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On using a core idea to foster the transition to advanced mathematics – transferring the idea of average to complex path integrals

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ABSTRACT

The transition from Riemann integrals to other real integrals has been discussed previously in mathematics education literature. In this transition, the interpretation of integrals as averages of the integrated functions generalises smoothly to measure and probability theory. In this paper, I address the transition to complex path integrals. Based on the premise that using core ideas throughout mathematics curricula may facilitate the transition from earlier to more advanced courses, I analyse epistemologically to what extent the idea of averaging may be transferred to complex path integrals. In this case, the transition poses special epistemological challenges. Using an example from an expert interview, in which the expert aims to apply the interpretation of integrals as averages to complex path integrals, I illustrate the aforementioned challenges empirically.

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Complex analysis; integrals; mean values; epistemology of mathematical objects; basic ideas

1. Introduction

Various kinds of transitions are dealt with in mathematics education research. Most often, a strong emphasis has been put on the transition from school to university or from university to the workplace (e.g. Biza et al., 2016; Di Martino et al., 2023a, 2023b; Gueudet, 2008, 2023; Gueudet et al., 2016). Only a handful of studies dealt with the transition from earlier to later courses in mathematics curricula (i.e. roughly from the second to third year onwards) though (e.g. Dray & Manogue, 2023; Hochmuth et al., 2021; Jones, 2020; Kondratieva & Winsløw, 2018). Notwithstanding, the sequencing of calculus and real analysis to multivariate real analysis, measure theory, or complex analysis offers immense potential for studying the transition to more advanced courses. One of the main reasons for this is that many mathematical concepts students encounter in earlier courses reappear in later ones, oftentimes conceptually enhanced but still recognisable as similar ('polysemous cross-curricular topics'; Kontorovich, 2018). It is commonly believed that students' previous conceptions of a mathematical concept affect their conceptions about them when they reappear and that such a 'domanial shift and the substantial change are potential sources for students' difficulties and mistakes' (Kontorovich, 2018, p. 6; cf. Biza, 2021; McGowen & Tall,

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2010). In this regard, researchers have considered calculus at the crossroads of disciplines (see e.g. Biza et al., 2022) and undertaken pioneering educational research on the transition from Riemann integrals to multivariate integrals and real path integrals (e.g. Jones, 2020) or complex path integrals (Hancock, 2018; Hanke, 2020, 2022a, 2022b; Oehrtman et al., 2019; Soto & Oehrtman, 2022) recently. In line with Jones' (2020, pp. 1–2) observation that 'it can be useful for calculus education to learn how students understand integrals across this entire progression [...] to help identify coherent ways of thinking about integrals' and his analysis of 'how possible ways of thinking about integrals described in the research literature might apply to [real path] integrals' (p. 2), I extend this inquiry to the case of complex analysis.[1]

This paper further contributes to the discussion about vertical coherence in the teaching of integrals. For that purpose, I analyse the benefits and constraints of a potential transfer of the idea of averaging, which has been identified previously as a possible interpretation for Riemann integrals, to complex path integrals. Gluchoff (1991) proposed such a transfer in response to students' demand for a simple interpretation of complex path integrals. Unfortunately, this idea has not been taken up further and seems to be missing in textbook literature on complex analysis (cf. Hanke, 2022a, 2022b). Therefore, it is the topic of the present contribution. Even though the idea of average generalises rather smoothly to real path integrals of the first kind and measure-theoretic integrals, this is only partly the case for complex path integrals. The arising challenges, however, supply stimulating points for discussion in classrooms and may foster students' holistic understanding of integrals throughout their curricula.

In this line of reasoning, the research questions for this paper are 'Which epistemological challenges do arise when the idea of averaging is transferred from Riemann integrals to complex path integrals?' and 'Which opportunities does this transfer have for a vertically coherent curriculum in mathematics?' This way, this paper extends the epistemological research presented in Hanke (2022a, 2022b) and contributes to the literature on advanced university mathematics education in two ways, namely

(1) with an *epistemological analysis* of the transfer of the idea of *averaging* to complex path integrals, and
(2) with an *example from an expert interview* with a mathematics lecturer to demonstrate the empirical presence of the idea of averaging as a cross-curricular interpretation for integrals and to illustrate the challenges for this transfer addressed in the epistemological analysis.

2. Outline

The paper is structured as follows. First, I embed the epistemological research undertaken in this paper into current literature on the transition from earlier to more advanced parts of mathematics curricula. Then, I outline what I understand with a *core idea* to be used for teaching a cross-curricular topic (Kontorovich, 2018) and proceed to review core ideas discussed in the literature on integrals in terms of the notion of *aspects* and *basic ideas* from German subject-matter didactics (Greefrath et al., 2016; vom Hofe, 1995; vom Hofe & Blum, 2016). I continue with previous epistemological and empirical research about the transfer of these core ideas but focus mostly on the interpretation of integrals as averages.

Then, I present the epistemological analysis of the transfer of the idea of averaging to complex path integrals. The case of a lecturer underlines that this idea may be used as a core idea for teaching integrals but is nevertheless challenging for an expert as well. Finally, in the discussion, I outline perspectives for further research on vertically aligned curricula on integrals.

3. Theoretical and methodological framework for the epistemological analysis

3.1. Embedding into the literature on the transition to advanced mathematical topics

3.1.1. Core ideas

For school mathematics, one aspect of vertical alignment between different school years and levels has oftentimes been discussed with the notions 'fundamental ideas', 'central ideas', or others (e.g. Bruner, 1960; Schweiger, 2006; Tietze et al., 2000; Vohns, 2016), with varying meaning depending on the author. Those 'ideas' categorise mathematical concepts or mathematical activities from different yet interconnected points of view. In one strand of this discussion the ideas should be overarching for mathematics as a whole and in another strand the ideas should be more specific to branches of mathematics such as calculus and real analysis. For integrals in calculus and real analysis, educators have identified – not unambiguously though – several ideas to be included in school and university teaching (see e.g. Biza et al., 2022; Ely & Jones, 2023; Greefrath et al., 2016; Thompson & Harel, 2021). Hence, it is likely that students enter university mathematics training with preconceptions about 'integrals' related to certain of those ideas and learn new interpretations for them. As university students, they may wonder how the interpretations they encountered previously may apply to 'integrals' in later parts of their curricula.

This paper belongs to the second of the previously mentioned strands of discussion. It deals with the orientation in teaching, in which a certain idea for a cross-curricular topic (Kontorovich, 2018), here 'integral', is used *multiply* whenever this cross-curricular topic reappears as another of its instances. In order not to be biased with a specific previous conceptualisation of a fundamental or central idea from the literature, I use the term *core idea* here. While fundamental ideas rather relate to mathematics at large (e.g. 'number', 'algorithm', or 'measuring'; Schweiger, 2006, pp. 66–67) or to largely grouped branches such as calculus/real analysis (e.g. 'functions and curves', 'integral and integrability', or 'functional equations' appear under the name 'Leitidee [guiding idea; EH.]'; Tietze et al., 2000, p. 184), a core idea shall be understood to be related to a mathematical concept more specifically (see 'Subject-matter didactical approach to interpretations of integrals: Aspects and basic ideas'). Nevertheless, core ideas may 'recur in different areas of mathematics', 'recur at different levels', 'help to design curricula', 'build up semantic networks between different areas' and 'improve memory' – properties Schweiger used to characterise fundamental ideas (Schweiger, 2006, p. 68; original bullet points omitted). That is, a core idea serves as an interpretation for a mathematical concept, which may also be used to interpret another mathematical concept, possibly subject to certain modifications. Hence, once a core idea for a particular instance of a cross-curricular concept is found, it may apply to different instances of that concept, too. Such an idea may have previously been identified as particularly valuable for teaching a certain instance of a cross-curricular concept and may then prove to be valuable for another instance of that cross-curricular concept, too. As illustrated in this paper, the idea of 'mean value' may be used to interpret and establish coherence among different integrals provided that it is properly adapted.[2]

3.1.2. Advanced mathematical topics in mathematics education

In university mathematics education with sequentially organised modules, explicit guidelines of subject-specific materials for the teaching of advanced topics are almost not yet available (Hochmuth, 2021). This is confirmed in the overview by Winsløw et al. (2021) who argue that

> [m]aterial that identifies fundamental or central ideas, provides insight into learning difficulties or obstacles for the students and that shows possible remedies [. . .] is available for teaching at school level, for instance to know about different ways to approach and organise the teaching of derivatives or integrals (cf. Greefrath et al., 2016). Similar expositions are inaccessible or unavailable when it comes to more advanced subjects (e.g. linear algebra) and their teaching at university level. (p. 74)

Accordingly, there is a need for guidance, at best supported by mathematics education research, for aligning teaching to core ideas. The curricularly organised sequence from calculus and real analysis in one variable to several variables, vector analysis, measure and probability theory, and complex analysis, in which integrals play a crucial role seems to be particularly suitable for a contribution to the discussion about vertically aligned teaching.

In this context, Kondratieva and Winsløw (2018) discussed the issue of a vertically aligned curriculum emanating from calculus in terms of *Klein's Plan B*. Klein's Plan B refers to the 'the organic combination of the partial fields [e.g. different courses in a modularized study programme; EH.], and upon the stimulation which these exert one upon another' and the formation of 'an understanding of several fields under a uniform point of view [... to comprehend] the sum total of mathematical science as a great connected whole' (Klein 1908/1932; cited by Kondratieva & Winsløw, 2018, p. 122). On the contrary, *Klein's Plan A* aims to 'divide[] the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields' (Klein, 1908/1932; cited by Kondratieva & Winsløw, 2018, p. 121). Kondratieva and Winsløw (2018) exemplified Plan B with trigonometric functions from the point of view of vector analysis and Fourier analysis both as successors of calculus.

Hochmuth (2021) discussed an even more mathematically advanced topic. He acknowledged that the first year of mathematics curricula had already been studied a lot in current mathematics education research. He further identified that many topics from the first mathematics courses at university are taught without emphasising their importance or future use and that the usefulness of many notions becomes apparent only much later when students reencounter these concepts in more advanced mathematics courses. To bridge this gap between early and more advanced courses, Hochmuth (2021) used a theorem from nonlinear approximation theory to construct a task for analysis students in their first year, in which they must activate several different concepts such as function spaces, their norms, and the notion of order of approximation. I agree with the author that advanced mathematical topics

> are an important part of the rationale for teaching mathematical concepts. Therefore, they should be investigated for their potential to provide stimuli for tasks that can be posed in first-year courses and, in the context of their completion, can lead to learning processes that could effectively contribute to overcoming compartmentalized knowledge, to the development of better interconnected knowledge, in this case also towards ideas of advanced mathematics and, in addition, to an expansion of acquired rationales. [...] Generally, advanced mathematics, as considered here, has not yet been didactically researched to the author's knowledge. Therefore, the question arises how respective transitions between content of first-year and advanced courses might be described to support the implementation of tasks by showing that they address relevant transitions aspects. (Hochmuth, 2021, p. 1114)

While Hochmuth (2021) and Kondratieva and Winsløw (2018) discuss how advanced topics may be initiated in courses on calculus or real analysis, I look the other way and ask what the core idea of integrals as averages must be when transferred to complex analysis, what challenges arise, and how they can be integrated into the teaching of complex analysis. This research aligns thus also with Dray and Manogue's (2023), Jones' (2020), Oehrtman et al.'s (2019), and Soto and Oehrtman's (2022) research about the transfer of interpretations for Riemann integrals to other integrals. Jones (2020) studied the transition from interpretations of Riemann integrals to real path integrals of the first and second kinds. Oehrtman et al. (2019) and Soto and Oehrtman (2022) investigated to what extent experts and students in complex analysis can make sense of interpretations for integrals from calculus and real analysis. Furthermore, Dray and Manogue (2023) analysed how real path integrals of a second kind are introduced in one mathematics and one physics textbooks.

3.2. Methodological remarks about epistemological analyses

The epistemological analysis presented in this paper may be positioned within a modern approach to subject-matter didactics (cf. Bergsten, 2020; Hußmann et al., 2016; Sträßer, 2020). The analysis contains connections between interpretations of mathematical concepts and theorems of calculus/real analysis and complex analysis in relation to averages. The overall aim is to foster epistemological awareness in mathematics lecturers who teach complex analysis about potential hurdles or overgeneralisations learners may potentially come across when trying to adapt the interpretation of integrals as averages to complex analysis. If these hurdles or overgeneralisations are not mediated with a guiding expert, it seems unlikely that learners connect complex path integrals to averages appropriately. After all, case studies have shown that both novices and experts are not very familiar with interpretations of complex path integrals (Hanke, 2022a, 2022b; Oehrtman et al., 2019; Soto & Oehrtman, 2022) and the interpretation of complex path integrals as averages (Gluchoff, 1991) seems practically absent from the literature. To what extent such epistemological awareness can be observed in lectures and to what extent learners can actually mediate between interpretations for integrals in one context as well as another, however, then largely remains the subject of future research, as is the case with the innovative epistemological research presented by Hochmuth (2021) or Dray and Manogue (2023). Nevertheless, I also present elements of a case study supporting the potential hurdles and overgeneralisations examined in the epistemological analysis.

This kind of epistemological analysis as presented here is important for advanced mathematics education. It contributes to putting university mathematics education researchers in the position to conduct in-depth empirical research with teachers and learners, as is already the case with school mathematics.

3.3. Subject-matter didactical approach to interpretations of integrals: aspects and basic ideas

Greefrath et al. (2016) established a twofold way of looking at mathematical concepts. In doing so, they complemented the notion of 'basic idea' (sometimes also translated with 'basic mental models' from the German 'Grundvorstellungen'; vom Hofe, 1995; vom Hofe & Blum, 2016), which has been frequently used in German subject-matter didactics for at least 30 years with the notion of 'aspect'.

A *basic idea* of a mathematical concept is a 'conceptual interpretation that gives it meaning' (Greefrath et al., 2016, p. 101). They connect the mathematical concept with 'familiar knowledge or experiences, or back to (mentally) represented actions' (vom Hofe & Blum, 2016, p. 230). They are 'an *idea* one simply has to get, to understand what the related mathematical content is essentially about and to make appropriate use of it' (Vohns, 2016, p. 127; emph. orig.). As such, basic ideas are often stated in a normative way, even though empirical research may also be conducted to find out about the '*individual images and explanatory models*' of individuals (vom Hofe & Blum, 2016, p. 232). On the other hand, Greefrath et al. (2016, p. 101) characterise an *aspect* of a mathematical concept as 'a subdomain of the concept that can be used to characterize it on the basis of mathematical content'. This comes closer to a mathematical definition, and it is usually grounded in mathematics only, rather than in an application or relation to real-life situations.

In the following, I will summarise aspects and basic ideas for Riemann integrals. Similar ideas are presented at numerous other places and within other frameworks than subject-matter didactics or with the aim to distinguish them in more detail in empirical research (e.g. recently Jones, 2020; Jones & Ely, 2023; Kouropatov & Dreyfus, 2013; Oehrtman & Simmons, 2023; Thompson & Harel, 2021). However, this list is concise and differentiates seven ideas in total, comprising many ideas specified in other literature.

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Greefrath et al. (2016) name three aspects of the Riemann integral $\int_a^b f(x)\mathrm{d}x$:

(1) The *product sum aspect*: Using this aspect, one may characterise $\int_a^b f(x)\mathrm{d}x$ as the limits of sums whose addends are composed of products of the lengths of subintervals of $[a, b]$ and function values or infima or suprema of the function on these subintervals. This idea emphasises the idea of the Riemann integral as a generalised sum. Using infinitesimals, one may also regard the products added up as products of function values and differentials (Ely, 2021).
(2) The *anti-derivative aspect*: This aspect characterises the Riemann integral as the difference between an anti-derivative F of f on $[a, b]$ at the upper and lower limits of integration, that is, $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$. Of course, this aspect is only appropriate if f is integrable and has an anti-derivative.

(3) The *measure aspect*: According to this aspect Riemann integrals satisfy 'fundamental properties of measure' when applied to areas, lengths, or volumes (Greefrath et al., 2016, p. 115). The proper foundation to make this aspect rigorous is measure theory.

The four basic ideas for Riemann integrals are (Greefrath et al., 2016, pp. 116–121):

(1) The *basic idea of area*: $\int_a^b f(x)\mathrm{d}x$ is the balance of the area enclosed by the graph of f, the vertical axes at $x = a$ and $x = b$, and the x-axis, where parts lying above the x-axis are weighted positively and the others negatively.
(2) The *basic idea of (re)construction*: With this basic idea one interprets $\int_a^b f(x)\mathrm{d}x$ as the change of a quantity between points a and b in space or time whose rate of change is f.
(3) *Basic idea of accumulation*: In this case, the integral $\int_a^b f(x)\mathrm{d}x$ is considered as the net accumulated sum of a quantity measure in terms of the values of f. Here, f is not necessarily interpreted as the rate of change of the quantity measured with the integral.
(4) The *basic idea of average*: According to this basic idea the Riemann integral $\int_a^b f(x)\mathrm{d}x$ corresponds to the value of a constant function with value m which has the same integral as f: $m = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$.

4. Previous results on the transition from Riemann integrals to other integrals

Already quite a lot of research has focused on the transfer of interpretations of Riemann integrals to other integrals. These transfers ground in the interpretations subsumed in the previous subsection (e.g. Jones & Dorko, 2015, and Martínez-Planell & Trigueros, 2021, for multivariate integrals; Dray & Manogue, 2023, Jones, 2020, and Ponce Campuzano et al., 2019, for real path integrals; Hanke (2020, 2022a), Oehrtman et al. (2019), Soto and Oehrtman (2022) for complex path integrals). But one has to keep in mind that the basic ideas and aspects are normative suggestions, not individual conceptions per se. For instance, Jones (2020, pp. 2–4) summarises his detailed literature review on students' conceptions of Riemann integrals with the list (1) *space underneath a graph*, (2) *anti-derivative*, (3) *adding up pieces*, (4) *accumulation from rate*, and (5) *averaging*, which are at least potentially generalisable to other integrals than Riemann integrals. In the next paragraphs, I will concentrate on recent studies related to real and complex path integrals in mathematics education.

4.1. Transferring the basic idea of average to real path integrals

Jones (2020) investigated the transfer of the basic idea of average to real path integrals. This idea transfers to the real path integral of the first kind rather smoothly. The real path integral of the first kind of the function h along a path γ may be interpreted as the mean value of h along $\mathrm{tr}(\gamma)$, $\bar{h}$, 'in which the varying values of [h] are understood to be averaged out evenly across the domain [$\mathrm{tr}(\gamma)$]', that is, $\int_\gamma h\mathrm{d}\mathbf{s} = \bar{h} \cdot L(\gamma)$ (Jones, 2020, p. 5). Formally, this means that one can either circularly define this average $\bar{h}$ as

$$\frac{1}{L(\gamma)}\int_\gamma h\mathrm{d}\mathbf{s}$$

or, avoiding circularity in the definition of average, it could also be defined as the limit of sums of the form $\frac{1}{n}\sum_{k=1}^{n} h(\gamma(t_k))$ for equipartitions $a = t_1 < \cdots < t_n = b$ of the domain of γ (where the notion of average is most readily visible in case that γ is a *simple* path, that is, has no self-intersections). In the second case, $\int_\gamma h\mathrm{d}\mathbf{s} = \bar{h} \cdot L(\gamma)$ then amounts to a theorem. For the real path integral of the second kind, Jones (2020, p. 6) states that 'it is unclear what exactly this average might look like for a vector field' and that the 'empirical evidence for this type of understanding [i.e. this basic idea; EH.] is fairly scant'. In fact, none of the 10 students he interviewed described real path integrals of the first kind in relation to averages. For real path integrals of the second kind, only the keywords 'average magnitude' and 'overall direction' were reported as empirical evidence from student answers (Jones, 2020, p. 15).

4.2. Transferring basic ideas to complex path integrals

It was investigated in Hanke (2022a) to what extent the aspects and basic ideas for Riemann integrals relate to complex path integrals. Four aspects (valid for continuous complex-valued functions and piecewise continuously differentiable paths) and four partial aspects (i.e. aspects that require more restricting conditions on the integrands or paths) for complex path integrals were found. I will only describe the aspects here because the partial aspects are not needed in the remainder. For this purpose, let $\Omega \subseteq \mathbb{C}$ denote a domain, $\gamma : [a, b] \to \Omega$ a piecewise continuously differentiable path, and $f = u + iv : \Omega \to \mathbb{C}$ a continuous function. The aspects are as follows (Hanke, 2022a, chs. 8–9):

(1) The *product sum aspect*: This aspect is a direct transfer from Riemann integrals, that is, given partitions $a = t_0 < \cdots < t_n = b$ and $\xi_k \in [t_{k-1}, t_k]$ for $1 \le k \le n$, $\int_\gamma f(z)\mathrm{d}z$ is the limit of

$$\sum_{k=1}^{n} f(\gamma(\xi_k)) \cdot (\gamma(t_k) - \gamma(t_{k-1})) \tag{1}$$

as $n \to \infty$ and $\min_{k=1,\ldots,n}|t_k - t_{k-1}| \to 0$.

(2) The *substitution aspect*: $\int_\gamma f(z)\mathrm{d}z$ is

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t \tag{2}$$

and can thus be defined without any reference to limits of sums except for that the second integral is a Riemann integral potentially previously defined with limits of sums.

(3) The *vector analysis aspect*: The complex path integral can be defined in terms of two real path integrals of the second kind:

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma (u, -v)\mathrm{d}\mathbf{T} + i\int_\gamma (v, u)\mathrm{d}\mathbf{T}.$$

(4) The *mean value aspect*: Let T denote the unit tangential vector attached to the path of integration, that is, $T(\gamma(t)) = \gamma'(t)/|\gamma'(t)|$ for $t \in [a, b]$ (without any significant

restriction, it can be assumed that $\gamma'(t) \neq 0$), then

$$\int_\gamma f(z)\mathrm{d}z = L(\gamma) \cdot \operatorname*{av}_{z \in \mathrm{tr}(\gamma)} [f(z)T(z)]. \tag{3}$$

Here, $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)T(z)]$ is the *average* of the function $f \cdot T$ on the domain of integration, but not the average of f. This aspect goes back to Gluchoff (1991). Note that for the case of complex path integrals, the mean value aspect is a full aspect, not only a basic idea. This is because the average in Equation (3) may be defined without circularly depending on $\int_\gamma f(z)\mathrm{d}z$; hence, Equation (3) is indeed a possible *definition* of $\int_\gamma f(z)\mathrm{d}z$. This mean value aspect and its derivation will be analysed further below in the epistemological analysis.

Additionally, Soto and Oehrtman (2022) have shown that students from a complex analysis course, who had not yet been taught complex integration, may transfer the basic idea of accumulation to complex path integrals, but were not sure what is accumulated. This resonates with findings by Hanke (2022a) and Oehrtman et al. (2019) who have conducted expert interviews about how lecturers interpret complex path integrals and showed that the majority of the interviewed experts struggled to interpret complex path integrals. Keeping this in mind, it is not surprising that Gluchoff (1991) described that, in his experience,

> students are mystified on first exposure to this concept, and working examples by the formula $\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$ can be a baffling experience; what sense is a beginning student to make of the results $\int_{|z|=1} \mathrm{Re}(z)\mathrm{d}z = \pi i$ or $\int_1^i \frac{i}{z}\mathrm{d}z = \frac{-\pi}{2}$? (pp. 641–642; notation adapted)[3]

5. Epistemological analysis: transferring the basic idea of average to complex analysis

In the following, I present the epistemological analysis of the transfer of the basic idea of average to complex path integrals. For that purpose, I examine the characteristics of this basic idea and look for their counterparts in the case of complex analysis. This is in line with subject-matter didactic analyses for school mathematics (e.g. Greefrath et al., 2016; Hußmann et al., 2016), but adapted to university-level mathematics, in which precise definitions and propositions play a much more dominant role. Therefore, I adhere to a great level of mathematical detail, akin to the expositions by Dray and Manogue (2023), Hochmuth (2021), or Kondratieva and Winsløw (2018). The epistemological analysis presented here partially belonged to the epistemological analysis with a much wider scope on the transfer of aspects and basic ideas to complex analysis in Hanke (2022a), for which more than fifty textbooks on complex analysis were examined.

5.1. Three features of the basic idea of average for Riemann integrals

The basic idea of average is little represented in mathematics education research literature when compared to the other interpretations referred to above (see e.g. Jones' (2020) survey). Yet it is a reasonable interpretation for several reasons, which students are likely to encounter in their calculus/real analysis courses.

(1) First, the constant function on $[a, b]$ with value $m := \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x$ has the same integral as g, that is,

$$\int_a^b g(x)\mathrm{d}x = \int_a^b m\mathrm{d}x = (b-a)\cdot m. \tag{4}$$

(2) Second, the *mean value theorem* belongs to the central theorems in calculus/real analysis. According to this theorem, if g is continuous, then there is a point $\xi \in [a, b]$ such that

$$\int_a^b g(x)\mathrm{d}x = g(\xi)(b-a). \tag{5}$$

(3) Third, the integral is like a continuous version of the arithmetic mean: If for each $n \in \mathbb{N}$ the points $\xi_k \in [t_{k-1}, t_k]$ $(k = 1, \ldots, n)$ are distributed equally on $[a, b]$ (i.e. each ξ_k is an element of the subintervals $[t_{k-1}, t_k]$ of an equidistant partition of $[a, b]$), the arithmetic means

$$\frac{1}{n}\sum_{k=1}^{n} g(\xi_k) = \frac{1}{b-a}\sum_{k=1}^{n}\frac{b-a}{n}g(\xi_k) \tag{6}$$

converge to $\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x$ as $n \to \infty$ if the integral exists.

These properties generalise directly to the case of measure-theoretic integrals. For instance, if Θ is a finite measure space with measure μ on a σ-algebra of Θ, $\mu(\Theta) \neq 0$, and $\rho : \Theta \to \bar{\mathbb{R}}$ is a μ-integrable function, then $M := \mu(\Theta)^{-1}\int_\Theta \rho\mathrm{d}\mu$ satisfies an obvious analogue to Equation (4). If Θ is a probability space (i.e. $\mu(\Theta) = 1$, the expected value $\mathrm{E}(\rho) = \int_\Theta \rho\mathrm{d}\mu$ of a random variable ρ may also be regarded as a generalised version of the mean value (Axler, 2020).

5.2. Similarities and differences between complex path integrals and real integrals

Complex path integrals share several formal properties to real integrals. For example, the mapping $f \mapsto \int_\gamma f(z)\mathrm{d}z$ is linear ($\mathbb{C}$-linear in this case), $\gamma \mapsto \int_\gamma f(z)\mathrm{d}z$ is additive (that is, the integrals along the juxtaposition of two paths equals the sum of the integrals along the two paths), and the inequality $\left|\int_\gamma f(z)\mathrm{d}z\right| \leq L(\gamma)\max_{z\in\mathrm{tr}(\gamma)}|f(z)|$ is generally satisfied (Lang, 1999). In addition, the structure present in the *product sum* and *substitution aspect* aligns with the definition of Riemann or real path integrals, the only difference is that the multiplication is from $\mathbb{C}$, not from $\mathbb{R}$ or a scalar product in $\mathbb{R}^2$. Accordingly, it might be expected that other properties or basic ideas as stated by Greefrath et al. (2016) transfer from the real to the complex case as well. It is likely that at least some students will assume that this is the case, in particular, if instructors emphasise these basic ideas in calculus/real analysis classes. Soto and Oehrtman's (2022) case study, in which students transferred the basic idea of accumulation to complex path integrals individually but were uncertain about what was accumulated, supports this hypothesis.

However, since the products appearing in Equations (1) and (2) are complex multiplication, complex path integrals are also fundamentally different from their real counterparts.

For instance, this is reflected in the *vector analysis aspect* because $\int_\gamma f(z)\mathrm{d}z$ is in general not equal to the formal analogues of real path integrals of the first and second kind $\int_\gamma f\mathrm{d}\mathbf{s} = \int_\gamma u\mathrm{d}\mathbf{s} + i\int_\gamma v\mathrm{d}\mathbf{s}$ and $\int_\gamma (u,v)\mathrm{d}\mathbf{T}$.

Finally, important integral theorems of complex analysis have no counterparts for real integrals. I recall them here in their simplest form for future reference. If $\Omega \subseteq \mathbb{C}$ is an open set, $f : \Omega \to \mathbb{C}$ *holomorphic*, $\omega \in \Omega$, and $r > 0$ such that the closure of $B_r(\omega) := \{z \in \mathbb{C} : |z-\omega| < r\}$ is completely contained in Ω; then

- *Cauchy's integral theorem states that* $\int_{\partial B_r(\omega)} f(z)\mathrm{d}z = 0$ and
- *Cauchy's integral formula states that* $\frac{1}{2\pi i}\int_{\partial B_r(\omega)} \frac{f(z)}{z-z_0}\mathrm{d}z = f(z_0)$ for all $z_0 \in B_r(\omega)$.

5.3. Average interpretations for complex path integrals

Since $\frac{1}{L(\gamma)}\int_\gamma h\mathrm{d}\mathbf{s}$ may be interpreted as the average of a real-valued function h on $\mathrm{tr}(\gamma)$, the *average* of a complex-valued function f on $\mathrm{tr}(\gamma)$ may be defined as

$$\underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}[f(z)] := \frac{1}{L(\gamma)}\int_\gamma \mathrm{Re}(f)\mathrm{d}\mathbf{s} + i\frac{1}{L(\gamma)}\int_\gamma \mathrm{Im}(f)\mathrm{d}\mathbf{s} = \frac{1}{L(\gamma)}\int_\gamma f\mathrm{d}\mathbf{s}. \tag{7}$$

In other words, the real and imaginary part of $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)]$ are the averages of the real and imaginary part of the function. In accordance with Equation (4), this average satisfies

$$\int_\gamma f\mathrm{d}\mathbf{s} = \int_\gamma \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}[f(z)]\mathrm{d}\mathbf{s} = \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}[f(z)]\cdot L(\gamma).$$

It was described above that there is a mean value aspect for complex path integrals by Gluchoff (1991). He derived it as follows: For each $n \in \mathbb{N}$ suppose that $z_0, z_1, \ldots, z_n$ are points on $\mathrm{tr}(\gamma)$ chosen such that $|z_k - z_{k-1}| = L(\gamma)/n$ for $k = 1, \ldots, n$, then,

$$\frac{1}{L(\gamma)}\int_\gamma f(z)\mathrm{d}z = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(z_{k-1})\frac{z_k - z_{k-1}}{|z_k - z_{k-1}|} = \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}[f(z)T(z)],$$

where the average corresponds to Equation (7) for $f\cdot T$ (recall $T(\gamma(t)) = \gamma'(t)/|\gamma'(t)|$ for $t \in [a,b]$). This heuristic derivation relies on the chosen equidistant partition, but a formal computation using Equations (2) and (7) also implies

$$\frac{1}{L(\gamma)}\int_\gamma f(z)\mathrm{d}z = \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}[f(z)T(z)]. \tag{8}$$

In summary, after dividing by $L(\gamma)$, the complex path integral is the average of the function $f\cdot T$ on the trace of γ, which may be interpreted as the function multiplied by a twist induced by the direction of the path.[4]

As a first example, consider $f(z) = z^2$ and a path γ traversing the unit circle once anticlockwise. Then, $T(z) = iz$ for $z \in \partial B_1(0)$ and Equation (8) yields

$$\int_{\partial B_1(0)} z^2\mathrm{d}z = 2\pi\cdot\underset{z\in\partial B_1(0)}{\mathrm{av}}[z^2\cdot iz] = 2\pi i\cdot\underset{z\in\partial B_1(0)}{\mathrm{av}}[z^3] = 2\pi i\cdot 0 = 0$$

(Gluchoff, 1991, p. 643).

It is worth noticing that the average value in Equation (8), unlike Equations (4) and (7), does not involve the integrand f itself, but the modified function $f \cdot T$ on the trace $\mathrm{tr}(\gamma)$. Thus, the naïve transfer from the basic idea of average, namely that the complex path integral $\frac{1}{L(\gamma)} \int_\gamma f(z)\mathrm{d}z$ was the average of f in the sense of $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)]$, is not endorsable in general. From the perspectives of learners, this may seem rather surprising because Equation (8) is derived similarly to Equation (6). The difference here is that the numbers $z_k - z_{k-1}$ or $\gamma(t_k) - \gamma(t_{k-1})$ in the product sum aspect of complex path integrals are complex numbers and do not necessarily agree with $|z_k - z_{k-1}|$ or $|\gamma(t_k) - \gamma(t_{k-1})|$. Similarly, in the substitution aspect $\gamma'(t)$ is complex multiplied to $f(\gamma(t))$. Notwithstanding, it is also perfectly reasonable that $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)]$ cannot be the right choice to determine $\frac{1}{L(\gamma)} \int_\gamma f(z)\mathrm{d}z$ because the complex path integral depends on the parametrization and not only the trace of γ itself. The dependence on the parametrization may be familiar to students however from real path integrals of second kind and indirectly from Riemann integrals: In the latter case, $\int_a^b g(x)\mathrm{d}x$ is first defined for intervals $[a, b]$ with $a < b$ and $\int_b^a g(x)\mathrm{d}x := -\int_a^b g(x)\mathrm{d}x$ is a *separate* definition.

I have already dealt with the analogue to Equation (6). Now, I describe counterparts to Equations (4) and (5), which were constitutive for the basic idea of average for Riemann integrals besides Equation (6). Let $m_1 := \mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)]$ and $m_2 := \mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)T(z)]$. On the one hand, replacing the difference $b - a$ in Equation (4) with the difference of the start and endpoint of γ, $\gamma(b) - \gamma(a)$, the first of these averages satisfies the second but in general not the first equality in Equation (4) because we have

$$\int_\gamma m_1 \mathrm{d}z = m_1 \cdot (\gamma(b) - \gamma(a))$$

and

$$\int_\gamma f(z)\mathrm{d}z \neq \int_\gamma m_1 \mathrm{d}z \text{ and } \frac{1}{L(\gamma)} \int_\gamma f(z)\mathrm{d}z \neq \int_\gamma m_1 \mathrm{d}z.$$

On the other hand, the second of these averages satisfies a different set of equations analogue to the two equalities in Equation (4), namely

$$\int_\gamma f(z)\mathrm{d}z = m_2 \cdot L(\gamma) \text{ and } \int_\gamma m_2 \mathrm{d}z = m_2 \cdot (\gamma(b) - \gamma(a)),$$

hence mixing proper placements of $L(\gamma)$ and $\gamma(b) - \gamma(a)$ as analogues for $b - a$. Hence, care must be taken when deriving equations involving the complex path integral and the averages m_1 and m_2. The mean value theorem as in Equation (5) is also no longer valid. To show this, assume that γ is the directed line segment from $c \in \mathbb{C}$ to $d \in \mathbb{C}$. Then, there does not need to be a ξ on that line segment such that $\int_\gamma f(z)\mathrm{d}z = f(\xi)(d - c)$: For instance, for $f(z) = e^{iz}$, $c = 0$, and $d = 1$, this would lead to the equation $0 = \int_0^1 e^{it}\mathrm{d}t = e^{i\xi}$, which has no solution in ξ. However, another form of the mean value theorem is true, which is based on the separation into real and imaginary parts: One can find $\xi_1, \xi_2 \in \mathrm{tr}(\gamma)$ such that

$$\int_\gamma f(z)\mathrm{d}z = (d - c)(\mathrm{Re}\,(f(\xi_1)) + i\,\mathrm{Im}\,(f(\xi_2)))$$

(Rodríguez et al., 2013, p. 109).

5.4. Averages and Cauchy's integral formula

Another close connection between averages of functions and complex path integrals is given by *Cauchy's integral formula* (see above). Let f be holomorphic on an open neighbourhood of $\partial B_r(\omega)$ and $z_0 = \omega$, then Cauchy's integral formula yields

$$f(\omega) = \frac{1}{2\pi i}\int_{\partial B_r(\omega)} \frac{f(z)}{z-\omega}\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(\omega + re^{it})\mathrm{d}t. \tag{9}$$

The right-hand side is simply the average of f on the boundary of the ball $B_r(\omega)$ in the usual sense of mean values for (complex-valued) functions. Noteworthy, the complex path integral $\int_{\partial B_r(\omega)} f(z)\mathrm{d}z$ is not involved here. It cannot even be further away from being involved because $\int_{\partial B_r(\omega)} f(z)\mathrm{d}z = 0$ by *Cauchy's integral theorem*. Equation (9) resonates well with the mean value property of harmonic functions: If u is a harmonic function on an open neighbourhood of $B_r(\omega)$ in $\mathbb{R}^2$, then $u(\omega) = \frac{1}{2\pi}\int_0^{2\pi} u(\omega + e^{it})\mathrm{d}t$ (where I identified vectors in $\mathbb{R}^2$ with elements of $\mathbb{C}$) (Lang, 1999, p. 261). As is the case with harmonic functions, several textbooks on complex analysis explicitly state equation (9) as a theorem with the name 'mean value property' (e.g. Remmert, 1998, p. 203).

Let me finish this analysis with another simple, yet illustrative example, suitable for a comparison of the mean value interpretations in teaching. Let $f(z) \equiv c \in \mathbb{C}$ be a constant function and consider the path traversing the boundary of the unit circle $\partial B_1(0)$ once anticlockwise. The previously considered quantities are now the following:

- Since f is constant, it is holomorphic, and thus $\int_{\partial B_1(0)} f(z)\mathrm{d}z = 0$ by Cauchy's integral theorem.
- Cauchy's integral formula applied to the centre of the unit circle yields $c = f(0) = \frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{c}{z}\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} c\mathrm{d}t = c$.
- The averages m_1 and m_2 of f and $f \cdot T$ compute to $m_1 = \mathrm{av}_{z\in\partial B_1(0)}[f(z)] = \mathrm{av}_{z\in\partial B_1(0)}[c] = c$ and $m_2 = \mathrm{av}_{z\in\partial B_1(0)}[f(z)T(z)] = \mathrm{av}_{z\in\partial B_1(0)}[ciz] = ci\,\mathrm{av}_{z\in\partial B_1(0)}[z] = 0$ because $T(z) = iz$ for $z \in \partial B_1(0)$.

These computations agree with the observations made before. An explicit example for the inequalities $\int_\gamma f(z)\mathrm{d}z \neq \int_\gamma m_1\mathrm{d}z \neq \frac{1}{L(\gamma)}\int_\gamma f(z)\mathrm{d}z$ is moreover given by $f(z) = \mathrm{Re}(z) + 2\mathrm{Im}(z) + i\mathrm{Im}(z)^2$ and with the same path along $\partial B_1(0)$ (Hanke, 2022a, pp. 134, 159).

Since the relationships between the different mean values (and their properties in real and complex analysis) and Cauchy's integral formula are likely very demanding for learners, they must be made explicit in teaching if one wants to establish a coherent understanding of integrals and averages throughout the curriculum. Even an example as simple as that of a constant function is likely suitable for a demonstration in class. Explicit tasks on calculating $\int_\gamma f(z)\mathrm{d}z$, Cauchy's integral formula, and the mean values $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)]$ and $\mathrm{av}_{z\in\mathrm{tr}(\gamma)}[f(z)T(z)]$ for various functions may not only help familiarise students with basic methods to determine complex path integrals. Such tasks may also help them see horizontal connections between the current topics of the complex analysis course and vertical connections by using the core idea of average for the teaching of the cross-curricular topic integral in the sense of Klein's Plan B. Table 1 summarises the mean value interpretations considered in this paper for the case of real-valued functions $g : [a, b] \to \mathbb{R}$ and

Table 1. Comparison between mean value interpretations for functions of one real and one complex variable.

	$g : [a, b] \to \mathbb{R}$ continuous	$f : \mathrm{tr}(\gamma) \to \mathbb{C}$ continuous, γ regular piecewise continuously differentiable path
Mean value for f	$m := \frac{1}{b-a}\int_a^b g(x)dx$	$m_1 := \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}(f(z)) = \frac{1}{L(\gamma)}\int_\gamma f\mathrm{d}s$ and $m_2 := \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}}(f(z)T(z)) = \frac{1}{L(\gamma)}\int_\gamma f\cdot T\mathrm{d}s$, where T is the unit tangential vector field induced by γ
Relationship to Riemann sums	$m = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} g(x_k)\right)$ 'generalization of arithmetic mean' for points $x_1, x_2, \ldots, x_n$ in each subinterval of an equidistant partition of $[a, b]$	$m_2 = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(\xi_k)\cdot\frac{z_k - z_{k-1}}{\lvert z_k - z_{k-1}\rvert}\right)$ for an equidistant partition $z_1, z_2, \ldots, z_n$ of $\mathrm{tr}(\gamma)$ and $\xi_1, \xi_2, \ldots, \xi_n$ on the respective segments on $\mathrm{tr}(\gamma)$
Relationship between mean value and integral	$\int_a^b g(x)\mathrm{d}x = \int_a^b m\mathrm{d}x = m\cdot(b-a)$	$\frac{1}{\star}\int_\gamma f(z)\mathrm{d}z \underbrace{\neq}_{i.g.} \int_\gamma m_1\mathrm{d}z = m_1(\gamma(b) - \gamma(a))$ for $\star = L(\gamma)$ or $\star = \gamma(b) - \gamma(a)$ if $\gamma(a) \neq \gamma(b)$ and $\int_\gamma f(z)\mathrm{d}z = m_2\cdot L(\gamma), \int_\gamma m_2\mathrm{d}z = m_2\cdot(\gamma(b) - \gamma(a))$
Mean value theorem	there is a $\xi \in [a; b]$ such that $\int_a^b g(x)\mathrm{d}x = g(\xi)\cdot(b-a)$	for the directed line segment γ from c to d in $\mathbb{C}$ there are $\xi_1, \xi_2 \in \mathrm{tr}(\gamma)$ such that $\int_\gamma f(z)\mathrm{d}z = (d-c)(\mathrm{Re}\,(f(\xi_1)) + i\,\mathrm{Im}\,(f(\xi_2)))$
	no analogue	*Cauchy's integral formula* $f(\omega) = \frac{1}{2\pi i}\int_{\partial B_r(\omega)}\frac{f(z)}{z-\omega}\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(\omega + re^{it})$ for f holomorphic on Ω and $\overline{B_r(\omega)} \subseteq \Omega$

the Riemann integral and complex-valued functions $f : \mathrm{tr}(\gamma) \to \mathbb{C}$ and the complex path integral.

6. The case of Sebastian

To support the epistemological analysis, I will now add an example of a lecturer's individual interpretations of the notion of complex path integral from a larger study of mine (Hanke, 2022a, 2022b). This example supports the epistemological analysis in two ways. First, it shows that the interpretation of integrals as averages is indeed present among experts. Second, it confirms that identifying the functions from which the averages are taken is a crucial moment in this transfer. The reader may find detailed analyses and methodological information in Hanke (2022a, 2022b). In retrospect, the following interview passage may in fact be seen as the original motivation for my epistemological analysis.

Sebastian (anonymized) is a professor of mathematics at a mid-sized German university with teaching experience in complex analysis. In his research, he encounters many different integrals. Prior to the interview, Sebastian has never attempted to transfer the basic idea of average to complex path integrals. This basic idea has also not been mentioned by the interviewer before. The interviewer asks Sebastian to interpret $\int_\gamma f(z)\mathrm{d}z$. Sebastian first rejects the basic idea of area (not shown here) and then describes integrals as mean values in general:

Sebastian: […] I would always tell my pupils: Actually, one should think about mean values, in particular when one has Lebesgue integration in mind, and measures. It is about measuring. And, uhm, this geometric intuition [of area] can destroy this higher dimensional situation. […] And therefore I find it much better if one imagines: integration is mean value formation.

He thus supports the core idea presented with the words 'mean value formation' as suitable for the cross-curricular topic 'integration'. In particular, he does not yet talk about complex path integrals specifically. Next, the interviewer wants to know what is being averaged here. Sebastian proceeds with two explanations in line with $\mathrm{av}_{z\in \mathrm{tr}(\gamma)}[f(z)]$.

Interviewer: And which/ mean value of what?

Sebastian: Yes, of what's, uh, in the integrand, so to speak. […] Yes, uh, for me this is simply the mean value of the complex numbers, which I grab along this path. Therefore this is again a complex number because it does not have a geometrical area meaning, but mean value formation over the objects, which one quasi sees along the path. And, uhm, in my view this has nothing to do with area. […]

This first explanation realises the mean value as one for 'the complex numbers […] along this path'. Since this yields a complex number, Sebastian does not consider the interpretation as an area as adequate. In my view, this description corresponds to the average of f along γ in the sense of $\mathrm{av}_{z\in \mathrm{tr}(\gamma)}[f(z)]$.

For the second realisation of mean value that Sebastian produces, recall that multiplication with a complex number $re^{i\varphi}$ ($r \geq 0$, $\varphi \in \mathbb{R}$) induces a dilation by the modulus r followed by a rotation by the argument φ measured in radian. Needham (1997) calls such a dilation-rotation an *amplitwist*. Accordingly, any function value $f(z)$, which Sebastian 'grab[s] along this path' can also be said to induce a dilation by its modulus and a rotation by its argument. In line with this geometric interpretation of complex numbers, Sebastian explains further:

Sebastian: […] This is actually the rotation that one measures on the plane. And this/ here we are again at what we discussed previously, that these, uh, complex numbers always have this character of an amplitwist. […] And on the other hand, I just have these values of the function f of z and, uh, f of z does now what we have seen previously, yes, this now maps some portion of what one has here with this [grid; see Figure 1] […]. And, uhm, geometrically speaking, this involves such a stretching and a twist probably, yes, so where the grid points are somehow distorted or so. […] And I average this effect along this path so to speak. [… a few turns later: …] So the number f of z really is a linear amplitwist for me. And this effect is averaged along this path and this is what the integral means to me.

Consequently, the mean value Sebastian addressed previously might be interpreted in an abstract sense as a mean value of the amplitwists induced by f on the trace of γ.[5] The

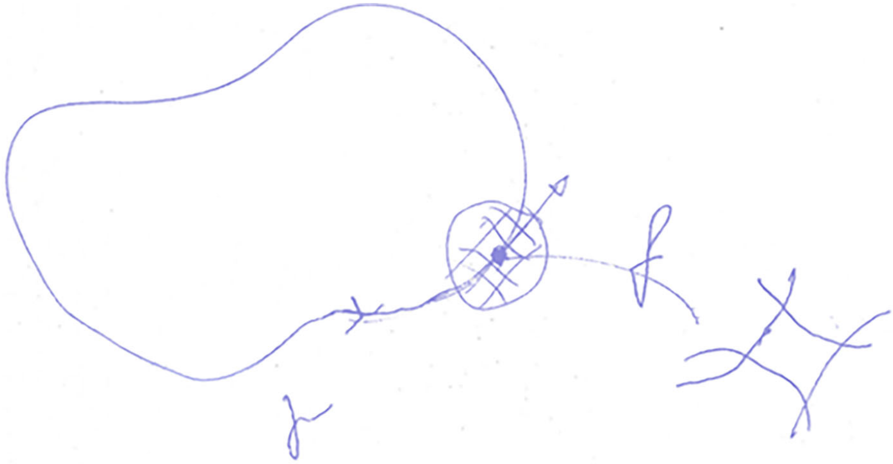

Figure 1. Sebastian's drawing of a path γ as well as the dilation and rotation induced by f (Figure 1 from Hanke, 2022b; see also Figure 15.3(b) from Hanke, 2022a).

influence of the parametrization of the path (e.g. γ' or T) is absent from Sebastian's explanation of how the average is formed. In the epistemological analysis, it was discussed in detail that the parametrization is not negligible and that the average of $f \cdot T$ must be used to get $\frac{1}{L(\gamma)} \int_\gamma f(z)\mathrm{d}z$.

In summary, two realisations for the mean value of f could be found in this interview excerpt as a response to interpret complex path integrals, one as a mean value of the values of f as complex numbers and one as the mean value of the amplitwists induced by f. While this observation supports that the basic idea of average is a useful core idea for the teaching of integrals, it also underlines the challenges identified in the epistemological analysis in substantiating that $\mathrm{av}_{z\in \mathrm{tr}(\gamma)}[f(z)T(z)]$ instead of $\mathrm{av}_{z\in \mathrm{tr}(\gamma)}[f(z)]$ is the appropriate choice of average needed to represent complex path integrals.

7. Discussion

Recently, mathematics educators have identified the gap that little support is available for structuring the teaching and learning of advanced mathematical topics (i.e. roughly from the second to third year onwards) (e.g. Hochmuth et al., 2021; Winsløw et al., 2021). One possibility to develop such materials is to focus on core ideas for cross-curricular concepts that may be adapted to each instance of these concepts. For the teaching of integrals, such ideas for Riemann integrals (see e.g. Greefrath et al., 2016; Jones, 2020) must be carefully adapted for a successful transfer to more advanced integrals as was shown mostly for the basic ideas of area, accumulation, or (re)construction from a rate of change (e.g. Dray & Manogue, 2023; Jones, 2020; Jones & Dorko, 2015; Jones & Ely, 2023; Oehrtman & Simmons, 2023; Soto & Oehrtman, 2022). Moreover, Jones (2020, p. 15) concluded that there is 'something uniquely challenging in constructing understandings for [real path] integral expressions'. In the present epistemological analysis, I contributed to clarifying what this challenge looks like for the transfer of the basic idea of average to complex path integrals. The main difficulty is to realise that $\frac{1}{L(\gamma)} \int_\gamma f(z)\mathrm{d}z$ is the average of the values of $f(z)$ rotated and dilated by $T(z)$ for points the $z \in \mathrm{tr}(\gamma)$, not simply of the values $f(z)$ (Gluchoff, 1991). Such a potential overgeneralisation of the basic idea of average to complex path integrals was underlined in a case study with one lecturer. Additionally, Cauchy's integral formula yields another interpretation of the value of a holomorphic function in the centre of a circle in terms of the average of the values on the boundary.

Considering the research presented in this article, as well as that of Jones (2020) and the special issue on the teaching of definite integrals (Ely & Jones, 2023), among others, further research is needed to identify and align potential core ideas and to find out how students and lecturers actually realise these ideas for themselves and in teaching. For example, how widespread is the basic idea of average among students and lecturers, and how do they attempt to transfer it from Riemann to complex path integrals?

The presence of such core ideas in the early stages of mathematics teaching may cause students to ask what these ideas may mean for integrals in advanced mathematics courses. In particular, it is likely that highlighting certain interpretations as particularly important will lead learners to believe that a transfer of these central ideas is generally possible. In the classroom, therefore, attention needs to be paid to the ways in which individual ideas are presented as particularly central, and then, in later courses, attention needs to be paid to how these ideas need to be adapted. In this sense, I have tried to contribute to the discussion about the possibility of a vertically aligned curriculum for teaching integrals. However, this discussion also raises the question of whether the possibility of using overarching core ideas should perhaps be abandoned in favour of mostly locally appropriate ideas, which then of course need to be developed individually. Therefore, it is necessary to develop and empirically investigate further proposals for teaching higher mathematical content.

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Notes

1. A path in $\mathbb{R}^2 \cong \mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{R}^2 \cong \mathbb{C}$ ($a < b$ are real numbers). For the sake of simplicity, it is assumed in most parts of this article that the paths are continuously differentiable (i.e. γ' exists and is continuous) and simple or simple closed (i.e. $\gamma_{|[a,b)}$ or $\gamma_{|[a,b]}$ is injective), and all functions to be integrated are continuous. Let $\mathrm{tr}(\gamma)$ denote the trace of γ, that is, the curve traversed by γ, and let $L(\gamma)$ denote the length of γ. I will use the signifier 'path integral' throughout this paper to denote what others may also call 'line integrals', 'contour integrals', 'curvilinear integrals' etc. I will reserve the use of f for complex-valued functions of one complex variable, g for real-valued functions of one real variable, h for real-valued functions of two real variables, and the tuple notation (u, v) for vector fields of two real variables. Three types of path integrals are relevant for the present study: (1) *real path integrals of first kind*, which take a path in $\mathbb{R}^2$ and a real-valued function of two real variables as inputs and may be defined as $\int_\gamma (u, v)\mathrm{d}\mathbf{T} := \int_a^b h(\gamma(t))|\gamma'(t)|\mathrm{d}t$; by means of analogy

also set $\int_\gamma f \mathrm{d}\mathbf{s} := \int_\gamma \mathrm{Re}(f)\mathrm{d}\mathbf{s} + i\int_\gamma \mathrm{Im}(f)\mathrm{d}\mathbf{s}$; (2) *real path integrals of second kind*, which take a path in $\mathbb{R}^2$ and an $\mathbb{R}^2$-valued function of two real variables as inputs and may be defined as $\int_\gamma (u, v)\mathrm{d}\mathbf{T} := \int_a^b (u(\gamma(t)),\ v(\gamma(t)))*\gamma'(t)\mathrm{d}t$, where $*$ represents the scalar product; and (3) *complex path integrals*, which take a path in $\mathbb{C}$ and a complex-valued function of one complex variable as inputs and may be defined as $\int_\gamma f(z)\mathrm{d}z := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t$, where $\cdot$ is complex multiplication.

2. For the case of integrals, I will later refer to the notion of 'basic ideas' (Greefrath et al., 2016; vom Hofe, 1995; vom Hofe & Blum, 2016; see 'Subject-matter didactical approach to interpretations of integrals: Aspects and basic ideas'). Basic ideas are intended to provide meaning for mathematical concepts within inner- and extra-mathematical situations learners are already familiar with or should be familiarised with. That is, using the term 'core idea' I intend to identify an idea suitable for interpreting a cross-curricular concept in general, and the term 'basic idea' refers specifically to the work by Greefrath et al. (2016). It may be the case that a basic idea for a particular instance of a cross-curricular concept proves to be transferable to other instances of the cross-curricular concept. If this is the case, a basic idea functions as a core idea in the sense discussed in this paper. In fact, this is exactly what the present investigation is about: finding out in which way the basic idea of average for Riemann integrals proves to be transferable to complex path integrals.
3. For example, in the left integral, the integrand is purely real and the path of integration is distributed uniformly around the origin, yet the integral is purely imaginary.
4. See also Pringsheim (1925) who used a mean value for complex functions to replace the notion of complex path integral in proofs of power and Laurent series developments of holomorphic functions. This approach, however, 'which is only integral-free insofar as its inner workings aren't examined, has not caught on' (Remmert, 1998, p. 352).
5. It must be hypothesised that Sebastian errs here. The geometric effect of f on a portion of the complex plane is approximately an amplitwist induced by the derivative f' (in case f is holomorphic), not f. This follows from the approximation $\Delta f \approx f' \cdot \Delta z$. The reader may consult Needham (1997) for details.

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