Analysis of a Soap Film Catenoid Driven by an Electrostatic Force

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Abstract

The interplay between surface tension and electrostatics is the underlying mechanism of many processes taking place on small length-scales. There are various examples in nature and technology, including the whole field of micro-electro-mechanical systems.

This thesis is devoted to a free boundary problem modelling a prototypical set-up in which both surface tension and electrostatics have the ability to break the set-up. The set-up consists of a conductive soap film spanned between two parallel rings, which are placed inside an outer metal cylinder. On the one hand, if the gap between the rings is not too big, surface tension forces the soap film to take the shape of a catenoid. In particular, surface tension pushes the film inwards. On the other hand, applying a voltage between the catenoid and the outer cylinder results in an electrostatic force pulling the film outwards. While a previous mathematical investigation focused on a simplified small aspect ratio model of the set-up and did not yet include time, we drop the small aspect ratio assumption which yields a completely different type of model. We also include time into our considerations.

In the first part of this thesis, we derive the new model for the soap film catenoid subjected to an electrostatic force. The model consists of a quasilinear parabolic equation for the evolution of the film coupled with an elliptic equation for the electrostatic potential in the unknown domain between outer cylinder and soap film catenoid. Then, for the rotationally symmetric case, we show local well-posedness of this free boundary problem by recasting it as a single quasilinear parabolic equation with a non-local source term. As the source term turns out to have slightly weaker regularity than required, the proof of local well-posedness contains a refinement of a classical fixed point argument based on semigroup theory.

In the second part of this thesis, we discuss different kinds of behaviour that the soap film displays depending on the strength of the applied voltage. For small voltages as well as voltages for which surface tension and electrostatics are balanced, we show the existence of stationary solutions and study their stability. Moreover, we prove that stable stationary solutions behave physically in the sense that they always deflect outwards if the applied voltage is increased. Finally, for large applied voltages, we show that solutions to the evolution problem do not exist globally for a large class of initial values. The proofs in the second part of this thesis mostly, but not exclusively, rely on the implicit function theorem, the principle of linearized stability, (anti-)maximum principles as well as positivity of a certain Fourier series.

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Contents

Introduction

Soap bubbles and soap films fascinate mathematicians and non-mathematicians alike. While the latter find the play of colours, originating from the reflection of light on the surface of a bubble, and the different shapes taken by a bubble floating in air optically appealing, the former are particularly attracted by the close connection between soap films and minimal surfaces [**17**]. If, for example, a soap film is spanned between two parallel rings with a small gap, its surface tension forces it to take the shape of a catenoid, a minimal surface which is depicted in Figure 0.1.

Less present but no less fascinating is the response of soap bubbles or films to electrostatics. Experiments dealing with the deformation and breaking of soap bubbles and films due to electrostatics can be found in [**56, 59, 68, 69**]. Particularly, Taylor's work [**68**] from the 1960's, in which he studied the breaking of drops and soap bubbles in an electric field, has to be stressed. While his actual interest was in the formation of thunderstorms, he used the experiments with soap as prototypical set-ups to explain processes driven by the interplay of surface tension and electrostatics.

In the same sense, this thesis is devoted to a new mathematical model for a prototypical set-up investigating the interaction of surface tension and electrostatics. The specific feature of the set-up is that both surface tension and electrostatics have the ability to break it. The set-up was suggested by Moulton [**58**], see also Moulton and Pelesko [**59, 60**], and consists of a soap film catenoid placed inside an outer metal cylinder. On the one hand, surface tension pulls the film inwards and might even break the film if the gap between the rings is too big. On the other hand, applying a voltage between the catenoid and the outer cylinder results in an electrostatic force pulling the film outwards, in the extreme case until it touches the outer metal cylinder. We aim at understanding the interplay between these forces pulling the film in opposite directions. In particular, we are led by the following questions:

- How does the film respond to an increase of the electrostatic force?
- Can breaking of the film be prevented by the electrostatic force?
- Can breaking of the film be triggered by the electrostatic force?

The answers are not only interesting for the set-up itself, but may also allow conclusions for applications of the interplay between surface tension and electrostatics in technology and nature. Examples of applications are the fabrication of microstructures [**40**], electrowetting [**9**] as well as the whole field of micro-electro-mechanical systems (MEMS). The latter includes tiny sensors and switches, used everywhere in modern technology.

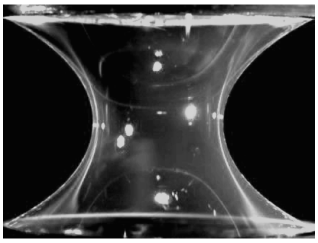

FIGURE 0.1. Picture of a soap film catenoid taken from the additional material of [**34**]. In this thesis, the soap film catenoid will be placed inside an outer metal cylinder with larger radius.

We now present the new model for the soap film catenoid driven by an electrostatic force and explain its main mathematical features. While our model describes the same set-up as [**58**], it has a completely different mathematical structure than the model used there. Instead, it is a variant of a class of models for MEMS [**46**]. Our model is a free boundary problem, meaning that it includes a partial differential equation posed in an a-priori unknown domain. In dimensionless and shifted form, it describes the evolution of two unknown functions. The first unknown $u : [0,T] \times (-1,1) \to (-1,1)$ models the film deflection, which is given by the surface of revolution with profile $u+1$. The case $u = -1$ corresponds to self-touching of the film, while $u = 1$ means that the film touches the outer cylinder. At fixed time, the second unknown $\psi : \overline{\Omega(u(t))} \to \mathbb{R}$ is the electrostatic potential defined on the closure of the a-priori unknown domain

$$\Omega(u(t)) := \big\{(z,r) \in (-1,1) \times (0,2) \,\big|\, u(t,z) + 1 < r < 2\big\}$$

between the soap film and the outer cylinder. Assuming that the evolution of (u,ψ) is quasi-static in ψ, it is governed by the following system of equations:

$$\left\{\begin{aligned} \partial_t u - \partial_z \mathrm{arctan}(\partial_z u) &= -\frac{1}{u+1} + \lambda\,(1+(\partial_z u)^2)^{1/2}\,\big|\nabla\psi(\,\cdot\,,u+1)\big|^2\,, && \\ u(t,\pm 1) &= 0\,, && \\ u(0,\,\cdot\,) &= u_0 \qquad \text{in} \quad (-1,1)\,, && \\ \Delta\psi &= 0 \qquad\ \, \text{in} \quad \Omega(u(t))\,, && \\ \psi &= h_u \qquad \text{on} \quad \partial\Omega(u(t))\,. && \end{aligned}\right. \tag{0.1}$$

Here, u_0 describes an initial film shape, which might not necessarily be the catenoid, and h_u is a certain fixed function, introduced rigorously later, which is 0 on the film and 1 on the outer metal cylinder. Finally, the control parameter $\lambda \in [0,\infty)$ is of great importance for the study of qualitative properties as it measures the relative strength of the applied voltage.

If no voltage is applied, that is $\lambda = 0$ in (0.1), then the soap film evolves according to rotationally symmetric mean curvature flow, see [**19**]. This yields a quasilinear parabolic equation for u only.

However, in the interesting case of an applied voltage $\lambda > 0$, the parabolic equation for u is coupled with an elliptic equation for ψ. The coupling is due to the electrostatic force

$$\big(1+(\partial_z u)^2\big)^{1/2}\big|\nabla\psi(\,\cdot\,,u+1)\big|^2$$

contained as a source term in the equation for u and due to the unknown domain $\Omega(u(t))$ on whose closure ψ is defined. The latter makes the coupling highly non-local, and thus it is challenging to treat (0.1), in particular its qualitative behaviour. The specific feature of the set-up, namely that both surface tension and electrostatics can break it, is reflected by u taking values in $(-1,1)$ and by the opposite signs of the source term

$$-\frac{1}{u+1}+\lambda\,(1+(\partial_z u)^2)^{1/2}\,\big|\nabla\psi(\,\cdot\,,u+1)\big|^2\,.$$

Each part can become singular, which follows for the second one from the fact that the electrostatic potential ψ has to jump immediately from 0 to 1 for $u=1$. Finally, we mention the technical, not yet apparent difficulty that ψ has limited regularity as $\Omega(u(t))$ is not a smooth domain, while defining the electrostatic force requires the trace of the squared gradient of ψ to be meaningful.

Before we describe the contributions of this thesis regarding the free boundary problem (0.1), let us have a closer look at the mathematical literature. There are numerous investigations of models dealing with the interplay of surface tension and electrostatics, including variational models for charged drops [**61, 62**] as well as different types of MEMS-models, like singular equations [**8, 28, 65**], free boundary problems [**23, 46**] or transmission problems [**49**]. In the following, we focus on the two most relevant previous investigations for our work. These are the MEMS-model [**24**] by Escher, Laurençot and Walker as well as the model [**58**] by Moulton.

First, the model from [**58**], see also [**60**], describes the same set-up as we do. However, it is restricted to the stationary case and consists of a singular but local elliptic equation. In particular, the model from [**58**] is no free boundary problem. It can be derived from the stationary version of (0.1) by assuming a small aspect ratio of the set-up. Besides modelling, [**58**] contains an investigation of the qualitative behaviour of solutions using numerical and formal methods, like formal asymptotics, stability analysis or bifurcation. Therefore, it serves as a valuable inspiration for our analytical results. Though the assumption of a small aspect ratio is crucial to derive the model from [**58**], it may be seen as a controversial assumption as there are other model parameters approximately of the same order as the aspect ratio.

Second, the model [**24**] belongs to a class of free boundary problems modelling MEMS, which were introduced and analytically investigated for the first time by Laurençot and Walker in [**42**] for the stationary case and shortly after by Escher, Laurençot and Walker in [**23**] for the evolution case. While an overview of further developments for free boundary problems modelling MEMS can be found in [**46**], we restrict ourselves to reviewing the model from the article [**24**] by Escher, Laurençot and Walker in more detail.

Mathematically, the model is a free boundary problem for two unknowns (u, ψ) consisting of a parabolic equation for u and Laplace equation for ψ in an a-priori unknown domain. Insofar, our model (0.1) has the same mathematical structure. However, the parabolic equation for u in [**24**] differs significantly from that in (0.1) and reads

$$\partial_t u - \frac{\partial_z^2 u}{\big(1 + (\partial_z u)^2\big)^{3/2}} = -\lambda \left|\nabla \psi(\,\cdot\,, u)\right|^2 , \tag{0.2}$$

where u is allowed to take values in the whole interval $(-1, \infty)$ and the electrostatic force pulls in the other direction. Since

$$\partial_z \mathrm{arctan}(\partial_z u) = \frac{\partial_z^2 u}{1 + (\partial_z u)^2}$$

in (0.1), we observe that the equation (0.2) contains an additional factor $\big(1+(\partial_z u)^2\big)^{-1/2}$. The main difference is now that the source term of (0.2) consists only of the electrostatic force and has therefore a fixed sign. This reflects the fact that only electrostatics has the ability to break the set-up in [**24**] and not both surface tension and electrostatics which is the special feature of our set-up. Finally, we mention that [**24**] is formulated in cartesian coordinates, while we use cylindrical coordinates, which require an adaptation of the boundary conditions for ψ.

In [**24**] well-posedness as well as qualitative properties of solutions are rigorously proven. However, the opposite signs in the source term of (0.1) yield a significantly different set of stationary solutions compared to [**24**], and proofs of qualitative properties of solutions to (0.1) require adaptation as well as new ingredients.

We end the literature review with a brief summary of qualitative properties previously established in free boundary problems for MEMS. Stationary solutions have been studied in [**14, 22, 24, 43, 44, 64**] and include the construction via implicit function theorem as well as the study of stability based on the principle of linearized stability. For results concerning finite time blow-up for large λ, which correspond to a strong electrostatic force, we refer to [**22, 23, 24, 47, 52**]. Another qualitative property is the direction in which u deflects. A first attempt to characterize the direction of deflection for an evolution problem, in which the source term may have terms of opposite signs, is contained in [**25, 52**], while the direction of deflection for stationary problems of 4-th order, but with fixed sign of the source term, is treated in [**43, 64**].

We now give an overview of the contributions of this thesis. In the first part, Chapters 1-3, we derive the free boundary problem (0.1) and show that it is locally well-posed in Sobolev spaces. More precisely, in Chapter 1, we state the modelling assumptions and derive the new model for the soap film catenoid in an electric field. Its dimensionless form coincides then, up to an additional parameter, with the free boundary problem (0.1). Next, Chapter 2 serves as a preparation for the proof of local well-posedness. It contains a complete argument showing elliptic regularity on non-smooth convex domains.

The first part of this thesis ends with Chapter 3, in which we establish local well-posedness of the free boundary problem (0.1) by reinterpreting it as a single quasilinear parabolic equation with a non-local source term. After a detailed investigation of the non-local term, this single equation is then solved with the aid of semigroup theory and Banach's fixed point theorem. The proof follows [**24**], and the reason for performing the whole fixed point argument by hand is that the non-local source term has slightly weaker regularity than usually required, see [**3, 5**].

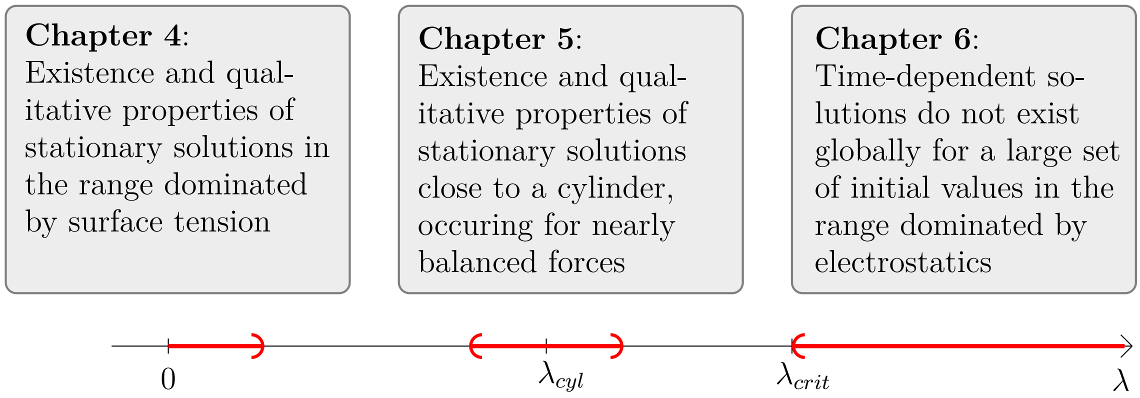

FIGURE 0.2. Schematic overview of the results from Chapters 4-6 in which we explore the behaviour of the soap film for λ reaching from 0 to infinity. Each chapter is devoted to a different range (depicted in red) of the control parameter λ. The values λ_{cyl} and λ_{crit} will be introduced in the corresponding chapters.

In the second part of this thesis, consisting of Chapters 4-6, we study qualitative properties of solutions to the free boundary problem (0.1). A common focus is set on the behaviour of the film deflection u in response to the control parameter λ which modulates the applied voltage and is thus responsible for the strength of the electrostatic force. We identify three ranges for the control parameter λ, in which different behaviours of the film deflection u are observable. To each range, a chapter is devoted, and the results are summarized in Figure 0.2. In Chapter 4, we study stationary solutions for small applied voltages, while in Chapter 5 we are concerned with stationary solutions in the situation where surface tension and electrostatic force are nearly balanced. Besides answering classical questions such as the ones about existence and stability, both chapters contain the following result as a highlight: Stable stationary solutions behave physically in the sense that they are always deflected outwards if the applied voltage λ is increased. Its strategy of proof appears to be new in the context of problems driven by surface tension and electrostatics. Finally, in Chapter 6, we consider solutions to the evolution problem and show that these cease to exist if λ is big enough and if the initial value satisfies an additional condition. This condition on the initial value is explicitly given and easy to verify.

CHAPTER 1

Modelling

In this chapter, we explain the precise set-up of our problem, state the modelling assumptions and use energy methods to formally derive a model for the soap film in an electric field. Finally, we present the dimensionless mathematical equations, whose analytical investigation is the main issue of this thesis.

1.1. Problem Set-Up

We study a tiny soap film spanned between two parallel metal rings of equal size and subjected to an external electrostatic force. The set-up was introduced in [**58, 60**] for the first time and is schematically depicted in Figure 1.1. To realize the electrostatic force, the soap film, which forms a tubular bridge, is placed inside a larger outer metal cylinder, and a voltage between this cylinder and the metal rings is applied. Since the rings and the outer cylinder are not connected and soap films are conductive, an electrostatic force, changing the film's shape, is induced. The top and bottom of the system are left open and the set-up is surrounded by air.

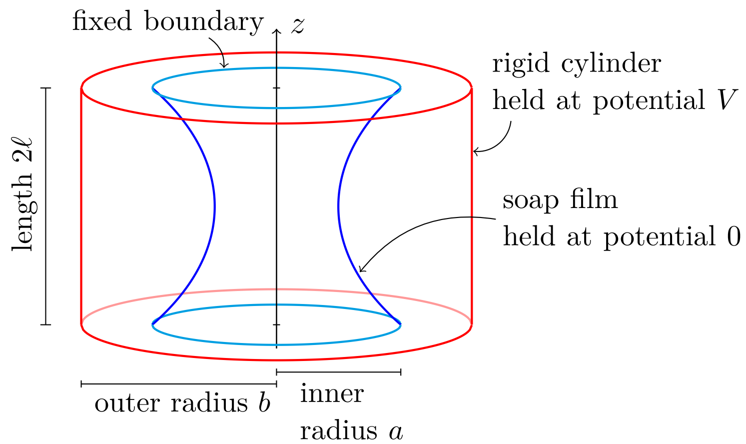

FIGURE 1.1. Depiction of the soap film (blue) inside a rigid outer cylinder (red). The film, which is surrounded by air, is fixed at two parallel rings of equal size (light blue) whereas the remaining part of the film is free to move (dark blue). Applying a voltage between the film and the rigid outer cylinder changes the shape of the film.

The problem includes an interesting competition of two dominant forces on the microscale: surface tension and electrostatic force, see [**65, 70**]. The surface tension pushes the film inwards, while the electrostatic force pulls the film outwards, in the extreme case until it touches the rigid outer cylinder and breaks. Experimental realizations of related set-ups may be found in [**34**] without electrostatic force and in [**59**] in the presence of a voltage but applied between the soap film bridge and an inner metal cylinder instead of an outer metal cylinder.

1.2. Model Derivation

In this section, we describe both the dynamical as well as the static behaviour of the film. The model derivation does not focus on mathematical rigour. Nevertheless, some assumptions are introduced only for the mathematical treatment later on. We start with the time-independent situation:

1.2.1. Static Case. We assume that the problem is entirely driven by surface tension and the electrostatic force coming from the applied voltage. All other effects, for example gravitational effects are neglected (as the film is tiny and has a light weight). Therefore, the problem can be described by two unknowns, the film deflection u as well as the electrostatic potential ψ. Compared to the introduction, the unknowns are not yet time-dependent, shifted or dimensionless. Moreover, we work with cylindrical coordinates, and, for simplicity, assume that the problem is rotationally symmetric. Then, we can look at the two-dimensional cross section, which is depicted in Figure 1.2, instead of the full set-up from Figure 1.1. The film deflection u, now describing the profile function of a surface of revolution, and the electrostatic potential ψ are then given by:

$$
\begin{aligned}
&\textbf{Film Deflection}: && u = u(z):\ [-\ell,\ell] \to (0,b)\,, && u(\pm\ell) = a \\
&\textbf{Electrostatic Potential}: && \psi = \psi(z,r):\ \overline{\Omega(u)} \to \mathbb{R}\,, && \psi(z,u) = 0\,, \\
& && && \psi(z,b) = V\,.
\end{aligned}
$$

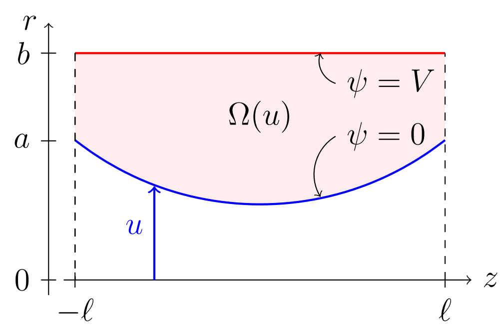

FIGURE 1.2. Cross section of the soap film bridge in an electric field. Compared to Figure 1.1, the picture is rotated by 90 degrees.

Note that we chose z as the first coordinate. Moreover, 2ℓ is the distance between the metal rings, a is their radius, b is the radius of the outer cylinder and V denotes the applied voltage. Finally, we assume that the electrostatic potential ψ is defined on $\overline{\Omega(u)}$ which denotes the space between film and outer cylinder.

Electrostatic Potential. The behaviour of the electrostatic potential ψ is captured by Gauss's law from electrostatics [**35**]. For a fixed film deflection u, it states that the corresponding potential ψ solves $\Delta\psi = 0$ as the charge distribution between film and outer cylinder vanishes. Moreover, $\psi = 0$ at the film and $\psi = V$ at the outer cylinder are prescribed.

Less obvious is how ψ should behave at the open top and bottom of the system. As ψ is not defined on the whole space, we cannot prescribe a certain behaviour of ψ at infinity. Instead a different approach is taken: Similar to the treatment of a cylindrical capacitor [**35**], we ignore fringing effects which allows to prescribe the behaviour of ψ at the top and bottom directly. To find the boundary condition without fringing, one imagines an infinity long cylindrical capacitor (i.e. we assume that the film forms a cylinder and extend the set-up at the top and bottom to infinity). In this setting, the electrostatic potential is

$$\psi_{cyl}(z,r) = V \, \frac{\ln\left(\dfrac{r}{a}\right)}{\ln\left(\dfrac{b}{a}\right)} \,,$$

and plugging in $z = \pm\ell$ yields the desired behaviour of ψ at the top and bottom without fringing.

In summary, the electrostatic potential ψ satisfies

$$\begin{cases} \Delta\psi & = \frac{1}{r}\partial_r\left(r\partial_r\psi\right) + \partial_z^2\psi = 0 \quad \text{in} \quad \Omega(u)\,, \\ \psi & = h_u \quad \text{on} \quad \partial\Omega(u) \end{cases}$$

with

$$\Omega(u) = \left\{(z,r) \in (-\ell,\ell)\times(0,b) \,\middle|\, r > u(z)\right\},$$

and

$$h_u(z,r) = V \, \frac{\ln\left(\dfrac{r}{u(z)}\right)}{\ln\left(\dfrac{b}{u(z)}\right)}$$

as a short hand for the boundary behaviour described above. Here, we recall that $u(\pm\ell) = a$. Finally, we mention that by [**35, 65**] the electrostatic energy is given by

$$E_e = -\pi\varepsilon_0 \int_{\Omega(u)} |\nabla\psi|^2 r \,\mathrm{d}(z,r)\,. \tag{1.1}$$

The constant $\varepsilon_0 > 0$ denotes the electric permittivity of the vacuum, which can also be used for air, and the factor π stems from the rotational symmetry.

Film Deflection. We use an energy consideration to find an equation for the film deflection u. More precisely, we first give a formula for the total energy of the system in dependence on u and then use the fact that an actual film deflection should minimize this energy. This approach is adapted from [**7, 12, 30**], see also [**48**].

Total Energy. As we take into account only surface tension and electrostatic force, the total energy is given by

$$E(u) = E_m(u) + E_e(u)\,. \tag{1.2}$$

It combines the mechanical energy

$$E_m(u) = 2\pi T \int_{-\ell}^{\ell} u\sqrt{1+(\partial_z u)^2}\,\mathrm{d}z\,,$$

which consists of the surface tension $T > 0$ of the film times its surface area and accounts for stretching, with the electrostatic energy

$$E_e(u) = -\pi\varepsilon_0 \int_{\Omega(u)} |\nabla\psi_u|^2 r\,\mathrm{d}(z,r)\,,$$

which is (1.1) with additional emphasis on the u-dependencies. In particular, it is worth mentioning that not only the domain $\Omega(u)$, but also the electrostatic potential ψ_u depend on the film deflection u.

Minimizers of the Total Energy. A necessary condition for the film deflection u to minimize the total energy is that the first variation $\delta E(u)$ vanishes. We will formally compute δE in the following. While δE_m is easy to compute, the non-local and complicated dependency of ψ_u on u via its domain of definition $\Omega(u)$ makes the computation of δE_e more intricate, and we provide only a formal reasoning.

First Variation of Mechanical Energy. Let u be a fixed film deflection and $v \in \mathcal{D}(-\ell,\ell)$ such that $u + sv$ satisfies $0 < u + sv < b$ for $|s| << 1$. Then, the mechanical energy of the perturbed film deflection is given by

$$e_m(s) = E_m(u+sv) = 2\pi T \int_{-\ell}^{\ell} (u+sv)\sqrt{1+\big(\partial_z(u+sv)\big)^2}\mathrm{d}z$$

with the derivative

$$e_m'(s) = 2\pi T \int_{-\ell}^{\ell} \Bigg(v\sqrt{1+\big(\partial_z(u+sv)\big)^2} + (u+sv)\,\frac{2\partial_z(u+sv)\,\partial_z v}{2\sqrt{1+\big(\partial_z(u+sv)\big)^2}} \Bigg)\,\mathrm{d}z\,.$$

Hence, the first variation of the mechanical energy is

$$\begin{aligned}\delta E_m(u)v = e_m'(0) &= 2\pi T \int_{-\ell}^{\ell} \left(v\sqrt{1+(\partial_z u)^2} + \frac{u\,\partial_z u\,\partial_z v}{\sqrt{1+(\partial_z u)^2}} \right) \mathrm{d}z \\ &= 2\pi T \int_{-\ell}^{\ell} \left(\sqrt{1+(\partial_z u)^2} - \partial_z\Big(\frac{u\,\partial_z u}{\sqrt{1+(\partial_z u)^2}} \Big) \right) v\,\mathrm{d}z\,,\end{aligned}$$

where we integrated by parts in the last step. Since

$$\begin{aligned}\partial_z\Big(\frac{u\partial_z u}{\sqrt{1+(\partial_z u)^2}} \Big) &= \frac{(\partial_z u)^2 + u\,\partial_z^2 u}{\sqrt{1+(\partial_z u)^2}} - (u\,\partial_z u)\,\frac{(\partial_z u)(\partial_z^2 u)}{\big(1+(\partial_z u)^2\big)^{3/2}} \\ &= \frac{(\partial_z u)^2 + u\,\partial_z^2 u + (\partial_z u)^4}{(1+(\partial_z u)^2)^{3/2}}\,,\end{aligned}$$

the result can be simplified to

$$\begin{aligned}\delta E_m(u)v &= 2\pi T \int_{-\ell}^{\ell} \left(\frac{1 + 2(\partial_z u)^2 + (\partial_z u)^4 - \big((\partial_z u)^2 + u\,(\partial_z^2 u) + (\partial_z u)^4\big)}{(1+(\partial_z u)^2)^{3/2}} \right) v\,\mathrm{d}z \\ &= 2\pi T \int_{-\ell}^{\ell} \left(\frac{1 + (\partial_z u)^2 - u\,(\partial_z^2 u)}{(1+(\partial_z u)^2)^{3/2}} \right) v\,\mathrm{d}z \\ &= 2\pi T \int_{-\ell}^{\ell} \left(\frac{1}{(1+(\partial_z u)^2)^{1/2}} - \frac{u}{(1+(\partial_z u)^2)^{3/2}}\,\partial_z^2 u \right) v\,\mathrm{d}z\,. \end{aligned} \tag{1.3}$$

First Variation of Electrostatic Energy. Let u be a fixed film deflection and $v \in \mathcal{D}(-\ell,\ell)$ such that $u+sv$ is still between 0 and b for $|s| << 1$. As u is fixed throughout the following computation, we introduce the abbreviations

$$\Omega = \Omega(u)\,, \qquad \Omega_s = \Omega(u+sv)\,, \qquad \psi = \psi_u\,, \qquad \psi_s = \psi_{u+sv}\,.$$

The electrostatic energy of the perturbed film deflection is then given by

$$e_e(s) = E_e(u+sv) = -(\varepsilon_0\,\pi) \int_{\Omega_s} \big(|\partial_z \psi_s|^2 + |\partial_r \psi_s|^2 \big)\, r\,\mathrm{d}(z,r)\,, \tag{1.4}$$

where both Ω_s and ψ_s depend on s. To compute the derivative of e_e, we transform Ω_s back to the s-independent domain Ω via the map $\Phi_s = (\Phi_s^1, \Phi_s^2) : \Omega \to \Omega_s$ defined by

$$\begin{aligned}\Phi_s^1(z,r) &= z\,, \\ \Phi_s^2(z,r) &= (u(z) + sv(z))\, \left[1 - \left(\frac{r-u(z)}{b-u(z)} \right) \right] + b\, \left(\frac{r-u(z)}{b-u(z)} \right) \\ &= r + sv(z) \left(\frac{b-r}{b-u(z)} \right)\,,\end{aligned}$$

and depicted in Figure 1.3. Since the second component $\Phi_s^2(z,\,\cdot\,)$ maps the line $[u(z),b]$ to the line $[u(z)+sv(z),b]$, we see that Φ_s is a diffeomorphism. Moreover, we note that

$$\partial_s \Phi_s(z,r) = \left(0, v(z) \left(\frac{b-r}{b-u(z)} \right) \right) \tag{1.5}$$

as well as

$$\det\left(\mathrm{D}_{(z,r)}\Phi_s\right) = \det\begin{pmatrix}1 & 0\\ * & 1-\frac{sv(z)}{b-u(z)}\end{pmatrix} = \frac{b-\big(u(z)+sv(z)\big)}{b-u(z)} > 0\,, \qquad |s| << 1\,.$$

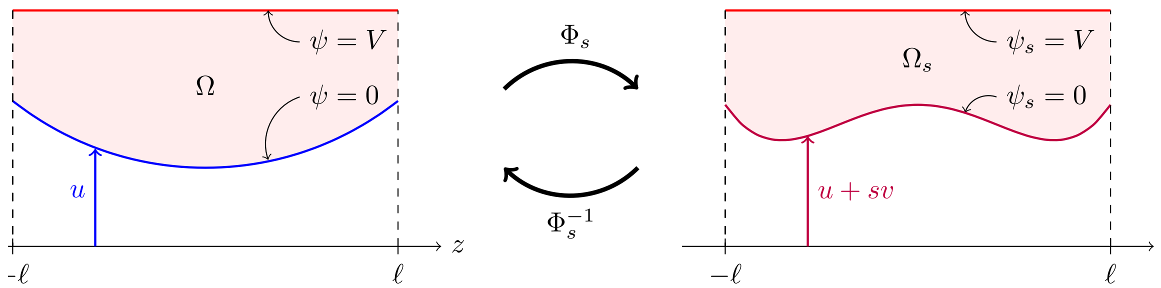

FIGURE 1.3. The diffeomorphism Φ_s^{-1} transforms the domain Ω_s between the perturbed film deflection $u+sv$ and the rigid outer cylinder to the fixed domain Ω between the fixed film deflection u and the rigid outer cylinder.

Therefore, Reynold's transport theorem [**20**, Proposition 5.4.] is applicable to (1.4) and yields

$$\begin{aligned}\delta E_e(u)v = e_e'(0) &= (-\varepsilon_0\pi)\int_\Omega \Big(r\,\partial_s\left(|\partial_z\psi_s|^2+|\partial_r\psi_s|^2\right)|_{s=0} \\ &\qquad + \operatorname{div}\left(\left(|\partial_z\psi|^2+|\partial_r\psi|^2\right)\, r\,\partial_s\Phi_s|_{s=0}\right)\Big)\,\mathrm{d}(z,r) \\ &= (-\varepsilon_0\pi)\int_\Omega\Big(2r\,\left(\nabla\psi_s\cdot\nabla\partial_s\psi_s\right)|_{s=0} \\ &\qquad + \operatorname{div}\left(\left(|\partial_z\psi|^2+|\partial_r\psi|^2\right)\, r\,\partial_s\Phi_s|_{s=0}\right)\Big)\,\mathrm{d}(z,r)\,,\end{aligned} \tag{1.6}$$

where the gradient and divergence are taken with respect to z and r. Recalling that ψ_s solves the Laplace equation in cylindrical coordinates, we see that

$$\begin{aligned}\operatorname{div}\big((\partial_s\psi_s)\,r\,(\partial_z\psi_s,\partial_r\psi_s)\big)\big|_{s=0} &= \Big((\partial_z\partial_s\psi_s)\,r\,\partial_z\psi_s + (\partial_s\psi_s)\partial_z^2\psi_s\,r \\ &\qquad + (\partial_r\partial_s\psi_s)\,r\,\partial_r\psi_s + (\partial_s\psi_s)\,\partial_r(r\,\partial_r\psi_s)\Big)\Big|_{s=0} \\ &= (\partial_z\partial_s\psi_s)|_{s=0}\,r\,\partial_z\psi + (\partial_r\partial_s\psi_s)|_{s=0}\,r\,\partial_r\psi \\ &= r\,\nabla\psi\cdot\nabla(\partial_s\psi_s)|_{s=0}\,.\end{aligned}$$

Hence, also the first term in (1.6) may be written in divergence form, and by Gauss's theorem the expression can be simplified to

$$\begin{aligned}\delta E_e(u)v = (-\varepsilon_0\pi)\int_{\partial\Omega}\Big(&2\,(\partial_s\psi_s)|_{s=0}\,r\,\nabla\psi\cdot\nu \\ &+\left(|\partial_z\psi|^2+|\partial_r\psi|^2\right) r\,\partial_s\Phi_s|_{s=0}\cdot\nu\Big)\mathrm{do}(z,r)\,,\end{aligned} \tag{1.7}$$

where ν denotes the outer normal on $\partial\Omega$. By (1.5) the expression $\partial_s\Phi_s|_{s=0}$ vanishes on the boundary parts $\{\pm\ell\}\times(a,b)$ and $[-\ell,\ell]\times\{b\}$.
Since Φ_s maps each point from these boundary parts onto itself and the boundary condition for ψ_s is independent of s, $(\partial_s\psi_s)|_{s=0}$ also vanishes at these three boundary parts. Therefore, only the boundary integral along graph(u) contributes to (1.7). At this part, the normal ν points towards the z-axis and is given by

$$\nu = \frac{1}{\sqrt{1+(\partial_z u)^2}}\left(\partial_z u, -1\right), \tag{1.8}$$

while the surface measure for integrating over the graph of u is

$$\mathrm{d}o(z) = \sqrt{1+(\partial_z u)^2}\,\mathrm{d}z\,. \tag{1.9}$$

Next, to compute $(\partial_s\psi_s)|_{s=0}$ on graph(u), we differentiate the equality

$$\begin{aligned}\psi_s\big(\Phi_s(z,r)\big) &= \psi_s\bigg(z, r+sv(z)\Big(\frac{b-r}{b-u(z)}\Big)\bigg)\\ &= 0 \quad \text{on} \quad \mathrm{graph}(u)\end{aligned}$$

with respect to s yielding

$$(\partial_s\psi_s)\big(\Phi_s(z,r)\big) + \nabla\psi_s\big(\Phi_s(z,r)\big)\cdot\partial_s\Phi_s(z,r) = 0\,,$$

so that

$$(\partial_s\psi_s)|_{s=0}(z,r) = -\nabla\psi(z,r)\cdot\partial_s\Phi_s|_{s=0}(z,r) \quad \text{on} \quad \mathrm{graph}(u)\,. \tag{1.10}$$

In view of (1.5), and (1.8) - (1.10), the expression for $\delta E_e(u)$ from (1.7) becomes

$$\begin{aligned}\delta E_e(u)v = (\varepsilon_0\pi)\int_{-\ell}^{\ell}\bigg(&2\,\nabla\psi\big(z,u(z)\big)\cdot\big(0,v(z)\big)\,u(z)\,\nabla\psi\big(z,u(z)\big)\cdot\big(\partial_z u(z),-1\big)\\ &-\Big(\big|\partial_z\psi\big(z,u(z)\big)\big|^2+\big|\partial_r\psi\big(z,u(z)\big)\big|^2\Big)\,u(z)\,\big(0,v(z)\big)\cdot\big(\partial_z u(z),-1\big)\bigg)\mathrm{d}z\\ = (\varepsilon_0\pi)\int_{-\ell}^{\ell}\bigg(&2\partial_r\psi\big(z,u(z)\big)\,v(z)\,u(z)\Big[\partial_z\psi\big(z,u(z)\big)\partial_z u(z)-\partial_r\psi\big(z,u(z)\big)\Big]\\ &+\Big(\big|\partial_z\psi\big(z,u(z)\big)\big|^2+\big|\partial_r\psi\big(z,u(z)\big)\big|^2\Big)\,u(z)\,v(z)\bigg)\mathrm{d}z\\ = (\varepsilon_0\pi)\int_{-\ell}^{\ell}\bigg(&2\partial_r\psi\big(z,u(z)\big)\,v(z)\,u(z)\,\partial_z\psi\big(z,u(z)\big)\partial_z u(z)\\ &+\Big(\big|\partial_z\psi\big(z,u(z)\big)\big|^2-\big|\partial_r\psi\big(z,u(z)\big)\big|^2\Big)\,u(z)\,v(z)\bigg)\mathrm{d}z\,.\end{aligned} \tag{1.11}$$

Finally, from

$$\psi\big(z,u(z)\big) = 0\,, \qquad z\in(-\ell,\ell)\,,$$

it follows that

$$\partial_z\psi\big(z,u(z)\big) + \partial_r\psi\big(z,u(z)\big)\,\partial_z u(z) = 0\,, \qquad z\in(-\ell,\ell)\,.$$

Consequently, (1.11) can be simplified to

$$\delta E_e(u)v = (-\varepsilon_0\pi)\int_{-\ell}^{\ell}\big|\nabla\psi\big(z,u(z)\big)\big|^2\, u(z)\, v(z)\,\mathrm{d}z\,. \tag{1.12}$$

First Variation of the Total Energy. From the necessary condition for the film deflection u

$$0 = \delta E(u)v = \delta E_m(u)v + \delta E_e(u)v\,, \qquad v\in\mathcal{D}(-\ell,\ell),$$

and the expressions for $\delta E_m(u)$ and $\delta E_e(u)$ in (1.3) and (1.12) respectively, it follows that u satisfies the equation

$$0 = 2\,T\left(\frac{1}{(1+\partial_z u^2)^{1/2}} - \frac{u}{(1+\partial_z u^2)^{3/2}}\,\partial_z^2 u\right) - \varepsilon_0\, u\,\big|\nabla\psi\big(z,u(z)\big)\big|^2\,. \tag{1.13}$$

Summary of Static Case. In this subsection, we derived a model for the static behaviour of a soap film exposed to an electric field. The complete model consists of the unknown film deflection $u:(-\ell,\ell)\to(0,b)$ given by

$$\begin{cases} 0 &= 2\,T\left(\dfrac{1}{(1+\partial_z u^2)^{1/2}} - \dfrac{u}{(1+\partial_z u^2)^{3/2}}\,\partial_z^2 u\right) - \varepsilon_0\, u\,\big|\nabla\psi\big(z,u(z)\big)\big|^2\,, \\ u(\pm\ell) &= a\,, \end{cases} \tag{1.14}$$

and the electrostatic potential $\psi:\overline{\Omega(u)}\to\mathbb{R}$ satisfying

$$\begin{cases} \Delta\psi &= \frac{1}{r}\partial_r\left(r\partial_r\psi\right) + \partial_z^2\psi = 0 \quad \text{in} \quad \Omega(u)\,, \\ \psi &= h_u \quad \text{on} \quad \partial\Omega(u) \end{cases} \tag{1.15}$$

with

$$\Omega(u) = \big\{(z,r)\in(-\ell,\ell)\times(0,b)\,\big|\, r>u(z)\big\}\,.$$

The boundary condition is given by

$$h_u(z,r) = V\,\frac{\ln\left(\dfrac{r}{u(z)}\right)}{\ln\left(\dfrac{b}{u(z)}\right)}\,. \tag{1.16}$$

1.2.2. Dynamical Case. Having described the static behaviour of the soap film, we now focus on its dynamical behaviour. Instead of treating the full time evolution within the theory of electrodynamics, we use the common approach for related problems on the microscale [**28, 46, 65**] and assume that the electric part of the problem behaves quasi-statically. So we ignore the magnetic field, induced by the film's dynamics, and treat the electric problem approximately, for frozen films, within the theory of electrostatics. Hence, our unknowns are again the film deflection $u(t)$, now depending on t, and the corresponding electrostatic potential ψ.

In the following, we take all assumptions and all simplifications from the static case as well as the new assumed quasi-static behaviour of ψ into account:

- The problem is rotationally symmetric,
- The only forces involved are due to the applied voltage and the surface tension of the film,
- For fixed time, the total energy E is given by (1.2),
- The electrostatic potential $\psi(t)$ satisfies (1.15) and (1.16) for fixed time.

Note that due to the quasi-static assumption, time naturally occurs in the equation for the electrostatic potential only as a parameter (and is therefore often suppressed) whereas the equation itself remains the same as in the static case. Consequently, we turn directly to the dynamical behaviour of $u(t)$.

Dynamical Behaviour of the Film. First, we review the situation in which no voltage is applied between soap film and rigid outer cylinder since the general case will follow from a similar consideration. Without applied voltage, the film's dynamics is determined by its surface tension only and can be described by mean curvature flow. As pointed out in [**34**], mean curvature flow is a simplified model for the evolution of the film in which inertial forces are neglected. In the situation of a surface of revolution, the mean curvature flow is given by

$$\beta V_u(t) = TH\big(u(t)\big)\,,$$

where

$$V_u = \frac{-\partial_t u}{(1+(\partial_z u))^{1/2}}$$

is the normal velocity of the surface of revolution with profile u,

$$H(u) = \left(\frac{1}{u\,(1+\partial_z u^2)^{1/2}} - \frac{1}{(1+\partial_z u^2)^{3/2}}\,\partial_z^2 u\right)$$

denotes its mean curvature, see [**19**], and $\beta > 0$ is a damping constant. Note that there is a close connection between mean curvature flow and the mechanical surface energy E_m. Namely, as $u(t,\pm\ell) = a$, we have $\partial_t u(t,\pm\ell) = 0$ and find, in view of (1.3), that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_m\big(u(t)\big) &= \delta E_m\big(u(t)\big)\partial_t u \\
&= 2\pi \int_{-\ell}^{\ell} T\left(\frac{1}{(1+\partial_z u^2)^{1/2}} - \frac{u}{(1+\partial_z u^2)^{3/2}}\,\partial_z^2 u\right)\partial_t u\,\mathrm{d}z \\
&= 2\pi \int_{-\ell}^{\ell} T\,H(u)\,u\,\partial_t u\,\mathrm{d}z \\
&= -\int_{\mathrm{graph(u)}} T\,H(u)\,V_u(t)\,\mathrm{d}o(z)\,.
\end{aligned}$$

In the last step, we also used $\mathrm{d}o(z) = 2\pi(1+(\partial_z u)^2)^{1/2}\,u\,\mathrm{d}z$ for a surface of revolution. So letting V_u evolve according to the mean curvature flow reduces the surface energy E_m most efficiently, see [**16, 31**].

Now we consider the general situation of an applied voltage between the film and the rigid outer cylinder. Then, according to (1.2), the total energy consists not only of surface energy but also of the electrostatic energy

$$E\big(u(t)\big) = E_m\big(u(t)\big) + E_e\big(u(t)\big)\,.$$

As before, we want to find a law for V_u such that the total energy is reduced most efficiently. Recalling the expression for δE_e from (1.12), we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E\big(u(t)\big) &= \delta E\big(u(t)\big)\partial_t u = \delta E_m\big(u(t)\big)\partial_t u + \delta E_e\big(u(t)\big)\partial_t u\\ &= \pi\int_{-\ell}^{\ell}\Big(2TH(u)\,u - \varepsilon_0\,\big|\nabla\psi\big(z,u\big)\big|^2\,u\Big)\,\partial_t u\,\mathrm{d}z\\ &= -\frac{1}{2}\int_{\mathrm{graph}(u)}\Big(2TH(u) - \varepsilon_0\,\big|\nabla\psi\big(z,u\big)\big|^2\Big)V_u(t)\,\mathrm{d}o(z)\,.\end{aligned}$$

Hence, the film deflection should evolve according to

$$\beta V_u(t) = 2TH(u) - \varepsilon_0\big|\nabla\psi(z,u)\big|^2\,.$$

Plugging in the expression for $H(u)$ and V_u, and rewriting the second order term in divergence form, we get

$$\beta\partial_t u - 2T\,\partial_z\mathrm{arctan}(\partial_z u) = -\frac{2T}{u} + \varepsilon_0\,(1+(\partial_z u)^2)^{1/2}\big|\nabla\psi(z,u)\big|^2\,.$$

Summary of Dynamical Case. In this subsection, we have derived a model for the dynamical behaviour of a soap film exposed to an electric field. The complete model consists of the unknown film deflection $u(t,\,\cdot\,):(-\ell,\ell)\to(0,b)$ given by

$$\left\{\begin{aligned}\beta\partial_t u - 2T\,\partial_z\mathrm{arctan}(\partial_z u) &= -\frac{2T}{u} + \varepsilon_0\,(1+(\partial_z u)^2)^{1/2}\big|\nabla\psi(z,u)\big|^2\,,\\ u(t,\pm\ell) &= a\,,\\ u(0,z) &= u_0\end{aligned}\right. \tag{1.17}$$

with initial shape u_0, and the electrostatic potential $\psi:\overline{\Omega(u)}\to\mathbb{R}$ satisfying

$$\left\{\begin{aligned}\Delta\psi &= \tfrac{1}{r}\partial_r\left(r\partial_r\psi\right) + \partial_z^2\psi = 0 \quad\text{in}\quad \Omega(u)\,,\\ \psi &= h_u \quad\text{on}\quad \partial\Omega(u)\end{aligned}\right. \tag{1.18}$$

with

$$\Omega(u) = \big\{(z,r)\in(-\ell,\ell)\times(0,b)\,\big|\,r>u(z)\big\}\,.$$

The boundary condition is given by

$$h_u(z,r) = V\,\frac{\ln\left(\dfrac{r}{u(z)}\right)}{\ln\left(\dfrac{b}{u(z)}\right)}\,. \tag{1.19}$$

In the equations (1.18)-(1.19) as well as in the evolving domain $\Omega(u)$, we suppressed the t-dependency.

Moreover, we recall the notation:

- T tension of soap film,
- V applied voltage between film and outer cylinder,
- 2ℓ distance of parallel metal rings,
- a inner radius of the metal rings and b radius of outer cylinder,
- ε_0 electric permittivity of the vacuum.

1.3. Dimensionless Equations

In the investigation of (1.17)-(1.19), we focus on the interplay between the electrostatic force and the surface tension measured in terms of V and T, as well as the distance of the metal rings 2ℓ compared to their radii a. To exclude further impacts, we fix the ratio between the radius of the rings and that of the rigid outer cylinder

$$b/a = 2$$

for the rest of this thesis. The choice of ratio 2 will eliminate further constants in the dimensionless system of equations. Denoting variables from the previous section with a subscribed *old*, we introduce the new dimensionless variables

$$z = \frac{z_{old}}{\ell}\,, \qquad \psi = \frac{\psi_{old}}{V}\,, \qquad r = \frac{r_{old}}{a}\,, \qquad u = \frac{u_{old}}{a} - 1\,, \qquad t = \frac{2T\,t_{old}}{a^2\beta}\,,$$

transforming (1.17) into

$$\begin{cases} \partial_t u - \sigma\,\partial_z \arctan(\sigma\partial_z u) \\ \qquad\qquad = -\dfrac{1}{u+1} + \lambda\,(1+\sigma^2(\partial_z u)^2)^{1/2}\left(\sigma^2\big|\partial_z\psi(z,u+1)\big|^2 + \big|\partial_r\psi(z,u+1)\big|^2\right), \\ u(t,\pm 1) \quad = 0\,, \qquad -1 < u < 1\,, \\ u(0,z) \quad = u_0\,, \qquad z \in (-1,1)\,, \end{cases} \tag{1.20}$$

with initial shape $-1 < u_0 < 1$, and new parameters

$$\sigma = \frac{a}{\ell} \qquad \text{and} \qquad \lambda = \frac{\varepsilon_0 V^2}{2Ta}\,.$$

The control parameter λ measures the relative strength of the applied voltage V between film and outer cylinder compared to the surface tension T of the film whereas σ is the relative distance of the parallel metal rings. For small σ, the rings are pulled further apart. The equation (1.18)-(1.19) for the electostatic potential ψ becomes

$$\begin{cases} \dfrac{1}{r}\partial_r\left(r\partial_r\psi\right) + \sigma^2\partial_z^2\psi \; = 0 \quad \text{in} \quad \Omega(u)\,, \\ \qquad\qquad\qquad\qquad \psi \; = h_u \quad \text{on} \quad \partial\Omega(u)\,, \end{cases} \tag{1.21}$$

where

$$\Omega(u) = \big\{(z,r) \in (-1,1)\times(0,2)\,|\,u(z)+1 < r < 2\,\big\}\,,$$

and

$$h_u(z,r) = \ln\Big(\frac{r}{u(z)+1}\Big)\Big/\ln\Big(\frac{2}{u(z)+1}\Big)\,. \tag{1.22}$$

Here, we suppressed, once again, the t-dependency of the equation and the t-dependency of the domain $\Omega(u)$.

The system (1.20)-(1.22) consists of the parabolic equation (1.20) for the film deflection u and the elliptic equation (1.21)-(1.22) for the electrostatic potential ψ. The equations are coupled through the source term of (1.20), and the unknown domain $\Omega(u)$, on whose closure the electrostatic potential ψ is defined. Hence, the coupled system (1.20)-(1.22) is a free boundary problem.
The analytical investigation of (1.20)-(1.22) as well as its time-independent version, which corresponds to the dimensionless version of the static case (1.14) -(1.16), is the main subject of this thesis.

We conclude this chapter with some remarks:

Remarks 1.1 (a) In [**58, 60**] a time-independent and simplified version of (1.20)-(1.22) is studied, in which the equation for the electrostatic potential ψ can be solved explicitly resulting in a single equation for the film deflection u. Including time, this equation reads

$$\begin{cases} \partial_t u - \sigma\, \partial_z \arctan(\sigma \partial_z u) &= -\dfrac{1}{u+1} + \lambda\, g_{sar}(u)\,, \\ \qquad\qquad u(t, \pm 1) &= 0\,, \qquad -1 < u < 1\,, \\ \qquad\qquad u(0,z) &= u_0\,, \qquad z \in (-1,1)\,, \end{cases} \tag{1.23}$$

with initial value $-1 < u_0 < 1$, and electrostatic force

$$g_{sar}(u) := (1 + \sigma^2 (\partial_z u)^2)^{1/2}\, \frac{1}{(u+1)^2 \ln^2\left(\dfrac{2}{u+1}\right)}\,, \tag{1.24}$$

depending only pointwise on u and $\partial_z u$. One refers to (1.23)-(1.24) as the small aspect ratio model since its formal derivation is based on the assumption that a certain model parameter is small. The derivation of the small aspect ratio model can be found in Appendix B.
(b) The electrostatic force $g_{sar}(u)$ in the small aspect ratio model (1.23)-(1.24) tends to infinity at $z \in (-1,1)$ either if $u(z) \to 1$, which corresponds to the soap film touching the outer cylinder, or if $u(z) \to -1$, which might be interpreted as a self-repulsion of the film. This asymptotic behaviour of $g_{sar}(u)$ follows easily from the fact that $g_{sar}(u) \geq \tilde{g}(u)$ on $(-1,1)$ where

$$\tilde{g}(s) := \frac{1}{(s+1)^2 \ln^2\left(\dfrac{2}{s+1}\right)}\,, \qquad s \in (-1,1)\,.$$

A plot of $\tilde{g}$ is depicted in Figure 1.4.
(c) In models for MEMS, where a deformable membrane or plate is suspended above a fixed ground plate, it is usually assumed that each point of the membrane or plate can move only perpendicular to the ground plate, see [**28**] for small aspect ratio models or [**46**] for models with a free boundary. Our modelling assumption is that the film always moves in normal direction.

(d) Since the computation for the first variation of the electrostatic energy δE_e is not rigorous, the derivation of (1.20)-(1.22) is only formal. However, it might be possible to adapt [**43**, Proposition 2.2] to our setting to make the computation rigorous.

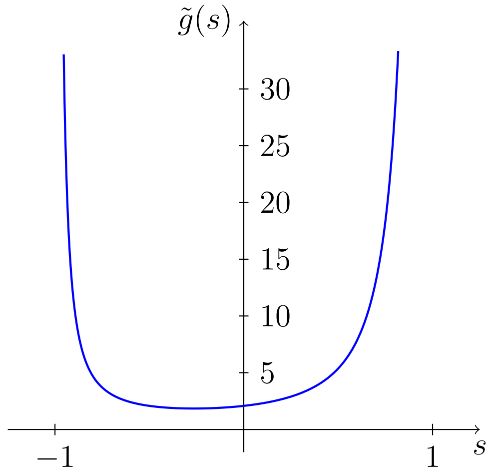

FIGURE 1.4. Plot of the function $\tilde{g}$ which bounds the electrostatic force in the small aspect ratio model from below.

CHAPTER 2

Preliminaries

In this chapter, we provide some preliminary material needed to show local well-posedness of the free boundary problem (1.20)-(1.22). In Section 2.1, we collect notations from analytic semigroup theory, which can be used to solve parabolic equations with general non-local right-hand sides. In Section 2.2, we prove a regularity result for elliptic equations on convex domains (including rough domains like a rectangle).

2.1. Notations from Semigroup Theory

For $q \in (2,\infty)$ and $s \in (0,2]$, we denote the fractional Sobolev spaces with Dirichlet boundary conditions by

$$W^s_{q,D}(-1,1) := \begin{cases} W^s_q(-1,1) & \text{for} \quad s \in (0,1/q)\,, \\ \big\{v \in W^s_q(-1,1)\,\big|\, v(\pm 1) = 0\big\} & \text{for} \quad s \in (1/q,2] \end{cases}$$

and refer to Appendix A for further properties of these spaces.

We then say that an operator $B : W^2_{q,D}(-1,1) \to L_q(-1,1)$ belongs to

$$\mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)$$

if $-B$ generates an analytic semigroup on $L_q(-1,1)$ with domain $W^2_{q,D}(-1,1)$. Moreover, we require a more quantitative characterisation of generators of analytic semigroups from [**5**]: For $\omega > 0$ and $k \geq 1$, an operator $B \in \mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)$ belongs to the class

$$\mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1), k, \omega\big) \tag{2.1}$$

if $\omega + B$ is an isomorphism from $W^2_{q,D}(-1,1)$ onto $L_q(-1,1)$ and if

$$\frac{1}{k} \leq \frac{\big\|(\mu + B)v\big\|_{L_q(-1,1)}}{|\mu|\|v\|_{L_q(-1,1)} + \|v\|_{W^2_{q,D}(-1,1)}} \leq k\,, \quad \mathrm{Re}\mu \geq \omega\,, \quad v \in W^2_{q,D}(-1,1) \setminus \{0\}\,.$$

The classes (2.1) make it possible to derive uniform estimates on semigroups, and hence to treat quasilinear parabolic equations with a non-local right-hand side.

2.2. Elliptic Regularity on Convex Domains

The goal of this section is to give a detailed and complete proof of the following elliptic regularity result on bounded, convex domains:

Theorem 2.1 *Suppose that*

- $\Omega_0 \subset \mathbb{R}^n$ *is convex, open and bounded,*
- $A \in \left[W_q^1(\Omega_0)\right]^{n\times n}$, *where* $q > n$, *is symmetric,*
- *there is* $\alpha > 0$ *with*

$$\xi^T A(x)\xi \geq \alpha|\xi|^2\,, \qquad x \in \overline{\Omega}_0\,, \qquad \xi \in \mathbb{R}^n\,.$$

Then, for each $F \in L_2(\Omega_0)$, *the problem*

$$\begin{cases} -\mathrm{div}\big(A(x)\nabla\phi\big) & = F \quad \text{in} \quad \Omega_0\,, \\ \qquad\qquad \phi & = 0 \quad \text{on} \quad \partial\Omega_0 \end{cases} \tag{2.2}$$

has a unique solution $\phi \in W_{2,D}^2(\Omega_0)$. *Moreover, there exists a constant* C *depending only on* q, n, Ω_0, *the* W_q^1*-norm of the coefficients of* A *and the ellipticity constant* α *such that*

$$\|\phi\|_{W_2^2(\Omega_0)} \leq C\,\|F\|_{L_2(\Omega_0)}\,. \tag{2.3}$$

Theorem 2.1 can also be found in [**41**, Theorem 3.10.1], but with a proof scattered over the book, while the special case $q = \infty$ is presented in [**37**, Theorem 3.2.1.2]. In the context of MEMS in [**45**, Proposition 2.7], a version of Theorem 2.1 for $n = 3$, $q \geq 3$ and coefficient matrix $A \in \left[W_q^1(\Omega_0)\right]^{n\times n}$ satisfying additional structural constraints is proven. Finally, another very special case of Theorem 2.1 in which $A(\,\cdot\,)$ is replaced by a simple scalar function $a \in W_q^1(\Omega_0)$ is contained in [**18**, Proposition 7.6].

We note that, thanks to the Riesz Representation Theorem, problem (2.2) has a unique weak solution $\phi \in W_{2,D}^1(\Omega_0)$, that is ϕ satisfies

$$\int_{\Omega_0} \nabla\phi^T A(x)\nabla\varphi\,\mathrm{d}x = \int_{\Omega_0} F\varphi\,\mathrm{d}x\,, \qquad \varphi \in W_{2,D}^1(\Omega_0)\,.$$

The difficult part is to improve its regularity because such an improvement of regularity is usually derived for bounded C^2-domains, see for example [**29**, Theorem 6.3.4]. But convex domains as assumed in Theorem 2.1 are merely Lipschitz domains [**37**, Corollary 1.2.2.3], for which, in general, an improvement of regularity may even fail.

The proof from [**37**] for the special case $q = \infty$ is based on an approximation of Ω_0 from the inside by a sequence of smooth convex domains (Ω_m) on which the elliptic problem possesses a sequence of unique W_2^2-solutions (ϕ_m). Furthermore, using the convexity of the domains, a remarkable W_2^2-a-priori estimate for ϕ_m independent of m is derived, which allows the author to extract a subsequence of (ϕ_m) converging to a W_2^2- solution to the original problem on Ω_0. We combine the idea of domain approximation from [**37**] with an approximation of the coefficients of $A(\,\cdot\,)$ to reduce the assumption from $q = \infty$ to $q > n$. To this end, we first study improved Sobolev embeddings for convex domains, then refine the a-priori-estimate from [**37**] and finally give a new and detailed proof of Theorem 2.1.

Non-Sharp Sobolev Embedding on Convex Domains. For convex domains, it is possible to give a precise characterization of the embedding constant in Sobolev's embedding theorem depending on the volume and diameter of the domain but not on other quantities of it such as the shape of its boundary. Though this fact is often used, it is difficult to find it stated explicitly in the standard literature. However, a more general version than needed for our purposes may be found in [**57**] for example. For convenience, we recall the proof of a special case here. First, we clarify some notations:

If $\Omega \subset \mathbb{R}^n$ is measurable, we denote its volume by $|\Omega|$ and its diameter by $\mathrm{diam}(\Omega)$. Moreover, if $v \in L_1(\Omega)$, we let

$$v_\Omega = |\Omega|^{-1} \int_\Omega v \,\mathrm{d}y$$

be the average of v over Ω.

Now we state two lemmata from which Sobolev's embedding theorem on convex domains will follow easily:

Lemma 2.2 *Let Ω be a bounded and measurable subset of $\mathbb{R}^n$, and $q > n$. Given $v \in L_2(\Omega)$, let*

$$R(v)(x) := \int_\Omega |x-y|^{1-n} v(y) \,\mathrm{d}y$$

be the Riesz potential. Then, there exists a constant $C > 0$ depending only on q, n and $|\Omega|$ such that

$$\|R(v)\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq C \, \|v\|_{L_2(\Omega)} \,, \qquad v \in L_2(\Omega) \,.$$

Proof. This is [**33**, Lemma 7.12.] applied for $0 \leq \frac{1}{2} - \frac{q-2}{2q} < \frac{1}{n}$. □

Lemma 2.3 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Then,*

$$|v(x) - v_\Omega| \leq \frac{\mathrm{diam}(\Omega)^n}{n|\Omega|} \, R\big(|\nabla v|\big)(x)$$

for a.a. $x \in \Omega$ and $v \in W_1^1(\Omega)$.

Proof. This is proven in [**33**, Lemma 7.16.]. □

Sobolev's embedding theorem on convex domains is now a direct consequence.

Proposition 2.4 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$, let $q > n$. Then, there exists a constant $C > 0$ depending only on q, n, $\mathrm{diam}(\Omega)$ and $|\Omega|$ such that*

$$\|v\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq C \, \|v\|_{W_2^1(\Omega)} \,, \qquad v \in W_2^1(\Omega) \,.$$

Proof. We estimate

$$\|v\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq \|v_\Omega\|_{L_{\frac{2q}{q-2}}(\Omega)} + \|v - v_\Omega\|_{L_{\frac{2q}{q-2}}(\Omega)} \,.$$

Because of

$$\|v_\Omega\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq |\Omega|^{-1/q}\|v\|_{L_2(\Omega)}$$

and

$$\begin{aligned}\|v - v_\Omega\|_{L_{\frac{2q}{q-2}}(\Omega)} &= \Big(\int_\Omega |v(x) - v_\Omega|^{\frac{2q}{q-2}}\,\mathrm{d}x\Big)^{\frac{q-2}{2q}} \\ &\leq C\big(n, \mathrm{diam}(\Omega), |\Omega|\big)\,\big\|R\big(|\nabla v|\big)\big\|_{L_{\frac{2q}{q-2}}(\Omega)} \\ &\leq C\big(q, n, \mathrm{diam}(\Omega), |\Omega|\big)\,\|\nabla v\|_{L_2(\Omega)}\end{aligned}$$

due to Lemma 2.2 and Lemma 2.3, the statement follows. □

We will need the above Proposition 2.4 in the following form:

Corollary 2.5 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Let $q > n$ and $\delta > 0$. Then, there exists a constant $C > 0$ depending only on q, n, δ,* diam(Ω) *and* $|\Omega|$ *such that*

$$\|v\|^2_{L_{\frac{2q}{q-2}}(\Omega)} \leq \delta\|v\|^2_{W^1_2(\Omega)} + C\,\|v\|^2_{L_2(\Omega)} \tag{2.4}$$

for all $v \in W^1_2(\Omega)$.

Proof. For fixed $\varepsilon \in (0, q-n)$, we deduce from

$$2 < \frac{2q}{q-2} < \frac{2(q-\varepsilon)}{(q-\varepsilon)-2}$$

and Hölder's inequality that

$$\|v\|^2_{L_{\frac{2q}{q-2}}(\Omega)} \leq \|v\|^{2(1-\theta)}_{L_2(\Omega)}\,\|v\|^{2\theta}_{L_{\frac{2(q-\varepsilon)}{(q-\varepsilon)-2}}(\Omega)}$$

for suitable $\theta = \theta(q, n) \in (0, 1)$ and all $v \in W^1_2(\Omega)$. Now an application of Proposition 2.4 together with the weighted Young's inequality completes the proof of (2.4). □

Remark 2.6 Note that for $n > 2$ the threshold value $q = n$ may be included in Proposition 2.4 by using the Hardy-Littlewood-Sobolev inequality, see [**57**] for example. However, we cannot utilize this sharp embedding result in Corollary 2.5 since in any case we loose some integrability.

Improved a-priori Estimate. With a suitable control over the constants in Sobolev's embedding theorem at hand, we can now turn to the improved a-priori estimate on smooth convex domains (here that means: $\partial\Omega \in C^2$). The starting point is [**37**, Theorem 3.1.2.1], in which an a-priori estimate for the Dirichlet-Laplacian on smooth convex domains is proven:

Theorem 2.7 [**37**, Theorem 3.1.2.1] *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2-boundary. Let $q > n$. Then, there exists a constant C depending only on* $\operatorname{diam}(\Omega)$ *such that*

$$\|\phi\|_{W_2^2(\Omega)} \leq C \|\Delta\phi\|_{L_2(\Omega)}, \qquad \phi \in W_2^2(\Omega) \cap W_{2,D}^1(\Omega). \tag{2.5}$$

Using a transformation to the principal axes, this result readily generalizes to elliptic operators with constant coefficients. We recall the proof here to point out the dependence of the estimate on the occuring parameters.

Corollary 2.8 [**37**, Lemma 3.1.3.2] *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2-boundary, and A a symmetric $n \times n$-matrix with each eigenvalue larger than $\alpha > 0$. Then, there exists a constant C_1 depending only on the diameter of Ω and α such that*

$$\|\phi\|_{W_2^2(\Omega)} \leq C_1 \left\|\operatorname{div}\bigl(A\nabla\phi\bigr)\right\|_{L_2(\Omega)}, \qquad \phi \in W_2^2(\Omega) \cap W_{2,D}^1(\Omega). \tag{2.6}$$

Proof. Let $\phi \in W_2^2(\Omega) \cap W_{2,D}^1(\Omega)$. Denote by λ_1 to λ_n the eigenvalues of A and let S be the orthogonal matrix with

$$S^T A S = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Put

$$R := (\sqrt{A})^{-1} = S^T \begin{pmatrix} 1/\sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\lambda_n} \end{pmatrix} S.$$

Then $R\Omega$ is again convex, open and bounded with C^2-boundary as well as

$$\operatorname{diam}(R\Omega) \leq \frac{1}{\sqrt{\alpha}} \operatorname{diam}(\Omega). \tag{2.7}$$

Note that the function v defined by $v(x) := \phi(R^{-1}x)$ for $x \in R\Omega$ belongs to $W_2^2(R\Omega) \cap W_{2,D}^1(R\Omega)$ again and solves

$$\Delta v = G, \qquad G(x) := \operatorname{div}\bigl(A\nabla\phi\bigl(R^{-1}x\bigr)\bigr).$$

Consequently, Theorem 2.7 together with (2.7) implies that

$$\|v\|_{W_2^2(R\Omega)} \leq C\bigl(\alpha, \operatorname{diam}(\Omega)\bigr) \|G\|_{L_2(R\Omega)},$$

and transforming this inequality back to Ω yields

$$\|\phi\|_{W_2^2(\Omega)} \leq C\bigl(n, \alpha, \operatorname{diam}(\Omega)\bigr) \left\|\operatorname{div}\bigl(A\nabla\phi\bigr)\right\|_{L_2(\Omega)}.$$

□

We can now generalize the a-priori estimate for elliptic operators with variable coefficients from [**37**, Lemma 3.1.3.2, Theorem 3.1.3.1]. As in [**37**, Lemma 3.1.3.2], we treat variable coefficient operators locally as a perturbation of constant coefficient operators. Our new ingredient is the Sobolev's embedding theorem for convex domains. It allows us to formulate an a-priori estimate in which the constant does not depend on the W^1_∞-norm of the coefficient matrix $A(\,\cdot\,)$ as in [**37**], but only on the W^1_q-norm of this matrix for $q > n$.

Theorem 2.9 *Suppose that*

- $\Omega_0 \subset \mathbb{R}^n$ *is convex, open and bounded,*
- $q > n$,
- $A \in \big[C^\infty(\overline{\Omega}_0)\big]^{n\times n}$ *is symmetric,*
- *there is* $\alpha > 0$ *with*

$$\xi^T A(x)\xi \geq \alpha|\xi|^2\,, \qquad x \in \overline{\Omega}_0\,, \qquad \xi \in \mathbb{R}^n\,.$$

Then, there exists a constant C depending only on the W^1_q-norm of the coefficients of A, the ellipticity constant α and Ω_0 such that for each convex and open $\Omega \subset \Omega_0$ with C^2-boundary and $|\Omega| \geq \frac{1}{2}|\Omega_0|$ the estimate

$$\|\phi\|_{W^2_2(\Omega)} \leq C\,\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|_{L_2(\Omega)}\,, \qquad \phi \in W^2_2(\Omega) \cap W^1_{2,D}(\Omega)\,, \tag{2.8}$$

holds.

Proof. **(i)** *Local Estimate:*
Near fixed $x_0 \in \overline{\Omega}_0$, our first goal is to prove a local version of (2.8). To this end, we interpret the operator locally as a perturbation of the constant coefficient operator $-\mathrm{div}\big(A(x_0)\nabla\phi\big)$ treated in Corollary 2.8. Assume that $\phi \in W^1_{2,D}(\Omega)$ with support contained in $\mathbb{B}(x_0, r) \cap \Omega$ (where $r > 0$ will be determined later). Writing $A(x_0) = [a_{ij}(x_0)]$ and $A(x) = [a_{ij}(x)]$ we deduce from

$$\begin{aligned}
&\mathrm{div}\big(A(x_0)\nabla\phi\big) - \mathrm{div}\big(A(x)\nabla\phi\big)\\
&\qquad = \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x_0)\partial_j\phi\big) - \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\partial_j\phi\big)\\
&\qquad = \sum_{i,j=1}^{n} \big(a_{ij}(x_0) - a_{ij}(x)\big)\partial_i\partial_j\phi - \sum_{i,j=1}^{n} \partial_i a_{ij}(x)\partial_j\phi
\end{aligned}$$

that

$$\begin{aligned}
&\big|\mathrm{div}\big(A(x_0)\nabla\phi\big) - \mathrm{div}\big(A(x)\nabla\phi\big)\big|^2\\
&\qquad \leq C(n)\Big(\sum_{i,j=1}^{n} |a_{ij}(x_0) - a_{ij}(x)|^2\,|\partial_i\partial_j\phi|^2 + \sum_{i,j=1}^{n} |\partial_i a_{ij}(x)|^2\,|\partial_j\phi|^2\Big)\,.
\end{aligned}$$

Integrating w.r.t. $x \in \Omega$ and using that $W^1_q(\Omega_0) \hookrightarrow C^s(\overline{\Omega}_0)$ with $s = 1 - n/q > 0$, we get

$$\begin{aligned}
&\big\|\mathrm{div}\big(A(x_0)\nabla\phi\big) - \mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)}\\
&\qquad \leq C\big(n, \|A\|_{W^1_q(\Omega_0)}, \Omega_0\big)\left(r^{2s}\,\|\phi\|^2_{W^2_2(\Omega)} + \sum_{i,j=1}^{n}\int_\Omega |\partial_i a_{ij}|^2\,|\partial_j\phi|^2\,\mathrm{d}x\right).
\end{aligned} \tag{2.9}$$

Applying Hölder's inequality with exponents $\frac{2}{q} + \frac{q-2}{q} = 1$ together with Corollary 2.5 to the second term in (2.9) gives

$$\begin{aligned}\int_\Omega |\partial_i a_{ij}|^2\,|\partial_j\phi|^2\,\mathrm{d}x &\le \|a_{ij}\|^2_{W^1_q(\Omega)}\,\|\partial_j\phi\|^2_{L_{\frac{2q}{q-2}}(\Omega)}\\ &\le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0\big)\Big(\delta\,\|\partial_j\phi\|^2_{W^1_2(\Omega)} + C(\delta)\,\|\partial_j\phi\|^2_{L_2(\Omega)}\Big)\end{aligned}$$

for each $\delta > 0$. Plugging this back into (2.9) yields

$$\begin{aligned}&\big\|\mathrm{div}\big(A(x_0)\nabla\phi\big) - \mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)}\\ &\qquad\le C\big(\|A\|_{W^1_q(\Omega_0)}, n, q, \Omega_0\big)\Big((r^{2s}+\delta)\,\|\phi\|^2_{W^2_2(\Omega)} + C(\delta)\|\phi\|^2_{W^1_2(\Omega)}\Big)\,.\end{aligned} \tag{2.10}$$

Now we infer from Corollary 2.8, the triangle inequality and (2.10) that there exists a constant $C_2 = C_2\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0\big)$ with

$$\begin{aligned}\|\phi\|^2_{W^2_2(\Omega)} &\le 2\,C_1^2\,\Big(\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)} + \big\|\mathrm{div}\big(A(x_0)\nabla\phi\big) - \mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)}\Big)\\ &\le C_2\,\Big(\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)} + (r^{2s}+\delta)\,\|\phi\|^2_{W^2_2(\Omega)} + C(\delta)\|\phi\|^2_{W^1_2(\Omega)}\Big)\,.\end{aligned}$$

Choosing r and $\delta > 0$ with

$$(r^{2s}+\delta) \le \frac{1}{2\,C_2}\,,$$

we arrive at

$$\|\phi\|^2_{W^2_2(\Omega)} \le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0\big)\,\Big(\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)} + \|\phi\|^2_{W^1_2(\Omega)}\Big) \tag{2.11}$$

for all $\phi \in W^2_{2,D}(\Omega)$ with support contained in $\mathbb{B}(x_0, r) \cap \Omega$. Finally, it follows from Friedrich's inequality, Gauss's theorem and the weighted Young's inequality that

$$\begin{aligned}\|\phi\|^2_{W^1_2(\Omega)} &\le C\big(\mathrm{diam}(\Omega_0)\big)\|\nabla\phi\|^2_{L_2(\Omega)}\\ &\le \tfrac{1}{\alpha}\,C\big(\mathrm{diam}(\Omega_0)\big)\int_\Omega \nabla\phi^T A(x)\nabla\phi\,\mathrm{d}x\\ &\le C\big(\mathrm{diam}(\Omega_0), \alpha\big)\,\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\|^2_{L_2(\Omega)} + \tfrac{1}{2}\|\phi\|^2_{W^1_2(\Omega)}\,,\end{aligned}$$

so that we can eliminate the W^1_2-norm of ϕ on the right-hand side of (2.11) and get

$$\|\phi\|^2_{W^2_2(\Omega)} \le C\,\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|^2_{L_2(\Omega)} \tag{2.12}$$

for all $\phi \in W^1_{2,D}(\Omega)$ with support contained in $\mathbb{B}(x_0, r) \cap \Omega$ where C and also r depend only on q, n , $\|A\|_{W^1_q(\Omega_0)}$, the ellipticity constant α, and Ω_0 (but not on Ω).

(ii) *Global Estimate:*
We aim for a global version of (2.12). To this end, note that $\overline{\Omega}_0$ is compact so that there are $x_1, \dots, x_m \in \overline{\Omega}_0$ with

$$\overline{\Omega}_0 \subset \bigcup_{i=1}^m \mathbb{B}(x_i, r).$$

Let $\{\theta_i \,|\, i = 1, \ldots, m\}$ be a smooth partition of unity on $\overline{\Omega}_0$ subordinated to $\bigcup_{i=1}^m \mathbb{B}(x_i, r)$ and note that it only depends on Ω_0 and r (and hence not on Ω). The situation is depicted in Figure 2.1. For $\phi \in W^2_{2,D}(\Omega)$, it follows that $\theta_i\phi \in W^2_{2,D}(\Omega)$ with support in $\mathbb{B}(x_i, r) \cap \Omega$ and

$$\begin{aligned}\|\phi\|_{W^2_2(\Omega)} &\le \sum_{i=1}^m \|\theta_i\phi\|_{W^2_2(\Omega)}\\ &\le C\big(\|A\|_{W^1_q(\Omega_0)}, n, q, \Omega_0, \alpha\big) \sum_{i=1}^m \big\|\mathrm{div}\big(A(\,\cdot\,)\nabla(\theta_i\phi)\big)\big\|_{L_2(\Omega)} \end{aligned} \tag{2.13}$$

thanks to (2.12). For the right-hand side, we compute

$$\begin{aligned}\mathrm{div}\big(A(x)\nabla(\theta_i\phi)\big) &= \theta_i\,\mathrm{div}\big(A(x)\nabla\phi\big) + \sum_{j=1}^n \Big((\partial_j\phi)\big[A(x)\nabla\theta_i\big]_j + (\partial_j\theta_i)\big[A(x)\nabla\phi\big]_j\Big)\\ &\quad + \phi\,\mathrm{div}\big(A(x)\nabla\theta_i\big)\\ &=: I + II + III\,.\end{aligned}$$

Since

$$\|\theta_i\|_{C^2(\overline{\Omega}_0)} \le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0, \alpha\big)$$

for each $i = 1, \ldots, m$, we find

$$\|I\|_{L_2(\Omega)} \le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0, \alpha\big)\,\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|_{L_2(\Omega)} \tag{2.14}$$

as well as

$$\begin{aligned}\|II\|_{L_2(\Omega)} &\le 2n^2\|\theta_i\|_{C^1(\overline{\Omega}_0)}\,\|A\|_{C(\overline{\Omega}_0)}\,\|\nabla\phi\|_{L_2(\Omega)}\\ &\le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0, \alpha\big)\,\|\nabla\phi\|_{L_2(\Omega)}\,,\end{aligned} \tag{2.15}$$

where we used that $W^1_q(\Omega_0) \hookrightarrow C(\overline{\Omega}_0)$. For III, we compute further

$$III = \sum_{j,k=1}^n \phi\,\partial_j a_{jk}\,\partial_k\theta_i + a_{jk}\,\phi\,\partial_j\partial_k\theta_i\,,$$

and hence

$$\|III\|_{L_2(\Omega)} \le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0, \alpha\big) \sum_{j,k=1}^n \Big(\|\phi\,\partial_j a_{jk}\|_{L_2(\Omega)} + \|\phi\|_{L_2(\Omega)}\Big)\,,$$

where we applied the embedding $W^1_q(\Omega_0) \hookrightarrow C(\overline{\Omega}_0)$ once more. Since

$$\begin{aligned}\|\phi\,\partial_j a_{jk}\|^2_{L_2(\Omega)} = \int_\Omega \phi^2(\partial_j a_{jk})^2\,\mathrm{d}x &\le \|\phi\|^2_{L_{\frac{2q}{q-2}}(\Omega)}\|A\|^2_{W^1_q(\Omega_0)}\\ &\le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0\big)\|\phi\|^2_{W^1_2(\Omega)}\end{aligned}$$

due to Hölder's inequality and Sobolev's embedding theorem 2.4, we arrive at

$$\|III\|_{L_2(\Omega)} \le C\big(\|A\|_{W^1_q(\Omega_0)}, q, n, \Omega_0, \alpha\big)\|\phi\|_{W^1_2(\Omega)}\,. \tag{2.16}$$

Plugging the estimates for I to III in (2.14)-(2.16) back into (2.13), we find that

$$\|\phi\|_{W_2^2(\Omega)} \leq C\big(\|A\|_{W_q^1(\Omega_0)}, q, n, \Omega_0, \alpha\big) \left(\big\|\mathrm{div}\big(A(\,\cdot\,)\nabla\phi\big)\big\|_{L_2(\Omega)} + \|\phi\|_{W_2^1(\Omega)}\right)$$

for $\phi \in W_{2,D}^2(\Omega)$. Eventually, we can apply the same steps which lead to (2.12) to eliminate the W_2^1-norm of ϕ on the right-hand side. $\square$

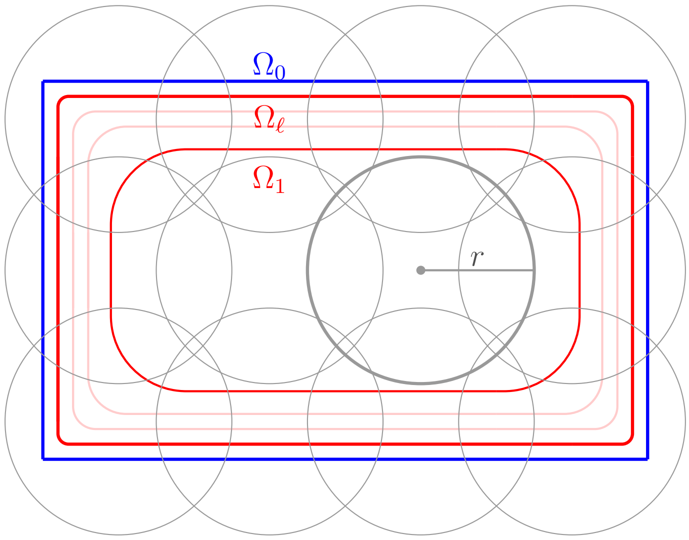

FIGURE 2.1. Depiction of Ω_0 (blue) covered by balls of radius r (grey), together with a sequence of different choices for Ω, denoted by $\Omega_1, \ldots, \Omega_\ell$ (red). Note that the covering of Ω_0 is independent of (Ω_ℓ).

Elliptic Regularity on Convex Domains: Proof of Theorem 2.1. Based on the improved a-priori estimate (2.8), we can complete the proof of Theorem 2.1 .

Proof of Theorem 2.1. (i) *Approximation of the domain:*
Suppose first that $A \in \big[C^\infty(\overline{\Omega}_0)\big]^{n\times n}$. Then, we can follow the lines of the proof of [**37**, Theorem 3.2.1.2] with [**37**, Equation (3.2.1.3)] replaced by the improved a-priori estimate from Theorem 2.9.[1] Hence, problem (2.2) has a unique solution $\phi \in W_{2,D}^2(\Omega_0)$ which additionially satisfies the estimate

$$\|\phi\|_{W_2^2(\Omega_0)} \leq C\big(q, n, \Omega_0, \|A\|_{W_q^1(\Omega)}, \alpha\big)\, \|F\|_{L_2(\Omega_0)}\,. \tag{2.17}$$

(ii) *Approximation of the coefficients:*
Now we treat the general case $A \in \big[W_q^1(\Omega_0)\big]^{n\times n}$ with $q > n$. Recall that Ω_0 has a

[1]It is not difficult to ensure that $|\Omega_m| \geq \frac{1}{2}|\Omega_0|$ where (Ω_m) denotes the approximative sequence of smooth convex subsets of Ω_0 with $\mathrm{dist}\big(\partial\Omega_m, \partial\Omega_0\big) \to 0$ as $m \to \infty$ from [**37**]. Indeed, we may assume without loss of generality that $0 \in \mathrm{int}(\Omega_0)$. Since $\frac{1}{\sqrt[n]{2}}\Omega_0 \subset \mathrm{int}(\Omega_0)$ with $\big|\frac{1}{\sqrt[n]{2}}\Omega_0\big| = \frac{1}{2}\big|\Omega_0\big|$ and $\mathrm{dist}\big(\frac{1}{\sqrt[n]{2}}\Omega_0, \partial\Omega_0\big) > 0$, we find $m_0 \in \mathbb{N}$ with $\frac{1}{\sqrt[n]{2}}\Omega_0 \subset \Omega_m$ for $m \geq m_0$ and consequently $|\Omega_m| \geq 1/2|\Omega_0|$ for $m \geq m_0$.

Lipschitz boundary so that we find a sequence $(A^{(m)}) \subset \big[C^\infty(\overline{\Omega}_0)\big]^{n\times n}$ such that each $A^{(m)}(x)$ is symmetric and $A^{(m)} \to A$ in $\big[W_q^1(\Omega_0)\big]^{n\times n}$. Moreover, we may assume that

$$\sup_m \|A^{(m)}\|_{W_q^1(\Omega_0)} \leq 2\|A\|_{W_q^1(\Omega_0)}\,.$$

It remains to arrange that the $(A^{(m)})$ have a common ellipticity constant. To this end, note that $q > n$ implies that

$$\begin{aligned}\big|\xi^T A(x)\xi - \xi^T A^{(m)}(x)\xi\big| &\leq |\xi^T|\,\|A - A^{(m)}\|_{C(\overline{\Omega}_0)}|\xi| \\ &\leq C\,\|A - A^{(m)}\|_{W_q^1(\Omega_0)} \to 0\end{aligned}$$

for $m \to \infty$ and for each $\xi \in \mathbb{R}^n$ with $|\xi| = 1$ and $x \in \overline{\Omega}_0$. Hence, we may assume without loss of generality that

$$\big|\xi^T A(x)\xi - \xi^T A^{(m)}(x)\xi\big| \leq \alpha/2\,, \qquad |\xi| = 1\,,\ x \in \overline{\Omega}_0\,,\ m \in \mathbb{N}\,,$$

which immediately implies that each $A^{(m)}$ is uniformly elliptic with a common ellipticity constant $\alpha/2$.

Now it follows from part (i) that there exists a unique solution $\phi_m \in W^2_{2,D}(\Omega_0)$ to the problem

$$\left\{\begin{aligned} -\mathrm{div}\big(A^{(m)}(x)\nabla\phi_m\big) &= F \quad \text{in} \quad \Omega_0\,, \\ \phi_m &= 0 \quad \text{on} \quad \partial\Omega_0 \end{aligned}\right.$$

with

$$\|\phi_m\|_{W_2^2(\Omega_0)} \leq C\big(q, n, \Omega_0, \|A\|_{W_q^1(\Omega)}, \alpha\big)\,\|F\|_{L_2(\Omega_0)}\,, \qquad m \in \mathbb{N}\,, \tag{2.18}$$

due to (2.17). Hence, we find a subsequence (ϕ_m) and $\phi \in W^2_{2,D}(\Omega_0)$ with $\phi_m \to \phi$ in $W_2^1(\Omega_0)$ and $\phi_m \rightharpoonup \phi$ in $W_2^2(\Omega_0)$. Letting $m \to \infty$ in the weak formulation

$$\int_{\Omega_0} \nabla\phi_m^T A^{(m)}(x)\nabla\varphi\,\mathrm{d}x = \int_{\Omega_0} F\varphi\,\mathrm{d}x\,, \qquad \varphi \in \mathcal{D}(\Omega_0)\,,$$

we see that ϕ is a solution to (2.2). It is unique due to the Riesz Representation Theorem. Finally, estimate (2.3) follows from (2.18) and the weak lower semi-continuity of $\|\cdot\|_{W_2^2(\Omega_0)}$. $\square$

CHAPTER 3

Local Well-Posedness

In this chapter, we establish local well-posedness of the free boundary problem (1.20)-(1.22). To this end, we want to recast (1.20)-(1.22) as a single parabolic equation for the film deflection u only,

$$\begin{cases} \partial_t u - \sigma\, \partial_z \arctan(\sigma \partial_z u) &= -\dfrac{1}{u+1} + \lambda\, g(u)\,, \\ u(t,\pm 1) &= 0\,, \qquad -1 < u < 1\,, \\ u(0,z) &= u_0\,, \qquad z \in (-1,1)\,, \end{cases}$$

with initial shape $-1 < u_0 < 1$ and electrostatic force

$$g(u) := (1 + \sigma^2 (\partial_z u)^2)^{3/2} \left| \partial_r \psi_u(z, u+1) \right|^2 . \tag{3.1}$$

Because the spatial derivative of the boundary condition $\psi_u\big(z, u(z)+1\big) \equiv 0$ gives

$$\partial_z \psi_u(z, u+1) = -(\partial_z u) \partial_r \psi_u(z, u+1)\,,$$

this single parabolic equation is exactly equivalent to equation (1.20). The difference is now that we view the electrostatic force as a non-local map $[u \mapsto g(u)]$ between suitable function spaces. More precisely, for fixed time, this map should first take the function u to the electrostatic potential ψ_u, which solves the elliptic equation (1.21)-(1.22) on the u-dependent domain $\Omega(u)$, and then do further manipulations with ψ_u resulting in (3.1). In this reinterpretation of (1.20)-(1.22), the electrostatic potential occurs no longer as an equal unknown, but only as a quantity completely subordinated to and determined by the film deflection u at fixed time. This is also the reason for using the notation ψ_u with subscribed u.

Of course, it is not clear if the non-local map $[u \mapsto g(u)]$ is meaningful. Therefore, Section 3.1 is devoted to the study of $[u \mapsto g(u)]$. In particular, we show that $[u \mapsto g(u)]$ is Lipschitz continuous in a suitable functional analytic setting. Then, in Section 3.2, we prove local well-posedness of (1.20)-(1.22) relying on its reinterpretation in terms of u, semigroup theory and Banach's fixed point theorem. The whole chapter follows [**24**], in which local well-posedness for a quasilinear free boundary problem modelling MEMS is established.

3.1. Elliptic Subproblem

We analyse the map $[v \mapsto g(v)]$, where $g(v)$ denotes the electrostatic force from (3.1), and $v \in W^2_{q,D}(-1,1)$ with $q > 2$ and $-1 < v(z) < 1$ is a time-independent film deflection. The key step in the analysis of $[v \mapsto g(v)]$ is the investigation of $[v \mapsto \psi_v]$ with ψ_v being the solution to the elliptic subproblem (1.21)-(1.22), i.e. to

$$\begin{cases} \frac{1}{r}\partial_r\left(r\partial_r\psi_v\right) + \sigma^2\partial_z^2\psi_v &= 0 \quad \text{in} \quad \Omega(v)\,, \\ \psi_v &= h_v \quad \text{on} \quad \partial\Omega(v) \end{cases}$$

with

$$\Omega(v) = \big\{(z,r) \in (-1,1)\times(0,2)\,|\,v(z)+1 < r < 2\big\}$$

and boundary condition

$$h_v(z,r) = \ln\left(\frac{r}{v(z)+1}\right)\Big/\ln\left(\frac{2}{v(z)+1}\right).$$

As in [**46**], we note that this elliptic equation has a unique weak solution $\psi_v \in W_2^1\big(\Omega(v)\big)$ by Lax-Milgram Theorem, but that this regularity is by no means sufficient to define the electrostatic force $g(v)$ as it contains the square of the trace of the derivative of ψ_v. In addition, the Lax-Milgram Theorem provides no information on the dependency of ψ_v on v. To make the dependency of ψ_v on v accessible, we transform the domain $\Omega(v)$, on whose closure ψ_v is defined, to an v-independent reference domain, and work out how the resulting equations depend on its coefficients.

More precisely, for a given film deflection $v \in W_{q,D}^2(-1,1)$ with $-1 > v(z) > 1$ and $q > 2$, we transform the domain $\Omega(v)$ to the fixed rectangle

$$\Omega = (-1,1)\times\big(1,2\big)$$

via $T_v : \overline{\Omega(v)} \to \overline{\Omega}$ defined by

$$T_v(z,r) := \left(z, \frac{r-2v(z)}{1-v(z)}\right), \qquad (z,r) \in \overline{\Omega(v)}\,. \tag{3.2}$$

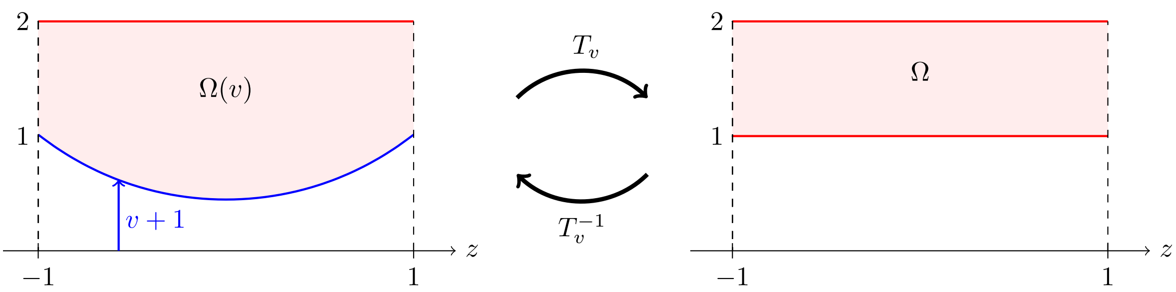

FIGURE 3.1. The diffeomorphism T_v transforms the domain $\Omega(v)$ to the fixed reference domain Ω, which is a rectangle.

Due to the chain rule as well as transformation results for Sobolev functions [**63**, Lemma 2.3.2], we get that the electrostatic potential ψ_v solves (1.21)-(1.22) weakly or strongly on $\Omega(v)$ if and only if $\phi_v := \psi_v \circ (T_v)^{-1}$ is a weak or strong solution to

$$\begin{cases} L_v\phi_v &= 0 \quad \text{in} \quad \Omega\,, \\ \phi_v &= \frac{\ln(r)}{\ln(2)} \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.3}$$

where the transformed v-dependent differential operator L_v is given by

$$\begin{aligned} L_v w := {} & \sigma^2(1-v)\partial_z^2 w - 2\sigma^2\,\partial_z v\,(2-r)\partial_r\partial_z w \\ & + \frac{1+\sigma^2(\partial_z v)^2(2-r)^2}{1-v}\,\partial_r^2 w \\ & + \left[-\sigma^2(2-r)\Big(\partial_z^2 v + \frac{2(\partial_z v)^2}{1-v}\Big) + \frac{1}{2v+(1-v)r}\right]\partial_r w\,. \end{aligned} \tag{3.4}$$

In divergence form this operator reads

$$L_v w = \operatorname{div}\left(A(v)\nabla w\right) + d(v)\cdot\nabla w \tag{3.5}$$

with

$$A(v) = [a_{ij}(v)]_{i,j=1}^2 := \begin{pmatrix} \sigma^2(1-v) & -\sigma^2\,\partial_z v\,(2-r) \\ -\sigma^2\,\partial_z v\,(2-r) & \dfrac{1+\sigma^2(\partial_z v)^2(2-r)^2}{1-v} \end{pmatrix},$$

$$d(v) = \begin{pmatrix} d_1(v) \\ d_2(v) \end{pmatrix} := \begin{pmatrix} 0 \\ \dfrac{1}{2v+(1-v)r} \end{pmatrix}.$$

On the one hand, if the soap film touches the outer rigid cylinder, i.e. $v(z_0) = 1$ for some $z_0 \in (-1,1)$, the coefficient $a_{11}(v)$ will vanish, while $a_{22}(v)$ will develop a singularity in z_0. On the other hand, self-touching of the film, i.e. $v(z_0) = -1$, would yield a singularity of the lower-order coefficient $d_2(v)$ in $(z_0, 1)$. To exclude these critical phenomena, we will study the dependency of (3.3) on v only on the sets

$$S(\kappa) := \left\{ v \in W^2_{q,D}(-1,1) \,\middle|\, \|v\|_{W^2_q(-1,1)} \le 1/\kappa\,,\ -1+\kappa \le v(z) \le 1-\kappa \right\}$$

for $\kappa > 0$ and fixed $q > 2$.

3.1.1. Solution Theory. The aim of this subsection is threefold: We present the weak and strong solution theory for the problem

$$\begin{cases} L_v \Phi_v &= F \quad \text{in} \quad \Omega\,, \\ \quad \Phi_v &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.6}$$

with F in $W^{-1}_{2,D}(\Omega)$ or $L_2(\Omega)$ respectively, which is closely related to the transformed problem (3.3), we derive a-priori estimates for Φ_v holding uniformly on $S(\kappa)$, and we use interpolation theory to improve these a-priori estimates. The applied methods are similar to those leading to [**24**, Lemma 2.2].

As a preliminary step, we check that the transformed operator $-L_v$ is again uniformly elliptic with ellipticity constant independent of $v \in S(\kappa)$.

Lemma 3.1 *There exists a constant $\alpha = \alpha(\kappa) > 0$ such that*

$$\alpha|\xi|^2 \le \xi^T A(v)\xi \le \frac{1}{\alpha}|\xi|^2\,, \qquad \xi \in \mathbb{R}^2\,, \quad (z,r) \in \Omega\,, \quad v \in S(\kappa)\,.$$

Proof. The real eigenvalues $\mu_\pm$ of $A(v)$ satisfy

$$\operatorname{tr}\big(A(v)\big) = \mu_+ + \mu_- \,, \qquad \det\big(A(v)\big) = \mu_+\mu_-$$

with

$$\det\big(A(v)\big) = \sigma^2 > 0$$

and

$$\operatorname{tr}\big(A(v)\big) = \frac{\sigma^2(1-v)^2 + 1 + \sigma^2(\partial_z v)^2(2-r)^2}{1-v} > 0$$

as well as

$$\operatorname{tr}\big(A(v)\big) \leq \frac{4\sigma^2 + 1 + \sigma^2\, C(q)^2\, \kappa^{-2}}{\kappa}\,,$$

where $C(q)$ denotes the embedding constant of $W_q^2(-1,1) \hookrightarrow C^1([-1,1])$. Thus, we find $\alpha(\kappa) > 0$ with

$$\frac{1}{\alpha(\kappa)} \geq \operatorname{tr}\big(A(v)\big) \geq \mu_+ \geq \mu_- \geq \frac{\det\big(A(v)\big)}{\operatorname{tr}\big(A(v)\big)} \geq \alpha(\kappa) > 0$$

for all $(z,r) \in \Omega$ and $v \in S(\kappa)$. □

Weak Solutions. We consider weak solutions to (3.6) and corresponding a-priori estimates. Though existence and uniqueness results for solutions are usually supplemented by a-priori estimates, see [**33**, Corollary 8.7, Lemma 9.17] and [**29**, Theorem 6.2.6], we have to repeat the arguments to include the v-dependency.

Lemma 3.2 *For each $v \in S(\kappa)$ and each $F \in W_{2,D}^{-1}(\Omega)$, there exists a unique weak solution $\Phi_v \in W_{2,D}^1(\Omega)$ to* (3.6)*, i.e. to the equation*

$$\begin{cases} L_v\Phi_v &= F \quad \text{in} \quad \Omega\,, \\ \quad \Phi_v &= 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

Moreover, there exists $C_1(\kappa) > 0$ (independent of F, Φ_v and v) such that

$$\|\Phi_v\|_{W_{2,D}^1(\Omega)} \leq C_1(\kappa)\|F\|_{W_{2,D}^{-1}(\Omega)}\,. \tag{3.7}$$

Proof. The existence of a unique weak solution to problem (3.6) is a consequence of [**33**, Theorem 8.3]. So it remains to prove (3.7):
(i) As a first step, we show the existence of $C(\kappa) > 0$ with

$$\|\Phi_v\|_{W_2^1(\Omega)} \leq C(\kappa)\left(\|\Phi_v\|_{L_2(\Omega)} + \|F\|_{W_{2,D}^{-1}(\Omega)}\right) \tag{3.8}$$

for each $v \in S(\kappa)$ and $F \in W_{2,D}^{-1}(\Omega)$. To this end, we test the weak formulation of (3.6) with $\Phi = \Phi_v$ resulting in

$$\int_\Omega \nabla\Phi^T A(v)\nabla\Phi \,\mathrm{d}(z,r) = \int_\Omega \big(d(v)\cdot\nabla\Phi\big)\Phi \,\mathrm{d}(z,r) - \langle F, \Phi\rangle_{W_2^1(\Omega)}\,.$$

Combining now the uniform ellipticity of $-L_v$ with v-independent ellipticity constant $\alpha(\kappa) > 0$ from Lemma 3.1 with Friedrich's inequality yields

$$\|\Phi\|^2_{W^1_2(\Omega)} \le C(\kappa)\left(\Big|\int_\Omega \big(d(v)\cdot\nabla\Phi\big)\Phi\,\mathrm{d}(z,r)\Big| + \|F\|_{W^{-1}_{2,D}(\Omega)}\|\Phi\|_{W^1_2(\Omega)}\right)$$

for some $C(\kappa) > 0$. Finally, the fact that $\|d(v)\|_\infty$ is uniformly bounded on $S(\kappa)$ together with Hölder's inequality and Young's inequality gives

$$\|\Phi\|^2_{W^1_2(\Omega)} \le C(\kappa)\big(\|\Phi\|^2_{L_2(\Omega)} + \|F\|_{W^{-1}_{2,D}(\Omega)}\|\Phi\|_{W^1_2(\Omega)}\big)$$

for some new $C(\kappa) > 0$, which is obviously equivalent to (3.8).

(ii) It remains to eliminate the L_2-norm of Φ_v on the right-hand side of (3.8). We proceed by contradiction and therefore assume that (3.7) is not true. Then, we find sequences $(v_k) \in S(\kappa)$, $(\Phi_k) \in W^1_{2,D}(\Omega)$ and $(F_k) \subset W^{-1}_{2,D}(\Omega)$ such that Φ_k is the unique weak solution to

$$\begin{cases} L_{v_k}\Phi_k &= F_k \quad \text{in} \quad \Omega\,, \\ \quad\Phi_k &= 0 \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

satisfying the estimate

$$\|\Phi_k\|_{W^1_2(\Omega)} \ge k\|F_k\|_{W^{-1}_{2,D}(\Omega)} > 0\,. \tag{3.9}$$

Putting $\tilde\Phi_k := \Phi_k/\|\Phi_k\|_{L_2(\Omega)}$ as well as $\tilde F_k := F_k/\|\Phi_k\|_{L_2(\Omega)} \in W^{-1}_{2,D}(\Omega)$, we see that $\tilde\Phi_k \in W^1_{2,D}(\Omega)$ is the unique weak solution to

$$\begin{cases} L_{v_k}\tilde\Phi_k &= \tilde F_k \quad \text{in} \quad \Omega\,, \\ \quad\tilde\Phi_k &= 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.10}$$

For $k \ge 2C(\kappa)$ with $C(\kappa)$ given in (3.8), it follows from (3.8) and (3.9) that

$$\|\tilde\Phi_k\|_{W^1_2(\Omega)} \le 2C(\kappa)\,, \qquad \|\tilde F_k\|_{W^{-1}_{2,D}(\Omega)} \le \frac{2C(\kappa)}{k}\,,$$

in particular $\tilde F_k \to 0$ in $W^{-1}_{2,D}(\Omega)$. Furthermore, the Theorem of Eberlein-Smulyan together with the Theorem of Rellich-Kondrachov implies the existence of a subsequence (not relabeled) $(\tilde\Phi_k)$ and $\Phi^* \in W^1_{2,D}(\Omega)$ with $\|\Phi^*\|_{L_2(\Omega)} = 1$ and $\tilde\Phi_k \rightharpoonup \Phi^*$ in $W^1_2(\Omega)$. Moreover, for $\varepsilon \in (0, 1/q')$, with q' denoting the dual exponent of q, we have

$$(v_k) \subset S(\kappa) \subset W^2_q(-1,1) \overset{c}{\hookrightarrow} W^{2-\varepsilon}_q(-1,1) \hookrightarrow C^1([-1,1])\,,$$

so that the convexity of $S(\kappa)$ implies the existence of another subsequence (v_k) and $v \in S(\kappa)$ with $v_k \to v$ in $C^1([-1,1])$. As a consequence, we find

$$a_{ij}(v_k) \to a_{ij}(v) \quad \text{in} \quad C([-1,1])\,, \qquad d_i(v_k) \to d_i(v) \quad \text{in} \quad C^1([-1,1])\,.$$

Using this fact together with $\tilde F_k \to 0$ in $W^{-1}_{2,D}(\Omega)$, $\tilde\Phi_k \rightharpoonup \Phi^*$ in $W^1_2(\Omega)$ and the triangle inequality, we may take the limit $k \to \infty$ in the weak formulation of (3.10). It follows that Φ^* is a weak solution to

$$\begin{cases} L_v\Phi^* &= 0 \quad \text{in} \quad \Omega\,, \\ \quad\Phi^* &= 0 \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

and the uniqueness of weak solutions implies that $\Phi^* = 0$, which already contradicts the fact that $\|\Phi^*\|_{L_2(\Omega)} = 1$. So the assumption was wrong and we find $C_1(\kappa)$ with

$$\|\Phi\|_{W_2^1(\Omega)} \le C_1(\kappa)\|F\|_{W_{2,D}^{-1}(\Omega)}\,, \qquad v \in S(\kappa)\,, \quad F \in W_{2,D}^{-1}(\Omega)\,,$$

as claimed. □

Regularity Step: Strong Solutions. We establish that Φ_v is a strong solution to (3.6) if the right-hand side F is more regular. Since Ω is a rectangle, i.e. a domain with corners, this result does not follow from standard elliptic regularity theory, but from Theorem 2.1, the refined version of [**37**, Theorem 3.2.1.2].

Lemma 3.3 *For each $v \in S(\kappa)$ and each $F \in L_2(\Omega)$, there exists a unique strong solution $\Phi_v \in W_{2,D}^2(\Omega)$ to* (3.6)*, i.e to the equation*

$$\begin{cases} L_v\Phi_v &= F \quad \text{in} \quad \Omega\,, \\ \quad \Phi_v &= 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

Moreover, there exists $C_2(\kappa) > 0$ (independent of F, Φ_v and v) such that

$$\|\Phi_v\|_{W_2^2(\Omega)} \le C_2(\kappa)\|F\|_{L_2(\Omega)}\,.$$

Proof. By Lemma 3.2, we find $\Phi_v \in W_{2,D}^1(\Omega)$ being the unique weak solution to

$$\begin{cases} \operatorname{div}\big(A(v)\nabla w\big) &= F - d(v)\cdot\nabla\Phi_v \quad \text{in} \quad \Omega\,, \\ \qquad w &= 0 \quad \text{on} \quad \Omega\,. \end{cases}$$

From the facts that $W_q^1(-1,1)$ is a Banach algebra and $W_q^1(-1,1) \hookrightarrow C([-1,1])$, we deduce that the coefficients of $-L_v$ satisfy

$$\sum_{i,j=1}^{2} \|a_{ij}(v)\|_{W_q^1(\Omega)} + \sum_{i=1}^{2} \|d_i(v)\|_\infty \le C(\kappa)\,, \qquad v \in S(\kappa)\,.$$

Because the ellipticity constant $\alpha(\kappa) > 0$ of $-L_v$ is independent of $v \in S(\kappa)$, see Lemma 3.1, we deduce from Theorem 2.1 that Φ_v belongs to $W_{2,D}^2(\Omega)$ and that there exists $C(\kappa) > 0$ with

$$\|\Phi_v\|_{W_2^2(\Omega)} \le C(\kappa)\big(\|\Phi_v\|_{W_2^1(\Omega)} + \|F\|_{L_2(\Omega)}\big)\,.$$

Since $\|F\|_{W_{2,D}^{-1}(\Omega)} \le \|F\|_{L_2(\Omega)}$, it follows from Lemma 3.2 that

$$\|\Phi_v\|_{W_2^2(\Omega)} \le C_2(\kappa)\|F\|_{L_2(\Omega)}$$

for some $C_2(\kappa) > 0$, and the proof is complete. □

In summary, the previous two Lemmata 3.2 and 3.3 ensure unique weak and strong solvability of (3.6). More precisely, for $v \in S(\kappa)$, the operator

$$L_D(v)\Phi := L_v\Phi\,, \qquad \Phi \in W_{2,D}^1(\Omega) \tag{3.11}$$

satisfies

$$L_D(v) \in \mathcal{L}_{is}(W_{2,D}^1(\Omega), W_{2,D}^{-1}(\Omega)) \cap \mathcal{L}_{is}(W_{2,D}^2(\Omega), L_2(\Omega)),$$

and its inverse $L_D(v)^{-1}$ is uniformly bounded for $v \in S(\kappa)$.

From a solution to (3.6) one easily obtains a solution to the transformed electrostatic problem: Noting that $f_v := L_v \frac{\ln(r)}{\ln(2)}$ belongs to $L_2(\Omega)$ one finds that

$$\phi_v := -L_D(v)^{-1} f_v + \frac{\ln(r)}{\ln(2)} \in W_2^2(\Omega) \tag{3.12}$$

is the unique strong solution to the transformed electrostatic problem (3.3). Thanks to

$$\|f_v\|_{L_2(\Omega)} \leq C(\kappa)\,, \qquad v \in S(\kappa)\,, \tag{3.13}$$

and the uniform estimates on $L_D(v)^{-1}$, the function ϕ_v also satisfies a uniform estimate

$$\|\phi_v\|_{W_2^2(\Omega)} \leq C(\kappa)\,, \qquad v \in S(\kappa)\,. \tag{3.14}$$

Remark 3.4 We briefly comment on the regularity of the original electrostatic potential $\psi_v = \phi_v \circ T_v$ solving

$$\begin{cases} \dfrac{1}{r}\partial_r(r\partial_r\psi_v) + \partial_z^2\psi_v &= 0 \quad \text{in} \quad \Omega(v)\,, \\ \psi_v &= h_v \quad \text{on} \quad \partial\Omega(v)\,, \end{cases}$$

where we set $\sigma = 1$ in this remark, and h_v is given by (1.21). Due to the corners of $\Omega(v)$, one might expect the regularity $\psi_v \in W_2^2(\Omega(v)) \cap C^\infty\big(\overline{\Omega(v)} \setminus \{(\pm 1, 1), (\pm 1, 2)\}\big)$ to be optimal in general. However, one can show that ψ_v is smooth up to the boundary in $(\pm 1, 2)$. In addition, if $v \in W_\infty^3(-1,1)$ with $v(\pm 1) = v_z(\pm 1) = v_{zz}(\pm 1) = 0$, then $\psi_v \in C^{2,\alpha}(\overline{\Omega(v)})$ for any $\alpha \in (0,1)$, i.e. ψ_v is a classical solution. This follows from the Schwarz reflection principle [**33**, Exercise 2.4] and Schauder Theory, see [**33**, Lemma 6.18].

Fine Tuning Via Interpolation. Finally, using interpolation theory, we get an improved norm estimate for the inverse of $L_D(v)$, which results in better estimates for $[v \mapsto \phi_v]$ in the next subsection. The proof is exactly the same as in [**24**].

Proposition 3.5 *Given $\theta \in [0,1] \setminus \{1/2\}$, there is a constant $C_3(\kappa) > 0$ such that*

$$\|L_D(v)^{-1}\|_{\mathcal{L}(W_{2,D}^{\theta-1}(\Omega), W_{2,D}^{\theta+1}(\Omega))} \leq C_3(\kappa)\,, \qquad v \in S(\kappa)\,.$$

Proof. See [**24**, Lemma 2.3]. □

3.1.2. Regularity of the Electrostatic Force. In this subsection, we prove Lipschitz continuity and analyticity of the electrostatic force $[v \mapsto g(v)]$, understood as a map in two different functional analytic settings. We start with the Lipschitz continuity, for which we follow [**24**], while the subsequently proven analyticity will be based on [**23**]. At the end of this subsection, we also explain the need for the different mapping properties of $[v \mapsto g(v)]$.

For convenience, we recall the notation

$$S(\kappa) = \Big\{ v \in W^2_{q,D}(-1,1) \,\Big|\, \|v\|_{W^2_q(-1,1)} \leq 1/\kappa\,,\ -1+\kappa \leq v(z) \leq 1-\kappa \Big\}$$

for $\kappa > 0$ and $q > 2$ while q' denotes the dual exponent of q.

The Lipschitz continuity of the electrostatic force $[v \mapsto g(v)]$ is proven in several steps. First, we derive continuity properties of $[v \mapsto L_v]$ where L_v is defined in (3.4). Subsequently, we establish continuity of $[v \mapsto \phi_v]$, and finally, we transfer the continuity properties to $[v \mapsto g(v)]$. The regularity of $[v \mapsto L_v]$ follows as in [**24**, Lemma 2.4].

Lemma 3.6 *Given $\xi \in [0, 1/q')$ and $\alpha \in (\xi, 1)$, there exists $C_4(\kappa)$ such that*

$$\|L_v - L_w\|_{\mathcal{L}(W^2_2(\Omega), W^{-\alpha}_{2,D}(\Omega))} \leq C_4(\kappa) \|v - w\|_{W^{2-\xi}_q(-1,1)}$$

for all $v, w \in S(\kappa)$.

Proof. Let $v, w \in S(\kappa)$ and $\Phi \in W^2_2(\Omega)$. Then, $L_v\Phi \in L_2(\Omega) \subset W^{-\alpha}_{2,D}(\Omega)$ where the critical term $-\sigma^2(2-r)\,\partial_z^2 v\,\partial_r\Phi$ belongs to $L_2(\Omega)$ thanks to Hölder's inequality and the embedding $W^1_2(\Omega) \hookrightarrow L_{\frac{2q}{q-2}}(\Omega)$. For $\psi \in W^\alpha_{2,D}(\Omega)$, the definition of L_v in non-divergence form yields

$$\begin{aligned}
&\int_\Omega \big[(L_v - L_w)\Phi\big]\psi\,\mathrm{d}(z,r)\\
&= \sigma^2 \int_\Omega [w - v]\,\partial_z^2\Phi\,\psi\,\mathrm{d}(z,r)\\
&\quad - 2\sigma^2 \int_\Omega (2-r)\,[\partial_z v - \partial_z w]\,\partial_z\partial_r\Phi\,\psi\,\mathrm{d}(z,r)\\
&\quad + \int_\Omega \left(\frac{1+\sigma^2(\partial_z v)^2(2-r)^2}{1-v} - \frac{1+\sigma^2(\partial_z w)^2(2-r)^2}{1-w} \right) \partial_r^2\Phi\,\psi\,\mathrm{d}(z,r)\\
&\quad - \sigma^2 \int_\Omega (2-r)\,[\partial_z^2 v - \partial_z^2 w]\,\partial_r\Phi\,\psi\,\mathrm{d}(z,r)\\
&\quad - 2\sigma^2 \int_\Omega (2-r) \left(\frac{(\partial_z v)^2}{1-v} - \frac{(\partial_z w)^2}{1-w} \right) \partial_r\Phi\,\psi\,\mathrm{d}(z,r)\\
&\quad + \int_\Omega \left(\frac{1}{2v + (1-v)r} - \frac{1}{2w + (1-w)r} \right) \partial_r\Phi\,\psi\,\mathrm{d}(z,r)\\
&=: I + II + \cdots + VI\,.
\end{aligned}$$

We point out again that IV is the critical term since the second weak derivatives of v and w occur. Now we estimate each integral, treating the difficult task IV at the end.

For I: It follows from Hölder's inequality and the embedding $W_q^{2-\xi}(-1,1) \hookrightarrow C^1([-1,1])$ that

$$\begin{aligned}
|I| &\leq \sigma^2 \int_\Omega |w-v||\partial_z^2\Phi|\,|\psi|\,\mathrm{d}(z,r) \\
&\leq \sigma^2 \|w-v\|_\infty \,\|\partial_z^2\Phi\|_{L_2(\Omega)}\,\|\psi\|_{L_2(\Omega)} \\
&\leq C\,\sigma^2 \|v-w\|_{W_q^{2-\xi}(-1,1)}\|\Phi\|_{W_2^2(\Omega)}\,\|\psi\|_{W_{2,D}^\alpha(\Omega)}\,.
\end{aligned}$$

For II: Similar to I, we find

$$\begin{aligned}
|II| &\leq 4\sigma^2 \int_\Omega |\partial_z v - \partial_z w|\,|\partial_z\partial_r\Phi|\,|\psi|\,\mathrm{d}(z,r) \\
&\leq 4\sigma^2 \|\partial_z v - \partial_z w\|_\infty \|\partial_z\partial_r\Phi\|_{L_2(\Omega)}\|\psi\|_{L_2(\Omega)} \\
&\leq C\,4\sigma^2 \|v-w\|_{W_q^{2-\xi}(-1,1)}\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)}\,.
\end{aligned}$$

For III: Noting that $1-v\geq\kappa$ for $v\in S(\kappa)$, we find

$$\begin{aligned}
|III| \leq{}& \left\|\frac{1+\sigma^2(\partial_z v)^2(2-r)^2}{1-v} - \frac{1+\sigma^2(\partial_z w)^2(2-r)^2}{1-w}\right\|_\infty \|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)} \\
\leq{}& \frac{1}{\kappa^2}\max\left\{1,4\sigma^2\right\}\Big(\big\|(\partial_z v)^2(1-w)-(\partial_z w)^2(1-v)\big\|_\infty \\
&+\|w-v\|_\infty\Big)\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)} \\
\leq{}& C(\kappa)\,\Big(\|\partial_z v\|_\infty^2\|w-v\|_\infty + \|1-v\|_\infty\|\partial_z v+\partial_z w\|_\infty\|\partial_z v-\partial_z w\|_\infty \\
&+\|w-v\|_\infty\Big)\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)} \\
\leq{}& C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)}\,.
\end{aligned}$$

For V: Similar to the previous integrals, we estimate

$$\begin{aligned}
|V| \leq{}& 4\sigma^2\left\|\frac{(\partial_z v)^2(1-w)-(\partial_z w)^2(1-v)}{(1-v)(1-w)}\right\|_\infty \|\partial_r\Phi\|_{L_2(\Omega)}\|\psi\|_{L_2(\Omega)} \\
\leq{}& \frac{4\sigma^2}{\kappa^2}\Big(\|\partial_z v\|_\infty^2\|w-v\|_\infty \\
&+\|1-v\|_\infty\|\partial_z v+\partial_z w\|_\infty\|\partial_z v-\partial_z w\|_\infty\Big)\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)} \\
\leq{}& C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)}\,.
\end{aligned}$$

For VI: Since $r\geq 1$, we have

$$2v+(1-v)r\geq\kappa\,,\qquad v\in S(\kappa)\,,$$

and hence

$$\begin{aligned}
|VI| &\leq \frac{2}{\kappa^2}\|v-w\|_\infty\|\partial_r\Phi\|_{L_2(\Omega)}\|\psi\|_{L_2(\Omega)} \\
&\leq C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\|\Phi\|_{W_2^2(\Omega)}\|\psi\|_{W_{2,D}^\alpha(\Omega)}\,.
\end{aligned}$$

For IV: The estimate of integral IV is special as second derivatives of v and w occur.

Therefore, a simple application of Hölder's inequality combined with Sobolev's embedding theorem only yields existence of the integral IV but not the desired estimate. Instead, we argue as follows: Due to Fubini's Theorem and the fact that $\partial_z^2 \in \mathcal{L}\big(W_q^{2-\xi}(-1,1), W_{q,D}^{-\xi}(-1,1)\big)$ by [**37**, Theorem 1.4.4.6] (as $1-\xi \neq 1/q$), we find

$$\begin{aligned}
|IV| &\le \sigma^2 \bigg| \int_\Omega (2-r)\, [\partial_z^2 v - \partial_z^2 w]\, \partial_r \Phi\, \psi \,\mathrm{d}(z,r) \bigg| \\
&= \sigma^2 \bigg| \int_{-1}^1 [\partial_z^2 v - \partial_z^2 w](z) \bigg(\int_1^2 (2-r) \partial_r \Phi(z,r) \psi(z,r)\,\mathrm{d}r \bigg)\,\mathrm{d}z \bigg| \\
&\le \sigma^2 \|\partial_z^2 v - \partial_z^2 w\|_{W_{q,D}^{-\xi}(-1,1)} \bigg\| \int_1^2 (2-r) \partial_r \Phi(\,\cdot\,,r) \psi(\,\cdot\,,r)\,\mathrm{d}r \bigg\|_{W_{q'}^{\xi}(-1,1)} \\
&\le C\,\sigma^2 \|v-w\|_{W_q^{2-\xi}(-1,1)} \bigg\| \int_1^2 (2-r) \partial_r \Phi(\,\cdot\,,r) \psi(\,\cdot\,,r)\,\mathrm{d}r \bigg\|_{W_{q'}^{\xi}(-1,1)} .
\end{aligned}$$

Here, we also used the fact that $W_{q',D}^{\xi}(-1,1) = W_{q'}^{\xi}(-1,1)$ due to the choice $\xi < 1/q'$ so that the dual space of $W_{q'}^{\xi}(-1,1)$ coincides with $W_{q,D}^{-\xi}(-1,1)$. Next, we deduce from Lemma A.2 that

$$|IV| \le C\,\sigma^2 \|v-w\|_{W_q^{2-\xi}(-1,1)} \big\|(2-r)\partial_r \Phi\, \psi\big\|_{W_{q'}^{\xi}(\Omega)} .$$

Finally, the Multiplication Theorem A.1 ensures

$$W_2^1(\Omega) \cdot W_2^1(\Omega) \cdot W_2^\alpha(\Omega) \hookrightarrow W_{q'}^{\xi}(\Omega) ,$$

and we arrive at

$$\begin{aligned}
|IV| &\le C \|v-w\|_{W_q^{2-\xi}(-1,1)} \|2-r\|_{W_2^1(\Omega)} \|\partial_r \Phi\|_{W_2^1(\Omega)} \|\psi\|_{W_{2,D}^\alpha(\Omega)} \\
&\le C \|v-w\|_{W_q^{2-\xi}(-1,1)} \|\Phi\|_{W_2^2(\Omega)} \|\psi\|_{W_{2,D}^\alpha(\Omega)} .
\end{aligned}$$

Summing up the estimates for I to VI, we have shown that

$$\bigg| \int_\Omega \big[(L_v - L_w)\Phi\big]\, \psi \,\mathrm{d}(z,r) \bigg| \le C_4(\kappa) \|v-w\|_{W_q^{2-\xi}(-1,1)}\, \|\Phi\|_{W_2^2(\Omega)}\, \|\psi\|_{W_{2,D}^\alpha(\Omega)} .$$

Taking the supremum over $\psi \in W_{2,D}^\alpha(\Omega)$ with $\|\psi\|_{W_{2,D}^\alpha(\Omega)} \le 1$, we get

$$\big\|(L_v - L_w)\Phi\big\|_{W_{2,D}^{-\alpha}(\Omega)} \le C_4(\kappa) \|v-w\|_{W_q^{2-\xi}(-1,1)} \|\Phi\|_{W_2^2(\Omega)} ,$$

and thus

$$\|L_v - L_w\|_{\mathcal{L}(W_2^2(\Omega), W_{2,D}^{-\alpha}(\Omega))} \le C_4(\kappa) \|v-w\|_{W_q^{2-\xi}(-1,1)}$$

as claimed. □

Next, we study the dependence of ϕ_v on v. The result is the analogue to [**24**, Lemma 2.6].

Lemma 3.7 *Let $\xi \in [0,1/q')$ and $\alpha \in (\xi,1)$ with $\alpha \neq 1/2$ be given. Then, there exists $C_5(\kappa)$ such that*

$$\|\phi_v - \phi_w\|_{W_{2,D}^{2-\alpha}(\Omega)} \le C_5(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\,, \qquad v,w \in S(\kappa)\,.$$

Proof. Let us recall from (3.12) that

$$\phi_v = -L_D(v)^{-1} f_v + \frac{\ln(r)}{\ln(2)}\,, \qquad f_v = L_v \frac{\ln(r)}{\ln(2)}\,.$$

First, we deduce from Lemma 3.6 that

$$\begin{aligned}\|f_v - f_w\|_{W_{2,D}^{-\alpha}(\Omega)} &\le \|L_v - L_w\|_{\mathcal{L}(W_2^2(\Omega), W_{2,D}^{-\alpha}(\Omega))} \left\| \frac{\ln(r)}{\ln(2)} \right\|_{W_2^2(\Omega)} \\ &\le C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\,. \end{aligned} \tag{3.15}$$

Next, we write

$$\phi_v - \phi_w = -L_D(v)^{-1}(f_v - f_w) + \big(L_D(w)^{-1} - L_D(v)^{-1}\big) f_w\,.$$

Then, a combination of (3.15) with Proposition 3.5 (for $\theta = 1-\alpha \neq 1/2$ and $\theta = 1$) as well as Lemma 3.6 yields

$$\begin{aligned}\|\phi_v - \phi_w\|_{W_{2,D}^{2-\alpha}(\Omega)} &\le \|L_D(v)^{-1}(f_v - f_w)\|_{W_{2,D}^{2-\alpha}(\Omega)} + \|(L_D(v)^{-1} - L_D(w)^{-1}) f_w\|_{W_{2,D}^{2-\alpha}(\Omega)} \\ &\le \|L_D(v)^{-1}\|_{\mathcal{L}(W_{2,D}^{-\alpha}(\Omega), W_{2,D}^{2-\alpha}(\Omega))} \|f_v - f_w\|_{W_{2,D}^{-\alpha}(\Omega)} \\ &\quad + \|L_D(v)^{-1}(L_w - L_v)L_D(w)^{-1} f_w\|_{W_{2,D}^{2-\alpha}(\Omega)} \\ &\le C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)} + \|L_D(v)^{-1}\|_{\mathcal{L}(W_{2,D}^{-\alpha}, W_{2,D}^{2-\alpha}(\Omega))} \\ &\quad \times \|L_w - L_v\|_{\mathcal{L}(W_2^2(\Omega), W_{2,D}^{-\alpha}(\Omega))} \|L_D(w)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,D}^2(\Omega))} \|f_w\|_{L_2(\Omega)} \\ &\le C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)} \big(1 + \|f_w\|_{L_2(\Omega)}\big)\,. \end{aligned}$$

Finally, estimate (3.13) ensures that the L_2-norm of f_w is uniformly bounded on $S(\kappa)$, and the assertion follows.

□

Having established continuity properties of $[v \mapsto \phi_v]$, we turn to the main issue of this section and provide Lipschitz continuity of the electrostatic force $[v \mapsto g(v)]$. The result is an adaptation of [**24**, Proposition 2.1]:

Proposition 3.8 *Let $q \in (2,\infty)$, $\kappa \in (0,1)$ and $\lambda, \sigma > 0$. For $\xi \in [0,1/2)$ and $\nu \in [0, 1/2-\xi)$, the map*

$$[v \mapsto g(v)] : \quad S(\kappa) \to W_{2,D}^{\nu}(-1,1)$$

is bounded, and there exists a constant $C_6(\kappa) > 0$ such that

$$\|g(v) - g(w)\|_{W_{2,D}^{\nu}(-1,1)} \le C_6(\kappa)\|v-w\|_{W_{q,D}^{2-\xi}(-1,1)} \tag{3.16}$$

as well as

$$\left\| \frac{1}{v+1} - \frac{1}{w+1} \right\|_{W_{2,D}^{\nu}(-1,1)} \le C_6(\kappa)\|v-w\|_{W_{q,D}^{2-\xi}(-1,1)}\,, \qquad v,w \in S(\kappa)\,. \tag{3.17}$$

Proof. **(i)** As a first step, we express the electrostatic force

$$g(v) = \big(1+\sigma^2(\partial_z v)^2\big)^{3/2}\,\big|\partial_r\psi_v\big(z,v+1\big)\big|^2$$

defined in (3.1) in terms of the transformed electrostatic potential ϕ_v. To this end, we recall from (3.2) that

$$\psi_v(z,r) = \phi_v\big(T_v(z,r)\big) = \phi_v\Big(z,\,\frac{r-2v(z)}{1-v(z)}\Big)\,,\qquad (z,r)\in\Omega(v)\,,$$

and consequently

$$\partial_r\psi_v\big(z,v(z)+1\big) = \frac{\partial_r\phi_v(z,1)}{1-v(z)}\,,\qquad z\in(-1,1)\,.$$

This yields

$$g(v) = \big(1+\sigma^2(\partial_z v)^2\big)^{3/2}\,\frac{|\partial_r\phi_v(\,\cdot\,,1)|^2}{(1-v)^2}\,,\qquad v\in S(\kappa)\,. \tag{3.18}$$

Moreover, as the second preliminary observation, we note that

$$\big\|\partial_r\phi_v(\,\cdot\,,1)\big\|_{W_2^{1/2}(-1,1)} \le C(\kappa)\,,\qquad v\in S(\kappa)\,. \tag{3.19}$$

Indeed, since ϕ_v belongs to $W_2^2(\Omega)$, the trace theorem [**37**, Theorem 1.5.1.2] yields

$$\|\partial_r\phi_v(\,\cdot\,,1)\|_{W_2^{1/2}(-1,1)} \le C\,\|\phi_v\|_{W_2^2(\Omega)}\,,\qquad v\in S(\kappa)\,,$$

for some constant $C>0$ independent of v. In combination with the fact that ϕ_v is uniformly bounded on $S(\kappa)$ due to (3.14), estimate (3.19) then follows.
(ii) We deduce from the representation of g in (3.18) and

$$W_q^1(-1,1)\cdot W_2^{1/2}(-1,1)\cdot W_2^{1/2}(-1,1)\hookrightarrow W_2^{\nu}(-1,1)\,,$$

due to the Multiplication Theorem A.1, that

$$\begin{aligned}\|g(v)\|_{W_{2,D}^{\nu}} &\le C\,\left\|\frac{\big(1+\sigma^2(\partial_z v)^2\big)^{3/2}}{(1-v)^2}\right\|_{W_q^1(-1,1)}\big\|\partial_r\phi_v(\,\cdot\,,1)\big\|^2_{W_2^{1/2}(-1,1)}\\ &\le C(\kappa)\end{aligned}$$

for $v\in S(\kappa)$. Here, the last inequality follows from (3.19). Consequently, g maps $S(\kappa)$ to $W_{2,D}^{\nu}(-1,1)$ and is bounded.
(iii) We present the main part of the proof. Namely, we derive the stated Lipschitz

continuity of g based on (3.18). To this end, we write

$$\begin{aligned}
&\|g(v)-g(w)\|_{W^\nu_{2,D}(-1,1)}\\
&\le \left\| \frac{(1+\sigma^2 w_z^2)^{3/2}}{(1-w)^2}\Big(|\partial_r\phi_v(\,\cdot\,,1)|^2-|\partial_r\phi_w(\,\cdot\,,1)|^2\Big)\right\|_{W^\nu_{2,D}(-1,1)}\\
&\quad+\left\|(1+\sigma^2 w_z^2)^{3/2}\left(\frac{1}{(1-w)^2}-\frac{1}{(1-v)^2}\right)|\partial_r\phi_v(\,\cdot\,,1)|^2\right\|_{W^\nu_{2,D}(-1,1)}\\
&\quad+\left\|\Big((1+\sigma^2 v_z^2)^{3/2}-(1+\sigma^2 w_z^2)^{3/2}\Big)\frac{1}{(1-v)^2}|\partial_r\phi_v(\,\cdot\,,1)|^2\right\|_{W^\nu_{2,D}(-1,1)}\\
&=: I+II+III\,,
\end{aligned}$$

and estimate each part separately:
For I: We let $\alpha\in(\xi,1/2-\nu)$, and write

$$I=\left\|\frac{(1+\sigma^2 w_z^2)^{3/2}}{(1-w)^2}\Big(\partial_r\phi_v(\,\cdot\,,1)+\partial_r\phi_w(\,\cdot\,,1)\Big)\Big(\partial_r\phi_v(\,\cdot\,,1)-\partial_r\phi_w(\,\cdot\,,1)\Big)\right\|_{W^\nu_{2,D}(-1,1)}.$$

From

$$W_q^1(-1,1)\cdot W_2^{1/2}(-1,1)\cdot W_2^{1/2-\alpha}(-1,1)\hookrightarrow W_2^\nu(-1,1),$$

which holds thanks to the Multiplication Theorem A.1, we deduce that

$$\begin{aligned}
I&\le \left\|\frac{(1+\sigma^2 w_z^2)^{3/2}}{(1-w)^2}\right\|_{W_q^1(-1,1)}\big\|\partial_r\phi_v(\,\cdot\,,1)+\partial_r\phi_w(\,\cdot\,,1)\big\|_{W_2^{1/2}(-1,1)}\\
&\qquad\qquad\times\big\|\partial_r\phi_v(\,\cdot\,,1)-\partial_r\phi_w(\,\cdot\,,1)\big\|_{W_2^{1/2-\alpha}(-1,1)}\\
&\le C(\kappa)\,\big\|\partial_r\phi_v(\,\cdot\,,1)-\partial_r\phi_w(\,\cdot\,,1)\big\|_{W_2^{1/2-\alpha}(-1,1)}\\
&\le C(\kappa)\,\|\partial_r\phi_v-\partial_r\phi_w\|_{W_2^{1-\alpha}(\Omega)}\,,\qquad v,w\in S(\kappa)\,.
\end{aligned}$$

In addition to the Multiplication Theorem, we applied (3.19), the fact that $W_q^1(-1,1)$ is a Banach algebra and the chain rule to derive the second estimate, while the third estimate follows from properties of the trace, see [**37**, Theorem 1.5.1.2]. Now using continuity of differentiation between fractional Sobolev spaces due to [**37**, Theorem 1.4.4.6] (which is applicable as $1-\alpha\neq 1/2$) and subsequently Lemma 3.7, we conclude that

$$\begin{aligned}
I&\le C(\kappa)\;\|\phi_v-\phi_w\|_{W_2^{2-\alpha}(\Omega)}\\
&\le C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)}\,,\qquad v,w\in S(\kappa)\,.
\end{aligned}$$

For II: We estimate

$$\begin{aligned}
II&\le \big\|(1+\sigma^2 w_z^2)^{3/2}\big\|_{W_q^1(-1,1)}\left\|\frac{1}{(1-w)^2}-\frac{1}{(1-v)^2}\right\|_{W_q^1(-1,1)}\|\partial_r\phi_v(\,\cdot\,,1)\|^2_{W_2^{1/2}(-1,1)}\\
&\le C(\kappa)\left\|\frac{1}{(1-w)^2}-\frac{1}{(1-v)^2}\right\|_{W_q^1(-1,1)}\,,\qquad v,w\in S(\kappa)\,,
\end{aligned}$$

where we use that $W_q^1(-1,1)$ is a Banach algebra and

$$W_q^1(-1,1)\cdot W_2^{1/2}(-1,1)\cdot W_2^{1/2}(-1,1)\hookrightarrow W_2^\nu(-1,1)\,,$$

thanks to the Multiplication Theorem A.1. Writing

$$\frac{1}{(1-w)^2}-\frac{1}{(1-v)^2}=\frac{2-w-v}{(1-w)^2\,(1-v)^2}\,(w-v)\,,$$

and using once more that $W_q^1(-1,1)$ is an algebra, we deduce further that

$$\begin{aligned} II &\le C(\kappa)\,\|v-w\|_{W_q^1(-1,1)}\\ &\le C(\kappa)\,\|v-w\|_{W_q^{2-\xi}(-1,1)}\,, \qquad v,w\in S(\kappa). \end{aligned}$$

For III: We rewrite

$$\begin{aligned} (1+\sigma^2v_z^2)^{3/2}-(1+\sigma^2w_z^2)^{3/2} &= (1+\sigma^2v_z^2)^{1/2}\big((1+\sigma^2v_z^2)-(1+\sigma^2w_z^2)\big)\\ &\quad+(1+\sigma^2w_z^2)\big((1+\sigma^2v_z^2)^{1/2}-(1+\sigma^2w_z^2)^{1/2}\big)\\ &= r(v,w)\,(v_z+w_z)(v_z-w_z) \end{aligned}$$

with

$$r(v,w):=\sigma^2\left((1+\sigma^2v_z^2)^{1/2}+\frac{(1+\sigma^2w_z^2)}{\sqrt{1+\sigma^2v_z^2}+\sqrt{1+\sigma^2w_z^2}}\right)\in W_q^1(-1,1)\,.$$

Then, we estimate III by

$$\begin{aligned} III &= \left\|\frac{r(v,w)(v_z+w_z)}{(1-v)^2}\,(v_z-w_z)\,|\partial_r\phi_v(\,\cdot\,,1)|^2\right\|_{W_{2,D}^\nu(-1,1)}\\ &\le C(\kappa)\|v_z-w_z\|_{W_q^{1-\xi}(-1,1)}\\ &\le C(\kappa)\|v-w\|_{W_q^{2-\xi}(-1,1)} \end{aligned}$$

using

$$W_q^1(-1,1)\cdot W_q^{1-\xi}(-1,1)\cdot W_2^{1/2}(-1,1)\cdot W_2^{1/2}(-1,1)\hookrightarrow W_{2,D}^\nu(-1,1)$$

due to the Multiplication Theorem, (3.8). Combining the estimates for I-III yields (3.16).

(iv) The second estimate (3.17) follows directly:

$$\begin{aligned} \left\|\frac{1}{v+1}-\frac{1}{w+1}\right\|_{W_{2,D}^\nu(-1,1)} &\le C\left\|\frac{w-v}{(v+1)(w+1)}\right\|_{W_q^1(-1,1)}\\ &\le C(\kappa)\|w-v\|_{W_{q,D}^{2-\xi}(-1,1)}\,, \qquad v,w\in S(\kappa)\,. \end{aligned}$$

□

Thanks to Sobolev's embedding theorem, we have the following L_q-L_q-version of Proposition 3.8:

Corollary 3.9 *Let $q \in (2,\infty)$, $\kappa \in (0,1)$ and $\lambda, \sigma > 0$. For $\xi \in [0,1/q)$ and $2\mu \in [0,1/q-\xi)$, there exists a constant $C_7(\kappa) > 0$ such that the map*

$$[v \mapsto g(v)] : \quad S(\kappa) \to W^{2\mu}_{q,D}(-1,1)$$

is bounded by $C_7(\kappa)$ and

$$\|g(v) - g(w)\|_{L_q(-1,1)} \le \frac{C_7(\kappa)}{2\lambda} \|v - w\|_{W^{2-\xi}_{q,D}(-1,1)} ,$$

as well as

$$\left\| \frac{1}{v+1} - \frac{1}{w+1} \right\|_{L_q(-1,1)} \le \frac{C_7(\kappa)}{2} \|v - w\|_{W^{2-\xi}_{q,D}(-1,1)} , \qquad v, w \in S(\kappa) .$$

Proof. Since

$$2\mu + 1/2 - 1/q < 1/q - \xi + 1/2 - 1/q = 1/2 - \xi ,$$

we can fix $\nu \in \big(2\mu + 1/2 - 1/q\, ,\, 1/2 - \xi\big)$. While Sobolev's embedding theorem ensures that

$$W^{\nu}_{2,D}(-1,1) \hookrightarrow W^{2\mu}_{q,D}(-1,1) ,$$

the choice of ξ and ν is compatible with Proposition 3.8. □

Note that Proposition 3.8 and the corresponding Corollary 3.9 establish Lipschitz continuity of g with respect to a weaker norm than the $\|\cdot\|_{W^2_{q,D}(-1,1)}$-norm, which will be essential to prove local existence in the quasilinear setting. Besides local existence, we will study stability of stationary solutions to the free boundary problem (1.20)-(1.22) via the principle of linearized stability as well. To this end, we require at least Fréchet-differentiability of the electrostatic force g. The next proposition shows that g, considered on an open subset of $W^2_{q,D}(-1,1)$ equipped with the usual $\|\cdot\|_{W^2_{q,D}(-1,1)}$-norm, is even analytic. The proof is similar to that of [**23**, Proposition 5].

Proposition 3.10 *Let $q \in (2,\infty)$ and put*

$$S := \{w \in W^2_{q,D}(-1,1) \mid -1 < w < 1\} .$$

Then, the electrostatic force g is analytic from S to $L_q(-1,1)$.

Proof. First, we note that the mappings

$$\left[v \mapsto \frac{1}{1-v}\right] , \quad \left[v \mapsto \big(1 + \sigma^2(\partial_z v)^2\big)^{3/2}\right]$$

are analytic from S to $W^1_q(-1,1)$, which follows from an adaptation of [**13**, Example 4.3.6] and from the fact that the composition of analytic maps is again analytic. Next, we deduce that the maps

$$[v \mapsto L_D(v)] : S \to \mathcal{L}\big(W^2_{2,D}(\Omega), L_2(\Omega)\big) , \qquad [v \mapsto f_v] : S \to L_2(\Omega)$$

are also analytic so that the definition of ϕ_v from (3.12) combined with the analyticity of the inversion map $[\ell \mapsto \ell^{-1}]$ for bounded linear operators implies that $[v \mapsto \phi_v]$ is analytic from S to $W^2_2(\Omega)$ as well. Finally, the representation of g in terms of ϕ_v in (3.18) and the Multiplication Theorem A.1 yield the analyticity of g from S to $L_q(-1,1)$. $\square$

3.2. Coupled System

We are now in a position to prove local well-posedness as well as a global existence criterion for the coupled free boundary problem (1.20)-(1.22). As already mentioned, the main idea is to reinterpret the system (1.20)-(1.22) as the following single quasilinear parabolic equation for the film deflection

$$\partial_t u - \sigma\partial_z \arctan(\sigma\partial_z u) = G(u) \tag{3.20}$$

with the non-local right-hand side $[u \mapsto G(u)]$ given by

$$G(u) := -\frac{1}{u+1} + \lambda g(u)\,. \tag{3.21}$$

Having established Lipschitz continuity of $[u \mapsto G(u)]$ between fractional Sobolev spaces in the last section, we now solve (3.20). Our proof follows [**24**] with smaller changes according to [**44**]. In particular, our proof is again based on semigroup theory (more precisely: its time-dependent counterpart [**5**]) and Banach's fixed point theorem, and strongly relies on arguments from the quasilinear theory [**5, 3**]. We point out that the regularity of $[u \mapsto G(u)]$ from Corollary 3.9 is slightly different to the one usually required for local well-posedness results in the quasilinear theory, see for example [**3**, Theorem 12.1] or also [**55**, Theorem 1.1]. This makes an adaptation of some of the arguments necessary.

Before we start, we introduce some notations: Let $q \in (2,\infty)$ and $\xi \in (0, 1/q')$ where q' denotes the dual exponent of q. For $\kappa \in (0,1)$, we put

$$Z(\kappa) := \left\{ v \in W_q^{2-\xi}(-1,1) \,\middle|\, \|v\|_{W_q^{2-\xi}(-1,1)} \le 1/\kappa\,,\ -1+\kappa \le v(z) \le 1-\kappa \right\}$$

and define

$$B(v)w := -\frac{\sigma^2}{(1+\sigma^2 v_z^2)} w_{zz}\,, \qquad w \in W^2_{q,D}(-1,1)\,,$$

for $v \in Z(\kappa)$, where the connection between $[v \mapsto B(v)]$ and (3.20) is given via

$$B(u)u = -\frac{\sigma^2 u_{zz}}{(1+\sigma^2 u_z^2)} = -\sigma\partial_z \arctan(\sigma\partial_z u)\,, \qquad u \in W^2_{q,D}(-1,1)^1\,,$$

which is the second order operator occuring on the left-hand side of (3.20). Furthermore, the choice of ξ and Sobolev's embedding theorem ensure that $-\dfrac{\sigma^2}{(1+\sigma^2 v_z^2)} \in C\big([-1,1]\big)$ so that each $B(v)$ is uniformly elliptic.

In the next two lemmata, we investigate properties of $[v \mapsto B(v)]$. More precisely, we show that $[v \mapsto B(v)]$ is globally Lipschitz continuous on $Z(\kappa)$ and that each $-B(v)$ generates an analytic semigroup satisfying uniform estimates for $v \in Z(\kappa)$.

---

[1] It is clear that $B(u)$ is also defined in this case.

Lemma 3.11 *Let $q \in (2,\infty)$, $\kappa \in (0,1)$ and $\xi \in (0,1/q')$. Then, there exists a constant $l(\kappa)$ such that*

$$\|B(w)-B(v)\|_{\mathcal{L}(W^2_{q,D}(-1,1),L_q(-1,1))} \le l(\kappa)\|w-v\|_{W^{2-\xi}_{q,D}(-1,1)}$$

for $v,w \in Z(\kappa)$.

Proof. It is clear that $B(v) \in \mathcal{L}\big(W^2_{q,D}(-1,1),L_q(-1,1)\big)$. The Lipschitz continuity follows from

$$\begin{aligned}
&\|B(w)-B(v)\|_{\mathcal{L}(W^2_{q,D}(-1,1),L_q(-1,1))}\\
&\qquad \le \sigma^2 \left\| \frac{1}{(1+\sigma^2 w_z^2)} - \frac{1}{(1+\sigma^2 v_z^2)} \right\|_\infty\\
&\qquad \le \sigma^4 \left\| \frac{1}{(1+\sigma^2 w_z^2)(1+\sigma^2 v_z^2)} \right\|_\infty \|w_z+v_z\|_\infty \|w_z-v_z\|_\infty\\
&\qquad \le l(\kappa)\|w-v\|_{W^{2-\xi}_{q,D}(-1,1)}\,,
\end{aligned}$$

where we made use of $Z(\kappa)$ being continuously embedded and bounded in $C^1\big([-1,1]\big)$ due to Sobolev's embedding theorem. □

Lemma 3.12 *Let $q \in (2,\infty)$, $\kappa \in (0,1)$ and $\xi \in (0,1/q')$. Moreover, let $\omega > 0$ be fixed. Then, there is a constant $k := k(\kappa) \ge 1$ such that for each $v \in Z(\kappa)$ one has*

$$B(v) \in \mathcal{H}\big(W^2_{q,D}(-1,1),L_q(-1,1),k,\omega\big)\,.$$

Proof. In [**5**, Remark I.1.2.1 (a)] a criterion for $B(v)$ to belong to one of the quantitative versions of $\mathcal{H}\big(W^2_{q,D}(-1,1),L_q(-1,1)\big)$, introduced in (2.1), is presented. The criterion is relatively easy to check and allows us to deduce the uniform statement of this lemma. We now give the precise criterion:

Assume that there are constants $C_i(\kappa) > 0$ for $i = 8,9$ such that for all $v \in Z(\kappa)$ one has:

(i) $\big\|B(v)\big\|_{\mathcal{L}(W^2_{q,D}(-1,1),L_q(-1,1))} \le C_8(\kappa)\,,$

(ii) $[\mathrm{Re}\mu \ge \omega] \in \rho(-B(v))$ and

$$\big\|[\mu+B(v)]^{-1}\big\|_{\mathcal{L}(L_q(-1,1))} \le \frac{C_8(\kappa)}{|\mu|}\,, \qquad \mathrm{Re}\mu \ge \omega\,,$$

(iii) $\big\|[\omega+B(v)]^{-1}\big\|_{\mathcal{L}(L_q(-1,1),W^2_{q,D}(-1,1))} \le C_9(\kappa).$

Then, the assertion of the lemma follows from [**5**, Remark I.1.2.1 (a)].

Thus, we only have to check (i)–(iii). We first define $V := (1+\sigma^2 v_z^2)/\sigma^2$ so that $B(v)w = -1/V\, w_{zz}$. Then,

$$1/\sigma^2 \le V \le C_{10}(\kappa)\,, \qquad v \in Z(\kappa)\,, \tag{3.22}$$

and the differential operator $B(v)$ satisfies

$$\big\|B(v)\big\|_{\mathcal{L}(W^2_{q,D}(-1,1),L_q(-1,1))} \le \sigma^2\,, \qquad v \in Z(\kappa)\,,$$

which is condition (i).

Next, we check condition (ii). To this end, we note that $B(v)$ is uniformly elliptic. Consequently, for $f \in L_q(-1,1)$, the equation

$$\begin{cases} B(v)u &= f\,, \\ u(\pm 1) &= 0 \end{cases}$$

is uniquely solvable in $W^2_{q,D}(-1,1)$ with $B(v)^{-1} \in \mathcal{L}\big(L_q(-1,1), W^2_{q,D}(-1,1)\big)$ due to [**33**, Theorem 9.15, Lemma 9.17]. It follows from the Theorem of Rellich-Kondrachov that $B(v)^{-1} \in \mathcal{L}\big(L_q(-1,1)\big)$ is compact, and [**39**, Theorem 6.29] implies that the spectrum $\sigma(-B(v))$ consists only of eigenvalues. Now we fix an eigenvalue μ of $-B(v)$ and a corresponding eigenfunction $\varphi \in W^2_{q,D}\big((-1,1),\mathbb{C}\big)$. Testing

$$\mu\varphi - \frac{1}{V}\partial_z^2\varphi = 0$$

with $V\,\overline{\varphi} \in W^2_{q',D}\big((-1,1),\mathbb{C}\big)$ and using integration by parts yields

$$\mu = \frac{-\int_{-1}^1 |\partial_z\varphi|^2\,\mathrm{d}z}{\int_{-1}^1 V\,|\varphi|^2\,\mathrm{d}z} < 0$$

so that

$$\big[\mathrm{Re}\mu > 0\big] \subset \rho(-B(v))\,, \qquad v \in Z(\kappa)\,.$$

Next, let $u \in W^2_{q,D}(-1,1)$ be the unique solution to

$$[\mu + B(v)]u = f\,, \qquad f \in L_q\big((-1,1),\mathbb{C}\big)\,,$$

for $\mu > 0$. Testing this equation with $V\,|u|^{q-2}\overline{u} \in L_{q'}\big((-1,1),\mathbb{C}\big)$ yields – along the lines of the proof of [**53**, Proposition 2.4.2] – the resolvent estimate (ii).

Finally, we turn to condition (iii). For $v \in Z(\kappa)$ and $u \in W^2_{q,D}(-1,1)$, we find

$$\begin{aligned} \|u\|^q_{W^2_{q,D}(-1,1)} &\le \|u\|^q_{W^1_q(-1,1)} + C_{10}(\kappa)^q\big\|B(v)u\big\|^q_{L_q(-1,1)} \\ &\le \frac{1}{2}\|u\|^q_{W^2_{q,D}(-1,1)} + C\,\|u\|^q_{L_q(-1,1)} + C_{10}(\kappa)^q\big\|[\omega + B(v)]u\big\|^q_{L_q(-1,1)} \end{aligned} \tag{3.23}$$

thanks to (3.22), the triangle inequality and Ehrling's lemma. Bringing the first term on the right-hand side of (3.23) to the left-hand side, we deduce from (ii) the existence of a constant $C_9(\kappa) > 0$ with

$$\|u\|_{W^2_{q,D}(-1,1)} \le C_9(\kappa)\big\|[\omega + B(v)]u\big\|_{L_q(-1,1)}\,, \quad v \in Z(\kappa)\,, \quad u \in W^2_{q,D}(-1,1)\,,$$

which is equivalent to condition (iii). Now everything is proven.

□

If v now depends on t, then $-B(v)$ generates a parabolic evolution operator (instead of an analytic semigroup), which satisfies regularity estimates holding uniformly on $Z(\kappa)$. The corresponding result is [**23**, Proposition 3.2].

Proposition 3.13 *Let $q \in (2,\infty)$, $\kappa \in (0,1)$, $\rho \in (0,1)$ and $\xi \in (0,1/q')$. For $\tau \in (0,1]$, we define*

$$\mathcal{V}_\tau(\kappa) := \Bigg\{ v : [0,\tau] \to W^{2-\xi}_{q,D}(-1,1) \,\Bigg|\, \|v(t)-v(s)\|_{W^{2-\xi}_{q,D}(-1,1)} \le |t-s|^\rho\,, \quad v(t) \in Z(\kappa)\,, \quad s,t \in [0,\tau] \Bigg\}\,.$$

Then, for each $v \in \mathcal{V}_\tau(\kappa)$, there exists a unique parabolic evolution operator

$$\big\{ U_{B(v)}(t,s) \,\big|\, 0 \le s \le t \le \tau \big\}$$

possessing $W^2_{q,D}(-1,1)$ as regularity subspace. Moreover, for fixed $2\mu \in (0,1/q)$, there exists a constant $C_{11}(\kappa) \ge 1$ independent of τ and $v \in \mathcal{V}_\tau(\kappa)$ such that

$$\big\| U_{B(v)}(t,s) \big\|_{\mathcal{L}(W^2_{q,D}(-1,1))} + \big(t-s\big)^{1-\mu} \big\| U_{B(v)}(t,s) \big\|_{\mathcal{L}(W^{2\mu}_{q,D}(-1,1), W^2_{q,D}(-1,1))} \le C_{11}(\kappa)$$

for $0 \le s < t \le \tau$.

Proof. Let $\omega > 0$ and put

$$\mathcal{B} := \Big\{ \big[t \mapsto B(v(t)) \big] \,\Big|\, v \in \mathcal{V}_\tau(\kappa) \Big\}.$$

From Lemma 3.11 and Lemma 3.12, we deduce that

$$\mathcal{B} \subset C^\rho\Big([0,\tau]\,, \mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1), k, \omega\big) \Big)$$

is bounded, which implies that $\mathcal{B}$ satisfies condition [**5**, Equation II (5.0.1)]. Here, $k = k(\kappa) \ge 1$ is the same as in Lemma 3.12. Since condition [**5**, Equation II (5.0.1)] is satisfied, we can use the uniform estimates for parabolic evolution operators from [**5**, Section II.5]. More precisely, the statement follows from [**5**, Theorem II.5.1.1, Lemma II.5.1.3] and the identification of interpolation spaces as fractional Sobolev spaces with Dirichlet boundary conditions based on [**3**, Theorem 5.2]. The latter originates from [**36, 66**]. □

Remark 3.14 The above proof ensures that the uniform estimates from [**5**, Section II.5] hold true. Together with the regularity estimates for the non-local operator $[u \mapsto G(u)]$ defined in (3.21), see Corollary 3.9, they form the basis for the upcoming fixed point argument.

We turn to the proof of the main result of this chapter:

Theorem 3.15 (Local Well-Posedness)
Let $q \in (2,\infty)$, $\lambda, \sigma > 0$ and $u_0 \in W^2_{q,D}(-1,1)$ with $1 > u_0(z) > -1$ for $z \in (-1,1)$. Then, there exists a unique maximal solution (u,ψ_u) to the coupled free boundary problem (1.20)-(1.22) on the maximal interval of existence $\big[0, T_{max}\big)$ in the sense that

$$u \in C^1\big([0,T_{max}), L_q(-1,1)\big) \cap C\big([0,T_{max}), W^2_{q,D}(-1,1)\big)$$

solves (1.20) and $\psi_{u(t)} \in W^2_2\big(\Omega(u(t))\big)$ solves (1.21)-(1.22) for each $t \in [0,T_{max})$.

Proof. It suffices to show the existence of a unique local solution u to (3.20), which may subsequently be extended to a unique maximal solution. We want to apply Banach's fixed point theorem:
(i) *Choice of a complete metric space*: Fix $\kappa > 0$ with

$$u_0 \in S(2\kappa) \cap Z(2\kappa)$$

as well as

$$\xi \in (0, 1/q)\,, \qquad \rho \in (0, \xi/4)\,, \qquad 2\mu \in (0, 1/q - \xi)\,, \qquad \tau \in (0, 1]\,.$$

Here, we recall that $u_0 \in S(2\kappa)$ is equivalent to

$$\|u_0\|_{W^2_{q,D}(-1,1)} \leq \frac{1}{2\kappa}\,, \qquad 1 - 2\kappa \geq u_0 \geq -1 + 2\kappa\,,$$

while $u_0 \in Z(2\kappa)$ is equivalent to

$$\|u_0\|_{W^{2-\xi}_{q,D}(-1,1)} \leq \frac{1}{2\kappa}\,, \qquad 1 - 2\kappa \geq u_0 \geq -1 + 2\kappa\,,$$

where different norms are used due to the fact that the analysis of the right-hand side of (3.20) requires control of the W^2_q-norm, while the arguments from [**5**, Section II.5] only apply for slightly weaker norms. Moreover, by Proposition 3.13, we find $C_{11}(\kappa) \geq 1$ independent of τ such that

$$\big\|U_{B(v)}(t, s)\big\|_{\mathcal{L}(W^2_{q,D}(-1,1))} + \big(t - s\big)^{1-\mu}\big\|U_{B(v)}(t, s)\big\|_{\mathcal{L}(W^{2\mu}_{q,D}(-1,1), W^2_{q,D}(-1,1))} \leq C_{11}(\kappa) \tag{3.24}$$

for each $v \in \mathcal{V}_\tau(\kappa)$ and $0 \leq s < t \leq \tau$. Now, we put $\tilde{\kappa} := \dfrac{\kappa}{C_{11}(\kappa)} \leq \kappa$ and define

$$\mathcal{V}_\tau(\kappa, \tilde{\kappa}) := \Bigg\{ v : [0, \tau] \to W^2_{q,D}(-1, 1) \;\Bigg|\; \|v(t) - v(s)\|_{W^{2-\xi}_{q,D}(-1,1)} \leq |t - s|^\rho\,, \;\; v(t) \in S(\tilde{\kappa}) \cap Z(\kappa)\,, \;\; s, t \in [0, \tau] \Bigg\}$$

with $\mathcal{V}_\tau(\kappa, \tilde{\kappa}) \subset \mathcal{V}_\tau(\kappa)$. Thanks to the Theorem of Eberlein-Smulyan, $\mathcal{V}_\tau(\kappa, \tilde{\kappa})$, equipped with the metric

$$d(v, w) := \sup_{t \in [0,\tau]} \|v(t) - w(t)\|_{W^{2-\xi}_{q,D}(-1,1)}\,,$$

is a complete metric space.
(ii) *Definition of the map* Λ: Recall from (3.21) that we use the abbreviation

$$G(v(t)) = \frac{-1}{1 + v(t)} + \lambda g(v)(t)\,, \qquad v \in \mathcal{V}_\tau(\kappa, \tilde{\kappa})\,, \qquad t \in [0, \tau]\,,$$

for the right-hand side of (3.20), and note that

$$[t \mapsto G(v(t))] \in C^\rho\big([0, \tau], L_q(-1, 1)\big)$$

due to Corollary 3.9. Hence, thanks to [**5**, Theorem II.1.2.1, Remark II.2.1.2 (b)], the variation-of-constant-formula

$$\Lambda(v)(t) := U_{B(v)}(t, 0)u_0 + \int_0^t U_{B(v)}(t, s)G(v(s))\,\mathrm{d}s\,, \qquad t \in [0, \tau]\,,$$

defines for each $v \in \mathcal{V}_\tau(\kappa,\tilde\kappa)$ the unique solution

$$\Lambda(v) \in C^1\big([0,\tau], L_q(-1,1)\big) \cap C\big([0,\tau], W^2_{q,D}(-1,1)\big)$$

to the linear problem

$$\partial_t u + B(v)u = G(v)\,, \qquad u(0)=u_0\,.$$

It remains to adjust $\tau \in (0,1]$ such that the map Λ possesses further properties:
(iii) Λ *is a self-mapping*: It follows from [**5**, Theorem II.5.3.1] (with $\alpha = 1-\xi/2+2\rho$ and $\beta = 1-\xi/2$) that

$$\begin{aligned}
&\|\Lambda(v)(t)-\Lambda(v)(s)\|_{W^{2-\xi}_{q,D}(-1,1)}\\
&\qquad \le C_{12}(\kappa)\,|t-s|^{2\rho}\left(\|u_0\|_{W^{2-\xi+4\rho}_{q,D}(-1,1)} + \|G(v(t))\|_{L_\infty((0,t),L_q(-1,1))}\right)\\
&\qquad \le C_{13}(\kappa)\left(\frac{1}{2\kappa}+C_7(\tilde\kappa)\right)\tau^\rho\,|t-s|^\rho \qquad v\in\mathcal{V}_\tau(\kappa,\tilde\kappa)\,,\ s,t\in[0,\tau]\,,
\end{aligned}$$

where we additionally used the choice of κ and Corollary 3.9. Making τ smaller, if necessary, we find, for arbitrary $v \in \mathcal{V}_\tau(\kappa,\tilde\kappa)$ and $s,t\in[0,\tau]$, that

$$\|\Lambda(v)(t)-\Lambda(v)(s)\|_{W^{2-\xi}_{q,D}(-1,1)} \le |t-s|^\rho\,. \tag{3.25}$$

Next, the triangle inequality and (3.25) imply that

$$\begin{aligned}
\|\Lambda(v)(t)\|_{W^{2-\xi}_{q,D}(-1,1)} &\le \|\Lambda(v)(t)-\Lambda(v)(0)\|_{W^{2-\xi}_{q,D}(-1,1)} + \|u_0\|_{W^{2-\xi}_{q,D}(-1,1)}\\
&\le \tau^\rho + \frac{1}{2\kappa}\,,
\end{aligned} \tag{3.26}$$

while (3.25) combined with Sobolev's embedding theorem gives

$$\begin{aligned}
\Lambda(v)(t) &\le u_0 + \|\Lambda(v)(t)-\Lambda(v)(0)\|_\infty\\
&\le 1-2\kappa + C\,\|\Lambda(v)(t)-\Lambda(v)(0)\|_{W^{2-\xi}_{q,D}(-1,1)}\\
&\le 1-2\kappa + C\,\tau^\rho\,.
\end{aligned} \tag{3.27}$$

A similar argument yields

$$\Lambda(v)(t) \ge -1+2\kappa - C\,\tau^\rho\,. \tag{3.28}$$

Moreover, we have

$$\begin{aligned}
\|\Lambda(v)(t)\|_{W^2_{q,D}(-1,1)} &\le C_{11}(\kappa)\|u_0\|_{W^2_{q,D}(-1,1)} + C_{11}(\kappa)\int_0^t (t-s)^{\mu-1}\|G(v(s))\|_{W^{2\mu}_{q,D}(-1,1)}\,\mathrm{d}s\\
&\le \frac{C_{11}(\kappa)}{2\kappa} + C_{11}(\kappa)C_7(\tilde\kappa)\int_0^t s^{\mu-1}\,\mathrm{d}s\\
&\le \frac{1}{2\tilde\kappa} + C_{11}(\kappa)C_7(\tilde\kappa)\,\frac{\tau^\mu}{\mu}\,,
\end{aligned} \tag{3.29}$$

where we applied (3.24) for the first inequality and Corollary 3.9 for the second one, while the last inequality follows from the choice of $\tilde\kappa$[2]. Making $\tau\in(0,1]$ smaller, if

[2] In (3.29), the role of $\tilde\kappa$ becomes clear as we can only show that $\|U_{B(v)}(t,0)u_0\|_{W^2_{q,D}(-1,1)}$ is bounded, but have no possibility to adjust the bound to be smaller than $\frac{1}{2\kappa}$.

necessary, equations (3.25)-(3.29) imply that Λ maps $\mathcal{V}_\tau(\kappa,\tilde{\kappa})$ into itself.
(iv) Λ *is a contraction*: Finally, for $v,w\in\mathcal{V}_\tau(\kappa,\tilde{\kappa})$ and $t\in[0,\tau]$, it follows from [**5**, Theorem II.5.2.1] (with $\alpha=1$, $\beta=1-\xi/2$ and $\gamma=\mu$) that

$$\begin{aligned}
&\|\Lambda(v)(t)-\Lambda(w)(t)\|_{W^{2-\xi}_{q,D}(-1,1)}\\
&\quad\le C_{14}(\kappa)\,\tau^{\xi/2}\bigg(\|G(v)-G(w)\|_{L_\infty((0,t),L_q(-1,1))}\\
&\qquad\quad+\|B(v)-B(w)\|_{C([0,t],\mathcal{L}(W^2_{q,D},L_q)}\cdot\Big(\|u_0\|_{W^2_{q,D}(-1,1)}+\|G(v)\|_{L_\infty((0,t),W^{2\mu}_{q,D}(-1,1))}\Big)\bigg)\\
&\quad\le C_{14}(\kappa)\,\tau^{\xi/2}\bigg(C_7(\tilde{\kappa})+\ell(\kappa)\Big(\frac{1}{2\kappa}+C_7(\tilde{\kappa})\Big)\bigg)\,d(v,w)\,.
\end{aligned}$$

Here, we have also applied Corollary 3.9 and Lemma 3.11 for the second inequality. Making $\tau\in(0,1]$ smaller, if necessary, and taking the supremum over $t\in[0,\tau]$, we find

$$d\big(\Lambda(v),\Lambda(w)\big)\le\frac{1}{2}\,d(v,w)\,,\qquad v,w\in\mathcal{V}_\tau(\kappa,\tilde{\kappa})\,,$$

i.e. Λ is a contraction.

Gathering (i)-(iv) together, Banach's fixed point argument yields the local existence of a unique solution $u\in C^1\big([0,\tau],L_q(-1,1)\big)\cap C\big([0,\tau],W^2_{q,D}(-1,1)\big)$ while for fixed time $t\in[0,\tau)$ the transformed electrostatic potential $\phi_{u(t)}$ belongs to $W^2_2(\Omega)$ with $\Omega=(-1,1)\times(1,2)$, see (3.12), which is equivalent to $\psi_{u(t)}\in W^2_2\big(\Omega(u(t))\big)$. Hence, everything is proven. □

Since τ in the above fixed point argument only depends on κ and $\tilde{\kappa}$ which itself only depends on κ, we can show the typical global existence criterion for solutions to parabolic equations:

Theorem 3.16 (Global Existence Criterion)
If for each $\tau>0$, there exists $\kappa(\tau)\in(0,1)$ such that the unique maximal solution u from Theorem 3.15 satisfies $u(t)\in S(\kappa(\tau))$ for all $t\in[0,T_{max})\cap[0,\tau]$, then u exists globally, that is, $T_{max}=\infty$.

Proof. This follows easily from Theorem 3.15 by a contradiction argument. □

Theorem 3.16 implies that if $T_{max}<\infty$, then the soap film touches itself, touches the outer metal cylinder or the $\|\cdot\|_{W^2_{q,D}(-1,1)}$-norm of u blows up. Here, all three cases may happen at once. While the first two cases possess a direct physical interpretation, the interpretation of the third one is less clear. The norm blow-up might indicate that the soap film can no longer be described by a graph of a function. For example, this could be due to a rupture of the film, or a non-physical behaviour of the film due to a limitation of the model.

Another consequence of the uniqueness of solutions is the following:

Corollary 3.17 (Symmetry)
If the initial value u_0 is even, i.e. $u_0(z) = u_0(-z)$, then the unique maximal solution u from Theorem 3.15 and the corresponding electrostatic potential ψ_u are even with respect to z at each time $t \in [0, T_{max})$.

Proof. First, let $v \in S(\kappa)$ for some $\kappa > 0$ and define $\tilde{v}(z) := v(-z)$. Then, the unique solvability of the electrostatic problem implies that $\psi_{\tilde{v}}(z,r) = \psi_v(-z,r)$ for all $(z,r) \in \Omega(\tilde{v})$ and consequently $g(\tilde{v})(z) = g(v)(-z)$ for $z \in (-1,1)$ by definition of the electrostatic force in (3.1). Now, if the initial value u_0 in Theorem 3.15 is even, i.e. $u_0(z) = u_0(-z)$, then the uniqueness of solutions implies that the maximal solution u is even in z, too. In particular, $\tilde{u}(t) = u(t)$ for each $t \in [0, T_{max})$ and consequently $\psi_{u(t)}(z,r) = \psi_{u(t)}(-z,r)$. □

We conclude this chapter with two remarks on the local existence result presented in Theorem 3.15:

Remarks 3.18 (a) Note that we do not provide conditions on λ, σ and the initial value u_0 which ensure global existence of solutions. This is due to the fact that the right-hand side of (3.20),

$$G(u) = \frac{-1}{1+u} + \lambda g(u)\,,$$

consists of terms of opposite signs with unknown growth of $[u \mapsto g(u)]$. Nevertheless, in later chapters, we achieve global existence results in the form of stable stationary solutions.
(b) We mention that the Fréchet-derivative of

$$-B(u)u = \frac{\sigma^2 u_{zz}}{(1+\sigma^2 u_z^2)}\,, \qquad u \in W^2_{q,D}(-1,1)\,,$$

evaluated at any initial value $u_0 \in W^2_{q,D}(-1,1)$, generates an analytic semigroup. This allows an alternative fixed point argument based on linearization around the initial value.

CHAPTER 4

Stationary Solutions near the Catenoids

In this chapter, we study existence and qualitative properties of stationary solutions to (1.20)-(1.22) for small values of λ and σ fixed above a certain critical value. As the electrostatic potential ψ_u can always be recovered from the film deflection u, we solely focus on u. The stationary version of (1.20)-(1.22) is then given by the non-local elliptic equation

$$\begin{cases} -\sigma\partial_z \text{arctan}(\sigma\partial_z u) = -\dfrac{1}{u+1} + \lambda\, g(u)\,, \\ u(\pm 1) = 0\,, \quad -1 < u < 1 \end{cases} \tag{4.1}$$

with non-local term $g(u)$ defined in (3.1) and capturing the impact of the electrostatic potential ψ_u. Throughout the whole chapter, we use the abbreviation

$$F(u) := \sigma\partial_z \text{arctan}(\sigma\partial_z u) - \frac{1}{u+1}\,. \tag{4.2}$$

4.1. Existence of Stationary Solutions near the Catenoids

We address the existence of stationary solutions for small λ and σ large enough. As a starting point, we recall the situation in which no voltage is applied, that is $\lambda = 0$ in (4.1):

$$\begin{cases} -\sigma\partial_z \text{arctan}(\sigma\partial_z u) = -\dfrac{1}{u+1}\,, \\ u(\pm 1) = 0\,, \quad -1 < u < 1\,. \end{cases} \tag{4.3}$$

This is the well-known minimal surface equation for a surface of revolution. It can be found in many textbooks, see for example [**38**, p. 282], that there exists a threshold value σ_{crit} such that (4.3) has:

- no solution for $\sigma < \sigma_{crit}$,
- exactly one solution for $\sigma = \sigma_{crit}$,
- exactly two solutions for $\sigma > \sigma_{crit}$.

The threshold value can be computed as $\sigma_{crit} = \frac{\cosh(c_{crit})}{c_{crit}} \approx 1.5$ with $c_{crit} \approx 1.2$ being the unique positive solution to

$$c_{crit}\sinh(c_{crit}) - \cosh(c_{crit}) \overset{!}{=} 0\,. \tag{4.4}$$

Moreover, all stationary solutions have the shape of a (translated) catenoid

$$u_*(z) := \frac{\cosh(cz)}{\cosh(c)} - 1\,, \qquad z \in (-1, 1)\,,$$

where the constant $c > 0$ satisfies the relation

$$\sigma = \frac{\cosh(c)}{c} . \tag{4.5}$$

For $\sigma > \sigma_{crit}$, there are two possible choices for this constant which we denote by c_{in} and c_{out}. They satisfy

$$c_{out} < c_{crit} < c_{in} \tag{4.6}$$

and result in an inner catenoid u_{in} corresponding to c_{in} and an outer catenoid u_{out} with $u_{out} > u_{in}$ in $(-1, 1)$. For $\sigma \searrow \sigma_{crit}$, the inner and outer catenoid merge into a single critical one u_{crit}, whereas for $\sigma \nearrow \infty$ the outer catenoid tends to the constant function 0, while the inner one touches itself at $z = 0$. The inner and outer catenoid together with the critical one are depicted in Figure 4.1.

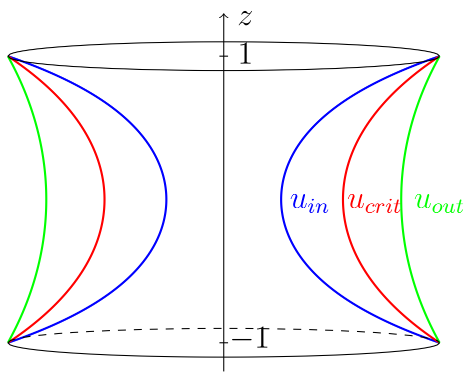

FIGURE 4.1. Depiction of the critical catenoid (red) for $\sigma = \sigma_{crit}$ together with the pair of inner catenoid (blue) and outer catenoid (green) for $\sigma \approx 1.9$. The corresponding c-values are $c_{in} \approx 2.0$ and $c_{out} \approx 0.6$.

Based on the analysis for $\lambda = 0$, we now prove the existence of at least two stationary solutions for small $\lambda > 0$ and $\sigma > \sigma_{crit}$ by applying the implicit function theorem. A similar result is proven in [**23**, Theorem 3 (i)] and [**24**, Theorem 1.2 (i)] in which the existence of at least one steady state in different models for MEMS-devices is shown. The main difference is that we have to solve an additional ordinary differential equation.

Theorem 4.1 *Let $q \in (2, \infty)$ and $\sigma > \sigma_{crit}$. Then, there exists $\delta = \delta(\sigma) > 0$ and analytic functions*

$$[\lambda \mapsto u_{in}^\lambda] : [0, \delta) \to W_{q,D}^2(-1, 1) , \qquad u_{in}^0 = u_{in} ,$$
$$[\lambda \mapsto u_{out}^\lambda] : [0, \delta) \to W_{q,D}^2(-1, 1) , \qquad u_{out}^0 = u_{out}$$

such that u_{in}^λ and u_{out}^λ are two different solutions to (4.1) for each $\lambda \in (0, \delta)$. Moreover, u_{in}^λ and u_{out}^λ as well as the corresponding electrostatic potentials $\psi_{u_{in}^\lambda} \in W_2^2\big(\Omega(u_{in}^\lambda)\big)$ and $\psi_{u_{out}^\lambda} \in W_2^2\big(\Omega(u_{out}^\lambda)\big)$ are symmetric with respect to the r-axis.

Proof. Put

$$S := \left\{ w \in W^2_{q,D}(-1,1) \;\middle|\; -1 < w < 1 \right\}.$$

In the following, we want to resolve equation (4.1), that is $F(w) + \lambda g(w) = 0$ with F from (4.2), locally around $(w, \lambda) = (u_{in}, 0)$ and $(w, \lambda) = (u_{out}, 0)$. Because F and g (see Proposition 3.10) are analytic from S to $L_q(-1,1)$, this is possible if and only if $DF(u_{in})$ and $DF(u_{out})$ are isomorphisms from $W^2_{q,D}(-1,1)$ to $L_q(-1,1)$. Using the definition of F and the relation

$$\sigma \partial_z \arctan(\sigma \partial_z w) = \frac{\sigma^2 \partial_z^2 w}{1 + \sigma^2 (\partial_z w)^2},$$

we compute

$$DF(w)v = \frac{\sigma^2}{(1 + \sigma^2 w_z^2)}\, v_{zz} - \frac{2\sigma^4 w_z w_{zz}}{(1 + \sigma^2 w_z^2)^2}\, v_z + \frac{1}{(w+1)^2}\, v\,, \qquad w \in S\,. \tag{4.7}$$

Evaluating DF at the generic catenoid

$$u_*(z) = \frac{\cosh(cz)}{\cosh(c)} - 1$$

with c being either c_{in} or c_{out} and using

$$\begin{aligned} \sigma &= \frac{\cosh(c)}{c}\,, & (u_*)_z &= \frac{\sinh(cz)}{\sigma}\,, \\ (u_*)_{zz} &= \frac{c\cosh(cz)}{\sigma}\,, & 1 + \sigma^2 (u_*)_z^2 &= \cosh^2(cz) \end{aligned} \tag{4.8}$$

several times, we find

$$\begin{aligned} DF(u_*)v &= \frac{\sigma^2}{\cosh^2(cz)}\, v_{zz} - \frac{2\sigma^2\, c \cosh(cz)\sinh(cz)}{\cosh^4(cz)}\, v_z + \frac{\cosh^2(c)}{\cosh^2(cz)}\, v \\ &= \frac{\sigma^2}{\cosh^2(cz)} v_{zz} - \frac{2\sigma^2\, c}{\cosh^2(cz)} \tanh(cz) v_z + \frac{\sigma^2\, c^2}{\cosh^2(cz)} v\,. \end{aligned} \tag{4.9}$$

Due to the Fredholm alternative and elliptic regularity theory, $DF(u_*)$ is an isomorphism if and only if $DF(u_*)v = 0$ has the unique solution $v = 0$ in $W^2_{q,D}(-1,1)$. Multiplying $DF(u_*)v = 0$ by $-\frac{\cosh^2(cz)}{\sigma^2}$ yields the equivalent condition that

$$\begin{cases} -v_{zz} + 2\, c \tanh(cz) v_z - c^2 v = 0\,, \\ v(\pm 1) = 0 \end{cases} \tag{4.10}$$

only possesses the trivial solution for c equal to c_{in} or c_{out}. This has already been shown in [**58**, p. 49] with the aid of the shooting method, which is briefly recalled here for the reader's convenience: First, one fixes $C_1, C_2 \in \mathbb{R}$ and observes that the initial value problem

$$\begin{cases} -v_{zz} + 2\, c \tanh(cz) v_z - c^2 v = 0\,, \\ v(0) = C_1\,, \qquad v_z(0) = C_2\, c \end{cases}$$

has the unique solution

$$v(z) = C_2 \sinh(cz) - C_1 \left(c\, z \sinh(cz) - \cosh(cz) \right). \tag{4.11}$$

Next, one tries to adjust C_1 and C_2 such that v satisfies the boundary conditions in (4.10),

$$v(1) = C_2 \sinh(c) - C_1 \big(c \sinh(c) - \cosh(c)\big) \stackrel{!}{=} 0\,,$$
$$v(-1) = -C_2 \sinh(c) - C_1 \big(c \sinh(c) - \cosh(c)\big) \stackrel{!}{=} 0\,,$$

or, equivalently,

$$C_2 \stackrel{!}{=} 0 \qquad \text{and} \qquad C_1 \big(c \sinh(c) - \cosh(c)\big) \stackrel{!}{=} 0\,.$$

As $c \sinh(c) - \cosh(c) = 0$ if and only if $c = c_{crit}$ by (4.4), it follows that $C_1 = C_2 = 0$ for $c \neq c_{crit}$. Hence,

$$\text{(4.10) has only the trivial solution for } c \neq c_{crit}, \tag{4.12a}$$

while

$$v(z) = C_1 \big(c_{crit}\, z \sinh(c_{crit} z) - \cosh(c_{crit} z)\big)\,, \quad C_1 \in \mathbb{R} \setminus \{0\}$$
$$\text{is a non-trivial solution to (4.10) for } c = c_{crit}. \tag{4.12b}$$

Since $c_{in} > c_{crit} > c_{out}$, we find that $DF(u_{in})$ as well as $DF(u_{out})$ are isomorphisms between $W^2_{q,D}(-1,1)$ and $L_q(-1,1)$. Hence, the implicit function theorem in the form [**13**, Theorem 4.5.4] yields some $\delta > 0$ and analytic functions

$$[\lambda \mapsto u^\lambda_{in}] : [0,\delta) \to W^2_{q,D}(-1,1)\,, \qquad u^0_{in} = u_{in}\,,$$
$$[\lambda \mapsto u^\lambda_{out}] : [0,\delta) \to W^2_{q,D}(-1,1)\,, \qquad u^0_{out} = u_{out}$$

such that u^λ_{in} and u^λ_{out} are two different solutions to (4.1) for each $\lambda \in (0,\delta)$ with

$$\|u^\lambda_{in} - u_{in}\|_{W^2_{q,D}(-1,1)} < \delta\,, \qquad \|u^\lambda_{out} - u_{out}\|_{W^2_{q,D}(-1,1)} < \delta\,.$$

Additionally, if u solves (4.1) for some $\lambda \in (0,\delta)$ with

$$\|u - u_{in}\|_{W^2_{q,D}(-1,1)} < \delta \quad \text{or} \quad \|u - u_{out}\|_{W^2_{q,D}(-1,1)} < \delta\,, \tag{4.13}$$

then $u = u^\lambda_{in}$ or $u = u^\lambda_{out}$. Because $\big[z \mapsto u^\lambda_{in}(-z)\big]$ is a second solution to (4.1), see the proof of Corollary 3.17, having the same W^2_q-distance to u_{in} as u^λ_{in}, we deduce from (4.13) that $u^\lambda_{in}(-z) = u^\lambda_{in}(z)$. As a consequence, the electrostatic potential $\psi_{u^\lambda_{in}}$ is also symmetric with respect to the r-axis. The symmetry of u^λ_{out} is shown similarly. □

Remark 4.2 For $\sigma = \sigma_{crit}$, the existence of stationary solutions for small λ seems to be unknown. Since in this case (4.10) possesses a non-trivial solution, it follows that $DF(u_{crit})$ is not an isomorphism, and the implicit function theorem is not applicable.

Remark 4.3 We deduce from (4.13) that for $\sigma > \sigma_{crit}$ and each $\lambda \in \big(0, \delta(\sigma)\big)$ no other solution $u \in W^2_{q,D}(-1,1)$ to (4.1), having sufficiently small $W^2_{q,D}$-distance to one of the catenoids, exists. However, based on the numerical analysis performed for the small aspect ratio model in [**58**], one would expect to find other, non-convex but oscillating steady states.

Remark 4.4 In agreement with Theorem 4.1 we denote from now on the inner catenoid u_{in} by u^0_{in}, the outer catenoid u_{out} by u^0_{out}, and the generic catenoid u_* by u^0_*.

4.2. Stability of Stationary Solutions near the Catenoids

We study stability of stationary solutions to (1.20)-(1.22) under rotationally invariant perturbations in the presence of a small voltage. Here, we restrict ourselves to the case $\sigma > \sigma_{crit}$, for which there are (at least) two stationary solutions u_{in}^{λ} and u_{out}^{λ} thanks to Theorem 4.1. Our main result reads as follows:

Theorem 4.5 *Let $q \in (2, \infty)$ and $\sigma > \sigma_{crit}$. Then, there exists $\delta = \delta(\sigma) > 0$ such that for each $\lambda \in [0, \delta)$:*
(i) *The stationary solution u_{in}^{λ} to (1.20)-(1.22) is unstable in $W_{q,D}^2(-1, 1)$.*
(ii) *The stationary solution u_{out}^{λ} to (1.20)-(1.22) is exponentially asymptotically stable in $W_{q,D}^2(-1, 1)$. More precisely, there exist numbers $\omega_0, m, M > 0$ such that for each initial value $u_0 \in W_{q,D}^2(-1, 1)$ with $\|u_0 - u_{out}^{\lambda}\|_{W_{q,D}^2} < m$, the solution u to (1.20)-(1.22) exists globally in time and the estimate*

$$\|u(t) - u_{out}^{\lambda}\|_{W_{q,D}^2(-1,1)} + \|\partial_t u(t)\|_{L_q(-1,1)} \le M\, e^{-\omega_0 t}\|u_0 - u_{out}^{\lambda}\|_{W_{q,D}^2(-1,1)}$$

holds for $t \ge 0$.

In second-order MEMS-models, see [**23**, Theorem 3 (ii)] and [**24**, Theorem 1.2 (ii)], the principle of linearized stability has been applied to prove asymptotic exponential stability of a steady state for small λ.

We pursue a similar approach for our set-up, but due to the λ-independent term on the right-hand side of (4.1), we face a more complicated situation, even for $\lambda = 0$. Thus, we postpone the proof of the full statement of Theorem 4.5 to the end of this section and first turn to the special case of $\lambda = 0$ in Theorem 4.5. That is, we discuss stability of the inner and outer catenoid u_{in}^0 and u_{out}^0, which are precisely the steady states for $\lambda = 0$. Note that the catenoids u_{in}^0 and u_{out}^0 are classically studied in the field of calculus of variations, in which another notion of stability is used. We comment on this in Remark 4.10.

Stability Analysis of the Inner and Outer Catenoid. Fix $\sigma > \sigma_{crit}$ and set $\lambda = 0$. For a uniform computation, we linearize (1.20)-(1.22) around

$$u_*^0(z) = \frac{\cosh(cz)}{\cosh(c)} - 1$$

with c being either c_{in} or c_{out}. For a solution $u \in W_{q,D}^2(-1, 1)$ to (1.20)-(1.22) with initial value u_0 close to u_*^0, we put $v := u - u_*^0$ and write

$$\partial_t v = \partial_t(u - u_*^0) = F(u_*^0 + v) - F(u_*^0)$$

with F given by (4.2) and being smooth in a $W_{q,D}^2$-neighbourhood of u_*^0. In (4.9), we already computed the derivative of F, which we now require in divergence form

$$\begin{aligned} DF(u_*^0)v &= \frac{\sigma^2}{\cosh^2(cz)} v_{zz} - \frac{2\sigma^2 c}{\cosh^2(cz)} \tanh(cz) v_z + \frac{\sigma^2 c^2}{\cosh^2(cz)} v \\ &= \sigma^2 \left[\partial_z \Big(\frac{1}{\cosh^2(cz)}\, v_z \Big) + \frac{c^2}{\cosh^2(cz)} v \right]. \end{aligned} \tag{4.14}$$

Thus, the linearization of (1.20)-(1.22) around the generic catenoid u_*^0 is given by

$$\partial_t v - DF(u_*^0)v = F(v+u_*^0) - F(u_*^0) - DF(u_*^0)v =: G(v)$$

with $DF(u_*^0)$ as above and $G \in C^\infty\big(\mathcal{O}, L_q(-1,1)\big)$ for a small neighbourhood $\mathcal{O}$ of 0 in $W^2_{q,D}(-1,1)$ satisfying $G(0)=0$ as well as $DG(0)=0$. Moreover, since $-DF(u_*^0)$ is a uniformly elliptic operator of second order with bounded smooth coefficients, $-DF(u_*^0)$ belongs to $\mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)$, see [**53**, Theorem 2.5.1 (ii)]. Letting $\mu_0(c)$ be the first eigenvalue of $DF(u_*^0)$ − for details on the spectrum of $DF(u_*^0)$, we refer to Lemma 4.6 below − the stability criterion [**54**, Theorem 9.1.2, Theorem 9.1.3] takes the following simple form:

- if $\mu_0(c) < 0$, then u_*^0 is exponentially asymptotically stable,
- if $\mu_0(c) > 0$, then u_*^0 is unstable.

As only the sign of this first eigenvalue is crucial, we multiply $0 = \big(\mu - DF(u_*^0)\big)v$ for $\mu \in \mathbb{C}$ by $1/\sigma^2 > 0$ and instead study the sign of the first eigenvalue of the resulting problem

$$\left\{\begin{array}{l} 0 = \mu v - \dfrac{c^2}{\cosh^2(cz)} v - \partial_z\Big(\dfrac{1}{\cosh^2(cz)} v_z\Big)\,, \\ v(\pm 1) = 0\,. \end{array}\right. \tag{4.15}$$

This is a regular Sturm-Liouville problem. Therefore, our analysis starts with the collection of some classical results about the spectrum of such problems:

Lemma 4.6 *For fixed $c \in (0,\infty)$, the spectrum of* (4.15) *consists only of countably infinitely many, algebraically simple eigenvalues*

$$\mu_0(c) > \mu_1(c) > \cdots > \mu_n(c) \to -\infty\,.$$

The eigenfunction v_n^c corresponding to $\mu_n(c)$ has exactly n zeroes in $(-1,1)$ and satisfies

$$v_n^c(-z) = (-1)^n\, v_n^c(z)\,, \qquad z \in (-1,1)\,.$$

Proof. For fixed c, we write $\mu_n = \mu_n(c)$ and $v_n = v_n^c$. By [**71**, p. 286], the spectrum of (4.15) consists only of countably infinitely many eigenvalues of geometric multiplicity 1 tending to $-\infty$. Moreover, the corresponding eigenfunction v_n has exactly n zeroes in $(-1,1)$. Note that with $v_n(z)$ also $v_n(-z)$ is a solution to (4.15). Since μ_n has geometric multiplicity 1, it follows that $v_n(z) = C\, v_n(-z)$ for some $C \in \mathbb{R}\setminus\{0\}$. In particular, it is

$$v_n(0) = C\, v_n(0)\,, \qquad \partial_z v_n(0) = -C\partial_z v_n(0).$$

If n is even, then $v_n(0) \neq 0$ and $C = 1$. If n is odd, then $\partial_z v_n(0) \neq 0$ and $C = -1$. It remains to check that each eigenvalue of (4.15) is semi-simple in the sense of [**54**, Definition A.2.3]. To this end, we introduce for $p \geq 2$ the closed operator

$$B_p v := \frac{1}{\sigma^2} DF(u_*^0)v = \frac{c^2}{\cosh^2(cz)} v + \partial_z\Big(\frac{1}{\cosh^2(cz)} v_z\Big)$$

on $L_p(-1,1)$ with domain $D(B_p) := W^2_{p,D}(-1,1)$ associated with (4.15). We have to check, first for the auxiliary case $p = 2$ and then for $p = q > 2$, that $\mathrm{im}(\mu_n - B_p)$ is closed in $L_p(-1,1)$ and

$$L_p(-1,1) = \ker(\mu_n - B_p) \oplus \operatorname{im}(\mu_n - B_p)\,. \tag{4.16}$$

For $p=2$, this follows directly from the Fredholm alternative [**29**, Theorem 6.2.4 (iii)] and elliptic regularity theory. The latter also implies that

$$\ker(\mu_n - B_q) = \ker(\mu_n - B_2)\,, \qquad \operatorname{im}(\mu_n - B_q) = \operatorname{im}(\mu_n - B_2) \cap L_q(-1,1)\,.$$

Hence, for $q>2$, $\operatorname{im}(\mu_n - B_q)$ is closed in $L_q(-1,1)$. Based on these observations, the decomposition (4.16) for $p=q$ follows from the one for $p=2$. □

The function $[c \mapsto \mu_0(c)]$ is called *first eigencurve* for (4.15). In [**10**], qualitative properties of eigencurves for Sturm-Liouville problems depending linearly on a parameter c are stated. Though (4.15) depends non-linearly on c, it is still possible to adapt the idea of [**10**, Section 2.1] to derive sufficient qualitative properties of the first eigencurve:

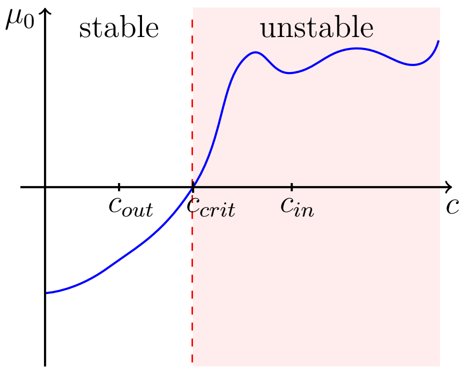

FIGURE 4.2. Qualitative behaviour of the first eigencurve $[c \mapsto \mu_0(c)]$ for the problem (4.15). The sign of $\mu_0(c)$ decides whether the catenoid corresponding to c is asymptotically exponentially stable or unstable.

Proposition 4.7 *The first eigencurve*

$$\mu_0 : (0,\infty) \to \mathbb{R}\,, \qquad c \mapsto \mu_0(c)$$

of (4.15) *is smooth and has exactly one zero. It is attained at* c_{crit} *with* $\mu_0'(c_{crit}) > 0$.

Proof. **(i)** *Smoothness:* Let $v(\,\cdot\,; c, \mu)$ be the unique non-trivial solution to

$$0 = \mu v - \frac{c^2}{\cosh^2(cz)} v - \partial_z\Big(\frac{1}{\cosh^2(cz)} v_z\Big) \tag{4.17}$$

supplemented with initial conditions

$$v(-1) = 0\,, \qquad v_z(-1) = 1\,, \tag{4.18}$$

and define

$$D(c,\mu) := v(1; c, \mu)\,. \tag{4.19}$$

As $v(\,\cdot\,;c,\mu)$ depends smoothly on the parameters (c,μ), see for example [**4**, Theorem 9.5, Remark 9.6 (b)], we have $D \in C^\infty\big((0,\infty)\times\mathbb{R},\mathbb{R}\big)$. Moreover, we note that μ and $v(\,\cdot\,;c,\mu)$ are a pair of eigenvalue and eigenfunction to (4.15) if and only if $D(c,\mu)=0$. We claim that it is further possible to characterize the first eigenvalue $\mu_0(c)$ via D and $v(\,\cdot\,;c,\mu)$:

$$D(c,\mu)=0 \text{ and } v(z;c,\mu)\neq 0 \text{ for } z\in(-1,1) \qquad \Longleftrightarrow \qquad \mu=\mu_0(c)\,. \tag{4.20}$$

Indeed, if $D(c,\mu)=0$ and $v(z;c,\mu)\neq 0$ for $z\in(-1,1)$, then $v(\,\cdot\,;c,\mu)$ is an eigenfunction of (4.15) corresponding to the eigenvalue μ and having no zero in $(-1,1)$. It then follows from Lemma 4.6 that $\mu=\mu_0(c)$. Otherwise, if μ coincides with the first eigenvalue $\mu_0(c)$ of (4.15), then the unique solvability of initial value problems yields a constant $C\in\mathbb{R}\setminus\{0\}$ with

$$v\big(\,\cdot\,,c,\mu_0(c)\big)=Cv_0^c\,,$$

where v_0^c denotes the first eigenfunction from Lemma 4.6. Thus, by Lemma 4.6 the function $v\big(\,\cdot\,,c,\mu_0(c)\big)$ satisfies Dirichlet boundary conditions and has no zero in $(-1,1)$. This proves (4.20).

For fixed $c>0$, we wish to resolve $D(c,\mu)=0$ for μ locally around $(c,\mu)=\big(c,\mu_0(c)\big)$. Recalling that $v=v(\,\cdot\,;c,\mu)$ depends smoothly on μ and c, we let v_μ and v_c be the derivatives of v with respect to μ and c. Moreover, we compute that the derivative of (4.17) with respect to μ is given by

$$0=v+\mu v_\mu-\frac{c^2}{\cosh^2(cz)}v_\mu-\partial_z\left(\frac{1}{\cosh^2(cz)}v_{z\mu}\right)\,. \tag{4.21}$$

Multiplying (4.17) by v_μ and subtracting the product of (4.21) and v, we find

$$0=-v^2-\partial_z\Big(\frac{1}{\cosh^2(cz)}v_z\Big)v_\mu+\partial_z\Big(\frac{1}{\cosh^2(cz)}v_{z\mu}\Big)v\,.$$

Integrating the previous identity over $(-1,1)$ yields

$$\begin{aligned}0<\int_{-1}^1 v^2\,\mathrm{d}z&=\int_{-1}^1\left(\partial_z\Big(\frac{1}{\cosh^2(cz)}v_{z\mu}\Big)v-\partial_z\Big(\frac{1}{\cosh^2(cz)}v_z\Big)v_\mu\right)\mathrm{d}z\\&=\left[\frac{1}{\cosh^2(cz)}(v_{z\mu}v-v_zv_\mu)\right]_{z=-1}^{z=1}\,.\end{aligned} \tag{4.22}$$

We want to evaluate (4.22) at $(c,\mu)=\big(c,\mu_0(c)\big)$. For $\mu=\mu_0(c)$, it follows from (4.20) that $v\big(\,\cdot\,;c,\mu_0(c)\big)$ is a first eigenfunction, and Lemma 4.6 yields that $v\big(\,\cdot\,;c,\mu_0(c)\big)$ is even with $v\big(\pm 1;c,\mu_0(c)\big)=0$. By symmetry and the initial conditions (4.18), we get $v_z(1;c,\mu_0(c))=-v_z(-1;c,\mu_0(c))=-1$. Moreover, applying the initial condition $v(-1;c,\mu)=0$ for all (c,μ), we find $v_\mu\big(-1;c,\mu_0(c)\big)=0$. Consequently, (4.22) can be reduced to

$$0<\int_{-1}^1 v^2\,\mathrm{d}z=\frac{v_\mu\big(1;c,\mu_0(c)\big)}{\cosh^2(c)}\,.$$

Recalling that $D(c,\mu)=v\big(1;c,\mu\big)$ by (4.19), we deduce further that

$$\partial_\mu D\big(c,\mu_0(c)\big)=v_\mu\big(1;c,\mu_0(c)\big)=\cosh^2(c)\int_{-1}^1 v^2>0\,. \tag{4.23}$$

Hence, for fixed $c>0$, the implicit function theorem yields some $\rho>0$ and a function $\tilde{\mu}\in C^\infty\big((c-\rho,c+\rho),\mathbb{R}\big)$ with $\tilde{\mu}(c)=\mu_0(c)$ and

$$D\big(\tilde{c},\tilde{\mu}(\tilde{c})\big)=D\big(c,\mu_0(c)\big)=0\,,\qquad \tilde{c}\in(c-\rho,c+\rho)\,. \tag{4.24}$$

In addition, by the smooth dependence of $v\big(\,\cdot\,,\tilde{c},\tilde{\mu}(\tilde{c})\big)$ on $\tilde{c}$, we may assume that $v\big(\,\cdot\,,\tilde{c},\tilde{\mu}(\tilde{c})\big)$ has no zero in $(-1,1)$ as the same holds true for $v\big(\,\cdot\,,c,\mu_0(c)\big)$. Thus, (4.20) implies

$$\mu_0(\tilde{c})=\tilde{\mu}(\tilde{c})\,,\qquad \tilde{c}\in(c-\rho,c+\rho)\,,$$

and the smoothness of $[c\mapsto\mu_0(c)]$ follows from that.

(ii) *Zeroes:* Rewriting (4.15) for $\mu=0$ in non-divergence form, we see that it is equivalent to (4.10). Hence, it follows from (4.12a) and (4.12b) that 0 is an eigenvalue of (4.15) if and only if $c=c_{crit}$. In this case, the corresponding eigenfunction is a multiple of

$$w(z):=c_{crit}\,z\sinh(c_{crit}z)-\cosh(c_{crit}z)\,.$$

Since w has no zeroes in $(-1,1)$, we deduce from Lemma 4.6 that 0 is the first eigenvalue of (4.15) for $c=c_{crit}$ so that c_{crit} is indeed the only zero of μ_0.

(iii) *Derivative at c_{crit}:* Since

$$\mu_0'(c_{crit})=-\frac{\partial_c D(c_{crit},0)}{\partial_\mu D(c_{crit},0)} \tag{4.25}$$

and $\partial_\mu D(c_{crit},0)>0$ thanks to (4.23), we have to check that $\partial_c D(c_{crit},0)<0$. Differentiating (4.17) with respect to c yields

$$\begin{aligned}0=\mu v_c&+\frac{2c^2\sinh(cz)z}{\cosh^3(cz)}v-\frac{2c}{\cosh^2(cz)}v-\frac{c^2}{\cosh^2(cz)}v_c\\&+\partial_z\Big(\frac{2\sinh(cz)z}{\cosh^3(cz)}v_z\Big)-\partial_z\Big(\frac{1}{\cosh^2(cz)}v_{zc}\Big)\,.\end{aligned} \tag{4.26}$$

Multiplying (4.26) by $v=v(\,\cdot\,;c,\mu)$ and subtracting the product of (4.17) and v_c yields

$$\begin{aligned}0=&\,\partial_z\Big(\frac{1}{\cosh^2(cz)}v_z\Big)v_c\\&+\frac{2c^2\sinh(cz)z}{\cosh^3(cz)}v^2-\frac{2c}{\cosh^2(cz)}v^2+\partial_z\Big(\frac{2\sinh(cz)z}{\cosh^3(cz)}v_z\Big)v-\partial_z\Big(\frac{1}{\cosh^2(cz)}v_{zc}\Big)v\,.\end{aligned} \tag{4.27}$$

Plugging $(c,\mu)=(c_{crit},0)$ into (4.27) and then integrating from -1 to 1 gives

$$\begin{aligned}&\int_{-1}^{1}\Big(\partial_z\Big(\frac{1}{\cosh^2(c_{crit}z)}v_z\Big)v_c-\partial_z\Big(\frac{1}{\cosh^2(c_{crit}z)}v_{zc}\Big)v\Big)\mathrm{d}z\\ &=\int_{-1}^{1}\frac{2c_{crit}}{\cosh^2(c_{crit}z)}\big(1-c_{crit}\tanh(c_{crit}z)z\big)v^2\mathrm{d}z+\int_{-1}^{1}\frac{2\sinh(c_{crit}z)z}{\cosh^3(c_{crit}z)}v_z^2\,\mathrm{d}z\,. \qquad (4.28)\end{aligned}$$

For the second integral on the right-hand side, we have used integration by parts and the fact that the boundary terms vanish due to $v(\pm1;c_{crit},0)=0$ by (4.20) and $\mu_0(c_{crit})=0$. From

$$\begin{aligned}1-c_{crit}\tanh(c_{crit}z)z&\geq 1-c_{crit}\tanh(c_{crit})\\ &=\frac{\cosh(c_{crit})-c_{crit}\sinh(c_{crit})}{\cosh(c_{crit})}=0\,,\qquad z\in(-1,1)\,,\end{aligned}$$

which is due to (4.4) combined with the positivity of the second integral on the right-hand side of (4.28), we deduce that

$$\begin{aligned}0&<\int_{-1}^{1}\Big(\partial_z\Big(\frac{1}{\cosh^2(c_{crit}z)}v_z\Big)v_c-\partial_z\Big(\frac{1}{\cosh^2(c_{crit}z)}v_{zc}\Big)v\Big)\mathrm{d}z\\ &=\Big[\frac{1}{\cosh^2(c_{crit}z)}\big(v_zv_c-v_{zc}v\big)\Big]_{z=-1}^{z=1}\\ &=\frac{-\partial_cD(c_{crit},0)}{\cosh^2(c_{crit})}\,,\end{aligned}$$

where we have used $v_z(1;c_{crit},0)=-v_z(-1;c_{crit},0)$ by symmetry, the initial values (4.18) and the definition of D. Finally, (4.23) and (4.25) yield $\mu_0'(c_{crit})>0$. □

Corollary 4.8 *The inequalities $\mu_0(c_{out})<0$ and $\mu_0(c_{in})>0>\mu_1(c_{in})$ hold true.*

Proof. This follows from Proposition 4.7 and the fact that $c_{out}<c_{crit}<c_{in}$, see (4.6). Note that similar arguments as in step (i) guarantee the smoothness of the second eigencurve $[c\mapsto\mu_1(c)]$, which always lies below the first eigencurve $[c\mapsto\mu_0(c)]$. Because the first eigencurve is sign-changing and the only eigencurve with a zero by step (ii) in the above proof, it follows that $0>\mu_1(c_{in})$. □

In particular, $DF(u^0_{out})$ has only strictly negative eigenvalues, while $DF(u^0_{in})$ has exactly one strictly positive eigenvalue and all other eigenvalues are strictly negative. Regarding the stability analysis of the catenoids, we end up with the following:

Corollary 4.9 *For $\sigma>\sigma_{crit}$ and $\lambda=0$, the inner catenoid u^0_{in} is unstable whereas the outer catenoid u^0_{out} is exponentially asymptotically stable in $W^2_{q,D}(-1,1)$.*

Finally, we come to our main purpose and show the corresponding properties of u^λ_{in} and u^λ_{out} for $\lambda>0$ sufficiently small:

Proof of Theorem 4.5. Letting u_*^λ be either u_{in}^λ or u_{out}^λ, the linearization of (1.20)-(1.22) around u_*^λ reads

$$\begin{aligned}\partial_t v - \big(DF(u_*^\lambda) + \lambda Dg(u_*^\lambda)\big)v &= F(u_*^\lambda + v) - F(u_*^\lambda) - DF(u_*^\lambda)v \\ &\quad + \lambda\big(g(u_*^\lambda + v) - g(u_*^\lambda) - Dg(u_*^\lambda)v\big) =: G_\lambda(v)\,, \end{aligned} \tag{4.29}$$

where F is given by (4.2). Thanks to Proposition 3.10, we find $G_\lambda \in C^\infty\big(\mathcal{O}, L_q(-1,1)\big)$ for a small neighbourhood $\mathcal{O}$ of 0 in $W^2_{q,D}(-1,1)$ satisfying $G_\lambda(0) = 0$ as well as $DG_\lambda(0) = 0$. Moreover, since

$$\begin{aligned}&\|DF(u_*^\lambda) + \lambda Dg(u_*^\lambda) - DF(u_*^0)\|_{\mathcal{L}(W^2_{q,D},L_q)} \\ &\qquad \le \|DF(u_*^\lambda) - DF(u_*^0)\|_{\mathcal{L}(W^2_{q,D},L_q)} + \lambda\|Dg(u_*^\lambda)\|_{\mathcal{L}(W^2_{q,D},L_q)} \to 0\,,\end{aligned}$$

as $\lambda \to 0$ by Theorem 4.1, and $-DF(u_*^0) \in \mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)$, we deduce from [**5**, Theorem 1.3.1 (i)] the existence of $\delta > 0$ such that

$$-\big(DF(u_*^\lambda) + \lambda Dg(u_*^\lambda)\big) \in \mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)\,, \qquad \lambda \in [0,\delta)\,.$$

We now investigate the stability of u_{in}^λ and u_{out}^λ separately:

(i) *Instability of* u_{in}^λ: Due to Corollary 4.8 and Lemma 4.6, the operator $DF(u_{in}^0)$ possesses a positive, isolated and algebraically simple eigenvalue so that the perturbation result [**54**, Proposition A.3.2] for such eigenvalues allows to make $\delta > 0$ smaller such that $DF(u_{in}^\lambda) + \lambda Dg(u_{in}^\lambda)$ also has an eigenvalue with positive real part for $\lambda \in [0,\delta)$. Moreover, since the embedding $W^2_{q,D}(-1,1) \hookrightarrow L_q(-1,1)$ is compact, the spectrum of $DF(u_{in}^\lambda) + \lambda Dg(u_{in}^\lambda)$ consists only of eigenvalues with no finite accumulation point, see [**39**, Theorem 6.29]. Thus, there is a constant $C > 0$ such that the strip $\big\{\mu \in \mathbb{C} \,\big|\, 0 < \mathrm{Re}\mu < C\big\}$ is contained in the resolvent set of $DF(u_{in}^\lambda) + \lambda Dg(u_{in}^\lambda)$. Applying now [**54**, Theorem 9.1.3] to (4.29) shows the instability of u_{in}^λ.

(ii) *Stability of* u_{out}^λ: Since the spectral bound of $DF(u_{out}^0)$ is negative due to Corollary 4.8, it follows from [**5**, Corollary 1.4.3] that we may take $\delta > 0$ so small that $DF(u_{out}^\lambda) + \lambda Dg(u_{out}^\lambda)$ also has a negative spectral bound for $\lambda \in [0,\delta)$. Hence, [**54**, Theorem 9.1.2] implies that u_{out}^λ is exponentially asymptotically stable.

□

We end this section with a discussion of two alternative approaches to the stability behaviour of the outer catenoid u_{out}^0:

Remarks 4.10 **(a)** For $c_{out} < c_{crit}$, it is possible to apply the comparison principle for eigenvalues of Sturm-Liouville problems [**71**, p. 294] to get that $\mu_0(c_{out}) < \mu_0(c_{crit}) = 0$, from which the stability of the outer catenoid u_{out}^0 follows.
(b) One might apply results from the Calculus of Variation. In the Calculus of Variation, one is concerned with critical points of energy functionals, which are sometimes called stable, see [**32**], if they are local minimizers of the energy functional.

Critical points of the surface energy

$$E_m(u) = \int_{-1}^{1} (u+1)\sqrt{1+\sigma^2 u_z^2}\,\mathrm{d}z\,, \qquad u(\pm 1) = 0$$

associated with a surface of revolution with profile $u+1$ are precisely the stationary solutions to (1.20)-(1.22) with $\lambda = 0$. For $\sigma > \sigma_{crit}$, the critical points are the inner and outer catenoid u_{in}^0 and u_{out}^0. It is known that u_{out}^0 is a local minimizer of the surface energy, while u_{in}^0 is not, see [**32**, Sections 5.2.4, 6.2.3]. The connection to our notion of stability stems from the necessary condition for a local minimizer that the second variation

$$\delta^2 E_m(u_{out}^0; v) := \frac{\mathrm{d}^2}{\mathrm{d}t^2} E_m(u_{out}^0 + tv)\bigg|_{t=0}$$

is non-negative. Assuming that v is an eigenfunction of $DF(u_{out}^0)$ corresponding to the eigenvalue μ, one might compute

$$0 \le \delta^2 E_m(u_{out}^0; v) = -\frac{1}{\cosh(c_{out})}\int_{-1}^{1} vDF(u_{out}^0)v\,\mathrm{d}z = -\frac{\mu}{\cosh(c_{out})}\|v\|_{L_2(-1,1)}^2 .$$

Since $\mu \neq 0$, which is due to (4.12a), it follows that $\mu < 0$ and consequently u_{out}^0 is stable in our notion.

4.3. Direction of Deflection

We investigate the directions in which the stationary solutions u_{out}^λ and u_{in}^λ, stemming from u_{out}^0 and u_{in}^0 respectively, are deflected for small λ. For the small aspect ratio model (1.23)-(1.24), formal asymptotic analysis is used in [**58**] to argue that increasing λ results in the outer catenoid u_{out}^0 being pulled outwards and (at least the middle part of) the inner catenoid u_{in}^0 being pulled inwards. For the free boundary problem (1.20)-(1.22), we will recover the same behaviour of the outer catenoid:

Theorem 4.11 *For fixed $\sigma > \sigma_{crit}$, there exists $\delta > 0$ such that*

$$u_{out}^{\bar\lambda}(z) < u_{out}^{\lambda}(z)\,, \qquad 0 \le \bar\lambda < \lambda < \delta\,, \quad z \in (-1,1)\,.$$

Note that Theorem 4.11 reflects a physically expected behaviour: A larger electrostatic force pulls stable configurations of the film outwards.
Regarding the unstable deflections u_{in}^λ, the situation is more complicated and we are only able to present some rigorously proven results in case of the small aspect ratio model formally analysed in [**58**], see Proposition 4.16 later.

Preliminary Considerations. For the moment, let u_*^λ be either u_{in}^λ or u_{out}^λ. Because u_*^λ was constructed in Theorem 4.1 by applying the implicit function theorem to the analytic function $\big[w \mapsto F(w) + \lambda g(w)\big]$, we may write

$$u_*^\lambda = u_*^0 + \lambda\,\partial_\lambda u_*^0 + o(\lambda)\,, \qquad \lambda \to 0$$

with

$$\partial_\lambda u_*^0 = -\big(DF(u_*^0)\big)^{-1} g(u_*^0) \tag{4.30}$$

in $W^2_{q,D}(-1,1)$. Here, g is the electrostatic force, u^0_* is either u^0_{in} or u^0_{out} and c will, in the following, denote the corresponding constant c_{in} or c_{out}. Moreover, we recall from (4.14) that

$$DF(u^0_*)v = \sigma^2 \left[\partial_z \Big(\frac{1}{\cosh^2(cz)} v_z \Big) + \frac{c^2}{\cosh^2(cz)}\, v \right] , \tag{4.31}$$

as well as $g(u^0_*)(z) \geq 0$, $z \in (-1,1)$ by (3.1). Thus, the sign of

$$u^\lambda_* - u^0_* = \lambda\big(- DF(u^0_*)\big)^{-1} g(u^0_*) + o(\lambda)\,, \qquad \lambda \to 0$$

for small λ is closely related to the question whether or not $-DF(u^0_*)$ satisfies a maximum principle, an anti-maximum principle in the spirit of [**15**] or none of them. Note that the scalar function $-\frac{c^2}{\cosh^2(cz)} < 0$ appearing in the definition of $-DF(u^0_*)$ has the wrong sign for the common weak and strong maximum principles [**29**, Theorem 6.4.2, Theorem 6.4.4] to apply.

4.3.1. Deflection from the Outer Catenoid: Proof of Theorem 4.11. Our goal is to prove Theorem 4.11, i.e. that u^λ_{out} deflects monotonically outwards for small $\lambda > 0$. To this end, we have seen that $-DF(u^0_{out})$ needs to satisfy a maximum principle. Since $-DF(u^0_{out})$ is of the form (4.31), it falls in the class of operators investigated in [**6**], and we can rely on a strong maximum principle from [**6**]. It is based on functional analysis and requires that $DF(u^0_{out})$ has a negative spectral bound, which is true thanks to Corollary 4.8.

Lemma 4.12 *Let $c = c_{out}$ and $f \in L_q(-1,1)$ with $f \geq 0$ a.e. and $f \not\equiv 0$. Then, the function $v := \big(- DF(u^0_{out})\big)^{-1} f \in W^2_{q,D}(-1,1)$ satisfies $v(z) > 0$ for $z \in (-1,1)$ as well as $v_z(-1) > 0$ and $v_z(1) < 0$.*

Proof. Recall that $q > 2$, hence $W^2_{q,D}(-1,1) \hookrightarrow C^1([-1,1])$, and that the spectrum of $DF(u^0_{out})$ is contained in $\big(- \infty, 0)$ thanks to Corollary 4.8. Now [**6**, Theorem 15] yields the assertion. □

Furthermore, we check that the right-hand side $g(u^0_{out})$ satisfies the conditions of the above lemma:

Lemma 4.13 *The function $g(u^0_{out})$ belongs to $L_q(-1,1)$ with $g(u^0_{out}) \geq 0$ a.e. and $g(u^0_{out}) \not\equiv 0$.*

Proof. This follows from the definition of g in (3.1) combined with an application of Hopf's Lemma to the electrostatic potential $\psi_{u^0_{out}}$ corresponding to u^0_{out}. □

Eventually, we show the main result of this subsection:

Proof of Theorem 4.11 From Lemma 4.12, Lemma 4.13 and (4.30), it follows that $\partial_z[\partial_\lambda u^0_{out}](1) < 0$ as well as $\partial_z[\partial_\lambda u^0_{out}](-1) > 0$. Thanks to the embedding of $W^2_{q,D}(-1,1)$ in $C^1([-1,1])$, we find $\varepsilon > 0$ such that

$$\begin{aligned} \partial_z[\partial_\lambda u^0_{out}](z) &\leq -4\varepsilon\,, && z \in (1-\varepsilon, 1]\,, \\ \partial_z[\partial_\lambda u^0_{out}](z) &\geq 4\varepsilon\,, && z \in (-1, -1+\varepsilon]\,. \end{aligned} \tag{4.32}$$

Furthermore, since $\partial_\lambda u^0_{out}$ is continuous and strictly positive on $[-1+\varepsilon, 1-\varepsilon]$ by Lemma 4.12, we find $\tilde{\varepsilon} > 0$ such that

$$\partial_\lambda u^0_{out}(z) \geq 4\tilde{\varepsilon}\,, \qquad z \in [-1+\varepsilon, 1-\varepsilon]\,. \tag{4.33}$$

Finally, the continuity of $\big[(z,\lambda) \to \partial_\lambda u^\lambda_{out}(z)\big]$ and $\big[(z,\lambda) \to \partial_z[\partial_\lambda u^\lambda_{out}](z)\big]$ allows us to extend (4.32) and (4.33) to

$$\begin{aligned} &\partial_z[\partial_\lambda u^\lambda_{out}](z) \leq -2\varepsilon\,, \qquad z \in (1-\varepsilon, 1]\,, \qquad \lambda \in [0,\delta_1]\,,\\ &\partial_z[\partial_\lambda u^\lambda_{out}](z) \geq \ 2\varepsilon\,, \qquad z \in [-1,-1+\varepsilon)\,, \quad \lambda \in [0,\delta_1]\,, \end{aligned} \tag{4.34}$$

and

$$\partial_\lambda u^\lambda_{out}(z) \geq 2\tilde{\varepsilon}\,, \qquad z \in [-1+\varepsilon, 1-\varepsilon]\,, \qquad \lambda \in [0,\delta_1]\,, \tag{4.35}$$

for suitably chosen $\delta_1 > 0$. Let us now write

$$u^\lambda_{out} = u^{\overline{\lambda}}_{out} + \partial_\lambda u^{\overline{\lambda}}_{out}\,(\lambda - \overline{\lambda}) + R(\lambda, \overline{\lambda}) \tag{4.36}$$

in $W^2_{q,D}(-1,1) \hookrightarrow C^1\big([-1,1]\big)$ with error term

$$R(\lambda,\overline{\lambda}) := \int_0^1 (1-t)\,\partial^2_\lambda\, u^{\overline{\lambda}+t(\lambda-\overline{\lambda})}_{out}\,\mathrm{d}t\,(\lambda-\overline{\lambda})^2$$

satisfying the uniform estimate

$$\frac{\|R(\lambda,\overline{\lambda})\|_{C^1}}{|\lambda-\overline{\lambda}|} \leq C\,|\lambda-\overline{\lambda}|$$

for some $C > 0$ independent of $\lambda\,,\,\overline{\lambda} \in [0,\delta_1]$. As a consequence, we find $\delta_2 > 0$ with

$$\frac{\|R(\lambda,\overline{\lambda})\|_{C^1}}{|\lambda-\overline{\lambda}|} \leq \min\big\{\varepsilon, \tilde{\varepsilon}\big\}\,, \qquad 0 < \lambda - \overline{\lambda} \leq \delta_2\,, \quad \lambda \leq \delta_1\,. \tag{4.37}$$

From (4.35)-(4.37), it follows that

$$\frac{u^\lambda_{out}(z) - u^{\overline{\lambda}}_{out}(z)}{\lambda-\overline{\lambda}} \geq \tilde{\varepsilon}\,, \qquad z \in [-1+\varepsilon, 1-\varepsilon]\,,$$

while (4.34) - (4.37) yield

$$\frac{\partial_z u^\lambda_{out}(z) - \partial_z u^{\overline{\lambda}}_{out}(z)}{\lambda-\overline{\lambda}} \geq \varepsilon\,, \qquad z \in [-1,-1+\varepsilon)\,,$$

as well as

$$\frac{\partial_z u^\lambda_{out}(z) - \partial_z u^{\overline{\lambda}}_{out}(z)}{\lambda-\overline{\lambda}} \leq -\varepsilon\,, \qquad z \in (1-\varepsilon, 1]\,.$$

Here, all three estimates above hold for $0 < \lambda - \overline{\lambda} \leq \delta_2$ and $\lambda \leq \delta_1$. From these estimates and the fact that

$$u^\lambda_{out}(\pm 1) = u^{\overline{\lambda}}_{out}(\pm 1) = 0\,,$$

we deduce

$$u^\lambda_{out}(z) > u^{\overline{\lambda}}_{out}(z)\,, \qquad z \in (-1,1)\,, \quad 0 < \lambda - \overline{\lambda} \leq \delta_2\,, \quad \lambda \leq \delta_1. \tag{4.38}$$

This is the assertion at least for $\overline{\lambda}$ close to λ. For the general case, put $\delta := \delta_1$. Then, for arbitrary $0 \leq \overline{\lambda} \leq \lambda < \delta$, the assertion follows after iterating (4.38) finitely many times. □

Remark 4.14 Similar results can be proven for u^0_{out} in the small aspect ratio model (1.23)-(1.24). More precisely, also in the small aspect ratio model we can construct a local curve of stable stationary solutions emanating from u^0_{out} and deflecting monotonically outwards. The reason is that we only used the positivity and analyticity of the electrostatic force $[u \mapsto g(u)]$ in the proofs above, which are also properties of the simplified electrostatic force $[u \mapsto g_{sar}(u)]$ in the small aspect ratio model.

4.3.2. Deflection from the Inner Catenoid in the Small Aspect Ratio Model. In this subsection, we focus on the simpler small aspect ratio model (1.23)-(1.24). All previous results of this chapter remain valid if $g(u)$ is replaced by $g_{sar}(u)$ from (1.24). In particular, there exists again a local curve of unstable stationary solutions $[\lambda \mapsto u^\lambda_{in}]$ emanating from u^0_{in} in the small aspect ratio model. Note that, in general, this curve differs from the curve of stationary solutions emanating from u^0_{in} in the full free boundary problem.

We aim at understanding in which direction u^λ_{in} deflects in the small aspect ratio model (1.23)-(1.24). Letting

$$g_{sar}(z) := g_{sar}(u^0_{in})(z) = \frac{\cosh^2(c_{in})}{\cosh(c_{in}z)} \frac{1}{\ln^2\Big(2\,\frac{\cosh(c_{in})}{\cosh(c_{in}z)}\Big)} > 0\,, \qquad z \in (-1,1)\,,$$

our starting point for the investigation of the direction of deflection is again the formula

$$\partial_\lambda u^0_{in} = \big(-DF(u^0_{in})\big)^{-1} g_{sar}\,,$$

which is analogue to (4.30), and we are interested in the sign of $\partial_\lambda u^0_{in}$. Since $c_{in} > c_{crit}$, Corollary 4.8 implies now that $DF(u^0_{in})$ has exactly one strictly positive eigenvalue and all other eigenvalues of $DF(u^0_{in})$ are strictly negative so that the maximum principle from [**6**] fails. Instead, we investigate whether an anti-maximum principle applies, which would yield negativity of $\partial_\lambda u^0_{in}$. In Appendix C, we present a criterion from [**67**] for such an anti-maximum principle to hold, which we complemented by a short argument that $\partial_\lambda u^0_{in}$ is sign-changing in case that the criterion fails.

In the following, we want to apply the criterion from Appendix C to $-DF(u^0_{in})$ and g_{sar}. To state it precisely, let

$$\varphi(z) := \cosh(c_{in}z) - c_{in}\, z \sinh(c_{in}z)$$

be the unique solution to the initial value problem

$$\begin{cases} 0 = -\partial_z\Big(\dfrac{1}{\cosh^2(c_{in}z)}\varphi_z\Big) - \dfrac{c_{in}^2}{\cosh^2(c_{in}z)}\varphi \quad \text{on } (-1,1)\,, \\ \varphi(0) = 1\,, \quad \varphi_z(0) = 0\,, \end{cases} \tag{4.39}$$

associated with the boundary value differential operator $-DF(u^0_{in})$.

The function φ, which is depicted in Figure 4.3, is symmetric, has exactly two zeroes $z = \pm c_{crit}/c_{in}$ in $(-1,1)$ and is sign-changing.

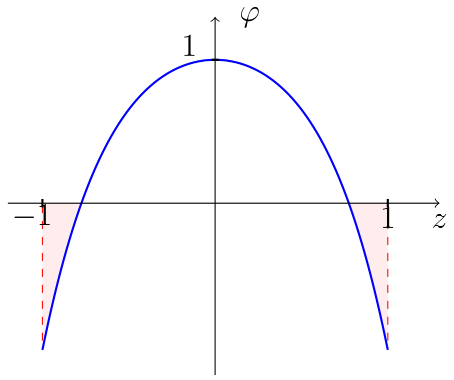

FIGURE 4.3. The solution φ to (4.39) for some $c_{in} > c_{crit}$.

With φ at hand, the criterion reads:

$$\int_{-1}^{1} g_{sar}(z)\varphi(z)\,\mathrm{d}z > 0 \qquad \Longrightarrow \qquad \partial_\lambda u_{in}^0 < 0 \text{ in } (-1,1)\,,$$
$$\int_{-1}^{1} g_{sar}(z)\varphi(z)\,\mathrm{d}z < 0 \qquad \Longrightarrow \qquad \partial_\lambda u_{in}^0 \text{ is sign-changing in } (-1,1)\,.$$

Dependent on the parameter σ, we get:

Lemma 4.15 **(i)** *There exists $\sigma_* > \sigma_{crit}$ such that for each $\sigma \in (\sigma_{crit}, \sigma_*)$ the corresponding deflection $[\lambda \mapsto u_{in}^\lambda]$ satisfies*

$$\partial_\lambda u_{in}^0(z) < 0\,, \qquad z \in (-1,1)\,, \qquad \partial_z[\partial_\lambda u_{in}^0](-1) < 0\,, \qquad \partial_z[\partial_\lambda u_{in}^0](1) > 0\,.$$

(ii) *There exists $\sigma^* > \sigma_{crit}$ such that for each $\sigma > \sigma^*$ and each corresponding deflection $[\lambda \mapsto u_{in}^\lambda]$ there exists $r_0 \in (0,1)$, depending on σ, such that $\partial_\lambda u_{in}^0 < 0$ on $(-r_0, r_0)$ and $\partial_\lambda u_{in}^0 > 0$ on $(-1,-r_0) \cup (r_0, 1)$ as well as*

$$\partial_z[\partial_\lambda u_{in}^0](-1) > 0\,, \quad \partial_z[\partial_\lambda u_{in}^0](-r_0) < 0\,,$$
$$\partial_z[\partial_\lambda u_{in}^0](r_0) > 0\,, \quad \partial_z[\partial_\lambda u_{in}^0](1) < 0\,.$$

Moreover, one has $\sigma^* \geq \sigma_* > \sigma_{crit}$.

Proof. For simplicity, we use the abbreviation $c = c_{in}$. We write

$$\begin{aligned}
&\int_{-1}^{1} g_{sar}(z)\,\varphi(z)\,\mathrm{d}z \\
&\quad = \int_{-1}^{1} \left(\frac{\cosh^2(c)}{\cosh(cz)} \, \frac{1}{\ln^2\!\left(2\,\frac{\cosh(c)}{\cosh(cz)}\right)} \big[\cosh(cz) - cz\sinh(cz)\big] \right) \mathrm{d}z \\
&\quad = \frac{\cosh^2(c)}{c} \int_{-c}^{c} \left(\frac{1}{\cosh(z)} \, \frac{1}{\ln^2\!\left(2\,\frac{\cosh(c)}{\cosh(z)}\right)} \big[\cosh(z) - z\sinh(z)\big] \right) \mathrm{d}z \\
&\quad = 2\,\frac{\cosh^2(c)}{c} \int_{0}^{c} \left(\frac{1}{\ln^2\!\left(2\,\frac{\cosh(c)}{\cosh(z)}\right)} \big[1 - z\tanh(z)\big] \right) \mathrm{d}z \\
&\quad =: 2\,\frac{\cosh^2(c)}{c}\, I_1(\sigma)\,,
\end{aligned} \tag{4.40}$$

where we recall from (4.5) and (4.6) that $c = c_{in}$ is completely determined by being the largest solution to $\sigma = \frac{\cosh(c)}{c}$. Moreover, note that $2\,\frac{\cosh^2(c)}{c} > 0$ is irrelevant for the sign of (4.40) and that

$$1 - z\tanh(z) \begin{cases} \geq 0\,, & z \in (0, c_{crit}]\,, \\ < 0\,, & z \in (c_{crit}, c)\,, \end{cases} \tag{4.41}$$

due to the choice of c_{crit} in (4.4). We first estimate $I_1(\sigma)$ from below and then from above:

(i) From (4.41) we deduce that $I_1(\sigma_{crit}) > 0$. Since the integral $I_1(\sigma)$ depends continuously on $c = c_{in}$, hence continuously on $\sigma \geq \sigma_{crit}$, we find $\sigma_* > \sigma_{crit}$ with

$$I_1(\sigma) > 0\,, \qquad \sigma \in (\sigma_{crit}, \sigma_*)\,.$$

Thus, (4.40) is positive for such σ and the assertion follows from Lemma C.1.

(ii) For the estimate from above, we write

$$\begin{aligned}
I_1(\sigma) &= \int_{0}^{c_{crit}} \left(\frac{1}{\ln^2\!\left(2\,\frac{\cosh(c)}{\cosh(z)}\right)} \big[1 - z\tanh(z)\big] \right) \mathrm{d}z \\
&\quad + \int_{c_{crit}}^{c} \left(\frac{1}{\ln^2\!\left(2\,\frac{\cosh(c)}{\cosh(z)}\right)} \big[1 - z\tanh(z)\big] \right) \mathrm{d}z \\
&=: I_2(\sigma) + I_3(\sigma)
\end{aligned}$$

and deduce from (4.41) that the integrand in $I_2(\sigma)$ is positive, while the integrand in $I_3(\sigma)$ is negative. Since $\ln\Big(2\,\frac{\cosh(c)}{\cosh(z)}\Big) > 0$ for all $z \in (0,c)$, we estimate

$$\begin{aligned} I_2(\sigma) + I_3(\sigma) &\leq \frac{1}{\ln^2\Big(2\,\frac{\cosh(c)}{\cosh(c_{crit})}\Big)} \int_0^{c_{crit}} \big[1 - z\tanh(z)\big]\,\mathrm{d}z \\ &\quad + \frac{1}{\ln^2\Big(2\,\frac{\cosh(c)}{\cosh(c_{crit})}\Big)} \int_{c_{crit}}^{c} \big[1 - z\tanh(z)\big]\,\mathrm{d}z \\ &= \frac{1}{\ln^2\Big(2\,\frac{\cosh(c)}{\cosh(c_{crit})}\Big)} \int_0^{c} \big[1 - z\tanh(z)\big]\,\mathrm{d}z\,. \end{aligned}$$

Now the right-hand side is negative if and only if

$$I_4(\sigma) := \int_0^c \big[1 - z\tanh(z)\big]\,\mathrm{d}z < 0\,.$$

Because $\sigma \nearrow \infty$ implies $c = c_{in} \nearrow \infty$, the integral $I_4(\sigma)$ diverges to $-\infty$ and we find $\sigma^* \geq \sigma_{crit}$ such that $I_4(\sigma) < 0$ for all $\sigma > \sigma^*$. Hence, (4.40) is negative for such values of σ and the assertion follows from Lemma C.1. □

Based on Lemma 4.15, we describe the qualitative behaviour of $[\lambda \mapsto u_{in}^{\lambda}]$ in the small aspect ratio model in case the parameter σ is either sufficiently close to σ_{crit} or sufficiently large. The results are depicted in Figure 4.4. In particular, for σ sufficiently close to σ_{crit}, we discover a contrary behaviour to u_{out}^{λ}: The deflection u_{in}^{λ} of u_{in}^{0} is directed inwards instead of outwards.

Proposition 4.16 *Let $\sigma > \sigma_{crit}$ be fixed and σ_*, σ^* be as in Lemma 4.15.*
(i) *If $\sigma < \sigma_*$, then there exists $\delta > 0$ such that*

$$u_{in}^{\overline{\lambda}}(z) > u_{in}^{\lambda}(z)\,, \qquad 0 \leq \overline{\lambda} < \lambda < \delta\,, \quad z \in (-1,1)\,.$$

(ii) *If $\sigma > \sigma^*$, then there exist $\delta > 0$, $r_0 \in (0,1)$ and $n \in \mathbb{N}$ with $2/n < \min\{r_0, 1 - r_0\}$ such that*

$$u_{in}^{\overline{\lambda}}(z) > u_{in}^{\lambda}(z)\,, \quad 0 \leq \overline{\lambda} < \lambda < \delta\,, \quad z \in [-r_0 + 1/n, r_0 - 1/n]$$

as well as

$$u_{in}^{\overline{\lambda}}(z) < u_{in}^{\lambda}(z)\,, \quad 0 \leq \overline{\lambda} < \lambda < \delta\,, \quad z \in (-1, -r_0 - 1/n] \cup [r_0 + 1/n, 1)\,.$$

Moreover, u_{in}^{λ} intersects $u_{in}^{\overline{\lambda}}$ on $(-1,1)$ in exactly two points z_1, z_2 with

$$z_1 \in (-r_0 - 1/n, -r_0 + 1/n)\,, \qquad z_2 \in (r_0 - 1/n, r_0 + 1/n)\,,$$

and u_{in}^{λ} is strictly decreasing on $[-r_0 - 1/n, -r_0 + 1/n]$ as well as strictly increasing on $[r_0 - 1/n, r_0 + 1/n]$.

Proof. **(i)** By Lemma 4.15 (i), this follows exactly as in the proof of Theorem 4.11.
(ii) The argument is again quite similar to the one in Theorem 4.11: First, we use Taylor's expansion as in Theorem 4.11 together with Lemma 4.15 (ii) to deduce the

existence of $r_0 \in (0,1)$ and $n \in \mathbb{N}$ with $1/n$ small enough (which replaces ε from the proof of Theorem 4.11) as well as $\delta > 0$ such that

$$u_{in}^{\lambda}(z) < u_{in}^{\overline{\lambda}}(z)\,, \qquad z \in [-r_0 + 1/n, r_0 - 1/n]\,, \tag{4.42}$$

$$u_{in}^{\lambda}(z) > u_{in}^{\overline{\lambda}}(z)\,, \qquad z \in [-1 + 1/n, -r_0 - 1/n] \cup [r_0 + 1/n, 1 - 1/n]\,, \tag{4.43}$$

$$\partial_z u_{in}^{\lambda}(z) > \partial_z u_{in}^{\overline{\lambda}}(z)\,, \qquad z \in [-1, -1 + 1/n] \cup [r_0 - 1/n, r_0 + 1/n]\,, \tag{4.44}$$

$$\partial_z u_{in}^{\lambda}(z) < \partial_z u_{in}^{\overline{\lambda}}(z)\,, \qquad z \in [-r_0 - 1/n, -r_0 + 1/n] \cup [1 - 1/n, 1]\,, \tag{4.45}$$

for $0 \leq \overline{\lambda} < \lambda < \delta$. Next, we deduce from (4.43) - (4.45) and the fact that u_{in}^{λ} as well as $u_{in}^{\overline{\lambda}}$ satisfy Dirichlet boundary conditions that

$$u_{in}^{\overline{\lambda}}(z) < u_{in}^{\lambda}(z)\,, \quad z \in (-1, -r_0 - 1/n] \cup [r_0 + 1/n, 1)\,, \tag{4.46}$$

for $0 \leq \overline{\lambda} < \lambda < \delta$. Moreover, since

$$u_{in}^{0}(z) = \frac{\cosh(c_{in} z)}{\cosh(c_{in})} - 1$$

with derivative

$$\partial_z u_{in}^{0}(z) = \frac{\sinh(c_{in} z)}{\sigma} \quad \begin{cases} \leq 0\,, & z \leq 0\,, \\ > 0\,, & z > 0\,, \end{cases}$$

we infer from (4.44) with $\overline{\lambda} = 0$ that u_{in}^{λ} is strictly increasing on $[r_0 - 1/n, r_0 + 1/n]$. Similarly, (4.45) yields that u_{in}^{λ} is strictly decreasing on $[-r_0 - 1/n, -r_0 + 1/n]$. It remains to study the intersection points of u_{in}^{λ} and $u_{in}^{\overline{\lambda}}$. To this end, we deduce from (4.42) and (4.46) that u_{in}^{λ} and $u_{in}^{\overline{\lambda}}$ may only intersect on

$$(-r_0 - 1/n, -r_0 + 1/n) \cup (r_0 - 1/n, r_0 + 1/n) \subset (-1, 1)\,.$$

Thanks to (4.42) and (4.46), we find

$$u_{in}^{\lambda}(-r_0 - 1/n) > u_{in}^{\overline{\lambda}}(-r_0 - 1/n)\,, \qquad u_{in}^{\lambda}(-r_0 + 1/n) < u_{in}^{\overline{\lambda}}(-r_0 + 1/n)$$

for $0 \leq \overline{\lambda} < \lambda < \delta$. Consequently, (4.45) yields that u^{λ} and $u^{\overline{\lambda}}$ have exactly one intersection point z_1 in $(-r_0 - 1/n, -r_0 + 1/n)$. Finally, note that the existence of the second intersection point z_2 in $(r_0 - 1/n, r_0 + 1/n)$ follows similarly. Now, everything is proven. □

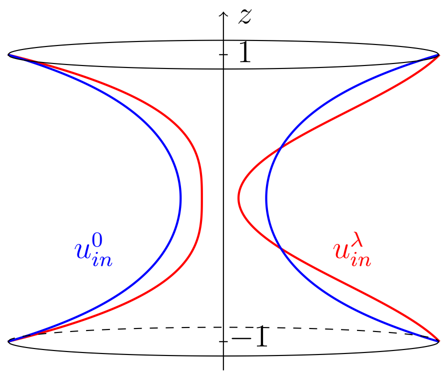

Figure 4.4. Qualitative behaviour of the deflection u_{in}^λ (red) of the inner catenoid u_{in}^0 (blue) for small applied voltages in the small aspect ratio model. On the left a possible deflection for $\sigma \in (\sigma_{crit}, \sigma_*)$ is depicted, while on the right a possible deflection for $\sigma \in (\sigma^*, \infty)$ is shown. For $\sigma \in (\sigma_*, \sigma^*)$, the qualitative behaviour of the deflection is unknown. The outer cylinder is not depicted in this graphic.

CHAPTER 5

Stationary Solutions near the Cylinder

We have seen that for small $\lambda > 0$ and $\sigma > \sigma_{crit}$ there exist at least two stationary solutions to the free boundary problem (1.20)-(1.22). However, for arbitrary σ, we find another interesting stationary solution: We may choose a larger $\lambda_{cyl} > 0$ such that electrostatic force and surface tension are perfectly balanced. Then, the soap film is time-independent and takes the shape of a cylinder $u = 0$. In this chapter, we will determine λ_{cyl}, do some preliminary considerations and then turn to the core issue – the study of existence and qualitative properties of stationary solutions close to $u = 0$. As in the previous chapter, these stationary solutions solve the non-local elliptic equation

$$\begin{cases} F(u) + \lambda\, g(u) = 0\,, \\ u(\pm 1) = 0\,, \quad -1 < u < 1 \end{cases} \tag{5.1}$$

with non-local electrostatic force $g(u)$ given by (3.1) and

$$F(u) := \sigma \partial_z \arctan(\sigma \partial_z u) - \frac{1}{u+1}\,. \tag{5.2}$$

However, this time the control parameter λ is not small, resulting in an impact of the free boundary on the linearization of (5.1) and thus making the exact computation of its spectrum difficult. Instead, we derive qualitative properties of it in two preliminary sections using a Fourier series ansatz. This ansatz is also used in [**1, 26, 27, 50**].

To determine λ_{cyl}, we plug $u = 0$ into (5.1). Then, the equation for the film deflection reads

$$\begin{aligned} 0 &= -\,1 + \lambda_{cyl}\, g(0) \\ &= -\,1 + \lambda_{cyl}\, |\partial_r \psi_0(z,1)|^2\,, \end{aligned} \tag{5.3}$$

while the equation for the electrostatic potential becomes

$$\begin{cases} \dfrac{1}{r}\partial_r\big(r\partial_r\psi_0\big) + \sigma^2\partial_z^2\psi_0 = 0 \quad \text{in} \quad \Omega\,, \\ \psi_0(z,r) = \dfrac{\ln(r)}{\ln(2)} \quad \text{on} \quad \partial\Omega \end{cases}$$

with $\Omega = (-1,1) \times (1,2)$. Its solution is explicitly given by

$$\psi_0(z,r) = \ln(r)/\ln(2)\,. \tag{5.4}$$

Inserting ψ_0 into (5.3) first yields

$$g(0) = \frac{1}{\ln(2)^2}\,, \tag{5.5}$$

and then

$$\lambda_{cyl} = \ln(2)^2 \,. \tag{5.6}$$

Remark 5.1 Note that λ_{cyl} is independent of σ. Also in the small aspect ratio model [**58**], the value λ_{cyl} does only depend on the ratio between outer radius and inner radius, which is 2 for simplicity in our model. Nevertheless, for both models it is surprising, that neither the length nor the inner radius of the set-up have a direct impact on λ_{cyl}.

5.1. Linearized Operator

In this preliminary section, we compute the linearization $DF(0) + \lambda_{cyl} Dg(0)$ for $\lambda_{cyl} = \ln(2)^2$ around the cylinder $u = 0$. We also check that $DF(0) + \lambda_{cyl} Dg(0)$ generates an analytic semigroup and derive first properties of its spectrum.

Recall that we already computed DF in (4.7) which we evaluate at $u = 0$ now:

$$DF(0)v = \sigma^2 \partial_z^2 v + v.$$

We note that this is the generator of an analytic semigroup, i.e.

$$-DF(0) \in \mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big) \,. \tag{5.7}$$

Introducing the notation

$$-\Delta_{cyl,D} : \ W^2_{2,D}(\Omega) \to L_2(\Omega) \,, \quad f \mapsto -\frac{1}{r}\partial_r\big(r\partial_r f\big) - \sigma^2 \partial_z^2 f \,, \tag{5.8}$$

we find the following expression for the remaining term:

Lemma 5.2 *The linearization of g around 0 is given by*

$$\lambda_{cyl}\, Dg(0)v = 2v + 2\,\partial_r(-\Delta_{cyl,D})^{-1}\Big[-\frac{2}{r^3}v - \sigma^2\,\frac{2-r}{r}\,v_{zz}\Big](\,\cdot\,,1)$$

for $v \in W^2_{q,D}(-1,1)$.

Proof. Using the definition of g from (3.1), (5.4) and the relation $\psi_u = \phi_u \circ T_u$ with T_u defined in (3.2), we compute

$$\begin{aligned} Dg(0)v &= D\Big[(1+\sigma^2 u_z^2)^{3/2}\,\big|\partial_r\psi_u(z,u+1)\big|^2\Big]\Big|_{u=0} v \\ &= 2\,\partial_r\psi_0(z,1)\,D\big[\partial_r\psi_u(z,u+1)\big]\Big|_{u=0} v \\ &= \frac{2}{\ln(2)}\,D\Big[\frac{\partial_r\phi_u(z,1)}{1-u}\Big]\Big|_{u=0} v \\ &= \frac{2}{\ln(2)}\Big(\partial_r\phi_0(z,1)\,v + D\big[\partial_r\phi_u(z,1)\big]\Big|_{u=0} v\Big) \,. \end{aligned} \tag{5.9}$$

At this point, we recall from (3.12) that ϕ_u, the transformation of ψ_u to the fixed domain Ω, is given by

$$\phi_u = -L_D(u)^{-1} L_u\Big(\frac{\ln(r)}{\ln(2)}\Big) + \frac{\ln(r)}{\ln(2)}$$

with L_u and $L_D(u)$ defined in (3.4) and (3.11) respectively. In particular,

$$L_0\Big(\frac{\ln(r)}{\ln(2)}\Big) = \frac{1}{\ln(2)}\Big(\frac{1}{r}\partial_r\big(r\partial_r \ln(r)\big) + \sigma^2\partial_z^2 \ln(r)\Big) = 0\,, \tag{5.10}$$

and hence

$$\phi_0 = \frac{\ln(r)}{\ln(2)}\,,$$

as well as

$$\partial_r\phi_0(z,1) = \frac{1}{\ln(2)}\,. \tag{5.11}$$

Moreover,

$$\begin{aligned} L_u\Big(\frac{\ln(r)}{\ln(2)}\Big) =& \frac{1+\sigma^2u_z^2(2-r)^2}{1-u}\,\partial_r^2\Big(\frac{\ln(r)}{\ln(2)}\Big) \\ &+ \Big(-\sigma^2(2-r)u_{zz} - 2\sigma^2\frac{2-r}{1-u}\,u_z^2 + \frac{1}{2u+(1-u)r}\Big)\,\partial_r\Big(\frac{\ln(r)}{\ln(2)}\Big) \\ =& \frac{1}{\ln(2)}\Big[-\frac{1+\sigma^2u_z^2(2-r)^2}{(1-u)}\frac{1}{r^2} \\ &\quad + \Big(-\sigma^2(2-r)u_{zz} - 2\sigma^2\frac{2-r}{1-u}\,u_z^2 + \frac{1}{2u+(1-u)r}\Big)\,\frac{1}{r}\Big] \end{aligned}$$

implies that

$$\begin{aligned} DL_u\Big(\frac{\ln(r)}{\ln(2)}\Big)\Big|_{u=0} v &= \frac{1}{\ln(2)}\Big[-\frac{1}{r^2}\,v + \Big(-\frac{1}{r^2}\,(2-r)\,v - \sigma^2(2-r)v_{zz}\Big)\,\frac{1}{r}\Big] \\ &= \frac{1}{\ln(2)}\,\Big[-\frac{2}{r^3}\,v - \sigma^2\frac{2-r}{r}\,v_{zz}\Big]\,. \end{aligned} \tag{5.12}$$

Combining (5.10) and (5.12) gives

$$\begin{aligned} D\phi_u\big|_{u=0}\,v &= D\Big[-L_D(u)^{-1}L_u\Big(\frac{\ln(r)}{\ln(2)}\Big)\Big]\,\Big|_{u=0} v \\ &= \frac{1}{\ln(2)}\,\big(-L_D(0)\big)^{-1}\,\Big[-\frac{2}{r^3}\,v - \sigma^2\frac{2-r}{r}\,v_{zz}\Big] \\ &= \frac{1}{\ln(2)}\,(-\Delta_{cyl,D})^{-1}\,\Big[-\frac{2}{r^3}\,v - \sigma^2\frac{2-r}{r}\,v_{zz}\Big]\,. \end{aligned} \tag{5.13}$$

For the last line, we used $\big(-L_D(0)\big)^{-1} = (-\Delta_{cyl,D})^{-1}$. Because the chain rule yields

$$D\big[\partial_r\phi_u(\,\cdot\,,1)\big]\Big|_{u=0} v = \partial_r\big(D\phi_u\big|_{u=0} v\big)(\,\cdot\,,1)\,,$$

it follows from (5.9), (5.11) and (5.13) that

$$Dg(0)v = \frac{2}{\ln(2)^2}\Big[v + \partial_r(-\Delta_{cyl,D})^{-1}\,\Big[-\frac{2}{r^3}\,v - \sigma^2\frac{2-r}{r}\,v_{zz}\Big](\,\cdot\,,1)\Big]\,.$$

Finally, the assertion follows as $\lambda_{cyl} = \ln(2)^2$ by (5.6). □

In summary, the linearized operator is given by

$$\big(DF(0)+\lambda_{cyl}Dg(0)\big)v = \sigma^2\partial_z^2 v + 3v + 2\,\partial_r(-\Delta_{cyl,D})^{-1}\Big[-\frac{2}{r^3}v - \sigma^2\,\frac{2-r}{r}\,v_{zz}\Big](\,\cdot\,,1)\,. \tag{5.14}$$

Next, we show that $\lambda_{cyl}Dg(0)$ is compact and consequently, in view of (5.7), that the perturbed operator $DF(0)+\lambda_{cyl}Dg(0)$ again is the generator of an analytic semigroup:

Proposition 5.3 *For $q>2$, we have*

$$-\big(DF(0)+\lambda_{cyl}\,Dg(0)\big)\in\mathcal{H}\big(W^2_{q,D}(-1,1),L_q(-1,1)\big)\,.$$

Proof. We note that the following composition of maps

$$L_2(\Omega)\stackrel{(-\Delta_{cyl,D})^{-1}}{\longrightarrow} W^2_{2,D}(\Omega)\stackrel{\partial_r}{\longrightarrow} W^1_2(\Omega)\stackrel{\text{tr}}{\longrightarrow} W_2^{1/2}(-1,1)\stackrel{c}{\hookrightarrow} L_q(-1,1)$$

defines a compact linear operator from $L_2(\Omega)$ to $L_q(-1,1)$. The notion tr denotes the trace operator with respect to the boundary part $r\equiv 1$. Because the map $\big[v\mapsto -2/r^3\,v-\sigma^2(2-r)/r\,v_{zz}\big]$ is bounded from $W^2_{q,D}(-1,1)$ to $L_2(\Omega)$, it follows that $\lambda_{cyl}Dg(0)\in\mathcal{L}\big(W^2_{q,D}(-1,1),L_q(-1,1)\big)$ is compact as the composition of a compact and a bounded operator. Now the assertion follows from (5.7) and the perturbation result [**54**, Proposition 2.4.3] (or [**5**, Theorem I.1.5.1]). □

Remark 5.4 We note that $DF(0)+\lambda_{cyl}Dg(0)$ can also be considered as a bounded linear operator (or even a generator of an analytic semigroup) from $W^2_{2,D}(-1,1)$ to $L_2(-1,1)$. This fact allows one to work with Fourier series.

We introduce eigenvalues and eigenfunctions of $-\Delta_{cyl,D}:\ W^2_{2,D}(\Omega)\to L_2(\Omega)$ defined in (5.8) with $\Omega=(-1,1)\times(1,2)$. For details on the following descriptions, we refer to Appendix D.

First, we observe that $-\Delta_{cyl,D}$ is not symmetric with respect to the standard scalar product. Therefore, we introduce the spaces

$$L_{2,r}(1,2):=\Big(L_2(1,2),(\,\cdot\,|\,\cdot\,)_{L_{2,r}(1,2)}\Big)$$

with weighted scalar product

$$(f|h)_{L_{2,r}(1,2)}:=\int_1^2 f(r)\,h(r)\,r\,\mathrm{d}r\,,\qquad f,h\in L_2(1,2)\,,$$

and, analogously,

$$L_{2,r}(\Omega):=\Big(L_2(\Omega),(\,\cdot\,|\,\cdot\,)_{L_{2,r}(\Omega)}\Big)$$

with weighted scalar product

$$(f|h)_{L_{2,r}(\Omega)}:=\int_{-1}^1\int_1^2 f(r)\,h(r)\,r\,\mathrm{d}r\,\mathrm{d}z\qquad f,h\in L_2(\Omega)\,.$$

These spaces are obviously isomorphic to $L_2(1,2)$ and $L_2(\Omega)$ respectively. We abbreviate both scalar products by $(\,\cdot\,|\,\cdot\,)_{L_{2,r}}$.

Second, we note that the operator $-\Delta_{cyl,D}$ splits into two parts acting on the z and r variable respectively:

The one-dimensional Dirichlet-Laplacian $-\partial_z^2 : W^2_{2,D}(-1,1) \to L_2(-1,1)$ acts on the first variable z. Its spectrum consists entirely of eigenvalues

$$\nu_j := \frac{(j+1)^2\pi^2}{4}\,, \qquad j \in \mathbb{N}\,,$$

with geometric multiplicity 1 and corresponding normalized eigenfunctions

$$\varphi_j(z) := \begin{cases} \cos\left(\dfrac{(j+1)\pi}{2}\,z\right) & \text{if } j \text{ is even}\,, \\ \sin\left(\dfrac{(j+1)\pi}{2}\,z\right) & \text{if } j \text{ is odd} \end{cases}$$

for $z \in (-1,1)$. The eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ form an orthonormal basis of $L_2(-1,1)$.

The operator $-\frac{1}{r}\partial_r(r\partial_r\,\cdot\,) : W^2_{2,D}(1,2) \to L_2(1,2)$ acts on the second variable r. Its spectrum consists entirely of eigenvalues

$$0 < \xi_0 < \xi_1 < \cdots < \xi_k \to \infty \tag{5.15}$$

with geometric multiplicity 1. The corresponding sequence of normalized eigenfunctions $\{\rho_k\}_{k\in\mathbb{N}}$ belongs to $C^\infty\big([1,2]\big) \cap W^2_{2,D}(1,2)$ and forms an orthonormal basis of $L_{2,r}(1,2)$.

As a consequence, the spectrum of the composite operator $-\Delta_{cyl,D}$ consists entirely of eigenvalues

$$\xi_k + \sigma^2\nu_j\,, \quad j,k \in \mathbb{N}\,,$$

and the corresponding eigenfunctions $\rho_k\varphi_j \in C^\infty(\overline{\Omega}) \cap W^2_{2,D}(\Omega)$ form an orthonormal basis of $L_{2,r}(\Omega)$. Moreover, for each $f \in W^2_{2,D}(\Omega)$, there exists $b_{jk} \in \mathbb{R}$ such that

$$f = \sum_{j,k} b_{jk}\rho_k\varphi_j\,,$$

where the sequence converges unconditionally in $W^2_2(\Omega)$. Based on this preparation, we can compute the Fourier representation of $DF(0) + \lambda_{cyl}Dg(0)$.

Lemma 5.5 *For $q > 2$ and $v \in W^2_{q,D}(-1,1)$, the linearized operator can be written as*

$$\big(DF(0) + \lambda_{cyl}Dg(0)\big)v = \sigma^2\partial_z^2 v + 3v + 2(B_1 + B_2)v$$

in $L_2(-1,1)$, where

$$B_1 v := \sum_{j,k} \frac{c_k}{\xi_k + \sigma^2\nu_j}\,\partial_r\rho_k(1)\,(v|\varphi_j)_{L_2}\,\varphi_j$$

with (unconditional) convergence in $L_2(-1,1)$ *and*

$$B_2 v := \sum_{j,k} \frac{\sigma^2 \nu_j\, d_k}{\xi_k + \sigma^2 \nu_j}\, \partial_r \rho_k(1)\, (v|\varphi_j)_{L_2}\, \varphi_j$$

with (unconditional) convergence in $L_2(-1,1)$*. The coefficients* (c_k)*,* $(d_k) \in \ell_2$ *are given by*

$$c_k := \Big(-\frac{2}{r^3} \,\Big|\, \rho_k \Big)_{L_{2,r}}, \qquad d_k := \Big(\frac{2-r}{r} \,\Big|\, \rho_k \Big)_{L_{2,r}}, \qquad k \in \mathbb{N}.$$

Proof. For $v \in W^2_{q,D}(-1,1) \hookrightarrow W^2_{2,D}(-1,1)$, we represent the corresponding solution $f \in W^2_{2,D}(\Omega)$ to

$$(-\Delta_{cyl,D}) f = -\frac{2}{r^3} v - \sigma^2 \frac{2-r}{r} v_{zz} \tag{5.16}$$

by its Fourier series

$$f = \sum_{j,k} b_{jk}\, \rho_k\, \varphi_j\,, \qquad b_{jk} \in \mathbb{R}\,,$$

with respect to the eigenfunctions of $-\Delta_{cyl,D}$ and aim to determine the b_{jk}. Recall that this series converges unconditionally in $W^2_2(\Omega)$. Expressing both sides of (5.16) by its Fourier series and using the orthogonality of $\{\rho_k\, \varphi_j\}_{j,k}$ in $L_{2,r}(\Omega)$, we find

$$\begin{aligned} b_{jk}\big(\xi_k + \sigma^2 \nu_j\big) &= \Big(-\frac{2}{r^3} \,\Big|\, \rho_k \Big)_{L_{2,r}} (v|\varphi_j)_{L_2} + \sigma^2 \Big(\frac{2-r}{r} \,\Big|\, \rho_k \Big)_{L_{2,r}} (-v_{zz}|\varphi_j)_{L_2} \\ &= \bigg[\Big(-\frac{2}{r^3} \,\Big|\, \rho_k \Big)_{L_{2,r}} + \sigma^2 \nu_j \Big(\frac{2-r}{r} \,\Big|\, \rho_k \Big)_{L_{2,r}} \bigg] (v|\varphi_j)_{L_2} \\ &= (c_k + \sigma^2 \nu_j\, d_k)\, (v|\varphi_j)_{L_2}\,. \end{aligned}$$

Hence, we have

$$f = \sum_{j,k} \frac{c_k + \sigma^2 \nu_j\, d_k}{\xi_k + \sigma^2 \nu_j}\, (v|\varphi_j)_{L_2}\, \varphi_j\, \rho_k \tag{5.17}$$

and

$$\partial_r f(\,\cdot\,,1) = \sum_{j,k} \frac{c_k + \sigma^2 \nu_j\, d_k}{\xi_k + \sigma^2 \nu_j}\, \partial_r \rho_k(1)\, (v|\varphi_j)_{L_2}\, \varphi_j \tag{5.18}$$

in $L_2(-1,1)$ with $(c_k)\,,\,(d_k) \in \ell_2$. Noting that

$$\big(DF(0) + \lambda_{cyl} Dg(0)\big) v = \sigma^2 \partial_z^2 v + 3v + 2\, \partial_r f(\,\cdot\,,1)$$

thanks to (5.14), the assertion follows from (5.18). □

Based on the Fourier representation of $DF(0) + \lambda_{cyl} Dg(0)$ in Lemma 5.5, we can prove that all eigenvalues of its complexification are real and that they are given by scaled versions of the same profile function:

Definition 5.6 We call $\mu : (0,\infty) \to \mathbb{R}$, given by

$$\mu(s) := -s + 3 + 2\Big[\sum_k \frac{c_k}{\xi_k + s}\partial_r \rho_k(1)\Big] + 2s\left[\sum_k \frac{d_k}{\xi_k + s}\partial_r \rho_k(1)\right], \qquad s > 0\,, \quad (5.19)$$

eigencurve profile for $DF(0) + \lambda_{cyl} Dg(0)$. The coefficients (c_k) and (d_k) are the same as in Lemma 5.5.

The well-definedness of $[s \mapsto \mu(s)]$ is a consequence of the next Lemma 5.7 in which we establish a connection between $[s \mapsto \mu(s)]$ and the eigenvalues of the linearization $DF(0) + \lambda_{cyl} Dg(0)$. Furthermore, we already mention that the eigencurve profile is defined also in $s = 0$, see Lemma 5.9 below.

Lemma 5.7 *The spectrum of $DF(0) + \lambda_{cyl} Dg(0)$ consists entirely of real eigenvalues with no finite accumulation point. These eigenvalues are given by*

$$\mu_j(\sigma) := \mu\big(\sigma^2 \nu_j\big)$$

for $j \in \mathbb{N}$ (at this stage possibly neither ordered nor distinct). An eigenfunction which coincides with the j-th eigenfunction φ_j of the one-dimensional Dirichlet-Laplacian corresponds to each eigenvalue.

Proof. Because $W^2_{q,D}(-1,1)$ is compactly embedded in $L_q(-1,1)$, the spectrum of the complexification of the linearized operator consists only of eigenvalues with no finite accumulation point, see [**39**, Theorem 6.29]. Moreover, Lemma 5.5 ensures that

$$\Big(\big(DF(0) + \lambda_{cyl} Dg(0)\big) w_1 \,\Big|\, w_2\Big)_{L_2} = \Big(w_1 \,\Big|\, \big(DF(0) + \lambda_{cyl} Dg(0)\big) w_2\Big)_{L_2}$$

for $w_1, w_2 \in W^2_{q,D}(-1,1)$. Consequently, all eigenvalues of the complexification of $DF(0) + \lambda_{cyl} Dg(0)$ are real, and eigenfunctions to different eigenvalues are orthogonal with respect to the L_2-scalar product. A short computation based on Lemma 5.5 shows that

$$\big(DF(0) + \lambda_{cyl} Dg(0)\big)\varphi_j = \mu_j(\sigma)\varphi_j\,, \qquad j \in \mathbb{N}\,,$$

for the j-th eigenfunction φ_j of the one-dimensional Dirichlet-Laplacian. Finally, as $\{\varphi_j\}_j$ forms an orthogonal basis of $L_2(-1,1)$, there are no other eigenvalues than the μ_j's. □

5.2. Qualitative Properties of the Eigencurve Profile

To further analyse the spectrum of the linearized operator $DF(0) + \lambda_{cyl} Dg(0)$, it suffices to investigate the eigencurve profile $[s \mapsto \mu(s)]$. In particular, we will show the following:

Proposition 5.8 *The eigencurve profile $[s \mapsto \mu(s)]$ is strictly decreasing on $[0,\infty)$ and there exists $s_0 \in (0,\infty)$ with $\mu(s_0) = 0$.*

The proof of Proposition 5.8 is given after some preparation. As a first step towards it, we present another representation of the eigencurve profile $[s \mapsto \mu(s)]$, which includes the case $s = 0$. We also compute the derivative of the eigencurve profile.

Lemma 5.9 *The eigencurve profile* μ *may equivalently be written as*

$$\mu(s) = -s + 3 + 2\,\partial_r h_s(1)\,, \qquad s \in (0,\infty)\,, \tag{5.20}$$

where $h_s \in W^2_{2,D}(-1,1)$ *solves*

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r h_s\big) + s\,h_s = \frac{-2}{r^3} + s\,\frac{2-r}{r}\,, \\ h_s(1) = h_s(2) = 0\,. \end{cases} \tag{5.21}$$

This representation holds even for $s > -\xi_0$ *with* $\xi_0 > 0$ *from* (5.15). *In particular,* $\mu \in C^\infty\big((-\xi_0,\infty),\mathbb{R}\big)$ *with*

$$\mu'(s) = -1 + 2\,\partial_r p_s(1)\,, \tag{5.22}$$

where $p_s \in W^2_{2,D}(-1,1)$ *solves*

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r p_s\big) + s\,p_s = \frac{2-r}{r} - h_s\,, \\ p_s(1) = p_s(2) = 0\,. \end{cases} \tag{5.23}$$

Proof. **(i)** We derive the alternative formula (5.20) for μ: For $s \in (0,\infty)$, we find $\sigma \in (0,\infty)$ such that $s = \sigma^2\nu_0$. Let us note that the solution f_s to

$$\begin{aligned}(-\Delta_{cyl,D})f_s &= -\frac{2}{r^3}\varphi_0 - \sigma^2\frac{2-r}{r}\partial_z^2\varphi_0 \\ &= \Big(-\frac{2}{r^3} + s\,\frac{2-r}{r}\Big)\,\varphi_0\end{aligned}$$

with φ_0 denoting the first eigenfunction of the one-dimensional Dirichlet-Laplacian, can be written in the form

$$f_s(z,r) = h_s(r)\,\varphi_0(z) \tag{5.24}$$

due to its Fourier representation (5.17) (with $v = \varphi_0$). Next, we compute

$$\begin{aligned}(-\Delta_{cyl,D})f_s &= -\frac{1}{r}\partial_r\big(r\,\partial_r h_s(r)\big)\varphi_0(z) - \sigma^2\,h_s(r)\partial_{zz}\varphi_0(z) \\ &= \Big(-\frac{1}{r}\partial_r(r\,\partial_r h_s(r)) + s\,h_s(r)\Big)\varphi_0(z)\,,\end{aligned}$$

and deduce that h_s has to solve

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r h_s\big) + s\,h_s = -\frac{2}{r^3} + s\,\frac{2-r}{r}\,, \quad r \in (1,2)\,, \\ h_s(1) = h_s(2) = 0\,. \end{cases}$$

By elliptic regularity theory, we even have $h_s \in C^\infty\big([1,2]\big)$. We now derive from the relation $s = \sigma^2\nu_0$, (5.14) and (5.24) that

$$\begin{aligned}\big(DF(0) + \lambda_{cyl}Dg(0)\big)\varphi_0 &= (-s+3)\varphi_0 + 2\,\partial_r f_s(\,\cdot\,,1) \\ &= \big(-s + 3 + 2\,\partial_r h_s(1)\big)\varphi_0\,.\end{aligned}$$

Combining this with Lemma 5.7 yields

$$\mu(s) = -s + 3 + 2\,\partial_r h_s(1)\,, \qquad s \in (0,\infty)\,,$$

which is formula (5.20).
(ii) Note that the operator $-\frac{1}{r}\partial_r\big(r\partial_r\,\cdot\,\big)+s$ is invertible for each $s\in(-\xi_0,\infty)$. Because the right-hand side of (5.21) depends smoothly on s, and taking the inverse is a smooth operation, it follows that $\mu\in C^\infty\big((-\xi_0,\infty),\mathbb{R}\big)$. Moreover, its derivative is given by

$$\mu'(s)=-1+2\,\partial_r p_s(1)\,,$$

where $p_s:=\partial_s h_s\in W^2_{2,D}(-1,1)$. Finally, we note that taking the derivative of both sides of (5.21) with respect to s results in (5.23). This shows the remaining formula (5.22) for μ'. □

Remark 5.10 Note that the solution h_s to (5.21) can be expressed in terms of Bessel functions of the first and second kind. Nevertheless, this expression is very lengthy, and we were not able to deduce properties of the eigencurve profile from it.

For the special case $s=0$, it is possible to give explicit formulas for $\mu(0)$ and $\mu'(0)$:

Lemma 5.11 *The values $\mu(0)$ and $\mu'(0)$ are given by*

$$\mu(0)=-1+\frac{2}{\ln(2)}>0$$

and

$$\mu'(0)=-2+\frac{3}{2\ln(2)^2}-\frac{1}{\ln(2)}<-\frac{3}{10}\,.$$

Proof. **(i)** For $\mu(0)$, we note that (5.21) with $s=0$ reads

$$\begin{cases} -\dfrac{1}{r}\partial_r\big(r\,\partial_r h_0\big)=\dfrac{-2}{r^3}\,,\\ h_0(1)=h_0(2)=0\,.\end{cases}$$

This equation is solved by

$$h_0(r)=\frac{2-r}{r}+\frac{\ln(r)}{\ln(2)}-1 \tag{5.25}$$

with derivative

$$\partial_r h_0(1)=-2+\frac{1}{\ln(2)}\,. \tag{5.26}$$

Hence, equation (5.20) gives

$$\mu(0)=-1+\frac{2}{\ln(2)}>0\,.$$

(ii) For $\mu'(0)$, we first recall from (5.22) that

$$\mu'(0)=-1+2\,\partial_r p_0(1)\,.$$

The function p_0 solves

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r p_0\big) = 1 - \dfrac{\ln(r)}{\ln(2)}\,, \\ p_0(1) = p_0(2) = 0\,, \end{cases}$$

which is (5.23) with $s = 0$ and inserted expression for h_0 from (5.25). This equation has the explicit solution

$$p_0(r) = \left(\frac{3 - \ln(2)}{4\ln(2)^2}\right)\ln(r) + \frac{1 + \ln(2)}{4\ln(2)} + \frac{r^2\,\ln(r/2) - r^2}{4\ln(2)}$$

with

$$\begin{aligned} \partial_r p_0(1) &= \frac{3 - \ln(2)}{4\ln(2)^2} + \frac{-2\ln(2) + 1 - 2}{4\ln(2)} \\ &= \frac{3}{4\ln(2)^2} - \frac{1}{2\ln(2)} - \frac{1}{2}\,. \end{aligned}$$

Plugging $\partial_r p_0(1)$ into the formula for $\mu'(0)$ yields the assertion. □

Based on this preparation, we provide a proof that $[s \mapsto \mu(s)]$ is strictly decreasing on $[0, \infty)$ and has exactly one zero.

Proof of Proposition 5.8. As μ is smooth with $\mu(0) > 0$ as well as $\mu'(0) < 0$ by Lemma 5.9 and Lemma 5.11, it is enough to show that $[s \mapsto \mu'(s)]$ is decreasing for $s \geq 0$. We will achieve that by applying the weak maximum principle several times.
(i) First, we apply it to $[s \mapsto h_s]$, where we recall from (5.21) that h_s solves

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r h_s\big) + s\,h_s = \dfrac{-2}{r^3} + s\,\dfrac{2-r}{r}\,, \\ h_s(1) = h_s(2) = 0\,. \end{cases}$$

Because $\frac{2-r}{r}$ solves

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r f(r)\big) = -\dfrac{2}{r^3}\,, \\ f(1) = 1\,, \quad f(2) = 0\,, \end{cases}$$

it follows that $\frac{2-r}{r} - h_s$ is a solution to

$$\begin{cases} -\frac{1}{r}\partial_r\big(r\,\partial_r f(r)\big) + s\,f(r) = 0\,, \\ f(1) = 1\,, \quad f(2) = 0\,. \end{cases}$$

An application of the weak maximum principle yields

$$h_s \leq \frac{2-r}{r}\,, \qquad s \geq 0\,. \tag{5.27}$$

For $s > \tilde{s} \geq 0$, the difference $h_s - h_{\tilde{s}}$ solves

$$\begin{cases} -\dfrac{1}{r}\partial_r\big(r\partial_r f(r)\big) + sf(r) = (s-\tilde{s})\Big(\dfrac{2-r}{r} - h_{\tilde{s}}(r)\Big)\,, \\ f(1) = f(2) = 0\,, \end{cases}$$

where the right-hand side is non-negative thanks to (5.27). Consequently, the weak maximum principle gives

$$h_s \geq h_{\tilde{s}}\,, \qquad s > \tilde{s} \geq 0\,. \tag{5.28}$$

(ii) Now we apply the weak maximum principle for $s \geq 0$ to the solution $p_s = \partial_s h_s$ to (5.23), i.e. to

$$\begin{cases} -\dfrac{1}{r}\partial_r\big(r\,\partial_r p_s\big) + s\,p_s = \dfrac{2-r}{r} - h_s\,, \\ p_s(1) = p_s(2) = 0\,. \end{cases}$$

Due to (5.27), the right-hand side of this equation is non-negative and the weak maximum principle yields $p_s \geq 0$. For $s > \tilde{s} \geq 0$, we then find that $p_s - p_{\tilde{s}}$ solves

$$\begin{cases} -\dfrac{1}{r}\partial_r\big(r\,\partial_r f\big) + s\,f = (h_{\tilde{s}} - h_s) + (\tilde{s} - s)p_{\tilde{s}}\,, \\ f(1) = f(2) = 0 \end{cases}$$

with non-positive right-hand side thanks to $p_{\tilde{s}} \geq 0$ and (5.28). Applying the weak maxiumum principle once more, we see that $p_s - p_{\tilde{s}}$ attains its maximum at $r = 1$ and hence

$$\partial_r p_s(1) \leq \partial_r p_{\tilde{s}}(1)\,, \qquad s > \tilde{s} \geq 0\,,$$

which shows $\mu'(s) \leq \mu'(\tilde{s})$ for $s > \tilde{s} \geq 0$ as claimed. □

Remark 5.12 Since $h_s(1) = h_{\tilde{s}}(1) = 0$ and $h_s - h_{\tilde{s}} \geq 0$ on $[1,2]$ for $s > \tilde{s} \geq 0$ by (5.28), it follows that $[s \mapsto \partial_r h_s(1)]$ is increasing. Hence, it is not useful for proving that $[s \mapsto \mu(s)]$ is decreasing, and we had to work with the derivative $[s \mapsto \mu'(s)]$ instead.

Through the relation

$$\mu_j(\sigma) = \mu(\sigma^2 \nu_j)\,, \qquad j \in \mathbb{N}\,,$$

where $\mu_j(\sigma)$ are the eigenvalues of the linearized operator $DF(0) + \lambda_{cyl} Dg(0)$, we can derive properties of its spectrum from properties of its eigencurve profile.

Lemma 5.13 *The eigenvalues of $DF(0) + \lambda_{cyl} Dg(0)$ are ordered*

$$\mu_0(\sigma) > \mu_1(\sigma) > \cdots > \mu_j(\sigma) > \mu_{j+1}(\sigma) > \dots\,, \qquad j \in \mathbb{N}\,,$$

and have geometric multiplicity 1.

Proof. Because $[s \mapsto \mu(s)]$ is strictly decreasing by Proposition 5.8 and the eigenvalues of the one-dimensional Dirichlet-Laplacian (ν_j) are strictly increasing, the eigenvalues $(\mu_j(\sigma))$ are strictly decreasing. In particular, they are distinct and each one has geometric multiplicity 1 where the corresponding eigenfunction is the j-th eigenfunction φ_j of the Dirichlet-Laplacian, see Lemma 5.7. □

Further properties of the eigenvalues $\mu_j(\sigma)$, following from properties of the eigencurve profile $[s \mapsto \mu(s)]$, are discussed in the next sections: First, to construct stationary solutions close to the cylinder in Section 5.3, we require $\mu_j(\sigma) \neq 0$ for all $j \in \mathbb{N}$. For the analysis of stability in Section 5.4, we need to know whether $\mu_0(\sigma)$ is positive or negative. If $\mu_0(\sigma)$ is negative, we also require that it is algebraically simple. Finally, in Section 5.5, we require upper and lower bounds on $\mu_j(\sigma)$ for $j \in \mathbb{N}$ to investigate in which direction the cylinder is deflected when the electrostatic force is increased.

5.3. Existence of Stationary Solutions near the Cylinder.

In this section, we show existence of stationary solutions for λ close to λ_{cyl} by applying the implicit function theorem. We recall that the stationary cylinder $u = 0$ occurs exactly for the parameter λ_{cyl} given by (5.6).

Theorem 5.14 *Let $q \in (2, \infty)$, and $s_0 > 0$ be the unique zero of the eigencurve profile $[s \mapsto \mu(s)]$ from Proposition 5.8. Then, for each $\sigma > 0$ with*

$$\sigma^2 \neq \frac{4\, s_0}{\pi^2\,(j+1)^2}\,, \qquad j \in \mathbb{N}\,,$$

there exists $\delta = \delta(\sigma) > 0$ and an analytic function

$$[\lambda \mapsto u^{\lambda}_{cyl}] : (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta) \to W^2_{q,D}(-1,1)\,, \qquad u^{\lambda_{cyl}}_{cyl} = 0$$

such that u^{λ}_{cyl} is a solution to (5.1) *for each $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$. Moreover, u^{λ}_{cyl} as well as the corresponding electrostatic potential $\psi_{u^{\lambda}_{cyl}} \in W^2_2\big(\Omega(u^{\lambda}_{cyl})\big)$ are symmetric with respect to the r-axis.*

Proof. Put

$$S := \big\{ w \in W^2_{q,D}(-1,1) \;\big|\; -1 < w < 1 \big\}\,.$$

In the following, we want to resolve equation (4.1), that is $F(w) + \lambda g(w) = 0$ with F from (5.2), locally around $(w, \lambda) = (0, \lambda_{cyl})$. Because F and g (see Proposition 3.10) are analytic from S to $L_q(-1,1)$ and the spectrum of $DF(0) + \lambda_{cyl} Dg(0)$ consists only of eigenvalues, this is possible if and only if 0 is no eigenvalue of $DF(0) + \lambda_{cyl} Dg(0)$. For $j \in \mathbb{N}$, we have

$$\sigma^2 \neq \frac{4\, s_0}{\pi^2\,(j+1)^2} \qquad \Longleftrightarrow \qquad \sigma^2 \nu_j \neq s_0 \qquad \Longleftrightarrow \qquad \mu_j(\sigma) = \mu(\sigma^2 \nu_j) \neq 0$$

by the properties of the eigencurve profile and by assumption. Consequently, the implicit function theorem (in the form [**13**, Theorem 4.5.4]) is applicable. It yields some $\delta > 0$ and an analytic function

$$[\lambda \mapsto u^{\lambda}_{cyl}] : (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta) \to W^2_{q,D}(-1,1)\,, \qquad u^{\lambda_{cyl}}_{cyl} = 0$$

such that u^λ_{cyl} is a solution to (5.1) for each $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$ with

$$\|u^\lambda_{cyl}\|_{W^2_{q,D}(-1,1)} < \delta\,.$$

Additionally, if u solves (5.1) for some $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$ with

$$\|u\|_{W^2_{q,D}(-1,1)} < \delta\,, \tag{5.29}$$

then $u = u^\lambda_{cyl}$. Because $\big[z \mapsto u^\lambda_{cyl}(-z)\big]$ is a second solution to (5.1), see the proof of Corollary 3.17, having the same W^2_q-distance to 0 as u^λ_{cyl}, it follows from (5.29) that $u^\lambda_{cyl}(-z) = u^\lambda_{cyl}(z)$. As a consequence, the electrostatic potential $\psi_{u^\lambda_{cyl}}$ is symmetric with respect to the r-axis. □

5.4. Stability of Stationary Solutions near the Cylinder.

We study stability of stationary solutions near the cylinder under rotationally symmetric perturbations. We find a sharp threshold value $\sigma_{cyl} > 0$ such that the stationary solution u^λ_{cyl} to (1.20)-(1.22), which was constructed for most $\sigma > 0$ in Theorem 5.14, is unstable for $\sigma < \sigma_{cyl}$ and stable for $\sigma > \sigma_{cyl}$.

Theorem 5.15 *Let $q \in (2, \infty)$ and $\sigma^2 \neq \dfrac{4s_0}{\pi^2(j+1)^2}$ for $j \in \mathbb{N}$ and s_0 being the unique zero of the eigencurve profile $[s \mapsto \mu(s)]$ from Proposition 5.8. Define*

$$\sigma_{cyl} := \sqrt{\frac{s_0}{\nu_0}}\,, \tag{5.30}$$

where $\nu_0 = \pi^2/4$. Then, there exists $\delta > 0$ such that for each $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$ the stationary solution u^λ_{cyl} to (1.20)-(1.22) satisfies:
(i) *If $\sigma < \sigma_{cyl}$, then u^λ_{cyl} is unstable in $W^2_{q,D}(-1,1)$.*
(ii) *If $\sigma > \sigma_{cyl}$, then u^λ_{cyl} is exponentially asymptotically stable in $W^2_{q,D}(-1,1)$. More precisely, there exist numbers $\omega_0, m, M > 0$ such that for each initial value $u_0 \in W^2_{q,D}(-1,1)$ with $\|u_0 - u^\lambda_{cyl}\|_{W^2_{q,D}} < m$, the solution u to (1.20)-(1.22) exists globally in time and the estimate*

$$\|u(t) - u^\lambda_{cyl}\|_{W^2_{q,D}(-1,1)} + \|\partial_t u(t)\|_{L_q(-1,1)} \leq M\, e^{-\omega_0 t}\|u_0 - u^\lambda_{cyl}\|_{W^2_{q,D}(-1,1)}$$

holds for $t \geq 0$.

The stability result will be complemented with a numerical approximation of σ_{cyl}, and a discussion of an important stabilizing effect of the electrostatic force.

As in Section 4.2, we want to apply the principle of linearized stability. We roughly follow [**1, 26, 27, 50**].

For a solution $u \in W^2_{q,D}(-1,1)$ to (1.20)-(1.22) with initial value u_0 close to u^λ_{cyl}, we put $v := u - u^\lambda_{cyl}$. Then, v satisfies the linearized equation

$$\begin{aligned}\partial_t v - \big(DF(u^\lambda_{cyl}) + \lambda Dg(u^\lambda_{cyl})\big)v &= F(u^\lambda_{cyl}+v) - F(u^\lambda_{cyl}) - DF(u^\lambda_{cyl})v \\ &\quad + \lambda\big(g(u^\lambda_{cyl}+v) - g(u^\lambda_{cyl}) - Dg(u^\lambda_{cyl})v\big) =: G_{cyl}(v).\end{aligned} \tag{5.31}$$

Thanks to Proposition 3.10, we have $G_{cyl} \in C^\infty\big(\mathcal{O}, L_q(-1,1)\big)$ for a small neighbourhood $\mathcal{O}$ of 0 in $W^2_{q,D}(-1,1)$ satisfying $G_{cyl}(0) = 0$ as well as $DG_{cyl}(0) = 0$.

First, we study the stability of the cylinder $u^{\lambda_{cyl}}_{cyl} = 0$:

Lemma 5.16 *Let $q \in (2,\infty)$ and $\lambda = \lambda_{cyl}$. Then, the following holds:*
(i) *If $\sigma < \sigma_{cyl}$, then the stationary solution $u = 0$ to* (1.20)-(1.22) *is unstable in $W^2_{q,D}(-1,1)$.*
(ii) *If $\sigma > \sigma_{cyl}$, then the stationary solution $u = 0$ to* (1.20)-(1.22) *is exponentially asymptotically stable in $W^2_{q,D}(-1,1)$.*

Proof. Because of (5.31) and the fact that $-\big(DF(0) + \lambda_{cyl}Dg(0)\big)$ belongs to $\mathcal{H}\big(W^2_{q,D}(-1,1), L_q(-1,1)\big)$ by Proposition 5.3, we can apply results from [**54**]. The choice of σ_{cyl} in (5.30) guarantees that the largest eigenvalue $\mu_0(\sigma)$ of $DF(0)+\lambda_{cyl}Dg(0)$ satisfies

$$\mu_0(\sigma) = \mu(\sigma^2\nu_0) \begin{cases} < 0\,, & \sigma > \sigma_{cyl}\,, \\ > 0\,, & \sigma < \sigma_{cyl}\,. \end{cases}$$

Hence, the assertion follows from [**54**, Theorem 9.1.2, Theorem 9.1.3]. □

Second, we transfer the (in-)stability of the cylinder to the stationary solutions u^λ_{cyl} going through the cylinder. To transfer the instability result, we require that $\mu_0(\sigma)$ is algebraically simple:

Lemma 5.17 *The eigenvalue $\mu_0(\sigma)$ of $DF(0) + \lambda_{cyl}Dg(0)$ is algebraically simple in the sense of* [**54**, Definition A.2.7].

Proof. For simplicity, we put $A := DF(0) + \lambda_{cyl}Dg(0)$ and write μ_j for the j-th eigenvalue $\mu_j(\sigma)$ of A throughout this proof. Since μ_0 has geometric multiplicity 1 due to Lemma 5.13, it remains to check that μ_0 is semi-simple. Because $W^2_{q,D}(-1,1)$ is compactly embedded in $L_q(-1,1)$, the operator A has a compact resolvent and [**21**, 1.19 Corollary] ensures that μ_0 is a pole of the resolvent of A. Consequently, [**54**, Remark A.2.4] shows that μ_0 is semi-simple if and only if

$$\ker(\mu_0 - A)^2 = \ker(\mu_0 - A) = \mathbb{R}\cdot\varphi_0\,,$$

where φ_0 is the first eigenvalue of the one-dimensional Dirichlet-Laplacian, see Lemma 5.7. Now let $f \in \ker(\mu_0 - A)^2$ and $C \in \mathbb{R}$ with $(\mu_0 - A)f = C\,\varphi_0$. It remains to show that $C = 0$: To this end, we write

$$f = \sum_{j=0}^\infty a_j\,\varphi_j\,, \qquad a_j \in \mathbb{R}\,,$$

with convergence in $W_2^2(-1,1)$. Moreover, we recall from Remark 5.4 that $\mu_0 - A$ can be considered as an operator from $W_{2,D}^2(-1,1)$ to $L_2(-1,1)$. Consequently, we find

$$(\mu_0 - A)f = \sum_{j=0}^{\infty} (\mu_0 - \mu_j)\, a_j \varphi_j$$

in $L_2(-1,1)$. Since Fourier series are unique, it follows that $a_j = 0$ for $j > 0$ and $C = (\mu_0 - \mu_0)a_0 = 0$. We conclude that μ_0 is a semi-simple eigenvalue of A and hence even a simple one. □

Proof of Theorem 5.15. For $\sigma \in (0,\infty)$ with $\sigma^2 \neq \dfrac{4s_0}{\pi^2(j+1)^2}$ for $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\|DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda) - DF(0) - \lambda_{cyl} Dg(0)\|_{\mathcal{L}(W_{q,D}^2, L_q)} \\
&\qquad \le \|DF(u_{cyl}^\lambda) - DF(0)\|_{\mathcal{L}(W_{q,D}^2, L_q)} + \lambda \|Dg(u_{cyl}^\lambda) - Dg(0)\|_{\mathcal{L}(W_{q,D}^2, L_q)} \\
&\qquad\quad + |\lambda - \lambda_{cyl}|\, \|Dg(0)\|_{\mathcal{L}(W_{q,D}^2, L_q)} \to 0\,,
\end{aligned}$$

as $\lambda \to \lambda_{cyl}$ by Theorem 5.14. Since the linearized operator $-\big(DF(0) + \lambda_{cyl} Dg(0)\big)$ belongs to $\mathcal{H}\big(W_{q,D}^2(-1,1), L_q(-1,1)\big)$, we deduce from [**5**, Theorem I.1.3.1 (i)] the existence of $\delta > 0$ such that

$$-\big(DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda)\big) \in \mathcal{H}\big(W_{q,D}^2(-1,1), L_q(-1,1)\big)\,, \qquad \lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)\,.$$

We now investigate the stability of $u_{cyl}^{\lambda_{cyl}}$ for $\sigma < \sigma_{cyl}$ and $\sigma > \sigma_{cyl}$ separately:

(i) *Instability for* $\sigma < \sigma_{cyl}$: In this case, we know that the first eigenvalue $\mu_0(\sigma)$ of the operator $DF(0) + \lambda_{cyl} Dg(0)$ is positive. Because it is also isolated and algebraically simple by Lemma 5.17, the perturbation result [**54**, Proposition A.3.2] for such eigenvalues allows to make $\delta > 0$ smaller such that $DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda)$ also has an eigenvalue with positive real part for $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$. Moreover, since the embedding $W_{q,D}^2(-1,1) \hookrightarrow L_q(-1,1)$ is compact, the spectrum of $DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda)$ consists only of eigenvalues with no finite accumulation point, see [**39**, Theorem 6.29]. Thus, there is a constant $C > 0$ such that the strip $\big\{\mu \in \mathbb{C} \,\big|\, 0 < \mathrm{Re}\mu < C\big\}$ is contained in the resolvent set of $DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda)$. Applying now [**54**, Theorem 9.1.3] to (5.31) shows the instability of u_{cyl}^λ for $\sigma < \sigma_{cyl}$.

(ii) *Stability for* $\sigma > \sigma_{cyl}$: Since the spectral bound of $DF(0) + \lambda_{cyl} Dg(0)$ is negative due to the choice $\sigma > \sigma_{cyl}$, it follows from [**5**, Corollary I.1.4.3] that we may take $\delta > 0$ so small that also $DF(u_{cyl}^\lambda) + \lambda Dg(u_{cyl}^\lambda)$ has a negative spectral bound for $\lambda \in (\lambda_{cyl} - \delta, \lambda_{cyl} + \delta)$. Hence, u_{cyl}^λ is exponentially asymptotically stable by [**54**, Theorem 9.1.2].

□

Note that Theorem 5.15 is in accordance with the stability analysis of the cylindrical solution for the small aspect ratio model in [**58**]. However, in the models [**23, 24**] no analogue steady state seems to exist.

Next, we derive an approximate upper bound for σ_{cyl}.

Remark 5.18 In Figure 5.1, a numerical plot of the eigencurve profile $[s \mapsto \mu(s)]$ is shown. We observe that the unique zero s_0 of the plot satisfies $s_0 \leq 4.2$. This indicates the following upper bound

$$\sigma_{cyl} = \frac{2\sqrt{s_0}}{\pi} \leq \frac{2\sqrt{4.2}}{\pi} \approx 1.3\,.$$

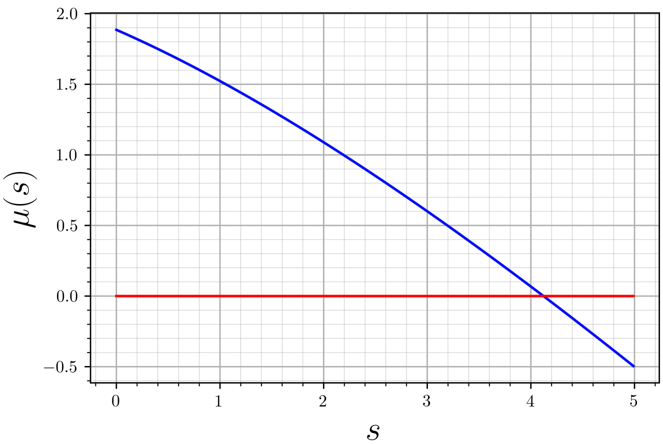

FIGURE 5.1. Numerical plot of the eigencurve profile $[s \mapsto \mu(s)]$ (blue) together with the constant $[s \mapsto 0]$ (red).

We want to explain an important physical implication of Remark 5.18. As preparation, we first recall the meaning of the parameter σ, and then have a closer look at the soap film bridge in absence of the electrostatic force.

The parameter σ gives the ratio between the radius of the metal rings between which the soap film is spanned divided by half of their distance. A smaller σ indicates that the metal rings are pulled farther apart. Without electrostatic force, we recall that the dynamics of the soap film are modelled by rotationally symmetric mean curvature flow, i.e. the film deflection satisfies (1.20)-(1.22) with $\lambda = 0$. Moreover, there exists the critical ratio $\sigma_{crit} \approx 1.5$ below which no stationary solutions to (1.20)-(1.22) with $\lambda = 0$ exist, see Chapter 4. Further, for $\sigma < \sigma_{crit}$ and arbitrary initial shape of the film, there exists no global solution for the film's dynamics at all. This follows from [**19**] and the parabolic comparison principle [**51**, Theorem 9.7][1]. After this overview we can explain the stabilizing effect of the electrostatic force indicated by Remark 5.18.

[1]Consider the rotationally symmetric mean curvature flow with Dirichlet boundary conditions, that is (1.20)-(1.22) with $\lambda = 0$, and arbitrary initial value u_0. First, one can compare a solution to this problem with a shrinking cylinder (depending on u_0 this might not be necessary) and subsequently with the solution to the flow with Dirichlet boundary conditions and cylinder as initial value. The latter pinches-off, that is touches itself, in finite time by [**19**].

Remark 5.19 Since $\sigma_{crit} \approx 1.5$ and $\sigma_{cyl} \approx 1.3$, at least numerically, we have $\sigma_{crit} > \sigma_{cyl}$. Here, the stability of the cylinder and also that of u_{cyl}^{λ} switches at σ_{cyl}. Hence, for $\sigma \in (\sigma_{cyl}, \sigma_{crit})$ and suitably scaled electrostatic force we find stability of the cylinder by Theorem 5.15 – in particular many solutions to (1.20)-(1.22) with λ close to λ_{cyl} do exist globally in time – while in absence of the electrostatic force, i.e. for $\lambda = 0$ in (1.20)-(1.22), all solutions cease to exist after a finite time. In other words, the electrostatic force might be used to avoid self-touching (or spontaneous breaking) of the soap film bridge, and hence to stabilize the film in a range where the two metal rings are pulled farther apart than the critical ratio σ_{crit}. This effect is observed also for the small aspect ratio model in [**59**].

Finally, we discuss an analytic estimate of σ_{cyl}:

Remark 5.20 Recalling that $[s \mapsto \mu'(s)]$ is a decreasing function, we find

$$\mu(s) = \mu(0) + \int_0^s \mu'(\tilde{s})\, \mathrm{d}\tilde{s} \le \mu(0) + s\mu'(0)\,, \qquad s \ge 0\,,$$

with $\mu(0) > 0$ and $\mu'(0) < 0$ explicitly given by Lemma 5.11. Consequently,

$$s > \frac{\mu(0)}{-\mu'(0)}$$

is a sufficient condition for $\mu(s) < 0$. In particular, the unique zero s_0 of $[s \mapsto \mu(s)]$ satisfies

$$s_0 \le \frac{\mu(0)}{-\mu'(0)}\,,$$

and we find

$$\sigma_{cyl} = \frac{2\sqrt{s_0}}{\pi} \le \frac{2}{\pi}\sqrt{\frac{\mu(0)}{-\mu'(0)}}\,.$$

Unfortunately, this upper bound turns out to be slightly bigger than σ_{crit}. Hence, it is not good enough to display the stabilizing effect of the electrostatic force.

5.5. Direction of Deflection

For the stable range $\sigma > \sigma_{cyl}$, we show that the stationary solutions u_{cyl}^{λ}, stemming from the cylinder $u = 0$, are deflected monotonically outwards with respect to λ:

Theorem 5.21 *For $\sigma > \sigma_{cyl}$, there exists $\delta > 0$ such that*

$$u_{cyl}^{\overline{\lambda}}(z) < u_{cyl}^{\lambda}(z)\,, \qquad \lambda_{cyl} - \delta < \overline{\lambda} < \lambda < \lambda_{cyl} + \delta\,, \quad z \in (-1,1)\,.$$

This reflects the physically expected behaviour that increasing the control parameter λ, i.e. increasing the impact of the electrostatic force, pulls the film farther outwards. For the simpler small aspect ratio model, formal asymptotic analysis has been used in [**58**] to establish a similar result.

Our proof is based on the linear approximation

$$\begin{aligned} u_{cyl}^{\lambda} &= u_{cyl}^{\lambda_{cyl}} + (\lambda - \lambda_{cyl})\,\partial_\lambda u_{cyl}^{\lambda_{cyl}} + o\big(\lambda - \lambda_{cyl}\big) \\ &= (\lambda - \lambda_{cyl})\,\partial_\lambda u_{cyl}^{\lambda_{cyl}} + o(\lambda - \lambda_{cyl})\,, \qquad \lambda \to \lambda_{cyl} \end{aligned} \tag{5.32}$$

with

$$\begin{aligned} \partial_\lambda u_{cyl}^{\lambda_{cyl}} &= -\big[DF(0) + \lambda_{cyl} Dg(0)\big]^{-1} g(0) \\ &= -\frac{1}{\ln(2)^2}\big[DF(0) + \lambda_{cyl} Dg(0)\big]^{-1}\,\mathbb{1} \end{aligned} \tag{5.33}$$

in $W^2_{q,D}(-1,1)$. Here, we inserted $g(0) = \ln(2)^{-2}$ from (5.5) and used the fact that $[\lambda \mapsto u_{cyl}^{\lambda}]$ was constructed via the implicit function theorem. We now ask for positivity of (5.33). Recall that the same ansatz has been used in Chapter 4 to analyse in which direction the stable catenoid is deflected. In Chapter 4, the question of positivity was answered by a maximum principle. However, the linearized operator under consideration includes the complicated non-local part $+\lambda_{cyl} Dg(0)$, which most likely precludes the use of maximum principles. Instead, we expand $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$ in a Fourier series and show positivity of this series. An essential ingredient for the positivity proof are estimates on the eigencurve profile $[s \mapsto \mu(s)]$.

We start with computing the Fourier series of the function $\mathbb{1} := [z \mapsto 1]$ occurring on the right-hand side of (5.33).

Lemma 5.22 *The Fourier series of* $\mathbb{1}$ *with respect to the* L_2*-eigenbasis of the one-dimensional Dirichlet-Laplacian is*

$$1 = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \cos\left(\frac{(2j+1)\pi}{2} z\right), \qquad z \in (-1,1)\,,$$

with (unconditional) convergence in $L_2(-1,1)$.

Proof. We write

$$\mathbb{1} = \sum_{i=0}^{\infty} (\mathbb{1}|\varphi_i)_{L_2}\varphi_i\,,$$

where we recall that the eigenfunctions $(\varphi_i)_{i\in\mathbb{N}}$ of the one-dimensional Dirichlet-Laplacian are given by suitable scaled sine- and cosine-functions. Since the eigenfunctions φ_{2j+1} are sine-functions, they are odd and we clearly have $(\mathbb{1}|\varphi_{2j+1})_{L_2} = 0$. Moreover, we find

$$\begin{aligned} (\mathbb{1}|\varphi_{2j})_{L_2} &= \int_{-1}^{1} \cos\left(\frac{(2j+1)\pi}{2} z\right) \mathrm{d}z \\ &= \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi}{2}\right) = \frac{4}{\pi}\frac{(-1)^j}{(2j+1)}\,, \qquad j \in \mathbb{N}\,. \end{aligned}$$

□

Next, we compute the Fourier series of $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$. This is possible because $DF(0) + \lambda_{cyl} Dg(0)$ can be extended to a bounded linear operator from $W^2_{2,D}(-1,1)$ to $L_2(-1,1)$, see Remark 5.4.

Lemma 5.23 *Let $\sigma > \sigma_{cyl}$. Then, the Fourier series of $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$ is*

$$\partial_\lambda u_{cyl}^{\lambda_{cyl}}(z) = \frac{4}{\pi \ln(2)^2} \sum_{j=0}^{\infty} a_j \cos\left(\frac{(2j+1)\pi}{2} z \right) \tag{5.34}$$

with coefficients

$$a_j := \frac{(-1)^j}{(2j+1)\big(-\mu_{2j}(\sigma)\big)}, \qquad j \in \mathbb{N},$$

and (unconditional) convergence in $C^1\big([-1,1]\big)$. Here, $\mu_{2j}(\sigma)$ denotes the $2j$-th eigenvalue of $DF(0) + \lambda_{cyl} Dg(0)$.

Proof. We write

$$\partial_\lambda u_{cyl}^{\lambda_{cyl}} = \sum_{j=0}^{\infty} b_j \varphi_j$$

with suitable $b_j \in \mathbb{R}$ and (unconditional) convergence in $W^2_2(-1,1) \hookrightarrow C^1\big([-1,1]\big)$. We have convergence in $W^2_2(-1,1)$ since $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$ belongs to $W^2_{2,D}(-1,1)$. We write

$$-\big[DF(0) + \lambda_{cyl} Dg(0)\big] \partial_\lambda u_{cyl}^{\lambda_{cyl}} = \sum_{j=0}^{\infty} -\mu_j(\sigma) b_j \varphi_j$$

with $\big(\mu_j(\sigma)\big)_j$ denoting the eigenvalues of $DF(0) + \lambda_{cyl} Dg(0)$, which are all strictly smaller than zero. Based on (5.33) and Lemma 5.22, a comparison of Fourier coefficients in $L_2(-1,1)$ yields the assertion. ☐

Since each cosine in the series (5.34) is scaled by an odd multiple of $\pi/2$, each of its partial sums is an odd cosine sum in the sense of Appendix E. A sufficient condition for such a sum to be positive is presented in Lemma E.2. More precisely, the condition yields that if

$$C_1 := a_0 - \sum_{j=1}^{\infty} (2j+1)|a_j| > 0$$

with coefficients a_j from Lemma 5.23, then

$$\partial_\lambda u_{cyl}^{\lambda_{cyl}}(z) \geq C_1 \frac{4}{\pi \ln(2)^2} \cos\left(\frac{\pi}{2} z \right), \qquad z \in (-1,1). \tag{5.35}$$

Inserting the a_j's, we are left with checking convergence and sign of

$$C_1 = \frac{1}{(-\mu_0(\sigma))} - \sum_{j=1}^{\infty} \frac{1}{(-\mu_{2j}(\sigma))}.$$

We recall that the eigenvalues $\mu_{2j}(\sigma)$ of $DF(0) + \lambda_{cyl} Dg(0)$ can be written as

$$\mu_{2j}(\sigma) = \mu(\sigma^2 \nu_{2j})$$

with eigencurve profile $[s \mapsto \mu(s)]$ and the eigenvalues of the Dirichlet-Laplacian

$$\nu_{2j} = \frac{(2j+1)^2}{4} \pi^2 , \qquad j \in \mathbb{N} .$$

Upper and lower bounds for the eigenvalues $\mu_{2j}(\sigma)$ are derived from properties of the eigencurve profile $[s \mapsto \mu(s)]$:

Lemma 5.24 *Let $\sigma > \sigma_{cyl}$. Then, the eigenvalues of $DF(0) + \lambda_{cyl} Dg(0)$ satisfy*

$$-\sigma^2 \frac{(2j+1)^2}{4} \pi^2 < \mu_{2j}(\sigma) < -\frac{3}{10} \frac{\pi^2}{4} \sigma^2 \left((2j+1)^2 - 1 \right) , \qquad j \in \mathbb{N}.$$

Proof. **(i)** *We derive the lower bound*: For $s \in [0, \infty)$, we have

$$\mu(s) = -s + 3 + 2\partial_r h_s(1)$$

with h_s being the solution to (5.21). Because $[s \mapsto \partial_r h_s(1)]$ is an increasing function by Remark 5.12 and $\partial_r h_0(1) = -2 + 1/\ln(2)$, see (5.26), we deduce that

$$\mu(s) \geq -s + 3 + 2\partial_r h_0(1) = -s + \frac{2}{\ln(2)} - 1 > -s .$$

Inserting $s = \sigma^2 \nu_{2j}$ results in the estimate from below.
(ii) *We derive the upper bound*: Since $\sigma > \sigma_{cyl}$, we find $\sigma^2 \nu_{2j} > s_0$ with s_0 being the unique zero of the eigencurve profile $[s \mapsto \mu(s)]$. Because $[s \mapsto \mu'(s)]$ is decreasing on $[0, \infty)$, see the proof of Proposition 5.8, with $\mu'(0) < -3/10$ by Lemma 5.11, it follows that

$$\mu_{2j}(\sigma) = \mu(\sigma^2 \nu_{2j}) = \int_{s_0}^{\sigma^2 \nu_{2j}} \mu'(\tilde{s}) \, \mathrm{d}\tilde{s} \leq \frac{-3}{10} (\sigma^2 \nu_{2j} - s_0) \leq \frac{-3}{10} \sigma^2 (\nu_{2j} - \nu_0) .$$

Inserting the expressions for ν_{2j} and ν_0 finishes off the proof. □

Now we check the sign of C_1 to establish (5.35).

Proposition 5.25 *For $\sigma > \sigma_{cyl}$, equation (5.35) holds. In particular, $\partial_\lambda u_{cyl}^{\lambda_{cyl}}(z) > 0$ for each $z \in (-1, 1)$, and $\partial_z[\partial_\lambda u_{cyl}^{\lambda_{cyl}}](-1) > 0$ as well as $\partial_z[\partial_\lambda u_{cyl}^{\lambda_{cyl}}](1) < 0$.*

Proof. Due to Lemma 5.24, we have

$$0 < \frac{1}{\big(-\mu_{2j}(\sigma) \big)} \leq \frac{10}{3} \frac{4}{\pi^2 \sigma^2} \frac{1}{(2j+1)^2 - 1} , \qquad j \in \mathbb{N} ,$$

where the infinite sum over the right-hand side converges. Consequently, the constant C_1 is finite. Moreover, a second application of Lemma 5.24 ensures that

$$\begin{aligned} C_1 &= \frac{1}{(-\mu_0(\sigma))} - \sum_{j=1}^{\infty} \frac{1}{(-\mu_{2j}(\sigma))} \\ &\geq \frac{4}{\pi^2\sigma^2}\left(1 - \frac{10}{3}\sum_{j=1}^{\infty}\frac{1}{\big((2j+1)^2-1\big)}\right) \\ &= \frac{4}{\pi^2\sigma^2}\left(1 - \frac{10}{3}\cdot\frac{1}{4}\right) = \frac{2}{3\pi^2\sigma^2} > 0\,, \end{aligned}$$

where the relation

$$\sum_{j=1}^{\infty}\frac{1}{\big((2j+1)^2-1\big)} = \lim_{n\to\infty}\frac{1}{4}\cdot\frac{n}{n+1} = \frac{1}{4}$$

follows by induction. Now an application of Lemma E.2 finishes off the proof. □

Finally, we prove Theorem 5.21, the main result of this section. It states that the local branch of stationary solutions $[\lambda \mapsto u_{cyl}^{\lambda}]$ going through the stable cylinder is deflected monotonically outwards if the applied voltage is increased.

Proof of Theorem 5.21. Recall that our ansatz in (5.32)-(5.33) is the same as in the analysis of the direction of deflection for the outer catenoid. Only for the positivity of $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$, we required a new argument presented in Proposition 5.25. Hence, after having established positivity of $\partial_\lambda u_{cyl}^{\lambda_{cyl}}$ in Proposition 5.25, the remaining steps of the proof are similar to the analysis of the outer catenoid, see the proof of Theorem 4.11. □

Remark 5.26 Let $\sigma < \sigma_{cyl}$ so that the cylinder is unstable. For the small aspect ratio model, also the direction of deflection of the unstable cylinder has been formally analysed in [**58**], see in particular [**58**, Equation (2.67)]. Translated to the full free boundary problem, one would hope to find the following: If the total number of strictly positive eigenvalues $\mu_j(\sigma)$ of $DF(0) + \lambda_{cyl}Dg(0)$ is even, the unstable cylinder is deflected outwards, while an odd number would imply a deflection directed inwards.

CHAPTER 6

Non-Existence of Global Solutions for Large Voltages

We study the dynamical behaviour of the film deflection u for large applied voltages, which corresponds to an increase of the control parameter λ. For such large λ, a dominance of the electrostatic force is expected and, as a consequence, non-existence of global solutions. The goal of the current chapter is to prove non-existence of global solutions for λ above a critical value λ_{crit} and initial film deflection u_0 lying above the inner catenoid u_{in}. The question of non-existence of global solutions for large λ in variants of MEMS models has been previously studied in [**22, 23, 47, 52**].

Before we state the precise theorem and outline its proof, we recall the equations for the dynamics: The dynamical film deflection u solves the parabolic equation

$$\begin{cases} \partial_t u - \sigma\, \partial_z \mathrm{arctan}(\sigma \partial_z u) & = -\dfrac{1}{u+1} + \lambda\, g(u)\,, \\ \qquad\qquad u(t,\pm 1) & = 0\,, \qquad -1 < u < 1\,, \\ \qquad\qquad u(0,z) & = u_0\,, \qquad z \in (-1,1)\,, \end{cases} \tag{6.1}$$

with initial value $u_0 \in W^2_{q,D}(-1,1)$ satisfying $-1 < u_0 < 1$, and electrostatic force

$$g(u) = (1+\sigma^2(\partial_z u)^2)^{3/2} |\partial_r \psi_u(z, u+1)|^2\,. \tag{6.2}$$

Moreover, the electrostatic potential ψ_u is given by

$$\begin{cases} \dfrac{1}{r}\partial_r\left(r \partial_r \psi_u\right) + \sigma^2 \partial_z^2 \psi_u & = 0 \quad \text{in} \quad \Omega(u)\,, \\ \qquad\qquad \psi_u & = h_u \quad \text{on} \quad \partial\Omega(u)\,, \end{cases} \tag{6.3}$$

where

$$\Omega(u) = \big\{ (z,r) \in (-1,1) \times (0,2) \,|\, u(z)+1 < r < 2 \,\big\}$$

with suppressed t-dependency and

$$h_u(z,r) = \frac{\ln\left(\dfrac{r}{u(z)+1}\right)}{\ln\left(\dfrac{2}{u(z)+1}\right)}\,. \tag{6.4}$$

We also recall from Chapter 4 that, in the range $\sigma > \sigma_{crit}$, there exist two stationary solutions to (6.1)-(6.4) with $\lambda = 0$, one being the inner catenoid u_{in}. Now we state the main result precisely:

Theorem 6.1 *Let $\sigma \geq \sigma_{crit}$*[1]*. There exists $\lambda_{crit}(\sigma) > 0$ such that for each $\lambda > \lambda_{crit}$ and each initial condition $u_0 \geq u_{in}$, the corresponding solution (u, ψ_u) to* (6.1)-(6.4) *has a finite maximal time of existence $T_{max}(u_0) < \infty$.*

For $t \nearrow T_{max}$, the film touches the rigid outer cylinder, as expected for a dominant electrostatic force, or $\|u(t)\|_{W_q^2(-1,1)}$ explodes, for example due to a spontaneous rupture of the film. This asymptotic behaviour of the film is a consequence of Theorem 3.16 and the fact that $u_0 \geq u_{in}$ will guarantee $u(t) \geq u_{in}$ as we shall see below.

To outline the proof of Theorem 6.1, we fix $\sigma \geq \sigma_{crit}$ and a solution (u, ψ_u) to (6.1)-(6.4) with initial value $u_0 \geq u_{in}$. We consider the functional

$$E(t) := -\int_{-1}^{1} \ln\big(u(t,z)+1\big)\,\mathrm{d}z\,, \qquad t \in [0, T_{max})\,, \qquad T_{max} = T_{max}(u_0)\,,$$

which is bounded from below

$$E(t) \geq -2\ln(2)\,, \qquad t \in [0, T_{max})\,,$$

while we aim at showing

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq -C < 0\,, \qquad t \in [0, T_{max})\,, \tag{6.5}$$

for λ above a critical threshold value. Obviously, this is only possible if $T_{max} < \infty$.

A related result is contained in [**22**], in which non-existence of global solutions to a quasilinear MEMS model for large λ is shown by deriving a more involved inequality for the functional $\tilde{E}(t) := \int_{-1}^{1} u(t,z)\,\mathrm{d}z$. Note that in the model (6.1)-(6.4) the right-hand side of (6.1) has two terms of opposite signs. The term $-1/(u+1)$ will be controlled by the restriction $u_0 \geq u_{in}$, while the positivity of the electrostatic force $+\lambda g(u)$ is accounted for by using the logarithm in the definition of E. Finally, the electrostatic force (6.2) contains an additional factor $\big(1+\sigma^2(\partial_z u)^2\big)^{1/2}$ stemming from the modelling assumption that the film always moves in normal direction. This factor makes it possible to derive the simpler functional inequality (6.5) for E.

The inequality (6.5) will follow from several auxiliary results. Similarly to [**43**], an important technical tool in some of the proofs will be the approximation of u by its smoother time averages. We recall their definition and smoothing properties in the next remark.

Remark 6.2 Let $T \in (0, T_{max})$ and $\delta \in (0, T_{max} - T)$. As u is in $C\big([0,T_{max}), W_q^2(-1,1)\big)$ and $C^1\big([0,T_{max}), L_q(-1,1)\big)$, its Steklov average

$$u_\delta(t,z) := \frac{1}{\delta}\int_t^{t+\delta} u(s,z)\,\mathrm{d}s\,, \qquad z \in (-1,1)\,,$$

belongs to $C^1\big([0,T], W_q^2(-1,1)\big) \subset C^1\big([0,T]\times[-1,1],\mathbb{R}\big)$ with

$$u_\delta \longrightarrow u \quad \text{in} \quad C\big([0,T], W_q^2(-1,1)\big)\,. \tag{6.6}$$

[1]For $\sigma = \sigma_{crit}$, we slightly abuse notation and denote the critical catenoid u_{cat} by u_{in} as well.

Moreover, we have

$$\partial_t u_\delta(t,z) = \frac{u(\,\cdot\, + \delta) - u(\,\cdot\,)}{\delta} \longrightarrow \partial_t u \quad \text{in} \quad C\big([0,T], L_q(-1,1)\big) \tag{6.7}$$

for $\delta \to 0$.

As a first auxiliary result, we compute the derivative of E.

Lemma 6.3 *The functional E belongs to $C^1\big([0,T_{max}),\mathbb{R}\big)$ with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\int_{-1}^{1} \frac{\partial_t u(t,z)}{u(t,z)+1}\,\mathrm{d}z\,. \tag{6.8}$$

Proof. Let $T \in (0,T_{max})$. The Steklov average u_δ of u satisfies

$$u_\delta(t,z) \geq \min_{(t,z)\in[0,T]\times[-1,1]} u(t,z) > -1\,,$$

because u is continuous and always stays above -1. Consequently,

$$E_\delta(t) := -\int_{-1}^{1} \ln\big(u_\delta(t)+1\big)\,\mathrm{d}z\,, \qquad t \in [0,T]\,,$$

is continuously differentiable with

$$E_\delta(t) - E_\delta(0) = -\int_0^t \int_{-1}^{1} \frac{\partial_t u_\delta(s,z)}{u_\delta(s,z)+1}\,\mathrm{d}z\;\mathrm{d}s\,.$$

In view of (6.6)-(6.7), passing to the limit $\delta \to 0$ yields

$$E(t) - E(0) = -\int_0^t \int_{-1}^{1} \frac{\partial_t u(s,z)}{u(s,z)+1}\,\mathrm{d}z\;\mathrm{d}s\,,$$

and hence $E \in C^1\big([0,T_{max}),\mathbb{R}\big)$ with derivative given by (6.8). $\square$

Next, we show that in the specific situation $\sigma \geq \sigma_{crit}$ under consideration, the parabolic comparison principle is applicable to (6.1)-(6.4). More precisely, since the non-local electrostatic force $g(u)$ is always positive, one can show that $u_0 \geq u_{in}$ implies $u(t) \geq u_{in}$ for all times $t \in [0,T_{max})$. Based on the book [**51**], we include a proof here. We point out that in the situation $\sigma < \sigma_{crit}$, where no steady states for $\lambda = 0$ exist, the comparison principle seems not applicable.

Proposition 6.4 *If $u_0 \geq u_{in}$, then $u(t) \geq u_{in}$ for all $t \in [0,T_{max})$.*

Proof. We fix $T \in (0,T_{max})$ and introduce, for better readability, the notation $v := u_{in}$. Since this is a steady state of (6.1) for $\lambda = 0$, the difference $w := v - u \in C\big([0,T], W_q^2(-1,1)\big) \cap C^1\big([0,T], L_q(-1,1)\big)$ satisfies

$$\partial_t w + \sigma\Big(\partial_z \arctan(\sigma\partial_z u) - \partial_z \arctan(\sigma\partial_z v)\Big) + \left(\frac{1}{v+1} - \frac{1}{u+1}\right) = -\lambda g(u) \leq 0\,.$$

Testing this equation for $t \in [0,T]$ against $w_+(t) \in W^1_{q,D}(-1,1)$, we find

$$\begin{aligned} 0 \geq & \int_{-1}^1 \partial_t w(t,z)\, w_+(t,z)\,\mathrm{d}z \\ & - \sigma \int_{-1}^1 \Big(\arctan\big(\sigma\partial_z u(t,z)\big) - \arctan\big(\sigma\partial_z v(t,z)\big)\Big)\,\partial_z w_+(t,z)\,\mathrm{d}z \\ & + \int_{-1}^1 \left(\frac{1}{v(t,z)+1} - \frac{1}{u(t,z)+1}\right) w_+(t,z)\,\mathrm{d}z =: I + II + III\,. \end{aligned} \tag{6.9}$$

We treat each term separately starting with the easiest one:
For III: We note that

$$III = \int_{-1}^1 \frac{u(t,z)-v(t,z)}{\big(v(t,z)+1\big)\big(u(t,z)+1\big)}\, w_+(t,z)\,\mathrm{d}z = \int_{-1}^1 b(t,z)\, w_+(t,z)^2\,\mathrm{d}z$$

with

$$b(t,z) := -\frac{1}{\big(v(t,z)+1\big)\big(u(t,z)+1\big)}\,, \qquad |b(t,z)| \leq C\,, \qquad (t,z) \in [0,T]\times[-1,1]\,,$$

for some constant $C > 0$.
For II: We compute that

$$\begin{aligned} & - \sigma\Big(\arctan\big(\sigma\partial_z u(t,z)\big) - \arctan\big(\sigma\partial_z v(t,z)\big)\Big) \\ & = \sigma\left(\int_0^1 \partial_z \arctan\big(s\sigma\partial_z v(t,z) + (1-s)\sigma\partial_z u(t,z)\big)\,\mathrm{d}s\right)\big(\partial_z v(t,z) - \partial_z u(t,z)\big) \\ & = \sigma\left(\int_0^1 \frac{1}{1+\sigma^2\big(s\partial_z v(t,z) + (1-s)\partial_z u(t,z)\big)^2}\,\mathrm{d}s\right)\partial_z w(t,z) \\ & =: a(t,z)\partial_z w(t,z)\,, \end{aligned}$$

and consequently

$$II = \int_{-1}^1 a(t,z)\partial_z w_+(t,z)^2\,\mathrm{d}z \geq 0\,.$$

For I: We use an approximation argument to show that

$$\int_0^t I\,\mathrm{d}\tau = \int_0^t \int_{-1}^1 \partial_t w\, w_+\,\mathrm{d}z\,\mathrm{d}\tau = \frac{1}{2}\int_{-1}^1 w_+(t)^2\,\mathrm{d}z\,, \qquad t \in [0,T]\,. \tag{6.10}$$

Let $w_\delta = u_{in} - u_\delta$ be the Steklov average of w. Then, $w_\delta \in C^1([0,T]\times[-1,1],\mathbb{R})$ with

$$\begin{aligned} \int_0^t \int_{-1}^1 \partial_t w_\delta\,(w_\delta)_+\,\mathrm{d}z\,\mathrm{d}\tau &= \int_{-1}^1 \int_0^t \partial_\tau\left(\int_0^{w_\delta(\tau,z)} s_+\,\mathrm{d}s\right)\mathrm{d}\tau\,\mathrm{d}z \\ &= \frac{1}{2}\int_{-1}^1 \left(\big(w_\delta(t)_+\big)^2 - \big(w_\delta(0)_+\big)^2\right)\mathrm{d}z\,. \end{aligned}$$

In view of (6.6), (6.7) as well as $w(0) = u_{in} - u_0 \leq 0$, the limit $\delta \searrow 0$ yields (6.10). Integrating (6.9) from 0 to t and inserting the results for I to III gives

$$0 \geq \frac{1}{2} \int_{-1}^{1} w_+(t)^2 \, \mathrm{d}z - C \int_0^t \int_{-1}^{1} w_+^2 \, \mathrm{d}z \, \mathrm{d}\tau \,,$$

or, equivalently,

$$2\,C \int_0^t \|w_+(\tau)\|^2_{L_2(-1,1)} \, \mathrm{d}\tau \geq \|w_+(t)\|^2_{L_2(-1,1)} \,, \qquad t \in [0,T] \,.$$

Since $w_+ \in C\big([0,T], L_q(-1,1)\big)$, also $\Big[t \mapsto \|w_+(t)\|^2_{L_2(-1,1)}\Big]$ is continuous on $[0,T]$, and Gronwall's lemma implies

$$\|w_+(t)\|^2_{L_2(-1,1)} = 0 \,, \qquad t \in [0,T] \,.$$

Because $w = u_{in} - u$ and $T < T_{max}$ was arbitrary, it follows that $u(t) \geq u_{in}$ for $t \in [0, T_{max})$ as claimed. □

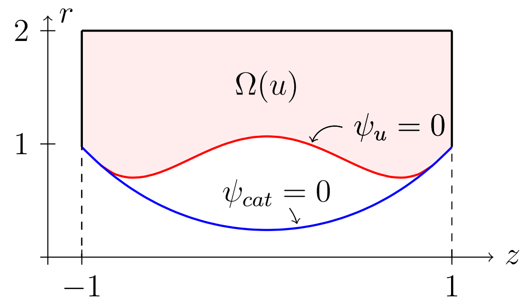

FIGURE 6.1. The situation in the proof of Proposition 6.5: The film deflection $u{+}1$ (red) lies above the inner catenoid $u_{in}{+}1$ (blue). Note that the electrostatic potentials ψ_u and ψ_{cat} coincide on the black boundary parts of $\Omega(u)$ and are positive there.

We want to derive suitable estimates for the right-hand side of (6.8), and, as the main step in this direction, we have to connect the electrostatic force with the L_1-norm of $\partial_z u$. The proof is based on Gauss's theorem and its idea is inspired by [**22**].

Proposition 6.5 *There exists a constant $C_1(\sigma) > 0$ (independent of t and u_0) such that*

$$\int_{-1}^{1} \big(1 + (\sigma \partial_z u)^2\big)^{3/2} |\partial_r \psi_u(z, u+1)|^2 \, \mathrm{d}z \geq \varepsilon \, C_1(\sigma) - \varepsilon^2 \int_{-1}^{1} \sqrt{1 + (\sigma \partial_z u)^2} \, \mathrm{d}z$$

for each $t \in [0, T_{max})$ and each $\varepsilon > 0$.

Proof. In the following, we fix $t \in [0, T_{max})$ and use the abbreviations $\psi_u := \psi_{u(t)}$ (as usual) and $\psi_{cat} := \psi_{u_{in}}$. The situation is depicted in Figure 6.1. Since u always stays above the inner catenoid u_{in}, we can consider the function $f := \psi_u - \psi_{cat}$ in $\Omega(u)$.

On the boundary of $\Omega(u)$, this function satisfies

$$\begin{aligned} f &= 0 \quad \text{on} \quad \{\pm 1\} \times [1,2]\,, \\ f &= 0 \quad \text{on} \quad [-1,1] \times \{2\}\,, \\ f &\leq 0 \quad \text{on} \quad \mathrm{graph}(u+1)\,. \end{aligned}$$

Therefore, the maximum principle implies that f attains its maximum on the whole boundary parts $\{\pm 1\} \times [1,2]$ and $[-1,1] \times \{2\}$. Hence, the outer normal derivative of f satisfies

$$\partial_\nu f \geq 0 \quad \text{on} \quad \{\pm 1\} \times (1,2) \quad \text{and on} \quad (-1,1) \times \{2\}\,,$$

which is equivalent to

$$\partial_\nu \psi_u \geq \partial_\nu \psi_{cat} \quad \text{on} \quad \{\pm 1\} \times (1,2) \quad \text{and on} \quad (-1,1) \times \{2\}\,.$$

Since ψ_u solves

$$0 = \mathrm{div}\left(r \begin{pmatrix} \sigma^2 \partial_z \psi_u \\ \partial_r \psi_u \end{pmatrix} \right) \quad \text{in} \quad \Omega(u)\,, \tag{6.11}$$

we deduce from Gauss's theorem and $\partial_z \psi_u = -\partial_z u \, \partial_r \psi_u$ on $\mathrm{graph}(u+1)$ that

$$\begin{aligned} &\int_{-1}^{1} (u+1)\big(1 + (\sigma \partial_z u)^2\big) \partial_r \psi_u(z, u+1)\, \mathrm{d}z \\ &\quad = -\int_{\mathrm{graph}(u+1)} r \begin{pmatrix} \sigma^2 \partial_z \psi_u \\ \partial_r \psi_u \end{pmatrix} \cdot \nu \, \mathrm{d}o(z,r) \\ &\quad = -\int_1^2 \sigma^2 r\, \partial_z \psi_u(-1,r)\, \mathrm{d}r + \int_{-1}^{1} 2\, \partial_r \psi_u(z,2)\, \mathrm{d}z + \int_1^2 \sigma^2\, r\, \partial_z \psi_u(1,r)\, \mathrm{d}r \\ &\quad = \int_1^2 \sigma^2\, r\, \partial_\nu \psi_u(-1,r)\, \mathrm{d}r + \int_{-1}^{1} 2\, \partial_\nu \psi_u(z,2)\, \mathrm{d}z + \int_1^2 \sigma^2\, r\, \partial_\nu \psi_u(1,r)\, \mathrm{d}r \\ &\quad \geq \int_1^2 \sigma^2\, r\, \partial_\nu \psi_{cat}(-1,r)\, \mathrm{d}r + \int_{-1}^{1} 2\, \partial_\nu \psi_{cat}(z,2)\, \mathrm{d}z + \int_1^2 \sigma^2\, r\, \partial_\nu \psi_{cat}(1,r)\, \mathrm{d}r \\ &\quad = -\int_1^2 \sigma^2 r\, \partial_z \psi_{cat}(-1,r)\, \mathrm{d}r + \int_{-1}^{1} 2\, \partial_r \psi_{cat}(z,2)\, \mathrm{d}z + \int_1^2 \sigma^2\, r\, \partial_z \psi_{cat}(1,r)\, \mathrm{d}r \\ &\quad = -\int_{\mathrm{graph}(u_{in}+1)} r \begin{pmatrix} \sigma^2 \partial_z \psi_{cat} \\ \partial_r \psi_{cat} \end{pmatrix} \cdot \nu \, \mathrm{d}o(z,r) \\ &\quad = \int_{-1}^{1} (u_{in}+1)\big(1 + (\sigma \partial_z u_{in})^2\big) \partial_r \psi_{cat}(z, u_{in}+1)\, \mathrm{d}z =: C_1(\sigma)\,. \end{aligned} \tag{6.12}$$

In the last step, we have used that ψ_{cat} solves (6.11) in $\Omega(u_{in})$. Next, we show that $C_1(\sigma) > 0$: Because ψ_{cat} attains its minimum on the whole $\mathrm{graph}(u_{in}+1)$, it follows from Hopf's Lemma that $\partial_\nu \psi_{cat} < 0$, and hence

$$\begin{aligned} 0 &> \partial_z \psi_{cat}\big(z, u_{in}(z)+1\big) \partial_z u_{in}(z) - \partial_r \psi_{cat}\big(z, u_{in}(z)+1\big) \\ &= -\big(1 + \partial_z u_{in}(z)^2\big)\, \partial_r \psi_{cat}\big(z, u_{in}(z)+1\big)\,. \end{aligned}$$

Consequently, $\partial_r\psi_{cat} > 0$ on graph$(u_{in}+1)$ and

$$\begin{pmatrix}\sigma^2\partial_z\psi_{cat}\\ \partial_r\psi_{cat}\end{pmatrix}\cdot\nu = \begin{pmatrix}\sigma^2\partial_z\psi_{cat}\\ \partial_r\psi_{cat}\end{pmatrix}\cdot\frac{1}{\sqrt{1+\partial_z u_{in}^2}}\begin{pmatrix}\partial_z u_{in}\\ -1\end{pmatrix}$$
$$= \frac{-\big(1+(\sigma\partial_z u_{in})^2\big)}{\sqrt{1+\partial_z u_{in}^2}}\,\partial_r\psi_{cat}\big(z,u_{in}(z)+1\big) < 0\,,$$

which implies $C_1(\sigma) > 0$. Now we are ready to finish off the proof: A combination of (6.12) with $u+1\in(0,2)$ and the weighted Young's inequality gives

$$\frac{C_1(\sigma)}{2} \le \int_{-1}^{1}\big(1+(\sigma\partial_z u)^2\big)^{3/4+1/4}|\partial_r\psi_u(z,u+1)|\,\mathrm{d}z$$
$$\le \frac{1}{2\varepsilon}\int_{-1}^{1}\big(1+(\sigma\partial_z u)^2\big)^{3/2}|\partial_r\psi_u(z,u+1)|^2\,\mathrm{d}z + \frac{\varepsilon}{2}\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z$$

for $\varepsilon > 0$, and multiplying this inequality by 2ε yields

$$\int_{-1}^{1}\big(1+(\sigma\partial_z u)^2\big)^{3/2}|\partial_r\psi_u(z,u+1)|^2\,\mathrm{d}z \ge \varepsilon\, C_1(\sigma) - \varepsilon^2\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z$$

as claimed. □

Finally, the last auxiliary result compares the integral of $\arctan(\sigma\partial_z u)\,\sigma\partial_z u$ with the L_1-norm of $\partial_z u$:

Lemma 6.6 *For each $t\in[0,T_{max})$, the estimate*

$$\int_{-1}^{1}\arctan(\sigma\partial_z u)\,\sigma\partial_z u\,\mathrm{d}z \ge \frac{\pi}{4}\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z - \pi$$

holds.

Proof. We recall that

$$\arctan(x)x \ge 0\,, \qquad \arctan(1) = \frac{\pi}{4}\,, \qquad \sqrt{x^2+y^2} \le |x|+|y|$$

for $x,y\in\mathbb{R}$ and introduce the set

$$A := \Big\{z\in[-1,1]\,\Big|\,\big|\arctan(\sigma\partial_z u(z))\big| \ge \frac{\pi}{4}\Big\}\,.$$

Noting that $\sigma|\partial_z u| \le 1$ on A^c, we estimate

$$\int_{-1}^{1}\arctan(\sigma\partial_z u)\,\sigma\partial_z u\,\mathrm{d}z \ge \int_A \arctan(\sigma\partial_z u)\,\sigma\partial_z u\,\mathrm{d}z$$
$$\ge \frac{\pi}{4}\int_A \sigma|\partial_z u|\,\mathrm{d}z + \frac{\pi}{4}\int_{A^c}\sigma|\partial_z u|\,\mathrm{d}z - \frac{\pi}{2}$$
$$= \frac{\pi}{4}\int_{-1}^{1}\sigma|\partial_z u|\,\mathrm{d}z + \frac{\pi}{4}\int_{-1}^{1}1\,\mathrm{d}z - \pi$$
$$\ge \frac{\pi}{4}\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z - \pi\,.$$

□

Based on Lemma 6.3−Lemma 6.6, we can prove the main result of this chapter:

Proof of Theorem 6.1. Let $\lambda > 0$ and (u, ψ_u) be a solution to (6.1)-(6.4) with $u_0 \geq u_{in}$. We have to show that $T_{max} < \infty$. Since the functional

$$E(t) = -\int_{-1}^{1} \ln\big(u(t)+1\big)\,\mathrm{d}z\,, \quad t \in [0, T_{max})$$

is bounded from below by $-2\ln(2)$, it suffices to show

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) < -C < 0\,, \quad t \in [0, T_{max})\,,$$

for some $C > 0$ independent of t, to exclude the possibility of global existence. Introducing the constant

$$C_2(\sigma) := \frac{1}{\min_{z\in[-1,1]} u_{in} + 1} \in (0,\infty)\,,$$

we note that

$$\frac{1}{2} \leq \frac{1}{u+1} \leq C_2(\sigma)\,,$$

as u always stays above the inner catenoid u_{in} by Proposition 6.4. Using (6.8) and (6.1), we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= -\sigma\int_{-1}^{1} \partial_z\mathrm{arctan}(\sigma\partial_z u)\frac{1}{u+1}\,\mathrm{d}z + \int_{-1}^{1}\frac{1}{(u+1)^2}\,\mathrm{d}z \\
&\quad - \lambda\int_{-1}^{1}\frac{1}{u+1}\big(1+(\sigma\partial_z u)^2\big)^{3/2}|\partial_r\psi_u(z,u+1)|^2\,\mathrm{d}z \\
&\leq -\int_{-1}^{1}\frac{\arctan(\sigma\partial_z u)\,\sigma\partial_z u}{(u+1)^2}\,\mathrm{d}z - \left[\frac{\sigma\arctan(\sigma\partial_z u)}{u+1}\right]_{-1}^{1} + 2\,C_2(\sigma)^2 \\
&\quad - \frac{\lambda}{2}\int_{-1}^{1}\big(1+(\sigma\partial_z u)^2\big)^{3/2}|\partial_r\psi_u(z,u+1)|^2\,\mathrm{d}z \\
&\leq -\frac{1}{4}\int_{-1}^{1}\arctan(\sigma\partial_z u)\sigma\partial_z u\,\mathrm{d}z + \sigma\pi + 2\,C_2(\sigma)^2 \\
&\quad - \frac{\lambda}{2}\int_{-1}^{1}\big(1+(\sigma\partial_z u)^2\big)^{3/2}|\partial_r\psi_u(z,u+1)|^2\,\mathrm{d}z\,.
\end{aligned}$$

Next, for $\varepsilon > 0$, Proposition 6.5 and Lemma 6.6 imply that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) &\leq -\frac{\pi}{16}\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z + \frac{\pi}{4} + \sigma\pi + 2C_2(\sigma)^2 \\
&\quad - \frac{\lambda}{2}\bigg(\varepsilon C_1(\sigma) - \varepsilon^2\int_{-1}^{1}\sqrt{1+(\sigma\partial_z u)^2}\,\mathrm{d}z\bigg)\,.
\end{aligned}$$

Choosing $\varepsilon = \sqrt{\dfrac{\pi}{8\lambda}}$, we reduce this inequality to

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \le \frac{\pi}{4}\sigma\pi - 2C_2(\sigma)^2 - \frac{\sqrt{\lambda\,\pi}\,C_1(\sigma)}{4\sqrt{2}}\,. \tag{6.13}$$

The right-hand side is strictly less than zero if $\lambda > \lambda_{crit}(\sigma)$ where

$$\lambda_{crit}(\sigma) := \frac{32}{\pi C_1(\sigma)^2}\left(\frac{\pi}{4} + \sigma\pi + 2C_2(\sigma)^2\right)^2. \tag{6.14}$$

Hence, for $\lambda > \lambda_{crit}(\sigma)$, the solution (u, ψ_u) cannot be global.

□

Remark 6.7 **(i)** Computing the smallest possible value for $\lambda_{crit}(\sigma)$ in Theorem 6.1 is of particular interest. An upper bound for $\lambda_{crit}(\sigma)$ is given by formula (6.14), where σ, the radius of the rings divided by their distance, is easy to determine, and the constants

$$C_1(\sigma) = \int_{-1}^{1} (u_{in}+1)\big(1 + (\sigma\partial_z u_{in})^2\big)\partial_r\psi_{cat}(z, u_{in}+1)\,\mathrm{d}z$$

from (6.12), and

$$C_2(\sigma) = \frac{1}{\min_{z\in[-1,1]} u_{in} + 1}\,, \qquad \min_{z\in[-1,1]} u_{in} = \frac{1}{\cosh(c_{in})} - 1\,,$$

where $c_{in} > c_{crit}$ solves $\frac{\cosh(c_{in})}{c_{in}} = \sigma$, may be accessible through numerical computations.

(ii) A consequence of the proof of Theorem 6.1 is that, for given $\lambda > \lambda_{crit}(\sigma)$, there exists a uniform upper bound on the blow-up time T_{max}: Abbreviating the right-hand side of (6.13) by

$$-C_3(\sigma,\lambda) := \frac{\pi}{4}\sigma\pi - 2C_2(\sigma)^2 - \frac{\sqrt{\lambda\,\pi}\,C_1(\sigma)}{4\sqrt{2}}\,,$$

we deduce from (6.13), the fact that $u_0 \ge u_{in}$, and the definition of E that

$$\begin{aligned} E(t) &= E(0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} E(\tau)\,\mathrm{d}\tau \\ &\le -\int_{-1}^{1} \ln\big(u_{in}(z)+1\big)\,\mathrm{d}z - t\,C_3(\sigma,\lambda)\,, \qquad t \in [0, T_{max})\,. \end{aligned}$$

Now, using $E(t) \ge -2\ln(2)$, we find

$$T_{max} \le \left(2\ln(2) - \int_{-1}^{1} \ln\big(u_{in}(z)+1\big)\,\mathrm{d}z\right) C_3(\sigma,\lambda)^{-1}\,,$$

where the right-hand side is independent of the initial value $u_0 \ge u_{in}$. As in (i), this upper bound may be accessible through numerical computations.

Conclusion

In this thesis, we have investigated a free boundary problem modelling a soap film spanned between two parallel rings and subjected to an electrostatic force. While we have established local well-posedness of the model in Sobolev spaces, a particular focus has been on qualitative behaviour of solutions. We have shown that the model is capable of displaying several aspects of the interplay of surface tension and electrostatics, and we restrict ourselves to summarizing the results regarding this interplay:

- Stable stationary film shapes, which are the physically most relevant ones, deflect monotonically outwards if the electrostatic force is increased. This response of the film to an increase of the electrostatic force has been proven in Section 4.3 and Section 5.5.

- For nearly balanced forces, that is λ close to λ_{cyl}, and most σ, there exists a stationary solution close to the cylinder. We have demonstrated numerically that this stationary solution remains stable even for some distances between the two metal rings at which the soap film would always break without electrostatic force. Hence, the electrostatic force may prevent the soap film from breaking. For details, we refer to Section 5.4.

- In the range dominated by electrostatics, many solutions cease to exist in finite time. More precisely, for $\sigma \geq \sigma_{crit}$, which means that the metal rings are close enough to each other so that the soap film can form a catenoid, and $\lambda \geq \lambda_{crit}$, all solutions with initial value lying above the inner (or critical) catenoid cease to exist in finite time. This result hints at a breaking of the film triggered by the electrostatic force. It has been shown in Chapter 6.

Let us now mention a few extensions that might be worth investigating. Based on our study of stability of stationary solutions close to the cylinder, it might be possible, for fixed voltage $\lambda = \lambda_{cyl}$, to perform a rigorous bifurcation analysis of the cylinder in dependence on the parameter σ. Of course, it would also be interesting to extend the result concerning non-existence of global solutions for large λ from the range $\sigma \in [\sigma_{crit}, \infty)$ to the whole range $\sigma \in (0, \infty)$. For $\sigma \in (0, \sigma_{crit})$, no parabolic comparison principle seems to apply, and it is unclear how a possible self-touching of the film can be treated alternatively. Finally, it would be interesting to study a three-dimensional version of the free boundary problem modelling the set-up without rotational symmetry. Here, the difficulty lies in the treatment of the electrostatic force due to weaker multiplication results for Sobolev functions.

Appendices

A. Fractional Sobolev Spaces

In this appendix, we recall the definition of fractional Sobolev spaces and collect two results on these spaces.

Let $U \subset \mathbb{R}^n$ be open and bounded with a Lipschitz boundary. Moreover, let $p \in (1,\infty)$ and $s \in (0,1)$. Then, the fractional Sobolev space

$$W_p^s(U) := \left\{ f \in L_p(U) \,\middle|\, \int_{U\times U} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y < \infty \right\} ,$$

equipped with the norm

$$\|f\|^p_{W_p^s(U)} := \|f\|^p_{L_p(U)} + \int_{U\times U} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,, \qquad f \in W_p^s(U)\,,$$

is a Banach space. Analogously, for $s \in (0,1)$, the fractional Sobolev space

$$W_p^{1+s}(U) := \left\{ f \in W_p^1(U) \,\middle|\, \partial^\alpha f \in W_p^s(U) \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| = 1 \right\} ,$$

equipped with the norm

$$\|f\|^p_{W_p^{1+s}(U)} := \|f\|^p_{W_p^1(U)} + \sum_{|\alpha|=1} \int_{U\times U} \frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,, \qquad f \in W_p^{1+s}(U)\,,$$

is a Banach space. Moreover, for $p \in (1,\infty)$ and $s \in (0,2]$ with $s \neq 1/p$, we set

$$W^s_{p,D}(U) := \begin{cases} W_p^s(U) & \text{for} \quad s \in (0,1/p)\,, \\ \left\{ f \in W_p^s(U) \,\middle|\, f = 0 \text{ on } \partial U \right\} & \text{for} \quad s \in (1/p,2]\,, \end{cases}$$

i.e. we include Dirichlet boundary conditions into the fractional Sobolev space whenever the trace on ∂U is defined. Finally, we let $W^{-s}_{p',D}(U)$ be the dual space of $W^s_{p,D}(U)$ where p' denotes the dual exponent of p.

Now we include two results on fractional Sobolev spaces. While the first one is an important general result on multiplication of fractional Sobolev functions from [**2**], the second one is a simple lemma. Both results are also used in [**22**].

Theorem A.1 (Multiplication Theorem)
Let $U \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $m \in \mathbb{N}$ with $m \geq 2$ and $p, p_j \in (1,\infty)$ as well as $s, s_j \in (0,\infty)$ for $1 \leq j \leq m$. If $s \leq \min\{s_j\}$ and

$$s - \frac{n}{p} < \begin{cases} \displaystyle\sum_{s_j < n/p_j} \left(s_j - \frac{n}{p_j} \right) & \text{if} \quad \displaystyle\min_{1\leq j\leq m} \left\{ s_j - \frac{n}{p_j} \right\} < 0\,, \\ \displaystyle\min_{1\leq j\leq m} \left\{ s_j - \frac{n}{p_j} \right\} & \text{otherwise}\,, \end{cases}$$

then pointwise multiplication

$$\prod_{j=1}^{m} W_{p_j}^{s_j}(U) \to W_p^s(U)$$

is continuous.

Proof. The statement follows from [**2**, Theorem 4.1, Remark 4.2 (d)]. □

Lemma A.2 *Let $\Omega = (-1,1) \times (1,2)$, $s \in [0,1]$ and $p \in (1,\infty)$. Then,*

$$\left[\zeta \mapsto \int_1^2 \zeta(\,\cdot\,, r)\,\mathrm{d}r \right] \in \mathcal{L}\big(W_p^s(\Omega), W_p^s(-1,1)\big)\,.$$

Proof. **(i)** For $\zeta \in L_p(\Omega)$, we have

$$\begin{aligned} \left\| \int_1^2 \zeta(\,\cdot\,, r)\,\mathrm{d}r \right\|_{L_p(-1,1)}^p &= \int_{-1}^1 \left| \int_1^2 \zeta(z,r)\,\mathrm{d}r \right|^p \mathrm{d}z \\ &\leq \int_{-1}^1 \left(\int_1^2 |\zeta(z,r)|^p\,\mathrm{d}r \right) \mathrm{d}z \\ &= \|\zeta\|_{L_p(\Omega)}^p\,. \end{aligned}$$

(ii) Next, let $\zeta \in W_p^1(\Omega)$. Thanks to (i), it remains to check that $\int_1^2 \partial_z \zeta(\,\cdot\,, r)\,\mathrm{d}r$ is the weak derivative of $\int_1^2 \zeta(\,\cdot\,, r)\,\mathrm{d}r$. To this end, let $\varphi \in \mathcal{D}(-1,1)$ and note that

$$-\int_{-1}^1 \left(\int_1^2 \zeta(z,r)\,\mathrm{d}r \right) \partial_z \varphi(z)\,\mathrm{d}z = -\int_{-1}^1 \left(\int_1^2 \zeta(z,r)\,\partial_z \underbrace{\varphi(z)}_{\notin \mathcal{D}(\Omega)}\,\mathrm{d}r \right) \mathrm{d}z\,. \tag{A.1}$$

Choose $\tilde{\varphi}_n \in \mathcal{D}(1,2)$ with $1 \geq \tilde{\varphi}_n \geq 0$ and $\tilde{\varphi}_n(r) \to 1$ as $n \to \infty$ for each $r \in (1,2)$. Applying Lebesgue's theorem to equation (A.1), we find

$$\begin{aligned} -\int_{-1}^1 \left(\int_1^2 \zeta(z,r)\,\mathrm{d}r \right) \partial_z \varphi(z)\,\mathrm{d}z &= -\lim_{n\to\infty} \int_{-1}^1 \left(\int_1^2 \zeta(z,r)\,\partial_z(\tilde{\varphi}_n(r)\varphi(z))\mathrm{d}r \right) \mathrm{d}z \\ &= \lim_{n\to\infty} \int_{-1}^1 \left(\int_1^2 \partial_z \zeta(z,r)\,(\tilde{\varphi}_n(r)\varphi(z))\mathrm{d}r \right) \mathrm{d}z \\ &= \int_{-1}^1 \left(\int_1^2 \partial_z \zeta(z,r)\mathrm{d}r \right) \varphi(z)\,\mathrm{d}z\,, \end{aligned}$$

and the assertion follows.

(iii) From (i) and (ii) we get

$$\Big[\zeta \mapsto \int_1^2 \zeta(\,\cdot\,, r)\,\mathrm{d}r\Big] \in \mathcal{L}\big(L_p(\Omega), L_p(-1,1)\big) \cap \mathcal{L}\big(W_p^1(\Omega), W_p^1(-1,1)\big)\,.$$

Since fractional Sobolev spaces can be characterized via real interpolation, i.e.

$$\big((L_p(-1,1), W_p^1(-1,1)\big)_{s,p} = W_p^s(-1,1)\,, \qquad \big(L_p(\Omega), W_p^1(\Omega)\big)_{s,p} = W_p^s(\Omega)\,,$$

interpolation finishes off the proof. □

B. Small Aspect Ratio Model

We briefly explain how the small aspect ratio model, described in Remark 1.1 (a) and introduced for the stationary case in [**58, 60**], can be obtained formally from the free boundary problem. To this end, we drop the assumption $b/a = 2$ for the moment and scale the free boundary problem (1.17)-(1.19) according to [**58**] by

$$z = \frac{z_{old}}{\ell}\,, \qquad \psi = \frac{\psi_{old}}{V}\,, \qquad r = \frac{r_{old}}{b-a}\,, \qquad u = \frac{u_{old}}{a} - 1\,, \qquad t = \frac{2T\, t_{old}}{a^2\beta}\,.$$

Here, we recall that the radius of the outer cylinder b is always strictly greater than the radius of the metal rings a. Moreover, we note that the scaling of r, with the inverse of the radii difference $b-a$, differs from Chapter 1 for $b/a \neq 2$, while z, ψ, u and t are scaled as before. Introducing the aspect ratio

$$\varepsilon = \frac{b-a}{\ell}\,,$$

the dimensionless model then becomes

$$\begin{cases} \partial_t u - \sigma\, \partial_z \arctan(\sigma \partial_z u) &= -\dfrac{1}{u+1} + \lambda \left(\dfrac{a}{b-a}\right)^2 g(u)\,, \\ u(t,\pm 1) &= 0\,, \qquad -1 < u < 1\,, \\ u(0,z) &= u_0\,, \qquad z \in (-1,1)\,, \\ \frac{1}{r}\partial_r\left(r \partial_r \psi\right) + \varepsilon^2 \partial_z^2 \psi &= 0 \quad \text{in} \quad \Omega(u)\,, \\ \psi &= h_u \quad \text{on} \quad \partial\Omega(u) \end{cases} \tag{B.1}$$

with

$$\Omega(u) = \left\{ (z,r) \in (-1,1) \times \left(0, \frac{b}{b-a}\right) \,\middle|\, \frac{a}{b-a}(u(z)+1) < r < \frac{b}{b-a} \right\},$$

and

$$\begin{aligned} g(u) := (1+\sigma^2(\partial_z u)^2)^{1/2} \Bigg(\varepsilon^2 \left| \partial_z \psi\left(z, \left(\frac{a}{b-a}\right)(u(z)+1)\right)\right|^2 \\ + \left| \partial_r \psi\left(z, \left(\frac{a}{b-a}\right)(u(z)+1)\right)\right|^2 \Bigg)\,, \end{aligned} \tag{B.2}$$

as well as

$$h_u(z,r) = \ln\left(\frac{(b-a)r}{a(u(z)+1)}\right) \Big/ \ln\left(\frac{b}{a(u(z)+1)}\right). \tag{B.3}$$

Here, the new parameter ε occurs in the equation for the electrostatic potential ψ and in the definition of the electrostatic force g.[2] Assuming a small aspect ratio, i.e. $\varepsilon << 1$, one formally considers the case $\varepsilon = 0$ in (B.1)-(B.3).

[2]For details on g, we refer to its rigorous introduction in Chapter 3.

Then, the electrostatic potential solves $\frac{1}{r}\partial_r(r\partial_r\psi) = 0$ and is explicitly given by $\psi(z,r) = h_u(z,r)$. Consequently, (B.1)-(B.3) reduces to the small aspect ratio model

$$\begin{cases} \partial_t u - \sigma\,\partial_z \arctan(\sigma\partial_z u) &= -\dfrac{1}{u+1} + \lambda\, g_{sar}(u)\,, \\ u(t,\pm 1) &= 0\,, \qquad -1 < u < 1\,, \\ u(0,z) &= u_0\,, \qquad z \in (-1,1)\,, \end{cases} \tag{B.4}$$

with explicitly given electrostatic force

$$g_{sar}(u) := (1+\sigma^2(\partial_z u)^2)^{1/2}\, \frac{1}{(u+1)^2 \ln^2\left(\dfrac{b}{a(u+1)}\right)}\,. \tag{B.5}$$

Note that this simplified electrostatic force $g_{sar}(u)$ in (B.5) depends only pointwise on u and $\partial_z u$, while the electrostatic force in the free boundary problem, see (B.2), depends on u and ψ. Choosing $b/a = 2$, which is allowed in [**58**], yields equations (1.23)-(1.24) from Remark 1.1 (a), while the stationary version of (B.4)-(B.5) coincides with the model from [**58, 60**].

Remark B.1 In second-order MEMS models, more precisely models for a membrane suspended above a fixed ground plate, a rigorous derivation of the small aspect ratio model from the free boundary problem is possible, see [**42**] for the stationary case and [**23, 24**] for the time-dependent one. However, the situation in (B.1)-(B.3) is different. The main reason lies in the relation

$$\varepsilon = \sigma\,\frac{b-a}{a}$$

which makes the passage to the limit $\varepsilon \searrow 0$ impossible without changing other parameters explicitly contained in the equations. Therefore, we refrain from a rigorous study of $\varepsilon \searrow 0$ in (B.1)-(B.3).

C. Anti-Maximum Principle

In this appendix, we present a criterion to decide whether or not an anti-maximum principle applies to the uniformly elliptic second order operator

$$-DF(u_{in}^0)v = -\sigma^2 \left[\partial_z\Big(\frac{1}{\cosh^2(c_{in}z)}\, v_z\Big) + \frac{c_{in}^2}{\cosh^2(c_{in}z)} v\right], \qquad v \in W_{q,D}^2(-1,1)$$

from (4.14) and a given function $f > 0$ on $(-1,1)$. We recall that $-DF(u_{in}^0)$ is the linearization of (4.2) around the inner catenoid u_{in}^0, and $c_{in} > 0$ denotes the constant corresponding to u_{in}^0. If an anti-maximum principle applies, it would yield that $v = \big(-DF(u_{in}^0)\big)^{-1} f$ is negative. The presented criterion stems from [**67**] but is slightly extended to include a statement for the case where no such principle applies.

To state the criterion precisely, let us recall from Subsection 4.3.2 that

$$\varphi(z) = \cosh(c_{in}z) - c_{in}\, z \sinh(c_{in}z)$$

is the unique solution to the initial value problem

$$\begin{cases} 0 = -\partial_z\Big(\dfrac{1}{\cosh^2(c_{in}z)}\varphi_z\Big) - \dfrac{c_{in}^2}{\cosh^2(c_{in}z)}\varphi \quad \text{on } (-1,1)\,, \\ \varphi(0) = 1\,, \quad \varphi_z(0) = 0 \end{cases}$$

associated with the boundary value differential operator $-DF(u_{in}^0)$. Now the criterion reads:

Lemma C.1 *Let $f \in C\big([-1,1]\big)$ with $f(z) = f(-z) > 0$ for each $z \in [-1,1]$ and consider the even function $v := \big(-DF(u_{in}^0)\big)^{-1} f \in W_{q,D}^2(-1,1) \cap C^2\big([-1,1]\big)$.*
(i) *If*

$$\int_{-1}^{1} f(z)\,\varphi(z)\,\mathrm{d}z > 0\,,$$

then v satisfies a strong anti-maximum principle meaning that $v < 0$ on $(-1,1)$ with $v_z(-1) < 0$ and $v_z(1) > 0$.
(ii) *If*

$$\int_{-1}^{1} f(z)\,\varphi(z)\,\mathrm{d}z < 0\,,$$

then v does not satisfy an anti-maximum (or maximum) principle. Instead, v is sign-changing: there exists $r_0 \in (0,1)$ such that $v < 0$ on $(-r_0, r_0)$ and $v > 0$ on $(-1,-r_0) \cup (r_0, 1)$ as well as

$$v_z(-1) > 0\,, \quad v_z(-r_0) < 0\,,$$
$$v_z(r_0) > 0\,, \quad v_z(1) < 0\,.$$

Proof. Throughout, we use the abbreviation $c = c_{in}$. Because f is strictly positive and Corollary 4.8 ensures that $DF(u_{in}^0)$ has exactly one strictly positive eigenvalue, while all other eigenvalues are strictly negative, we can rely on the anti-maximum principle from [**67**] and its proof:

(i) This follows directly from [**67**, Theorem 2.3].
(ii) In the proof of [**67**, Theorem 2.3], it is shown that the set

$$I_- := \big\{ z \in (-1,1) \,\big|\, v(z) < 0 \big\}$$

coincides either with $(-1,1)$ or with $(-r_0, r_0)$ for some $0 < r_0 < 1$. Because v satisfies Dirichlet boundary conditions, $v_z(0) = 0$ due to symmetry, and $\varphi_z(0) = 0$ as initial data, integration by parts yields

$$\begin{aligned}
&\frac{1}{\cosh^2(c)}\varphi(1)\, v_z(1) \\
&= \Big[\frac{1}{\cosh^2(cz)}\varphi(z)\, v_z(z) - \frac{1}{\cosh^2(cz)}\varphi_z(z)\, v(z) \Big]_{z=0}^{z=1} \\
&= \int_0^1 \bigg(\partial_z\Big(\frac{1}{\cosh^2(cz)} v_z(z) \Big)\varphi(z) - \partial_z\Big(\frac{1}{\cosh^2(cz)}\varphi_z(z)\Big) v(z) \bigg)\, \mathrm{d}z\,. \qquad \text{(C.1)}
\end{aligned}$$

Adding $\pm \dfrac{c^2}{\cosh^2(cz)} v(z)\varphi(z)$ to the integrand in (C.1) and using the differential equation for φ as well as the relation $-DF(u_{in}^0)v = f$, we see that

$$\begin{aligned}
\frac{1}{\cosh^2(c)}\varphi(1)\, v_z(1) &= \frac{1}{\sigma^2} \int_0^1 \big(DF(u_{in}^0)v \big)(z)\, \varphi(z)\, \mathrm{d}z \\
&= -\frac{1}{\sigma^2} \int_0^1 f(z)\varphi(z)\, \mathrm{d}z \\
&= -\frac{1}{2\sigma^2} \int_{-1}^1 f(z)\varphi(z)\, \mathrm{d}z > 0\,. \qquad \text{(C.2)}
\end{aligned}$$

The last integral is positive by assumption. Now we deduce from (C.2) and $\varphi(1) < 0$, where the negativity of $\varphi(1)$ can be seen in Figure 4.3 on p. 68, that also $v_z(1) < 0$. Since v satisfies Dirichlet boundary conditions, it follows that v is non-negative close to $z = 1$, and consequently I_- cannot be the whole interval $(-1,1)$. Hence, we have $I_- = (-r_0, r_0)$ for some $0 < r_0 < 1$. Moreover, we note that also $v_z(-1) > 0$ by symmetry. To show that v is strictly positive on $(-1,-r_0) \cup (r_0, 1)$, we assume for contradiction that $v(z_0) = 0$ for some z_0 with $r_0 < |z_0| < 1$. Since $v \geq 0$ close to z_0, it follows that v has a local minimum at z_0 and hence necessarily $v_z(z_0) = 0$. But then we find that

$$\begin{aligned}
0 &> -\frac{1}{\sigma^2} f(z_0) \\
&= \frac{1}{\sigma^2} \big[DF(u_{in}^0)v \big](z_0) \\
&= \partial_z\Big(\frac{1}{\cosh^2(cz)} \Big)\Big|_{z=z_0} v_z(z_0) + \frac{1}{\cosh^2(cz_0)} v_{zz}(z_0) \\
&\quad + \frac{c^2}{\cosh^2(cz_0)} v(z_0) \\
&= \frac{1}{\cosh^2(cz_0)} v_{zz}(z_0)\,, \qquad \text{(C.3)}
\end{aligned}$$

i.e. v has a strict local maximum at z_0 which is not possible. Hence, v is indeed strictly positive on $(-1,-r_0) \cup (r_0, 1)$. Since v is negative on $(-r_0, r_0)$, we have $v(r_0) = 0$ and $v_z(r_0) \geq 0$. Finally, assume for contradiction that $v_z(r_0) = 0$. Then, a similar computation as in (C.3) shows that $v_{zz}(r_0) < 0$, i.e. v has a local maximum at r_0, which is impossible. Hence, we have established $v_z(r_0) > 0$, and also $v_z(-r_0) < 0$ by symmetry. $\square$

D. Eigenbasis of the Laplacian in Cylindrical Coordinates

In this appendix, we recall some facts about the Dirichlet-Laplacian in cylindrical coordinates

$$-\Delta_{cyl,D} : \ W^2_{2,D}(\Omega) \to L_2(\Omega)\,, \quad f \mapsto -\frac{1}{r}\partial_r\big(r\partial_r f\big) - \sigma^2\partial_z^2 f$$

on $\Omega = (-1,1)\times(1,2)$ under rotational symmetry. Therefore, we introduce the spaces

$$L_{2,r}(1,2) := \Big(L_2(1,2), (\,\cdot\,|\,\cdot\,)_{L_{2,r}(1,2)}\Big)$$

with scalar product

$$(f|h)_{L_{2,r}(1,2)} := \int_1^2 f(r)\,h(r)\,r\,\mathrm{d}r\,, \qquad f,h\in L_2(1,2)\,,$$

and

$$L_{2,r}(\Omega) := \Big(L_2(\Omega), (\,\cdot\,|\,\cdot\,)_{L_{2,r}(\Omega)}\Big)$$

with scalar product

$$(f|h)_{L_{2,r}(\Omega)} := \int_{-1}^1\int_1^2 f(r)\,h(r)\,r\,\mathrm{d}r\,\mathrm{d}z \qquad f,h\in L_2(\Omega)\,.$$

These spaces are obviously isomorphic to $L_2(1,2)$ and $L_2(\Omega)$ respectively.

The operator $-\Delta_{cyl,D}$ splits into two parts acting on the z and r variable respectively:

The one-dimensional Dirichlet-Laplacian $-\partial_z^2 : W^2_{2,D}(-1,1) \to L_2(-1,1)$ acts on the first variable z. Its spectrum consists entirely of eigenvalues

$$\nu_j := \frac{(j+1)^2\pi^2}{4}\,, \qquad j\in\mathbb{N}\,,$$

with geometric multiplicity 1 and corresponding normalized eigenfunctions

$$\varphi_j(z) := \begin{cases} \cos\left(\dfrac{(j+1)\pi}{2}\,z\right) & \text{if } j \text{ is even}\,,\\[2ex] \sin\left(\dfrac{(j+1)\pi}{2}\,z\right) & \text{if } j \text{ is odd}\end{cases}$$

for $z\in(-1,1)$. The eigenfunctions $\{\varphi_j\}_j$ form an orthonormal basis of $L_2(-1,1)$.

The operator $-\frac{1}{r}\partial_r(r\partial_r\,\cdot\,) : W^2_{2,D}(1,2)\to L_2(1,2)$ acts on the second variable r. Its spectrum consists entirely of eigenvalues

$$0<\xi_0<\xi_1<\cdots<\xi_k\to\infty$$

with geometric multiplicity 1. The corresponding sequence of normalized eigenfunctions $\{\rho_k\}_k$ belongs to $C^\infty\big([1,2]\big)\cap W^2_{2,D}(1,2)$ and forms an orthonormal basis of $L_{2,r}(1,2)$, see [**71**, §27].

The next proposition shows that the spectrum and eigenfunctions of the composite operator $-\Delta_{cyl,D}$ are entirely determined by the spectrum and eigenfunctions of its two parts. A related computation for a more complicated operator splitting in two parts is contained in [**1**].

Proposition D.1 *The spectrum of*

$$-\Delta_{cyl,D} : W^2_{2,D}(\Omega) \to L_2(\Omega)$$

consists entirely of eigenvalues

$$\xi_k + \sigma^2 \nu_j \, , \quad j,k \in \mathbb{N}.$$

The corresponding eigenfunctions $\rho_k \varphi_j \in W^2_{2,D}(\Omega) \cap C^\infty(\overline{\Omega})$ form an orthonormal basis of $L_{2,r}(\Omega)$. Moreover, for each $f \in W^2_{2,D}(\Omega)$, there exist $b_{jk} \in \mathbb{R}$ such that

$$f = \sum_{j,k} b_{jk} \rho_k \varphi_j \, ,$$

where the sequence converges unconditionally in $W^2_2(\Omega)$.

Proof. **(i)** First, we check that $\{\rho_k \, \varphi_j\}_{j,k}$ forms an orthonormal basis of $L_{2,r}(\Omega)$. Since

$$\big(\rho_k \, \varphi_j \,\big|\, \rho_l \, \varphi_i\big)_{L_{2,r}(\Omega)} = (\varphi_j \,|\, \varphi_i)_{L_2(-1,1)} \, (\rho_k \,|\, \rho_l)_{L_{2,r}(1,2)} = \delta_{ij} \delta_{kl} \, ,$$

the set $\{\rho_k \, \varphi_j\}_{j,k}$ is orthonormal in $L_{2,r}(\Omega)$. Because $\mathcal{D}(\Omega)$ is dense in $L_{2,r}(\Omega)$, we only have to show that

$$0 = \big(h \,\big|\, \rho_k \, \varphi_j\big)_{L_{2,r}(\Omega)} \quad \forall \, j,k \in \mathbb{N} \qquad \Longrightarrow \qquad h \equiv 0$$

for $h \in \mathcal{D}(\Omega)$. Because the scalar product may be written as

$$\begin{aligned} 0 = \big(h \,\big|\, \rho_k \, \varphi_j\big)_{L_{2,r}(\Omega)} &= \int_{-1}^1 \left(\int_1^2 h(z,r) \, \rho_k(r) \, r \, \mathrm{d}r \right) \varphi_j(z) \, \mathrm{d}z \\ &= \left(\int_1^2 h(\,\cdot\,, r) \rho_k(r) \, r \, \mathrm{d}r \,\middle|\, \varphi_j \right)_{L_2(-1,1)} , \qquad j,k \in \mathbb{N} \end{aligned}$$

and $\{\varphi_j\}_j$ is an orthonormal basis of $L_2(-1,1)$, it follows that

$$0 = \int_1^2 h(z,r) \rho_k(r) \, r \, \mathrm{d}r = \big(h(z,\,\cdot\,) \,\big|\, \rho_k\big)_{L_{2,r}(1,2)} \, , \qquad k \in \mathbb{N} \, , \quad z \in (-1,1) \, .$$

But $\{\rho_k\}_k$ is again an orthonormal basis of $L_{2,r}(1,2)$, and hence $h \equiv 0$. Consequently, $\{\rho_k \varphi_j\}_{j,k}$ is indeed an orthonormal basis of $L_{2,r}(\Omega)$.

(ii) We investigate the eigenvalues and eigenfunctions of $-\Delta_{cyl,D}$: Because $W^2_{2,D}(\Omega)$ is compactly embedded in $L_2(\Omega)$ and $-\Delta_{cyl,D} \in \mathcal{L}\big(W^2_{2,D}(\Omega), L_2(\Omega)\big)$ is invertible by Lemma 3.3, the spectrum of $-\Delta_{cyl,D}$ consists only of eigenvalues with no finite accumulation point, see [**39**, Thorem 6.29]. Moreover, a direct computation gives

$$\Big(- \Delta_{cyl,D} w_1 \,\Big|\, w_2 \Big)_{L_{2,r}(\Omega)} = \Big(w_1 \,\Big|\, - \Delta_{cyl,D} w_2 \Big)_{L_{2,r}(\Omega)}$$

for $w_1, w_2 \in W^2_{2,D}(\Omega)$. Consequently, all eigenvalues of $-\Delta_{cyl,D}$ are real, and eigenfunctions to different eigenvalues are orthogonal with respect to the $L_{2,r}(\Omega)$-scalar product. Moreover

$$-\Delta_{cyl,D}\rho_k\varphi_j = (\xi_k + \sigma^2\nu_j)\rho_k\varphi_j\,, \qquad j,k \in \mathbb{N}\,,$$

and by step (i) the corresponding eigenfunctions $\{\rho_k\,\varphi_j\}_{j,k}$ form an orthonormal basis of $L_{2,r}(\Omega)$. Hence, we have found all eigenvalues of $-\Delta_{cyl,D}$.
(iii) Finally, for $f \in W^2_{2,D}(\Omega)$, there exists $h \in L_{2,r}(\Omega)$ with $f = (-\Delta_{cyl,D})^{-1}h$. Writing h as

$$h = \sum_{j,k} \big(h\,\big|\,\rho_k\varphi_j\big)_{L_{2,r}(\Omega)}\rho_k\varphi_j$$

with unconditional convergence in $L_{2,r}(\Omega)$ and applying $(-\Delta_{cyl,D})^{-1}$ to both sides gives

$$f = \sum_{j,k} b_{jk}\rho_k\varphi_j\,, \qquad b_{jk} = \frac{\big(h\,\big|\,\rho_k\varphi_j\big)_{L_{2,r}(\Omega)}}{\xi_k + \sigma^2\nu_j}\,, \qquad j,k \in \mathbb{N}\,,$$

with unconditional convergence in $W^2_{2,D}(\Omega)$. Now everything is proven. □

E. Odd Cosine Sums

We present a sufficient condition for an odd cosine sum to be positive on $(-1,1)$. In the following, a cosine sum is odd if each cosine is scaled by an odd multiple of $\pi/2$.

As the first step, we reduce odd cosine sums to even ones by applying trigonometric identities. The ansatz is taken from [**11**, Lemma 5]. However, we require a slightly different formula for the coefficients of the even cosine sum.

Lemma E.1 *Let $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. Consider the sum of odd cosines*

$$f(z) := \sum_{j=0}^{n} a_j \cos\left(\frac{(2j+1)\pi}{2}\, z\right), \qquad z \in (-1,1)\,.$$

Then

$$\frac{f(z)}{\cos(\frac{\pi}{2}z)} = \sum_{j=0}^{n} b_j \cos(j\pi z)\,, \qquad z \in (-1,1)\,,$$

with coefficients

$$b_0 := \sum_{k=0}^{n} (-1)^k a_k\,, \qquad b_j := 2\sum_{k=j}^{n} (-1)^{k-j} a_k\,, \qquad j = 1, \ldots, n\,.$$

Proof. We proceed by induction. For $n = 0$, the assertion is obviously true. And for $n = 1$, the assertion follows directly from the trigonometric identity

$$\cos\left(\frac{(2n+1)\pi}{2} z\right) + \cos\left(\frac{(2n-1)\pi}{2} z\right) = 2\cos(n\pi z)\cos\left(\frac{\pi}{2} z\right).$$

For the step from $n-1$ to n, we now apply the above identity to rewrite f as

$$\begin{aligned} f(z) = &\sum_{j=0}^{n-2} a_j \cos\left(\frac{(2j+1)\pi}{2}\, z\right) + (a_{n-1} - a_n)\cos\left(\frac{(2(n-1)+1)\pi}{2}\, z\right) \\ &+ 2a_n \cos(n\pi z)\cos\left(\frac{\pi}{2} z\right), \end{aligned}$$

which implies that the coefficients b_0 to b_{n-1} of $\frac{f(z)}{\cos(\frac{\pi}{2}z)}$ are given by[3]

$$b_0 = \sum_{k=0}^{n-2} (-1)^k a_k + (-1)^{n-1}(a_{n-1} - a_n) = \sum_{k=0}^{n} (-1)^k a_k\,,$$

$$b_j = 2\sum_{k=j}^{n-2} (-1)^{k-j} a_k + 2(-1)^{n-1-j}(a_{n-1} - a_n) = 2\sum_{k=j}^{n} (-1)^{k-j} a_k\,, \qquad j = 1, \ldots, n-1,$$

while

$$b_n = 2a_n = 2\sum_{k=n}^{n} (-1)^{k-n} a_k$$

is also fulfilled. □

[3] For b_{n-1}, we use the convention $\sum_{k=n-1}^{n-2} (-1)^{k-(n-1)} a_k = 0$.

As the second and final step, we prove a sufficient condition for an odd cosine sum to be positive on $(-1,1)$.

Lemma E.2 *Let $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. Moreover, let $C > 0$ with*

$$a_0 - \sum_{j=1}^{n} (2j+1)\,|a_j| \geq C > 0\,.$$

Then, the sum of odd cosines

$$f(z) := \sum_{j=0}^{n} a_j \cos\left(\frac{(2j+1)\pi}{2}\, z\right), \qquad z \in (-1,1)\,.$$

satisfies

$$f(z) \geq C\, \cos\left(\frac{\pi}{2} z\right), \qquad z \in (-1,1)\,.$$

In particular, $f(z) > 0$ for each $z \in (-1,1)$ and $f_z(\pm 1) \neq 0$.

Proof. We prove the equivalent statement

$$\frac{f(z)}{\cos(\frac{\pi}{2} z)} \geq C\,, \qquad z \in (-1,1).$$

Lemma E.1 yields

$$\frac{f(z)}{\cos(\frac{\pi}{2} z)} = \sum_{j=0}^{n} b_j \cos(j\pi z)$$

with coefficients

$$b_0 = \sum_{k=0}^{n} (-1)^k a_k\,, \quad b_j = 2\sum_{k=j}^{n} (-1)^{k-j} a_k\,, \qquad j = 1, \ldots, n\,,$$

so that

$$\begin{aligned} \frac{f(z)}{\cos(\frac{\pi}{2} z)} &\geq b_0 - \sum_{j=1}^{n} |b_j| \geq \sum_{k=0}^{n} (-1)^k a_k - 2\sum_{j=1}^{n} \sum_{k=j}^{n} |a_k| \\ &\geq a_0 - \sum_{k=1}^{n} |a_k| - 2\sum_{j=1}^{n} j\,|a_j| = a_0 - \sum_{j=1}^{n} (2j+1)|a_j| \geq C\,. \end{aligned}$$

□

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