi + \tfrac{2\,\mathrm{i}}{3}\,(a_1 + a_2) . \tag{2.16}$$

The implications of the conditions (2.15) for the Riemannian geometry of $T^{1,1}$ is as follows. Since our geometry consists of two copies of $\mathbb{C}P^1 \simeq S^2$, the round Kähler metric [10]

$$g_{S^2\times S^2} = 4R_1^2\,\beta_1\otimes\bar{\beta}_1 + 4R_2^2\,\beta_2\otimes\bar{\beta}_2 \tag{2.17}$$

parameterized by two radii R_l appears. In our case, the Sasaki–Einstein condition fixes these radii. Recalling the orthonormality of the forms e^μ, we obtain the metric

$$g = \delta_{\mu\nu}\,e^\mu\otimes e^\nu = \tfrac{2}{3}\,\beta_1\otimes\bar{\beta}_1 + \tfrac{2}{3}\,\beta_2\otimes\bar{\beta}_2 + \eta\otimes\eta \tag{2.18}$$

with implicit summation throughout over repeated upper and lower indices. This shows that imposing this condition requires $R_1^2 = R_2^2 = \frac{1}{6}$ (see also the metric in [22]). In particular, we cannot rescale the radii as for Kähler structures on the coset space $\mathbb{C}P^1\times\mathbb{C}P^1$.

2.3. *Canonical connection*

We proceed to the definition of the canonical connection on $T^{1,1}$ and its curvature. Recall that the structure equations relate the connection 1-forms $\Gamma^\mu_{\ \nu}$, the torsion 2-form T^μ, and the differentials of the basis 1-forms e^μ by

$$\mathrm{d}e^\mu = -\Gamma^\mu_{\ \nu}\wedge e^\nu + T^\mu . \tag{2.19}$$

Hence the Levi-Civita connection, determined by the requirement $T^\mu = 0$, is expressed by the non-vanishing components (see Appendix A for details)

$$\begin{aligned} \Gamma^1_{\ 2} &= -\tfrac{1}{2}\,e^5 - 2\,\mathrm{i}\,a , \quad \Gamma^1_{\ 5} = e^2 , \quad \Gamma^2_{\ 5} = -e^1 , \\ \Gamma^3_{\ 4} &= -\tfrac{1}{2}\,e^5 + 2\,\mathrm{i}\,a , \quad \Gamma^3_{\ 5} = e^4 , \quad \Gamma^4_{\ 5} = -e^3 , \end{aligned} \tag{2.20}$$

with the antisymmetry $\Gamma^\mu_{\ \nu} = -\Gamma^\nu_{\ \mu}$. The curvature of the Levi-Civita connection yields the Ricci tensor

[3] The 1-form η is dual to the R-symmetry generator of the AdS/CFT dual superconformal gauge theory.

$$\mathrm{Ric}_g \;=\; 4\delta_{\mu\nu}\, e^\mu \otimes e^\nu \;=\; 4g\ , \tag{2.21}$$

which confirms that the space with the chosen metric is also Einstein. Moreover, the structure has generic holonomy, i.e. the entire Lie algebra $\mathfrak{so}(5)$.

In dealing with special geometries, it is useful to consider adapted connections that are compatible with the given structure. Declaring all terms in (2.20), apart from those containing the form a, to be torsion, we obtain the U(1) connection

$$\Gamma^1{}_2 \;=\; -2\,\mathrm{i}\,a\ ,\quad \Gamma^2{}_1 \;=\; 2\,\mathrm{i}\,a\ ,\quad \Gamma^3{}_4 \;=\; 2\,\mathrm{i}\,a\ ,\quad \Gamma^4{}_3 \;=\; -2\,\mathrm{i}\,a\ , \tag{2.22}$$

which is the *canonical connection* on $T^{1,1}$ viewed as a homogeneous space $\mathrm{Spin}(4)/\mathrm{U}(1)$. This connection coincides with the canonical connection on $T^{1,1}$ viewed as a Sasaki–Einstein manifold, as introduced in [25].[4] We shall use the canonical connection as a starting point for our investigation because it is an instanton, according to the generalized definition in [25]. For a five-dimensional Sasaki–Einstein manifold, the instanton equation is given by

$$\star R \;=\; -\star Q \wedge R \qquad \text{with} \qquad Q \;=\; \tfrac{1}{2}\,\omega^3 \wedge \omega^3 \;=\; e^{1234} \tag{2.23}$$

for a curvature 2-form R, where $\star$ is the Hodge operator associated to the Sasaki–Einstein metric. The curvature of the U(1) connection (2.22) reads

$$R^1{}_2 \;=\; -R^2{}_1 \;=\; 3\left(e^{12} - e^{34}\right) \;=\; R^4{}_3 \;=\; -R^3{}_4 \tag{2.24}$$

and it indeed solves the instanton equation (2.23).

3. Quiver gauge theory

3.1. Quiver bundles

Since the Sasaki–Einstein manifold in our discussion is realized as a coset space G/H, a very natural condition to impose is G-equivariance of the vector bundles over $T^{1,1}$ which carry a gauge connection. A detailed mathematical description of this *equivariant dimensional reduction* can be found in [2], whereas brief reviews of the procedure and the resulting quiver gauge theories can be found e.g. in [3,4]. Given a Hermitian vector bundle $\mathcal{E} \to M^d \times G/H$ of rank k such that the group G acts trivially on M^d, equivariance with respect to G means that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "G\curvearrowright"] \arrow[d, "\pi"] & \mathcal{E} \arrow[d, "\pi"] \\ M^d \times G/H \arrow[r, "G\curvearrowright"] & M^d \times G/H \end{tikzcd} \tag{3.1}$$

commutes, and the action of G on the bundle $\mathcal{E}$ induces an isomorphism between the fibres $\mathcal{E}_x$ and $\mathcal{E}_{g\cdot x}$ for all $x \in M^d \times G/H$. Since the group H acts trivially on the base space, equivariance of the bundle induces a representation of H on the fibres $\mathcal{E}_x \simeq \mathbb{C}^k$. Consequently to obtain G-equivariant bundles of a given rank k, one has to study (smaller) H-representations inside the group $\mathrm{U}(k)$ which is the generic structure group of the bundle acting on the fibres. This representation can be taken to descend from the restriction of an irreducible G-representation $\mathcal{D}$

[4] For a Sasaki–Einstein five-manifold, the torsion of the canonical connection is given by [25] $T^5 = P_{5\mu\nu}\, e^{\mu\nu}$ and $T^a = \frac{3}{4}\, P_{a\mu\nu}\, e^{\mu\nu}$ for $a = 1, 2, 3, 4$ with $P = \eta \wedge \omega^3$.

comprised of irreducible SU(2)-representations on $\mathbb{C}^{m_1+1}$ and $\mathbb{C}^{m_2+1}$ as $\mathcal{D}|_H = \bigoplus_{i,\alpha} \rho_{i\alpha}$, which implies that the structure group U(k) is reduced to the subgroup

$$\prod_{i=0}^{m_1}\prod_{\alpha=0}^{m_2} \mathrm{U}(k_{i\alpha}) \subset \mathrm{U}(k) , \qquad \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2} k_{i\alpha} = k . \tag{3.2}$$

The labelling by *two* indices is due to the special choice of G here as a product of two Lie groups. Any G-equivariant bundle $\mathcal{E} \to M^d \times G/H$ restricts to an H-equivariant bundle $E = \mathcal{E}|_{M^d}$ over M^d; inversely, any H-equivariant bundle $E \to M^d$ induces a G-equivariant bundle $\mathcal{E} = G \times_H E$ over $M^d \times G/H$ [2].

Example. To clarify the resulting structures of the bundles involved and for later comparisons, let us briefly review this construction for the Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ [10], i.e. $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $H = \mathrm{U}(1) \times \mathrm{U}(1)$ (instead of U(1)), which will be used as reference in the remainder of this paper. The isotopical decomposition of an H-equivariant vector bundle $E \to M^d$ of rank k reads [10]

$$E = \bigoplus_{i=0}^{m_1}\bigoplus_{\alpha=0}^{m_2} E_{i\alpha} \otimes \underline{S}^{(1)}_{m_1-2i} \otimes \underline{S}^{(2)}_{m_2-2\alpha} , \tag{3.3}$$

where one uses a Levi decomposition of the complexified group $G_{\mathbb{C}}$. The vector spaces $\underline{S}^{(l)}_{p_l} \simeq \mathbb{C}$ are irreducible U(1)-representations of weight p_l, and the remaining generators of G act non-trivially only on the bundles $E_{i\alpha} \to M^d$ of rank $k_{i\alpha}$, providing ladder operators due to the SU(2) commutation relations. By induction of bundles, a G-equivariant bundle $\mathcal{E} \to M^d \times G/H$ admits the decomposition

$$\mathcal{E} = \bigoplus_{i=0}^{m_1}\bigoplus_{\alpha=0}^{m_2} E_{i\alpha} \otimes \mathcal{L}^{m_1-2i}_{(1)} \otimes \mathcal{L}^{m_2-2\alpha}_{(2)} , \tag{3.4}$$

where

$$\mathcal{L}^{p_l}_{(l)} = \mathrm{SU}(2) \times_{\mathrm{U}(1)} \underline{S}^{(l)}_{p_l} \tag{3.5}$$

are the monopole line bundles over $\mathbb{C}P^1$ with monopole charge p_l. For a rigorous mathematical treatment of SU(2)-equivariant bundles over $\mathbb{C}P^1$ see [7].

The structural features of the decomposition of G-equivariant vector bundles for a chosen G-module $\mathcal{D}$ can be encoded in quivers. For this, one draws a vertex for each irreducible H-representation $\rho_{i\alpha}$ and depicts by arrows the homomorphisms between two representations $\rho_{i\alpha} \to \rho_{j\beta}$ induced by the action of the entire group G. Thus the restriction of the representation $\mathcal{D}$ leads to an oriented graph which encodes the field content of the gauge theory. After dimensional reduction of pure Yang–Mills theory on $M^d \times G/H$ to the spacetime M^d, the arrows of the quiver constitute a scalar potential for the gauge theory on M^d; for this reason the corresponding fields are sometimes referred to as *Higgs fields*, and we shall adapt this nomenclature in the following.

3.2. Representations of Spin(4)

Following the approach outlined above, we have to study the irreducible representations of the group $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and we shall start from the defining representation of SU(2) on $\mathbb{C}^2$.

Since we need in particular the H-representation inside that of the entire structure group, it is convenient to choose the representation by the diagonal Pauli matrix and the two ladder operators

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{3.6}$$

with the usual commutation relations

$$\big[\sigma_3, \sigma_\pm\big] = \pm 2\sigma_\pm , \qquad \big[\sigma_+, \sigma_-\big] = \sigma_3 . \tag{3.7}$$

For any positive integer m one obtains the generalization to an irreducible representation on $\mathbb{C}^{m+1}$ given by the matrices

$$I^+_{(m)} = \begin{pmatrix} 0 & \gamma_0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{m-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} , \qquad I^-_{(m)} = \big(I^+_{(m)}\big)^\dagger ,$$

$$I^3_{(m)} = \mathrm{diag}\,(m, m-2, \dots, -m+2, -m) , \tag{3.8}$$

with $\gamma_j^2 := (j+1)\,(m-j)$ for $j = 0, 1, \dots, m-1$, yielding the relations (3.7). Irreducible representations of the group G are then given by the tensor product of two single SU(2)-representations on $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$, and the six generators read

$$I^\pm_{(m_1)} \otimes \mathbf{1}_{m_2+1} , \quad I^3_{(m_1)} \otimes \mathbf{1}_{m_2+1} , \quad \mathbf{1}_{m_1+1} \otimes I^\pm_{(m_2)} , \quad \mathbf{1}_{m_1+1} \otimes I^3_{(m_2)} . \tag{3.9}$$

Since $T^{1,1}$ is a reductive homogeneous space, one has the splitting

$$\mathfrak{g} := \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathfrak{m} =: \mathfrak{h} \oplus \mathfrak{m} \tag{3.10}$$

with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, where the Lie algebra $\mathfrak{h}$ is generated by the difference of the two diagonal operators, and the five-dimensional complement $\mathfrak{m}$ can be identified with the cotangent space of $T^{1,1}$. By definition and construction of the basis $\{e^\mu\}$ and the monopole form a, the ladder operators on the two copies of SU(2) are dual to the complex forms $e^1 \pm \mathrm{i}\,e^2$ and $e^3 \pm \mathrm{i}\,e^4$, while the forms a and $\frac{3\,\mathrm{i}}{4}\,e^5$ correspond to the difference and sum, respectively, of the diagonal operators.

Since the existence of a G-equivariant structure on a vector bundle is accompanied by a reduction of its structure group according to (3.2), the direct sum of H-representations $\mathbb{C}^{k_{i\alpha}} := \mathbb{C}^{k_i} \otimes \mathbb{C}^{k_\alpha}$,

$$\mathbb{C}^k = \bigoplus_{i=0}^{m_1} \bigoplus_{\alpha=0}^{m_2} \mathbb{C}^{k_{i\alpha}} , \qquad k = \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} k_{i\alpha} , \tag{3.11}$$

must be studied under the action of the group G. Due to the block form of the broken structure group and in the spirit of how vertices of the quiver arise, it is convenient to interpret vectors in the space $\mathbb{C}^k$ as vectors of length $(m_1+1)\,(m_2+1)$ whose entries are vectors in the spaces $\mathbb{C}^{k_{i\alpha}}$ rather than complex numbers, as dictated in the decomposition (3.11). Then each entry of the vector corresponds exactly to one vertex $v_{i\alpha}$ in the quiver, and arrows occur if there is a non-vanishing homomorphism in $\mathrm{Hom}\big(\mathbb{C}^{k_{i\alpha}}, \mathbb{C}^{k_{j\beta}}\big)$ as an entry in the block matrices which describe the action of G. The representation of the group action in terms of the generators given above can be adapted to the vector space $\mathbb{C}^k$ by keeping the general form of (3.8) and substituting the complex numbers as entries by matrices. We will mostly assume this convention implicitly in the following.

Finally, for a more convenient description of the generator I_6 of $\mathfrak{h}$ on $\mathbb{C}^k$, we introduce natural projection operators on $\mathbb{C}^{m_1+1}$ and $\mathbb{C}^{m_2+1}$, respectively, by [10]

$$\Pi_i : \mathbb{C}^{m_1+1} \longrightarrow \mathbb{C} \ , \qquad \Pi_i \ = \ \mathrm{diag}(0,\dots,0,\underset{\substack{\uparrow \\ \text{i-th slot}}}{1},0,\dots,0) \ , \quad i \ = \ 0,1,\dots,m_1 \ ,$$

$$\Pi_\alpha : \mathbb{C}^{m_2+1} \longrightarrow \mathbb{C} \ , \qquad \Pi_\alpha \ = \ \mathrm{diag}(0,\dots,0,\underset{\substack{\uparrow \\ \alpha\text{-th slot}}}{1},0,\dots,0) \ , \quad \alpha \ = \ 0,1,\dots,m_2 \ , \tag{3.12}$$

where Latin indices always refer to the first copy of SU(2) and Greek indices to the second copy. The projection from the tensor product $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$ to the component with indices i and α is thus given by the operator

$$\Pi_{i\alpha} : \mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1} \longrightarrow \mathbb{C} \ , \qquad \Pi_{i\alpha} \ := \ \Pi_i \otimes \Pi_\alpha \tag{3.13}$$

and thus by the diagonal square matrix

$$\Pi_{i\alpha} \ = \ \big(\delta_{ij}\,\delta_{\alpha\beta}\,\delta_{ik}\,\delta_{\alpha\gamma}\big)^{j,k=0,1,\dots,m_1}_{\beta,\gamma=0,1,\dots,m_2} \tag{3.14}$$

of size $[(m_1+1)\,(m_2+1)]^2$. Furthermore, we introduce the operators

$$\Pi^{(1)}_i := \sum_{\alpha=0}^{m_2} \Pi_{i\alpha} \ = \ \Pi_i \otimes \sum_{\alpha=0}^{m_2} \Pi_\alpha \ = \ \Pi_i \otimes \mathbf{1}_{m_2+1} \ ,$$

$$\Pi^{(2)}_\alpha := \sum_{i=0}^{m_1} \Pi_{i\alpha} \ = \ \sum_{i=0}^{m_1} \Pi_i \otimes \Pi_\alpha \ = \ \mathbf{1}_{m_1+1} \otimes \Pi_\alpha \ , \tag{3.15}$$

which project on all components with a fixed value of the first or second index, respectively. Interpreting (implicitly) all entries 1 as the identity operator $\mathbf{1}$ of the pertinent dimension, one obtains a representation of the generators of the maximal torus of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on the vector space (3.11) by the diagonal matrices

$$\Upsilon^{(1)} := \sum_{i=0}^{m_1} (m_1 - 2i)\,\Pi^{(1)}_i \ = \ I^3_{(m_1)} \otimes \mathbf{1}_{m_2+1} \ ,$$

$$\Upsilon^{(2)} := \sum_{\alpha=0}^{m_2} (m_2 - 2\alpha)\,\Pi^{(2)}_\alpha \ = \ \mathbf{1}_{m_1+1} \otimes I^3_{(m_2)} \ . \tag{3.16}$$

In particular, the Lie algebra $\mathfrak{h}$ is generated by

$$I_6 \ = \ \Upsilon^{(1)} - \Upsilon^{(2)} \ = \ \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} (m_1 - m_2 - 2i + 2\alpha)\,\Pi_{i\alpha} \ . \tag{3.17}$$

3.3. Representations of quivers

Based on the previous algebraic description of the generators of G and their representation on the vector space $\mathbb{C}^k$, the forms of the Higgs fields and the quivers can already be deduced to some extent, but to actually construct the most general connection that is compatible with the equivariance of the bundles we have to formulate the construction more precisely. For this, we start from the canonical connection $\Gamma = I_6 \otimes a$ and recall that the space $T^{1,1}$ is reductive, i.e. one has the commutation relations

$$\big[I_6, I_\mu\big] = f_{6\mu}^{\nu}\, I_\nu \,, \qquad \big[I_\mu, I_\nu\big] = f_{\mu\nu}^{6}\, I_6 + f_{\mu\nu}^{\rho}\, I_\rho \,, \qquad \mu,\nu,\rho = 1,\dots,5\,, \tag{3.18}$$

according to the decomposition of $\mathfrak{g}$ into the Lie algebra $\mathfrak{h}$ of $\mathrm{U}(1)$ and its complement $\mathfrak{m}$. A connection $\mathcal{A}$ on the G-equivariant bundle $\mathcal{E} \to M^d \times T^{1,1}$ can generically be written as

$$\begin{aligned}\mathcal{A} &= A + \Gamma + X_\mu \otimes e^\mu \qquad\qquad (3.19)\\ &=: A + \Gamma + \phi^{(1)} \otimes \Big(e^1 - \mathrm{i}\, e^2\Big) - \phi^{(1)\,\dagger} \otimes \Big(e^1 + \mathrm{i}\, e^2\Big) + \phi^{(2)} \otimes \Big(e^3 - \mathrm{i}\, e^4\Big)\\ &\quad - \phi^{(2)\,\dagger} \otimes \Big(e^3 + \mathrm{i}\, e^4\Big) + \phi^{(3)} \otimes e^5\,,\end{aligned}$$

where A is a connection on the corresponding H-equivariant bundle $E \to M^d$, and where we have combined the bundle endomorphisms, expressed by the skew-Hermitian matrices X_μ, into the quantities

$$\phi^{(1)} := \tfrac{1}{2}\,(X_1 + \mathrm{i}\, X_2)\,, \quad \phi^{(2)} := \tfrac{1}{2}\,(X_3 + \mathrm{i}\, X_4) \quad \text{and} \quad \phi^{(3)} := X_5\,, \tag{3.20}$$

which we call Higgs fields. The Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ accompany the anti-holomorphic 1-forms[5] $\bar\Theta^1 := e^1 - \mathrm{i}\, e^2$ and $\bar\Theta^2 := e^3 - \mathrm{i}\, e^4$. The tensor product symbol between the endomorphism part and the form part will be omitted from now on.

For the connection (3.19) to be compatible with the equivariance of the underlying bundle, we have to impose two conditions that can be directly gleamed from the structure of the field strength. Setting $A = 0$ for the time being, the curvature of the connection reads

$$\begin{aligned}\mathcal{F} &= \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\\ &= \big(\big[I_6, X_1\big] - 2\,\mathrm{i}\, X_2\big)\, a\wedge e^1 + \big(\big[I_6, X_2\big] + 2\,\mathrm{i}\, X_1\big)\, a\wedge e^2 + \big(\big[I_6, X_3\big] + 2\,\mathrm{i}\, X_4\big)\, a\wedge e^3\\ &\quad + \big(\big[I_6, X_4\big] - 2\,\mathrm{i}\, X_3\big)\, a\wedge e^4 + \big[I_6, X_5\big] a\wedge e^5 + \big(\big[X_1, X_2\big] - 2X_5 + \tfrac{3}{2}\,\mathrm{i}\, I_6\big)\, e^{12}\\ &\quad + \big(\big[X_3, X_4\big] - 2X_5 - \tfrac{3}{2}\,\mathrm{i}\, I_6\big)\, e^{34} + \big(\big[X_1, X_5\big] + \tfrac{3}{2}\, X_2\big)\, e^{15}\\ &\quad + \big(\big[X_2, X_5\big] - \tfrac{3}{2}\, X_1\big)\, e^{25} + \big(\big[X_3, X_5\big] + \tfrac{3}{2}\, X_4\big)\, e^{35} + \big(\big[X_4, X_5\big] - \tfrac{3}{2}\, X_3\big)\, e^{45}\\ &\quad + \big[X_1, X_3\big] e^{13} + \big[X_1, X_4\big] e^{14} + \big[X_2, X_3\big] e^{23} + \big[X_2, X_4\big] e^{24} + \mathrm{d}X_\mu \wedge e^\mu\,. \end{aligned} \tag{3.21}$$

The G-equivariance is spoiled by terms involving a mixture of 1-forms in $\mathfrak{g}^*$ and $\mathfrak{h}^*$, i.e. by the occurrence of 2-forms $a\wedge e^\mu$. Therefore, firstly, one assumes that the Higgs fields do not depend on the coordinates of the coset space but only on those of the spacetime M^d, which ensures that the sum in the very last term does not contain incompatible 2-forms. Moreover, as an additional benefit, this condition greatly simplifies the dimensional reduction of the gauge theory.

Secondly, the first five terms in (3.21) must vanish, and this requirement determines the features of the quiver gauge theory. Supposing that the sum of the H-representations stems from an irreducible representation of G, for compatibility one may demand that the endomorphisms X_μ act in the same way on the fibres $\mathbb{C}^k$ as the generators (3.18) of the Lie algebra $\mathfrak{g}$ do, i.e.

[5] The action of the complex structure J induced by the contact structure on the leaf spaces is given by $J e^1 = -e^2$, $J e^2 = e^1$, $J e^3 = -e^4$, and $J e^4 = e^3$. The corresponding Kähler form is given by $\omega\,(\cdot,\cdot) := g\,(J\cdot,\cdot) = e^{12} + e^{34} = \omega^3$.

$$\left[I_6, X_\mu\right] = f_{6\mu}^{\nu} X_\nu \,. \tag{3.22}$$

Imposing this *equivariance condition* on the Higgs fields is equivalent to the vanishing of the terms that could spoil the equivariance. This condition can also be motivated [30] by recalling that a G-invariant connection of a vector bundle over a reductive homogeneous space can be parameterized [38] by linear maps $\Lambda : \mathfrak{g} \to \mathfrak{m}$ such that

$$\Lambda\big(\left[W, Y\right]\big) = \left[W, \Lambda(Y)\right] \tag{3.23}$$

for all $W \in \mathfrak{h}$ and $Y \in \mathfrak{m}$. By putting $X_\mu = \Lambda(I_\mu)$, one directly obtains the conditions (3.22) from the relations (3.18). It forces the underlying graph of the quiver to coincide with the weight diagram of the given representation of $\mathfrak{g}$ if $\mathfrak{h}$ is the Cartan subalgebra. Enlarging the subspace $\mathfrak{m}$ leads to fewer restrictions among the Higgs fields. For the Higgs fields associated to $T^{1,1}$ one obtains the necessary conditions

$$\begin{aligned} &\left[\Upsilon^{(1)} - \Upsilon^{(2)}, \phi^{(1)}\right] = 2\phi^{(1)} \,, \quad \left[\Upsilon^{(1)} - \Upsilon^{(2)}, \phi^{(2)}\right] = -2\phi^{(2)} \,, \\ &\left[\Upsilon^{(1)} - \Upsilon^{(2)}, \phi^{(3)}\right] = 0 \,. \end{aligned} \tag{3.24}$$

To evaluate these relations explicitly, let $E_{i\alpha, j\beta}$ be the square matrix of size $[(m_1+1)\,(m_2+1)]^2$ with the entry 1 (again interpreted as an identity operator) at the position $(i\alpha,\, j\beta)$ and zero otherwise, yielding the commutation relations

$$\left[E_{i\alpha,j\beta}, E_{k\gamma,l\delta}\right] = \delta_{jk}\,\delta_{\beta\gamma}\,E_{i\alpha,l\delta} - \delta_{il}\,\delta_{\alpha\delta}\,E_{k\gamma,j\beta} \,, \tag{3.25}$$

and in particular

$$\left[\Pi_{i\alpha}, E_{j\beta,k\gamma}\right] = \delta_{ij}\,\delta_{\alpha\beta}\,E_{i\alpha,k\gamma} - \delta_{ik}\,\delta_{\alpha\gamma}\,E_{j\beta,i\alpha} \,. \tag{3.26}$$

We decompose the Higgs fields according to their block structure as

$$\phi^{(a)} = \sum_{j,k=0}^{m_1} \sum_{\beta,\gamma=0}^{m_2} \phi^{(a)}_{j\beta,k\gamma}\, E_{j\beta,k\gamma} \qquad \text{for} \quad a = 1,2,3 \,, \tag{3.27}$$

where $\phi^{(a)}_{j\beta,k\gamma} \in \mathrm{Hom}(E_{k\gamma}, E_{j\beta})$. Then using (3.26) the commutators read

$$\left[\Upsilon^{(1)} - \Upsilon^{(2)}, \phi^{(a)}\right] = \sum_{j,k=0}^{m_1} \sum_{\beta,\gamma=0}^{m_2} 2\,(k - j - \gamma + \beta)\,\phi^{(a)}_{j\beta,k\gamma}\, E_{j\beta,k\gamma} \,. \tag{3.28}$$

Hence the conditions (3.24) restrict the component homomorphisms of the Higgs fields to be of the form

$$\begin{aligned} &\phi^{(1)}_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,1}\,\phi^{(1)}_{j\beta,k\gamma} \,, \qquad \phi^{(2)}_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,-1}\,\phi^{(2)}_{j\beta,k\gamma} \,, \\ &\phi^{(3)}_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,0}\,\phi^{(3)}_{j\beta,k\gamma} \,. \end{aligned} \tag{3.29}$$

This implies that $\phi^{(1)}$ and $\phi^{(2)}$ act as ladder operators which increase or decrease, respectively, by one unit the *relative* quantum number which is the difference of monopole charges at the vertices of the quiver given by

$$c_{i\alpha} := m_1 - m_2 - 2i + 2\alpha \,. \tag{3.30}$$

In particular, we cannot associate the action of any one of these Higgs fields to only a single copy of SU(2) in the tensor product as in the case of the quiver gauge theories associated to

$\mathbb{C}P^1 \times \mathbb{C}P^1$, as here they act simultaneously on both components. Since we allow for arbitrary entries of the Higgs fields in quiver gauge theory, there are generically *two* arrows (with opposite orientations) between vertices whose relative quantum numbers $c_{i\alpha}$ differ by one unit. Note that $\phi^{(1)}$ is not related to the adjoint bundle morphism of the field $\phi^{(2)}$, and so the pertinent quiver is generically the *double* of an underlying quiver; in this sense the resulting quiver gauge theories are analogous to those obtained via dimensional reduction over quasi-Kähler coset spaces [39]. On the other hand, the endomorphism $\phi^{(3)}$ represents the contribution from the vertical components of the Sasakian fibration and yields arrows conserving the relative quantum number $c_{i\alpha}$. This induces a loop at each vertex as well as arrows between vertices carrying the same $c_{i\alpha}$ value, which realize different partitions of a given difference of the indices i and α. This less restrictive property of the Higgs fields is caused by factoring out a smaller subalgebra $\mathfrak{h}$. Nonetheless, the equivariance conditions still rule out many possible arrows.

3.4. Examples

The general form of the Higgs fields and the quivers obtained by imposing G-equivariance over $T^{1,1}$ are completely dictated by the conditions (3.29). In order to gain a better insight into the structures obtained from these relations, we shall consider three explicit examples.

$(m_1, m_2) = (m, 0)$. In this case the representation labelled by the second index acts trivially and the tensor product reduces to a representation of the first SU(2) factor. The conditions (3.29) require the Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ to connect adjacent vertices (acting in opposite directions), while $\phi^{(3)}$ creates a loop at each vertex. Therefore this choice of representation yields the double of the A_m quiver [10,5,6] with vertex loops

$$\begin{tikzcd} (0,0) \arrow[loop above, dotted] \arrow[r, dashed, bend left] & (1,0) \arrow[loop above, dotted] \arrow[l, bend left] \arrow[r, dashed, bend left] & \cdots \arrow[l, bend left] \arrow[r, dashed, bend left] & (m-1,0) \arrow[loop above, dotted] \arrow[l, bend left] \arrow[r, dashed, bend left] & (m,0) \arrow[loop above, dotted] \arrow[l, bend left] \end{tikzcd} \tag{3.31}$$

where the vertices are labelled by their indices (i, α), and the arrows represent $\phi^{(1)}$ (solid lines), $\phi^{(2)}$ (dashed lines), and $\phi^{(3)}$ (dotted lines). For the equivariant connection in the simplest case $m = 1$ one obtains the explicit form

$$\mathcal{A} = \begin{pmatrix} a + \Phi^{(3)}_{0,0} & \Phi^{(1)}_{0,1} - \Phi^{(2)\,\dagger}_{1,0} \\ -\Phi^{(1)\,\dagger}_{0,1} + \Phi^{(2)}_{1,0} & -a + \Phi^{(3)}_{1,1} \end{pmatrix} , \tag{3.32}$$

where we have defined

$$\Phi^{(1)}_{i,j} := \phi^{(1)}_{i,j} \left(e^1 - \mathrm{i}\, e^2 \right) , \quad \Phi^{(2)}_{i,j} := \phi^{(2)}_{i,j} \left(e^3 - \mathrm{i}\, e^4 \right) \quad \text{and} \quad \Phi^{(3)}_{i,j} := \phi^{(3)}_{i,j}\, e^5 , \tag{3.33}$$

omitting Greek indices which refer to the second trivial factor of the tensor product. We notice that even for this simple case, by the weaker conditions on the Higgs fields, there are *two* contributions to the off-diagonal components of the connection.

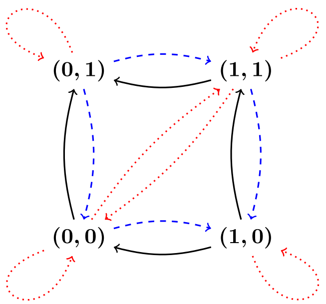
$(m_1, m_2) = (1, 1)$. In this representation the U(1) generator I_6 and its commutator with an arbitrary 4×4 matrix $(\bullet)$ are given by

$$I_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [I_6, (\bullet)] = \begin{pmatrix} 0\bullet & -2\bullet & 2\bullet & 0\bullet \\ 2\bullet & 0\bullet & 4\bullet & 2\bullet \\ -2\bullet & -4\bullet & 0\bullet & -2\bullet \\ 0\bullet & -2\bullet & 2\bullet & 0\bullet \end{pmatrix}, \tag{3.34}$$

which forces, according to (3.24), the Higgs fields to be of the form

$$\phi^{(1)} = \begin{pmatrix} 0 & 0 & * & 0 \\ * & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & * \\ 0 & * & 0 & 0 \end{pmatrix},$$
$$\phi^{(3)} = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix}, \tag{3.35}$$

in accordance with (3.29). The corresponding quiver



(3.36)

is a square lattice with double arrows as underlying graph and, additionally, the vertex loop modifications we have already encountered in the previous example. However, there is also a further arrow induced by $\phi^{(3)}$ because the vertices $(0,0)$ and $(1,1)$ realize the same relative quantum number $c_{i\alpha} = 2\,(\alpha - i)$. The compatible connection then reads

$$\mathcal{A} = \begin{pmatrix} \Phi^{(3)}_{00,00} & -\Psi_{01,00}{}^{\dagger} & \Psi_{00,10} & \Phi^{(3)}_{00,11} \\ \Psi_{01,00} & 2a + \Phi^{(3)}_{01,01} & 0 & \Psi_{01,11} \\ -\Psi_{00,10}{}^{\dagger} & 0 & -2a + \Phi^{(3)}_{10,10} & -\Psi_{11,10}{}^{\dagger} \\ -\Phi^{(3)}_{00,11}{}^{\dagger} & -\Psi_{01,11}{}^{\dagger} & \Psi_{11,10} & \Phi^{(3)}_{11,11} \end{pmatrix}, \tag{3.37}$$

where we have set $\Psi_{i\alpha,j\beta} := \Phi^{(1)}_{i\alpha,j\beta} - \Phi^{(2)}_{i\alpha,j\beta}{}^{\dagger}$, using again the definitions introduced above.

$\boldsymbol{(m_1, m_2) = (2, 1)}$**.** With the representation $I_6 = \mathrm{diag}(1, 3, -1, 1, -3, -1)$ one obtains the commutator

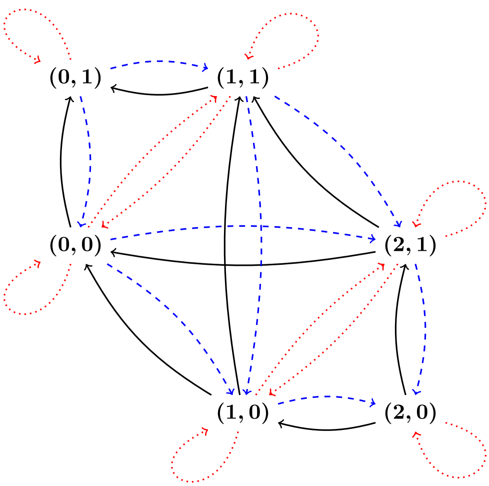
$$[I_6, (\bullet)] = \begin{pmatrix} 0\bullet & -2\bullet & 2\bullet & 0\bullet & 4\bullet & 2\bullet \\ 2\bullet & 0\bullet & 4\bullet & 2\bullet & 6\bullet & 4\bullet \\ -2\bullet & -4\bullet & 0\bullet & -2\bullet & 2\bullet & 0\bullet \\ 0\bullet & -2\bullet & 2\bullet & 0\bullet & 4\bullet & 2\bullet \\ -4\bullet & -6\bullet & -2\bullet & -4\bullet & 0\bullet & -2\bullet \\ -2\bullet & -4\bullet & 0\bullet & -2\bullet & 2\bullet & 0\bullet \end{pmatrix}, \tag{3.38}$$

which leads to endomorphisms of the form

$$\phi^{(1)} = \begin{pmatrix} 0 & 0 & * & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 \end{pmatrix},$$

$$\phi^{(3)} = \begin{pmatrix} * & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & * \end{pmatrix}. \tag{3.39}$$

The corresponding quiver



(3.40)

shows that, besides the additional arrows between vertices of the same relative quantum number, new arrows between the vertices $(0, 0)$ and $(2, 1)$ occur.

Given that the equivariance conditions (3.29) and, consequently, the structure of the quivers depend only on the relative quantum number $c_{i\alpha}$ rather than on the two U(1) charges separately, one may combine vertices with the same relative quantum number and define the homomorphisms between them by appropriate combinations of the Higgs fields. This repackaging implies that the quiver parameterized by (m_1, m_2) is reconsidered as an $(m_1 + m_2, 0)$ quiver by an equivalence relation on the vertices $(i, \alpha) \sim (i + \delta, \alpha + \delta)$ with integral δ, as long as the first entries remain in the interval $[0, m_1]$ and the second entries in $[0, m_2]$. Geometrically, this means that we project along lines with unit slope in the rectangular graph of the quiver. This turns all $\phi^{(3)}$ arrows into vertex loops. Hence the quiver gauge theory associated to $T^{1,1}$ for the decomposition (m_1, m_2) may be interpreted as that of the double of an $A_{m_1+m_2}$ quiver with one loop at each vertex, in terms of combinations of fields (with suitable multiplicities). This projection is comparable to the collapsing method applied for obtaining SU(3)-equivariant quiver gauge theories [11,3] starting from weight diagrams of SU(3).

3.5. Reduction to $A_{m_1} \oplus A_{m_2}$ quiver gauge theory

Let us now briefly pause to compare the quiver gauge theory obtained for the internal manifold $T^{1,1}$ with the quiver gauge theory associated to $\mathbb{C}P^1 \times \mathbb{C}P^1$, which is included as a special case in our framework. Starting from scratch, one may consider the equivariant dimensional reduction for the splitting $\mathfrak{g} = \mathfrak{m} \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1))$ by choosing, for instance, $\hat{I}_5 = \Upsilon^{(1)}$ and $\hat{I}_6 = \Upsilon^{(2)}$ as generators of the subalgebra $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, see (3.16). Then the resulting equivariance conditions are

$$
\begin{aligned}
&\left[\Upsilon^{(1)}, \phi^{(1)}\right] = 2\phi^{(1)} , \qquad \left[\Upsilon^{(2)}, \phi^{(1)}\right] = 0 = \left[\Upsilon^{(1)}, \phi^{(2)}\right] , \\
&\left[\Upsilon^{(2)}, \phi^{(2)}\right] = 2\phi^{(2)} ,
\end{aligned} \tag{3.41}
$$

and they uniquely determine the Higgs fields to be the ladder operators of the individual copies of SU(2), recovering the correct result of [10]; in particular we recover the rectangular lattice of the $A_{m_1} \oplus A_{m_2}$ quiver

$$
\begin{tikzcd}
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
(0,1) \arrow[d, dashed] & (1,1) \arrow[l] \arrow[d, dashed] & (2,1) \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
(0,0) & (1,0) \arrow[l] & (2,0) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{3.42}
$$

generated by $\phi^{(1)}$ (solid lines) and $\phi^{(2)}$ (dashed lines), where the vertices are labelled by their indices (i, α). The rectangular weight diagram (without double arrows) emerges here because $H = \mathrm{U}(1) \times \mathrm{U}(1)$ is the maximal torus of G in this case.

On the other hand, starting from our previous construction, the $\mathbb{C}P^1 \times \mathbb{C}P^1$ conditions are included in the more general $T^{1,1}$ framework. To recover this special case, we have to identify the second generator of $\mathfrak{h}$ in (3.19) and (3.21). Taking X_5 proportional to the generator $I_5 = \Upsilon^{(1)} + \Upsilon^{(2)}$, one has to interpret e^5 as the monopole field strength corresponding to the second generator and to demand a vanishing of the mixed terms, i.e. those containing the forms $e^\mu \wedge e^5$ in (3.21), which yields the additional equivariance conditions

$$
\begin{aligned}
&\left[X_5, X_1\right] = \tfrac{3}{2} X_2 , \quad \left[X_5, X_2\right] = -\tfrac{3}{2} X_1 , \quad \left[X_5, X_3\right] = \tfrac{3}{2} X_4 , \\
&\left[X_5, X_4\right] = -\tfrac{3}{2} X_3 .
\end{aligned} \tag{3.43}
$$

A comparison with the structure constants in (A.4) (see Appendix A) indicates that for the limit one should set $X_5 = -\frac{3\,\mathrm{i}}{4}\left(\Upsilon^{(1)} + \Upsilon^{(2)}\right)$, so that the further conditions read

$$
\left[\Upsilon^{(1)} + \Upsilon^{(2)}, \phi^{(1)}\right] = 2\phi^{(1)} , \qquad \left[\Upsilon^{(1)} + \Upsilon^{(2)}, \phi^{(2)}\right] = 2\phi^{(2)} . \tag{3.44}
$$

Together with (3.24), we obtain again the defining relations for the ladder operators of the two SU(2) Lie algebras. Thus the quiver gauge theory associated to $\mathbb{C}P^1 \times \mathbb{C}P^1$ (for the correct values of the radii R_l) is contained in the quiver gauge theory associated to $T^{1,1}$ by taking the limit $X_5 = -\frac{3\,\mathrm{i}}{4}\left(\Upsilon^{(1)} + \Upsilon^{(2)}\right)$.

3.6. Dimensional reduction of Yang–Mills theory

The G-equivariance constraints have strongly restricted the form of compatible gauge connections $\mathcal{A}$ on the bundle $\mathcal{E} \to M^d \times T^{1,1}$. We shall now study the action functional for pure Yang–Mills theory on the product manifold $M^d \times T^{1,1}$ and then perform the dimensional reduction to an effective quiver gauge theory on M^d, which is a Yang–Mills–Higgs theory with the internal manifold providing the non-trivial contributions to the Higgs potential.

After implementation of the equivariance conditions, the non-vanishing components of the field strength $\mathcal{F} = \frac{1}{2}\,\mathcal{F}_{\hat\mu\hat\nu}\, e^{\hat\mu} \wedge e^{\hat\nu}$ read[6]

$$
\begin{aligned}
&\mathcal{F}_{ab} = F_{ab} := \big(\mathrm{d}A + A\wedge A\big)_{ab} \qquad \text{for} \quad a,b = 1,\dots,d\,,\\
&\mathcal{F}_{a\mu} = \big(\mathrm{d}X_\mu\big)_a + \big[A_a, X_\mu\big] =: D_a X_\mu \qquad \text{for} \quad \mu = 1,\dots,5\,,\\
&\mathcal{F}_{12} = \big[X_1,X_2\big] - 2X_5 + \tfrac{3}{2}\,\mathrm{i}\, I_6\,, \quad \mathcal{F}_{13} = \big[X_1,X_3\big]\,, \quad \mathcal{F}_{14} = \big[X_1,X_4\big]\,,\\
&\mathcal{F}_{15} = \big[X_1,X_5\big] + \tfrac{3}{2}\,X_2\,, \qquad \mathcal{F}_{23} = \big[X_2,X_3\big]\,, \quad \mathcal{F}_{24} = \big[X_2,X_4\big]\,,\\
&\mathcal{F}_{25} = \big[X_2,X_5\big] - \tfrac{3}{2}\,X_1\,, \qquad \mathcal{F}_{34} = \big[X_3,X_4\big] - 2X_5 - \tfrac{3}{2}\,\mathrm{i}\, I_6\,,\\
&\mathcal{F}_{35} = \big[X_3,X_5\big] + \tfrac{3}{2}\,X_4\,, \qquad \mathcal{F}_{45} = \big[X_4,X_5\big] - \tfrac{3}{2}\,X_3\,,
\end{aligned}
\tag{3.45}
$$

where we denote by D_a the covariant derivatives (still assuming that the Higgs fields depend only on the coordinates of the manifold M^d). The resulting Yang–Mills Lagrangian is given by

$$
\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}\sqrt{\hat g}\ \mathrm{tr}\, \mathcal{F}_{\hat\mu\hat\nu}\,\mathcal{F}^{\hat\mu\hat\nu} = -\tfrac{1}{4}\sqrt{\hat g}\ \mathrm{tr}\, g^{\hat\sigma\hat\mu}\, g^{\hat\rho\hat\nu}\, \mathcal{F}_{\hat\mu\hat\nu}\,\mathcal{F}_{\hat\sigma\hat\rho}\,,
\tag{3.46}
$$

where we denote $\hat g = \det(g_{M^d})\det(g)$ with g_{M^d} the metric on M^d. As the metric g on the homogeneous space $T^{1,1}$ is given in terms of an orthonormal coframe, the Lagrangian simply reads

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\sqrt{\hat g}\ \mathrm{tr}\,\Big(&\frac{1}{2}\,F_{ab}\,F^{ab} + \sum_{\mu=1}^{5}\big(D_a X_\mu\big)\big(D^a X_\mu\big)\\
&+ \big(\big[X_1,X_3\big]\big)^2 + \big(\big[X_1,X_4\big]\big)^2 + \big(\big[X_2,X_3\big]\big)^2 + \big(\big[X_2,X_4\big]\big)^2\\
&+ \big(\big[X_1,X_2\big] - 2X_5 + \tfrac{3}{2}\,\mathrm{i}\,I_6\big)^2 + \big(\big[X_3,X_4\big] - 2X_5 - \tfrac{3}{2}\,\mathrm{i}\,I_6\big)^2\\
&+ \big(\big[X_1,X_5\big] + \tfrac{3}{2}\,X_2\big)^2 + \big(\big[X_2,X_5\big] - \tfrac{3}{2}\,X_1\big)^2 + \big(\big[X_3,X_5\big] + \tfrac{3}{2}\,X_4\big)^2\\
&+ \big(\big[X_4,X_5\big] - \tfrac{3}{2}\,X_3\big)^2\Big)\,.
\end{aligned}
\tag{3.47}
$$

The corresponding action functional is given by

$$
S_{\mathrm{YM}} = \int_{M^d\times T^{1,1}} \mathrm{d}^{d+5}x\ \mathcal{L}_{\mathrm{YM}}\,.
\tag{3.48}
$$

Since the Higgs fields do not depend on coordinates on $T^{1,1}$, the integral over the coset space simply yields its volume $\mathrm{Vol}\big(T^{1,1}\big) = \frac{16\pi^3}{27}$ in the chosen metric g. Hence dimensional reduction over $T^{1,1}$ of the Yang–Mills Lagrangian on $M^d \times T^{1,1}$ becomes

[6] Here indices $a,b,\dots$ refer to M^d and $\mu,\nu,\dots$ are indices along $T^{1,1}$, while hatted indices $\hat\mu,\hat\nu,\dots$ refer to cobasis directions on $M^d \times T^{1,1}$.

$$\mathcal{L}_{\mathrm{r}} = \frac{16\pi^3}{27}\,\mathcal{L}_{\mathrm{YM}} \qquad \text{and} \qquad S_{\mathrm{r}} = \int\limits_{M^d} \mathrm{d}^d x\ \mathcal{L}_{\mathrm{r}}\ , \tag{3.49}$$

which describes a Yang–Mills–Higgs theory on M^d. Of course, the result we are interested in is the concrete form of the Higgs contributions induced by the internal manifold. We have seen that imposing the compatibility condition of equivariance has ruled out many contributions to the connection and has fixed the form of the Lagrangian of the gauge theory as in (3.47). After choosing the decomposition of the representation (m_1, m_2) of the structure group, the only freedom that remains is the concrete realization of the allowed endomorphisms, represented by the arrows in the quiver. The instanton equations will require further relations among them, which shall be studied in the next section.

Often the restrictions of G-equivariance are so strong that the explicit evaluation of the Higgs contributions is significantly simplified. For this, we consider as an example the special solution of the $T^{1,1}$ quiver constraints given by the rectangular $A_{m_1} \oplus A_{m_2}$ quiver

$$\begin{tikzcd}
(0,m_2) \arrow[loop above, dotted] \arrow[d, dashed] & (1,m_2) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] & (2,m_2) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] & \cdots \arrow[l] & (m_1,m_2) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] \\
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & & \vdots \arrow[d, dashed] \\
(0,1) \arrow[loop above, dotted] \arrow[d, dashed] & (1,1) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] & (2,1) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] & \cdots \arrow[l] & (m_1,1) \arrow[l] \arrow[loop above, dotted] \arrow[d, dashed] \\
(0,0) \arrow[loop above, dotted] & (1,0) \arrow[l] \arrow[loop above, dotted] & (2,0) \arrow[l] \arrow[loop above, dotted] & \cdots \arrow[l] & (m_1,0) \arrow[l] \arrow[loop above, dotted]
\end{tikzcd} \tag{3.50}$$

with one loop at each vertex from the extra vertical component. This quiver arises if (3.29) is solved by imposing (3.41) and by demanding that the Higgs field $\phi^{(3)}$ is diagonal. It is then possible to exploit a grading of the connection, similarly to that of [10], which greatly reduces the number of contributions. With the abbreviations $\phi^{(1)}_{i+1\,\alpha} := \phi^{(1)}_{i\,\alpha,i+1\,\alpha}$, $\phi^{(2)}_{i\,\alpha+1} := \phi^{(2)}_{i\,\alpha,i\,\alpha+1}$ and $\phi^{(3)}_{i\alpha} := \phi^{(3)}_{i\alpha,i\alpha}$, the non-vanishing block components of the connection read

$$\begin{aligned}
\mathcal{A}^{i\alpha,i\alpha} &= A^{i\alpha} + \phi^{(3)}_{i\alpha}\, e^5 + c_{i\alpha}\, \mathbf{1}_{k_{i\alpha}}\, a \qquad \text{with} \quad c_{i\alpha} = m_1 - m_2 - 2i + 2\alpha\ , \\
\mathcal{A}^{i\,\alpha,i+1\,\alpha} &= \phi^{(1)}_{i+1\,\alpha}\left(e^1 - \mathrm{i}\,e^2\right) = -\left(\mathcal{A}^{i+1\,\alpha,i\,\alpha}\right)^\dagger\ , \\
\mathcal{A}^{i\,\alpha,i\,\alpha+1} &= \phi^{(2)}_{i\,\alpha+1}\left(e^3 - \mathrm{i}\,e^4\right) = -\left(\mathcal{A}^{i\,\alpha+1,i\,\alpha}\right)^\dagger\ ,
\end{aligned} \tag{3.51}$$

where $A^{i\alpha}$ is a connection on the vector bundle $E_{i\alpha} \to M^d$ with curvature $F^{i\alpha} = \mathrm{d}A^{i\alpha} + A^{i\alpha} \wedge A^{i\alpha}$. From the field strength $\mathcal{F}^{i\alpha,j\beta} = \mathrm{d}\mathcal{A}^{i\alpha,j\beta} + \sum_{l,\gamma} \mathcal{A}^{i\alpha,l\gamma} \wedge \mathcal{A}^{l\gamma,j\beta}$ we compute the reduced action functional and obtain[7] (see Appendix A for details)

[7] Alternatively, we may substitute directly into (3.47).

$$
\begin{aligned}
S_{\mathrm{r}} = \frac{16\pi^3}{27} \int\limits_{M^d} \mathrm{d}^d x \, \sqrt{\det(g_{M^d})} \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} \operatorname{tr}\Big\{ & \tfrac{1}{4} F^{i\alpha\,\dagger}_{ab} F^{i\alpha\,ab} + \tfrac{1}{2} \big(D_a \phi^{(3)}_{i\alpha}\big) \big(D^a \phi^{(3)}_{i\alpha}\big)^\dagger \\
& + \big(D_a \phi^{(1)}_{i\alpha}\big)^\dagger \big(D^a \phi^{(1)}_{i\alpha}\big) + \big(D_a \phi^{(1)}_{i+1\,\alpha}\big) \big(D^a \phi^{(1)}_{i+1\,\alpha}\big)^\dagger + \big(D_a \phi^{(2)}_{i\alpha}\big)^\dagger \big(D^a \phi^{(2)}_{i\alpha}\big) \\
& + \big(D_a \phi^{(2)}_{i\,\alpha+1}\big) \big(D^a \phi^{(2)}_{i\,\alpha+1}\big)^\dagger + 2 \big| \phi^{(1)\,\dagger}_{i\alpha} \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i+1\,\alpha} \phi^{(1)}_{i+1\,\alpha}{}^\dagger + \mathrm{i}\, \phi^{(3)}_{i\alpha} + \tfrac{3}{4} c_{i\alpha} \mathbf{1}_{k_{i\alpha}} \big|^2 \\
& + 2 \big| \phi^{(2)\,\dagger}_{i\alpha} \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i+1\,\alpha} \phi^{(2)}_{i+1\,\alpha}{}^\dagger + \mathrm{i}\, \phi^{(3)}_{i\alpha} - \tfrac{3}{4} c_{i\alpha} \mathbf{1}_{k_{i\alpha}} \big|^2 \\
& + 2 \big| \phi^{(1)}_{i+1\,\alpha} \phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1} \phi^{(1)}_{i+1\,\alpha+1} \big|^2 + 2 \big| \big(\phi^{(1)}_{i\,\alpha-1} \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i-1\,\alpha} \phi^{(1)}_{i\alpha} \big)^\dagger \big|^2 \\
& + 2 \big| \phi^{(2)\,\dagger}_{i\alpha} \phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha} \phi^{(2)}_{i+1\,\alpha}{}^\dagger \big|^2 + 2 \big| \big(\phi^{(2)}_{i-1\,\alpha+1}{}^\dagger \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i\,\alpha+1} \phi^{(2)}_{i\,\alpha+1}{}^\dagger \big)^\dagger \big|^2 \\
& + \big| \big(\phi^{(1)}_{i\alpha} \phi^{(3)}_{i\alpha} - \phi^{(3)}_{i-1\,\alpha} \phi^{(1)}_{i\alpha} - \tfrac{3}{2} \mathrm{i}\, \phi^{(1)}_{i\alpha} \big)^\dagger \big|^2 \\
& + \big| \phi^{(1)}_{i+1\,\alpha} \phi^{(3)}_{i+1\,\alpha} - \phi^{(3)}_{i\alpha} \phi^{(1)}_{i+1\,\alpha} - \tfrac{3}{2} \mathrm{i}\, \phi^{(1)}_{i+1\,\alpha} \big|^2 \\
& + \big| \big(\phi^{(2)}_{i\alpha} \phi^{(3)}_{i\alpha} - \phi^{(3)}_{i\,\alpha-1} \phi^{(2)}_{i\alpha} - \tfrac{3}{2} \mathrm{i}\, \phi^{(2)}_{i\alpha} \big)^\dagger \big|^2 \\
& + \big| \phi^{(2)}_{i\,\alpha+1} \phi^{(3)}_{i\,\alpha+1} - \phi^{(3)}_{i\alpha} \phi^{(2)}_{i\,\alpha+1} - \tfrac{3}{2} \mathrm{i}\, \phi^{(2)}_{i\,\alpha+1} \big|^2 \Big\} \,,
\end{aligned}
\tag{3.52}
$$

where we have used the abbreviation $|\Phi|^2 := \Phi\,\Phi^\dagger$ and the covariant derivatives from (3.45) take the form

$$
\begin{aligned}
D\phi^{(1)}_{i+1\,\alpha} &= \mathrm{d}\phi^{(1)}_{i+1\,\alpha} + A^{i\alpha} \phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha} A^{i+1\,\alpha} \,, \\
D\phi^{(2)}_{i\,\alpha+1} &= \mathrm{d}\phi^{(2)}_{i\,\alpha+1} + A^{i\alpha} \phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1} A^{i\,\alpha+1} \,, \\
D\phi^{(3)}_{i\alpha} &= \mathrm{d}\phi^{(3)}_{i\alpha} + A^{i\alpha} \phi^{(3)}_{i\alpha} - \phi^{(3)}_{i\alpha} A^{i\alpha} \,.
\end{aligned}
\tag{3.53}
$$

The most prominent difference between this reduced action functional and that of [10] is, of course, the appearance of the third Higgs field $\phi^{(3)}$ due to the remaining vertical component. As mentioned before, the radii of the two spheres, which occur as moduli in the action of [10], have been fixed here to numerical values by the Sasaki–Einstein condition. From (3.52) one recovers indeed the action functional corresponding to dimensional reduction over the Kähler coset space[8] $\mathbb{C}P^1 \times \mathbb{C}P^1$ by taking the limit $\phi^{(3)} = -\frac{3\,\mathrm{i}}{4}\big(\Upsilon^{(1)} + \Upsilon^{(2)}\big)$. Note that naively setting the additional Higgs field to zero leads to a different action functional.

The next task is to study the equations of motion, and in particular determine the vacua that are described by the Lagrangian (3.47). For this, we have to solve the Yang–Mills equations on $M^d \times T^{1,1}$, which is simplified by the existence of Killing spinors on the Sasaki–Einstein manifold $T^{1,1}$ because solutions of the instanton equation (2.23) also satisfy the Yang–Mills equations on $T^{1,1}$ in this instance [25]. Furthermore, it is even more convenient to work in even dimensions over the corresponding Calabi–Yau metric cone $C(T^{1,1})$, as one can then solve the Hermitian Yang–Mills equations, which imply the instanton equations. This is the subject of the remainder of this paper.

[8] For the correctly fixed values of the radii, see Appendix A.

4. Instantons on the conifold

4.1. Geometry of the cone $C(T^{1,1})$

As a metric cone over a Sasaki–Einstein manifold, the conifold $C(T^{1,1})$ is by construction a (non-compact) Calabi–Yau manifold, so that its Riemannian holonomy is contained in SU(3) and it is Ricci-flat. In contrast to the common framework of compactifications on orbifolds in string theory, the conifold cannot be described as a global quotient $\mathbb{C}^3/\Gamma$ by a discrete subgroup $\Gamma \subset \mathrm{SU}(3)$ because it is not flat. On the other hand, it also admits a description as a toric variety (see e.g. [40]) which is described in Cox homogeneous coordinates as the quotient space[9]

$$C\big(T^{1,1}\big) \simeq \big(\mathbb{C}^4 \setminus \mathcal{Z}\big) \big/ \mathbb{C}^* \tag{4.1}$$

with weights $(1, 1, -1, -1)$, i.e. one identifies $(z_1, z_2, z_3, z_4) \sim (\lambda\, z_1, \lambda\, z_2, \lambda^{-1}\, z_3, \lambda^{-1}\, z_4)$ for $\lambda \in \mathbb{C}^*$, where $\mathcal{Z}$ is the union of the loci of points $(z_1, z_2, 0, 0) \neq (0, 0, 0, 0)$ and $(0, 0, z_3, z_4) \neq (0, 0, 0, 0)$. Therefore one cannot study translationally invariant instantons in this case, which would lead to a quiver gauge theory generated by equivariance conditions with respect to the discrete group Γ, see e.g. [41]. Hence we shall briefly consider the geometry of the metric cone before we proceed to the description of instanton solutions on it.

By definition, the metric of the conifold is the warped product

$$g_{\mathrm{con}} = r^2\, g + \mathrm{d}r \otimes \mathrm{d}r = r^2 \sum_{\mu=1}^{6} e^\mu \otimes e^\mu = \mathrm{e}^{2\tau} \sum_{\mu=1}^{6} e^\mu \otimes e^\mu \tag{4.2}$$

with radial coordinate $r \in \mathbb{R}_{>0}$, where this equation establishes a conformal equivalence between the cone metric and the metric on the cylinder by setting

$$e^6 = \tfrac{1}{r}\,\mathrm{d}r = \mathrm{d}\tau \qquad \text{with} \quad \tau := \log(r)\ . \tag{4.3}$$

The Kähler form $\Omega(\cdot,\cdot) = g_{\mathrm{con}}(J\cdot,\cdot)$ (using the cylinder metric[10]) is given by

$$\Omega = r^2\, \omega^3 + r^2\, e^5 \wedge e^6 = r^2 \left(e^{12} + e^{34} + e^{56}\right)\ , \tag{4.4}$$

which is closed due to the defining Sasaki–Einstein relations (see Appendix A for details), and the holomorphic 1-forms are

$$\Theta^a := e^{2a-1} + \mathrm{i}\, e^{2a} \qquad \text{with} \quad J\Theta^a = \mathrm{i}\,\Theta^a \qquad \text{for} \quad a = 1, 2, 3\ . \tag{4.5}$$

By rescaling the forms

$$\tilde{e}^\mu := r\, e^\mu\ , \qquad \mathrm{d}\tilde{e}^\mu = r\, \mathrm{d}e^\mu - e^\mu \wedge \mathrm{d}r\ , \tag{4.6}$$

one obtains an orthonormal cobasis and structure equations with respect to the cone metric. The connection matrix for the Levi-Civita connection is given by

9 In this description, naive dimensional reduction of Yang–Mills theory over the conifold yields a quiver gauge theory based on the Klebanov–Witten quiver [17].

10 In what follows the descriptions with cone metric and with cylinder metric are considered equivalent.

$$\mathrm{d}\begin{pmatrix} \tilde{e}^1 - \mathrm{i}\,\tilde{e}^2 \\ \tilde{e}^3 - \mathrm{i}\,\tilde{e}^4 \\ \tilde{e}^5 - \mathrm{i}\,\tilde{e}^6 \end{pmatrix} = -\begin{pmatrix} 2a - \frac{1}{2}\,\mathrm{i}\,e^5 & 0 & \mathrm{i}\left(e^1 - \mathrm{i}\,e^2\right) \\ 0 & -2a - \frac{1}{2}\,\mathrm{i}\,e^5 & \mathrm{i}\left(e^3 - \mathrm{i}\,e^4\right) \\ \mathrm{i}\left(e^1 + \mathrm{i}\,e^2\right) & \mathrm{i}\left(e^3 + \mathrm{i}\,e^4\right) & \mathrm{i}\,e^5 \end{pmatrix} \wedge \begin{pmatrix} \tilde{e}^1 - \mathrm{i}\,\tilde{e}^2 \\ \tilde{e}^3 - \mathrm{i}\,\tilde{e}^4 \\ \tilde{e}^5 - \mathrm{i}\,\tilde{e}^6 \end{pmatrix} . \tag{4.7}$$

Being a Calabi–Yau threefold, the conifold has holonomy group SU(3), and a calculation of the curvature of the Levi-Civita connection (see Appendix A) shows that it is valued in the Lie subalgebra $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ and solves the instanton equation (2.23).[11] On the other hand, declaring again all terms apart from those containing the form a as torsion, one obtains a U(1) connection which is simply the lift of the canonical connection on $T^{1,1}$ to the cone; it is clearly still an instanton. Consequently, we have now two instanton solutions to start from in our construction.

For this, let us adapt the approach used on $T^{1,1}$ more generally to the metric cone $C(T^{1,1})$. Given an instanton $\Gamma = \Gamma^i\, I_i$ with generators $I_i \in \mathfrak{su}(2) \subset \mathfrak{u}(k)$ for $i = 6, 7, 8$, the ansatz for the connection reads

$$\mathcal{A} = \Gamma + X_\mu\, e^\mu \, . \tag{4.8}$$

From the structure equations $\mathrm{d}e^\mu = -\Gamma^\mu_\nu \wedge e^\nu + \frac{1}{2}\, T^\mu_{\rho\sigma}\, e^{\rho\sigma}$ and the Maurer–Cartan equation $\mathrm{d}e^\mu = -\frac{1}{2}\, f^\mu_{\rho\sigma}\, e^{\rho\sigma}$, it follows that $\left(\Gamma^\mu_\nu\right)_i = f^\mu_{i\nu}$ and the curvature yields

$$\mathcal{F} = \mathcal{F}_\Gamma + \Gamma^i \left(\left[I_i, X_\mu\right] - f^\nu_{i\mu}\, X_\nu\right) \wedge e^\mu + \tfrac{1}{2}\left(\left[X_\mu, X_\nu\right] + T^\sigma_{\mu\nu}\, X_\sigma\right) e^{\mu\nu} + \mathrm{d}X_\mu \wedge e^\mu \, , \tag{4.9}$$

where $\mathcal{F}_\Gamma := \mathrm{d}\Gamma + \Gamma \wedge \Gamma$. Now choose the endomorphisms X_μ such that the differentials $\mathrm{d}X_\mu$ do not yield contributions containing the forms Γ^i; in particular, this holds for the case of constant matrices and for spherically symmetric matrices $X_\mu = X_\mu(r)$ as instanton solutions that we consider below. Then the equivariance condition is exactly expressed by the vanishing of the second term,

$$\left[I_i, X_\mu\right] = f^\nu_{i\mu}\, X_\nu \, , \tag{4.10}$$

which generates the quiver.

Given the compatible connection and its curvature (4.9), one can obtain instanton solutions by using the Kähler form Ω for the formulation of the *Hermitian Yang–Mills equations* [35]

$$\mathcal{F}^{2,0} = 0 = \mathcal{F}^{0,2} \, , \qquad \Omega \lrcorner \mathcal{F}^{1,1} = 0 \, , \tag{4.11}$$

where $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$ refers to the decomposition into holomorphic and antiholomorphic parts with respect to the complex structure J, so that the first equation means that the field strength is invariant under the action of J. These equations can be regarded as stability conditions[12] on holomorphic vector bundles and are sometimes referred to as *Donaldson–Uhlenbeck–Yau equations* [43,44]; they imply the instanton equation (2.23).

[11] The existence of an SU(2)-structure on a (real) six-dimensional Calabi–Yau manifold implies that it has an almost product structure, see e.g. [42]. The conifold locally looks like a product of $\mathbb{R}_{>0} \times S^1$ with $\mathbb{C}P^1 \times \mathbb{C}P^1$.

[12] This pertains to compact Calabi–Yau manifolds; on the conifold they are simply a set of additional real differential equations imposed on a Hermitian connection.

4.2. Yang–Mills flows

When applying this construction to the lifted canonical connection $\Gamma = I_6\, a$ on $C(T^{1,1})$, the only difference from before that one has to take into account is the additional radial coordinate giving rise to one further endomorphism X_6,

$$\mathcal{A} \;=\; I_6\, a + \sum_{\mu=1}^{5} X_\mu\, e^\mu + X_6\, e^6 \;. \tag{4.12}$$

We are interested in spherically symmetric instanton solutions, i.e. those endomorphisms $X_\mu = X_\mu(\tau)$ which depend only on the radial coordinate $r = \mathrm{e}^\tau$. After implementing the equivariance conditions[13]

$$\big[I_6, \phi^{(1)}\big] \;=\; 2\phi^{(1)}\;, \qquad \big[I_6, \phi^{(2)}\big] \;=\; -2\phi^{(2)}\;, \qquad \big[I_6, \phi^{(3)}\big] \;=\; 0\;, \tag{4.13}$$

which are identical to those over $T^{1,1}$, the field strength is given by

$$\mathcal{F} \;=\; \mathcal{F}_{T^{1,1}} + \sum_{\mu=1}^{5} \Big(\big[X_\mu, X_6\big] - \frac{\mathrm{d}X_\mu}{\mathrm{d}\tau} \Big)\, e^{\mu 6}\;, \tag{4.14}$$

where $\mathcal{F}_{T^{1,1}}$ denotes the curvature we have already derived over $T^{1,1}$ with components $\mathcal{F}_{\mu\nu}$ from (3.45).

Evaluating the holomorphicity condition of the Hermitian Yang–Mills equations (4.11) in terms of the holomorphic 1-forms Θ^a leads to four first order ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}X_1}{\mathrm{d}\tau} &= -\frac{3}{2}\, X_1 + \big[X_1, X_6\big] + \big[X_2, X_5\big]\;, & \frac{\mathrm{d}X_2}{\mathrm{d}\tau} &= -\frac{3}{2}\, X_2 + \big[X_2, X_6\big] - \big[X_1, X_5\big]\;,\\
\frac{\mathrm{d}X_3}{\mathrm{d}\tau} &= -\frac{3}{2}\, X_3 + \big[X_3, X_6\big] + \big[X_4, X_5\big]\;, & \frac{\mathrm{d}X_4}{\mathrm{d}\tau} &= -\frac{3}{2}\, X_4 + \big[X_4, X_6\big] - \big[X_3, X_5\big]\;,
\end{aligned} \tag{4.15}$$

together with the constraints

$$\big[X_1, X_3\big] \;=\; \big[X_2, X_4\big] \qquad \text{and} \qquad \big[X_1, X_4\big] \;=\; -\big[X_2, X_3\big]\;. \tag{4.16}$$

The remaining stability condition $\Omega \lrcorner \mathcal{F} = 0$ yields the flow equation for X_5 given by

$$\frac{\mathrm{d}X_5}{\mathrm{d}\tau} \;=\; -4X_5 + \big[X_1, X_2\big] + \big[X_3, X_4\big] + \big[X_5, X_6\big]\;. \tag{4.17}$$

Inserting these equations into the action functional (3.48) over the cylinder $\mathbb{R}_{>0} \times T^{1,1}$ leads to cancellations of many contributions involving the Higgs potential, as to be expected from a vacuum solution on the Higgs branch of the quiver gauge theory. Moreover, one can see that the conditions imposed by the Hermitian Yang–Mills equations induce relations on the quiver: The constraints (4.16) are cast into the quiver relation

$$\big[\phi^{(1)}, \phi^{(2)}\big] \;=\; 0\;, \tag{4.18}$$

[13] Here we write $\phi^{(a)} := \frac{1}{2}\big(X_{2a-1} + \mathrm{i}\, X_{2a}\big)$ for $a = 1, 2, 3$.

i.e. the commutativity of the Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ follows naturally as a consequence of the Hermitian Yang–Mills equations. In the simplified example of the $A_{m_1} \oplus A_{m_2}$ quiver with vertex loops, which admits a grading of the connection, this implies commutativity of the quiver arrows around the rectangular lattice, $\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} = \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}$. One can also directly observe the vanishing of the corresponding contributions to the Higgs potential in the action functional (3.52).

Before we describe the general solutions to these flow equations under the given constraints, we consider the case of constant endomorphisms. When the matrices X_μ do not depend on r, the radial coordinate enters the framework just as a parameter that labels different copies of $T^{1,1}$ (as a foliation of the six-dimensional cone into copies of the underlying Sasaki–Einstein manifold along the preferred direction r). Therefore the examination of constant endomorphisms corresponds to studying instanton solutions for the original five-dimensional situation.[14] The flow equations turn exactly into the conditions (3.43) which arose as additional equivariance relations in the limit where the total space of the Sasakian fibration $T^{1,1}$ degenerates to its base $\mathbb{C}P^1 \times \mathbb{C}P^1$; they lead to the vanishing of most terms in the Lagrangian (3.47) of the quiver gauge theory on $T^{1,1}$. As discussed before, a solution to these equations is given by the choice $X_5 = -\frac{3\,\mathrm{i}}{4}\big(\Upsilon^{(1)} + \Upsilon^{(2)}\big)$, which shows that the quiver gauge theory on $\mathbb{C}P^1 \times \mathbb{C}P^1$ from [10] for the appropriate values of the radii R_l is not only contained in our description but even automatically realizes a solution of the Hermitian Yang–Mills equations on the conifold.

4.3. *Instanton moduli spaces*

In solving the generic case of spherically symmetric instantons given by solutions to the flow equations (4.15) and (4.17) under the derived constraints, one encounters Nahm-type equations describing the radial dependence of the matrices X_μ. Hence one can apply techniques similar to those that have been used for the description of the hyper-Kähler structure of the moduli space of the (original) Nahm equations, see in particular [45,46]. In [33], instantons arising from the Hermitian Yang–Mills equations on Calabi–Yau cones of any dimension have been studied using these methods, and it was shown that the equations which describe the moduli space do not depend on the concrete Sasaki–Einstein manifold under consideration but only on its dimension; thus our equations for the moduli space of Hermitian Yang–Mills instantons on the conifold can be included in that treatment.

The flow equations can be brought to a form similar to the Nahm equations by setting

$$X_i \;=\; \mathrm{e}^{-\frac{3}{2}\tau}\, W_i \ , \quad i \;=\; 1,2,3,4 \qquad \text{and} \qquad X_j \;=\; \mathrm{e}^{-4\tau}\, W_j \ , \quad j \;=\; 5,6 \tag{4.19}$$

in order to eliminate the linear terms. Changing again the coordinate to

$$s \;=\; \tfrac{1}{4}\,\mathrm{e}^{-4\tau} \;=\; \tfrac{1}{4}\,r^{-4} \;\in\; \mathbb{R}_{>0} \tag{4.20}$$

and writing

$$Z_1 \;:=\; \tfrac{1}{2}\,(W_1 + \mathrm{i}\,W_2) \ , \quad Z_2 \;:=\; \tfrac{1}{2}\,(W_3 + \mathrm{i}\,W_4) \ , \quad Z_3 \;:=\; \tfrac{\mathrm{i}}{2}\,(W_5 + \mathrm{i}\,W_6) \ , \tag{4.21}$$

we arrive at the set of equations

[14] The matrix X_6 can always be set to zero via a real gauge transformation, see e.g. [31].

$$\frac{\mathrm{d}Z_1}{\mathrm{d}s} = 2\big[Z_1, Z_3\big]\,, \qquad \frac{\mathrm{d}Z_2}{\mathrm{d}s} = 2\big[Z_2, Z_3\big]\,, \tag{4.22}$$

$$\frac{\mathrm{d}Z_3}{\mathrm{d}s} + \frac{\mathrm{d}Z_3^\dagger}{\mathrm{d}s} = 2(-s)^{-5/4}\left(\big[Z_1, Z_1^\dagger\big] + \big[Z_2, Z_2^\dagger\big]\right) - 2\big[Z_3, Z_3^\dagger\big] \tag{4.23}$$

together with the constraints

$$\big[I_6, Z_1\big] = 2Z_1\,, \qquad \big[I_6, Z_2\big] = -2Z_2\,, \qquad \big[I_6, Z_3\big] = 0\,, \qquad \big[Z_1, Z_2\big] = 0\,. \tag{4.24}$$

We therefore turn our attention to the moduli space of solutions to the equations (4.22)–(4.23) subject to the equivariance constraints and quiver relations from (4.24); we refer to the two equations (4.22) as the *complex equations* and to the equation (4.23) as the *real equation*. The complex equations and the constraints are invariant under the complex gauge transformations [46]

$$\begin{aligned} Z_a &\longmapsto g\cdot Z_a = g\,Z_a\,g^{-1}\,, \quad a = 1,2\,, \\ Z_3 &\longmapsto g\cdot Z_3 = g\,Z_3\,g^{-1} + \frac{1}{2}\,\frac{\mathrm{d}g}{\mathrm{d}s}\,g^{-1} \end{aligned} \tag{4.25}$$

for arbitrary smooth functions $g:\mathbb{R}_{>0}\to\mathcal{G}_{\mathbb{C}}\subset \mathrm{GL}(k,\mathbb{C})$, where $\mathcal{G}$ is the subgroup of $\mathrm{U}(k)$ which stabilizes the generator I_6 under the adjoint action. This observation motivates a description of the moduli space of the flow equations in terms of a Kähler quotient construction, involving an infinite-dimensional space of connections and an infinite-dimensional gauge group, or equivalently in terms of adjoint orbits. For instantons on Calabi–Yau cones the two approaches have been carried out in [33], whose results we will adapt to our setting together with the subsequent implementations of the equivariance constraints considered in [32,31].

For this, let $\mathbb{A}^{1,1}$ be the space of endomorphisms Z_a satisfying the complex equations (4.22) and the constraints (4.24); the complexified gauge group $\widehat{\mathcal{G}}_{\mathbb{C}}$ acts on $\mathbb{A}^{1,1}$ according to (4.25). This space is naturally an infinite-dimensional Kähler manifold with a gauge-invariant metric and symplectic form [33,31]. The corresponding moment map $\mu:\mathbb{A}^{1,1}\to\widehat{\mathfrak{g}}$ is defined by

$$\mu\big(Z, Z^\dagger\big) = \frac{\mathrm{d}Z_3}{\mathrm{d}s} + \frac{\mathrm{d}Z_3^\dagger}{\mathrm{d}s} - 2(-s)^{-5/4}\left(\big[Z_1, Z_1^\dagger\big] + \big[Z_2, Z_2^\dagger\big]\right) + 2\big[Z_3, Z_3^\dagger\big]\,, \tag{4.26}$$

where $\widehat{\mathfrak{g}}$ is the Lie algebra of $\widehat{\mathcal{G}}$ consisting of infinitesimal (real) gauge transformations which commute with the generator I_6. The moment map thus connects the space of solutions to the complex equations with the remaining real equation, so that the moduli space $\mathcal{M}$ of Hermitian Yang–Mills instantons is obtained by taking the Kähler quotient

$$\mathcal{M} = \mu^{-1}(0)\,\big/\,\widehat{\mathcal{G}}\,. \tag{4.27}$$

This quotient can be related [33,31] to the action of the complexified gauge group on the set of stable points $\mathbb{A}^{1,1}_{\mathrm{st}}\subset\mathbb{A}^{1,1}$ whose $\widehat{\mathcal{G}}_{\mathbb{C}}$-orbits intersect the zeroes of the moment map, so that the moduli space is realized as the GIT quotient

$$\mathcal{M} \simeq \mathbb{A}^{1,1}_{\mathrm{st}}\,\big/\,\widehat{\mathcal{G}}_{\mathbb{C}}\,. \tag{4.28}$$

Following again the treatments of the Nahm equations from [45–47] and their extensions to our setting of six-dimensional conical instantons from [33,31], we rewrite the solutions of the flow equations by applying a complex gauge transformation (4.25) which locally trivializes the matrix Z_3 as

$$Z_3 \;=\; -\frac{1}{2}\,g^{-1}\,\frac{\mathrm{d}g}{\mathrm{d}s}\;. \tag{4.29}$$

In this gauge, the complex equations (4.22) are solved by gauge transformations of constant matrices U_1 and U_2 as

$$Z_1 \;=\; g^{-1}\,U_1\,g\;, \qquad Z_2 \;=\; g^{-1}\,U_2\,g\;. \tag{4.30}$$

In order to fulfil the constraints (4.24), the matrices U_1 and U_2 must be mutually commuting and satisfy the equivariance conditions $\big[I_6, U_1\big] = 2U_1$ and $\big[I_6, U_2\big] = -2U_2$. By generalizing Donaldson's treatment of the ordinary Nahm equations [46], one can show that these gauge fixed solutions fulfil the remaining real equation (4.23), i.e. there exists a unique path $g(s)$ for $s \in \mathbb{R}_{>0}$ which satisfies the real equation [33].

Finally, we need to impose suitable boundary conditions. One can adapt Kronheimer's asymptotics [45] for the solutions to the flow equations on the six-dimensional cone [33,31] as

$$\lim_{s\to\infty}\,W_\mu(s) \;=\; g_0\,T_\mu\,g_0^{-1} \qquad \text{for} \quad \mu \;=\; 1,\dots,5\;, \tag{4.31}$$

where $g_0 \in \mathcal{G}$ and we have gauged away the scalar field X_6. Defining $V_a := \frac{1}{2}\,(T_{2a-1} + \mathrm{i}\,T_{2a})$ for $a = 1, 2$, the constant boundary matrices satisfy

$$\big[I_6, V_1\big] \;=\; 2V_1\;, \qquad \big[I_6, V_2\big] \;=\; -2V_2\;, \qquad \big[I_6, T_5\big] \;=\; 0\;, \qquad \big[V_1, V_2\big] \;=\; 0\;. \tag{4.32}$$

The asymptotic boundary conditions (4.31) determine the singular behaviour of the instanton connections X_μ as one approaches the conical singularity at $r = 0$ ($\tau \to -\infty$). On the other hand, one can choose boundary conditions such that $X_\mu(\tau) \to 0$ as $\tau \to +\infty$, giving instantons that are *framed* at infinity in $\mathbb{R}_{\geq 0}$, which implies that $W_\mu(s)$ has a limit as $s \to 0$ whose value is completely determined by the solution of the first order flow equations with the boundary conditions (4.31).

Following [33,31], such solutions identify the moduli space $\mathcal{M}$ of the Hermitian Yang–Mills equations on the metric cone in terms of adjoint orbits of the initial data T_μ. This follows from (4.22) which shows that the solutions $Z_a(s)$ for $a = 1, 2$ each lie respectively in the same adjoint orbit under the action of the complex Lie algebra $\mathfrak{gl}(k, \mathbb{C})$ for all $s \in \mathbb{R}_{\geq 0}$; by (4.31) they are contained in the closures of the adjoint $\mathcal{G}_{\mathbb{C}}$-orbits $\mathcal{O}_{V_a}$ of V_a. By the above construction of local solutions to the flow equations, for regular orbits $\mathcal{O}_{V_a}$ the map $Z_a(s) \mapsto Z_a(0)$ establishes a bijection

$$\mathcal{M} \;\simeq\; \mathcal{O}_{V_1} \times \mathcal{O}_{V_2} \tag{4.33}$$

which preserves the holomorphic symplectic structures. However, as discussed in [32,31], the orbits $\mathcal{O}_{V_a}$ are generally not regular and their closures generically coincide with nilpotent cones consisting of nilpotent Lie algebra elements; that such singular loci of fields arise is evident from the solutions we found to the equivariance constraints in terms of graded connections, for which the Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ are given by nilpotent matrices in $\mathfrak{gl}(k, \mathbb{C})$, i.e. $\big(\phi^{(1)}\big)^{m_1+1} = 0 = \big(\phi^{(2)}\big)^{m_2+1}$ [10].

The moduli space $\mathcal{M}$ also parameterizes certain BPS configurations of D-branes wrapping the conifold in Type IIA string theory. For this, we recall that the Hermitian Yang–Mills equations (4.11) arise as BPS equations for the (topologically twisted) maximally supersymmetric Yang–Mills theory in six dimensions, which is obtained by (naive) dimensional reduction of ten-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory to $C(T^{1,1})$, see e.g. [48]. In this way the equations (4.11) describe BPS bound states of D0–D2–D6 branes on the conifold, and the moduli space $\mathcal{M}$ parameterizes spherically symmetric and equivariant configurations thereof. In this context, the singularities of the moduli space $\mathcal{M}$ of Spin(4)-equivariant instantons corresponding to non-regular nilpotent orbits is reminiscent of those of the moduli spaces of Hermitian Yang–Mills instantons on the (resolved) conifold which are equivariant with respect to the maximal torus of the SU(3) holonomy group, see e.g. [48,49].

On a more speculative front, we recall that moduli spaces of solutions to the ordinary Nahm equations with Kronheimer's boundary conditions also appear as Higgs moduli spaces of supersymmetric vacua in $\mathcal{N}=4$ supersymmetric Yang–Mills theory on the half-space $\mathbb{R}^{1,2}\times\mathbb{R}_{\geq 0}$ with generalized Dirichlet boundary conditions [50]; these boundary conditions are realized by brane configurations in which D3-branes transversally intersect D5-branes at the boundary of $\mathbb{R}_{\geq 0}$, which is the simple pole at $s=0$ of the solutions to the Nahm equations. The flow equations in this case govern the evolution of the Higgs fields of the $\mathcal{N}=4$ gauge theory along the direction $s\in\mathbb{R}_{\geq 0}$, which represent the transverse fluctuations of the D3-branes. It would be interesting to determine whether the generalized Nahm equations (4.22)–(4.23) can be derived analogously in terms of intersecting (pairs of) D3-branes and D5-branes, with corresponding supersymmetric boundary conditions in the worldvolume gauge theory, and hence if the instanton moduli space $\mathcal{M}$ also parameterizes half-BPS states of certain D-brane configurations in type II string theory.

5. Conclusions

In this paper we examined dimensional reduction of Spin(4)-equivariant gauge theory over the coset space $T^{1,1}$ and characterized the compatible gauge connections in terms of representations of certain quivers. Special emphasis was placed on a comparison with the quiver gauge theory obtained from dimensional reduction over the Kähler coset space $\mathbb{C}P^1\times\mathbb{C}P^1$ [10], whose quiver representations are included as special solutions in the more general framework over $T^{1,1}$. We showed that the Higgs fields depend on only one combined quantum number $c_{i\alpha}$ rather than on two individual monopole charges separately. In the corresponding quivers we find more general arrows than the expected vertex loop modifications of the rectangular $A_{m_1}\oplus A_{m_2}$ quiver. In addition, this feature suggests an interpretation of the quiver gauge theory as that of the double of an $A_{m_1+m_2}$ quiver with suitable combinations of Higgs fields including multiplicities. The generic occurrence of doubles of quivers resembles the situation which occurs in dimensional reduction over quasi-Kähler coset spaces [39].

We studied the dimensional reduction of Yang–Mills theory and also compared it to that associated to $\mathbb{C}P^1\times\mathbb{C}P^1$ [10]. To study the Higgs branch of vacua of the quiver gauge theory, we made use of the special geometric structure of Sasaki–Einstein manifolds and formulated a generalized instanton equation on the metric cone $C(T^{1,1})$ by considering the Hermitian Yang–Mills equations. It was shown that the quiver gauge theory on the Kähler manifold $\mathbb{C}P^1\times\mathbb{C}P^1$ (for the correct fixed values of the radii) is contained as an instanton solution in the more general $T^{1,1}$ framework. The description of the moduli space of Hermitian Yang–Mills instantons led to Nahm-type equations, which we treated in terms of Kähler quotients and (nilpotent) adjoint orbits, and argued to have a natural interpretation in terms of BPS states of D-branes on the conifold.

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Appendix A. Connections and curvatures

A.1. Connections on $T^{1,1}$

The structure equations (2.11) of the coset space $T^{1,1}$, with parameters fixed by the Sasaki–Einstein condition in (2.15), can be expressed as

$$\mathrm{d}\begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \end{pmatrix} = \begin{pmatrix} 0 & 2\,\mathrm{i}\,a + \frac{1}{2}e^5 & 0 & 0 & -e^2 \\ -2\,\mathrm{i}\,a - \frac{1}{2}e^5 & 0 & 0 & 0 & e^1 \\ 0 & 0 & 0 & -2\,\mathrm{i}\,a + \frac{1}{2}e^5 & -e^4 \\ 0 & 0 & 2\,\mathrm{i}\,a - \frac{1}{2}e^5 & 0 & e^3 \\ e^2 & -e^1 & e^3 & -e^4 & 0 \end{pmatrix} \wedge \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \end{pmatrix}, \tag{A.1}$$

from which we obtain the connection 1-forms given in (2.20). The curvature tensor $R^\mu_{\ \nu} = \mathrm{d}\Gamma^\mu_{\ \nu} + \Gamma^\mu_{\ \sigma} \wedge \Gamma^\sigma_{\ \nu}$ has the non-vanishing contributions

$$\begin{aligned} &R^1_{\ 2} = 3e^{12} - 2e^{34}\,, \quad R^1_{\ 3} = -e^{24}\,, \quad R^1_{\ 4} = e^{23}\,, \quad R^1_{\ 5} = e^{15}\,, \quad R^2_{\ 3} = e^{14}\,, \\ &R^2_{\ 4} = -e^{13}\,, \quad R^2_{\ 5} = e^{25}\,, \quad R^3_{\ 4} = -2e^{12} + 3e^{34}\,, \quad R^3_{\ 5} = e^{35}\,, \quad R^4_{\ 5} = e^{45}\,, \end{aligned} \tag{A.2}$$

and hence $\mathfrak{so}(5)$ holonomy. Expressing the curvature in components $R^\mu_{\ \nu\lambda\kappa}$ and contracting to $R_{\lambda\kappa} = R^\mu_{\ \lambda\mu\kappa}$ yields the Ricci tensor (2.21).

The structure equations for the holomorphic forms $\Theta^1 = e^1 + \mathrm{i}\,e^2$ and $\Theta^2 = e^3 + \mathrm{i}\,e^4$ are

$$\begin{aligned} \mathrm{d}\Theta^1 &= -2\Theta^1 \wedge \theta^6 + 2\Theta^1 \wedge \theta^5\,, & \mathrm{d}\bar\Theta^1 &= 2\bar\Theta^1 \wedge \theta^6 - 2\bar\Theta^1 \wedge \theta^5\,, \\ \mathrm{d}\Theta^2 &= 2\Theta^2 \wedge \theta^6 + 2\Theta^2 \wedge \theta^5\,, & \mathrm{d}\bar\Theta^2 &= -2\bar\Theta^2 \wedge \theta^6 - 2\bar\Theta^2 \wedge \theta^5\,, \\ \mathrm{d}\theta^5 &= -\tfrac{3}{4}\left(\bar\Theta^1 \wedge \Theta^1 + \bar\Theta^2 \wedge \Theta^2\right), & \mathrm{d}\theta^6 &= \tfrac{3}{4}\left(\bar\Theta^1 \wedge \Theta^1 - \bar\Theta^2 \wedge \Theta^2\right), \end{aligned} \tag{A.3}$$

where we denote $\theta^5 := \frac{3\,\mathrm{i}}{4}\,e^5$ and $\theta^6 := a$. This yields the structure constants

$$\begin{aligned} &f^1_{61} = -2\,, \quad f^1_{51} = 2\,, \quad f^{\bar 1}_{6\bar 1} = 2\,, \quad f^{\bar 1}_{5\bar 1} = -2\,, \quad f^6_{1\bar 1} = \tfrac{3}{4}\,, \quad f^5_{1\bar 1} = -\tfrac{3}{4}\,, \\ &f^2_{62} = 2\,, \quad f^2_{52} = 2\,, \quad f^{\bar 2}_{6\bar 2} = -2\,, \quad f^{\bar 2}_{5\bar 2} = -2\,, \quad f^6_{2\bar 2} = -\tfrac{3}{4}\,, \quad f^5_{2\bar 2} = -\tfrac{3}{4}\,. \end{aligned} \tag{A.4}$$

A.2. Graded connections

For the graded connection (3.51), the non-vanishing contributions to its field strength are given by

$$\begin{aligned} \mathcal{F}^{i\alpha,i\alpha} &= \mathrm{d}\mathcal{A}^{i\alpha,i\alpha} + \mathcal{A}^{i\alpha,i\alpha} \wedge \mathcal{A}^{i\alpha,i\alpha} + \mathcal{A}^{i\,\alpha,i+1\,\alpha} \wedge \mathcal{A}^{i+1\,\alpha,i\,\alpha} + \mathcal{A}^{i\,\alpha,i-1\,\alpha} \wedge \mathcal{A}^{i-1\alpha,i\alpha} \\ &\quad + \mathcal{A}^{i\,\alpha,i\,\alpha+1} \wedge \mathcal{A}^{i\,\alpha+1,i\,\alpha} + \mathcal{A}^{i\,\alpha,i\,\alpha-1} \wedge \mathcal{A}^{i\,\alpha-1,i\,\alpha} \\ &= F^{i\alpha} + D\phi^{(3)}_{i\alpha} \wedge e^5 \end{aligned}$$

$$
\begin{aligned}
&+ \big(\phi^{(1)}_{i+1\,\alpha}\,\phi^{(1)}_{i+1\,\alpha}{}^{\dagger} - \phi^{(1)\,\dagger}_{i\alpha}\,\phi^{(1)}_{i\alpha} - \mathrm{i}\,\phi^{(3)}_{i\alpha} - \tfrac{3}{4}\,c_{i\alpha}\,\mathbf{1}_{k_{i\alpha}}\big)\,\Theta^1 \wedge \bar\Theta^1 \\
&+ \big(\phi^{(2)}_{i\,\alpha+1}\,\phi^{(2)}_{i\,\alpha+1}{}^{\dagger} - \phi^{(2)\,\dagger}_{i\alpha}\,\phi^{(2)}_{i\alpha} - \mathrm{i}\,\phi^{(3)}_{i\alpha} + \tfrac{3}{4}\,c_{i\alpha}\,\mathbf{1}_{k_{i\alpha}}\big)\,\Theta^2 \wedge \bar\Theta^2\ ,
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha} &= \mathrm{d}\mathcal{A}^{i\,\alpha,i+1\,\alpha} + \mathcal{A}^{i\alpha,i\alpha}\wedge\mathcal{A}^{i\,\alpha,i+1\,\alpha} + \mathcal{A}^{i\,\alpha,i+1\,\alpha}\wedge\mathcal{A}^{i+1\,\alpha,i+1\,\alpha} \\
&= D\phi^{(1)}_{i+1\,\alpha}\wedge\bar\Theta^1 \\
&\quad + \big(\phi^{(1)}_{i+1\,\alpha}\,\phi^{(3)}_{i+1\,\alpha} - \phi^{(3)}_{i\,\alpha}\,\phi^{(1)}_{i+1\,\alpha} - \tfrac{3\,\mathrm{i}}{2}\,\phi^{(1)}_{i+1\,\alpha}\big)\,\bar\Theta^1\wedge e^5 = -\Big(\mathcal{F}^{i+1\,\alpha,i\,\alpha}\Big)^{\dagger}\ ,
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\mathcal{F}^{i\,\alpha,i\,\alpha+1} &= \mathrm{d}\mathcal{A}^{i\,\alpha,i\,\alpha+1} + \mathcal{A}^{i\alpha,i\alpha}\wedge\mathcal{A}^{i\,\alpha,i\,\alpha+1} + \mathcal{A}^{i\,\alpha,i\,\alpha+1}\wedge\mathcal{A}^{i\,\alpha+1,i\,\alpha+1} \\
&= D\phi^{(2)}_{i\,\alpha+1}\wedge\bar\Theta^2 \\
&\quad + \big(\phi^{(2)}_{i\,\alpha+1}\,\phi^{(3)}_{i\,\alpha+1} - \phi^{(3)}_{i\alpha}\,\phi^{(2)}_{i\,\alpha+1} - \tfrac{3\,\mathrm{i}}{2}\,\phi^{(2)}_{i\,\alpha+1}\big)\,\bar\Theta^2\wedge e^5 = -\Big(\mathcal{F}^{i\,\alpha+1,i\,\alpha}\Big)^{\dagger},
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha+1} &= \mathcal{A}^{i\,\alpha,i+1\,\alpha}\wedge\mathcal{A}^{i+1\,\alpha,i+1\,\alpha+1} + \mathcal{A}^{i\,\alpha,i\,\alpha+1}\wedge\mathcal{A}^{i\,\alpha+1,i+1\,\alpha+1} \\
&= \big(\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}\big)\,\bar\Theta^1\wedge\bar\Theta^2 \ = \ -\Big(\mathcal{F}^{i+1\,\alpha+1,i\,\alpha}\Big)^{\dagger}\ ,
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha-1} &= \mathcal{A}^{i\,\alpha,i+1\,\alpha}\wedge\mathcal{A}^{i+1\,\alpha,i+1\,\alpha-1} + \mathcal{A}^{i\,\alpha,i\,\alpha-1}\wedge\mathcal{A}^{i\,\alpha-1,i+1\,\alpha-1} \\
&= \big(\phi^{(2)\dagger}_{i\alpha}\,\phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)\dagger}_{i+1\,\alpha}\big)\,\bar\Theta^1\wedge\Theta^2 \ = \ -\Big(\mathcal{F}^{i+1\,\alpha-1,i\,\alpha}\Big)^{\dagger}\ .
\end{aligned} \tag{A.9}
$$

In order to evaluate the sum $\mathcal{F}_{\hat\mu\hat\nu}\,\mathcal{F}^{\hat\mu\hat\nu}$, note that the metric on $T^{1,1}$ with respect to the forms $(\Theta^1, \bar\Theta^1, \Theta^2, \bar\Theta^2, e^5)$ reads $g = \delta_{\mu\nu}\,e^\mu\otimes e^\nu = \Theta^1\otimes\bar\Theta^1 + \Theta^2\otimes\bar\Theta^2 + e^5\otimes e^5$. Consequently, one obtains

$$
\begin{aligned}
\mathcal{F}_{\hat\mu\hat\nu}\,\mathcal{F}^{\hat\mu\hat\nu} &= \mathcal{F}_{ab}\,\mathcal{F}^{ab} + 2\big(\mathcal{F}_{a1}\,\mathcal{F}^{a1} + \mathcal{F}_{a\bar1}\,\mathcal{F}^{a\bar1} + \mathcal{F}_{a2}\,\mathcal{F}^{a2} + \mathcal{F}_{a\bar2}\,\mathcal{F}^{a\bar2} + \mathcal{F}_{a5}\,\mathcal{F}^{a5} \\
&\quad + \mathcal{F}_{1\bar1}\,\mathcal{F}^{1\bar1} + \mathcal{F}_{12}\,\mathcal{F}^{12} + \mathcal{F}_{1\bar2}\,\mathcal{F}^{1\bar2} + \mathcal{F}_{15}\,\mathcal{F}^{15} + \mathcal{F}_{\bar12}\,\mathcal{F}^{\bar12} + \mathcal{F}_{\bar1\bar2}\,\mathcal{F}^{\bar1\bar2} \\
&\quad + \mathcal{F}_{\bar15}\,\mathcal{F}^{\bar15} + \mathcal{F}_{2\bar2}\,\mathcal{F}^{2\bar2} + \mathcal{F}_{25}\,\mathcal{F}^{25} + \mathcal{F}_{\bar25}\,\mathcal{F}^{\bar25}\big) \\
&= \mathcal{F}_{ab}\,\mathcal{F}^{ab} + 4g^{ab}\,\big(\mathcal{F}_{a1}\,\mathcal{F}_{b\bar1} + \mathcal{F}_{a\bar1}\,\mathcal{F}_{b1} + \mathcal{F}_{a2}\,\mathcal{F}_{b\bar2} + \mathcal{F}_{a\bar2}\,\mathcal{F}_{b2}\big) + 2g^{ab}\,\mathcal{F}_{a5}\,\mathcal{F}_{b5} \\
&\quad + 8\,\big(\mathcal{F}_{1\bar1}\,\mathcal{F}_{\bar11} + \mathcal{F}_{12}\,\mathcal{F}_{\bar1\bar2} + \mathcal{F}_{1\bar2}\,\mathcal{F}_{\bar12} + \mathcal{F}_{\bar12}\,\mathcal{F}_{1\bar2} + \mathcal{F}_{\bar1\bar2}\,\mathcal{F}_{12} + \mathcal{F}_{2\bar2}\,\mathcal{F}_{\bar22}\big) \\
&\quad + 4\,\big(\mathcal{F}_{15}\,\mathcal{F}_{\bar15} + \mathcal{F}_{\bar15}\,\mathcal{F}_{15} + \mathcal{F}_{25}\,\mathcal{F}_{\bar25} + \mathcal{F}_{\bar25}\,\mathcal{F}_{25}\big)\ .
\end{aligned} \tag{A.10}
$$

Inserting the expressions for the field strength into this term and taking care of the correct index structure leads to the action functional (3.52). Setting $\phi^{(3)} = -\frac{3\,\mathrm{i}}{4}\,\big(\Upsilon^{(1)} + \Upsilon^{(2)}\big)$ then yields

$$
\begin{aligned}
S_{\mathrm{r}} = \frac{16\pi^3}{27}\int_{M^d} \mathrm{d}^d x\,\sqrt{\det(g_{M^d})}\,\sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2}\mathrm{tr}\Big\{&\tfrac14\,F^{i\alpha\,\dagger}_{ab}\,F^{i\alpha\,ab} + \big(D_a\phi^{(1)}_{i\alpha}\big)^{\dagger}\,\big(D^a\phi^{(1)}_{i\alpha}\big) \\
+ \big(D_a\phi^{(2)}_{i\alpha}\big)^{\dagger}\,\big(D^a\phi^{(2)}_{i\alpha}\big) &+ \big(D_a\phi^{(1)}_{i+1\,\alpha}\big)\,\big(D^a\phi^{(1)}_{i+1\,\alpha}\big)^{\dagger} + \big(D_a\phi^{(2)}_{i\,\alpha+1}\big)\,\big(D^a\phi^{(2)}_{i\,\alpha+1}\big)^{\dagger} \\
+ 2\big|\phi^{(1)\,\dagger}_{i\alpha}\,\phi^{(1)}_{i\alpha} &- \phi^{(1)}_{i+1\,\alpha}\,\phi^{(1)}_{i+1\,\alpha}{}^{\dagger} + \tfrac32\,(m_1-2i)\,\mathbf{1}_{k_{i\alpha}}\big|^2 \\
+ 2\big|\phi^{(2)\,\dagger}_{i\alpha}\,\phi^{(2)}_{i\alpha} &- \phi^{(2)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha}{}^{\dagger} + \tfrac32\,(m_2-2\alpha)\,\mathbf{1}_{k_{i\alpha}}\big|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\big|\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha+1}-\phi^{(2)}_{i\,\alpha+1}\,\phi^{(1)}_{i+1\,\alpha+1}\big|^2+2\big|\left(\phi^{(1)}_{i\,\alpha-1}\,\phi^{(2)}_{i\alpha}-\phi^{(2)}_{i-1\,\alpha}\,\phi^{(1)}_{i\alpha}\right)^\dagger\big|^2\\
&+2\big|\phi^{(2)\,\dagger}_{i\alpha}\,\phi^{(1)}_{i+1\,\alpha-1}-\phi^{(1)}_{i+1\,\alpha}\,\phi^{(2)}_{i+1\,\alpha}{}^\dagger\big|^2+2\big|\left(\phi^{(2)}_{i-1\,\alpha+1}{}^\dagger\,\phi^{(1)}_{i\alpha}-\phi^{(1)}_{i\,\alpha+1}\,\phi^{(2)}_{i\,\alpha+1}{}^\dagger\right)^\dagger\big|^2\Big\}\ .
\end{aligned}
\tag{A.11}
$$

This result coincides with the action functional derived in [10] if one rescales the Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ by the factor $\sqrt{3/2}$, which is necessary for the comparison as $\Theta^l=\sqrt{2/3}\,\beta_l$ for $l=1,2$ in the limit $\varphi=0$. Then one sees again that the Sasaki–Einstein condition has fixed the radii to $R_1^2=R_2^2=\frac{1}{6}$.

A.3. Connections on the conifold

From the Sasaki–Einstein condition $\mathrm{d}e^5=-2\omega^3=-2\left(e^{12}+e^{34}\right)$ it follows that the 2-form Ω is closed, as a simple calculation shows

$$
\begin{aligned}
\mathrm{d}\Omega&=\mathrm{d}\left[r^2\left(\omega^3+e^{56}\right)\right]\\
&=2r\,\mathrm{d}r\wedge\omega^3+r^2\left(-2\omega^3\right)\wedge e^6\\
&=2r\,\mathrm{d}r\wedge\omega^3-2r\,\omega^3\wedge\mathrm{d}r\ =\ 0\ .
\end{aligned}
\tag{A.12}
$$

It induces the complex structure with $Je^5=-e^6$, which yields the holomorphic 1-form $\Theta^3:=e^5+\mathrm{i}\,e^6$. The structure equations for the rescaled forms on the cone $\tilde{e}^\mu=r\,e^\mu$ read

$$
\begin{aligned}
\mathrm{d}\tilde{e}^1&=-\tfrac{3}{2r}\,\tilde{e}^{25}-2\,\mathrm{i}\,\tilde{e}^2\wedge a-\tfrac{1}{r}\,\tilde{e}^{16}\ , &\mathrm{d}\tilde{e}^2\ &=\ \tfrac{3}{2r}\,\tilde{e}^{15}+2\,\mathrm{i}\,\tilde{e}^1\wedge a-\tfrac{1}{r}\,\tilde{e}^{26}\ ,\\
\mathrm{d}\tilde{e}^3&=-\tfrac{3}{2r}\,\tilde{e}^{45}+2\,\mathrm{i}\,\tilde{e}^4\wedge a-\tfrac{1}{r}\,\tilde{e}^{36}\ , &\mathrm{d}\tilde{e}^4\ &=\ \tfrac{3}{2r}\,\tilde{e}^{35}-2\,\mathrm{i}\,\tilde{e}^3\wedge a-\tfrac{1}{r}\,\tilde{e}^{46}\ ,\\
\mathrm{d}\tilde{e}^5&=-\tfrac{2}{r}\,\tilde{e}^{12}-\tfrac{2}{r}\,\tilde{e}^{34}-\tfrac{1}{r}\,\tilde{e}^{56}\ , &\mathrm{d}\tilde{e}^6\ &=\ 0\ ,
\end{aligned}
\tag{A.13}
$$

which can be expressed in terms of complex forms as in (4.7). The curvature of the corresponding Levi-Civita connection Γ is given by

$$
R\ =\ \mathrm{d}\Gamma+\Gamma\wedge\Gamma\ =\ \begin{pmatrix} 2\,\mathrm{i}\left(e^{12}-e^{34}\right) & -\left(e^{13}+e^{24}\right)+\mathrm{i}\left(e^{23}-e^{14}\right) & 0\\ \left(e^{13}+e^{24}\right)+\mathrm{i}\left(e^{23}-e^{14}\right) & -2\,\mathrm{i}\left(e^{12}-e^{34}\right) & 0\\ 0 & 0 & 0\end{pmatrix}\ ,
\tag{A.14}
$$

and hence it is valued in $\mathfrak{su}(2)\subset\mathfrak{su}(3)$. This curvature solves the instanton equation and confirms the Ricci-flatness of the metric cone $C(T^{1,1})$ (by contracting the components of R to the Ricci tensor).

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