Singular K3 surfaces with class number 4 and fundamental discriminant

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ABSTRACT

In this thesis, we classify certain singular K3 surfaces over $\mathbb{Q}$. The Shioda-Inose theorem establishes a one-to-one correspondence between the set of isomorphism of singular K3 surfaces over $\mathbb{C}$ and the isomorphism classes of positive definite even oriented lattices of rank 2. As a result, we know that every singular K3 surface can be defined over a number field. Therefore a natural question consists in determining which singular K3 surfaces can be defined over the rational numbers. It is already known that singular K3 surfaces with class number 1 and 2 have a rational model. We prove that singular K3 surfaces with fundamental discriminant $|d| \leq 408$, whose class group is two torsion and class number 4 also has a rational model. To exhibit this, we find a Weierstrass model over $\mathbb{Q}$ for each of the four possible singular K3 surfaces with fundamental discriminant d. To achieve this, we will investigate the obstructions coming from the field of definition of singular K3 surfaces using the theory of lattices and Mordell-Weil lattices as well as of elliptic fibrations and class group theory. Additionally, we will utilize an important technique over finite fields, which consists in using the moduli theory of complex K3 surfaces to get specializations over $\mathbb{Q}$ of families of K3 surfaces X_λ with $\rho(X_\lambda) \geq 19$. Furthermore, we use p-adic multivariable Newton iteration, to solve algebraic equations over $\mathbb{Q}$ simultaneously .

Key words: Elliptic surfaces, K3 surfaces, singular K3 surfaces, Lattices, Mordell-Weil lattices, Weierstrass model.

ZUSAMMENFASSUNG

In dieser Arbeit klassifizieren wir bestimmte (Klassen von) singulären K3 Flächen. Das Shioda-Inose-Theorem stellt eine Eins-zu-Eins-Korrespondenz zwischen der Menge der Isomorphismen von singulären K3 Flächen über $\mathbb{C}$ und den Isomorphieklassen von positiv definiten, orientierten Gittern vom Rang 2 her. Deshalb wissen wir, dass jede singuläre K3 Fläche über einem Zahlkörper definiert werden kann. Daher stellt sich die natürliche Frage, welche singulären K3 Flächen über den rationalen Zahlen definiert werden können. Es ist bereits bekannt, dass singuläre K3 Flächen mit Klassenzahl 1 und 2 ein rationales Modell haben. Wir beweisen, dass singuläre K3 Flächen mit fundamentaler Diskriminante $|d| \leq 408$ und Klassenzahl $h(d) = 4$ ebenfalls ein rationales Modell haben.Um dies zu zeigen, finden wir ein Weierstraß-Modell über $\mathbb{Q}$ für jede der vier möglichen singulären K3-Flächen mit Fundamentaldiskriminante d haben. Um dies zu erreichen, untersuchen wir die Obstruktionen des zugrundeliegenden Körpers von singulären K3 Flächen, indem wir die Theorie der Gitter und Mordell-Weil-Gitter sowie der elliptischen Fibrationen und der Klassengruppentheorie verwenden. Darüber hinaus werden wir eine wichtige Technik über endlichen Körpern verwenden, indem wir die Modulitheorie der komplexen K3 Flächen verwenden, um Spezialisierungen über $\mathbb{Q}$ von Familien von K3-Flächen X_λ mit $\rho(X_\lambda) \geq 19$ zu erhalten. Des Weiteren verwenden wir die p-adische Mehrvariablen-Newton-Iteration, um algebraische Gleichungen über $\mathbb{Q}$ simultan zu lösen.

Schlagwörter: Elliptische Flächen, K3 Flächen, singuläre K3 Flächen, Gitter, Mordell-Weil-Gitter, Weierstrass-Modell.

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Introduction

K3 surfaces have been one of the central themes in algebraic geometry. They appear naturally in different areas of mathematics such as algebraic geometry, number theory, Lie groups, differential geometry and more, in addition to this, K3 surfaces also appear in modern areas of physics such as particle physics or string theory. In algebraic geometry K3 surfaces are a really special kind of objects because they have a trivial canonical bundle and irregularity zero. They can be considered as a 2 dimensional generalization of elliptic curves.

In this dissertation we investigate certain aspects of the arithmetic of singular K3 surfaces, i.e. K3 surfaces with maximum Picard number over $\mathbb{C}$. In particular we investigate the field of definition of singular K3 surfaces and we find explicit models over $\mathbb{Q}$ for certain singular K3 surfaces. One of the reasons these singular K3 surfaces are interesting to study is that, as established by Shioda-Inose work, they are two-dimensional analogues of elliptic curves with complex multiplication (CM). This will be explored in great detail throughout this thesis.
A classic result states that every elliptic curve with CM is defined over a number field. And given a fixed positive integer $n \in \mathbb{N}$,

$$\#\{E \text{ elliptic curve with CM defined over } K; [K:\mathbb{Q}] < n\}/_{\cong_{\mathbb{C}}} < \infty.$$

Additionally, it is possible to understand the elliptic curves with CM through the use of class field theory. By considering $\mathcal{O}$ the order associated with an elliptic curve with CM, it is a well-established result that there is a one-to-one correspondence between the class group $Cl(\mathcal{O})$ and the isomorphism classes of elliptic curves with $End(E) = \mathcal{O}$.

It is possible in a natural way to obtain an analogous version of these (and many more) results for singular K3 surfaces over $\mathbb{C}$. In groundbreaking work by Shioda and Inose [SI77], they proved that there is a one-to-one correspondence from the isomorphism classes of singular K3 surfaces to the set of isomorphism classes of positive definite even oriented (given by the choice of the order of a base) lattices of rank 2, given an abelian surface A, a Shioda-Inose structure associates with it a K3 surface X, which serves as a 2:1 covering of $Km(A)$—the Kummer surface associated with A. This construction ensures that the transcendental lattices of X and A are isomorphic, denoted as $T(X) \cong T(A)$. From the Shioda-Inose construction, it follows that singular K3 surfaces behave like elliptic curves with complex multiplication. For instance, they are defined over number fields.

Moreover Shafarevich [Sha96] proved a result for singular K3 surfaces with bounded field of definition. Given $n \in \mathbb{N}$

$$\#\{S \text{ singular K3 surface defined over } K : [K : \mathbb{Q}] < n\}/_{\cong_{\mathbb{C}}} < \infty.$$

This result tells us that there is a finite number of singular K3 surfaces defined over $\mathbb{Q}$.

By the results of Shioda-Inose every one of these singular K3 surfaces corresponds to a lattice given by a primitive quadratic form $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. And given a discriminant d, the number of primitive, positive definite quadratic forms of discriminant d is $h(d)$ known as the class number. So given a fundamental discriminant d, there are exactly $h(d)$ singular K3 surfaces with discriminant d (in this case we say that singular K3 surfaces have class number $h(d)$).

A singular K3 surface defined over $\mathbb{Q}$ is associated to an imaginary quadratic field K with class group with at is $Cl(K) \cong (\mathbb{Z}/2\mathbb{Z})^l$. There are 65 imaginary quadratic fields whose class groups are at most two-torsion and their class numbers are 1, 2, 4, 8, 16.

Singular K3 surfaces of class number 1 have models over $\mathbb{Q}$ by lemma 2.2.2, because the corresponding CM elliptic curves have j-invariants in $\mathbb{Q}$. However, for class number 2, the classification problem is more difficult and it was solved by Schütt and Schulze in [SSed]. Then the next question would be, what happens with the singular K3 surfaces with class number 4. Therefore, the main result of this work is as follows.

Theorem. Every singular K3 surface with class number 4 and fundamental discriminant $|d| \leq 408$ whose class group $Cl(d)$ is two torsion, has a model defined over $\mathbb{Q}$.

We will get this result by finding explicit Weierstrass models over $\mathbb{Q}$ for every respective singular K3 surface. In order to find these models over $\mathbb{Q}$, we have to study the fields of definition of singular K3 surfaces. Let X be a singular K3 surface of discriminant d. Take a subfield $L \subset H(d)$ such that X admits a model over some number field $L \subset H(d)$, such L exists by [Sch07b].

1. The degree of the extension $L/\mathbb{Q}$ is a multiple of the number of quadratic forms in the genus of $T(X)$ by [Sch07b].

2. Assuming that $NS(X)$ is generated by divisors defined over L, then the extension $L(\sqrt{d})$ contains the ring class field $H(d)$ by [Sch10].

These obstructions will be approached by working with elliptic fibrations and Weierstrass models. The crucial difficulty for the field of definition of singular K3 surfaces lies in dealing with the complexities of both obstructions above. That is to say, in order to obtain singular K3 surface over a small number field, with given discriminant d and transcendental lattice T, it has to admit a certain Galois action by obstruction 2. In consequence, this complicates the potential surfaces.

As a consequence of the obstruction 1 the genus of the transcendental lattice T (of a singular K3 surface defined over $\mathbb{Q}$) consists of a single class, which in particular implies that for a fundamental discriminant d, the class group $Cl(K)$ for $K = \mathbb{Q}(\sqrt{d})$ is at most two torsion i.e. $Cl(K) \cong (\mathbb{Z}/2\mathbb{Z})^n$, as stated in [Sch07b].

To classify singular K3 surfaces over $\mathbb{Q}$ with small class numbers, we must first perform lattice-theoretic calculations to determine which singular K3 surfaces meet the necessary criteria to be defined over $\mathbb{Q}$ (or other subfields of $H(d)$). This will help us to determine if a singular K3 surface can be defined over $\mathbb{Q}$.

For a fixed fundamental discriminant d such that $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and for each of the 4 elements of $Q \in Cl(d)$, we search for a singular K3 surface X over $\mathbb{Q}$ with $T(X) \cong Q$. In numerous instances, we are able to derive a suitable one-dimensional family of K3 surfaces with high Picard rank X_λ and subsequently specialize it over $\mathbb{Q}$ through computations over finite field. As the class number or absolute value of the discriminant increases, it may no longer be sufficient to consider one-dimensional families in order to find singular K3 surfaces over $\mathbb{Q}$. This necessitates the examination of higher-dimensional families to identify the singular K3 surfaces that occur. Problems related to this topic have been addressed in [ES13], [Elk08].

Structure of the dissertation

The present thesis is divided into three chapters.

Chapter 1 is meant to recall classical results of lattice theory, class field theory, K3 surfaces and elliptic surfaces. This chapter is divided into 4 sections: the first section corresponds to the study of lattices, we will recall basics of ADE lattices and some important results related to root lattices and overlattices of Kondo [Kon20] and Nikulin [Nik80]. In the second section we will recall essential definitions of class field theory such as quadratic forms, quadratic fields and Hilbert class field. In the third section we will review the main objects of study of this work, K3 surfaces and elliptic surfaces. The fourth section is mainly concerned with the study of singular fibers and elliptic fibrations through Tate's algorithm

[Tat75a], which is divided into two cases, multiplicative or additive reduction. Tate's algorithm is of particular importance in this work to find families of K3 surfaces as elliptic surfaces because as an application, it deduces the behavior of Kodaira types of singular fibers. At the end of this section we will recall some basic notions of quadratic twists and Mordell-Weil lattices to finish the chapter.

Chapter 2. We will look into the field of definition of singular K3 surfaces. We will examine the two main obstructions to their field of definition and explore how these obstructions relate to singular K3 surfaces. To begin, we will review some properties of elliptic curves with complex multiplication. This will provide us with a foundation for understanding these obstructions in greater detail. This chapter is divided into three sections. The first section covers the study of the Shioda-Inose structure, singular abelian surfaces, and Kummer surfaces. The second section focuses on the field of definition of singular K3 surfaces and its obstructions, including results from Schütt and Shimada [Sch07b], [Shi09]. The final section reviews the concept of modularity for elliptic curves (including results from [Wil95], [TW95], [Bre+01]) and singular K3 surfaces in [Liv95].

Chapter 3. We will focus on singular K3 surfaces with fundamental discriminant d whose $Cl(d)$ is two torsion and $\#Cl(d) = 4$. In the first section, we will summarize the conditions and obstructions that apply to singular K3 surfaces. After doing this, we will define the problem and outline the steps we will take to solve it. To ensure that the model of a singular K3 surface allows for sufficient Galois actions, we will examine the Galois action on the Néron-Severi group. We will also explore the possible ways in which the Galois action can act, particularly in cases where the surface X has an elliptic fibration. By examining the Galois action on the singular fibers and sections, we can determine conditions that need to be met in our search for singular K3 surfaces. In the last subsection we start our search for lattices for singular K3 surfaces.

For the remainder of chapter 3, we will divide the singular K3 surfaces to be studied based on the rank of the Mordell-Weil group. We will begin by examining the simplest case: singular K3 surfaces with an elliptic fibration and a Mordell-Weil group of rank zero. This type of singular K3 surface is referred to as an extremal K3 surface in [SZ01]. Table 3.3 lists the extremal K3 surfaces that are singular K3 surfaces with fundamental discriminant and class number 4, along with their transcendental lattice and configuration of singular fibers. We will then use this information to identify the corresponding Weierstrass model for some of the extremal K3 surfaces listed in Table 3.3.

Before we begin to work with singular K3 surfaces with Mordell-Weil rank 1, we will need to introduce some techniques over finite fields that will assist us in obtaining models over $\mathbb{Q}$. The first one consists in working with the moduli theory of K3 surfaces and the Weil conjectures. In many cases, for a given fundamental discriminant d and $Q \in Cl(d)$, we can obtain a family of K3 surfaces, X_λ, with $\rho(X_\lambda) \geq 19$, from which we can try to achieve a specialization with the desired discriminant and transcendental lattice. To specialize this family over $\mathbb{Q}$, we will work with reductions of X_λ over $\mathbb{F}_p$ for multiple primes that allow for good reduction. We will then search for specializations of this family over $\mathbb{F}_p$ that fulfill a condition derived from the Lefschetz fixed point formula. By finding enough of these specializations, we can lift them over $\mathbb{Q}$ using Chinese remainder theorem and euclidean algorithm. Then we probe if one of them corresponds to the specialization of X_λ over $\mathbb{Q}$ that we are looking for. The second technique discussed in this chapter is p-adic multivariate Newton iteration. It will assist us in solving systems of equations over $\mathbb{Q}$ that are not directly solvable with Tate algorithm and the previous mentioned technique.

The last two sections of this chapter focus on the study of singular K3 surfaces with Mordell-Weil rank 1 and 2, respectively. In section four, we will develop algorithm 3.4.1. This algorithm takes a fundamental discriminant d with $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and a quadratic form Q in $Cl(d)$ as inputs, and returns a collection of candidates N. If a K3 surface X has $NS(X) \cong U \oplus N$, it will be a singular K3 surface with $T(X) \cong Q$. Table 3.6 lists all the singular K3 surfaces with Mordell-Weil rank 1 that were solved in this project. The table includes information such as the discriminant of the Néron-Severi lattice, the configuration of singular fibers, the transcendental lattice, and the height pairing of the section that generates the Mordell-Weil group. For each of these singular K3 surfaces we found a Weierstrass model of type $y^2 = x^3 + Ax^2 + Bx + C$ with a section $P = (x_p, y_p)$. We have listed the associated Weierstrass models in tables 3.7, 3.8, 3.9, 3.10 and 3.11, which includes the coefficients A, B, C of the Weierstrass model and the first component x_P of the section P.

We obtained rational models for each singular K3 surface with class number 4 and fundamental discriminant $|d| \leq 408$, whose class group is at most two torsion, as singular K3 surfaces with Mordell-Weil rank 0 or 1, with the exception of three cases. One case has discriminant of $d = -195$, another has a discriminant $d = -228$, and the last one has discriminant of $d = -340$. These three cases will be addressed as singular K3 surfaces with Mordell-Weil rank 2 in the final section of this chapter.

For the singular K3 surfaces with class number 4 and fundamental discriminant $|d| > 408$ (with $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$), we have successfully found models over $\mathbb{Q}$ for some of them, namely those with discriminants $-520, -532, -555, -708, -760$. However, we didn't find all the four possible singular K3 surfaces in each discriminant. In this work, we were unable to find rational models for any new singular K3 surfaces with the desired discriminant and transcendental lattice for the remaining fundamental discriminants with class group $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$, which are $d = -795, -1012, -1435$. The task proved to be too challenging due to the large size of the discriminants.

1 | Preliminaries

1.1 Lattices

In this section we recall some definitions and properties about lattices that we are going to use throughout this thesis, as main references for this topic we use: [Nik80], [SS19], [CS99], [Ebe13], [Shi90].

Definitions 1.1.1. A lattice L is a finitely generated free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear pairing

$$L \times L \to \mathbb{Q}, \qquad (x, y) \to \langle x, y \rangle.$$

If it is integer valued, then the lattice is called integral. It is called even if $x^2 = \langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. The rank of L is the rank of L as a $\mathbb{Z}$-module.

Let L be a lattice of rank n and take a basis $\{\xi_1, ..., \xi_n\}$ of L. Given $x, y \in L$ we can write

$$x = \sum_{i=1}^{n} x_i \xi_i, \qquad y = \sum_{i=1}^{n} y_i \xi_i \qquad x_i, y_i \in \mathbb{Z}$$

hence we get a bilinear form

$$\langle x, y \rangle = \sum_{i,j}^{r} c_{i,j} x_i y_i, \qquad c_{i,j} = \langle \xi_i, \xi_j \rangle,$$

the matrix $I = (c_{i,j}) = (\langle \xi_i, \xi_j \rangle)$, is called the *Gram matrix* of L, with respect to the basis ξ_i. It is a real symmetric invertible matrix. Various properties of invariants of a lattice can be defined in terms of its Gram matrix I, for example the determinant $\det(L)$ of L is the determinant of any matrix representing the bilinear product $\langle , \rangle$ on L ($\det(L) = \det(I)$) so the value is independent of the choice of basis.
In particular if we choose an orthogonal basis (ξ_i) of the induced for on $L \otimes \mathbb{Q}$, the Gram matrix $I = (c_j)$ is going to be a diagonal matrix, the number of positive (n_+) and negative (n_-) c_j are independent of the choice of orthogonal basis; these invariants constitute the signature (n_+, n_-) of the pairing $\langle \cdot, \cdot \rangle$, with $n = n_+ + n_-$, we call this also the signature

of the lattice. L is called *positive* (respectively *negative*) definite if the signature of L is of the form $(rank(L), 0)$ (resp. $(0, rank(L))$.

Example 1.1.2. The integer lattice U of rank 2 with intersection matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is called the *hyperbolic plane*. It has $|det(U)| = 1$ and signature $(1, 1)$.

Definition 1.1.3. The pairing induces an isomorphism

$$Hom_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee = \{x \in L \otimes \mathbb{Q} \mid \langle x, L \rangle \subset \mathbb{Z}\}$$

with the dual lattice $L^\vee$, $L^\vee$ is a lattice with the same rank as L, so the quotient group $A_L = L^\vee/L$ is a finite, abelian group and is called the discriminant group. Its order is equal to $\mid \det L \mid$.

As A_L is a finite abelian group, the minimum number of generators of A_L is called the length $l(A_L)$ of A_L (the *p-length* $l_p(A_L)$ of A_L is the length of its p-part), note that $l(A_L) \leq rk(L)$. If $A_L = 0$, or equivalently $|det(L)| = 1$, we call the lattice L unimodular.

The discriminant group is equipped with a fractional form

$$f : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}, \qquad \langle \bar{x}, \bar{y} \rangle \to \langle x, y \rangle mod(\mathbb{Z}). \tag{1.1}$$

On an even lattice there is the quadratic form q_L given by

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \qquad \bar{x} \to \langle x, x \rangle mod(2\mathbb{Z}). \tag{1.2}$$

It is called the *discriminant form* of the even lattice L. The discriminant form has an important relation with the genus of a lattice, we will elaborate on it in section 1.1.3. Given two even lattices L and M, we say that $q_L \cong q_M$ if the following diagram commutes:

$$\begin{tikzcd} A_L \arrow[r, "\cong"] \arrow[d, "q_L"] & A_M \arrow[d, "q_M"] \\ \mathbb{Q}/2\mathbb{Z} \arrow[r] & \mathbb{Q}/2\mathbb{Z} \end{tikzcd}$$

1.1.1 Root lattices

In these paragraphs, we consider root lattices, in particular the root lattices of type A, D and E, because they arise naturally in the context of $K3$ and elliptic surfaces.

Definition 1.1.4. Let L be a definite even integral lattice:

- Elements $x \in L$ with $\langle x, x \rangle = \pm 2$ are called root vectors or more simply roots.
- The set of roots of L will be denoted by $\mathcal{R}(L)$.
- A definite even integral lattice is called root lattice if it is generated by roots.

Let V be a finite dimensional vector space, with an inner product $\langle , \rangle$, a *root system* is a finite set R of vectors in V, possessing the following properties: R does not contain the null vector, and it generates V; for every $\alpha \in R$ there exists an element $\alpha^\vee$ of the dual space $V^\vee$ such that $\alpha^\vee(\alpha) = 2$ and such that the endomorphism $s_\alpha : x \to x - \alpha^\vee(x)\alpha$ of V maps R into itself; for all $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Let L be a definite even integral lattice and V be the subspace of $L \otimes \mathbb{R}$ spanned by $\mathcal{R}(L)$. Then the set of roots $\mathcal{R}(L)$ forms a root system. It is known that if a root system is irreducible, then it is a root system of type A_r, D_r or E_r.

Definition 1.1.5. A lattice L of rank r is a root lattice of type $A_r(r \geq 1)$, $D_r(r \geq 4)$ or $E_r(r = 6, 7, 8)$, if there exists a basis $\{\alpha_1, \ldots, \alpha_r\} \subset \mathcal{R}(L)$ of L such that the following holds: for $1 \leq i < j \leq r$ we have the pairing $\langle \alpha_i, \alpha_j \rangle = 0$ unless

(A_r) $\langle \alpha_i, \alpha_j \rangle = -1 \Leftrightarrow i + 1 = j$

(D_r) $\langle \alpha_i, \alpha_j \rangle = -1 \Leftrightarrow i + 1 = j < r$, or $i = r - 2, j = r$

(E_r) $\langle \alpha_i, \alpha_j \rangle = -1 \Leftrightarrow i + 1 = j < r$, or $i = 3, j = r$

The importance of the root lattices of type A_r, D_r and E_r came from their big relation with different areas of math, in particular of our interest, the theory of elliptic surfaces and Mordell-Weil Lattices. The classification of root lattices follows from the following theorem:

Theorem 1.1.6 ([Kon20],Proposition 1.12)**.** Any positive definite even integral root lattice L, is isometric to an orthogonal sum of root lattices of type A_n, D_n or E_n.

We finish this section by enlisting the basic properties of the A, D, E lattices in the following table 1.1, such as determinant, discriminant group and discriminant form (determined on generators of the discriminant group), we refer to [Kon20, Section 1.1] for a precise study of ADE lattices.

L	A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
$\lvert det\rvert$	$n+1$	4	4	3	2	1
A_L	$\mathbb{Z}/n\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	-
q_L	$\frac{n}{n+1}$	$\begin{pmatrix} 1 & 1/2 \\ 1/2 & n/2 \end{pmatrix}$	$\frac{2n+1}{4}$	4/3	3/2	-

Table 1.1: Basic properties of root lattices

1.1.2 Overlattices

Definition 1.1.7. Let L be an even lattice, an overlattice of L is an even lattice L' containing L such that the quotient L'/L is a finite group. We denote by $[L' : L]$ the index of the overlattice, i.e. the index of L as a subgroup of L'.

Definition 1.1.8. Let L be a lattice, sublattice T of a lattice L is called primitive if the quotient L/T is torsion-free.

Lemma 1.1.9 ([Nik80], Section 1.4)**.** Let L be a lattice, and L' be an overlattice of L. Then the following equality holds:

$$\frac{|det(L)|}{|det(L')|} = [L' : L]^2$$

Let L be an even lattice, a subgroup $G \subset A_L$ is called isotropic if $q_L|_G = 0$, the isotropic subgroups of A_L characterize the overlattices of L.

Proposition 1.1.10 ([Nik80], Proposition 1.4.1)**.** There is a 1:1 correspondence between overlattices of L and isotropic subgroups of A_L.

For a lattice L and an overlattice L' given by an isotropic subgroup $G \subset A_L$, the discriminant group of L' is $A_{L'} = G^{\perp}/G$ with the quadratic form reduced to the quotient.

Definition 1.1.11. Let L be a lattice, L is called root-overlattice if one of the following conditions hold:

1. L is an overlattice of a root lattice.
2. L admits a $\mathbb{Q}$-basis given by root vectors on L.
3. The root part L_{root} has the same rank as L.

Otherwise, we say L is not a root-overlattice.

1.1.3 Genus of a lattice

There are some important properties about the genus of a lattice that we are going to need in this thesis, we are going to give a brief review about them, mostly we will be concerned with integral even lattices.

Definition 1.1.12. Two lattices M, N are in the same genus if $M \otimes \mathbb{Z}_p \cong N \otimes \mathbb{Z}_p$ are isometric over the p-adic integers for all primes p and $M \otimes \mathbb{R} \cong N \otimes \mathbb{R}$ are isometric over the real numbers.

The discriminant form and the quadratic form of a lattice have an important relation with the genus of a lattice, and the next two theorems show us their relevance for the rest of this work.

Theorem 1.1.13 ([Nik80], Proposition 1.6.1)**.** Let N be an even integral unimodular lattice, L a primitive non-degenerate sublattice of N and $M = L^{\perp}$. Then the discriminant groups are isomorphic, $A_L \cong A_M$, and the discriminant forms satisfy:

$$q_L = -q_M.$$

Theorem 1.1.14 ([Nik80], Corollary. 1.9.4)**.** The genus of an even lattice L is determined by its signature (n_+, n_-) and discriminant form q_L and vice versa.

When the genus of a lattice L is trivial, i.e. consists of a single class (which is determined uniquely by its signature and discriminant form), we will say that L is unique in its genus.

1.2 Class field theory

In this section we recall some definitions and properties about quadratic forms and class field theory that we are going to use in this thesis, as main references for this topic we use [Cox22], [Shi94], [Lan94].

1.2.1 Quadratic forms and form class group

We will denote a quadratic form in two variables $f(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$ as

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{1.3}$$

and its discriminant $d = b^2 - 4ac$. A quadratic form is primitive if its coefficients a, b, c are relatively prime.

Theorem 1.2.1. Let $d < 0$ be a fixed integer. There is a finite number of classes (up to action of $SL_2(\mathbb{Z})$) of primitive positive definite forms of discriminant d and it is denoted with $h(d)$ which is called the class number.

Let $d \equiv 0, 1 \bmod 4$ be negative, we denote the set of classes of primitive positive definite forms of discriminant d as $Cl(d)$. Dirichlet composition induces a well-defined binary operation on $Cl(d)$ which makes $Cl(d)$ into a finite Abelian group whose order is $h(d)$.
The group $Cl(d)$ is called the form class group to distinguish it from the ideal class group.

Remark 1.2.2. A primitive positive definite form $ax^2 + bxy + cy^2$ is said to be reduced if

$$|b| \leq a \leq c, \text{ and } b \geq 0 \text{ if } |b| = a \text{ or } a = c.$$

An integer $m \in \mathbb{Z}$ is represented by a form $f(x, y)$, if the equation

$$m = f(x, y) \tag{1.4}$$

has an integer solution in x and y. If the x and y in are relative prime in (1.4), we say that m is properly represented by $f(x, y)$. We say that two forms $Q = f(x, y)$ and $Q' = g(x, y)$ are equivalent if there are integers p, q, r, s such that

$$f(x, y) = g(px + qy, rx + sy) \quad \text{with} \quad ps - qr = \pm 1, \tag{1.5}$$

or in another words, there exists a matrix $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $det(M) = \pm 1$, such that $Q' = M^T Q M$. This means that M is in the group of 2×2 invertible integer matrices $GL(2, \mathbb{Z})$, and it follows that equivalence of forms is an equivalence relation. We say that an equivalence is a proper equivalence if $ps - qr = 1$, i.e $M \in SL(2, \mathbb{Z})$.

Lemma 1.2.3. Every positive definite form is properly equivalent to a unique reduced form.

An important method of separating reduced forms of the same discriminant is given by genus theory, the basic idea of genus is due to Lagrange who used quadratic forms to prove Euler and Fermat conjectures.

Definition 1.2.4. Two primitive positive definite forms $f(x, y), g(x, y)$ of discriminant d are in the same *genus* if they represent the same values in $(\mathbb{Z}/d\mathbb{Z})$.

Remark 1.2.5. Equivalent forms are in the same genus, but the converse does not hold. For example, $x^2 + 82y^2$ and $2x^2 + 41y^2$ are in the same genus but not equivalent over $\mathbb{Z}$. In particular each genus consists of a finite number of classes of forms.

In Section 1.1.3, we introduced the concept of the genus of a lattice. When working with rank 2 lattices, these two definitions of lattice coincide. We will further explore this concept in the context of even, positive-definite lattices of rank 2 when examining singular K3 surfaces.

Theorem 1.2.6. Let $f(x, y), g(x, y)$ primitive forms of discriminant $d \neq 0$, then these conditions are equivalent:

- $f(x, y), g(x, y)$ are in the same genus i.e., they represent the same values in $(\mathbb{Z}/d\mathbb{Z})$.
- $f(x, y), g(x, y)$ are equivalent modulo m for all nonzero integers m.
- $f(x, y), g(x, y)$ are equivalent over the p-adic integers $\mathbb{Z}_p$ for all primes p.

A proof of this theorem is found in [Hua82; War51].

1.2.2 Ideal class group

Given a field K we denote by $\mathcal{O}_K$ the ring of algebraic integers of K.

Theorem 1.2.7. Let $\mathcal{O}_K$ be the ring of integers of a number field K, then $\mathcal{O}_K$ is a Dedekind domain.

As one of the most important properties of a Dedekind domain we recall that $\mathcal{O}_K$ has a unique factorization on the level of ideals.

Corollary 1.2.8. If K is a number field, then any nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_K$ can be written uniquely as a product

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

of prime ideals, and the decomposition is unique up to order.

Besides ideals of $\mathcal{O}_K$, we are interested in fractional ideals, which are the non-zero finitely generated $\mathcal{O}_K$-submodules of K. The name fractional comes from the fact that these ideals are of the form $\alpha\mathfrak{a}$ where $\alpha \in K$ and $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ (more details about fractional ideas can be found in [Mar18, Chapter 3]).

Notation 1.2.9. We will denote by I_K the group of all fractional ideals of K, and by P_K the subgroup of of principal fractional ideals (i.e. those of the form $\alpha\mathcal{O}_K$ for some $\alpha \in K$).

Definition 1.2.10. The quotient I_K/P_K is called the ideal class group and it is denoted by $C(\mathcal{O}_K)$.

An important fact is that $C(\mathcal{O}_K)$ is a finite group, but it has a big importance in the context of quadratic fields.

Remarks 1.2.11. We need to mention some of the properties of primes in finite extensions. Let K be a number field and let L be a finite extension of K. If $\mathfrak{p}$ is a prime of $\mathcal{O}_K$, then $\mathfrak{p}\mathcal{O}_L$ is an ideal of $\mathcal{O}_L$, and has a prime factorization

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}.$$

$\mathfrak{P}_i$ are distinct primes of L containing $\mathfrak{p}$ and e_i is an integer number called the ramification index of $\mathfrak{p}$ over $\mathfrak{P}_i$. Each prime $\mathfrak{P}_i$ containing $\mathfrak{p}$ also gives a residue field extension $\mathcal{O}_K/\mathfrak{p} \subset \mathcal{O}_L/\mathfrak{P}_i$, the degree of this extension is called the inertia degree of $\mathfrak{p}$ in $\mathfrak{P}_i$ and it is denoted by f_i.

Theorem 1.2.12. Let $K \subset L$ be a Galois extension of a number field, and $\mathfrak{p}$ be a prime ideal in K then

- The Galois group $Gal(L/K)$ acts transitively on the primes of L containing $\mathfrak{p}$.
- The primes of L containing $\mathfrak{p}$ have the same ramification index e and inertial degree f.

Given a Galois extension $K \subseteq L$ of a number field, we say that an ideal $\mathfrak{p}$ of K ramifies if the ramification index $e > 1$, it is unramified if $e = 1$ and it splits completely if the ramification index and inertia degree $e = 1, f = 1$.

1.2.3 Hilbert class group

The Hilbert class field of a number field K is defined in terms of the unramified Abelian extensions of K. First an extension $K \subset L$ is called abelian if it is Galois and $Gal(L/K)$ is an abelian group. And an extension $K \subset L$ is unramified if it is unramified at all primes.

Theorem 1.2.13. Given a number field K, there is a finite Galois extension L of K such that:

i) L is an unramified Abelian extension of K.

ii) Any unramified abelian extension of K lies in L.

The field L of theorem 1.2.13 is called the *Hilbert class field* of K. The Hilbert class field of K is the maximal unramified Abelian extension of K and due to ii) theorem 1.2.13, the Hilbert class field of a field is unique.

Lemma 1.2.14 ([Cox22] Lemma 5.19)**.** Given a number field K, a Galois extension field L, and prime ideals $\mathfrak{p}$ of K and $\mathfrak{P}$ of L unramified over $\mathfrak{p}$, there exists a unique element $\sigma \in Gal(L/K)$ such that for every element $\alpha \in L$,

$$\sigma(\alpha) \equiv \alpha^{N(\mathfrak{p})} \bmod \mathfrak{P} \tag{1.6}$$

where $N(\mathfrak{p}) = \mid \mathcal{O}_K/\mathfrak{p} \mid$ is the norm of $\mathfrak{p}$

The unique element σ of lemma 1.2.14 is called the *Artin symbol* and it is denoted $((L/K)/\mathfrak{P})$, since it depends on the prime $\mathfrak{P}$ of L.

Remark 1.2.15. As a consequence of lemma 1.2.14, of the uniqueness of the Artin symbol, we have that

$$\left(\frac{L/K}{\sigma(\mathfrak{P})}\right) = \sigma \left(\frac{L/K}{\mathfrak{P}}\right) \sigma^{-1}$$

When $K \subset L$ is an Abelian extension of K, the Artin symbol only depends on the prime $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$ lying under $\mathfrak{P}$, to see this, let $\mathfrak{P}'$, be another prime containing $\mathfrak{p}$, we have $\mathfrak{P}' = \sigma(\mathfrak{P})$ for some $\sigma \in Gal(L/K)$, hence by remark 1.2.15 $((L/K)/\mathfrak{P}) = ((L/K)/\mathfrak{P}')$, since $Gal(L/K)$ is Abelian. It follows that whenever $K \subset L$ is Abelian, the Artin symbol can be written as $\left(\frac{L/K}{\mathfrak{p}}\right)$.

When $K \subset L$ is an unramified extension, the Artin symbol is defined for all the prime ideals of $\mathcal{O}_K$. So we can make use of the basic properties of fractional ideals. Any fractional ideal

$\mathfrak{a} \in I_K$ has a prime factorization

$$\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i^{r_i}$$

then we define the Artin symbol to be the product over the prime ideals, so the Artin symbol help us to define a homomorphism.

Definition 1.2.16. Let $K \subset L$ be an unramified Abelian extension, the map

$$\left(\frac{L/K}{\cdot}\right) : I_K \to Gal(L/K)$$

defines a homomorphism and it is called the *Artin map*.

Theorem 1.2.17. Let $K \subset L$ be the Hilbert class field of a number field K, then the Artin map is surjective, its kernel is the subgroup P_K of principal fractional ideals and the map induces an isomorphism between the ideal class group and $Gal(L/K)$,

$$Cl(\mathcal{O}_K) \xrightarrow{\sim} Gal(L/K).$$

1.2.4 Quadratic fields

To summarize the theory previously sketched, let's apply it to the case of most interest in this thesis, the case of quadratic fields. A quadratic field can be written uniquely in the form $K = \mathbb{Q}(\sqrt{N})$.

Definitions 1.2.18. Let $K = \mathbb{Q}(\sqrt{N})$ where $N \neq 0, 1$ is a square free integer, be a quadratic field, the discriminant of K is defined to be

$$d_K = \begin{cases} N & \text{if } N \equiv 1 \bmod 4 \\ 4N & \text{otherwise} \end{cases} \tag{1.7}$$

The ring of integers of K is given by

$$\mathcal{O}_K = \mathbb{Z}\left[\frac{d_k + \sqrt{d_k}}{2}\right] \tag{1.8}$$

We can see from the definition that $d_k \equiv 0, 1 \text{mod} 4$ and $K = \mathbb{Q}(\sqrt{d_k})$, the quadratic field is completely determined by its discriminant.

It is important to recall the Legendre symbol, let p be an odd prime number, an integer a is a quadratic residue modulo p if it is congruent to a perfect square modulo p and is a quadratic non-residue modulo p otherwise. The Legendre symbol is a function of a and p

defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \bmod p \\ -1 & \text{if } a \text{ is a non-quadratic residue modulo } p \\ 0 & \text{if } a \equiv 0 \bmod p \end{cases} \tag{1.9}$$

Theorem 1.2.19. Let K be a imaginary quadratic field with discriminant $d_K < 0$, $f = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ is a primitive positive quadratic form of discriminant d_K. Then the map

$$Cl(d_K) \to Cl(\mathcal{O}_K), \qquad f = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \to [a, (-b + \sqrt{d_K})/2].$$

induces an isomorphism between the form class group $Cl(d_K)$ and the ideal class group $Cl(\mathcal{O}_K)$.

Putting together Theorems 1.2.17 and 1.2.19, we have that the Galois group $Gal(L/K)$ of the Hilbert class field of an imaginary quadratic field K is isomorphic to the form class group $Cl(d_K)$. Thus the *class* refers to the class of properly equivalent quadratic forms $Cl(d_k)$.

1.3 K3 Surfaces and singular K3 surfaces

In this section we are going to review some fundamental properties of $K3$ surfaces, elliptic fibrations and singular $K3$ surfaces which are the main object of study in this thesis. The main references for this chapter are [Huy16; SS19; Mir89; Bar+04]. We fix an algebraically closed field $k = \bar{k}$, we will mention which results are only valid when k is the field of complex numbers $\mathbb{C}$.

Definition 1.3.1. A $K3$ surface over k is a smooth irreducible projective surface X with trivial canonical bundle and vanishing first cohomology:

$$\omega_X = \mathcal{O}_X \qquad h^1(X, \mathcal{O}_X) = 0.$$

Remark 1.3.2. In Chapter 2 and 3 we will be interested in the field of definition of K3 surfaces, in particular in K3 surfaces defined over number fields. A (geometrically smooth) projective surface X over a number field is a K3 surface if the base change to the algebraic closure $\bar{K}$, $X_{\bar{K}}$ is a K3 surface.

Example 1.3.3. Common examples of K3 surfaces over k of $char(k) \neq 2$ are:

- Double covers of $\mathbb{P}^2$ branched over a smooth sextic,
- Smooth quartics in $\mathbb{P}^3$,

- Smooth intersections of a quadric and a cubic in $\mathbb{P}^4$,
- Kummer surfaces, this is the minimal resolution $\widetilde{A/\langle\pm\rangle}$ of the quotient of an abelian surface by the involution (the -1 map on A).

The *Néron-Severi group* $NS(X)$ is the group of algebraic divisors defined over k modulo algebraic equivalence. For a K3 surface X over a field of characteristic 0 this is a free abelian group whose rank is called the *Picard number* $\rho(X)$ and is an integer such that $1 \leq \rho(X) \leq 20$. On a K3 surface over k algebraic and numerical equivalence coincide, and $NS(X)$ is torsion-free. Equipped with the intersection form, it becomes an even lattice of signature $(1, \rho(X) - 1)$, the Néron-Severi lattice.
Let X be a complex K3 surface. Its second cohomology group $H^2(X, \mathbb{Z})$ is torsion-free, and when endowed with the cup-product form it is isomorphic to the $K3$-lattice

$$\Lambda_{K3} := E_8(-1)^2 \oplus U^3.$$

Here U is the hyperbolic lattice and E_8 is the unique (up to isometry) unimodular, negative definite even rank 8 lattice. The signature of the lattice $H^2(X, \mathbb{Z})$ is $(3, 19)$.

Definition 1.3.4. For any surface X over $\mathbb{C}$, we define the transcendental lattice $T(X)$ as the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$:

$$T(X) = NS(X)^{\perp} \subset H^2(X, \mathbb{Z}).$$

For a complex K3 surface, $T(X)$ is an even lattice with signature $(2, 20 - \rho(X))$.

1.3.1 Singular K3 surfaces

Definition 1.3.5. A complex $K3$ surface X is called singular if its Picard number attains the Lefschetz bound

$$\rho(X) = h^{1,1}(X)$$

In other words a complex $K3$ surface is singular if $\rho(X) = 20$. An important aspect about singular $K3$ surfaces is that they involve no moduli, so the term "singular" should be understood in the sense of exceptional but not non-smooth. In many ways singular K3 surfaces behave like elliptic curves with complex multiplication (CM), i.e. elliptic curves E with $\mathbb{Z} \subsetneq End(E)$.
If X is a singular $K3$ surface, then the transcendental lattice $T(X)$ is a positive definite even oriented (given by choosing the order of a basis) lattice of rank two. Using the intersection form, we will identify the transcendental lattice with a 2×2 matrix $T(X) \Leftrightarrow \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$.

Definition 1.3.6. We will denote by d the discriminant $d(X)$ of a singular K3 surface X, i.e. the discriminant of the intersection form on the Néon–Severi lattice,

$$d = d(X) = disc(NS(X)). \tag{1.10}$$

Example 1.3.7. (Fermat Quartic) A classical example of a singular $K3$ surface S is the Fermat quartic in $\mathbb{P}^3$ defined as

$$S : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

In [PS72] an argument was given which proved that S contains 48 lines and these generated the $NS(S)$, with $disc(NS(S)) = -64$ and also that the intersection form of S is $T_S = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$. However the argument relied on a statement that turned out to be incorrect. The proof was later completed by Cassels in [Cas78].

For a more detailed study of Fermat surfaces, see [SSL10].

1.3.2 Elliptic fibrations

Let X be a $K3$ surface over an algebraically closed field $k = \bar{k}$ of characteristic $\neq 2, 3$ (in roder to avoid quasi-elliptic fibrations).

Definition 1.3.8. A genus one fibration on X consists of a morphism $\pi : X \to \mathbb{P}^1$ whose generic fiber is a smooth curve of genus one over the base. An elliptic fibration is a genus one fibration equipped with a distinguished section $O : \mathbb{P}^1 \to X$ with $\pi \circ O = id_{\mathbb{P}^1}$, which is relatively minimal in the sense that no fibre contains a (-1)-curve.

For a smooth irreducible curve C on a surface X, we can compute C^2 through the adjunction formula : $2g(C) - 2 = C^2 + C.K_X$, in particular on a $K3$ surface the adjunction formula has the form $C^2 = 2(g(C) - 1)$, so any elliptic curve has 0 self-intersection, on the other hand by the Riemann–Roch theorem we have that $dim|C| = g(C)$, then any smooth elliptic curve E on a $K3$ surface induces a genus 1 fibration by [PS72, Theorem 1, section 3] which said that a K3 surface X admits a genus one fibration if and only if there is a divisor $0 \neq D \in NS(X)$ with $D^2 = 0$.
There is a close relation between an elliptic surface $\pi : X \to \mathbb{P}^1$ and its generic fiber denoted E/K (which is an elliptic curve over the function field $K = k(\mathbb{P}^1)$).

Proposition 1.3.9. Let X be an elliptic fibration, there is a one-to-one correspondence between sections of $\phi : X \to \mathbb{P}^1$ and the K-rational points of its generic fiber E.

Proof. For any K-rational point of E, $P \in E(K)$, the closure of P in X is a curve $D \subseteq X$, if we take the restriction of π to D, $\pi|_D : D \to \mathbb{P}^1$ we have a finite, birational map, that induces an isomorphism of curves, then taking the inverse of this map, we have that

$\sigma : \mathbb{P}^1 \to X$ gives a unique section with $Im(\sigma) = D$. In the opposite direction, any section $\sigma : \mathbb{P}^1 \to X$ defines a curve $\sigma(\mathbb{P}^1)$ inside X that meets every fiber transversally in a single point. The curve D extends naturally to the underlying scheme of X by taking the Zariski closure, thus it meets the generic fibre in a K-rational point.

□

As a consequence of proposition 1.3.9, we can identify K-rational points of the generic fiber E/K with sections of the elliptic fibration X and vice versa.

Notation 1.3.10. Following the identification above, we will use the same notation $E(K)$ to denote the group of sections of an elliptic fibration $f : X \to \mathbb{P}^1$, also we will use the terms section and rational points in the same way. For $P \in E(K)$ (a K-rational point) we denote by (P) the curve on X which is the image of the section $P : \mathbb{P}^1 \to X$.

Let X be an elliptic surface, the zero section O defines a rational point on the genus one curve $E/k(\mathbb{P}^1)$, the generic fiber turns into an elliptic curve with zero given by O. By the Riemann–Roch theorem, the elliptic curve has a Weierstrass equation. In general we have the following:
Let K be a local field, complete with respect to a discrete valuation v. Let E/K be an elliptic curve, and let:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{1.11}$$

a Weierstrass equation for E/K where Δ denotes its discriminant.

Definition 1.3.11. Let E/K be an elliptic curve. A Weierstrass equation for E as (1.11), is called a minimal Weierstrass equation for E if $v(\Delta)$ is minimized subject to the condition that $a_1, a_2, a_3, a_4, a_6 \in R$ (where $R = \{x \in K : v(x) \geq 0\}$).This minimal value of $v(\Delta)$ is called the valuation of the minimal discriminant of E at v.

Remark 1.3.12. Having an elliptic curve E/K, with a Weierstrass form as (1.11), the substitution $(x, y) \to (u^{-2}x, u^{-3}y)$ leads to a new equation in which a_i is replaced by $u^i a_i$, by carefully choosing the value of u, we can obtain a Weierstrass equation where all coefficients are in R. Then, the discriminant Δ satisfies $v(\Delta) \geq 0$. Finally, since v is discrete, among all Weierstrass equations for E with coefficients in R, we can choose one that minimizes the value of $v(\Delta)$ and we obtain a minimal Weierstrass equation.

Lemma 1.3.13. Any elliptic surface over $\mathbb{P}^1$ admits a globally minimal Weierstrass form with polynomial coefficients $a_i(t) \in k[t]$:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

For any field of characteristic different from 2 , we can write the Weierstrass equation as

$$y^2 = x^3 + Ax^2 + Bx + C. \tag{1.12}$$

We can choose a generator t for the regular functions on $\mathbb{P}^1$ excluding a point, allowing us to work with polynomials in t. If X is an elliptic $K3$ surface, A, B, $C \in k[t]$ have degrees 4, 8, and 12, respectively, while the discriminant Δ has a degree of 24, accounting for the contribution at ∞. According to the identification in Proposition 1.3.9, sections of a fibration $\pi : X \to \mathbb{P}^1$ correspond to $k(t)$-rational points of the equation (1.12). The distinguished zero section is located at the point at infinity $(0 : 1 : 0)$. Let $P = (u(t), v(t))$ denote the section P of π corresponding to the $k(t)$-rational point $(u(t), v(t))$ of equation (1.12). With the above description of (1.12) for a K3 surface, $u(t)$ and $v(t) \in k(t)$ are rational functions of degrees 4 and 6, respectively, considering the contributions of the section P at $t = \infty$ (we rewrite the Weierstrass equation in terms of $s = 1/t$, $x' = x/t^2$, $y' = y/t^3$, the coordinates of the other chart).

Remark 1.3.14. Let X be a K3 surface, denote by F, $O \in NS(X)$ the algebraic equivalence classes of the general fiber F and the zero section O. Their intersection numbers are $O^2 = -2$, $F^2 = 0$ and $O.F = 1$. Thus they span a hyperbolic plane

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = U \subseteq NS(X). \tag{1.13}$$

Since U is unimodular, we have an orthogonal decomposition $NS(X) = U \oplus L$, where L is a negative definite even lattice of rank $\rho(X) - 2$.

Lemma 1.3.15 ([Huy16], Chap. 14, Corollary 3.8)**.** Every complex K3 surface of Picard number at least 12 admits an elliptic fibration.

It is possible to read off an elliptic fibration from the Néron-Severi lattice. A K3 surfaces X admits a genus one fibration if and only if there is an isotropic divisor $0 \neq D \in NS(X)$ (i.e. $D^2 = 0$). If in addition to this, there exists a divisor $E \in NS(X)$ with $(D.E) = 1$, this gives a section of the fibration, and the fibration is an elliptic fibration by [PS72, Theorem 1, section 3].

1.3.3 Singular fibers

Let $f : S \to \mathbb{P}^1$ be an elliptic surface over k, an algebraically closed field of characteristic $p \neq 2, 3$. In this thesis we will always assume that the fibrations have at least one singular fiber to exclude the trivial case of product of two curves.
Now we will give a review of the possible singular fibers of an elliptic surface, given by the Kodaira-Néron classification in [Kod60] and [Nér64].

Notation 1.3.16. Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface and $F_v = \pi^{-1}(v)$ a singular fiber ($v \in k(\mathbb{P}^1)$), we can write F_v as a sum of divisors with multiplicity

$$F_v = \sum_{i=0}^{m_v - 1} \mu_{i,v} \Theta_{v,i}$$

where:

- m_v: is the number of distinct irreducible components in F_v.
- $\Theta_{v,i}$ are the irreducible components $(0 \leq i \leq m_v - 1)$.
- $\mu_{i,v}$ is the multiplicity of $\Theta_{v,i}$ in F_v.
- $m_v^{(1)}$ the number of simple components.

We arrange the irreducible fiber components $\Theta_{v,i}$ of a singular fiber as specified in the following theorems.

Theorem 1.3.17. Let F_v be a singular fiber on an elliptic surface, then

1. There exists a unique simple component of F_v with a simple intersection with the zero section O; it is called the identity component, we can denote it by $\Theta_{v,0}$.
2. If F_v is an irreducible singular fibre (i.e. $F_v = \Theta_{v,0}$), then F_v is either a rational curve with a node or a rational curve with a cusp.
3. If F_v is a reducible singular fibre (i.e. it has several irreducible fiber components), then every component $\Theta_{v,i}$ of F_v is a smooth rational curve with self-intersection number $(\Theta_{v,i})^2 = -2$.

Theorem 1.3.18. (Kodaira[Kod60], Neron[Nér64]) The possible reducible fibers that can appear in an elliptic fibration are classified in the following types

$$I_m, I_b^*, III, IV, II^*, III^*, IV^*$$

for $n > 1$ and $b \geq 0$ (To simplify the notation we write $m_v = m$ and $\Theta_i = \Theta_{v,i}$).

I_m : $F_v = \Theta_0 + \cdots \Theta_{m-1}$, with $m \geq 3$, $(\Theta_i.\Theta_{i+1}) = 1$ for all $i = 0, \cdots m-2$ and $(\Theta_{m-1}.\Theta_0) = 1$, forming a cycle. For $m = 2$, the two components intersect tangentially at two points so that $(\Theta_0.\Theta_1) = 2$.

I_b^* :$F_v = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + \cdots 2\Theta_{b+4}$, with $m = b+5$, $b \geq 0$ and $(\Theta_0.\Theta_4) = (\Theta_1.\Theta_4) = (\Theta_2, \Theta_{b+4}) = (\Theta_3.\Theta_{b+4}) = 1$, and $(\Theta_4.\Theta_5) = \cdots = (\Theta_{b+3}, \Theta_{b+4}) = 1$.

III : $F_v = \Theta_0 + \Theta_1$, with $(\Theta_0.\Theta_1) = 2$, where the two components intersect tangentially at a single point and $m = 2$.

IV : $F_v = \Theta_0 + \Theta_1 + \Theta_2$, with $(\Theta_0.\Theta_1) = (\Theta_1.\Theta_2) = (\Theta_2.\Theta_0) = 1$, all the components meet in a single point and $m = 3$.

II^* : $F_v = \Theta_0 + 2\Theta_7 + 3\Theta_6 + 4\Theta_5 + 5\Theta_4 + 6\Theta_3 + 4\Theta_2 + 2\Theta_1 + 3\Theta_8$, with $m = 9$ where $(\Theta_0.\Theta_7) = (\Theta_7.\Theta_6) = (\Theta_6, \Theta_5) = (\Theta_5.\Theta_4) = (\Theta_4.\Theta_3) = (\Theta_3.\Theta_2) = (\Theta_2.\Theta_1) = (\Theta_3.\Theta_8) = 1$.

III^* :$F_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6 + 2\Theta_7$, with $m = 8$, where $(\Theta_0.\Theta_1) = (\Theta_1.\Theta_2) = (\Theta_2, \Theta_3) = (\Theta_3.\Theta_4) = (\Theta_4.\Theta_5) = (\Theta_5.\Theta_6) = (\Theta_3.\Theta_7) = 1$.

IV^* : $F_v = \Theta_0 + \Theta_1 + 2\Theta_2 + 3\Theta_3 + 2\Theta_4 + \Theta_5 + 2\Theta_6$, with $m = 7$ where $(\Theta_1.\Theta_2) = (\Theta_2.\Theta_3) = (\Theta_3, \Theta_4) = (\Theta_4.\Theta_5) = (\Theta_3.\Theta_6) = (\Theta_6.\Theta_0) = 1$.

For any (Θ_i, Θ_j) not given explicitly, we have $(\Theta_i, \Theta_j) = 0$ if $i \neq j$.

Singular fibers of type I_n in an elliptic fibration are commonly known as "multiplicative fibers," while all other singular fibers are referred to as "additive fibers". The smooth locus $F^{\#}$ of a singular fiber F is obtained by removing all multiple components and nodes (i.e., intersection points of fiber components). Similar to any algebraic group, $F^{\#}$ possesses a normal subgroup $F_0^{\#}$ with a finite quotient group $G(F)$. In general, one may choose the normal subgroup $F_0^{\#}$ as the smooth locus on the identity component Θ_0 of F. This leads to the following:

$$\textbf{Additive: } F_0^{\#} \cong \mathbb{G}_a \qquad F_0/F_0^{\#} \cong G(F)$$

$$\textbf{Multiplicative: } F_0^{\#} \cong \mathbb{G}_m \qquad F_0/F_0^{\#} \cong G(F) \cong (\mathbb{Z}/n\mathbb{Z}).$$

The elements of the group $G(F)$ can be identified with the simple fibre components; $G(F)$ is an abelian group of order equal to the determinant of the restricted Dynkin diagram of the singular fibre.

Notation 1.3.19.

- Among the simple components of the fiber F_v , we call the component Θ_0 meeting the zero section O the identity component. All other simple fiber components are called non-identity components.
- On a fiber of Kodaira type $I_{2n}(n \geq 1)$, the component Θ_n is called the opposite component.
- On a fiber of Kodaira type $I_n^*(n \geq 1)$, the component Θ_1 is called the near (simple) component, while the other simple components Θ_2, Θ_3 are called far components.
- Over a non algebraically closed field k_0, the singular fiber F_v is called "split singular fiber", if all the components are defined over the base field $k_0(v)$, and "non-split singular fiber" otherwise. We can notice that the identity component Θ_0 is automatically $k_0(v)-$rational.

There is an important relation of singular fibers and Dynkin diagrams of root lattices, for a complete reference see [Mir89, Table II.3.1], [SS19, Section 5.5]. We define the dual graph of a singular fiber F_v to be the graph whose vertices are the irreducible components of F_v, and such that vertices Θ_i and Θ_j are joined by one edge for each singular point lying on both of the corresponding irreducible components. And by the restricted dual graph of a reducible fibre F_v ($v \in k(\mathbb{P}^1)$), we mean the graph obtained in the same way as the dual graph, but starting from the $m-1$ non-identity components Θ_i for $(i = 1, ..., m-1)$. It is the subgraph of the dual graph obtained by deleting the component Θ_0.

Proposition 1.3.20. The restricted dual graph of a reducible fibre is the same as the Dynkin diagram of the root lattice of type A, D, E, thus we have

Fiber type F_v	I_n	I_b^*	III	IV	II^*	III	IV^*
Dynkin type T_v	A_{n-1}	D_{b+4}	A_1	A_2	E_8	E_7	E_6

Table 1.2: Dynkin type of singular fibers

And we have an important relation involving the discriminant group A_{T_v} of the lattice T_v , and the number of simple fibre components

$$|A_{T_v}| = det(T_v) = m_v^1$$

Proposition 1.3.21 ([Mir89], Lemma IV.3.3)**.** Let $k = \mathbb{C}$, let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration. Then

$$e(X) = \sum_{t \in \mathbb{P}^1} e(\pi^{-1}(t)) \tag{1.14}$$

where e denotes the Euler characteristic. And we have

$$e(F_v) = \begin{cases} 0 & \text{if } F_v \text{ is smooth .} \\ m_v & \text{if } F_v \text{ is a multiplicative fiber.} \\ m_v + 1 & \text{if } F_v \text{ is a additive fiber.} \end{cases} \tag{1.15}$$

Particularly if X is an elliptic K3 surface, and $L = U^\perp \subset NS(X)$ the orthogonal complement of the hyperbolic plane as in remark 1.3.14, if the root part of L (the sublattice of L generated by the root vectors of L) decomposes as $L_{root} = \bigoplus_{i \in I} L_i$ where the lattices L_i are ADE lattices, we have

$$\sum_{i \in I} e(L_i) \leq 24.$$

1.4 Tate's algorithm

In this section we are going to describe Tate's algorithm which computes, among other things, the reduction type of an special fiber on an elliptic surface given by a Weierstrass equation. We can find the original exposition of this algorithm in [Tat75a] or [Sil94, Chapter IV]. We are going to restrict our attention to the case of perfect fields k with characteristic $\neq 2, 3$, this restriction allows us to work with a Weierstrass equation of the form (1.16).

Notation 1.4.1. 1. R a discrete valuation ring with maximal ideal $\mathfrak{p}$, uniformizing element π, fractional field K, and perfect residue field k.

2. X an elliptic surface over k with generic fiber E/K (an elliptic curve over K), given by the Weierstrass equation

$$E : y^2 = x^3 + a_2 x^2 + a_4 x + a_6. \tag{1.16}$$

3. The discriminant of E is given by

$$\Delta = -27a_6^2 + 18a_2a_4a_6 + a_2^2a_4^2 - 4a_2^3a_6 - 4a_4^3. \tag{1.17}$$

The idea of Tate's algorithm is to start with an arbitrary Weierstrass equation for E/R and manipulate it to obtain a minimal proper regular model.

We will restrict to the case $R = \mathcal{O}_{\mathbb{P}^1, P}$ (i.e., the local ring of functions on $\mathbb{P}^1$ that are regular at the point P). And we will choose the coordinates of $\mathbb{P}^1$ in such a way that $P = 0$.

Algorithm 1.4.2. 1. In order to have a singular fibre on E, the discriminant has to vanish. We can work locally so we fix the parameter $0 \in \mathbb{P}^1$ with the normalized valuation v.

2. If $t \nmid \Delta$, then the special fiber is an elliptic curve, we have a fiber of type I_0 with $m = 1$ (number of components).

   Assuming that t divides Δ, we can perform a change of variables to move the singular point of E to $(0, 0)$. After this transformation, the Weierstrass equation of E takes the form:

$$E : y^2 = x^3 + a_2' x^2 + a_4' t x + a_6' t, \tag{1.18}$$

   where a_2', a_4', a_6' are coefficients that depend on the original coefficients of the Weierstrass equation of E. We have two possibilities:

   - if $t \nmid a_2'$ then the above equation describes a nodal rational curve at t, we have a multiplicative reduction.
   - if $t \mid a_2'$ then the above equation describes a cuspidal rational curve at t. We have an additive reduction.

1.4.1 Multiplicative reduction

If we have that $t \nmid a_2'$ in (1.18), then we have a fiber of type I_n with $n = v(\Delta)$. If $n = 1$, then $(0,0)$ is only a singularity of the fiber but not of the surface, and we have a fiber of type I_1 (a nodal rational curve).

If $n = v(\Delta) > 1$, there is a surface singularity at $(0,0,0)$. In order to resolve this we can make a translation on x, to make $t^m \mid a_4'$ in (1.18) with $m = \left[\frac{n}{2}\right]$ (the integer part of $\frac{n}{2}$) by translating x by t times a polynomial of degree $m-1$. We can see in the equation (1.17) from the summands $a_2^2 a_4^2$ and $4a_2^3 a_6$ of Δ that $v(\Delta) = n$ is equivalent to $v(a_6) = n$ and $v(a_4) = \left[\frac{n}{2}\right] + 1$. We have an equation of E of shape:

$$E : y^2 = x^3 + a_2''x^2 + a_4''t^{m+1}x + a_6''t^n.$$

Now we have to do a succession of m blow-ups of the surface at $(0,0)$, after every of the first $(m-1)$ blow ups the exceptional divisor locally has the form

$$y^2 = a_2''(0)x^2.$$

Every one of the first $m-1$ blow-ups introduces two rational components, which are conjugated over $k(\sqrt{a_2''(0)})/k$. More precisely, if $\sqrt{a_2''(0)} \in k$ we have a split multiplicative reduction over k, because $a_2''(0)$ is a square and the two rational components $y = \pm\sqrt{a_2''(0)}x$ are defined over k, if $\sqrt{a_2''(0)} \notin k$ we have a non-split multiplicative reduction (for $n > 2$).

After each blow-up $(1, ..., m-2)$, a surface singularity persists at $(0,0)$. In response, we proceed with an additional blow-up at $(0,0)$, specifically at the intersection point of the two most recently created exceptional curves.
After the final blow-up, so the local equation of the special fibre is

$$y^2 = a_2''(0)x^2 + (a_6''/t^{2m})(0)$$

if $n = 2m$, the local equation of the special fiber has only one rational component and,

$$y^2 = a_2''(0)x^2$$

if $n = 2m+1$, so that it has two rational components.

After performing the prescribed blow-ups, the surface becomes locally smooth at $(0,0)$. Specifically, upon completion of the final blow-up, the equation describing the surface near $(0,0)$ is nonsingular. Upon combining all of the blow-ups, $n-1$ rational components with self-intersection -2 are added to the surface. We can verify the intersection points of these rational components in the charts obtained after each blow-up, resulting in a cycle with n

rational components. In this way, we obtain a fiber of type I_n composed of n irreducible components.

1.4.2 Additive reduction

If $t \mid a_2'$ in (1.18), then the corresponding Weierstrass equation becomes

$$E : y^2 = x^3 + a_2'tx^2 + a_4'tx + a_6't. \tag{1.19}$$

Furthermore, we have $v(\Delta) \geq 2$, and $(0, 0)$ is a surface singularity when $v(\Delta) > 2$ (since from (1.17), if $t \mid a_2, a_4, a_6$, then $t^2 \mid \Delta$). The additive reduction fibers have types $II, III, IV, I_n^*, II^*, III^*, IV^*$.

1. If $t \nmid a_6'^2$, then we have a fiber of type II (a cuspidal rational curve) with $v(\Delta) = 2$ and one irreducible component.

2. Now if we assume $t^2 \mid a_6$ and $t^2 \nmid a_4$, we have an equation of shape

   $$E : y^2 = x^3 + a_2''tx^2 + a_4''\pi x + a_6''t^2$$

   then we have that $v(\Delta) = 3$, so $(0, 0)$ is a surface singularity, in order to resolve this, we make a blow up at $t = x = y = 0$ (and choose the chart that contains all components of the exceptional divisor of the blow up), we obtain that the $y^2 = \pi x^3 + a_2''tx^2 + a_4''x + a_6''$. Taking $t = 0$ to obtain that the exceptional divisor locally has the form

   $$(y^2 - a_4''(0)x - a_6''(0)) = 0$$

   thus the blow-up of E consist of a rational curve of degree two, meeting the strict transform of the cuspidal curve tangentially in one point, then we have a fiber of type III, with $v(\Delta) = 3$ and two irreducible components.

3. If we assume now that $t \mid a_4'$ but $t^2 \nmid a_6'$ in equation (1.19), we have an equation of shape

   $$E : y^2 = x^3 + a_2''tx^2 + a_4''t^2a_4x + a_6''t^2.$$

   Carrying out the same process as in the previous case. We obtain that the exceptional divisor locally has the form

   $$(y^2 - a_6''(0)) = 0$$

   consisting of two lines conjugate over $k(\sqrt{a_6''(0)})$ (when $\sqrt{a_6''(0)} \notin k$), meeting the strict transform of the cuspidal curve in one point. Then we have a fiber of type IV, with $v(\Delta) = 4$, and three irreducible components.

4. Now we assume that $t^2 \mid a_6'$ in (1.19). We obtain an equation with the form

   $$E : y^2 = x^3 + a_2''tx^2 + a_4''t^2x + a_6''t^3.$$

As before, we obtain a generic fiber $E' : y^2 = tx^3 + tx^2a_2'' + ta_4''x + ta_6''$, and the exceptional divisor locally takes the form:

$$y^2 = 0,$$

which consists of a double line that intersects the strict transform of the cuspidal curve in one point. However, the surface is still singular at $(0, 0)$, specifically at the intersection point of the double line and the strict transform of the cuspidal curve. To continue the desingularization process, we perform a blow-up along the double line $t = y = 0$ to E'. The exceptional divisor of this blow-up takes the form

$$P(x) = x^3 + a_2''(0)x^2 + a_4''(0)x + a_6''(0) = 0 \tag{1.20}$$

so the blow-up is given by gluing the strict transform of the cuspidal curve, the previous double line $y^2 = 0$, and the exceptional divisor E'. We now have three cases to consider, depending on the number of distinct roots in $\bar{k}$ of the polynomial (1.20).

- If $P(x)$ has three distinct roots in $\bar{k}$, then the exceptional divisor E' consists of three distinct rational lines, and our special fiber consist of a double line $y^2 = 0$ together with four rational components intersecting the double line. This indicates that we have a fiber of type I_0^* with $v(\Delta) = 6$, as stated in 1.3.21, and a total of five irreducible components.
- If $P(x)$ has one simple root and one double root, there is one singularity at the intersection with the double line that we need to resolve. We can make a translation over x to assume that the double root of $P(x)$ is located at $x = 0$, so that $P(x) = x^2(x + a_2''(0)) = 0$. In this case, we have $t^2 \mid a_4'$ and $t^3 \mid a_6'$, which means that the Weierstrass equation takes the form

$$E : y^2 = x^3 + a_2''tx^2 + a_4''t^3x + a_6''t^4. \tag{1.21}$$

  The exceptional divisor after this blow-up locally has the form $y^2 - a_6''(0) = 0$. If this rational quadratic equation has distinct roots in $\bar{k}$, then our special fiber consists of two double lines and four rational lines, and we have a fiber of type I_1^* with $v(\Delta) = 7$ and $m = 6$.
- If this exceptional divisor consists of a double line, we make a translation on y, allows to take the double root to be $y^2 = 0$, this implies $t^4 \mid a_6'$, we make another blow-up along this line and get an exceptional divisor with the form $a_2(0)x^2 + a_4(0)x + a_6(0)$. If it consists of two simple lines over $\bar{k}$, then we finish the desingularization and we have a fiber of type I_2^*, with $v(\Delta) = 8$ and $m = 7$.
- This procedure will eventually finish, since Δ has finite vanishing order every-

where. And at each two steps of the algorithm we force the coefficients a'_2, a'_4 and a'_6 on equation (1.19) to be divisible by an additional power of t and also for Δ.

If we have $v(\Delta) = n + 6$, we have a fiber of type I_n^* with $n + 5$ irreducible components.

5. If $P(x)$ has a triple root over $\bar{k}$, we can make a translation in x, to locate the triple root of $P(x)$ at $x = 0$. If we have the conditions $t^2 \mid a_2, t^3 \mid a_4, t^2 \mid a_6$ but $t^4 \nmid a_6$ in (1.16), we get a Weierstrass equation of the form

$$E : y^2 = x^3 + a''_2 t^2 x^2 + a''_4 t^3 x + a''_6 t^4.$$

Resolving this singularity, we introduce another two double lines, and two simple lines. And we get a IV^* fiber with 7 irreducible components and $v(\Delta) = 8$.

6. If we have the conditions $t^2 \mid a_2, t^3 \mid a_4, t^5 \mid a_6$ but $t^4 \nmid a_4$ in (1.16), we get a Weierstrass equation as

$$E : y^2 = x^3 + a''_2 t^2 x^2 + a''_4 t^3 x + a''_6 t^5.$$

After solving the singularity we are going to get a fiber of type III^* with 8 irreducible components and $v(\Delta) = 9$.

7. Now if we have the conditions $t^2 \mid a_2, t^4 \mid a_4, t^5 \mid a_6$ but $t^6 \nmid a_6$ in (1.16) , resolving the singularity, we get a fiber of type II^* with 9 irreducible components and $v(\Delta) = 10$. With Weierstrass equation:

$$y^2 = x^3 + a''_2 t^2 x^2 + a''_4 t^4 x + a''_6 t^5.$$

8. Finally if $t^2 \mid a_2, t^4 \mid a'_4, t^6, \mid a_6$ in (1.16), the Weierstrass equation was not minimal, and we can make a substitution $(x, y) = (t^2 x', t^3 y')$. We can notice each time we pass through this step, the original discriminant will be multiplied by t^{-12}, and it will end eventually since Δ has a finite order.

1.4.3 Quadratic twist

Let K a field, assuming that $char(K) \neq 2$, an elliptic curve over K has Weierstrass equation of the form

$$E : y^2 = x^3 + Ax^2 + Bx + C,$$

and associated with an elliptic curve we have the discriminant Δ and the $j-$invariant. An important property of the $j-$invariant is that it remains constant under transformations that preserve the given shape of Weierstrass form.

Definition 1.4.3. The only change of variables preserving the short Weierstrass equation $y^2 = x^3 + Bx + C$ is

$$x \to ux \quad \text{and} \quad y \to u^{3/2}y \quad \text{for some } u \in K^* \text{ a non-square} \tag{1.22}$$

this is called a quadratic twist. Usually we only consider $u \notin K^2$.

In particular, this will apply to any elliptic curve over the function field $K = k(C)$ of an algebraic curve C. If $char(k) \neq 2$, from (1.22) we can understand the quadratic twist at $u \in \bar{K}$ in an elliptic curve E/K with Weierstrass equation $E : y^2 = x^3 + Ax^2 + Bx + C$ as:

$$E : y^2 = x^3 + uAx^2 + u^2Bx + u^3C \tag{1.23}$$

This has the following effect on the fibres, both smooth and singular:

- The fiber at $v \in \bar{k}(C)$ remains the same if and only if u has even vanishing order (or pole order) at v, in particular this applies to all points on C where u doesn't have a zero or a pole.
- The fibre at $v \in \bar{k}(C)$ changes if and only if u has odd vanishing or pole order at v, in this case, the fibre changes according to the following pattern:

$$I_n \leftrightarrow I_n^*(n \geq 0), \quad II \leftrightarrow IV^*, \quad III \leftrightarrow III^*, \quad IV \leftrightarrow II^* \tag{1.24}$$

From 1.23 and Tate algorithm we can easily read these conditions.

Lemma 1.4.4. Two elliptic curves over any field K with the same j-invariant and $j \neq 0, 12^3$ are either isomorphic or quadratic twists of each other.

One can find the proof of this lemma in [Sil09]. In particular, we have the following corollary, concerning the quadratic twist and singular fibers of an elliptic surface.

Corollary 1.4.5. Any two elliptic surfaces with the same j-invariant $j \neq 0, 12^3$ have the same singular fibres up to a quadratic twist. In particular, the smooth fibres over some point $v \in k(C)$ are isomorphic as long as we assume that k is algebraically closed.

1.4.4 Mordell-Weil lattices

In this section we will recall some basic concepts of Mordell-Weil lattices, as main references we used [SS19] and [Mir89]. In this section we consider surfaces over an algebraically closed field k.

Let X be an elliptic surface with section and at least one singular fibre over some algebraic curve C defined over k, an algebraically closed field of characteristic $p \geq 0$. The sublattice of $NS(X)$ generated by the general fiber F and its zero section (O) is

$$U = \langle O, F \rangle \cong \begin{pmatrix} -\chi & 1 \\ 1 & 0. \end{pmatrix}$$

This matrix U has determinant -1 and signature $(1, 1)$. It is isometric to $\langle 1 \rangle \oplus \langle -1 \rangle$ if χ is odd, or to the hyperbolic plane $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If χ is even. Since U is unimodular, we have an orthogonal decomposition $NS(X) = U \oplus L$, where L is a lattice of $rank(L) = \rho(X)-2$.

Remark 1.4.6. For any section $P \in E(K)$, we have $P^2 = -\chi(X)$ where $\chi(X)$ is the Euler characteristic of X, by adjunction formula on elliptic surfaces.

Definition 1.4.7. Given an elliptic surface X with fibration $\pi : X \to \mathbb{P}^1$, the trivial lattice $Triv(X)$ is the sublattice of $NS(X)$ spanned by the components of F_v not intersection the zero section and the zero section

$$Triv(X) = \langle O, F \rangle \oplus \bigoplus_{v \in R} T_v.$$

Where T_v denotes the restricted dual graph of a singular reducible fibre F_v (as we saw in secion 1.3.3), and the sum runs over all the singular reducible fibers $R = \{v \in k(\mathbb{P}^1) \mid T_v \neq 0\}$.

Definition 1.4.8. Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration with generic fiber E over $K = k(\mathbb{P}^1)$. The Mordell-Weil group of $E(K)$, is the group of sections of the elliptic fibration, and we denote it by $MW(X)$.

The Néron-Severi lattice contains the information not only of sections but multisections of any degree, while the Trivial lattice includes all the fibre components and the zero section. From the following theorem it follows that modulo the trivial lattice everything can be understood in terms of sections.

Theorem 1.4.9 ([Shi90])**.** Let $f : X \to \mathbb{P}^1$ an elliptic surface, the map $P \to (P)$ mod $Triv(X)$ defines an isomorphism of abelian groups

$$MW(X) \cong NS(X)/Triv(X).$$

From theorem 1.4.9 follows the Shioda-Tate formula:

Corollary 1.4.10 (Shioda-Tate formula)**.** Let $\pi : X \to \mathbb{P}^1$ an elliptic surface, then

$$\rho(X) = 2 + \sum_{v \in R} (m_v - 1) + rank(MW(X)). \tag{1.25}$$

The sum on the right side is running again over all the singular fibers. We have that $rank(MW(X)) \leq \rho(X) - 2$, the equality holds if and only if the elliptic fibration has only irreducible fibers (i.e., all the fibers are either smooth elliptic curves, or nodal or cuspidal rational cubic curves).

Remark 1.4.11. For a reducible fiber F_v located at $v \in k(\mathbb{P}^1)$, we denote by A_v the Gram matrix of non-identity fiber components. By proposition 1.3.20, we have that $det(-A_v) = det(T_v) = m_v^{(1)}$.

Definition 1.4.12. Let $P, Q \in MW(X)$ be two sections such that (P) intersects the component $\Theta_{v,i}$ of a singular fiber T_v, and (Q) intersects the component $\Theta_{v,j}$ of the same singular fiber. Then we define the local contribution from the singular (reducible) fibre at $v \in k(\mathbb{P}^1)$, denoted by $contr_v(P, Q)$, as follows:

$$contr_v(P, Q) = \begin{cases} -(A_v^{-1})_{i,j} & \text{if } i \geq 1 \text{ and } j \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.26}$$

where the first possibility refers to the (i, j)-entry of the matrix $-A_v^{-1}$. And we set

$$contr_v(P) = contr_v(P, P).$$

In [Shi90], Shioda introduced an embedding of $MW(X)$ into $NS(X_{\mathbb{Q}})$ in order to define a good pairing on $MW(X)$. For any $P \in MW(X)$, they showed that there exists a unique element $\varphi(P) \in NS(X_{\mathbb{Q}})$ that satisfies two conditions: $\varphi(P) \equiv (P) \bmod Triv(X)_{\mathbb{Q}}$ and $\varphi(P) \perp Triv(X)$ and the kernel of this map was $MW(X)_{tors}$. They also showed that this map is a group homomorphism. Then they present the following lemma.

Lemma 1.4.13. For any $P, Q \in MW(X)$, let

$$\langle P, Q \rangle = -(\varphi(P).\varphi(Q)).$$

Then this defines a $\mathbb{Q}$-valued symmetric bilinear pairing on $MW(X)$.

We denote by $(P.Q)$ the intersection number of given sections (P) and (Q).

Theorem 1.4.14 (Height pairing)**.** Let $P, Q \in MW(X)$ be any two sections $P \neq Q$, we have

$$\langle P, Q \rangle = \chi + (P.O) + (Q.O) - (P.Q) - \sum_v contr_v(P, Q). \tag{1.27}$$

And in the case $P = Q$,

$$h(P) = 2\chi + 2(P.O) - \sum_v contr_v(P, P). \tag{1.28}$$

The height paring defines a $\mathbb{Q}$-valued symmetric bilinear pairing in $MW(X)$, which induces the structure of a positive-definite lattice on $MW(X)/MW(X)_{tors}$.

We can find a proof of theorem 1.4.14 in [SS19, section 6.5].

In particular, if X is a K3 surface, the equation (1.27) has the form

$$\langle P, Q\rangle = 2 + (P.O) + (Q.O) - (P.Q) - \sum_v contr_v(P, Q)$$

and the equation (1.28) has the form

$$h(P) = 4 + 2(P.O) - \sum_v contr_v(P, Q)$$

We can write down the explicit values of $contr_v$, if we have two sections P, Q. Supposing that P intersects $\Theta_{v,i}$ and Q intersects $\Theta_{v,j}$ and that $i \geq 1, j \geq 1$. Then the contribution terms are given in Table 1.3.

Type	III^*	IV^*	I_n	I_n^*
T_v	E_7	E_6	A_{n-1}	D_{n+4}
$contr_v(P)$	3/2	4/3	$\frac{i(n-i)}{n}$	$\begin{cases} 1 & \text{if } i = 1 \\ 1 + \frac{n}{4} & \text{if } i > 1 \end{cases}$
$contr_v(P, Q)$	-	2/3	$\frac{i(n-j)}{n}$	$\begin{cases} \frac{1}{2} & \text{if } i = 1 \\ \frac{1}{2} + \frac{n}{4} & \text{if } i > 1 \end{cases}$

Table 1.3: Local contributions from singular fibers

The third line corresponds to $contr_v(P, Q)$ when the sections P and Q intersect the same fiber component (i.e. $i = j$) or $P = Q$ ($contr_v(P)$), the fourth line corresponds to the case when $i < j$ (we can interchange P and Q if it is necessary).

Torsion sections are an important type of section to consider, we will recall some of their basic properties. And as a main reference to this we refer to [Sil94], [SS19] and [Cas67].

Proposition 1.4.15. Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface.

1. Let n be the order of the torsion subgroup $MW(X)_{tors}$. Then

$$n^2 \mid det(Triv(X)) = \prod_{v \in R} det(T_v).$$

2. Let $P \in MW(X)$, then P is a torsion section if and only if $h(P) = 0$, i.e.

$$2\chi + 2(P.O) - \sum_v contr_v(P) = 0.$$

3. For any additive fibre F_v, the torsion subgroup of $MW(X)_{tors}$ injects into the discriminant group of the fibre $T_v^\vee / T_v$.

4. For any torsion element P of order not divisible by $char(k)$, the section P and the zero section O are disjoint, i.e., $(P.O) = 0$. In particular, if $char(k) = 0$, then for any torsion section P, we have that

$$2\chi = \sum_{v \in R} contr_v(P).$$

We conclude this section by presenting the determinant formula for elliptic surfaces, which provides a precise method for computing the determinant of the Néron-Severi lattice.

Corollary 1.4.16 (Determinant formula)**.** Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with $r = rank(MW(X))$, then

$$\det(NS(X)) = (-1)^r \det(Triv(X)) \cdot \det(MW(X)) / \mid MW_{tors}(X) \mid^2 . \tag{1.29}$$

Remark 1.4.17. In the particular case of corollary 1.4.16, where $r = 1$ and the $MW(X)$ is torsion free, the determinant formula can be written as:

$$\det(NS(X)) = -\det(Triv(X)) \cdot h(P)$$

where P is a section that generates the Mordell-Weil group.

2 | Fields of definitions of K3 surfaces and singular K3 surfaces

Throughout this work we are mainly interested in the field of definition of singular $K3$ surfaces. In the first section we are going to recall and study some of the main related results that follow from the ideas of Shioda and Inose in [Ino77], [Shi06], [SI77], as well as some ideas for the proofs of these results, such as the Shioda-Inose structure, Inose's pencil, singular abelian surfaces and Kummer surfaces. In the last sections we will show the obstructions on the field of definition of singular $K3$ surfaces that we need to face, that we can find in [Sch07b], [HS12],[Sch10].

2.1 Arithmetic of singular $K3$ surfaces

On singular $K3$ surfaces the term "singular" is actually referring to exceptional (just like for singular j-invariants of elliptic curves with complex multiplication, which we'll see are closely related).

We denote

$$\mathcal{M} = \left\{ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \mid a, b, c \in \mathbb{Z}, a \geq 0, c \geq 0, 4ac - b^2 \geq 0 \right\}$$

on which $GL(2, \mathbb{Z})$ acts by $(g, M) \to g^T M g$, where $M \in \mathcal{M}$ and $g \in GL(2, \mathbb{Z})$.
The set of isomorphism classes of even positive-definite lattices of rank 2 is equal to

$$\mathcal{L} = \mathcal{M}/GL(2, \mathbb{Z})$$

while the set of isomorphism classes of even positive-definite oriented (given by choosing the order of a basis) lattices of rank 2 is equal to

$$\widetilde{\mathcal{L}} = \mathcal{M}/SL(2, \mathbb{Z}).$$

For any singular K3 surface X, we say that an order basis (t_1, t_2) of the transcendental lattice $T(X)$ is positive if $Im(p_X(t_1)/p_X(t_2)) > 0$, where p_X is the period map of X and

$T(X)$ with a choice of a positive basis is said to be positively oriented (as described in [SI77] by Shioda and Inose)

Theorem 2.1.1. (Shioda-Inose [SI77]) The map that sends a singular K3 surface to its transcendental lattice

$$X \mapsto T(X)$$

induces a one-to-one correspondence from the set of isomorphism classes of singular $K3$ surfaces to the set of isomorphism classes of positive definite even oriented lattices of rank 2.

The injectivity of this map comes from Torelli's Theorem on [PS72] and the surjectivity was proved by Shioda and Inose [SI77] where they first introduce the idea of singular $K3$ surface.

To establish surjectivity, Shioda and Inose initiated with a singular abelian surface A ($\rho(A) = 4$). By analyzing the intersection form of the transcendental lattice $Q = T(A)$, they observed that the Kummer surface of A becomes a singular K3 surface with an intersection form of $2Q$.In order to get a K3 surface with the intersection form Q, Shioda and Inose constructed of a new elliptic fibration on $Km(A)$. Through a quadratic base change to the base curve, they transformed it into another K3 surface denoted as X' with $\rho(X') = 20$ and intersection form $T(X') = Q$.
We are going to study in detail the Shioda-Inose structure. Before this we will recall some properties about elliptic curves with complex multiplication.

2.1.1 Elliptic curves with complex multiplication

We will recall some basic notions of elliptic curves with complex multiplication. Due to their close relation with singular $K3$ surfaces, this will help us understand the behavior of singular $K3$ surfaces. As main references for this topic we use [Sil09], [Sil94] and [Shi94].

Definition 2.1.2. An order $\mathcal{O}$ in a quadratic field K is a subset $\mathcal{O} \subset K$ such that

(i) $\mathcal{O}$ is a subring containing 1.

(ii) $\mathcal{O}$ is a finitely generated $\mathbb{Z}$ module.

(iii) $\mathcal{O}$ contains a $\mathbb{Q}-$basis of K.

Since an order $\mathcal{O}$ is torsion free, (ii) and (iii) are equivalent to $\mathcal{O}$ being a free $\mathbb{Z}$-module of rank 2. The ring of integers $\mathcal{O}_K$ of a field K is always an order in K, and in particular (i) and (ii) of the previous definition implies that for any order $\mathcal{O}$ of K, $\mathcal{O} \subset \mathcal{O}_K$, thus $\mathcal{O}_K$ is the maximal order of K.

Example 2.1.3. For an easy example of an order, consider $\mathbb{Z}[\sqrt{-n}] \subset K = \mathbb{Q}(\sqrt{-n})$.

Notation 2.1.4. We are going to use the following notation:

Q	:	binary quadratic form with discriminant d as in 1.2.1.
$K = Q(\sqrt{d})$	:	quadratic imaginary field associated to Q.
d_K	:	discriminant of K.
$Cl(Q)$	:	class group of Q.
f	:	conductor of Q: $d = f^2 d_K$.
$Cl(\mathcal{O})$	:	class group of $\mathcal{O}$.
$H(d)$	:	ring class field.

Definition 2.1.5. Let E be an elliptic curve over $\mathbb{C}$, then E has complex multiplication (CM) if $\mathbb{Z} \subsetneq End(E)$.

In this case, $End(E)$ is an order $\mathcal{O}$ in some quadratic imaginary field of K. For any order $\mathcal{O}$, there is an isomorphism between the ideal class group $Cl(\mathcal{O})$ and the form class group $Cl(d)$ consisting of positive definite, primitive, integral quadratic forms

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{2.1}$$

with discriminant $d = b^2 - 4ac$ up to the standard action of SL_2. We refer to [Cox22, Chapter 2, section 7] for more details about orders in quadratic fields. To the order $\mathcal{O}$ we can associate the ring class field $H(\mathcal{O}) = K(j(\mathcal{O}))$, that is an abelian extension of K (where $j(\mathcal{O})$ is the j-invariant of the associated lattice). By theorem 1.2.17, we have that $Gal(H(\mathcal{O}_K)/K)$ is isomorphic to $Cl(d)$.
We have a map in the other direction, sending a quadratic form $Q \in Cl(d)$ to a complex torus

$$\psi : Cl(d) \to \{\text{Elliptic curves with CM by } \mathcal{O}\} \tag{2.2}$$
$$Q \to E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \tag{2.3}$$

with

$$\tau = \frac{-b + \sqrt{d}}{2a} \tag{2.4}$$

We have got some of the ideas on the proof of the following proposition.

Proposition 2.1.6. There is a one-to-one correspondence between the class group $Cl(\mathcal{O}) \cong Cl(d)$ and isomorphism classes of elliptic curves with $End(E) = \mathcal{O}$.

Actually, we can say quite a bit more about the j-invariant of an elliptic curve having complex multiplication. As a corollary of the previous proposition we obtain two important properties of elliptic curves with CM.

Corollary 2.1.7.
1. There are only finitely many isomorphism classes of elliptic curves with CM by $\mathcal{O}_K$.
2. Let E be an elliptic curve with CM and $End(E) = \mathcal{O}_K$. Then $j(E)$ is algebraic over $\mathbb{Q}$.

Proof. Part 1 is clear since $Cl(O_K)$ is finite.
Let $\sigma \in End(E)$, then $End(E^\sigma) \cong End(E) \cong \mathcal{O}_K$. It follows from 1 that $\{E^\sigma | \sigma \in Aut(\mathbb{C}/\mathbb{Q})\}$ contains only finitely many isomorphism classes of elliptic curves. Since $j(E^\sigma) = j(E)^\sigma$, the set $\{j(E)^\sigma | \sigma \in Aut(\mathbb{C}/\mathbb{Q})\}$ is finite. It follows that $j(E)$ is algebraic over $\mathbb{Q}$.

□

Now we can start working with abelian surfaces obtained as a product of two CM elliptic curves.

2.1.2 Singular Abelian Surfaces and Kummer surfaces

This section is concerned with complex abelian and Kummer surfaces. In order to understand the Shioda-Inose structure, to sketch the proof of Theorem 2.1.1, we can start with Abelian and Kummer surfaces over $\mathbb{C}$. A singular abelian surface is an abelian surface A over $\mathbb{C}$ with maximal Picard rank $\rho(A) = 4$.

Remark 2.1.8. Let A be an abelian surface. Denote the involution by $i = [-1]$ (that is the inversion map with respect to the group structure), then the quotient A/i has 16 A_1 singularities corresponding to the 2-division points on A. The minimal resolution of A/i is a K3 surface, called Kummer surface and is denoted by $Km(A)$.

If we consider two elliptic curves E, E', their product is an abelian surface $A = E \times E'$ and we can get the Kummer surface $X' = Km(E \times E')$, the Picard rank of A has three possibilities which depend on whether E and E' are isogenous or have complex multiplication (CM):

$$\rho(A) = \begin{cases} 2 & \text{if } E \not\sim E' \\ 3 & \text{if } E \sim E' \text{ without CM} \\ 4 & \text{if } E \sim E' \text{ with CM} \end{cases} \tag{2.5}$$

and the Kummer surface X' has $\rho(X') = \rho(A) + 16$, in particular we have the Shioda-Mitani theorem.

Theorem 2.1.9. (Shioda-Mitani [SM74]) Let $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ be a quadratic form as in (1.3) and

$$\tau_1 = \frac{-b + \sqrt{d}}{2a} \quad \text{and} \quad \tau_2 = \frac{b + \sqrt{d}}{2} \tag{2.6}$$

then E_{τ_1} and E_{τ_2} are isogenous elliptic curves with CM in K, and $A = E_{\tau_1} \times E_{\tau_2}$ is a singular abelian surface with intersection form $Q = T(A)$.

If we take the Kummer surface of the singular Abelian surface constructed on Theorem 2.1.9, it gives us a singular $K3$ surface, however not all singular $K3$ surfaces are Kummer surfaces, because the transcendental lattice of a Kummer surface is always two divisible as an even lattice (as it was shown in [SM74]), we have

$$T(Km(A)) = T(A)(2). \tag{2.7}$$

In order to find a K3 surface X with $T(X) = Q$ we introduce the Shioda-Inose structure, this construction first appeared in [SI77].

2.1.3 Shioda-Inose structure

We will recall some of the ideas of [SM74; SI77], because of their fundamental importance to understand singular K3 surfaces.
Let X be a singular $K3$ surface, the transcendental lattice $T(X)$ is a positive definite even oriented lattice of rank 2, which can be represented by its intersection form $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ up to the action of $SL_2(\mathbb{Z})$. We refer to d, minus the discriminant of Q, as the discriminant of X and we can obtain $K = \mathbb{Q}(\sqrt{d})$ (the quadratic field associated to Q and X).

We saw in the previous section that we can construct a singular abelian surface A such that $T(A) = Q$, however $T(Km(A)) = 2Q$. To obtain a K3 surface with the original intersection form Q, in the singular case, starting with $A = E_{\tau_1} \times E_{\tau_2}$, Shioda and Inose investigated the double Kummer pencil on $Km(A)$, it produces a specific elliptic fibration. Then after a quadratic base change, it produces a singular K3 surface X with a transcendental lattice with the original intersection form $T(X) = Q$.

The previous Abelian surface A and the constructed surface X with a Nikulin involution i (i.e. an involution with eight isolated fixed points that leaves the holomorphic 2-form invariant) have a degree 2 rational map to the surface $Km(A)$, this construction is often called the Shioda-Inose structure.

$$\begin{tikzcd} X \arrow[dr, dashed] & & A \arrow[dl, dashed] \\ & Km(A) & \end{tikzcd}$$

Definition 2.1.10. A complex K3 surface X admits a Shioda–Inose structure if it admits a rational map of degree two to some Kummer surface $Km(A)$ such that $T(X) \cong T(A)$.

Theorem 2.1.11 (Shioda-Inose [SI77])**.** Any singular K3 surface X admits a Nikulin involution whose quotient has a Kummer surface $Km(A)$ as minimal resolution. And particularly any singular K3 surface X admits a Shioda-Inose structure.

The Shioda-Inose structures extend to K3 surface with Picard rank 19, 18 and 17, such that there is a primitive embedding of the transcendental lattice $T(X)$ in U^3. This was studied by Morrison in [Mor84].

The elliptic curves E_{τ_1}, E_{τ_2} of the previous construction can both be defined over the ring class field $H(d)$, since their j-invariants lie in $H(d)$, which is an abelian Galois extension of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$. By the properties of the field of definition of elliptic curves with complex multiplication, the Shioda-Inose construction implies that any singular K3 surface can be defined over a number field.

Corollary 2.1.12. Any complex singular K3 surface can be defined over some number field.

An explicit model for X over $H(d)$ was given by Inose in [Ino77] and by Schütt in [Sch07b] and we will see it in the next section, where we will recall some important result on the fields of definition of singular K3 surfaces.

2.2 Fields of definition

As any complex singular K3 surface can be defined over a number field, we will aim for the model defined over a number field with smallest degree over $\mathbb{Q}$. We will try to classify some singular K3 surfaces such that the field of definition hast fixed degree over $\mathbb{Q}$. First we will recall some important results related to the field of definition of singular K3 surfaces.

Theorem 2.2.1 (Shafarevich [Sha96])**.** Given $n \in \mathbb{N}$. Up to isomorphism over $\mathbb{C}$, there is a finite number of complex singular K3 surfaces that can be defined over a number field of degree at most n over $\mathbb{Q}$.

A basic property of an elliptic curve E is that it can be defined over $\mathbb{Q}(j(E))$. As a consequence of the theorem 2.1.11 singular K3 surfaces behave in a similar way, as can be seen from the following analogous result.

Lemma 2.2.2 (Schütt [Sch07b])**.** Let X be a singular K3 surface with intersection form Q, discriminant d. Let τ_1, τ_2 be as in (2.6). Then X has a model over the ring class field $H(d) = \mathbb{Q}(j(\tau_1), j(\tau_2))$.

Proof. The proof of this lemma was given in [Sch07b], based on the previous construction of Inose in [Ino77]. For a singular K3 surface X, Inose constructed a defining equation of X as a quartic in $\mathbb{P}^3$. Shioda refers to it as Inose's Pencil [Shi06]:

$$y^2 = x^3 + 3\alpha x t^4 + t^5(t^2 - 2\beta t + 1) \tag{2.8}$$

where

$$\alpha^3 = j(\tau_1)j(\tau_2) \qquad \beta^2 = (1 - j(\tau_1))(1 - j(\tau_2)). \tag{2.9}$$

If $\alpha\beta \neq 0$, then a twist of the above fibration defined over the smaller field was exhibited:

$$t \to \beta t \qquad x \to \frac{\beta}{\alpha} x \qquad y \to \sqrt{\frac{\beta^3}{\alpha^3}} y$$

Making $A = \alpha^3$, $B = \beta^2$, the transformation results in a new fibration given by the Weierstrass equation:

$$y^2 = x^3 - 3ABt^4x + ABt^5(Bt^2 - 2Bt + 1)$$

which means our surface X is defined over $\mathbb{Q}(j(\tau_1), j(\tau_2))$. $\square$

Later the following theorem was proved by Schütt and Hulek, extending Lemma 2.2.2 to the Néron-Severi group.

Theorem 2.2.3 (Schütt, Hulek [HS12])**.** Let X be a singular K3 surface of discriminant d, and let $H(d)$ the ring class field. Then X admits a model over $H(d)$ with $NS(X)$ defined over $H(d)$.

The Inose fibration (2.8), allows us to define a singular K3 surface X over a number field. However this field is not necessarily the field with the minimal degree of extension over $\mathbb{Q}$ such that X has a model over the field. We will see in the next chapter cases where the degree of extension of the ring class field is bigger than 1, but the surface can be defined over $\mathbb{Q}$.

2.2.1 Bounds on the field of definition of singular K3 surfaces.

We are going to recall an important result concerning to a lower bound on the degree of the field of definition of a singular K3 surface X. This result comes from lattice theory, and it involves the genus of a lattice. The result was proved by Shimada [Shi09] in the case of fundamental discriminant (i.e. $d = d_k$), and later by Schütt [Sch07b] in the general case.

Theorem 2.2.4 (Schütt, Shimada)**.** Let X be a singular K3 surface with discriminant d over a number field, and $K = \mathbb{Q}(\sqrt{d})$, then

$$\{T(X^\sigma) | \sigma \in Aut(\mathbb{C}/K)\} = \textit{genus of}\, T(X). \tag{2.10}$$

With this result we have two important corollaries for this work.

Corollary 2.2.5. Let X be a singular K3 surface, defined over some number field L such that $K \subset L$.The number of classes per genus divides $[L : K]$.

This gives us a lower bound for the field of definition.

Corollary 2.2.6. Let X be a singular K3 surface such that X is defined over $\mathbb{Q}$, then the genus of $T(X)$ consist of a single class. In particular this implies that the every element $Q \in Cl(K)$ is equivalent to its inverse on the class group, then the class group is at most two-torsion

$$Cl(K) = Gal(H(K)/K) \cong (\mathbb{Z}/2\mathbb{Z})^m. \tag{2.11}$$

With the notation 2.1.4, if we have the quadratic field $K = \mathbb{Q}(\sqrt{d})$, the discriminant of the field $d_K = d$ is a fundamental discriminant, we obtain

$$Cl(d) = Gal(H(d)/K) \cong (\mathbb{Z}/2\mathbb{Z})^m.$$

This corollary is particularly important for this work because it allows us to identify fields and fundamental discriminants where we can find singular K3 surfaces defined over $\mathbb{Q}$.

Remark 2.2.7. When studying singular K3 surfaces with a fixed discriminant $d = d_K$, it is necessary for the discriminant to satisfy the condition that the class group is at most two torsion. There are 65 known imaginary quadratic fields with this property, and by the work of Weinberger [Wei73], there exists at most one more such field. The possible class numbers $h(d_K)$ for the know 65 imaginary quadratic fields will be 1, 2, 4, 8, or 16. In *Disquisitiones*, Gauss listed the 65 discriminants that satisfy this condition, but they were actually discovered earlier by Euler, who called them "convenient numbers".

For the case of class number 1, the answer is given independently by Baker [Bak67], and Stark [Sta67], the only possible fundamental discriminants d_K for class number 1 are $-3, -4, -7, -8, -11, -19, -43, -67$ and -163.

2.2.2 Ring class field action on singular K3 surfaces.

Another important property of the field of definition of a singular surface $K3$ is related to the structure of the Néron-Severi group and the ring class field. This property was discovered by Schütt in [Sch10] for singular $K3$ surfaces where the Néron-Severi group is defined over $\mathbb{Q}$.

Theorem 2.2.8 (Elkies, Schütt [Sch10])**.** Let X be a singular K3 surface of discriminant d and L be a number field such that the divisors generating the $NS(\bar{X})$ are defined over L. Let $H(d)$ be the ring class field of d . Then

$$H(d) \subseteq L(\sqrt{d}). \tag{2.12}$$

Theorem 2.2.8 basically says that when we can define a singular K3 surface over some number field, the Néron-Severi group still carries the structure of the ring class field $H(d)$ through the Galois action on the Néron-Severi group. And by theorem 2.2.3 for any singular K3 surface X it is always possible to define a singular $K3$ surface over $H(d)$ with $NS(X)$ also defined over $H(d)$.

Therefore, Theorem 2.2.8 implies that the field of definition $H(d)$ in Theorem 2.2.3 is close to optimal. However, the problem of classifying singular K3 surfaces over $\mathbb{Q}$ is still open. Even in the case of singular K3 surfaces over $\mathbb{Q}$, we only know that there are finitely many of them by Theorem 2.2.1.

Remark 2.2.9. Theorem 2.2.8 implies that, as the class number $h(d)$ of a fundamental discriminant d increases, the problem of constructing singular K3 surfaces over $\mathbb{Q}$ with discriminant d becomes more challenging. This is because the Galois group of the ring class field $H(d)$ acts non-trivially on the Néron–Severi group $NS(X)$ of such surfaces. In particular, if X admits an elliptic fibration, a large value of $h(d)$ can result in a fibration with a greater number of possible reducible fibers or sections.

The particular case when a singular K3 surface X is defined over $\mathbb{Q}$ with $NS(\bar{X})$ generated by divisors over $\mathbb{Q}$ was studied by Elkies [Elk07].

Theorem 2.2.10. Let X be a singular K3 surface over $\mathbb{Q}$, with $NS(\bar{X})$ generated by divisors over $\mathbb{Q}$. Then X has discriminant d of class number one.

2.3 Modularity

K3 surfaces could be considered the equivalent of elliptic curves in two dimensions, so before we start with modularity of singular K3 surfaces, we will recall the property of modularity of elliptic curves.

2.3.1 Modularity of elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$ and $p \neq 2$ a prime number, we can choose a minimal Weierstrass equation at p (as on definition 1.3.11)

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with $a_2, a_4, a_6 \in \mathbb{Z}$. This Weierstrass model can be reduced modulo p by simply mapping a_i to $\bar{a}_i := a_1(\text{mod } p) \in \mathbb{Z}/p\mathbb{Z} =: \mathbb{F}_p$, this mapping gives a Weierstrass model and defines a curve $\widetilde{E}$ over the finite field $\mathbb{F}_p$. It turns out (the isomorphism class of) this curve $\widetilde{E}/\mathbb{F}_p$ is independent of the choice of minimal Weierstrass model for E. In particular, its number of points over $\mathbb{F}_p$, i.e. $\#\widetilde{E}(\mathbb{F}_p)$ is an invariant of E, so we can define

$$a_p(E) = p + 1 - \#\widetilde{E}(\mathbb{F}_p). \tag{2.13}$$

If $\widetilde{E}/\mathbb{F}_p$ is nonsingular, then we say that E has good reduction at p, otherwise we say that E has bad reduction at p. In the latter case, we say that E has multiplicative reduction if $\widetilde{E}/\mathbb{F}_p$ has a node, and we say that E has additive reduction at p if $\widetilde{E}/\mathbb{F}_p$ has a cusp.

For elliptic curves over $\mathbb{Q}$ the modularity was proven in the Taniyama-Shimura-Weil conjecture by Wiles, Taylor et al. [Wil95], [TW95], [Bre+01].

Theorem 2.3.1. Any elliptic curve over $\mathbb{Q}$ is modular, there exists a newform f of weight 2 with coefficients a_p such that for almost all p, $a_p = a_p(E)$.

Remark 2.3.2. In general, given a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index, a modular form of weight k is a holomorphic function on the upper half plane in $\mathbb{C}$ that satisfies the transformation law

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for any $\gamma \in \Gamma$, and it is holomorphic at infinity.

A newform is a cusp form that is 'new' at a specific level N, where the levels are defined by the nested congruence subgroups $\Gamma_0(N)$ of the modular group. These levels are ordered by divisibility; in other words, if M divides N, then $\Gamma_0(N)$ is a subgroup of $\Gamma_0(M)$. Oldforms for $\Gamma_0(N)$ are the span of all the subspaces obtained from modular forms of the form $g(d\tau)$ where g is a modular form of level M and M is a proper divisor of N, and d divides N/M. For a more detailed definition and study of old and newforms, please refer to [DS05, Section 5.6].
The term 'old' is used for these forms because, although they have level N, they actually originate from a smaller level, namely M. Therefore, from the perspective of level N, they are considered 'old.' The space of newforms represents the orthogonal complement with respect to the Petersson inner product (see [DS05, Section 5.4]) of the space of oldforms. Newforms are genuinely new to level N, hence their name.

By the transformation law, a modular form f always has a Fourier expansion

$$f = f(\tau) = \sum_n a_n q^n \text{ with } q = e^{2\pi i\tau}.$$

In general the twist of a newform $f = \sum a_n q^n$ by a Dirichlet character χ gives us a newform

$$f \otimes \chi = \sum_n a_n \chi(n) q^n$$

with possibly different level. For more details of modular forms we refer to [Shi94].

2.3.2 Modularity of singular K3 surfaces

Definition 2.3.3. Let K be a commutative ring of characteristic $p > 0$. The Frobenius endomorphism $Frob_p$ on K is defined by

$$Frob_p(x) = x^p \text{ for all } x \in K$$

It is possible to induce a Frobenius map on $K[x]$. For any polynomial $f \in K[x]$, let f^p the polynomial obtained from f by raising each coefficient of f to the p^{th} power.

If a singular K3 surface X is defined over a number field L, the transcendental lattice $T(X)$ gives rise to a system of two-dimensional $l-$adic Galois representation ϱ over L. In particular, if X is defined over $\mathbb{Q}$, modularity was proven by Livné.

Theorem 2.3.4. (Livné [Liv95]) Let X be a singular K3 surface over $\mathbb{Q}$ with discriminant d. Then X is modular: There is a newform f of weight 3 with CM by $\mathbb{Q}(\sqrt{d})$ and Fourier coefficients a_p such that for almost all p

$$trace(Frob_p^*) = a_p. \tag{2.14}$$

Here $Frob_p^* = \varrho(Frob)$ is the induced Frobenius map.

If we consider a K3 surface X, then X has good reduction X_p at almost all primes p. This reduction induces an embedding of lattices $NS(X) \hookrightarrow NS(X_p)$. The eigenvalues of $Frob_p^*$ on $H^2(X)$ (the sencond l-adic étale cohomology group) are encoded in the characteristic polynomial $\psi(x)$ of $Frob_p^*$. We can obtain the Fourier coefficients a_p by determining $\psi(x)$. The characteristic polynomial $\psi(x)$ of $Frob_p^*$ can, at least in principle, be computed explicitly using Lefschetz fixed point formula, after counting points over $\mathbb{F}_q$ with $q = p^s$ and $s = 1, ..., 11$. The Lefschetz fixed point formula simplifies to $tr((Frob_p^*)^i(H^2(X))) = \#X(\mathbb{F}_{p^i}) - 1 - p^{2i}$. We will go deeper into this topic in the section 3.3.

We are in a similar situation to the case of elliptic curves, where it is possible to associate to any newform of weight two with Fourier coefficients $a_p \in \mathbb{Z}$ an elliptic curve over $\mathbb{Q}$. we have the property that any newform of weight 3 with Fourier coefficients $a_p \in \mathbb{Z}$ has CM by a result of Ribet [Rib77]. And more important to us, there is a bijection up to twisting of newforms with imaginary quadratic fields whose class group is only two-torsion, given by a classification by Schütt in [Sch09].

Theorem 2.3.5 (Schütt [Sch09]). For a fixed weight $k+1$ there is a bijection between,

$$\left\{ \begin{array}{c} \text{CM newforms of weight } k+1 \\ \text{with rational eigenvalues} \\ \text{up to twisting} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Imaginary quadratic fields } K \\ \text{with class group } Cl(K) \subseteq (\mathbb{Z}/k\mathbb{Z})^g \\ \text{for certain } g \in \mathbb{N} \end{array} \right\}$$

There are 65 known imaginary quadratic fields whose class group is at most two torsion. Their class numbers go from 1 to 16, and the discriminant from -3 to -5460 as we can see in Table 2.1. By a work of Weinberger [Wei73], there exists at most one more such field, under the assumption on Siegel-Landau zeroes of L-series of odd real Dirichlet characters (which would follow from the extended Riemann hypothesis), the known list is complete. We enlist these discriminants d_k and their class numbers.

$h(d_K)$	d_K
1	$-3, -4, -7, -8, -11, -19, -43, -, 67, -163$
2	$-15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148, -187,$ $-232, -235, -267, -403, -427$
4	$-84, -120, -132, -168, -195, -228, -280, -312, -340, -372, -408, -435,$ $-483, -520, -532, -555, -595, -627, -708, -715, -760, -95, -1012, -1435$
8	$-420, -660, -840, -1092, -1155, -1320, -1380, -1428, -1540, -1848, -1995,$ $-3003, -3315$
16	-5460

Table 2.1: Fundamental discriminants whose class groups are at most two torsion

Theorem 2.3.6 (Elkies, Schütt [ES13]). Every known newform of weight 3 with Fourier coefficients $a_p \in \mathbb{Z}$ is associated to a singular $K3$ surface over $\mathbb{Q}$.

The proof of this theorem is constructive, for each of the 65 known imaginary quadratic fields in table 2.1 (whose class groups are at most two torsion), Elkies and Schütt gave a singular K3 surface over $\mathbb{Q}$.

To construct some of these singular K3 surfaces, Elkies and Schütt considered one-dimensional families of K3 surfaces over $\mathbb{Q}$ with Picard rank $\rho \geq 19$. By moduli theory of singular K3 surfaces, such a family has infinitely many specializations with $\rho = 20$ over k but only a finite number of specializations with $\rho = 20$ over $\mathbb{Q}$ (by theorem 2.2.1, there is only a finite number of singular K3 surfaces over $\mathbb{Q}$), then they looked for a specialization over $\mathbb{Q}$ with the desired discriminant. We will apply some of their methods to compute Weierstrass equations too in the next chapters.

Remark 2.3.7. Elkies and Schütt construct a singular K3 surfaces for every one of the 65 imaginary quadratic fields d_K in 2.1, but due to theorem 2.1.1, if we have a discriminants d_K with class number $h(d_K) = 4$, there exist 4 non isomorphic singular K3 surfaces over $\mathbb{C}$ with given discriminant.

3 | Models for singular K3 surfaces with class number 4

In this chapter, we will build on the results from the previous chapter to develop algorithms that allow us to construct explicit models of singular K3 surfaces over $\mathbb{Q}$. Some of these techniques have been previously used to address problems related to the modularity of singular K3 surfaces. For example, in [ES13], Elkies and Schütt showed that every Hecke eigenform of weight 3 with rational eigenvalues is associated with a K3 surface over $\mathbb{Q}$, and they provided explicit equations for singular K3 surfaces over every field $\mathbb{Q}(\sqrt{d})$ with d listed in Table 2.1. We will build upon their work and use these techniques to find explicit models of singular K3 surfaces.

In [Elk08], Elkies computed Shimura curves using K3 surfaces with Néron-Severi rank at least 19, demonstrating interesting techniques for computing Weierstrass equations over $\mathbb{Q}$ for K3 surfaces with large Picard rank. Furthermore, in [EK14], Elkies and Kumar and in [BM08], Beukers and Montanus obtained many explicit defining equations for K3 surfaces with high Picard rank ($\rho \geq 18$). We will make use of some of the techniques from these works to develop algorithms for computing explicit models of singular K3 surfaces.

In the initial section, our focus will be on the main objective of classifying singular K3 surfaces with fundamental discriminant d such that $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$. We aim to identify explicit models of singular K3 surfaces that are defined over $\mathbb{Q}$. In earlier chapters, we have discussed techniques for studying K3 surfaces with elliptic fibrations and utilizing Weierstrass models. In this chapter, we will further explore additional techniques for achieving our goal.

In the second section, we present techniques for finding families of K3 surfaces with high Picard rank, which can be challenging. We draw upon techniques over $\mathbb{F}_p$ from works such as [Vár17] and [Elk06]. One useful technique involves reducing mod p a family X_λ of K3 surfaces with $\rho \geq 19$ and checking for necessary conditions to obtain a specialization of the family with $\rho = 20$ over $\mathbb{Q}$. Another approach is p-adic Newton iteration in several variables.

In the following sections, we will organize our study of singular K3 surfaces with an elliptic fibration, according to the rank of their Mordell-Weil group, which determines the complexity of finding explicit models of these surfaces. We begin with the simplest case of extremal K3 surfaces as described in [SZ01], which have maximal Picard rank and trivial Mordell-Weil group. Next, we turn our attention to singular K3 surfaces with Mordell-Weil rank 1. For most of these surfaces, we will search for families of K3 surfaces with Picard rank 19 and attempt to obtain their specialization over $\mathbb{Q}$ with rank 20. The last and most challenging case that we cover in this work involves calculating Mordell-Weil rank 2 singular K3 surfaces.

3.1 Singular K3 surface with class number 4

The problem of determining the field of definition of a singular K3 surface is a challenging one, and currently, there is no general algorithm to solve it. However, in recent work [Sch07b; Sch10], Schütt provided some useful bounds on the fields of definition for singular K3 surfaces. In particular, in Theorem 2.2.6, we saw that for any singular K3 surface X defined over $\mathbb{Q}$ with discriminant d a fundamental discriminant, the class group $Cl(d)$ is two-torsion, that is,

$$Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^g.$$

In this work, we focus on the problem of determining the field of definition of singular K3 surfaces with a fixed class number, given this additional condition.

Definition 3.1.1. Let X be a singular K3 with discriminant d. We define the class number of X as the class number $h(d)$ of the associated class group $Cl(d)$.

The cases of singular K3 surfaces with class number 1 and 2 have already been solved, as demonstrated in [Sch07b] and [SSed], respectively. Therefore, we will now focus on the next case, namely finding the field of definition of singular K3 surfaces with fundamental discriminant with class number 4 and two torsion.

As we know from Theorem 2.1.1, for every fundamental discriminant d with class number 4, there exist four singular K3 surfaces (up to isomorphism) with the given discriminant. The fundamental discriminants with class number 4 and two torsion can be summarized in Table 3.1 as follows:

$h(d)$	d
4	-84,-120,-132, -168, -195, -228, -280, -312, -340, -372, -408, -435, -483, -520, -532, -555, -595, -627, -708, -715, -760, -795, -1012, -1435

Table 3.1: Fundamental discriminants with class number 4, whose class groups are two-torsion.

Problem 3.1.2. Which singular K3 surfaces with a fundamental discriminant d such that $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$ can be defined over $\mathbb{Q}$?

By Torelli's theorem for singular K3 surfaces, theorem 2.1.1, X correspond to an element $T(X) \cong Q \in Cl(d)$. And by Nikulin's theorem 1.1.13, there is a relation between the discriminant form of the Néron-Severi lattice and the discriminant form of the transcendental lattice

$$q_{NS(X)} = -q_{T(X)}. \tag{3.1}$$

Thus, we must obtain a model over $\mathbb{Q}$ for a singular K3 surface with $q_{NS(X)} = -q_{T(X)}$.

We know that the genus of an integral lattice is determined by its signature and discriminant form, as stated in Theorem 1.1.14. By Corollary 2.2.6, if X is a singular K3 surface defined over $\mathbb{Q}$, the genus of the transcendental lattice $T(X)$ consists of a single class. Therefore, the discriminant form of the Néron-Severi lattice $NS(X)$ uniquely determines $T(X)$.

Remark 3.1.3. As X is a singular K3 surface, by Theorem 2.2.8, the Néron-Severi group still carries the structure of the ring class field $H(d)$ through the Galois action on the Néron-Severi group. In the case of class number 4, by corollary 2.2.6 we have

$$Gal(H(d)/K) \cong (\mathbb{Z}/2\mathbb{Z})^2. \tag{3.2}$$

If X has an elliptic fibration over $\mathbb{Q}$, then the Galois group of the ring class field $H(d)$ acts nontrivially on the Néron-Severi group, and the action occurs on the reducible fibers or sections.

In the next section we are going to see how the Galois action can occur in the Néron-Severi group.

3.1.1 Galois action on the Néron-Severi group

Let X be a singular K3 surface, then a necessary condition for X to be defined over $\mathbb{Q}$ is that $NS(X)$ carries the structure of the ring class field $H(d)$, through a Galois action on it by theorem 2.2.8.

One of the advantages of working with elliptic surfaces is that the Shioda-Tate formula (1.25) reveals that the Néron-Severi group of the surface is generated by horizontal and vertical divisors, which correspond to fiber components and sections respectively by Theorem 1.4.9. This property is particularly useful when studying singular K3 surfaces over $\mathbb{Q}$. According to Theorem 2.2.8, the Galois group of the ring class field $H(d)$ acts nontrivially on the Néron-Severi group $NS(X)$. This means that with a suitable configuration of reducible fibers and sections, the fibration can be preserved under this nontrivial Galois action, which may affect the reducible fibers in various ways.

- A reducible fiber is fixed under the Galois action, but it has type I_n for $n \geq 3$, I_n^* for $n \geq 0$ or IV, IV^*, allowing for an involution of the components (or for I_0^* even an S_3 action) that preserves incidence relations and identity component.
- There are several reducible fibers of the same type lying over points of the base curve $\mathbb{P}^1$ that are Galois conjugate.

When studying a singular K3 surface X over $\mathbb{Q}$ with $rank(MW(X)) > 1$, the Galois action on the Néron-Severi group may become more intricate, since it increases the potential intersections between the sections that contribute to $MW(X)$, and the fiber components. The action may affect the number of intersections of the sections that generate $MW(X)$ and the fiber components. However, at the same time, there are also additional restrictions imposed by the automorphism group of the Mordell-Weil lattice of X. Therefore, in order to analyze this action, it is crucial to first comprehend the Galois action on $MWL(X)$. Understanding this action will provide valuable insights into how the sections and fiber components behave under Galois transformations.

In the particular case of surfaces X with $rank(MW(X)) = 2$ and torsion-free Mordell-Weil group (which we will explore later in this chapter), to understand the possible Galois actions on $MWL(X)$, we must observe the automorphisms of such lattices. In this case we can express the automorphisms on $MWL(X)$ in terms of the corresponding binary quadratic form Q, and we refer to [HS12] where the authors conducted a similar study. Multiplication by ± 1 gives the trivial automorphism of Q, any other automorphism will be called non-trivial. The problem whether a quadratic form Q admits non-trivial automorphisms depends on the order of Q in the class group of even positive definite quadratic forms with given discriminant.

Let f a binary quadratic form with integer coefficients and non-zero discriminant, represented by $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. For $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z})$ from (1.5), we have $f_M(x, y) = f(px + qy, rx + sy)$, we put

$$Aut(f) = \{M \in GL_2(\mathbb{Z}) | f_M = f\}.$$

Assuming that Q is positive definite and reduced as in Remark 1.2.2, we can compute the number of automorphisms through a straightforward computation of the automorphism group. Several distinct cases arise: $\#Aut(f) = 12$ when $a = b = c$, $\#Aut(f) = 8$ when $a = c$ and $b = 0$, $\#Aut(f) = 4$ when $a = b$ or $a = c$ or $b = 0$, and $\#Aut(f) = 2$ otherwise.

Lemma 3.1.4. A reduced quadratic form $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ is two-torsion in its class group if and only if

$$b = 0 \quad \text{or} \quad a = b \quad \text{or} \quad a = c.$$

We find a proof of this lemma in [Dic57] theorem 90. It is easy to check that the condition of lemma 3.1.4 is equivalent to the property that the quadratic form Q is equivalent to $\begin{pmatrix} 2a & -b \\ -b & 2c \end{pmatrix}$. Putting all this together we have the following lemma.

Lemma 3.1.5. The positive-definite quadratic form Q admits a non-trivial automorphism if and only if it is two-torsion in its class group.

In table 3.2 we summarize the lemma above and the previous computations on binary quadratic forms. We have the non-trivial automorphism groups of reduced quadratic forms, where D_{2n} denoted the dihedral group of order $2n$:

Q	$\begin{pmatrix} 2a & 0 \\ 0 & 2c \end{pmatrix}$	$\begin{pmatrix} 2a & a \\ a & 2c \end{pmatrix}$	$\begin{pmatrix} 2a & b \\ b & 2a \end{pmatrix}$	$\begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}$	$\begin{pmatrix} 2a & a \\ a & 2a \end{pmatrix}$
$Aut(Q)$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	D_8	D_{12}

Table 3.2: Reduced binary quadratic forms with non-trivial automorphism group

Depending on the configuration of the singular fibers and sections, a nontrivial action may occur on the sections. In particular, if we assume that the Mordell-Weil group is torsion-free, the action could manifest in the following way: if the Mordell-Weil lattice $MWL(X)$ admits a non-trivial automorphism, and there exists an involution of two sections that preserves the incidence relations with the fiber components when also applied as Galois action to the fiber.

3.1.2 Lattices for singular K3 with class number 4

We now resume our search for singular K3 surfaces over $\mathbb{Q}$, for fundamental discriminant with class number 4 and two torsion. Our approach to problem 3.1.2 is almost the opposite to the one mentioned in the beginning of this chapter (assuming we have a singular K3 surface over $\mathbb{Q}$). We will choose a fundamental discriminant d in Table 3.1, then we can take one of the four elements in $Cl(d)$. We have a brief description of the steps to follow to obtain explicit defining equations for singular K3 surfaces.

Algorithm 3.1.6. Let d be a fundamental discriminant, such that $h(d) = 4$, and let $Q \in Cl(d)$. The following steps are intended to generate a model X_s over $\mathbb{Q}$ for singular K3 surface X over $\mathbb{C}$, with a discriminant $disc(X) = d$ represented by a Weierstrass model, if such a surface with the specified characteristics exists. Additionally, these steps are designed to establish an isomorphism between $T(X)$ and Q.

1. Find a lattice L that fulfills the following conditions:

    - $det(U \oplus L) = -d$
    - $rank(L) = 18$.
    - $-q_L \cong q_Q$.

    Then there is a complex K3 surface X with $NS(X) = U \oplus L$ since the lattice $U \oplus L$ admits a primitive embedding into the K3 lattice (cf. [Mor84, Corollary 1.9]).

2. As the surface X comes with an elliptic fibration with a section such that $NS(X) = U \oplus L$, where L encodes fiber components and non-zero sections (by Theorem 1.4.9), verify that after accounting for quadratic twists, the Néron-Severi group $NS(X)$ allows for a Galois action isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

3. Apply Tate's algorithm and perform computations involving the discriminant of the Weierstrass equation to compute a Weierstrass model X_s over $\mathbb{Q}$ for the K3 surface X found in step 2, where the Néron-Severi group satisfies $NS(X) = U \oplus L$.

If the algorithm 3.1.6 yields a model over $\mathbb{Q}$, the resulting singular K3 surface X will have $T(X) \cong Q$, by corollary 1.1.14. Up to isomorphism, X will be the singular K3 surface associated with the quadratic form Q by Theorem 2.1.1.

The level of detail in Algorithm 3.1.6 appears somewhat limited, particularly in steps 2 and 3, which will feature significant variations based on the rank of the Mordell-Weil lattice $MWL(X)$. To execute step 3 in Algorithm 3.1.6, we will initiate the application of Tate's algorithm, as described in section 1.4.

One approach for constructing a singular K3 surface X over $\mathbb{Q}$ with a specified discriminant d is to initially identify a family of K3 surfaces X_λ and specialize this family to obtain X. However, it can prove to be quite challenging to find a family of K3 surfaces with a Picard rank exceeding 18. In scenarios where this becomes particularly difficult, alternative approaches, such as the construction of K3 surfaces over finite fields $\mathbb{F}_p$ may become necessary.

Throughout this work, we concentrate on elliptic K3 surfaces where the configuration of singular fibers ensures that the Mordell-Weil group is torsion-free, as demonstrated by the properties of torsion sections in Proposition 1.4.15. For every potential rank of the Mordell-Weil group for the singular K3 surfaces in this study, we present a method to compute a lattice L, leading to an isomorphism $NS(X) \cong U \oplus L$, as elaborated in Step 1 of Algorithm 3.1.6. Additionally, we guarantee that the Néron-Severi group possesses a Galois action isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

3.2 Extremal elliptic $K3$ surfaces

We will now begin our search for singular K3 surfaces over $\mathbb{Q}$ with class number 4, starting with the simplest case of singular K3 surfaces that has been studied previously: extremal elliptic K3 surfaces.

Definition 3.2.1. Let X be a singular K3 surface, X is called extremal elliptic if X admits an elliptic fibration with $rank(MW(X)) = 0$, i.e. a finite group of sections .

Extremal K3 surfaces were classified by Shimada and Zhang in [SZ01] and there is a finite number of them.

Up to torsion section, such surfaces are determined by the configuration of singular fibers and more in particular if they have no torsion they are completely determined by the configuration of singular fibers, also by Corollary 1.4.16 the discriminant of the Néron-Severi lattice is :

$$disc(NS(X)) = -\prod_v disc(F_v). \tag{3.3}$$

Remark 3.2.2. Since the classification of extremal K3 surfaces is already known and there is a finite number of them, we only need to search for a surface in [SZ01] with the desired discriminant and quadratic form of the transcendental lattice. No additional steps are required for the application of Algorithm 3.1.6 in this case.

Some explicit equations for extremal K3 surface have been found by Schütt in [Sch07a], and by Beuker and Montanus in [BM08]. We are going to list some of the extremal K3 surfaces found in these previous works, and also those found by us. The next table lists the discriminant d of a possible configuration of singular fibers, and the transcendental lattice $T(X)$ with intersection form $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ but we can denote it as $[2a, b, 2c]$. All the surfaces in Table 3.3 have a trivial Mordell-Weil group, then we can omit it.
We will write the step-by-step procedure to derive an explicit model of an extremal K3 surface, specifically the one identified as No. 2 in Table 3.3. For this specific instance, our approach initiates with a RES (rational elliptic surface), and by executing a quadratic twist on the Weierstrass model of this surface, we successfully yield a K3 surface.

No	discriminant	Reducible fibers configuration	T(X)
1	-84	I_7, I_3, I_6^*	$[2,0,42]$
2	-84	I_7, IV^*, I_2^*	$[4,2,22]$
3	-120	I_3, I_{10}, I_3^*	$[2,0,60]$
4	-120	I_5, I_6, I_5^*	$[4,0,30]$
5	-120	I_2, I_3, I_5, I_7^*	$[6,0,20]$
6	-168	I_2, I_3, I_7, I_5^*	$[4,0,42]$
7	-168	I_2, I_7, IV^*, I_1^*	$[2,0,84]$
8	-195	I_3, I_5, I_{13}	$[6,3,34]$
9	-280	$I_2.I_5, I_7, I_3^*$	$[4,0,70]$
10	-312	I_2, I_3, I_4, I_{13}	$[6,0,52]$

Table 3.3: Extremal elliptic K3 surface with class number 4

For discriminant $d = -84$ we have:

$$Cl(-84) = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix} \right\}$$

We start with the element $\begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix} \in Cl(-84)$. Looking in [SZ01], there exist an extremal K3 surface X with transcendental lattice $T(X) = \begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix}$. The associated surface to this transcendental lattice has a Néron-Severi group $NS(X) = U + A_6 + D_6 + E_6$ as stated in [SZ01, Table 2].

It is possible to induce a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois action on X up to a quadratic twist, as we saw in Section 3.1.1. After applying a quadratic twist to X, we can ensure that all fiber components of A_6 are defined over $\mathbb{Q}$. Hence there has to exists a Galois action as an involution on the simple components of the singular fibers D_6 and E_6 that preserves incidence relations and the identity component.

Example 3.2.3. Discriminant $d = -84$ and $T(X) = [4, 2, 22]$.
In this case we search for extremal K3 surface with three singular fibers of type I_7, I_2^*, IV^*, we work with an extended Weierstrass equation as in (1.12). We start with a rational elliptic surface (RES) with two singular fibers of type I_7 and II located at 0 and ∞ respectively:

$$\begin{aligned} &y^2 = x^3 + (-3ct^2 + a_1 t - c^3)x^2 + t^3(3c^2 t + 2c^3)x - c^3 t^6 \\ &\Delta = t^7 p(t) \end{aligned}$$

Here $p(t)$ is a polynomial of degree 3 over t, and we have two parameters in the Weierstrass equation. After a suitable normalization over x, y we obtain $c = 1$. At this point we need solve for a I_2 fiber, so we need a double root in the polynomial $p(t)$. In this case we can solve it in the discriminant of $p(t)$, $Disc_t(p(t)) = 0$. We have an extra I_2 fiber if and only if we take $a_1 = 6$. This gives us a singular fiber of type I_2 located at $t = 3/8$ and the Weierstrass equation takes the form:

$$y^2 = x^3 + (3t^2 - 6t + 1)x^2 + t^3(3t + 2)x + t^6$$
$$\Delta = (-1/262144)(27t - 4)t^7(8t - 3)^2$$

We have now a Weierstrass equation with three singular fibers of type I_7, I_2, II over $0, 3/8, \infty$ respectively. We apply a quadratic twist over $(t - 3/8)$ and ∞ to obtain an extremal K3 surface X with three singular fibers of type I_7, I_2^* and IV^* over $0, 3/8$ and ∞ respectively (see subsection 1.4.3), and the final equation has the form:

$$y^2 = x^3 + (t - 3/8)(-3t^2 + 6t + 1)x^2 + t^3(t - 3/8)^2(3t + 2)x - t^6(t - 3/8)^3$$
$$\Delta = (-1/262144)(27t - 4)t^7(8t - 3)^8$$

The surface X has $\rho(X) = 20$ and $disc(X) = 84$ with $NS(X) = U + A_6 + D_6 + E_6$ and $T(X) = \begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix}$. In this example the Galois action act on the fiber components of the singular fibers. The Galois action on the Néron–Severi group is encoded in the fields where the singular fibers with at least three components split.

Fiber	I_7	I_2^*	IV^*
Cusp	0	$3/8$	∞
Splitting field	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{10})$	$\mathbb{Q}(\sqrt{15})$

Table 3.4: Splitting field of singular fibers of extremal surface

In table 3.3, two of four singular K3 surfaces with class number 4 and discriminant 84 can be constructed as extremal K3 surfaces. We can consider another example, the element $\begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix} \in Cl(-84)$.

We can see in extremal complex K3 surface in [SZ01], there exist a extremal complex K3 surface X with this transcendental lattice with $NS(X) = U + A_2 + A_6 + D_{10}$. Because in this case we only have one singular fiber with additive reduction, we will solve it in a slightly different way in the following example.

Example 3.2.4. Discriminant $d = -84$ and $T(X) = [2, 0, 42]$.
In this case, we aim to find an extremal $K3$ surface with three singular fibers of type I_3, I_7, I_6^*. We begin by working with an extended Weierstrass form, as given in (1.12). To simplify the calculations, it is often useful to perform coordinate transformations such that

the reducible singular fibers are located at ∞ or 0 and the singularity of the Weierstrass model is at $(0,0)$.

Using Tate's algorithm (1.4), we perform five normalizations and obtain three singular fibers of type I_2, I_4, I_2^* located at $1, \infty, 0$ respectively. We can then use Tate's algorithm to promote the fibers I_4 and I_2^* to I_5 and I_3^* respectively. This yields a family of elliptic $K3$ surfaces with fibers of type I_2, I_5, I_3^* located at $1, \infty, 0$ respectively.

$$y^2 = x^3 + (a_3^2t^3 + a_2t^2 + a_1t + c_0)tx^2 + 2t^3(t-1)(a_3c_1t^2 + b_1t + b_0)x + t^5(t-1)^2(c_1^2t + b_0^2/c_0)$$
$$\Delta = (t-1)^2t^9p(t)$$

Here $p(t)$ is a polynomial of degree 8 over t, we have here 6 free parameters to choose. We work with $p(t)$ in order to to promote the singular fibers. We solved on the discriminant to make $t^3(t-1)|p(t)$ and the coefficients of degree 9 and 8 equal to 0 in order to promote the fibers I_2 to I_3, I_5 to I_7 and I_3^* to I_6^* to obtain the following Weierstrass form:

$$\begin{aligned}
A &= -\frac{1}{64}(27t^3 - 102t^2 + 132t - 8)t \\
B &= \frac{1}{16}(t-2)(t-1)(9t-2)t^3 \\
C &= -\frac{1}{16}(3t-2)(t-1)^2t^5 \\
y^2 &= x^3 + Ax^2 + Bx + C \\
\Delta &= \frac{1}{1024}(t-1)^3t^{12}(81t^2 - 396t - 28).
\end{aligned}$$

By Tate algorithm the above Weierstrass form guarantees that fiber types are I_3, I_7, I_6^* at $t = 1, \infty, 0$ respectively.

The cases number 8 and 10 were already solved in [BM08]. In the last two examples we made the explicit computations of the model over $\mathbb{Q}$ for the cases 1 and 2 in table 3.3.
The same ideas can be applied for the remaining cases in table 3.3, we enlist the remaining cases with their respective model over $\mathbb{Q}$.

Case $d = -120$ *and* $T(X) = [2, 0, 60]$.
For this case we got an extremal K3 surface with three singular fibers of type I_3, I_{10}, I_3^*

over $1, 0$ and ∞ respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (-1/8)(t-2)(25t^2-4t+4)\\
B &= (-2)(t-1)t^2(5t^2-4t+4)\\
C &= (-8)(t-2)(t-1)^2t^4\\
f &= x^3+Ax^2+Bx+C\\
\Delta &= 16(t-1)^3t^{10}(125t^2+12t-12).
\end{aligned}
$$

Case $d=-120$ *and* $T(X)=[4,0,30]$.
For this case we got an extremal K3 surface with three singular fibers of type I_5, I_6, I_5^* over $1, 0$ and ∞ respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (-1/4)t(7t^2-4t-4)\\
B &= (-2)(t-2)(t-1)^2t^3\\
C &= (4)(t-1)^4t^5\\
f &= x^3+Ax^2+Bx+C\\
\Delta &= (-2)(t-1)^5t^{11}(25t^2-30t+4).
\end{aligned}
$$

Case $d=-120$ *and* $T(X)=[6,0,20]$.
For this case we got an extremal K3 surface with four singular fibers of type I_2, I_3, I_5, I_7^* over $-5/49, 1, 0$ and ∞ respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (-1/800)(343t^3+1105t^2-675t-125)\\
B &= (1/400)(t-1)t^2(343t^2+150t-25)\\
C &= (-1/800)(343t-5)(t-1)^2t^4\\
f &= x^3+Ax^2+Bx+C\\
\Delta &= (16000000)(t-5)(49t+5)^2(t-1)^3t^5.
\end{aligned}
$$

Case $d=-168$ *and* $T(X)=[4,0,42]$.
For this case we got an extremal K3 surface with four singular fibers of type I_2, I_3, I_7, I_5^* over $-5/49, 1, \infty, 0$ respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (1/800)t(343t^3+1105t^2-675t-125)\\
B &= (1/400)(t-1)t^4(343t^2+150t-25)\\
C &= (1/800)(343t-5)(t-1)^2t^7\\
\Delta &= (16000000)(t-5)(49t+5)^2(t-1)^3t^{11}.
\end{aligned}
$$

Case $d=-168$ *and* $T(X)=[2,0,84]$.

For this case we got an extremal K3 surface with four singular fibers of type I_2, I_7, I_1^*, IV^* over $49/81, \infty, 0$ and 1 respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (-3/4)t(9t-1)(t-1)^2 \\
B &= (-2/27)(27t+1)t^2(t-1)^3 \\
C &= (-4/729)(27t+5)t^3(t-1)^4 \\
f &= x^3 + Ax^2 + Bx + C \\
\Delta &= (-16/19683)(81t-49)^2t^7(t-1)^8.
\end{aligned}
$$

Case $d = -280$ *and* $T(X) = [4, 0, 70]$.
For this case we got an extremal K3 surface with four singular fibers of type I_2, I_5, I_7, I_3^* over $-5/49, 0, \infty$ and 1 respectively, with an associated Weierstrass equation and discriminant:

$$
\begin{aligned}
A &= (1/800)(t-1)(343t^3 + 1105t^2 - 675t - 125) \\
B &= (1/400)t^2(t-1)^3(343t^2 + 150t - 25) \\
C &= (1/800)(343t-5)t^4(t-1)^5 \\
\Delta &= (16000000)(t-5)(49t+5)^2t^5(t-1)^9.
\end{aligned}
$$

Before starting with our search of singular K3 surfaces with $rank(MW) = 1$, we will explain some techniques to find specializations over $\mathbb{F}_p$ to exhibit singular K3 surfaces.

Remark 3.2.5. We can notice that the cases for $d = -120, -168, -280$ have a really similar Weierstrass model. Nanely, starting with the generic fiber of the surface with $d = -168$, a quadratic twist over t and $t-1$ yields the model with $d = -120$ and $d = -280$ respectively. In the construction of this Weierstrass model we started with a Weierstrass equation with five singular fibers of type I_2, I_3, I_7, I_5 at $-5/9, 1, 0, \infty$ respectively and a fiber of type I_0^* at $t = a$, then merge the fiber I_0^* with one of the I_3, I_7, I_5 respectively to obtain the desired model.

3.3 Techniques over $\mathbb{F}_p$

Our objective is to construct singular K3 surfaces over $\mathbb{Q}$ with a fixed discriminant d and transcendental lattice $T(X) \cong Q \in Cl(d)$. To this end, we make use of the Weierstrass equation, as given in Equation (1.12), and apply Tate's algorithm. In many cases, we can compute a family X_λ of elliptic K3 surfaces with a high Picard rank (18 or 19), given by a Weierstrass equation over $\mathbb{Q}$, such that this family can be specialized to yield the desired discriminant and Néron–Severi lattice (meaning that the K3 surface we are seeking is in this family, at least over $\mathbb{C}$). However, obtaining a specialization of this family over $\mathbb{Q}$

with Picard rank 20 can be challenging due to the increased complexity of the Weierstrass equation and discriminant. It becomes difficult to obtain a specialization of Picard rank 20 working only with the Weierstrass model and Tate's algorithm as we did in the cases of extremal elliptic K3 surfaces. Therefore, we will also employ some of the primary techniques described in [ES13] to address this challenge.

The strategy here is that if X is a smooth projective surface over a number field, then we can use information at a prime of good reduction for X to understand $NS(X)$ and bound $\rho(X)$. In addition to this, we will take advantage of the moduli theory of complex K3 surfaces, as well as the modularity results for singular K3 surfaces outlined in Theorems 2.3.4 and 2.3.5.

Any one dimensional family of $K3$ surfaces of Picard rank $\rho \geq 19$ has infinitely many specializations with $\rho = 20$ by [Ogu03, Corollary 1.6], however we are only interested in those specializations over $\mathbb{Q}$, which are finite in number due to Shafarevich's Theorem 2.2.1.

Here we work with a base change of $\bar{X}$, the reduction of X mod p (a prime with good reduction) consider as a surface over an algebraic closure of $\mathbb{F}_p$. Then we have an important condition that is based on the Lefschetz fixed point formula at a good prime p. This condition is written in terms of l-adic étale cohomology at some other prime $l \neq p$, $H^i_{et}(X, \mathbb{Q}_l)$. We just write $H^i(X)$ to abbreviate.

The cohomology groups $H^i(X)$ are equipped with an action of the Frobenius map $Frob_p^*$. The number of eigenvalues of $Frob_p^*$ that are of the form $p\zeta$ (where ζ is some root of unity) can be determined by examining the characteristic polynomial $\psi_p(x)$ of the linear operator $Frob_p^*$. To compute the characteristic polynomial, we can use two key ideas. First, the characteristic polynomial of a linear operator on a finite dimensional vector space can be recovered from knowing the traces of powers of the linear operator, in the following way:

Theorem 3.3.1. (Newton's identities) Let T be a linear operator on a vector space V of finite dimension n. Let t_i be the trace of the i-th composition T^i, and define

$$a_1 := -t_1 \quad \textit{and} \quad a_k := -\frac{1}{k}\left(t_k + \sum_{j=1}^{k-1} a_j t_{k-j}\right) \quad \textit{for} \quad k = 2, ..., n. \tag{3.4}$$

Then the characteristic polynomial of T is equal to

$$det(x \cdot I - T) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n. \tag{3.5}$$

Secondly in our cases the traces of powers of $Frob_p^*$ operating on $H^2(X)$ can be recovered from the Lefschetz fixed point formula in the case of K3 surfaces

$$tr((Frob_p^*)^i(H^2(X))) = \#X(\mathbb{F}_{p^i}) - 1 - p^{2i} \tag{3.6}$$

see [Man86] for the proof of this formula in the regular surface case. When X is a K3 surface, Theorem 3.3.1 tells us that we need to count points over $\mathbb{F}_p^i$ for $i = 1, \ldots, 22$. However, the characteristic polynomial $\psi_p(x)$ of $Frob_p^*$ satisfies a functional equation, as predicted by the Weil conjectures:

$$p^2\psi_p(x) = \pm x^{22}\psi_p(p^2/x). \tag{3.7}$$

This equation tells us that counting points over $\mathbb{F}_p^i$ for $i = 1, \ldots, 11$ is enough to determine the characteristic polynomial $\psi_p(x)$ up to sign on equation (3.7). The following theorem enables us to determine the trace of the induced Frobenius map $Frob_p^*$ on $H^2(X)$ in the case of singular K3 surfaces, and is a consequence of Weil's conjectures.

Theorem 3.3.2 ([Del72], Theorem 1)**.** Let X be a smooth variety over a finite field $\mathbb{F}_p$ of cardinality $q = p^r$ with p prime, and let $Frob_p \in Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ denote the Frobenius automorphism. For a prime $l \neq p$, let $Frob_p^*$ denote the automorphism of $H^2(\bar{X}, \mathbb{Q}_l)$ induced by $Frob_p$. Then $\rho(\bar{X})$ is bounded by the number of eigenvalues of $Frob_p^*$ counted with multiplicity that are of the form ζp (where ζ is some root of unity). Furthermore, the characteristic polynomial $\psi_p(x)$ of $Frob_p^*$ lies in $\mathbb{Q}[x]$.

Suppose we have a family of K3 surfaces X_λ over $\mathbb{Q}$ with $\rho(X_\lambda) \geq 19$ for every surface in the family. Since $\rho(X_\lambda) \geq 19$, we can predict 19 of the 22 eigenvalues of $Frob_p^*$ on $H^2(X_\lambda)$. By Theorem 3.3.1 and equation (3.7), we only need to count points over $\mathbb{F}_p$ to obtain a condition from equation (3.6) at a prime p with good reduction.

$$\#X(\mathbb{F}_p) = 1 + tr(Frob_p^*(H^2(X))) + p^2. \tag{3.8}$$

By Weil conjectures, the non-real eigenvalues of $Frob_p^*$ on $H^2(X)$ come in complex conjugate pairs. Hence in addition of the 19 eigenvalues mentioned above, we know that there is one extra eigenvalue of $Frob_p^*$, which has the form $\pm p$. And the remaining two eigenvalues (α_p, β_p) are algebraic integers of absolute value p. In particular the pair of eigenvalues (α_p, β_p) is determined by (3.8) and the sign of the other eigenvalue. If $\lambda_0 \in \mathbb{Q}$ is a specialization of X_λ over $\mathbb{Q}$ such that $\rho(\mathbb{F}_p(X_{\lambda_0})) = 20$, the two eigenvalues satisfied $\alpha_p = \bar{\beta}_p$ (i.e the reduction of X_{λ_0} mod p is not a supersingular K3 surface).

If the specialization at some $\lambda_0 \in \mathbb{Q}$ is a singular K3 surface over $\mathbb{Q}$, by Theorem 2.3.4, this specialization is modular. Furthermore, we choose the sign for the eigenvalue $\pm p$ in a manner that guarantees all possible eigenvalues have an absolute value of p, so we have

$$\alpha_p + \beta_p = a_p \tag{3.9}$$

where a_p is the Fourier coefficient of the corresponding newform f of weight 3 of Theorem

2.3.4. Since f has CM, both α_p and β_p lie in the imaginary quadratic extension K associated to f by Theorem 2.3.5. The associated field K has class group exponent 2, and this field is fixed when we change p. We can vary the prime p and use equation (3.9), when the specialization at λ_0 is a singular K3 surface. This provides a criterion for λ_0 such that X_{λ_0} could be a specialization over $\mathbb{Q}$ with $\rho(X_{\lambda_0}) = 20$.

Remark 3.3.3. It is important to note that the choice of prime p is crucial in the computations mentioned above. Specifically, we have two possibilities depending on whether p splits or not in the fixed quadratic field K.
Suppose we have a singular K3 surface X over a number field L. In this case, we can predict the geometric Picard number of the reductions. Let p be a prime number and $\mathfrak{p}$ a prime of L above p. If X has good reduction modulo $\mathfrak{p}$, we write $X_{\mathfrak{p}}$ for the reduced K3 surface. The reductions fall into two categories, which we find in [Shi09, Theorem 1]:

$$\rho(X_{\mathfrak{p}}) = \begin{cases} 20, & \text{if } p \text{ splits in } K, \\ 22, & \text{if } p \text{ is inert or ramified in } K. \end{cases} \tag{3.10}$$

At an inert prime p in K, the Fourier coefficient a_p of the newform is zero. This results in p and $-p$ being eigenvalues of $Frob_p^*$. The Tate conjecture in [Tat75b], predicts that the reduction $X_{\mathfrak{p}}$ has additional algebraic cycles. In the case of elliptic surfaces, these extra cycles would change the configuration of the reducible fibers, or they would occur as extra sections on the Mordell-Weil group.

Remark 3.3.4. Let X be a (geometrically irreducible) smooth projective variety defined over some finite field $k = \mathbb{F}_p$. Then we can consider the subgroup of the geometric Néron-Severi group $NS(X \otimes \bar{k})$, which is generated by divisor classes defined over k. Fixing some prime l relatively prime to p, there is an l-adic cycle class map

$$NS(X/k) \otimes \mathbb{Q}_l \hookrightarrow NS(X \otimes \bar{k}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{et}(X \otimes \bar{k}, \mathbb{Q}(1)).$$

Here we have applied a Tate twist to the second l-adic cohomology in order to make the embedding compatible with the natural Galois action on the Néron–Severi group. More precisely, the Frobenius morphism $Frob_p$ acts trivially on $NS(X/k)$, and there is an induced action on $NS(X \otimes \bar{k})$ which factors through a finite group, since $NS(X \otimes \bar{k})$ is always generated by divisor classes defined over some finite extension of the ground field.

Let's consider a singular K3 surface X over the finite field $\mathbb{F}_q$. The generators of the lattice $L \subset NS(X)$ provide sufficient eigenvalues of the Frobenius operator, allowing us to deduce the complete characteristic polynomial on $H^2(X)$, with the exception of a potential ambiguity, through a simple point count over $\mathbb{F}_p$:

$$a_p = \#X(\mathbb{F}_p) - 1 - tr(Frob_p^*(NS(X))) - p^2. \tag{3.11}$$

Here a_p would account for the trace of $Frob_p$ in the Galois representation associated with the rank two transcendental lattice $T(X)$ as discussed in section 2.3.2, and the sign depends on the Galois action on the remaining divisor class $\pm p$ generating $NS(X)$ together with L. Furthermore, here a_p is the Fourier coefficient of the corresponding newform f of weight 3.

For the sake of simplifying computations, we utilize an appropriate affine Weierstrass model. We have the following lemma.

Lemma 3.3.5 ([GT95] Lemma 3.3)**.** Suppose X has a minimal, elliptic surfaced defined over $\mathbb{F}_q$, corresponding to a minimal affine Weierstrass model X_{aff}, with stable fibers over each point in $\mathbb{P}^1$. Put $A_l(X) \subset H^2(Y, \mathbb{Q}_l)$ to be the subspace spanned by the generic fiber, the zero section, and all components of singular fibers not meeting the zero section. Let $H^2_{tr}(X)$ be the orthogonal complement of $A_l(X)$ in $H^2(X, \mathbb{Q}_l)$. Then

$$trace(H^2_{tr}(X)) = \#X_{aff}(\mathbb{F}_p) - q^2 - q.$$

Thus, for a singular elliptic K3 surface X with $rank(MW(X)) = 1$, considering point counts over $\mathbb{F}_p$ allows us to derive the following equation:

$$\#X_{aff}(\mathbb{F}_p) - 1 - 2p - p^2 \pm p = \alpha_p + \beta_p = a_p, \tag{3.12}$$

where $\#X_{aff}(\mathbb{F}_p)$ denotes the number of $\mathbb{F}_p$-rational points on a associated affine Weierstrass model X_{aff} of X.

To obtain a specialization over $\mathbb{Q}$ of a family of K3 surfaces X_λ with Picard rank 20 and the desired discriminant d, we employ an algorithm that searches for parameters $\lambda_i \in \mathbb{F}_{p_i}$ such that, for each prime p_i, the surface X_{λ_i} reduced modulo p_i has Picard rank 20 and it satisfies (3.9).

Once we have found a collection of parameters $\lambda_1 \bmod p_1, \ldots, \lambda_i \bmod p_i$ that satisfy these conditions, we lift them to a rational number λ_0 and from all the possible rational numbers, we can look for a specialization of our family X_λ over $\mathbb{Q}$ with desired discriminant and transcendental lattice. The algorithm proceeds as follows:

Algorithm 3.3.6. Let a family of K3 surfaces X_λ over $\mathbb{Q}$ with Picard number $\rho(X_\lambda) \geq 19$, we look for a specialization X_{λ_0} over $\mathbb{Q}$ with discriminant d (fundamental discriminant). To do this, we can follow these steps:

1. For several primes $p = p_i$ $(i = 1, ..., n)$ such that $(\frac{d}{p_i}) = 1$, use (3.8) and (3.12) to compute α_p, β_p for every $\lambda \in \mathbb{F}_p$.

2. Select those λ such that a choice of the sign of the eigenvalue $\pm p$, gives us that $\alpha_p + \beta_p = a_p$ as in (3.9), if there is any. Here a_p is the Fourier coefficient of the corresponding newform f of weight 3 up to twist.

3. For every collection $\{\lambda_i \bmod p_i\}_i$ such that every λ_i fulfills (3.12), compute a $\bar{\lambda_0} \bmod \prod_i p_i$ s.t. $\bar{\lambda_0} = \lambda_i \bmod p_i$ using the Chinese Remainder Theorem.

4. For every $\bar{\lambda_0}$ of the previous step, lift $\bar{\lambda_0}$ to a rational number λ_0 using the euclidean algorithm as
$$\lambda_0 = \frac{q_1}{q_2} \prod_i (p_i - \bar{\lambda_0})$$
where $\lambda_0 = \frac{s_1}{s_2}$.

5. Of all λ_0, select the rational number with the smallest height $(max(|s_1|, |s_2|))$.

For each family X_λ and a discriminant d, this procedure returns a candidate parameter $\lambda_0 \in \mathbb{Q}$, such that the specialization X_{λ_0} might have $disc(X_{\lambda_0}) = d$ and $\rho(X_{\lambda_0}) = 20$.

Assuming we have a one dimensional family of K3 surfaces X_λ over $\mathbb{Q}$ with $\rho(X_\lambda) \geq 19$, given by a Weierstrass equation $X_\lambda : y^2 = x^3 + A(t,\lambda)x^2 + B(t,\lambda)x + C(t,\lambda)$ we summarize algorithm 3.3.6 in the following pseudo-code:

Algorithm 1 $\mathbb{F}_p$ reduction and counting points

procedure
  for i in range(30) **do**
    $p = Primes()[i] \leftarrow i$
    $X_\lambda = \mathbb{F}_p(X_\lambda)$
    if $(\frac{d}{p}) = 1$ **then**
      for $\beta \in \mathbb{F}_p$ **do**
        $X_\beta = X_\lambda(\lambda = \beta) \leftarrow \beta$
        $points[\beta] = \#X_\beta(\mathbb{F}_p) \leftarrow X_\beta$
        if $points[\beta] - 1 - 2p - p^2 \pm p = \pm a_p$ **then**
          $Solutions[p] \leftarrow Solutions[p] + [\beta, p]$
  for $p \in Primes()[0:30]$ **do**
    if $Solutions[p]$ **then**
      for S in $Solutions[p]$ **do**
        for s in S **do**
          $collections \leftarrow collections + s$
  for C in $Collections$ **do**
    $c_{temp} = crt(C[0], C[1]) \leftarrow C$
    $results \leftarrow results + Euclidean\ algorithm(c_{temp})$
 return results

Algorithm 3.3.6 can be run with any number of primes. In most cases, 20 to 30 primes are sufficient to obtain a good parameter λ_0. In the following example, we illustrate the application of the algorithm on the remaining case with $d = -120$.

Example 3.3.7. Discriminant $d = -120$ and $T(X) = [10, 0, 12]$.
Let X be a complex K3 surface with trivial transcendental lattice $Triv(X) = U \oplus A_5 + D_6 + E_6$, and a rank one Mordell-Weil group $MW(X)$ generated by a section P with a height pairing $h(P) = 4 - 4/3 - 1 = 5/3$.
The section P intersects Θ_2, the second component of I_6, and the near component Θ_1 of I_2^*. Using the determinant formula (1.29), we find that X has $disc(X) = -120$, and by the Shioda-Tate formula (1.4.10), we find that $\rho(X) = 20$. With this information, we can compute the Néron-Severi lattice and the transcendental lattice $T(X)$, which is isomorphic to $\begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}$.
Moreover, it is possible to perform a quadratic twist so that the fiber components of the A_5 singular fibers are defined over $\mathbb{Q}$. And the Néron-Severi group of the resulting K3 surface admits a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois action, which corresponds to an involution of the simple far fiber components of the D_6 and the simple components of E_6 singular fibers, which is compatible with intersection numbers with the section P. Given these properties, we can proceed to compute a Weierstrass model for the K3 surface in question.

We applied Tate's algorithm to obtain the Weierstrass equation for X and obtained a one-dimensional family X_λ with $\rho(X_\lambda) \geq 19$. This family includes three singular fibers of type I_6, I_2^*, and IV^* over 0, ∞, and 1, respectively. The Weierstrass equation for X_λ is as follows:

$$A = \lambda(t-1)^2(t\lambda - 3t - \lambda)$$
$$B = (-1)t^2(t-1)^3(2t\lambda^2 - 6t\lambda - 2\lambda^2 + 3t + 2\lambda)$$
$$C = (\lambda - 1)t^4(t-1)^4(t\lambda - 2t - \lambda + 1)$$
$$f = x^3 + Ax^2 + Bx + C$$
$$\Delta = -t^6(t-1)^8(3t^2\lambda^4 - 18t^2\lambda^3 - 6t\lambda^4 + 45t^2\lambda^2 + 22t\lambda^3 + 3\lambda^4 - 54t^2\lambda - 18t\lambda^2 - 4\lambda^3 + 27t^2)$$

We aim to obtain a specialization of this family with $\lambda_0 \in \mathbb{Q}$, such that the family contains the desired section as described above. To achieve this, we utilized algorithm 3.3.6 for the first 30 primes, 14 of which satisfy $(\frac{-120}{p}) = 1$. We calculated $\#X_\lambda(\mathbb{F}_p)$ for $p \in \{11, 13, 17, 23, 29, 31, 37, 43, 47, 59, 67, 79, 101, 113\}$ using the given Weierstrass model. This process yielded approximately 5,000 parameters $\overline{\lambda_0}$ mod $\prod_i p_i$, after all the computations, which were then lifted to $\mathbb{Q}$ to obtain all possible $\lambda_0 \in \mathbb{Q}$. These rational numbers were subsequently ordered by height, and the value with the smallest height was chosen as our desired $\lambda_0 = -2/3$.

To obtain the Weierstrass equation for the candidate elliptic surface X with $\rho(X) = 20$ and $disc(X) = -120$, we specialize a family at $\lambda_0 = -2/3$. The resulting equation is:

$$f = x^3 + (-2/9)(11t-2)(t-1)^2x^2 + (-1/9)(71t-20)t^2(t-1)^3x + (-5/9)(8t-5)t^4(t-1)^4$$
$$\Delta = (-1/27)t^6(t-1)^8(2401t^2 - 424t + 48)$$

To confirm that we have indeed found the surface we are looking for, we need to find a section P that intersects the I_6 fiber on the second component at $t = 1$ and the I_2^* fiber on its nearby component at $t = \infty$. We construct examples with $(P \cdot O) = 0$ to facilitate the computations of the components of the section P as polynomials. This has consequences for the possible height of P, particularly implying that P can be expressed as $P = (x_P, y_P)$ with polynomials x_P and y_P of degree at most 4 and 6, respectively, in the coordinate t of the base curve. We can express y_P in terms of x_P by substituting x_P into the Weierstrass equation of X. Specifically, we obtain:

$$y_P^2 = f(x_P).$$

In order for the fibers to intersect as we described, according to Tate's algorithm (the way we construct the Weierstrass equation for this singular K3 surface), they must satisfy the divisibility conditions:

$$t^2 \mid x_P \quad \text{and} \quad t^4 \mid y_P^2.$$

At $t = \infty$, we rewrite the Weierstrass equation in terms of $s = 1/t, x' = x/t^2, y' = y/t^3$ (the coordinates of the other chart). The section P intersects the I_2^* fiber at ∞ on its near component, the degree of x_P over t is reduced by 1 and the degree of y_P^2 over t is reduced by 2 . Tate's algorithm shows that this fiber gives us an additional linear relation in the coefficients of x_P, coming from the intersection with the I_2^* fiber. Since x_P has degree three, there is one degree of freedom remaining. We can solve for this last coefficient using the equation $y_P^2 = f(x_P)$, which gives us the desired section:

$$x_P = (8/9)(5t+3)t^2 \qquad y_P^2 = (1/729)t^4(483t^2 + 26t + 3)^2.$$

The Galois action on the Néron-Severi group is encoded in the fields where the singular fibers with at least three components split, so here we have:

Fiber	I_6	I_2^*	IV^*
Cusp	0	∞	1
Splitting field	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{10})$	$\mathbb{Q}(\sqrt{15})$

Table 3.5: Splitting field of singular fibers

We can observe that the surface admits the expected $\mathbb{Z}/2\mathbb{Z}$ Galois action.

3.3.1 p-adic multivariate Newton iteration

Sometimes, we may encounter a system of equations over $\mathbb{Q}$ that cannot be solved directly using the techniques we have studied so far. This can happen, for instance, when attempting to resolve a singular fiber or when searching for sections in a family of K3 surfaces. In such situations, we can resort to p-adic Newton iteration in multiple variables as an alternative approach. Before describing this algorithm, it is important to note a classic lemma:

Theorem 3.3.8 (Hensel's Lemma)**.** Let $f(x)$ be a polynomial with integer coefficients. Let k be a positive integer, and r an integer such that $f(r) \equiv 0 \bmod p^k$. Let m be a positive integer. Then if $f'(r) \not\equiv 0 \bmod p$, there is an integer s such that $f(s) \equiv 0 \bmod p^{k+m}$ and $s \equiv r \bmod p^k$. So s is a "lifting" of r to a root mod p^{k+m}. Moreover, s is unique mod p^{k+m}.

We can find a proof of Hensel's Lemma in [Eis95]. Like Newton's method, Hensel's lemma works for systems of equations in several variables too.

Theorem 3.3.9. Let $f = (f_1, ..., f_n)$ a system of polynomials in n variables $x_1, ..., x_n$ with coefficients in $\mathbb{Z}$, and $\alpha \in \mathbb{Z}^n$ a simultaneous solution modulo p^k (i.e, $f_i(\alpha) \equiv 0 \bmod p^k$). Let m a positive integer. Assume

$$det(J(\alpha)) \not\equiv 0 \bmod p$$

where $J(\alpha)$ denotes the evaluation of the Jacobian matrix at α . Then there exists a point $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$ such that $f_i(\beta) \equiv 0 \bmod p^{k+m}$ and $\beta_i \equiv \alpha_i \bmod p^k$

Remark 3.3.10. Note that from the lift s of r in theorem 3.3.8 we can obtain the formula

$$s = r - \frac{f(r)}{f'(r)} \bmod p^{k+m} \tag{3.13}$$

This bears a striking resemblance to the formula used in Newton's method for approximating real roots of polynomials.

This can also be applied in the case of multiple variable as the multivariable Newton's method, and it can help us to find a solution of a system of rational equations.

Algorithm 3.3.11. Given a system $f = (f_i = 0)$ $(i = 1, ..., n)$ of algebraically independent polynomial equations over $\mathbb{Z}$ in n variables $z_1, ..., z_n$. The following steps can be used to search for solutions of the system over $\mathbb{Q}$, if any:

1. With a exhaustive search, find a solution $z = (z_1, ..., z_n) \in (\mathbb{Z}/p\mathbb{Z})^n$, if any.

2. Compute the Jacobian matrix J of f; if $det(J(z)) \not\equiv 0 \bmod p$ continue, if $J(z) \equiv 0 \bmod p$, return to step 1 (to look for another possible solution in $(\mathbb{Z}/p\mathbb{Z})^n$ if another solution exists).

3. Using Hensel's Lemma double the p-adic accuracy of the solution (i.e, if we have a solution $z \in (\mathbb{Z}/p^m\mathbb{Z})^n$ applying Hensel's Lemma we can get a solution $z \in (\mathbb{Z}/p^{2m}\mathbb{Z})^n$).

4. Compute $X = (x_1, ..., x_n)$ a lift of the solution in $\mathbb{Q}$ using Chinese remainder theorem and Euclidean algorithm (as in algorithm 3.3.6).

5. Check whether the lift solves the system of equations over $\mathbb{Q}$. If it does not, return to step 3, and increase the p-adic precision, and try steps 4 and 5 again.

The core step of Hensel's lemma is to define a sequence (z_n) of values that converge to the unique solution in $\mathbb{Q}_p$. On the step 3 in algorithm 3.3.11 to converge, we need some regularity assumptions on algorithm 3.3.11 for the system of polynomials f. It will converge if the Jacobian determinant $|det(J(z)|$ does not vanish at $z = (z_1, ..., z_n)$, as we can see in [Lew19; Eis95], this is why step 2 is included before applying Hensel's lemma.

We provide a small pseudo-code implementation of Algorithm 3.3.11 that runs for a fixed number of iterations. The algorithm takes as input a list of functions with rational coefficients $F_c = [f_i]_i$, the prime number p to work with, and the number of iterations n to run.

Algorithm 2 p-adic multi-variable Newton iteration method

procedure : $(F_c = \{f_i\}, p, iterations)$
    exhaustive search to get $(z_1, ..., z_n)$ with $z_i \in \mathbb{F}_p$
    $x_{old} \leftarrow (z_1, ..., z_n)$
    $F_c = \mathbb{F}_p(F_c) \leftarrow F_c$
    $J = Jacobian(F_c) \leftarrow F_c$
    if $det(J) \neq 0$ **then**
        $Res = J^{-1} * F_c$
        for $i < iterations$ **do**
            $Res = Res(x = x_{old}) \leftarrow x_{old}$
            $x_{old} = \mathbb{F}_p^i(x_{old} - Res)$
            $x_{old} = \mathbb{Z}(x_{old})$
    return x_{old}

With these techniques at hand, we can continue our search of singular K3 surfaces.

3.4 Singular K3 surfaces with $MW - rank\ 1$

We now shift our focus to the next case in our investigation of singular K3 surfaces with class number 4: those with $rank(MW) = 1$ as in example 3.3.7. To analyze such surfaces, the first step in Algorithm 3.1.6 is to obtain a suitable lattice L. Thankfully, lattice methods, aided by computer calculations, can be employed to systematically generate lattices that fulfill the constraints outlined in Theorems 2.2.10 and 2.2.4, as detailed in the following algorithm.

Algorithm 3.4.1. Let d be a fixed fundamental discriminant with $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $Q = \begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix} \in Cl(d)$ a binary quadratic form. The following steps provide candidate lattices (N, T), if any exist. These lattices satisfy the condition that, should a K3 surface X exist with $NS(X) \cong U \oplus N$ and $Triv(X) \cong T$, it would meet the following criteria: $\rho(X) = 20$, $NS(X)$ would possess a Galois action by $Gal(H(d)/\mathbb{Q})$, and $T(X) \cong Q$.

1. By performing an exhaustive search, we can obtain a collection of root lattices $\mathcal{R}(n)$ for any natural number n. Each lattice $L \in \mathcal{R}(n)$ can be expressed as a sum $L = \sum_i L_i$ of positive definite ADE root lattices L_i, such that the rank of L is n, i.e., $\sum_i \operatorname{rank}(L_i) = n$.

2. We consider all lattices L in $\mathcal{R}(17)$ and take their associated Gram matrix I_L, which has rank 17.

3. Construct a torsion-free lattice N of rank 18 such that L is a sublattice of N with the following properties: the Gram matrix I_N of N has a block form with I_L as one of the blocks; furthermore, the last row and column of I_N correspond to the possible intersections of a section P with the simple components of singular fibers of type L_i. Specifically, for each L_i, there is at most one entry equal to -1 in the last row and column of I_N that corresponds to the intersection of P with a simple component of the singular fiber of type L_i.

4. Compute the quadratic forms q_Q and q_N of Q and N respectively, if the quadratic forms are equivalent $-q_N \cong q_Q$ (if they are in the same genus), we can proceed.

5. We begin by considering a complex K3 surface X with $NS(X) = U \oplus N$, which is always possible since the lattice $U \oplus N$ admits a primitive embedding into the K3 lattice (see [Mor84]). This allows us to express $Triv(X) = U \oplus \sum_i L_i$, where L_i denotes the types of singular fibers. We can apply Corollary 1.4.10 (the Shioda-Tate formula) to conclude that X possesses a rank-1 Mordell-Weil group, generated by a single section P. Assuming the existence of a model for X over $\mathbb{Q}$, to ensure that $NS(X)$ admits a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois action, we perform a quadratic twist to ensure that the components of one singular fiber L_i (for a I_n or IV^* singular fiber) are defined over $\mathbb{Q}$.

- A reducible fiber of type I_n for $n \geq 3$, I_n^* for $n \geq 0$ or IV, IV^*, allowing for an involution of the components that preserves incidence relations and identity component.
- There are several reducible fibers of the same type that are Galois conjugates, in this case they need to preserve the incidence with the zero section, and the section P.

6. We verify that the Mordell-Weil group $MW(X) \cong N/L$ is torsion-free because the configuration of the singular fibers ensures that there are no torsion sections (see proposition 1.4.15).
7. Return the pair of lattices (N, L) as one of our candidates.

Remark 3.4.2. In Step 3 of Algorithm 3.4.1, the approach by which we construct the lattice N determines the behavior of the section P, its interaction with the zero section, and the local contributions of the singular fibers. The intersection with the zero section is dictated by the element $N_{18,18}$. In our specific cases, where this value equals 4, it results in the section P having no intersection with the zero section.

Remark 3.4.3. In step 5 of Algorithm 3.4.1, our focus lies on examining the Galois action with regard to singular fibers. This is a result of the earlier mentioned quadratic twist, which has a specific purpose: ensuring that the individual components of a singular fiber F_v are defined over $\mathbb{Q}$. This quadratic twist, in particular, affect the section P. Its effect depend of the section's intersection with the singular fiber F_v and the other singular fibers. The reason for this dependency is that the fiber components of F_v are defined over $\mathbb{Q}$, while the same is not necessarily true for the other singular fibers. Therefore, focusing on the Galois action on singular fibers could simplify the search of the desired singular K3 surface.

Algorithm 3.4.1 may return multiple candidate K3 surfaces. In such cases, we need to choose the best candidate, but determining what constitutes the "best" candidate can be unclear. Nonetheless, we can consider a few criteria to make this choice. For instance, one option is to choose the candidate with the smallest number of singular fibers, as this simplifies computations. Alternatively, we can choose the section with the smallest height pairing, as it also simplifies calculations.

We have here a small pseudo-code of our implementation of this algorithm in Sagemath.

Algorithm 3 Lattices candidates for $NS(X)$

procedure (n, d, Q)
  $P(n)$ = *Partitions of* $n \leftarrow n$
  $\mathcal{R}(n)$ = *Root lattices of rank* $n \leftarrow P(n)$
  for $L \in \mathcal{R}(n)$ **do**
    Gram matrix $I_L \leftarrow L$
    for Possible section P **do**
      I_N − Gram Matrix $\leftarrow (L, P)$
      N − Lattice $\leftarrow I_N$
      if *Discriminant group*$(N) \cong Q$ and $disc(L_s) = d$ **then**
        Lattices $\leftarrow$ *Lattices* $+ (L, N)$
  for $S \in$ *Lattices* **do**
    if S admits enough Galois action **then**
      LatGalois $\leftarrow$ *LatGalois* $+ S$
  for $G \in$ *LatGalois* **do**
    $MWL \leftarrow G$
    if MWL is torsion free **then**
      Candidates $\leftarrow$ *Candidates* $+ G$
  return *Candidates*

To illustrate how Algorithm 3.4.1 works, we present an example of the computations carried out to obtain the lattice for $d = -120$ and $Q \cong \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}$.

Example 3.4.4. In Example 3.3.7, we computed the Weierstrass model for a K3 surface with discriminant and transcendental lattice. Then, using Algorithm 3.4.1, we obtain the lattice N and a configuration of singular fibers satisfying the necessary conditions for a singular K3 surface with $NS(X) \cong U \oplus N$ to have $T(X) \cong \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}$ and be defined over $\mathbb{Q}$.

After an exhaustive search, Algorithm 3.4.1 returned several lattice candidates, among which is N, a lattice with rank 18. The lattice N contains a sublattice L isomorphic to $A_5 + D_6 + E_6$. A K3 surface with Néron-Severi isomorphic to N is a singular K3 surface with three singular fibers, of types I_6, I_2^* and IV^*. Moreover, this K3 surface has a Mordell-Weil rank of one, and the generator of the Mordell-Weil group is a section P and a height $h(P) = 5/3$. Notably, P intersects the Θ_1 simple component of the I_2^* fiber and the Θ_1 simple component of the IV^* fiber.
It is worth noting that after a quadratic twist, the fiber components of the I_6 fiber may be defined over $\mathbb{Q}$, and the lattice N admits a Galois action of $(\mathbb{Z}/2\mathbb{Z})^2$. This action corresponds to an permutation of the simple far fiber components of the I_2^* fiber and an permutation of the two simple fiber components Θ_2 and Θ_3 of the IV^* fiber. We have $disc(L) = 6 \cdot 4 \cdot 2$ and $disc(N) = disc(L) \cdot h(P) = -120$.

In the construction of the Gram matrix I_N for the lattice N, we adhered to the instructions details in Step 3 of Algorithm 3.4.1. It's noteworthy to mention that we reversed the sign to ensure that the lattice N becomes positive-definite. The lattice N would represent the Frame lattice which is the orthogonal projection of $U = \langle O, F \rangle$.

The matrix is constructed as a block matrix where each block represents the Gram matrix of the root lattices A_5, D_6, and E_6. The last element of the matrix, which is 4, represents the self-intersection of the section P (after the orthogonal projection with respect to U). The matrix is embedded into the frame lattice W. For any section P, its projection in W satisfies $(P)^2 = 4 + 2(P \cdot O)$. We have that $(P \cdot O) = 0$, the projection in W simplifies to $(P)^2 = 4$. We have the Gram matrix I_N for the lattice N:

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}$$

Now, we can compute the discriminant group of the lattice N and the lattice given by the matrix $Q = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}$. If we start with the dual lattice of Q, we get a lattice given by the Gram matrix $\begin{pmatrix} 1/10 & 0 \\ 0 & 1/12 \end{pmatrix}$, from it we can compute the discriminant group of Q, we obtain: The discriminant group A_Q is a finite module over $\mathbb{Z}$, it is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/60\mathbb{Z})$ and the Gram matrix of the quadratic form q_Q with values in $\mathbb{Q}/2\mathbb{Z}$: $\begin{pmatrix} 1/10 & 0 \\ 0 & 1/12 \end{pmatrix}$.

We compute the discriminant group and quadratic form of N, the discriminant group A_N is a finite module over $\mathbb{Z}$ given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$ and Gram matrix of the quadratic form q_N with values in $\mathbb{Q}/2\mathbb{Z}$: $\begin{pmatrix} 1/10 & 0 \\ 0 & 1/12 \end{pmatrix}$.

Having that the two discriminant groups A_N and A_Q are isomorphic, and the two quadratic forms q_Q, q_N are equivalent. By theorem 1.1.13, a K3 surface X with $NS(X) = U + N^-$, is a singular K3 surface with $disc(X) = -120$, and $T(X) = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}$.

Table 3.6 presents a list of all cases of singular K3 surfaces with fundamental discriminant d and $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$, all of which we have found a model over $\mathbb{Q}$ as singular K3 surfaces with Mordell-Weil group of rank 1.

No	$disc(NS(X))$	$\mathcal{R}(W)$	$T(X)$	$h(P)$
1	-84	$A_2 + D_5 + D_{10}$	$[6, 0, 14]$	$7/4$
2	-84	$D_5 + D_6 + E_6$	$[10, 4, 10]$	$7/4$
3	-120	$A_5 + D_6 + E_6$	$[10, 0, 12]$	$5/3$
4	-132	$A_6 + D_5 + E_6$	$[2, 0, 66]$	$11/7$
5	-132	$A_2 + A_4 + D_5 + D_6$	$[4, 2, 34]$	$11/20$
6	-132	$A_2 + A_4 + D_5 + E_7$	$[6, 0, 22]$	$11/6$
7	-132	$A_1 + A_6 + D_4 + E_6$	$[12, 6, 14]$	$11/14$
8	-168	$A_1 + A_5 + D_5 + E_6$	$[2, 0, 84]$	$7/6$
9	-168	$A_1 + A_2 + A_4 + D_4 + E_6$	$[6, 0, 28]$	$7/15$
10	-168	$A_1 + A_3 + D_7 + E_6$	$[12, 0, 14]$	$7/4$
11	-195	$A_1 + A_4 + E_6 + E_6$	$[14, 1, 14]$	$13/6$
12	-195	$A_2 + A_4 + A_4 + D_7$	$[10, 5, 22]$	$13/20$
13	-228	$A_2 + A_4 + D_5 + D_6$	$[2, 0, 114]$	$19/20$
14	-228	$A_1 + A_2 + A_4 + A_6 + D_4$	$[6, 0, 34]$	$19/70$
15	-228	$A_1 + A_2 + A_4 + D_4 + E_6$	$[12, 6, 22]$	$19/20$
16	-280	$A_1 + A_6 + D_4 + E_6$	$[2, 0, 140]$	$5/3$
17	-280	$A_3 + A_9 + D_5$	$[10, 0, 28]$	$7/4$
18	-280	$A_1 + A_4 + D_5 + D_7$	$[14, 0, 20]$	$7/4$
19	-312	$A_1 + A_1 + A_4 + D_5 + E_6$	$[2, 0, 156]$	$13/10$
20	-312	$A_1 + A_2 + A_4 + D_4 + E_6$	$[4, 0, 78]$	$13/15$
21	-312	$A_2 + A_4 + D_5 + D_6$	$[6, 0, 52]$	$13/10$
22	-312	$A_1 + A_5 + D_5 + E_6$	$[12, 0, 26]$	$13/6$
23	-340	$A_1 + A_1 + A_4 + D_5 + E_6$	$[2, 0, 170]$	$17/12$
24	-340	$A_4 + A_7 + D_6$	$[20, 10, 22]$	$17/8$
25	-340	$A_4 + A_4 + A_5 + D_4$	$[10, 0, 34]$	$17/30$

Continued on next page

Table 3.6 – *Continued from previous page*

No	$disc(NS(X))$	$\mathcal{R}(W)$	$T(X)$	$h(P)$
26	-372	$A_2 + A_5 + A_6 + D_4$	$[2, 0, 186]$	$31/42$
27	-372	$A_1 + A_2 + A_4 + D_4 + E_6$	$[4, 2, 94]$	$31/30$
28	-372	$A_1 + A_2 + A_4 + A_6 + D_4$	$[6, 0, 62]$	$31/70$
29	-372	$A_1 + A_1 + A_4 + D_5 + E_6$	$[12, 6, 34]$	$31/20$
30	-408	$A_2 + A_4 + D_5 + D_6$	$[2, 0, 204]$	$17/10$
31	-408	$A_1 + A_1 + A_4 + D_5 + E_6$	$[4, 0, 102]$	$17/10$
32	-408	$A_1 + A_2 + A_4 + A_6 + D_4$	$[6, 0, 68]$	$17/35$
33	-408	$A_1 + A_2 + A_4 + D_4 + E_6$	$[12, 0, 34]$	$17/15$
34	-435	$A_2 + A_4 + A_6 + D_5$	$[2, 1, 218]$	$29/28$
35	-435	$A_1 + A_4 + A_6 + E_6$	$[22, 7, 22]$	$29/14$
36	-483	$A_1 + A_4 + A_6 + E_6$	$[2, 1, 242]$	$23/10$
37	-520	$A_1 + A_2 + A_4 + D_5 + D_5$	$[4, 0, 130]$	$13/12$
38	-520	$A_3 + A_9 + D_5$	$[10, 0, 52]$	$13/4$
39	-520	$A_1 + A_1 + A_4 + D_5 + E_6$	$[20, 0, 26]$	$7/6$
40	-532	$A_2 + A_3 + A_6 + D_6$	$[2, 0, 266]$	$19/12$
41	-532	$A_1 + A_2 + A_4 + A_6 + D_4$	$[4, 2, 134]$	$19/30$
42	-555	$A_2 + A_4 + A_6 + D_5$	$[6, 3, 94]$	$37/28$
43	-555	$A_1 + A_4 + A_6 + E_6$	$[26, 11, 26]$	$37/14$
44	-708	$A_1 + A_2 + A_4 + A_6 + D_4$	$[6, 0, 118]$	$59/70$
45	-760	$A_1 + A_4 + A_8 + D_4$	$[2, 0, 380]$	$19/9$
46	-760	$A_1 + A_2 + A_4 + A_6 + D_4$	$[20, 0, 38]$	$19/21$
47	-795	$A_2 + A_4 + A_6 + D_5$	$[6, 3, 134]$	$53/28$

Table 3.6: Singular $K3$ surfaces with class number 4 and Mordell-Weil group of rank 1.

Remark 3.4.5. It is worth notice that the lattices with transcendental lattice $[2, 0, 84]$ and $[6, 0, 52]$, having discriminants of -168 and -312 respectively, appear in both Table 3.6 and Table 3.3. it mean the existence of two distinct fibrations on the same surface.

Each of the surfaces listed in Table 3.6 has a model over $\mathbb{Q}$ that can be written in the form $y^2 = x^3 + A(t)x^2 + B(t)x + C(t)$. Additionally, each surface has a section given by two polynomials $P = (u(t), v(t))$, where $u(t)$ has degree at most 4 and $v(t)$ has degree at most 6. And we have the condition that $v(t)^2 = u(t)^3 + A(t)u(t)^2 + B(t)u(t) + C(t)$, so we only need the first polynomial $u(t)$ to fully describe the section on our surface. We will provide the polynomials $A(t)$, $B(t)$, $C(t)$, and $u(t)$ that are required to describe the Weierstrass models over $\mathbb{Q}$ for every singular K3 surface listed in Table 3.6.

No	$A(t)$	$B(t)$	$C(t)$	$u(t)$
1	$(1/256)t(16t-21)^2$	$(1/256)t(16t-21)^2$	$(117649/1048576)(t-1)^2t^3$	$(-1/64)t(64t^2-105t+49)$
2	$(-9)t(t-1)$	$(-15)t^2(t-1)^3$	$(-7)t^3(t-1)^5$	$(1/9)t(t-1)(64t^2+8t+9)$
3	$(2/9)(11t-2)(t-1)^2$	$(-1/9)(71t-20)t^2(t-1)^3$	$(5/9)(8t-5)t^4(t-1)^4$	$(-8/9)(5t+3)t^2$
4	$(-2/15)(7t-25)(t-1)^2$	$(1/25)(t+50)t^2(t-1)^3$	$(3/250)(8t+25)t^4(t-1)^4$	$(-4/15)(t-3)t^2$
5	$(1/22528)(t-33)(t-1)$ $(49t^2+22t+121)$	$(1/131072)(t-33)$ $(t-1)(49t^2+22t+121)$	$(11/33554432)t^2$ $(t-33)^3(t-1)^5$	$(-1/1408)t(t-33)$ $(t-1)(t+11)$
6	$(1/500)(t+9)$ $(49t-529)(t-1)^2$	$(-2/125)t(7t-23)$ $(t+9)^2(t-1)^3$	$(4/125)t^2$ $(t+9)^3(t-1)^5$	$(-1/100)t(13t+112)(t-1)^2$
7	$(-1/9)(t+3)(27t^2-36t+16)$	$(1/3)(t-1)(9t-8)t^2(t+3)^2$	$(-1)(t-1)^2(t+3)^3t^4$	$(-3/4)(t-1)(t+3)t^2$
8	$(-1/9)(48t+1)(243t^2-27t+1)$	$9(t-1)(27t-2)t^2(48t+1)^2$	$(-729)(t-1)^2(48t+1)^3t^4$	$(-48)(t-1)(48t+1)t^2$
9	$(-3/512)(32t-7)(16t^2+8t+3)$	$(3/2048)t(2t+1)$ $(t-1)^2(32t-7)^2$	$(-1/32768)t^2$ $(32t-7)^3(t-1)^4$	$(1/32)(32t-7)(t-1)^2$
10	$(-3/4)t(4t^2-28t+27)$	$3(t+3)(t-1)^2t^3$	$(-1)(t-1)^4t^5$	$(-1/900)t(t-1)$ $(16384t^2-34564t+18225)$
11	$(t-1)^2$	$(1/72)t(15t-16)(t-1)^3$	$(1/186624)t^2(t-1)^4$ $(1625t^2-4064t+2304)$	$(-1/540)(125t-144)$ $(125t-114)(t-1)^2$
12	$-1/384t(2t-1)(92t^2+4t+23)$	$-(1/4608)t^2(4t^2+12t+1)$ $(4t^2+1)^2$	$(1/24576)(2t-1)t^3(4t^2+1)^4$	$(1/120)t(6t-5)(4t^2+1)$
13	$(4/3798613)(t-1)(64t-57)$ $(4624t^2-6308t+361)$	$(-128/815730721)t(68t-19)$ $(64t-57)^2(t-1)^3$	$(136192/23298085122481)t^2$ $(64t-57)^3(t-1)^5$	$(4/542659)t(t-1)$ $(64t-57)(512t-323)$
14	$(27/2000)(2t-5)$ $(32768t^3-17520t^2-13800t-125)$	$(19683/1250)(t-1)t^2$ $(2t-5)^2(8192t^2-7150t-475)$	$(14348907/3125)(2048t-1805)$ $(t-1)^2(2t-5)^3t^4$	$(-729/40)t(2t-5)(8t-7)$
15	$(-3/512)(32t-81)$ $(16t^2+8t+3)$	$(3/2048)t(2t+1)$ $(t-1)^2(32t-81)^2$	$(-1/32768)t^2$ $(32t-81)^3(t-1)^4$	$(1/9)t(32t-81)$ $(t-1)^2$

Table 3.7: Equations for singular K3 surfaces of Table 3.6, 1 to 14.

No	$A(t)$	$B(t)$	$C(t)$	$u(t)$
16	$(-1/3375)(375t-32)$ $(27t^2-36t+16)$	$(1/421875)(t-1)(9t-8)$ $t^2(375t-32)^2$	$(-1/52734375)(t-1)^2$ $(375t-32)^3t^4$	$(1/4687500)(375t-32)$ $(12500t^2-625t+32)$
17	$(-1/23901220)(62143172t^3-$ $-21899640t^2+110200t-125)$	$(1/58870)(t-1)t^3$ $(129514t^2-11890t+25)$	$(-1/580)(348t-5)$ $(t-1)^2t^6$	$(1/1421)(t-1)$ $(1421t^2-630t+50)$
18	$(-4/84035)(27t-20)$ $(2916t^2-1520t+25)$	$(-128/5764801)t(t-1)$ $(54t-5)(27t-20)^2$	$(-5120/1977326743)t^2$ $(t-1)^2(27t-20)^3$	$(16/61261515)(27t-20)$ $(531441t^2-1094445t+563200)$
19	$(-2/25)(675t+1)$ $(1200t^2+424t+1)$	$(512/5)(t-1)$ $(30t-1)t^2(675t+1)^2$	$(-32768)(t-1)^2$ $(675t+1)^3t^4$	$(-675/2)(t-1)$ $(675t+1)t^2$
20	$(-3/16)(t+49)(16t^2+8t+3)$	$(3/2)t(2t+1)(t-1)^2(t+49)^2$	$(-1)t^2(t+49)^3(t-1)^4$	$(1/32)(t-1)(t+49)(32t+343)$
21	$(1/77440)(t-1)(2197t+363)$ $(48841t^2-101882t+14641)$	$(-1/512)t(221t-121)$ $(2197t+363)^2(t-1)^3$	$(605/8192)t^2$ $(2197t+363)^3(t-1)^5$	$(-605/24)t$ $(55t+9)(t-1)^2$
22	$(-1/74287161)(204t+2197)$ $(70227t^2-77571t+28561)$	$(9/13867422257)(t-1)$ $(459t-338)t^2(204t+2197)^2$	$(-729/23298085122481)$ $(t-1)^2(204t+2197)^3t^4$	$(-3/8833547977709)t(204t+2197)$ $(75759616t^2-636240215t+135150652)$
23	$(-27/937024)(187t+27)$ $(5808t^2+3256t+729)$	$(81/9977431552)t$ $(22t+9)(187t+27)^2$ $(52272t^2+11044t-5103)$	$(-27/424958764662784)t^2$ $(187t+27)^3$ $(52272t^2+11044t-5103)^2$	$(1/3982352)(187t+27)$ $(222156t^2-65263t+3456)$
24	$(-1/3600)t(3125t^3-19050t^2+$ $+23868t-7992)$	$(-2/625)t^2(t-1)^2$ $(625t^2-2940t+2196)$	$(-144/15625)(125t-414)$ $t^3(t-1)^4$	$(-9/8000)$ $(1024t^2-620t+625)$
25	$(-1/1600)(t-2)$ $(25t-1)(t^2-100t+100)$	$(-1/20000)(t-1)t^2$ $(25t-1)^2(t^2-20t+20)$	$(-1/1000000)(t-2)$ $(t-1)^2(25t-1)^3t^4$	$(-1/250000000000)(t-1)^5t^6$ $(25t-1)^6(3t^2+500t-500)$
26	$(27/1600)(3t-2)(25t-24)$ $(9t^2+36t+4)$	$(729/20000)(t-1)(3t+2)$ $(729/20000)(t-1)(3*t+2)$	$(19683/1000000)(27t-2)$ $(t-1)^2(25t-24)^3t^4$	$(-27/1600)t$ $(25t-24)(64t-63)$
27	$(25/995328)(864t+361)$ $(16t^2+8t+3)$	$(625/23219011584)t(2t+1)$ $(t-1)^2(864t+361)^2$	$(15625/6499837226778624)$ $t^2(864t+361)^3(t-1)^4$	$(1/248832)t(t-1)$ $(288t+55)(864t+361)$

Table 3.8: Equations for singular K3 surfaces of Table 3.6, 15 to 26.

No	$A(t)$	$B(t)$	$C(t)$	$u(t)$
28	$(27/512000)(512t-605)$ $(32768t^3-17520t^2$ $-13800t-125)$	$(19683/81920000)$ $(t-1)t^2(512t-605)^2$ $(8192t^2-7150t-475)$	$14348907/52428800000$ $(2048t-1805)(t-1)^2$ $(512t-605)^3t^4$	$(-729/6400)$ $(512t-605)t^2$
29	$(1/153664)(567t-961)$ $(2352t^2+3640t+961)$	$(1/13071274496)t$ $(42t+31)(567t-961)^2$ $(24304t^2+58772t+35557)$	$(1/4447579574910976)$ $t^2(567t-961)^3$ $(24304t^2+58772t+35557)^2$	$(-1/5068283248)t$ $(567t-961)(4064256t^2+$ $28657020t+34170277)$
30	$(1/8794907264)(t-1)$ $(4913t-3249)$ $(1075369t^2-1724858t+130321)$	$(-1/109751747072)t$ $(1037t-361)(4913t-3249)^2$ $(t-1)^3$	$(4693/25710005262098432)$ $t^2(4913t-3249)^3$ $(t-1)^5$	$(-361/117128)t$ $(76t-89)(t-1)^2$
31	$(3/9800)(175t-256)$ $(1225t^2-8512t+6912)$	$(3/2240)(t-1)(35t+96)$ $t^2(175t-256)^2$	$(1/512)(t-1)^2$ $(175t-256)^3t^4$	$(-5/3528)t(t-1)$ $(175t-256)(2240t-3267)$
32	$(3/1000)(9t-10)(32768t^3$ $-17520t^2-13800t-125)$	$(486/625)(t-1)t^2$ $(9t-10)^2(8192t^2-7150t-475)$	$(157464/3125)(2048t-1805)$ $(t-1)^2(9t-10)^3t^4$	$(-81/80)t$ $(9t-10)(32t-27)$
33	$(-1/144)(27t-196)$ $(16t^2+8t+3)$	$(1/486)t(2t+1)$ $(t-1)^2(27t-196)^2$	$(-1/19683)t^2$ $(27t-196)^3(t-1)^4$	$(1/64)t$ $(27t-196)(t-1)^2$
34	$(27/5070025688)(841t-91)$ $(3048625t^3+10024287t^2$ $-7129941t-753571)$	$(19683/48501766991041)$ $(t-1)t^2(841t-91)^2$ $(3048625t^2+1205386t-587951)$	$(28697814/3711888729591358771)$ $(3048625t-458731)(t-1)^2$ $(841t-91)^3t^4$	$(-2916/9332572795)$ $(841t-91)$ $(121945t^2-45414t+3094)$
35	$(-1/27)$ $(29008t^2-420t-27)$	$(-43904/19683)(t-1)$ $t^2(92512t^2-5992t-837)$	$(-481890304/14348907)(t-1)^2$ $t^4(250880t^2+32116t-25947)$	$(1372/455625)t^2(29360128t^2$ $+7936000t+490075)$
36	$(1/9)(400t^2+620t+9)$	$(-80000/19683)(t-1)$ $t^2(12320t^2-3032t-27)$	$(1600000000/14348907)t^4$ $(t-1)^2(23552t^2+4204t+27)$	$(500/36756909)(t-1)$ $(20971520t^3-7018496t^2$ $-1757943t-94041)$

Table 3.9: Equations for singular K3 surfaces of Table 3.6, 27 to 35.

No	$A(t)$	$B(t)$	$C(t)$	$u(t)$
37	$(-1/18078415936)$ $(3375t-392)(19683t-5096)$ $(164025t^2-297136t+153664)$	$(-1/2712892291396)$ $(405t-392)t^2(3375t-392)^2$ $(19683t-5096)^2$	$(-1/1628413597910449)$ $(3375t-392)^3$ $(19683t-5096)^3t^4$	$(8/3001814235)(1269t-98)$ $(3375t-392)$ $(19683t-5096)$
38	$(-1/206082144)(521645427t^3$ $-134292870t^2+67940t-8)$	$(1/266256)(t-1)t^3$ $(549153t^2-17716t+4)$	$(-1/1376)(731t-2)$ $(t-1)^2t^6$	$(1372/45625)(t-1)(+6596t^3+$ $+29601t^2+764600t+49075)$
39	$(1/2795584)(62073t-2048)$ $(131043t^2-69920t+4096)$	$(1/856064)(t-1)(627t-128)$ $t^2(62073t-2048)^2$	$(1/262144)(t-1)^2$ $(62073t-2048)^3t^4$	$(-11/1216)(t-1)$ $(107217t^2-194560t+1048576)$
40	$(1/16335)(5t-11)(4096t^3$ $+2211t^2-726t+1331)$	$(32/81675)(t-1)t^2$ $(5t-11)^2(256t^2+55t+121)$	$(256/408375)(16t+11)$ $(t-1)^2(5t-11)^3t^4$	$(-1/375)(t-1)$ $(5t-11)(75t-11)$
41	$(27/608000)(608t-245)$ $(32768t^3-17520t^2$ $-13800t-125)$	$(19683/115520000)(t-1)$ $t^2(608t-245)^2$ $(8192t^2-7150t-475)$	$(14348907/87795200000)$ $(2048t-1805)(t-1)^2$ $(608t-245)^3t^4$	$(-486/475)(t-1)$ $(608t-245)t^2$
42	$(27/16656623000)(1369t-$ $345)(1367631t^3-6381465t^2$ $5673525t-1520875)$	$(-1458/327791930250625)t^2$ $(t-1)(1369t-345)^2(1367631t^2$ $-3694950t+1679575)$	$(157464/51605922539007146875)$ $(1367631t-1854835)(t-1)^2$ $(1369t-345)^3t^4$	$(27/15949110110)(1369t-345)$ $(810448t^2+86247t+127305)$
43	$(-1/1125)$ $(353648t^2+38440t-1125)$	$(-3813248/307546875)$ $(t-1)t^2$ $(2671580t^2+878137t-31050)$	$3635215077376/28025208984375$ $(-1)(t-1)^2t^4$ $(4444625t^2+6397036t-285660)$	$(t)(486+198t$ $+38420t^2+50212t^3)$
44	$(27/98000)(98t-605)$ $(32768t^3-17520t^2$ $-13800t-125)$	$(19683/3001250)(t-1)$ $t^2(98t-605)^2$ $(8192t^2-7150t-475)$	$(14348907/367653125)$ $(2048t-1805)(t-1)^2$ $(98t-605)^3t^4$	$(-27/480200)t$ $(98t-605)(53176t-46585)$
45	$(-1/30752)(961t-486)$ $(t^3-257t^2+459t-243)$	$(-1/3694084)(t-1)$ $t^2(961t-486)^2$ $(t^2-78t+81)$	$(-1/1775007362)$ $(t-27)(t-1)^2$ $(961t-486)^3t^4$	$(-1/124545600)$ $t(961t-486)(28629151t^2$ $-30047166t+7971615)$

Table 3.10: Equations for singular K3 surfaces of Table 3.6, 36 to 44.

No	$A(t)$	$B(t)$	$C(t)$	$u(t)$
46	$(27/1568000)(1568t+19)$ $(32768t^3-17520t^2$ $-13800t-125)$	$(19683/81920000)(t-1)$ $t^2(1568t+19)^2$ $(8192t^2-7150t-475)$	$(14348907/52428800000)$ $(2048t-1805)(t-1)^2$ $(1568t+19)^3t^4$	$(-243/98000)(t-1)$ $(75t+1)(1568t+19)$
47	$(27/780856112176)$ $(5618t-6993)(4019679000t^3$ $-2063615652t^2$ $-1822859094t-17373979)$	$(970299/142024257036879364)$ $(t-1)t^2(5618t-6993)^2$ $(2009839500t^2-1775836608t$ $-125240227)$	$(34869635163/$ $1033270497462978805702840)$ $(1004919750t-902793451)$ $(t-1)^2(5618t-6993)^3t^4$	$(-594/20037837047095)$ $(5618t-6993)$ $(16165064660t^2$ $-32807178981t+16787374821)$

Table 3.11: Equations for singular K3 surfaces of Table 3.6, 45 and 46.

Proposition 3.4.6. All the surfaces enlisted in Table 3.6 have a Néron-Severi group generated by fibers components and the sections O and P and a Weierstrass model with coefficients over $\mathbb{Q}$.

Proof. By 3.4.1, we confirm that all the singular K3 surfaces listed in Table 3.6 are indeed singular K3 surfaces with the desired discriminant and quadratic form. Through the application of the Tate algorithm (described in Section 1.4), we verify that the surfaces listed in Tables 3.7 to 3.11 have the singular fibers and sections as described in Table 3.6
Therefore, the fiber components and sections generate a lattice M of rank 20, which we need to show is isomorphic to $NS(X)$. The discriminant of M is given by equation (3.3), and the height pairing of the section (P) is given by equation (1.27). If the index of M in $NS(X)$ were greater than 1, then $NS(X)$ would have discriminant divided by the square greater than 1 of the index. By determinant formula (1.29), the discriminant is given by the discriminant of the trivial lattice and the pairing of the section, and for all the surfaces in table 3.6, the quotient $disc(M)/disc(NS(X))$ is a square if it is equal to 1, so the only possible index is 1, so $disc(M) = disc(NS(X))$ and $M = NS(X)$, completing the proof. $\square$

We have computed models over $\mathbb{Q}$ for all the singular K3 surfaces with fundamental discriminant $|d| \leq 408$, as well as for other surfaces with discriminants $-435, -483, -520, -532, -555, -760$, and -795. We have summarized our results in Table 3.6. While we were successful in finding models for most of the surfaces with fundamental discriminant $|d| \leq 408$, there were a few exceptions. In the following section, we will discuss these exceptional cases.

3.5 Singular K3 surfaces with class number 4 and $MW-rank$ 2.

So far, we have successfully computed explicit models over $\mathbb{Q}$ for all singular K3 surfaces with fundamental discriminant d up to 408 and $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$, with the exception of three special cases:

1. $d = -195$ and $T = [2, 1, 98]$.

2. $d = -228$ and $T = [4, 2, 58]$.

3. $d = -340$ and $T = [4, 2, 86]$.

We were unable to compute models over $\mathbb{Q}$ for these 3 cases as a singular K3 surfaces with Mordell-Weil rank 1. To resolve these three remaining cases, we will utilize singular K3 surfaces with Mordell-Weil rank 2. To do this, we need to make some modifications to the algorithms outlined in the previous sections. Specifically, we need to find a suitable

lattice N with a root lattice $L = \mathcal{R}(N)$ of rank 16, along with two sections. We can adapt algorithm 3.1.6 to this setting by modifying step 1 to require $rank(L) = 16$.
By modifying algorithm 3.4.1, we can construct a root lattice L of rank 16 and two sections P and Q that generate the Mordell-Weil group. We can then compute a discriminant d of the form

$$d = -\operatorname{disc}(L) \cdot \det \begin{pmatrix} h(P) & \langle P, Q \rangle \\ \langle P, Q \rangle & h(Q) \end{pmatrix} \tag{3.14}$$

using corollary 1.4.16. To ensure that the lattice N admits a possible $(\mathbb{Z}/2\mathbb{Z})^2$, we consider the automorphism group of the Mordell-Weil group, as discussed in section 3.1.1 and the Galois action on the singular fibers.

Assuming that the method mentioned above was successful, we obtained a root lattice L with rank 16. By using this lattice, we can generate a family of K3 surfaces via a Weierstrass equation of the form

$$y^2 = x^3 + A(t)x^2 + B(t)x + C(t) \qquad \text{with } A, B, C \in \mathbb{Q}[t]$$

as in equation (1.12), utilizing the Tate algorithm discussed in Section 1.4. Specifically, we construct a 2-dimensional family $X_{\alpha,\lambda}$ that satisfies $\rho(X_{\alpha,\lambda}) \geq 18$ for all α, λ, this family will have the required number and type of singular fibers, which is determined by L.
We specialize the 2-dimensional family $X_{\alpha,\beta}$ to obtain a 1-dimensional family X_λ while keeping the singular fibers fixed (we can do this with the methods we have study until this point). This family satisfies $\rho(X_\lambda) \geq 19$, providing a higher Picard rank than the original family.

We introduce the section with the smallest height of the two; we denote it by P. Utilizing $P = (u, v)$ where $u, v \in \mathbb{C}[t]$, we substitute u into the Weierstrass equation of $X_{\alpha,\lambda}$, as $y^2 = u^3 + A(t)u^2 + B(t)u + C(t)$. This allows us to obtain the desired 1-dimensional family X_λ with the desired singular fibers and a section P.
Now we can apply the methods that we studied in Section 3.3 to the 1-dimensional family X_λ to obtain the section Q. In this case, we must account for both sections P and Q to derive the Lefschetz fixed point formula, it has the equation:

$$tr(Frob_p^*(T(X))) = \#X(\mathbb{F}_p) - 1 - tr(NS(X)) - p^2.$$

Here we have to take into consideration the section P. So for a prime p with a good reduction, the equation (3.12) changes to get:

$$\#X_s(\mathbb{F}_p) - 1 - 2p - p^2 \pm p \pm p = \alpha_p + \beta_p = a_p. \tag{3.15}$$

The first $\pm p$ on (3.15) is determined by the section P, specifically by its field of definition. Given that we already know P at this point, we can determine the sign of $\pm p$. And the second $\pm p$ term on (3.15) indicates the dependence on the field of definition of the section Q, which remains undetermined. With these considerations we can apply algorithm 3.3.6 to search the parameter λ_0 such X_{λ_0} has a model over $\mathbb{Q}$ and $\rho(X_{\lambda_0}) = 20$, if we obtain any from the algorithm.

To construct families of singular K3 surfaces with Mordell-Weil rank 2, we modified algorithm 3.4.1 to obtain a root lattice L of rank 16 as a sublattice of a lattice N of rank 18, where N is constructed from L and two sections P and Q, and the discriminant is given by equation (3.14). We computed three cases as singular K3 surfaces with Mordell-Weil rank 2, which are listed in table 3.12, along with the transcendental lattice $T(X)$ and the height of sections P and Q.

$disc(NS(X))$	$\mathcal{R}(W)$	$T(X)$	$h(P)$	$h(Q)$	$\langle P, Q\rangle$
-195	$A_1 + A_9 + E_6$	$[2, 1, 98]$	1	$7/2$	$1/2$
-228	$A_1 + D_4 + D_5 + E_6$	$[4, 2, 58]$	$3/2$	$9/4$	1
-340	$A_2 + A_2 + D_6 + D_6$	$[4, 2, 86]$	$5/3$	$11/6$	$5/6$

Table 3.12: Singular K3 surfaces with Mordell-Weil group of rank 2.

We will focus on one of these three cases in the next example, providing a detailed solution that includes the Weierstrass equation and sections. For the remaining two cases, we will only list the corresponding Weierstrass equations and sections.

Example 3.5.1. Discriminant $d = -340$ and $T(X) = [4, 2, 86]$.

Using the method described above, we found an elliptic K3 surface X with singular fibers I_3, I_3, I_2^*, I_2^* and two sections P and Q with pairing $5/3$ and $11/6$, respectively. The height pairing of P and Q was $\langle P, Q\rangle = 2 - 2/3 - 1/2 = 5/6$.
First we applied the Tate algorithm (as outlined in Section 1.4) to obtain a family of K3 surfaces $X_{\alpha,\beta}$ with $\rho(X_{\alpha,\beta}) \geq 18$ and the four desired singular fibers of types I_3, I_3, I_2^*, I_2^* located at the roots of the polynomial $t^2\beta^2 + t\beta^2 + 1$,0,$\infty$ respectively. The Weierstrass equation for these surfaces is given by:

$$
\begin{aligned}
A &= t(\alpha t^2\beta^3 + 8t^2\beta^4 + 8\alpha t^2\beta^2 + \alpha t\beta^3 + 46t^2\beta^3 + 4t\beta^4 + 16\alpha t^2 * \beta + 9\alpha t\beta^2 + 65t^2\beta^2 \\
&\quad + 26\beta^3 + 24\alpha t\beta + 40t\beta^2 + 16\alpha t + \beta^2 + 8\beta + 16) \\
B &= (1/4)\beta^{-1}t^3(\alpha\beta + 4\beta^2 + 4\alpha + 10\beta)(t^2\beta^2 + t\beta^2 + 1)(\alpha t\beta^3 + 20t\beta^4 + 8\alpha t\beta^2 + \alpha\beta^3 \\
&\quad + 106t\beta^3 + 4\beta^4 + 16\alpha t\beta + 10\alpha\beta^2 + 140t\beta^2 + 26\beta^3 + 32\alpha\beta + 40\beta^2 + 32\alpha) \\
C &= (1/4)(2\beta + 5)^2t^5(\alpha\beta + 4\beta^2 + 4\alpha + 10\beta)^2(t^2\beta^2 + t\beta^2 + 1)^2 \\
f &= x^3 + Ax^2 + Bx + C \\
\Delta &= (1/16)\beta^{-3}t^8(\alpha\beta + 4\beta^2 + 4\alpha + 10\beta)^3(t^2\beta^2 + t\beta^2 + 1)^3p(\alpha, \beta, t).
\end{aligned}
$$

Here $p(\alpha, \beta, t)$ is a polynomial of degree 2 in t. From this family we can solve for the section P to obtain a 1-dimensional family X_λ with $\rho(X_\lambda) \geq 19$ with 4 singular fibers of type I_3, I_3, I_2^*, I_2^*, where the two I_3 fibers are located at the roots of the polynomial $(t^2\lambda^2 + t\lambda^2 + 1)$, the fibers I_2^*, I_2^* are located at $0, \infty$ and a section P with height pairing $h(P) = 4 - 2/3 - 2/3 - 1$ intersecting the first components of the fibers I_3, I_3 at the roots of the polynomial $(t^2\lambda^2 + t\lambda^2 + 1)$ and I_2^* at $t = 0$. We can write the Weierstrass model for this family of K3 surfaces as:

$$\begin{aligned}
A &= t(4t^2\lambda^4 + 36t^2\lambda^3 + 65t^2\lambda^2 + 12t\lambda^3 + 30t\lambda^2 + \lambda^2 + 8\lambda + 16) \\
B &= 8\lambda^2(2\lambda + 5)^2 t^3(2t\lambda + 7t + 1)(t^2\lambda^2 + t\lambda^2 + 1) \\
C &= 16\lambda^2(2\lambda + 5)^4 t^5(t^2\lambda^2 + t\lambda^2 + 1)^2 \\
f &= x^3 + Ax^2 + Bx + C \\
\Delta &= 256\lambda^2(2\lambda + 5)^4(t^2\lambda^2 + t\lambda^2 + 1)^3 t^8 p(t)
\end{aligned}$$

where $p(t)$ is a polynomial of degree 2 over t. We compute x_P with the conditions that $x_p = t(t^2\lambda^2 + t\lambda^2 + 1)(x_0 + x_1 t)$ and $y_P^2 = f(x_P)$, solving for x_0 and x_1. The x-component of the section P is given by

$$x_P = -(\lambda + 2)^{-1} t(t\lambda^3 + 2t\lambda^2 + \lambda^3 + t\lambda + 10\lambda^2 + 32\lambda + 32)(t^2\lambda^2 + t\lambda^2 + 1)$$

Applying algorithm 3.3.6 we obtain the desired specialization of X_λ at $\lambda = -21/2$, then we get the final singular K3 surface X and its discriminant:

$$\begin{aligned}
A &= \frac{1}{169} t(56448t^2 - 42336t + 169) \\
B &= \frac{-903168}{28561}(14t - 1)t^3(441t^2 + 441t + 4) \\
C &= \frac{462422016}{4826809} t^5(441t^2 + 441t + 4)^2 \\
f &= x^3 + Ax^2 + Bx + C \\
\Delta &= \frac{-7398752256}{23298085122481} t^8(1018886400t^2 + 469844928t - 485537)(441t^2 + 441t + 4)^3
\end{aligned}$$

Now we need to solve for the section Q, it should intersect the first component Θ_1 of a I_3 fiber (located at a root of the polynomial $(441t^2+441t+4)$) and a far component of the I_2^* at $t = 0$. The x-component of the Q section is not defined over $\mathbb{Q}$, due to the intersection with one I_3 fiber at a root of $(441t^2 + 441t + 4)$. We have that $x_Q = (x_0 + x_1 t)\left(t + \frac{-5}{42}i + \frac{1}{2}\right)t^2$. Moreover, the fiber at $t = 0$ gives one more linear relation in the coefficients of x_Q, so there is only one degree of freedom left, which can be solved from the equation $y^2 = f(x_Q)$.

Solving for the section Q, we have the x-components of the two sections P and Q, given by :

$$
\begin{aligned}
x_P &= \frac{-1}{11492} t(7581t + 2873)(441t^2 + 441t + 4) \\
x_Q &= \left(\frac{-79507685376}{28561} i - \frac{327826953216}{28561}\right)\left(t + \frac{-5}{42} i + \frac{1}{2}\right)\left(t + \frac{-715}{15162} i + \frac{983}{5054}\right) t^2.
\end{aligned}
$$

In this case, we can see that the corresponding quadratic form of the Mordell-Weil lattice $MW(X)$ of X has the form $\begin{pmatrix} 5/3 & 5/6 \\ 5/6 & 11/6 \end{pmatrix}$, by lemma 3.1.4, the Mordell-Weil group admits a non-trivial automorphism group, in other words, the complex conjugate of Q in terms of P and Q is given by $Q \to P + Q$. Then the Néron-Severi group admits a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois actions, as conjugation of the two I_3 fibers and as involution of the simple non-identity components of the I_2^* fiber at ∞. By (3.14), we confirm that the elliptic surface X has discriminant

$$
disc(NS(X)) = -3 \cdot 3 \cdot 4 \cdot 4 \cdot det \begin{pmatrix} 5/3 & 5/6 \\ 5/6 & 11/6 \end{pmatrix} = -340.
$$

We write down the final equations of the other two singular K3 surfaces of table 3.12.

Example 3.5.2. *Case $d = -195$ and $T(X) = [2, 1, 98]$.*
For this case, we obtained a singular K3 surface with three singular fibers over $1, 0$ and ∞ of types I_2, I_{10}, IV^*, respectively. The associated Weierstrass equation and discriminant are given by:

$$
\begin{aligned}
A &= -\frac{1}{36}(72t^2 + 4t + 39) \\
B &= -\frac{1}{18}(t-1)t^2(30t^2 + 121t + 39) \\
C &= -\frac{1}{36}(t-1)^2 t^4(375t^2 + 238t + 39) \\
f &= x^3 + Ax^2 + Bx + C \\
\Delta &= (-3888)(t-1)^2 t^{10}(1265625t^4 - 662750t^3 - 118569t^2 - 10062t + 54756)
\end{aligned}
$$

We also have two sections P and Q, where the section P has a height of 1 (intersecting the first component Θ_1 of the I_2 fiber and the fith component Θ_5 of the I_{10} fiber), and the section Q has a height of $7/2$ (intersecting the first component Θ_1 of I_2). The assumption $(P, Q) = 1$ implies $\langle P, Q \rangle = 1/2$. We obtained the equations for these two sections by working with Tate's algorithm with the Weiestrass equation and the equations $y^2 = f(x_P)$ and $y^2 = f(x_Q)$ as in the previous example, we obtain that, the x-components of each of them is given by:

$$x_P = -(t-1)(3t+1)t^2$$
$$x_Q = -\frac{1}{44161200}(t-1)(t^3+13800t^2-10448t-149041)$$

We can verify that the sections intersect in one point, after a blowing up of the Weierstrass form at $t=1$, the sections P, Q meet the same non-identity component of the special fiber. The surface X does allow a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois action, which corresponds to an involution of the simple non-identity components of the I_{10} and IV^* fibers. By (3.14), we confirm that the elliptic K3 surface X has discriminant

$$disc(NS(X)) = -3\cdot 4\cdot 4\cdot 2\cdot det\begin{pmatrix}1 & 1/2\\ 1/2 & 7/2\end{pmatrix} = -195.$$

Example 3.5.3. *Case* $d=-228$ *and* $T(X)=[4,2,58]$.
For this case we got an singular K3 surface with four singular fibers of type I_2, I_0^*, I_1^*, IV^* over $1, -675, \infty$ and 0 respectively, with an associated Weierstrass equation and discriminant :

$$\begin{aligned}
A &= (2484)(t+675)t^2\\
B &= (2041200)(t-1)(t+675)^2t^3\\
C &= (554040000)(t-1)^2(t+675)^3t^4\\
f &= x^3+Ax^2+Bx+C\\
\Delta &= (918330048000000)(32t-9025)(t-1)^2(t+675)^6t^8
\end{aligned}$$

With two sections P and Q where the section P has height $h(P) = 4-1/2-1-1 = 3/2$ (intersecting the first component Θ_1 of I_2, a simple component of I_0^*, and the near component of I_1^*), the section Q has height $h(Q) = 4-1/2-5/4 = 9/4$ (intersecting the first component Θ_1 of I_2 and a far component of I_1^*) with $(P,Q)=0$ we have $\langle P,Q\rangle = 1$ and the x-component of each of them given by:

$$x_P = -9(t-1)(t+675)(76t+675)$$
$$x_Q = -100(t-1)(9t^2+5000t+40000)$$

In this case, the surface X, the corresponding quadratic form of the Mordell-Weil lattice has the form $MW(X) = \begin{pmatrix}3/2 & 1\\ 1 & 9/4\end{pmatrix}$.
According to Lemma 3.2, the Mordell-Weil lattice has only trivial automorphism group.

However, the surface X allows for a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois action, with correspond to the involution of the simple non-identity components of the I_0^* fibers that do not intersect the section Q, and the involution of the non-identity components of the IV^* fiber.

Having resolved these three cases of singular K3 surfaces with Mordell-Weil rank 2, we can get a similar proposition to proposition 3.4.6

Proposition 3.5.4. The surfaces enlisted in Table 3.12 have a Néron-Severi group generated by fibers components and the given sections and a Weierstrass model with coefficients over $\mathbb{Q}$.

The proof of this proposition is analogous to that of proposition 3.4.6 in the previous section.

3.6 Conclusions

According to Shioda-Inose's theorem, every singular K3 surface is equivalent to an oriented lattice of rank 2. Utilizing this result, we can demonstrate that for each fundamental discriminant d such that $Cl(d) \cong (\mathbb{Z}/2\mathbb{Z})^2$, there exist four singular K3 surfaces with that discriminant, up to isomorphism. In Sections 3.3, 3.4, and 3.5, we have shown that any of the four possible singular K3 surfaces (up to isomorphism) with fundamental discriminant d listed in Table 3.1 (up to -408 with class number 4) have a model over $\mathbb{Q}$.

We used Weierstrass fibrations to construct explicit singular K3 surfaces for each discriminant d and binary quadratic form $Q \in Cl(d)$, in order to demonstrate their existence. These surfaces were then classified based on the rank of their Mordell-Weil group, starting with the simplest case of extremal elliptic K3 surfaces and moving on to those with Mordell-Weil rank 1. In total, we identified 46 different surfaces in this category, which are listed in Table 3.6.

Next, we extended our analysis to the more complex cases, we also investigated singular elliptic K3 surfaces with Mordell-Weil rank 2, completing our analysis for all possible cases up to discriminant $|d| = 408$. Additionally, we discovered models over $\mathbb{Q}$ for several other, including some with singular K3 surfaces, including those with discriminants $-435, -483, -520, -532, -555, -708, -760$, and -795.

Theorem 3.6.1. Every singular K3 surface with class number 4 and fundamental discriminant $|d| \leq 408$ and whose class group $Cl(d)$ is two torsion has a model defined over $\mathbb{Q}$. This includes the singular K3 surfaces listed in Table 3.3, Table 3.6, and Table 3.12.

Proof. The extremal singular K3 surfaces cases were previously studied and can be found in [SZ01]. In section 3.2, we obtained Weierstrass models for the singular K3 surfaces listed in Table 3.3.

We also analyzed singular K3 surfaces with Mordell-Weil rank one or two. The surfaces with Mordell-Weil rank one are listed in Table 3.6 and their Néron-Severi groups are described in Proposition 3.4.6. For surfaces with Mordell-Weil rank two, we listed them in Table 3.12 and provided additional details in Proposition 3.5.4. □

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A | Appendix A Sagemath Lattices MWR-1 Code

In this appendix we included the code in Python/Sage-math for algorithm 3.4.1, which is divided in different functions.

```

#Function partition, takes to integers n,m and it returns #
#a list with the possible partition of n in a sum of m elements#
def Partition(n,m):
    if m==1:
        P=[[n]]
    else:
        P=[]
        for i in range(0,floor(n/m)+1):
            Q=Partition(n-i*m,m-1)
            P1=deepcopy(Q)
            for j in range(0,len(Q)):
                P1[j].extend([i])
            P.extend(P1)
    return P

#Function RootPart, it takes a list of integers v, and it return a list
   which elements are list of two elements#
#the first element is a list of root lattices wich discriminant which
   rank coincide with the numbers of the original v#
#The secon element is an integer that represent how many times occurt
   the correspond root lattice##
def RootPart(v):

    n=len(v)
    M=[j for j in range(0,len(v)) if v[j]>0]
    if(len(M)==1 and max(v)==1):
        ind=M[0]
        Latt=[[[L],[1]] for L in r[ind+1]]
    else:
        ind=M[0]
        w=v
        w[ind]=w[ind]-1
```

```
        Latt2=RootPart(w)
        Latt=[]
        for L in r[ind+1]:
            for k in range(0,len(Latt2)):
                if L in Latt2[k][0]:
                    Post=[i for i in range(0,len(Latt2[k][0])) if Latt2[
    k][0][i]==L][0]
                    Ftemp=deepcopy(Latt2[k][1])
                    Ftemp[Post]=Ftemp[Post]+1
                    Latt.append([Latt2[k][0],Ftemp])
                else :
                    Ltemp=deepcopy(Latt2[k][0])
                    Ltemp.append(L)
                    Ftemp=deepcopy(Latt2[k][1])
                    Ftemp.append(1)
                    Latt.append([Ltemp,Ftemp])
    return Latt

#Function OrdinaLatt, it takes a list of two elements, the first is a
    list of root lattices, and te second is a list of integers numbers
    with len(Latt[0])=len(Latt[1])
#OrdinaLat return a list of two elements with the same conditions that
    the imput, but the lattices are order#
#First the type A lattices, then type D lattices and in the end type E
    lattices and also in decresing order acording to the rank##
 A=[A1,A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12,A13,A14,A15,A16,A17,A18]
 D=[D4,D5,D6,D7,D8,D9,D10,D11,D12,D13,D14,D15,D16,D17,D18]
 E=[E6,E7,E8]
def OrdinaLat(Latt):
    indA=[i for i in range(0,len(Latt[0])) if Latt[0][i] in A]
    indD=[i for i in range(0,len(Latt[0])) if Latt[0][i] in D]
    indE=[i for i in range(0,len(Latt[0])) if Latt[0][i] in E]
    Latt2=deepcopy(Latt)

    RankA=sorted([rank(Latt[0][i]) for i in indA])
    RankD=sorted([rank(Latt[0][i]) for i in indD])
    RankE=sorted([rank(Latt[0][i]) for i in indE])

    for j in range(0,len(indA)):
        k=[i for i in indA if rank(Latt[0][i])==RankA[j]][0]
        Latt2[0][j]=Latt[0][k]
        Latt2[1][j]=Latt[1][k]

    for j in range(len(indA),len(indA)+len(indD)):
        k=[i for i in indD if rank(Latt[0][i])==RankD[j-len(indA)]][0]
        Latt2[0][j]=Latt[0][k]
        Latt2[1][j]=Latt[1][k]

    for j in range(len(indA)+len(indD),len(indA)+len(indD)+len(indE)):
```

```
        k=[i for i in indE if rank(Latt[0][i])==RankE[j-len(indA)-len(
    indD)]][0]
        Latt2[0][j]=Latt[0][k]
        Latt2[1][j]=Latt[1][k]
    return Latt2

#Function RootLattice#
#input an integer number#
#Ouput ListLatt a list with len(LissLatt[i])=2 for every i in range(len(
    LissLatt)) with len(ListLatt[i][0])=len(ListLatt[i][1])#
#and sum_over_j(LissLat[i][0][j]*LissLat[i][1][j])=n for every i
def RootLattice(n):
    P=Partition(n,n)
    ListLatt=[]
    for p in P :
        Latt=RootPart(p)
        for j in range(0,len(Latt)):
            Latt1=OrdinaLat(Latt[j])
            if Latt1 not in ListLatt:
                ListLatt.append(Latt1)
    return ListLatt

##Function heightParing#
#input List of the form Latt=[L1,L2]#
#L1 is a list of root lattice, L2 list of integers numbers with len(L1)=
    len(L2)#
#The function compute all the possible Height Paring  for a section "P"
    that does not intersect the "0" section, given a lattice compose of
    root latice of ADE type"
#Ouput two list L1,L1c
#L1[i] is a list of the local contributions of the section "P" on the
    fiber "Latt[0][i]"
#L1c[i] is the list of components of the fibers "Latt" which intersect
    the "P" section to get the local contributions L1#
def HeightParing(Latt):
    L=[]
    Lc=[]
    for i in range(0, len(Latt[0])):
        for j in range(0, Latt[1][i]):
            if Latt[0][i] in A:
                n=rank(Latt[0][i])
                Ltemp=deepcopy(L)
                Lctemp=deepcopy(Lc)
                L=[]
                Lc=[]
                for s in range(0,floor((n+1)/2)+1):
                    Ltemp1=deepcopy(Ltemp)
                    Lctemp1=deepcopy(Lctemp)
                    if len(Ltemp)==0:
```

```
                    Ltemp1.append(s*(n+1-s)/(n+1))
                    Lctemp1.append(s)
                    L.append(Ltemp1)
                    Lc.append(Lctemp1)
                else :
                    for k in range(0,len(Ltemp)):
                        Ltemp1[k].append(s*(n+1-s)/(n+1))
                        L.append(Ltemp1[k])
                        Lctemp1[k].append(s)
                        Lc.append(Lctemp1[k])
        if Latt[0][i] in D:
            n=rank(Latt[0][i])
            Ltemp=deepcopy(L)
            Lctemp=deepcopy(Lc)
            L=[]
            Lc=[]
            if Latt[0][i]==D4:
                for s in range(0,2):
                    Ltemp1=deepcopy(Ltemp)
                    Lctemp1=deepcopy(Lctemp)
                    if len(Ltemp)==0:
                        Ltemp1.append(s)
                        L.append(Ltemp1)
                        Lctemp1.append(s)
                        Lc.append(Lctemp1)
                    else:
                        for k in range(0,len(Ltemp)):
                            Ltemp1[k].append(s)
                            L.append(Ltemp1[k])
                            Lctemp1[k].append(s)
                            Lc.append(Lctemp1[k])
            else:
                for s in range(0,3):
                    Ltemp1=deepcopy(Ltemp)
                    Lctemp1=deepcopy(Lctemp)

                    if len(Ltemp)==0:
                        Ltemp1.append(s+(s*(s-1)*(n/4-s))/2)
                        L.append(Ltemp1)
                        Lctemp1.append(s+s*(s-1)*(n/2-1))
                        Lc.append(Lctemp1)
                    else:
                        for k in range(0,len(Ltemp)):
                            Ltemp1[k].append(s+(s*(s-1)*(n/4-s))/2)
                            L.append(Ltemp1[k])
                            Lctemp1[k].append(s+s*(s-1)*(n/2-1))
                            Lc.append(Lctemp1[k])
        if Latt[0][i]==E6:
            Ltemp=deepcopy(L)
```

```
                L=[]
                Lctemp=deepcopy(Lc)
                Lc=[]
                for s in range(0,2):
                    Ltemp1=deepcopy(Ltemp)
                    Lctemp1=deepcopy(Lctemp)
                    if len(Ltemp)==0:
                        Ltemp1.append(s*3/4)
                        L.append(Ltemp1)
                        Lctemp1.append(s)
                        Lc.append(Lctemp1)
                    else :
                        for k in range(0,len(Ltemp)):
                            Ltemp1[k].append(s*4/3)
                            L.append(Ltemp1[k])
                            Lctemp1[k].append(s)
                            Lc.append(Lctemp1[k])
            if Latt[0][i]==E7:
                n=rank(Latt[0][i])
                Ltemp=deepcopy(L)
                L=[]
                Lctemp=deepcopy(Lc)
                Lc=[]
                for s in range(0,2):
                    Ltemp1=deepcopy(Ltemp)
                    Lctemp1=deepcopy(Lctemp)
                    if len(Ltemp)==0:
                        Ltemp1.append(s*3/2)
                        L.append(Ltemp1)
                        Lctemp1.append(s*(n))
                        Lc.append(Lctemp1)
                    else:
                        for k in range(0,len(Ltemp)):
                            Ltemp1[k].append(s*3/2)
                            L.append(Ltemp1[k])
                            Lctemp1[k].append(s*(n))
                            Lc.append(Lctemp1[k])
            if Latt[0][i]==E8:
                Ltemp=deepcopy(L)
                L=[]
                Ltemp1=deepcopy(Ltemp)
                Lctemp=deepcopy(Lc)
                Lc=[]
                Lctemp1=deepcopy(Lctemp)
                if len(Ltemp)==0:
                    Ltemp1.append(0)
                    L.append(Ltemp1)
                    Lctemp1.append(0)
                    Lc.append(Lctemp1)
```

```
                else:
                    for k in range(0,len(Ltemp)):
                        Ltemp1[k].append(0)
                        L.append(Ltemp1[k])
                        Lctemp1[k].append(0)
                        Lc.append(Lctemp1[k])
    L1=[]
    for l in L:
        s=0
        for k in range(0,len(Latt[1])):
            if Latt[1][k]>1:
                M=sorted([l[s+j] for j in range(0,Latt[1][k])])
                for j in range(0,Latt[1][k]):
                    l[s+j]=M[j]
            s=s+Latt[1][k]
        if l not in L1:
            L1.append(l)

    L1c=[]
    for l in Lc:
        s=0
        for k in range(0,len(Latt[1])):
            if Latt[1][k]>1:
                M=sorted([l[s+j] for j in range(0,Latt[1][k])])
                for j in range(0,Latt[1][k]):
                    l[s+j]=M[j]
            s=s+Latt[1][k]
        if l not in L1c:
            L1c.append(l)

    return L1,L1c

#Function Lattice_r18
#Input two inteders "r" and "d" and a List of lattices "ListLatt"
#Ouput return all the lattices of
def Lattice_r18(r,d,ListLatt):
    LattK3=[]
    for i in range(0,len(ListLatt)):
        Hp, Hpc=HeightParing(ListLatt[i])
        detTriv=1

        for k in range(0,len(ListLatt[i][0])):
            for j in range(0,ListLatt[i][1][k]):
                detTriv=detTriv*ListLatt[i][0][k].determinant()

        for l in range(0,len(Hp)):
            h=4-sum(Hp[l])
            if d==h*detTriv:
                M=[ListLatt[i][0],ListLatt[i][1],Hp[l], Hpc[l]]
```

```
                LattK3.append(M)

    return LattK3

#Function Lattice_Galois
#Input a list L, every element of L is a list consisting of 4 elements, a list of root lattices, a list of integer numbers of the same lenght,
#A list of local contributions(rational numbers) of a section P and a list of the component the intersect the  section P
#Ouput A sub-list of L, such that the lattice of generated by L[i] admidt ZZ/2*ZZ Galois action.

def Lattice_Galois(L):
    G=[A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12,A13,A14,A15,A16,A17,A18,D4,D5,D6,D7,D8,D9,D10,D11,D12,D13,D14,D15,D16,D17,D18,E6]
    A=[A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12,A13,A14,A15,A16,A17,A18]
    D=[D4,D5,D6,D7,D8,D9,D10,D11,D12,D13,D14,D15,D16,D17,D18]
    s=0
    Lg=[]

    for  l in L:
        s=0
        F1=[l[1][j] for j in range(0,len(l[0])) if l[0][j] in G]
        k1=0
        t=0
        for s1 in range(0,len(l[0])):
            for s2 in range(0,l[1][s1]):

                if l[2][t+s2]!=0:
                    if l[0][s1] in A and floor(rank(l[0][s1])/2) + 1 != l[3][t+s2]:
                        k1 += 1
                    elif l[0][s1] in D and l[0][s1] != D4 and l[2][t+s2]!=1:
                        k1 += 1
                    elif l[0][s1]==E6:
                        k1 += 1
            t=t+l[1][s1]
        Sp=[0]
        t=0
        for s1 in range(0,len(l[0])):
            if l[1][s1]>1:
                S=sorted([l[2][t+j] for j in range(0,l[1][s1])])
                St=[0 for j in range(0,l[1][s1])]
                k=0
                for i in range(0, l[1][s1]-1):
                    if S[i]==S[i+1]:
                        St[k]=St[k]+1
```

```
                    else:
                        k=k+1
                Sp.extend(St)
            t=t+l[1][s1]
        F3=[Sp[j]*(Sp[j]+1)/2 for j in range(0,len(Sp))]
        s=sum(F1)-k1+sum(F3)
        if s>=2:
            k=0
            Hp,Hpc=HeightParing([l[0],l[1]])
            for p in Hp:
                h=4-sum(p)
                if h==0:
                    k=k+1
            if k!=0:
                Lg.append(l)
    return Lg

#Function Discriminant_group
#Input a list L , every element of L is a list consisting of 4 elements, a list of root lattices, a list of integer numbers of the same lenght,
#A list of local contributions(rational numbers) of a section P and a list of the component the intersect the  section P
#and a discriminant group T
#Output sub-list of L, such that de discriminant group of L[i] is equal to T under isomorphism.
def Discriminant_gr(L,T):
    r_inv=T.invariants()
    if len(r_inv)==2:
        [r1,r2]=T.invariants()
        m=T.gram_matrix_quadratic()
        M=[]

        for i1 in range(0,r1):
            for j1 in range(0,r2):
                for i2 in range(0,r1):
                    for j2 in range(0,r2):
                        if [i1,j1]!=[i2,j2] and [i1,j1]!=[0,0] and [i2,j2]!=[0,0]:
                                b1=i1*i1*m[0][0]+j1*j1*m[1][1]+2*i1*j1*m[1][0]
                                b2=i2*i2*m[0][0]+j2*j2*m[1][1]+2*i2*j2*m[1][0]
                                d=i1*i2*m[0][0]+i1*j2*m[1][0]+i2*j1*m[1][0]+j1*j2*m[1][1]

                                b1=b1*r1*r2%(r1*r2*2)
                                b2=b2*r1*r2%(r1*r2*2)
                                d=d*r1*r2%(r1*r2*2)
```

```

                                b1=b1/(r1*r2)
                                b2=b2/(r1*r2)
                                d=d/(r1*r2)

                                l=Matrix([[b1,d],[d,b2]])
                                l2=Matrix([[b1,2-d],[2-d,b2]])
                                M.append(l)
                                M.append(l2)
    if len(r_inv)==1:
        r=r_inv[0]
        m=T.gram_matrix_quadratic()
        M=[]
        for i in  range(1,r):
            b=i^2*m[0]
            b=(b*r)%(2*r)
            b=b/r
            l=Matrix([b])
            l2 = Matrix([-b])
            M.append(l)
    Ld=[]
    for l in L:
        Lt=IntegralLattice(matrix(0))
        for i in range(0,len(l[0])):
            for j in range(0,l[1][i]):
                Lt=Lt.direct_sum(l[0][i])
        m=Lt.gram_matrix()
        s,s1=m.dimensions()
        H=matrix(s+1)

        for i in range(0,s):
            for j in range(0,s):
                H[i,j]=m[i,j]
        k=0
        k1=0
        for i in range(0,len(l[0])):
            for j in range(0,l[1][i]):
                if l[3][k1]!=0:
                    H[k+l[3][k1]-1,s]=-1
                    H[s,k+l[3][k1]-1]=-1
                k=k+rank(l[0][i])
                k1=k1+1
        H[s,s]=4
        Lh=IntegralLattice(H)
        Di=Lh.discriminant_group().gram_matrix_quadratic()
        if Di in M:
            Ld.append(l)
    return Ld
```

B | Appendix B Sagemath $\mathbb{F}_p$ reduction

In this appendix we included the code in Python/Sage-math for algorithm 3.3, which is divided in different functions.

```
def fourier_coeff(d,p):
    if d%4 == 1 :
        N=d
    else:
        N=d/4
    if N%4 == 1:
        fourier_c = []
        for a in range(-2*p,2*p):
            for b in range(-p,p):
                if p^2 ==(a+1/2*b)^2 - N*b^2/4 and a!=0 and b!=0:
                    if 2*(a+1/2*b) not in fourier_c:
                        fourier_c.append(2*(a+1/2*b))
    else:
        fourier_c = []
        for a in range(-p,p):
            for b in range(-p,p):
                if p^2 == a^2 - N*b^2 and a !=0 and b!=0:
                    if 2*a not in fourier_c:
                        fourier_c.append(2*a)
    return fourier_c

def merge_list (M):
    if len(M) == 1:
        merge_1 = []
        for m in M[0]:
            merge_1.append([m])
        return merge_1
    else:
        M_temp =merge_list(M[:-1])

    M_merge =[]
    for m in M_temp:
        for l in M[-1]:
```

```
            m_temp = copy(m)
            m_temp.append(l)
            M_merge.append(m_temp)
    return M_merge

def flat_list(M):
    M = merge_list(M)
    flat_M = []
    for elem in M:
        L1 = []
        L2 = []
        for i in  range (len(elem)):
            L1.append(elem[i][0])
            L2.append(elem[i][1])
        flat_M.append([L1,L2])
    return flat_M
```

```
def Counting_points(A,B,C,p):
    A1 = A.numerator()
    A2 = A.denominator()

    B1 = B.numerator()
    B2 = B.denominator()

    C1 = C.numerator()
    C2 = C.denominator()

    try:
        F1.<a,t> = GF(p)[]
        F.<x> = F1[]
        A1 = F(A1)
        A2 = F(A2)
        A_s = A1/A2

        B1 = F(B1)
        B2 = F(B2)
        B_s = B1/B2

        C1 = F(C1)
        C2 = F(C2)
        C_s = C1/C2
    except:
        print("Oops!", sys.exc_info()[0], "occurred.")
        print("For prime %s the denomiator of the Weistrass quation is zero " %p)
        return None
    else:
        if A_s == 0 or B_s==0 or C_s==0:
            print("For prime %s the reduction is not good !"%p)
```

```
            return None
        else:
            f_surface = x^3 + A_s*x^2 + B_s*x + C_s
            L_points = []
            for pr in GF(p):
                if A2(a=pr) !=0 and B2(a=pr) !=0 and C2(a=pr) != 0:
                    f_r = copy(f_surface)
                    A_r = copy(A_s)
                    B_r = copy(B_s)
                    C_r = copy(C_s)
                    count = 0
                    f_r = f_r(a = pr)
                    A_r = A_r(a = pr)
                    B_r = B_r(a = pr)
                    C_r = C_r(a = pr)

                    d = f_r.numerator().discriminant()
                    if d==0:
                        for el in GF(p):
                            f1 = f_r(t=el)
                            count += 1
                            for a in GF(p):
                                for b in GF(p):
                                    if b^2 == f1(x=a):
                                        count += 1

                        F3.<a,t,x> = GF(p)[]
                        f_inf1 = F3(f_r)
                        f_inf1= t^12*f_inf1(x=x/t^4,t=1/t)
                        f_inf1 = f_inf1(t=0)
                        count += 1
                        for a in GF(p):
                            for b in GF(p):
                                if b^2 ==f_inf1(x=a):
                                    count +=1
                    else:
                        F2.<t> =GF(p)[]
                        d = F2(d)
                        roots_temp = d.roots()
                        roots = [roots_temp[i][0] for i in range(len(roots_temp))]
                        for root in roots:
                            f1 = f_r(t = root)
                            count += 1
                            for a in GF(p):
                                for b in GF(p):
                                    if b^2 == f1(x=a):
                                        count += 1

```

```
                        for l in GF(p):
                            if l not in roots:
                                A_el = A_r(t=l)
                                B_el = B_r(t=l)
                                C_el = C_r(t=l)
                                count += EllipticCurve(GF(p),[0,A_el,0,
B_el,C_el]).cardinality()
                        F4.<t,x,a> = GF(p)[]
                        f_r = F4(f_r)
                        f_i = t^12*f_r(x=x/t^4,t=1/t)
                        f_inf = f_i(t=0)

                        count +=1
                        for a in GF(p):
                            for b in GF(p):
                                if b^2 ==f_inf(x=a):
                                    count += 1
                    L_points.append([pr,count])
    return L_points
```

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Curriculum vitae

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