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The $\mathcal{N} = 4$ effective action of type IIA supergravity compactified on $\mathrm{SU}(2)$-structure manifolds

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Abstract: We study compactifications of type IIA supergravity on six-dimensional manifolds with SU(2) structure and compute the low-energy effective action in terms of the non-trivial intrinsic torsion. The consistency with gauged $\mathcal{N} = 4$ supergravity is established and the gauge group is determined. Depending on the structure of the intrinsic torsion, antisymmetric tensor fields can become massive.

Keywords: Flux compactifications, Extended Supersymmetry, Supergravity Models, Supersymmetric Effective Theories

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Contents

1 Introduction

Compactification of ten-dimensional supergravities on generalized manifolds with G-structure has been studied for some time.[1] These manifolds are characterized by a reduced structure group G which, when appropriately chosen, preserves part of the original ten-dimensional supersymmetry [4, 5]. Furthermore, they generically have a non-trivial torsion which physically corresponds to gauge charges or mass parameters for some anti-symmetric tensor gauge potentials. Therefore, the low-energy effective action is a gauged or massive supergravity with a scalar potential which (partially) lifts the vacuum degeneracy present in conventional Calabi-Yau compactifications. The critical points of this scalar potential can further spontaneously break (some of) the left-over supercharges. As a consequence of this, such backgrounds are of interest both from a particle physics and a cosmological perspective.

[1] For reviews on this subject see, for example, [1–3] and references therein.

Most studies so far concentrated on six-dimensional manifolds with SU(3) or more generally SU(3) $\times$ SU(3) structure. Compactifying the ten-dimensional heterotic/type I supergravity on such manifolds leads to an $\mathcal{N}=1$ effective theory in four dimensions [6–11], while compactifying type II supergravity results in an $\mathcal{N}=2$ theory [12–17]. By employing an appropriate orientifold projection [18, 19] or by means of spontaneous supersymmetry breaking [20, 21], this $\mathcal{N}=2$ can be further broken to $\mathcal{N}=1$ (or $\mathcal{N}=0$).

A similar study for six-dimensional manifolds with SU(2) or SU(2) $\times$ SU(2) structure which generalize Calabi-Yau compactifications on $K3 \times T^2$ has not been completed yet. In refs. [5, 22, 23], geometrical properties of such manifolds were studied and the scalar field space was determined. Furthermore, it was shown in ref. [23] that manifolds with SU(2) $\times$ SU(2) structure cannot exist and therefore we only discuss the case of a single SU(2) in this paper. In ref. [24], the heterotic string was then compactified on manifolds with SU(2) structure and the $\mathcal{N}=2$ low-energy effective action was derived. In [25], type IIA compactifications on SU(2) orientifolds were studied and again the corresponding $\mathcal{N}=2$ effective action was determined. Finally in refs. [26, 27], preliminary studies of the $\mathcal{N}=4$ effective action for type IIA compactification on manifolds with SU(2) structure were conducted.[2]

The purpose of this paper is to continue these studies and in particular determine the bosonic $\mathcal{N}=4$ effective action of the corresponding gauged supergravity. One of the technical difficulties arises from the fact that frequently in these compactifications magnetically charged multiplets and/or massive tensors appear in the low-energy spectrum. Fortunately, the most general $\mathcal{N}=4$ supergravity covering such cases has been determined in ref. [33] using the embedding tensor formalism of ref. [34]. We therefore rewrite the action obtained from a Kaluza-Klein (KK) reduction in a form which is consistent with the results of [33]. As we will see, this amounts to a number of field redefinitions and duality transformations in order to choose an appropriate symplectic frame.

The organization of this paper is as follows: In section 2 we briefly review the relevant geometrical aspects of SU(2)-structure manifolds and set the stage for carrying out the compactification. Section 3.1 deals with the reduction of the NS-sector, which in fact coincides with the heterotic analysis carried out in [24] and therefore we basically recall their results. In section 3.2 we compactify the RR-sector and give the effective action in the KK-basis. In section 4 we perform the appropriate field redefinitions and duality transformations in order to compare the action with the results of ref. [33]. This allows us to determine the components of the embedding tensor parametrizing the $\mathcal{N}=4$ gauged supergravity action in terms of the intrinsic torsion. From the embedding tensor we then can easily compute the gauge group in section 4.3. Section 5 contains our conclusions and some of the technical material is supplied in the appendices A and B.

[2] The effective action for IIA compactified on $K3 \times T^2$ has been given in [28, 29]. $\mathcal{N}=4$ flux compactifications have been discussed for example in [30–32].

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2 SU(2) structures in six-manifolds

2.1 General setting

In this paper, we study type IIA space-time backgrounds of the form

$$M_{1,3} \times Y\,, \tag{2.1}$$

where $M_{1,3}$ denotes a four-dimensional Minkowski space-time and Y a six-dimensional compact manifold.[3] Furthermore, we focus on manifolds which preserve sixteen supercharges or in other words $\mathcal{N}=4$ supersymmetry in four space-time dimensions. This implies that Y admits two globally-defined nowhere-vanishing spinors η^i, $i=1,2$, that are linearly independent at each point of Y. The necessity for this requirement can be most easily seen by considering the two ten-dimensional supersymmetry generators ϵ^1, ϵ^2, which are Majorana-Weyl and thus reside in the representation **16** of the Lorentz group SO(1, 9). For backgrounds of the form (2.1), the Lorentz group is reduced to SO(1, 3) × SO(6) and the spinor representation decomposes as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})\,, \tag{2.2}$$

where **2** and **4** denote respectively four- and six-dimensional Weyl-spinor representations, while $\bar{\mathbf{2}}$ and $\bar{\mathbf{4}}$ are the corresponding conjugates. In terms of spinors we thus have

$$\begin{aligned} \epsilon^1 &= \sum_{i=1}^{2} (\xi^1_{i+} \otimes \eta^i_+ + \xi^1_{i-} \otimes \eta^i_-)\,, \\ \epsilon^2 &= \sum_{i=1}^{2} (\xi^2_{i+} \otimes \eta^i_- + \xi^2_{i-} \otimes \eta^i_+)\,, \end{aligned} \tag{2.3}$$

where the $\xi^{1,2}_i$ are the four $\mathcal{N}=4$ supersymmetry generators of $M_{1,3}$ and the subscript $\pm$ indicates both the four- and six-dimensional chiralities.

The existence of two nowhere-vanishing spinors η^i forces the structure group of Y to be SU(2). This can be seen as follows. Recall that the spinor representation for a generic six-dimensional manifold is the fundamental representation **4** of SU(4) ≃ SO(6). The existence of two singlets implies the decomposition

$$\mathbf{4} \to \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}\,, \tag{2.4}$$

which in turn leads to the fact that the structure group of the manifold is reduced to the subgroup acting on this **2**, namely SU(2).

2.2 Algebraic structure

Let us now briefly review the algebraic properties of SU(2)-structure manifolds. For a more detailed discussion, see [23].

[3] Note that we do not consider warped compactifications in this work. For discussions of a non-trivial warp factor, see for instance [19, 35].

Instead of using the spinors η^i, we can parametrize the SU(2) structure on a six-dimensional manifold by means of a complex one-form K, a real two-form J and a complex two-form Ω [5, 22]. The two-forms satisfy the relations

$$\Omega \wedge \bar{\Omega} = 2J \wedge J \neq 0\,, \qquad \Omega \wedge J = 0\,, \qquad \Omega \wedge \Omega = 0\,, \tag{2.5}$$

while the one-form is such that

$$K \cdot K = 0\,, \qquad \bar{K} \cdot K = 2\,, \qquad \iota_K J = 0\,, \qquad \iota_K \Omega = \iota_{\bar{K}} \Omega = 0\,. \tag{2.6}$$

These forms can be expressed in terms of the spinors as follows,

$$K_m = \bar{\eta}_2^c \gamma_m \eta_1\,, \tag{2.7}$$

$$J_{mn} = \tfrac{1}{2}\,\mathrm{i}\,(\bar{\eta}_1 \gamma_{mn} \eta_1 + \bar{\eta}_2 \gamma_{mn} \eta_2)\,, \qquad \Omega_{mn} = \bar{\eta}_2 \gamma_{mn} \eta_1\,, \tag{2.8}$$

where γ_m, $m = 1, \ldots, 6$, are SO(6) gamma-matrices and $\gamma_{mn} = \frac{1}{2}(\gamma_m \gamma_n - \gamma_n \gamma_m)$. By using Fierz identities and assuming that each η^i satisfies $\bar{\eta}^i \eta^i = 1$, it can be checked that these definitions for K, J and Ω indeed fulfill the relations (2.5) and (2.6).

The existence of the one-form K allows one to define an almost product structure $P_m{}^n$ on the manifold through the expression

$$P_m{}^n = K_m \bar{K}^n + \bar{K}_m K^n - \delta_m{}^n\,. \tag{2.9}$$

Using (2.6), it is easy to check that $P_m{}^n$ does square to the identity, that is

$$P_m{}^n P_n{}^p = \delta_m{}^p\,. \tag{2.10}$$

From the definition (2.9) and the first two relations in (2.6), it can be seen that K_m and $\bar{K}_m$ are eigenvectors of $P_m{}^n$ with eigenvalue $+1$. Also, all vectors simultaneously orthogonal to K_m and $\bar{K}_m$ have eigenvalue -1. Thus K_m and $\bar{K}_m$ span the $+1$ eigenspace and as a consequence the tangent space of Y splits as

$$TY = T_2 Y \oplus T_4 Y\,, \tag{2.11}$$

where $T_2 Y$ has a trivial structure group and is spanned by Re K^m and Im K^m. We can then choose a basis of one-forms v^i, $i = 1, 2$ on $T_2 Y$ normalized as

$$v^i \wedge v^j = \epsilon^{ij}\, \mathrm{vol}_2\,, \tag{2.12}$$

where vol_2 is the volume form on $T_2 Y$.

From the last constraints in (2.6), it follows that the two-forms J and Ω have 'legs' only along $T_4 Y$. The three real two-forms $J^1 = \mathrm{Re}\,\Omega$, $J^2 = \mathrm{Im}\,\Omega$ and $J^3 = J$ form a triplet of symplectic two-forms on $T_4 Y$ and from (2.5) we infer that

$$J^\alpha \wedge J^\beta = 2\delta^{\alpha\beta} \mathrm{vol}_4\,, \qquad \alpha, \beta = 1, 2, 3\,, \tag{2.13}$$

where vol_4 denotes the volume form on $T_4 Y$. Eq. (2.13) states that the J^α span a space-like three-plane in the space of two-forms on $T_4 Y$. The triplet J^α therefore defines an SU(2)

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structure on T_4Y. Finally, note that any pair of spinors $\tilde{\eta}^i$ which is related to η^i by an SU(2) $\simeq$ SO(3) transformation defines the same SU(2) structure [26]. The one-form K is invariant under this rotation but the two-forms J^α transform as a triplet.[4] Thus there is an SU(2) freedom in the parametrization of the SU(2) structure. This SU(2) is a subgroup of the R-symmetry group SU(4) of $\mathcal{N} = 4$ supergravity.

The case when all forms K, J and Ω (or equivalently v^i and J^α) are closed corresponds to a manifold Y having SU(2) holonomy. This can be seen from eq. (2.7) and (2.8), since these forms being closed translates into the spinors η^i being covariantly constant with respect to the Levi-Civita connection. The only such manifold in six dimensions is the product manifold $K3 \times T^2$, that is the product of a $K3$ manifold with a two-torus. In that case, the almost product structure P is trivially realized by the Cartesian product.

2.3 Kaluza-Klein data

So far, we analyzed the parametrization of an SU(2) structure over a single point of Y. This gives all deformations of the SU(2) structure. But in order to find the low-energy effective action we have to perform a Kaluza-Klein truncation of the spectrum and thereby eliminate all modes with a mass above the compactification scale. This we do in two steps. First, we have to ensure that there are no massive gravitino multiplets in the $\mathcal{N} = 4$ theory. It can be shown that these additional gravitino multiplets are SU(2) doublets which must therefore be projected out [13, 23]. This also automatically removes all one- and three-forms in the space of forms acting on tangent vectors in T_4Y. Furthermore, the splitting (2.11) becomes rigid, since a variation of this splitting is parametrized by a two-form with one leg on T_2Y and the other on T_4Y over each point of Y, but one-forms acting on T_4Y are projected out.

In the following, we will make the additional assumption that the almost product structure (2.9) is integrable. This means that every neighborhood U of Y can be written as a product $U_2 \times U_4$ such that T_2Y and T_4Y are tangent to U_2 and U_4, respectively. In other words, local coordinates $z^i, i = 1, 2$ and $y^a, a = 1, \dots, 4$ can be introduced on Y such that T_2Y is generated by $\partial/\partial z^i$ and T_4Y by $\partial/\partial y^a$. The metric on Y can therefore be written in block-diagonal form as

$$\mathrm{d}s^2 = g_{ij}(z, y)\, \mathrm{d}z^i \mathrm{d}z^j + g_{ab}(z, y)\, \mathrm{d}y^a \mathrm{d}y^b \ . \tag{2.14}$$

In a second step, we truncate the infinite set of differential forms on Y to a finite-dimensional subset. This chooses the light modes out of an infinite tower of (heavy) KK-states. This has to be done in a consistent way, i.e. such that only (but also all) scalars with masses below a chosen scale are kept in the low-energy spectrum.

Let us denote by $\Lambda^2 T_4Y$ the space of two-forms on Y that vanish identically when acting on tangent vectors in T_2Y. The Kaluza-Klein truncation means that we only need to consider an n-dimensional subspace $\Lambda^2_{\mathrm{KK}} T_4Y$ having signature $(3, n-3)$ with respect to the wedge product. The two-forms J^α span a space-like three-plane in $\Lambda^2_{\mathrm{KK}} T_4Y$ and therefore parametrize the space [23]

$$\mathcal{M}_{J^\alpha} = \frac{\mathrm{SO}(3, n-3)}{\mathrm{SO}(3) \times \mathrm{SO}(n-3)} \tag{2.15}$$

[4] Note also that the phase of K corresponds to the overall phase of the pair η^i.

with dimension $3n-9$. Together with the volume $\mathrm{vol}_4 \sim \mathrm{e}^{-\rho}$ this gives $3n-8$ geometric scalar fields on T_4Y. Let us choose a basis ω^I, $I = 1, \ldots, n$ on $\Lambda^2_{\mathrm{KK}} T_4Y$ such that

$$\omega^I \wedge \omega^J = \eta^{IJ} \mathrm{e}^\rho \mathrm{vol}_4 \,, \tag{2.16}$$

with η^{IJ} being the (symmetric) intersection matrix with signature $(3, n-3)$. The factor e^ρ was introduced in order to keep ω^I and η^{IJ} independent of the volume modulus.

The remaining geometric scalars are parametrized by K. The latter is a complex one-form acting on T_2Y which can be expanded in terms of the v^i fulfilling eq. (2.12). The overall real factor of K is proportional to the square root of vol_2, while the overall phase of K is not physical.[5] The other two degrees of freedom in K parametrize the complex structure on T_2Y. This gives altogether three geometric scalars on T_2Y.

On a generic manifold with SU(2) structure, the one- and two-forms are not necessarily closed. On the truncated subspace we just introduced, one can generically have [26, 27]

$$\begin{aligned} \mathrm{d}v^i &= t^i v^1 \wedge v^2 + t^i_I \omega^I \,, \\ \mathrm{d}\omega^I &= \tilde{T}^I_{iJ} v^i \wedge \omega^J \,, \end{aligned} \tag{2.17}$$

where the parameters t^i, t^i_I and $\tilde{T}^I_{iJ}$ are constant. Indeed, eqs. (2.17) state that J^α and K are in general not closed, their differential being related to the torsion classes of the manifold [5]. The parameters in the r.h.s. of (2.17) play the role of gauge charges in the low-energy effective supergravity, as we will see in section 3.1.

One can show that demanding integrability of the almost product structure (2.9) forces t^i_I to vanish [24]. The reason is that in such a case it is impossible to generate a form in $\Lambda^2 T_4Y$ like ω^I by differentiating a one-form v^i that acts non-trivially only on vectors in T_2Y. We will therefore restrict the discussion in the following to this case and set $t^i_I = 0$.

On the other hand, the parameters t^i and $\tilde{T}^I_{iJ}$ are not completely arbitrary but constrained by Stokes' theorem and nilpotency of the d-operator. Acting with d on eqs. (2.17) and using $\mathrm{d}^2 = 0$ leads to

$$t^i \tilde{T}^I_{iJ} - \epsilon^{ij} \tilde{T}^I_{iK} \tilde{T}^K_{jJ} = 0 \,, \tag{2.18}$$

where we choose $\epsilon^{12} = 1$. On the other hand, Stokes' theorem implies the vanishing of $\int_Y \mathrm{d}(v^i \wedge \omega^I \wedge \omega^J)$ for any compact Y, which yields

$$t^i \eta^{IJ} - \epsilon^{ij} \tilde{T}^I_{jK} \eta^{KJ} - \epsilon^{ij} \tilde{T}^J_{jK} \eta^{KI} = 0 \,. \tag{2.19}$$

This in turn implies that $\tilde{T}^I_{iJ}$ can be written as

$$\tilde{T}^I_{iJ} = \tfrac{1}{2} \epsilon_{ij} t^j \delta^I_J + T^I_{iJ} \,, \tag{2.20}$$

with $\epsilon_{12} = -1$ and T^I_{iJ} satisfying

$$T^I_{iK} \eta^{KJ} = -T^J_{iK} \eta^{KI} \,. \tag{2.21}$$

[5] The overall phase of K corresponds to the overall phase of the spinor pair η^i, which is of no physical relevance.

It will be useful to define two $n \times n$ matrices $T_i = (T_i)^I_J$, which due to (2.21) are in the algebra of $\mathrm{SO}(3, n-3)$. Finally, substituting $t^i_I = 0$ and (2.20) into the expressions (2.17) we are left with

$$\begin{aligned} \mathrm{d}v^i &= t^i v^1 \wedge v^2 \,, \\ \mathrm{d}\omega^I &= \tfrac{1}{2}\, t^i \epsilon_{ij} v^j \wedge \omega^I + T^I_{iJ} v^i \wedge \omega^J \,, \end{aligned} \tag{2.22}$$

where, according to eq. (2.18), the matrices T_i satisfy the commutation relation

$$[T_1, T_2] = t^i T_i \ . \tag{2.23}$$

If all parameters t^i and T^I_{iJ} vanish, we recover the case with closed forms v^i and J^α and consequently the manifold is $K3 \times T^2$. In this case, the two-forms ω^I are harmonic and span the second cohomology of $K3$, their number being fixed to $n = 22$.

3 The low-energy effective action

3.1 The NS-NS sector

As already mentioned in the introduction, the reduction of the NS-NS sector is completely similar to that performed in ref. [24] for the heterotic string, therefore we will essentially only recall the results.

The massless fields arising from the NS-NS sector in type IIA supergravity are the metric g_{MN}, the two-form $\mathcal{B}_2$ and the dilaton Φ. The ten-dimensional action governing the dynamics of these fields is given by

$$S_{\mathrm{NS}} = \tfrac{1}{2} \int_{M_{1,3} \times Y} \mathrm{e}^{-2\Phi} \big(\mathcal{R} + 4\mathrm{d}\Phi \wedge *\mathrm{d}\Phi - \tfrac{1}{2} \mathcal{H}_3 \wedge *\mathcal{H}_3 \big) \,, \tag{3.1}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{H}_3 = \mathrm{d}\mathcal{B}_2$ is the field-strength of the two-form $\mathcal{B}_2$. A KK ansatz for these fields can be written as

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{ij} \mathcal{E}^i \mathcal{E}^i + g_{ab} \mathrm{d}y^a \mathrm{d}y^b \,, \\ \mathcal{B}_2 &= B + B_i \wedge \mathcal{E}^i + b_{12} \mathcal{E}^1 \wedge \mathcal{E}^2 + b_I \omega^I \,, \end{aligned} \tag{3.2}$$

where we have defined the 'gauge-invariant' one-forms $\mathcal{E}^i = v^i - G^i_\mu \mathrm{d}x^\mu$. The expansion of the ten-dimensional two-form $\mathcal{B}_2$ leads to a set of four-dimensional fields: a two-form B, two vectors or one-forms B_i and $n+1$ scalar fields b_I and b_{12}.[6] In computing the low-energy effective action, one has to express the variation of the metric components g_{ab} in terms of the $3n-8$ geometric moduli on T_4Y or, more precisely, one needs an expression for the line element $g^{ac} g^{bd} \delta g_{ab} \delta g_{cd}$. As a first step one expands the two-forms J^α parametrizing the $\mathrm{SU}(2)$ structure in terms of the basis ω^I according to

$$J^\alpha = \mathrm{e}^{-\frac{\rho}{2}} \zeta^\alpha_I \omega^I \ . \tag{3.3}$$

[6] Note that in this paper we do not consider background flux for $\mathcal{H}_3$. This situation has been discussed for example in [30–32] where it was shown that, as usual, the background fluxes appear as gauge charges in the effective action which gauge specific directions in the $N = 4$ field space.

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However, the $3n$ parameters ζ_I^α are not all independent. Inserting the expansion (3.3) into eq. (2.13), and using the relation (2.16), one obtains the six independent constraints

$$\eta^{IJ}\zeta_I^\alpha\zeta_J^\beta = 2\delta^{\alpha\beta} \,. \tag{3.4}$$

Moreover, an SO(3) rotation acting on the upper index of ζ_I^α gives new two-forms J^α that are linear combinations of the old ones, defining therefore the same three-plane and leaving us at the same point of the moduli space. Altogether, we end up with the right number of $3n-9$ geometric moduli parametrizing $\mathcal{M}_{J^\alpha}$ in eq. (2.15). Furthermore, ref. [24] derived the line element to be

$$g^{ac}g^{bd}\delta g_{ab}\delta g_{cd} = \delta\rho^2 + (2\eta^{IJ} - \zeta^{\alpha I}\zeta^{\beta J})\delta\zeta_I^\alpha\delta\zeta_J^\beta \,, \tag{3.5}$$

where $\zeta^{\alpha I} = \eta^{IJ}\zeta_J^\alpha$. Note that this expression is indeed the metric on the coset

$$\mathbb{R}^+ \times \frac{\mathrm{SO}(3,n-3)}{\mathrm{SO}(3)\times\mathrm{SO}(n-3)} \,. \tag{3.6}$$

With the last result at hand, it is straightforward to insert the ansatz (3.2) into the action (3.1) and obtain the effective four-dimensional action

$$\begin{aligned} S_{\mathrm{NS}} = \tfrac{1}{2}\int_{M_{1,3}} \Big[& R*1 - \tfrac{1}{2}\mathrm{e}^{-4\phi}\big|\mathcal{D}B\big|^2 - \tfrac{1}{2}\mathrm{e}^{-2\phi-\eta}\tilde{g}_{ij}\mathcal{D}G^i\wedge*\mathcal{D}G^j \\ & - \tfrac{1}{2}\mathrm{e}^{-2\phi+\eta}\tilde{g}^{ij}\big(\mathcal{D}B_i - b_{12}\epsilon_{ik}\mathcal{D}G^k\big)\wedge*\big(\mathcal{D}B_j - b_{12}\epsilon_{jl}\mathcal{D}G^l\big) \\ & - |\mathrm{d}\phi|^2 - \tfrac{1}{2}\mathrm{e}^{2\eta}\big(|\mathcal{D}b_{12}|^2 + |\mathcal{D}\mathrm{e}^{-\eta}|^2\big) - \tfrac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\mathcal{D}\tilde{g}_{ij}\wedge*\mathcal{D}\tilde{g}_{kl} \\ & - \tfrac{1}{4}|\mathcal{D}\rho|^2 - \tfrac{1}{4}(H^{IJ}-\eta^{IJ})\mathcal{D}\zeta_I^\alpha\wedge*\mathcal{D}\zeta_J^\beta - \tfrac{1}{2}\mathrm{e}^\rho H^{IJ}\mathcal{D}b_I\wedge*\mathcal{D}b_J \\ & - \tfrac{5}{4}\mathrm{e}^{2\phi+\eta}\tilde{g}_{ij}t^it^j + \tfrac{1}{8}\mathrm{e}^{2\phi+\eta}\tilde{g}^{ij}[H,T_i]^I{}_J[H,T_j]^J{}_I \\ & - \tfrac{1}{8}\mathrm{e}^{2\phi-\eta+\rho}\tilde{g}_{ij}t^it^jH^{IJ}b_Ib_J - \tfrac{1}{2}\mathrm{e}^{2\phi+\eta+\rho}\tilde{g}^{ij}H^{IJ}T_{iI}^KT_{jJ}^Lb_Kb_L\Big] \,, \end{aligned} \tag{3.7}$$

where R denotes the Ricci scalar in four-dimensions and we have introduced the notation $|f|^2 = f\wedge*f$ for any form f. Moreover, the symmetric matrix H^{IJ} is defined according to $\omega^I\wedge*\omega^J = H^{IJ}\mathrm{e}^\rho\mathrm{vol}_4$, which can be expressed in terms of the parameters ζ_I^α by [24][7]

$$H^{IJ} = -\eta^{IJ} + \zeta^{\alpha I}\zeta^{\alpha J} \,. \tag{3.8}$$

(The commutators in (3.7) use $H^I{}_J = H^{IK}\eta_{KJ}$.) In the two-dimensional metric g_{ij} defined in (2.14) we separated the overall volume $\mathrm{e}^{-\eta}$ from the other two independent (complex structure) degrees of freedom by introducing the rescaled metric $\tilde{g}_{ij} = \mathrm{e}^\eta g_{ij}$. It satisfies $\det\tilde{g} = 1$ and can be expressed in terms of a complex-structure parameter κ as

$$\tilde{g}_{ij} = \frac{1}{\mathrm{Im}\,\kappa}\begin{pmatrix} 1 & \mathrm{Re}\,\kappa \\ \mathrm{Re}\,\kappa & |\kappa|^2 \end{pmatrix} \,. \tag{3.9}$$

[7]This expression can be derived by using the fact that the two-forms J^α are self-dual, $J^\alpha = *J^\alpha$, with all other orthogonal linear combinations of the ω^I being anti-self dual.

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In order to write the action in the Einstein frame, we also performed the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^{2\phi} g_{\mu\nu}$ of the four-dimensional metric, where $\phi = \Phi + \frac{1}{2}(\eta + \rho)$ is the four-dimensional dilaton. Finally, the various non-Abelian field-strengths and covariant derivatives in (2.14) are given by

$$
\begin{aligned}
\mathcal{D}B &= \mathrm{d}B + B_i \wedge \mathcal{D}G^i \,, && (3.10a)\\
\mathcal{D}G^i &= \mathrm{d}G^i - t^i G^1 \wedge G^2 \,, && (3.10b)\\
\mathcal{D}B_i &= \mathrm{d}B_i + \epsilon_{ij} t^k G^j \wedge B_k \,, && (3.10c)\\
\mathcal{D}\tilde{g}_{ij} &= \mathrm{d}\tilde{g}_{ij} + (\epsilon_{il}\tilde{g}_{jk} + \epsilon_{jl}\tilde{g}_{ik} - \epsilon_{kl}\tilde{g}_{ij}) t^k G^l \,, && (3.10d)\\
\mathcal{D}\mathrm{e}^{-\eta} &= \mathrm{d}\mathrm{e}^{-\eta} - \epsilon_{ij} t^j \mathrm{e}^{-\eta} G^i \,, && (3.10e)\\
\mathcal{D}b_{12} &= \mathrm{d}b_{12} - \epsilon_{ij} t^j b_{12} G^i - t^i B_i \,, && (3.10f)\\
\mathcal{D}\rho &= \mathrm{d}\rho - \epsilon_{ij} t^j G^i \,, && (3.10g)\\
\mathcal{D}\zeta_I^\alpha &= \mathrm{d}\zeta_I^\alpha + T_{iI}^J \zeta_J^\alpha G^i \,, && (3.10h)\\
\mathcal{D}b_I &= \mathrm{d}b_I + \tilde{T}_{iI}^J b_J G^i \,. && (3.10i)
\end{aligned}
$$

As a next step let us turn to the R-R sector.

3.2 The R-R sector

So far, we have reduced the kinetic term for the NS fields. The remaining part of the ten-dimensional action for type IIA supergravity consists of the kinetic terms for the R-R fields and the Chern-Simons term,

$$S_{\mathrm{RR}} = -\tfrac{1}{4} \int_{M_{1,3}\times Y} \left(\mathcal{F}_2 \wedge *\mathcal{F}_2 + \tilde{\mathcal{F}}_4 \wedge *\tilde{\mathcal{F}}_4 \right) , \tag{3.11}$$

$$S_{\mathrm{CS}} = -\tfrac{1}{4} \int_{M_{1,3}\times Y} \mathcal{B}_2 \wedge \mathcal{F}_4 \wedge \mathcal{F}_4 \,, \tag{3.12}$$

where $\mathcal{F}_2 = \mathrm{d}\mathcal{A}_1$ and $\mathcal{F}_4 = \mathrm{d}\mathcal{C}_3$. $\tilde{\mathcal{F}}_4$ is the modified field strength of $\mathcal{C}_3$ defined as

$$\tilde{\mathcal{F}}_4 = \mathrm{d}\mathcal{C}_3 - \mathcal{A}_1 \wedge \mathrm{d}\mathcal{B}_2 . \tag{3.13}$$

Analogously to the KK ansatz (3.2), we expand the ten-dimensional RR fields in the set of internal one-forms $\mathcal{E}^i$ and two-forms ω^I as follows,

$$
\begin{aligned}
\mathcal{A}_1 =& A + a_i \mathcal{E}^i \,,\\
\mathcal{C}_3 =& (C - A \wedge B) + (C_i - A \wedge B_i) \wedge \mathcal{E}^i \\
&+ (C_{12} - b_{12} A) \wedge \mathcal{E}^1 \wedge \mathcal{E}^2 + (C_I - b_I A) \wedge \omega^I + c_{iI} \mathcal{E}^i \wedge \omega^I \,.
\end{aligned} \tag{3.14}
$$

In terms of four-dimensional fields we thus have a three-form C, two two-forms C_i, $2+n$ vectors or one-forms A, C_{12} and C_I, and finally $2n+2$ scalars a_i and c_{iI}.[8] In the expansion

[8] As for the B-field, we also do not consider background fluxes for the RR field strengths in this paper. Their effect is similar to an $\mathcal{H}_3$ flux in that additional directions in the $N = 4$ field space become gauged [30–32].

of the three form $\mathcal{C}_3$, it is convenient to introduce some mixing with the four-dimensional components from $\mathcal{A}_1$ and $\mathcal{B}_2$. The reason for this is that in this case the four-dimensional field strengths $\mathrm{d}C$, $\mathrm{d}C_i$, $\mathrm{d}C_{12}$ and $\mathrm{d}C_I$ remain invariant under the gauge transformations

$$\begin{aligned} \mathcal{A}_1 &\to \mathcal{A}_1 + \mathrm{d}\Lambda\,, \\ \mathcal{B}_2 &\to \mathcal{B}_2 + \mathrm{d}\Lambda_1\,, \\ \mathcal{C}_3 &\to \mathcal{C}_3 + \mathrm{d}\Lambda_2 + \Lambda \mathrm{d}\mathcal{B}_2\,, \end{aligned} \tag{3.15}$$

which is a symmetry of type IIA supergravity, as can be seen from the modified field-strength (3.13).

Before we continue, let us pause and count the total number of light modes arising from the KK ansatz in the NS-NS plus RR-sector. From eq. (3.2) (and the subsequent analysis) we learn that the spectrum in the NS-sector contains the graviton, a two-form B, four vectors G^i, B_i and $4n-3$ scalars. From eq. (3.14), we see that two two-forms, $2+n$ vectors and $2n+2$ scalars arise in the RR-sector. After dualizing the three two-forms to scalars we thus have a total spectrum of a graviton, $6+n$ vectors and $6n+2$ scalars. As we review in the next section, this is indeed the spectrum of an $\mathcal{N}=4$ supergravity with n vector multiplets.

Substituting this expansion for the ten-dimensional fields into the action (3.11) and performing at the end the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^{2\phi} g_{\mu\nu}$, we obtain

$$\begin{aligned} S_{\mathrm{RR}} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[& \mathrm{e}^{-\eta-\rho}\big|\mathrm{d}A - a_i \mathcal{D}G^i\big|^2 + \mathrm{e}^{-4\phi-\eta-\rho}\big|\mathcal{D}C - \mathrm{d}A\wedge B\big|^2 \\ & + \mathrm{e}^{-2\phi-\rho}\tilde{g}^{ij}\big(\mathcal{D}C_i - \mathrm{d}A\wedge B_i + a_i\mathcal{D}B\big)\wedge \\ & \quad \wedge *\big(\mathcal{D}C_j - \mathrm{d}A\wedge B_j + a_j\mathcal{D}B\big) \\ & + \mathrm{e}^{\eta-\rho}\big|\mathcal{D}C_{12} - b_{12}\mathrm{d}A - a_i(\epsilon^{ij}\mathcal{D}B_j - b_{12}\mathcal{D}G^i)\big|^2 \\ & + \mathrm{e}^{-\eta}H_{IJ}\big(\mathcal{D}C^I - b^I\mathrm{d}A - c_i^I\mathcal{D}G^i\big)\wedge *\big(\mathcal{D}C^J - b^J\mathrm{d}A - c_j^J\mathcal{D}G^j\big) \\ & + \mathrm{e}^{2\phi}\tilde{g}^{ij}H_{IJ}\big(\mathcal{D}c_i^I + a_i\mathcal{D}b^I\big)\wedge *\big(\mathcal{D}c_j^J + a_j\mathcal{D}b^J\big) \\ & + \mathrm{e}^{2\phi-\rho}\tilde{g}^{ij}\mathcal{D}a_i\wedge *\mathcal{D}a_j + \mathrm{e}^{4\phi+\eta-\rho}(t^i a_i)^2 * 1 \\ & + \mathrm{e}^{4\phi+\eta}H_{IJ}\big[\epsilon^{ij}T_{iK}^I(c_j^K + a_j b^K) - t^i(c_i^I - a_i b^I)\big]\cdot \\ & \quad \cdot\big[\epsilon^{kl}T_{kL}^J(c_l^L + a_l b^L) - t^k(c_k^J - a_k b^J)\big] * 1\Big]\,. \end{aligned} \tag{3.16}$$

On the other hand, the Chern-Simons term (3.12) gives the following contribution

$$\begin{aligned} S_{\mathrm{CS}} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[& 2\epsilon^{ij}c_i^J\tilde{T}_{jJ}^I b_I\big(\mathcal{D}C - \mathrm{d}A\wedge B\big) \\ & - 2\big(\mathcal{D}C_i - \mathrm{d}A\wedge B_i\big)\wedge\epsilon^{ij}b_I\mathcal{D}c_j^I + b_{12}\eta_{IJ}\mathcal{D}C^I\wedge\mathcal{D}C^J \\ & + 2\big(\mathcal{D}C_{12} - b_{12}\mathrm{d}A\big)\wedge b_I\big(\mathcal{D}C^I - \tfrac{1}{2}b^I\mathrm{d}A - c_i^I\mathcal{D}G^i\big) \\ & - \mathcal{D}B\wedge\epsilon^{ij}c_{iI}\big(\mathcal{D}c_j^I - \tilde{T}_{jJ}{}^{I}C^J\big) + 2B_i\wedge\epsilon^{ij}\tilde{T}_{jIJ}C^I\wedge\mathcal{D}C^J \\ & - 2\big(\mathcal{D}B_i - b_{12}\epsilon_{ik}\mathcal{D}G^k\big)\wedge\epsilon^{ij}c_{jI}\big(\mathcal{D}C^I - \tfrac{1}{2}c_l^I\mathcal{D}G^l\big)\Big]\,. \end{aligned} \tag{3.17}$$

The non-Abelian field-strengths and covariant derivatives of all four-dimensional RR-fields are given by

$$\mathcal{D}C = \mathrm{d}C - C_i \wedge \mathcal{D}G^i \,, \tag{3.18a}$$

$$\mathcal{D}C_i = \mathrm{d}C_i + \epsilon_{ij} t^k G^j \wedge C_k + \epsilon_{ij} C_{12} \wedge \mathcal{D}G^j \,, \tag{3.18b}$$

$$\mathcal{D}C_{12} = \mathrm{d}C_{12} + t^i C_i - \epsilon_{ij} t^j G^i \wedge C_{12} \,, \tag{3.18c}$$

$$\mathcal{D}C^I = \mathrm{d}C^I + \tilde{T}_{iJ}{}^I G^i \wedge C^J \,, \tag{3.18d}$$

$$\mathcal{D}a_i = \mathrm{d}a_i + \epsilon_{ij} t^k a_k G^j \,, \tag{3.18e}$$

$$\mathcal{D}c_i^I = \mathrm{d}c_i^I + \epsilon_{ij} t^k c_k^I G^j - \tilde{T}_{jJ}{}^I c_i^J G^j + \tilde{T}_{jJ}{}^I C^J \,. \tag{3.18f}$$

Let us summarize. The bosonic part of the low-energy four-dimensional effective action arising from the compactification of type IIA supergravity on SU(2)-structure manifolds is given by the sum of the contribution from the NS-NS sector, eq. (3.7), and the contribution from the RR sector, eqs. (3.16) and (3.17), that is

$$S_{\text{eff}} = S_{\text{NS}} + S_{\text{RR}} + S_{\text{CS}} \,. \tag{3.19}$$

The covariant derivatives and field strengths corresponding to the various four-dimensional fields are given in eqs. (3.10) and (3.18).

The next step is to establish the consistency of this action with four-dimensional $\mathcal{N} = 4$ supergravity. To do this, we will bring the action into the canonical form proposed in ref. [33] by performing a series of field redefinitions.

4 Consistency with $\mathcal{N} = 4$ supergravity

The gravity multiplet of $\mathcal{N} = 4$ supergravity in four dimensions contains as bosonic degrees of freedom the metric, six massless vectors and two real scalars while a vector multiplet consist of a massless vector field and six real scalars. $\mathcal{N} = 4$ supergravity coupled to n vector multiplets has a global symmetry $\mathrm{SL}(2) \times \mathrm{SO}(6, n)$ and the scalar fields of the theory assemble into a complex field τ describing an $\mathrm{SL}(2)/\mathrm{SO}(2)$ coset and a $(6 + n) \times (6 + n)$ matrix M_{MN} parametrizing the coset

$$\frac{\mathrm{SO}(6, n)}{\mathrm{SO}(6) \times \mathrm{SO}(n)} \,. \tag{4.1}$$

In ref. [33], the action of the most general gauged $\mathcal{N} = 4$ supergravity is given using the embedding tensor formalism. All possible gaugings are encoded in two tensors, $f_{\alpha MNP}$ and $\xi_{\alpha M}$, where α is an SL(2) index taking the values $+$ and $-$. As it turns out, for the effective action (3.19) both f_{-MNP} and ξ_{-M} vanish, and therefore we choose to start with the formulas of ref. [33] adapted to this case. In order to simplify the notation, we omit the $\alpha = +$ index in the couplings f_{+MNP} and ξ_{+M} and write simply f_{MNP} and ξ_M for the non-trivial couplings. With this in mind, the action for gauged $\mathcal{N} = 4$ supergravity can be divided in three parts,

$$S_{\mathcal{N}=4} = S_{\text{kin}} + S_{\text{top}} + S_{\text{pot}} \,, \tag{4.2}$$

that is kinetic, topological and potential terms. The part of the action containing the kinetic terms reads

$$
\begin{aligned}
S_{\text{kin}} = \tfrac{1}{2}\int_{M_{1,3}} \Big[& R * 1 + \tfrac{1}{8}\mathcal{D}M_{MN} \wedge *\mathcal{D}M^{MN} - \tfrac{1}{2}(\operatorname{Im}\tau)^{-2}\mathcal{D}\tau \wedge *\mathcal{D}\bar{\tau} \\
& -(\operatorname{Im}\tau)\, M_{MN}\mathcal{D}V^{M+} \wedge *\mathcal{D}V^{N+} + (\operatorname{Re}\tau)\, \eta_{MN}\mathcal{D}V^{+} \wedge \mathcal{D}V^{N+} \Big] , \qquad (4.3)
\end{aligned}
$$

where the constant matrix η_{MN} is an $\mathrm{SO}(6,n)$ metric and the non-Abelian field-strengths for the electric vector fields V^{M+} are given by the expression

$$
\mathcal{D}V^{M+} = \mathrm{d}V^{M+} - \tfrac{1}{2}\hat{f}_{NP}{}^{M} V^{N+} \wedge V^{P+} + \tfrac{1}{2}\xi^{M} B^{++} , \qquad (4.4)
$$

where B^{++} is an auxiliary two-form whose role we soon explain.[9] The covariant derivatives of the scalar fields are defined as

$$
\mathcal{D}\tau = \mathrm{d}\tau + \xi_M \tau V^{M+} + \xi_M V^{M-} , \qquad (4.5)
$$

$$
\mathcal{D}M_{MN} = \mathrm{d}M_{MN} + \Theta_{PM}{}^{Q} M_{NQ} V^{P+} + \Theta_{PN}{}^{Q} M_{MQ} V^{P+} . \qquad (4.6)
$$

In these expressions, the following useful shorthands were used,

$$
\hat{f}_{MNP} = f_{MNP} - \tfrac{1}{2}\xi_M \eta_{PN} + \tfrac{1}{2}\xi_P \eta_{MN} - \tfrac{3}{2}\xi_N \eta_{MP} , \qquad (4.7)
$$

$$
\Theta_{MNP} = f_{MNP} - \tfrac{1}{2}\xi_N \eta_{PM} - \tfrac{1}{2}\xi_P \eta_{NM} . \qquad (4.8)
$$

As we can see, the presence of an auxiliary two-form field B^{++} is related to the fact that the complex scalar τ is charged with respect to the magnetic duals V^{M-} of the electric vector fields V^{M+}. The two-form B^{++} acts as a Lagrange multiplier, in the sense that its equation of motion merely ensures that V^{M-} and V^{M+} are related by an electric-magnetic duality. This follows from the last term in the topological part of the $\mathcal{N} = 4$ supergravity action

$$
\begin{aligned}
S_{\text{top}} = -\tfrac{1}{2}\int_{M_{1,3}} \Big[& \xi_M \eta_{NP} V^{M-} \wedge V^{N+} \wedge \mathrm{d}V^{P+} - \tfrac{1}{4}\hat{f}_{MNR}\hat{f}_{PQ}{}^{R} V^{M+} \wedge V^{N+} \wedge V^{P+} \wedge V^{Q-} \\
& - \xi_M B^{++} \wedge \big(\mathrm{d}V^{M-} - \tfrac{1}{2}\hat{f}_{QR}{}^{M} V^{Q+} \wedge V^{R-}\big) \Big] .
\end{aligned}
\qquad (4.9)
$$

Finally, there is also a potential energy that contributes to the action as

$$
\begin{aligned}
S_{\text{pot}} = -\tfrac{1}{16}\int_{M_{1,3}} & (\operatorname{Im}\tau)^{-1} \Big[3\xi^M \xi^N M_{MN} \\
& + f_{MNP} f_{QRS} \big(\tfrac{1}{3} M^{MQ} M^{NR} M^{PS} + (\tfrac{2}{3}\eta^{MQ} - M^{MQ})\eta^{NR}\eta^{PS}\big) \Big] .
\end{aligned}
\qquad (4.10)
$$

[9] As noted above, we omit the + index of ref. [33] in the couplings f_{MNP} and ξ_M, but we do keep it for the gauge fields and denote the electric vectors by V^{M+} while the magnetic vectors are V^{M-}.

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4.1 Field dualizations

The action S_{eff} that was obtained in (3.19) does not have the same structure as the action given in eq. (4.2). Most obviously, the spectrum currently contains two-form fields, which we must replace by their dual scalar fields. Furthermore, as can be easily verified, the quadratic couplings of the vector field-strengths are not of the simple form seen in eq. (4.3), which implies that also some of the vector fields must be traded for their dual fields.

Our strategy will be the following. First we remove the (non-dynamical) three-form field C from the theory and dualize the two-forms B and C_i to scalars β and γ^i, respectively. In a second step, we determine the correct electric-magnetic duality frame in which the action for the vector fields takes the form (4.3). This we can do by setting to zero the parameters T^I_{iJ} and t^i determining the charges, which makes it easier to perform electric-magnetic duality transformations on the vector fields. Once we have identified the correct electric-magnetic duality frame, we can read off the $\mathrm{SO}(6,n)$ coset matrix M_{MN}, the complex scalar τ and the metric η_{MN}. The final step is then to turn on the charges and use the information obtained in the previous steps to determine the components of the embedding tensor. Using the embedding tensor, we can then find the full expressions for the electric field strengths in the canonical action (4.3), as well as the correct topological terms (4.9). We can then verify that the action obtained in this way is equivalent to S_{eff} by elimination of the extra two-form B^{++} introduced by the embedding tensor formalism.

As already mentioned, the four-dimensional three-form C carries no degrees of freedom. We can integrate it out using its equation of motion. From the part of the effective action S_{eff} that depends on C, namely

$$S_C = -\tfrac{1}{4}\int_{M_{1,3}} \Big[\mathrm{e}^{-4\phi-\eta-\rho}\big|\mathcal{D}C - \mathrm{d}A\wedge B\big|^2 - 2\epsilon^{ij} b_I \tilde{T}^I_{iJ} c^J_j \big(\mathcal{D}C - \mathrm{d}A\wedge B\big)\Big]\,, \tag{4.11}$$

follows the equation of motion

$$\mathcal{D}C - \mathrm{d}A\wedge B = -e^{4\phi+\eta+\rho}\epsilon^{ij} b_I \tilde{T}^I_{iJ} c^J_j \ast 1\ . \tag{4.12}$$

Substituting this back into the action (4.11), we obtain the potential term

$$S'_C = -\tfrac{1}{4}\int_{M_{1,3}} \mathrm{e}^{4\phi+\eta+\rho}\big(\epsilon^{ij} b_I \tilde{T}^I_{iJ} c^J_j\big)^2 \ast 1\ . \tag{4.13}$$

Next, we trade the two-forms C_i and B for their dual scalars. In contrast to the three-form C, the two-forms C_i do not appear in the Lagrangian exclusively in the form $\mathrm{d}C_i$. As can be seen in the expression (3.18c) for the covariant field strength $\mathcal{D}C_{12}$, they are also present as a Stückelberg-like mass term $t^i C_i$, making it necessary to dualize the vector field C_{12} as well. Therefore, we dualize the C_i into scalar fields γ^i while at the same time dualizing the vector field C_{12} to a vector field $\tilde{C}$. As already mentioned, the scalar field dual to B will be called β. We present the details of this calculation in appendix A.

After these steps, we arrive at an action S'_{eff} containing only scalar and vector fields (apart from the metric). The total action can be split into three components

$$S'_{\text{eff}} = S_{\text{scalar}} + S_{\text{vector}} + S_{\text{potential}}\,, \tag{4.14}$$

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where the kinetic terms for the scalar fields (and the four-dimensional metric) are

$$
\begin{aligned}
S_{\text{scalar}} = -\tfrac{1}{2}\int_{M_{1,3}} \Big[& R*1 + |\mathrm{d}\phi|^2 + \tfrac{1}{2}\mathrm{e}^{2\eta}\big(|\mathcal{D}b_{12}|^2 + |\mathcal{D}\mathrm{e}^{-\eta}|^2\big) + \tfrac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\mathcal{D}\tilde{g}_{ij}\wedge *\mathcal{D}\tilde{g}_{kl} \\
& + \tfrac{1}{4}|\mathcal{D}\rho|^2 + \tfrac{1}{4}(H^{IJ} - \eta^{IJ})\mathcal{D}\zeta_I^\alpha \wedge *\mathcal{D}\zeta_J^\beta + \tfrac{1}{2}\mathrm{e}^\rho H^{IJ}\mathcal{D}b_I\wedge *\mathcal{D}b_J \\
& + \mathrm{e}^{2\phi-\rho}\tilde{g}^{ij}\mathcal{D}a_i \wedge *\mathcal{D}a_j + \mathrm{e}^{2\phi}\tilde{g}^{ij}H_{IJ}(\mathcal{D}c_i^I + a_i\mathcal{D}b^I)\wedge *(\mathcal{D}c_j^J + a_j\mathcal{D}b^J) \\
& + \mathrm{e}^{2\phi+\rho}\tilde{g}^{ij}(\mathcal{D}\gamma_i + b^I\mathcal{D}c_{iI})\wedge *(\mathcal{D}\gamma_j + b^J Dc_{jJ}) \\
& + \mathrm{e}^{4\phi}\big|\mathcal{D}\beta - \epsilon^{ij}\big(a_i\mathcal{D}\gamma_j + a_i b_I\mathcal{D}c_j^I - \tfrac{1}{2}c_{iI}\mathcal{D}c_j^I\big)\big|^2\Big] \,.
\end{aligned} \tag{4.15}
$$

The covariant derivatives $D\gamma_i$ and $D\beta$ are given by

$$
\mathcal{D}\gamma_i = \mathrm{d}\gamma_i - \epsilon_{ij}t^j(\gamma_k G^k + \tilde{C})\,, \tag{4.16a}
$$

$$
\mathcal{D}\beta = \mathrm{d}\beta + \tfrac{1}{2}c_{iJ}\tilde{T}_{iI}{}^J C^I \,. \tag{4.16b}
$$

The kinetic and topological terms for the vector fields are

$$
\begin{aligned}
S_{\text{vector}} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[& \mathrm{e}^{-2\phi-\eta}\tilde{g}_{ij}\mathcal{D}G^i \wedge *\mathcal{D}G^j + \mathrm{e}^{-\eta-\rho}|\mathrm{d}A - a_i\mathcal{D}G^i|^2 \\
& + \mathrm{e}^{-2\phi+\eta}\tilde{g}^{ij}(\mathcal{D}B_i - b_{12}\epsilon_{ik}\mathcal{D}G^k)\wedge *(\mathcal{D}B_j - b_{12}\epsilon_{jl}\mathcal{D}G^l) \\
& + \mathrm{e}^{-\eta+\rho}\big|\mathcal{D}\tilde{C} - \gamma_i\mathcal{D}G^i + b_I\big(\mathcal{D}C^I - \tfrac{1}{2}b^I\mathrm{d}A - c_k^I\mathcal{D}G^k\big)\big|^2 \\
& + \mathrm{e}^{-\eta}H_{IJ}\big(\mathcal{D}C^I - b^I\mathrm{d}A - c_i^I\mathcal{D}G^i\big)\wedge *\big(\mathcal{D}C^J - b^J\mathrm{d}A - c_j^J\mathcal{D}G^j\big) \\
& + b_{12}\eta_{IJ}\mathcal{D}C^I \wedge \mathcal{D}C^J + 2b_{12}\mathrm{d}A\wedge\mathcal{D}\tilde{C} \\
& - 2(\mathcal{D}B_i - b_{12}\epsilon_{il}\mathcal{D}G^l)\wedge\epsilon^{ij}\big[(c_{jI} + a_j b_I)\mathcal{D}C^I + \big(\gamma_j - \tfrac{1}{2}a_j b_I b^I\big)\mathrm{d}A \\
& \quad + a_j\mathcal{D}\tilde{C} - \big(\epsilon_{jk}\beta + a_j\gamma_k + \tfrac{1}{2}c_{jI}c_k^I + a_j b_I c_k^I\big)\mathcal{D}G^k\big] \\
& + 2B_i \wedge \big(\epsilon^{ij}\tilde{T}_{jIJ}C^I\wedge\mathcal{D}C^J + t^i\tilde{C}\wedge\mathrm{d}A\big)\Big] \,.
\end{aligned} \tag{4.17}
$$

Here, the non-Abelian field-strength for the vector field $\tilde{C}$ is

$$
\mathcal{D}\tilde{C} = \mathrm{d}\tilde{C} + \epsilon_{ij}t^j G^i \wedge \tilde{C} \,. \tag{4.18}
$$

Finally, the total potential reads

$$
\begin{aligned}
S_{\text{potential}} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[& \mathrm{e}^{4\phi+\eta+\rho}\big(\epsilon^{ij}b_I\tilde{T}_{iJ}^I c_j^J\big)^2 + \tfrac{5}{2}\mathrm{e}^{2\phi+\eta}\tilde{g}_{ij}t^i t^j + \tfrac{1}{4}\mathrm{e}^{2\phi-\eta+\rho}\tilde{g}_{ij}t^i t^j H^{IJ}b_I b_J \\
& + \tfrac{1}{4}\mathrm{e}^{2\phi+\eta}\tilde{g}^{ij}[H,T_i]^I{}_J[H,T_j]^J{}_I + \mathrm{e}^{2\phi+\eta+\rho}\tilde{g}^{ij}H^{IJ}T_{iI}^K T_{jJ}^L b_K b_L \\
& + \mathrm{e}^{4\phi+\eta}H_{IJ}\big[\epsilon^{ij}T_{iK}^I(c_j^K + a_j b^K) - t^i(c_i^I - a_i b^I)\big]\cdot \\
& \quad \cdot\big[\epsilon^{kl}T_{kL}^J(c_l^L + a_l b^L) - t^k(c_k^J - a_k b^J)\big]\Big] *1 \,.
\end{aligned} \tag{4.19}
$$

4.2 Determination of the embedding tensor

At this point, we can identify which vector fields in the effective action (4.14) correspond to the electric vector fields V^{M+} in the canonical action (4.2) and which vector fields should be dualized. Setting the parameters T_{iJ}^I and t^i to zero in the action (4.14), we can very easily

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trade vector fields for their electric-magnetic duals via the usual dualization procedure. It turns out that exchanging the vector fields B_i with their dual fields $B^{\bar{\imath}}$ suffices to bring the (ungauged) Lagrangian into the form (4.3).[10] The computation of the action for the fields $B^{\bar{\imath}}$ is given in section A.2 of the appendix.

From the action for the dualized fields we can determine the $\mathrm{SO}(6,n)$ metric η_{MN} as well as the complex scalar τ and the coset matrix M_{MN} which determine the canonical action (4.3). If we choose to arrange the electric vectors into the fundamental representation of $\mathrm{SO}(6,n)$ as

$$V^{M+} = (G^i, B^{\bar{\imath}}, A, \tilde{C}, C^I) \tag{4.20}$$

we find that the $\mathrm{SO}(6,n)$ metric η_{MN} is given by

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_{i\bar{\jmath}} & 0 & 0 & 0 \\ \delta_{\bar{\imath}j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{IJ} \end{pmatrix} , \tag{4.21}$$

and that the scalar factor in the topological vector field couplings is given by

$$\mathrm{Re}\,\tau = -\tfrac{1}{2} b_{12} \ . \tag{4.22}$$

We can find the imaginary part of τ by checking the kinetic term for b_{12} in the action (3.7), since according to (4.3) this should contain a factor $(\mathrm{Im}\,\tau)^{-2}$. In this way, we determine that the complex scalar τ is given by

$$\tau = \tfrac{1}{2}(-b_{12} + \mathrm{i}\, \mathrm{e}^{-\eta}) \ . \tag{4.23}$$

For completeness, the matrix M_{MN} is given in appendix B.

We now have enough information to determine the embedding tensor from the covariant derivatives and the non-Abelian field strengths in the action (4.14). We start by determining the components $\xi_{\alpha M}$ from the covariant derivative of τ. Comparing eqs. (3.10e) and (3.10f) with the general formula (4.5) we conclude that

$$\xi_i = -\epsilon_{ij} t^j \ , \tag{4.24}$$

and $\xi_{\bar{\imath}} = \xi_5 = \xi_6 = \xi_I = 0$. On the other hand, the components f_{MNP} of the embedding tensor are most easily determined from the non-Abelian field strengths of the vector fields V^{M+}. It turns out that setting

$$f_{ij\bar{\imath}} = -\tfrac{1}{2}\epsilon_{ij}\delta_{\bar{\imath}k} t^k \ , \tag{4.25a}$$

$$f_{i56} = \tfrac{1}{2}\epsilon_{ij} t^j \ , \tag{4.25b}$$

$$f_{iIJ} = -T_{iIJ} \ , \tag{4.25c}$$

[10] Note that turning off the parameters T^I_{iJ} and t^i corresponds to compactifications on $K3 \times T^2$. The effective action for this case has been determined in [27–29].

in the general formula (4.4) leads to an agreement with the field-strengths computed in (3.10b), (3.18d) and (4.18). Moreover, it can be checked that (4.24) and (4.25) satisfy the following quadratic constraints described in ref. [33],

$$\xi^M \xi_M = 0\,, \qquad \xi^M f_{MNP} = 0\,, \qquad 3 f_{R[MN} f_{PQ]}{}^R - 2\xi_{[M} f_{NPQ]} = 0\,, \tag{4.26}$$

where square brackets denote antisymmetrization of the corresponding indices. That the first two constraints are satisfied follows trivially from the expressions (4.24) and (4.25) with a metric (4.21). The third one follows from the commutation relation satisfied by the matrices T^I_{iJ} given in eq. (2.23), which as we saw is a consequence of demanding nilpotency of the exterior differential acting on the two-forms ω^I.

We now have all the information we need in order to write down the action with charged fields in the electric frame. The total field-strength for the electric vector field $B^{\bar{\imath}}$ in the action (4.3) is then

$$F^{\bar{\imath}+} = \mathrm{d}B^{\bar{\imath}} + \tfrac{1}{2}\delta^{i\bar{\imath}}\big[\epsilon_{ik}t^k(\delta_{j\bar{\jmath}}G^j \wedge B^{\bar{\jmath}} - A \wedge \tilde{C}) + T_{iIJ}C^I \wedge C^J - \epsilon_{ij}t^j B^{++}\big]\,, \tag{4.27}$$

while the topological term is given by

$$\begin{aligned} S_{\rm top} = \tfrac{1}{4}\int_{M_{1,3}} \Big[& B^{++} \wedge t^j \mathcal{D}B_j - t^i \mathcal{D}B_i \wedge (\delta_{j\bar{\jmath}} B^{\bar{\jmath}} \wedge \mathcal{D}G^j + \tilde{C} \wedge \mathrm{d}A) \\ & + 2t^i B_i \wedge (\delta_{j\bar{\jmath}} G^j \wedge B^{\bar{\jmath}} + A \wedge \tilde{C} + \tfrac{1}{2}\eta_{IJ} C^I \wedge \mathcal{D}C^J)\Big]\,. \end{aligned} \tag{4.28}$$

Using the expressions for f_{MNP}, M_{MN} and η_{MN}, it can be shown that the potential in (4.10) agrees with the potential (4.19) obtained from the KK reduction.

Summarizing, we have obtained an action of the form given in (4.3), (4.9) and (4.10). In order to write the action in this form, we had to introduce extra vector fields $B^{\bar{\imath}}$, as well as a tensor field B^{++}, which appears in the field strength $F^{+\bar{\imath}}$. To see that this form of the action is equivalent to the action given in equations (4.15), (4.17) and (4.19), one can use the equations of motion for B^{++} to eliminate B^{++} and $B^{\bar{\imath}}$. This reduces the action for the vector fields to the one in (4.17).

4.3 Killing vectors and gauge algebra

Finally let us determine the gauge group which arises from the compactifications studied in this paper. It will be useful to collectively denote all $(6n+2)$ scalar fields in the effective action by

$$\varphi^\Lambda = (b_{12}, \eta, \phi, \tilde{g}_{ij}, \rho, \zeta^x_I, a_i, \gamma_i, c^I_i, \beta, b_I)\,, \qquad \Lambda = 1, \ldots, 6n+2\,. \tag{4.29}$$

Then the Killing vectors $k_{M\alpha} = k^\Lambda_{M\alpha}(\varphi)\frac{\partial}{\partial\varphi^\Lambda}$ can be read off from the covariant derivatives of these fields in eqs. (3.10), (3.18) and (4.16) by comparing with the general formula

$$\mathcal{D}\varphi^\Lambda = \mathrm{d}\varphi^\Lambda - k^\Lambda_{M\alpha}(\varphi)V^{M\alpha}\,. \tag{4.30}$$

Doing this, we obtain the following expressions for the Killing vectors,

$$\begin{aligned}
k_{i+} &= \epsilon_{ij} t^j \left(b_{12} \frac{\partial}{\partial b_{12}} - \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \rho} \right) - T_{iI}^J \zeta_J^x \frac{\partial}{\partial \zeta_I^x} + t^j (\epsilon_{ik} \tilde{g}_{jl} + \epsilon_{il} \tilde{g}_{jk} - \epsilon_{ij} \tilde{g}_{kl}) \frac{\partial}{\partial \tilde{g}_{kl}} \\
&\quad + \epsilon_{ij} t^k a_k \frac{\partial}{\partial a_j} + \epsilon_{jk} t^k \gamma_i \frac{\partial}{\partial \gamma_j} + \left(\epsilon_{ij} t^k \delta_I^J - \delta_j^k \tilde{T}_{iI}{}^J \right) c_k^I \frac{\partial}{\partial c_j^J} - \tilde{T}_{iI}^J b_J \frac{\partial}{\partial b_I} , \\
k_{6+} &= \epsilon_{ij} t^j \frac{\partial}{\partial \gamma_i} , \qquad k_{I+} = \tilde{T}_{iI}{}^J \left(\frac{\partial}{\partial c_i^J} - \tfrac{1}{2} \epsilon^{ij} c_{jJ} \frac{\partial}{\partial \beta} \right) , \qquad k_{i-} = \epsilon_{ij} t^j \frac{\partial}{\partial b_{12}} , \\
k_{\bar{\imath}\pm} &= k_{5\pm} = k_{6-} = k_{I-} = 0 \ .
\end{aligned} \tag{4.31}$$

Now we can compute the Lie brackets for this set of vectors to obtain

$$\begin{aligned}
[k_{i+}, k_{j+}] &= -\epsilon_{ij} t^k k_{k+} , \qquad [k_{i+}, k_{6+}] = -\epsilon_{ij} t^j k_{6+} , \\
[k_{i+}, k_{I+}] &= -\tilde{T}_{iI}{}^J k_{J+} , \qquad [k_{i+}, k_{j-}] = \epsilon_{jk} t^k k_{i-} ,
\end{aligned} \tag{4.32}$$

with the all other brackets vanishing. Inspecting (2.17) we see that by choosing appropriate linear combinations of v^1 and v^2 we can set $t^1 = 0$ without loss of generality and then rename $t^2 \equiv t$. If we do this, k_{2-} is zero, and the non-vanishing Lie brackets (4.32) read

$$\begin{aligned}
[k_{1+}, k_{2+}] &= t k_{2+} , \qquad [k_{1+}, k_{1-}] = -t k_{1-} , \\
[k_{1+}, k_{6+}] &= t k_{6+} , \qquad [k_{1+}, k_{I+}] = \tfrac{1}{2} t k_{I+} + T_{1I}^J k_{J+} , \\
[k_{2+}, k_{I+}] &= T_{2I}^J k_{J+} \ .
\end{aligned} \tag{4.33}$$

This corresponds to the solvable algebra $(\mathbb{R}_{k_{6+}} \times \mathbb{R}_{k_{1-}} \times (\mathbb{R}^n_{k_{I+}} \rtimes \mathbb{R}_{k_{2+}})) \rtimes \mathbb{R}_{k_{1+}}$, where in the first semi-direct product, $\mathbb{R}_{k_{2+}}$ acts on $\mathbb{R}^n_{k_{I+}}$ by means of the matrix T_{2I}^J, while in the second, $\mathbb{R}_{k_{1+}}$ acts on $\mathbb{R}_{k_{6+}} \times \mathbb{R}_{k_{1-}} \times (\mathbb{R}^n_{k_{I+}} \rtimes \mathbb{R}_{k_{2+}})$ through the matrix

$$\mathrm{diag}\left(t, -t, \tfrac{1}{2} t \delta_I^J + T_{1I}^J, t\right) \ . \tag{4.34}$$

That the algebra (4.32) is indeed consistent with gauged $\mathcal{N} = 4$ supergravity we see by defining the following matrices [33]

$$X_{M+} = \begin{pmatrix} X_{M+N+}{}^{P+} & 0 \\ 0 & X_{M+N-}{}^{P-} \end{pmatrix} , \qquad X_{M-} = \begin{pmatrix} 0 & X_{M-N+}{}^{P-} \\ 0 & 0 \end{pmatrix} , \tag{4.35}$$

with non-vanishing entries given in terms of the embedding tensors by

$$\begin{aligned}
X_{M+N+}{}^{P+} &= -f_{MN}{}^P - \tfrac{1}{2} (\delta_M^P \xi_N - \delta_N^P \xi_N - \eta_{MN} \xi^P) , \\
X_{M+N-}{}^{P-} &= -f_{MN}{}^P - \tfrac{1}{2} (\delta_M^P \xi_N + \delta_N^P \xi_N - \eta_{MN} \xi^P) , \\
X_{M-N+}{}^{P-} &= -\delta_N^P \xi^M \ .
\end{aligned} \tag{4.36}$$

As discussed in ref. [33], the non-Abelian gauge algebra of the $\mathcal{N} = 4$ supergravity should be reproduced by the commutators

$$\begin{aligned}
[X_{M+}, X_{N+}] &= X_{M+N+}{}^{P+} X_{P+} , \\
[X_{M+}, X_{N-}] &= X_{M+N-}{}^{P-} X_{P-} = -X_{N-M+}{}^{P-} X_{P-} , \\
[X_{M-}, X_{N-}] &= 0 ,
\end{aligned} \tag{4.37}$$

And indeed, by using the expressions (4.24) and (4.25) for the embedding tensor in the formulas (4.35) to (4.37), the algebra (4.32) is recovered.

5 Conclusions

In this paper we considered type IIA supergravity compactified on a specific class of six-dimensional manifolds which have SU(2) structure. Such manifolds admit a pair of globally defined spinors and they can be further characterized by their non-trivial intrinsic torsion. Among the SU(2)-structure manifolds one also finds the Calabi-Yau manifold $K3 \times T^2$ for which the intrinsic torsion vanishes. Furthermore, the entire class of six-dimensional SU(2)-structure manifolds necessarily has an almost product structure of a four-dimensional component times a two-dimensional component which also generalizes the Calabi-Yau case. However, in order to simplify the analysis in this paper, we confined our attention to torsion classes which lead to an integrable almost product structure.

For this class of compactifications (with the additional requirement of the absence of massive gravitino multiplets) we determined the resulting four-dimensional $\mathcal{N} = 4$ low-energy effective action by performing a Kaluza-Klein reduction. By appropriate dualizations of one- and two-forms it was possible to go from the 'natural' field basis of the KK reduction to a supergravity field basis where the consistency with the 'standard' $\mathcal{N} = 4$ form as given in [33] could be established. In that process, we determined the components of the embedding tensor or in other words the couplings of the $\mathcal{N} = 4$ action in terms of the intrinsic torsion. The resulting gauge group is solvable, as usually is the case for these compactifications.

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A Dualizations

In this appendix we give some of the calculational steps involved in the field dualizations from sections 4.1 and 4.2. The purpose of a field dualization is to obtain an equivalent theory, where a (massless) p-form field is exchanged for a $(2-p)$-form field.

A.1 Dualization of two-forms

The Lagrangian obtained from the compactification still contains the two-forms B and C_i, which we can exchange for vector and scalar fields by performing the appropriate dualizations [36].

We start by dualizing the two-form fields C_i and the one-form C_{12}. Collecting all the relevant terms obtained from the action (3.16) and (3.17) for the RR fields, we have

$$\begin{aligned} S_{C_i,C_{12}} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[&\mathrm{e}^{-2\phi-\rho}\tilde{g}^{ij}(H_i+J_i)\wedge *(H_i+J_i) + \mathrm{e}^{\eta-\rho}|F_{12}+J_{12}|^2 \\ &- 2H_i\wedge\epsilon^{ij}b_I\mathcal{D}c_j^I + 2F_{12}\wedge K\Big]\,, \end{aligned} \tag{A.1}$$

where, for simplicity, we have introduced the following abbreviations,

$$\begin{aligned} H_i &\equiv \mathcal{D}C_i = \mathrm{d}C_i + \epsilon_{ij}t^kG^j\wedge C_k - \epsilon_{ij}C_{12}\wedge\mathcal{D}G^j\,, \\ F_{12} &\equiv \mathcal{D}C_{12} = \mathrm{d}C_{12} + t^iC_i + \epsilon_{ij}t^jG^i\wedge C_{12}\,, \\ J_i &= a_i\mathcal{D}B - \mathrm{d}A\wedge B_i\,, \\ J_{12} &= -b_{12}\mathrm{d}A - a_i(\epsilon^{ij}\mathcal{D}B_j - b_{12}\mathcal{D}G^i)\,, \\ K &= b_I(\mathcal{D}C^I - \tfrac{1}{2}b^I\mathrm{d}A - c_k^I\mathcal{D}G^k)\;. \end{aligned} \tag{A.2}$$

The fact that the bare p-form potential C_i also appears in the field strength F_{12} makes it impossible to replace C_i by dual scalar fields γ_i without also replacing F_{12} by the field strength of a dual vector field $\tilde{C}$. We can do this by constructing an action which is equivalent to (A.1), where the field strengths H_i and F_{12} are treated as independent fields. The equivalence to the original Lagrangian is guaranteed by introducing Lagrange multipliers γ_i and $\tilde{C}$ which enforce the correct Bianchi identities for H_i and F_{12}, namely

$$\begin{aligned} \mathrm{d}H_i &= -\epsilon_{ij}t^kG^j\wedge H_k + \epsilon_{ik}\mathcal{D}G^k\wedge F_{12}\,, \\ \mathrm{d}F_{12} &= t^iH_i + \epsilon_{ij}t^jG^i\wedge F_{12}\;. \end{aligned} \tag{A.3}$$

The modified action thus becomes

$$\begin{aligned} S'_{C_i,C_{12}} &= S_{C_i,C_{12}} - \tfrac{1}{2}\int_{M_{1,3}}\Big[\gamma_i\epsilon^{ij}(\mathrm{d}H_j + \epsilon_{jk}t^lG^k\wedge H_l - \epsilon_{jk}\mathcal{D}G^k\wedge F_{12}) \\ &\qquad + \tilde{C}\wedge(\mathrm{d}F_{12} - t^iH_i - \epsilon_{ij}t^jG^i\wedge F_{12})\Big] \\ &= S_{C_i,C_{12}} + \tfrac{1}{2}\int_{M_{1,3}}\Big[H_i\wedge\epsilon^{ij}(\mathrm{d}\gamma_j + \epsilon_{jk}t^k\gamma_lG^l + \epsilon_{jk}t^k\tilde{C}) \\ &\qquad - F_{12}\wedge(\mathrm{d}\tilde{C} + \epsilon_{ij}t^jG^i\wedge\tilde{C} - \gamma_i\mathcal{D}G^i)\Big]\;. \end{aligned} \tag{A.4}$$

Integrating out the fields H_i and F_{12} by using their equations of motion leads to the following action for the dual fields γ_i and $\tilde{C}$,

$$\begin{aligned} S_{\tilde{C},\gamma_i} = -\tfrac{1}{4}\int_{M_{1,3}}\Big[&\mathrm{e}^{2\phi+\rho}\tilde{g}^{ij}(\mathcal{D}\gamma_i + b_I\mathcal{D}c_i^I)\wedge *(\mathcal{D}\gamma_j + b^J\mathcal{D}c_j^J) \\ &+ e^{-\eta+\rho}|\mathcal{D}\tilde{C} - \gamma_i\mathcal{D}G^i + K|^2 \\ &+ 2\epsilon^{ij}(D\gamma_i + b_IDc_i^I)\wedge J_j - 2(\mathcal{D}\tilde{C} - \gamma_i\mathcal{D}G^i + K)\wedge J_{12}\Big]\,, \end{aligned} \tag{A.5}$$

where we have defined the covariant derivatives $\mathcal{D}\gamma_i$ and the non-Abelian field-strength $\mathcal{D}\tilde{C}$ as

$$\mathcal{D}\gamma_i = \mathrm{d}\gamma_i - \epsilon_{ij}t^j(\gamma_kG^k + \tilde{C})\,, \tag{A.6}$$

$$\mathcal{D}\tilde{C} = \mathrm{d}\tilde{C} + \epsilon_{ij}t^jG^i\wedge\tilde{C}\;. \tag{A.7}$$

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The dualization of the two-form B is much simpler, due to the simpler nature of its couplings. After the dualization of the two-forms C_i, the action for B, written in terms of its field strength $H \equiv \mathcal{D}B = \mathrm{d}B + B_i \wedge \mathcal{D}G^i$ and introducing a Lagrange multiplier β to enforce $\mathrm{d}^2B = \mathrm{d}(H - B_i \wedge \mathcal{D}G^i) = 0$, is given by

$$S_B = -\tfrac{1}{4}\int_{M_{1,3}} \left[\mathrm{e}^{-4\phi}|H|^2 + 2H\wedge W + 2\beta \wedge \mathrm{d}(H - B_i \wedge \mathcal{D}G^i)\right], \tag{A.8}$$

with the shorthand

$$W = \epsilon^{ij}\left(a_i D\gamma_j + a_i b_I \mathcal{D}c_j^I - \tfrac{1}{2}c_{iI}\mathcal{D}c_j^I + \tfrac{1}{2}c_{iJ}\tilde{T}_{iI}{}^J C^I\right) . \tag{A.9}$$

Eliminating H by using its equations of motion, we obtain the action for the dual scalar field β,

$$S_\beta = -\tfrac{1}{4}\int_{M_{1,3}} \left[\mathrm{e}^{4\phi}\left|\mathcal{D}\beta - \epsilon^{ij}\left(a_i\mathcal{D}\gamma_j + a_i b_I\mathcal{D}c_j^I - \tfrac{1}{2}c_{iI}\mathcal{D}c_j^I\right)\right|^2 - 2\beta\wedge\mathcal{D}B_i\wedge\mathcal{D}G^i\right], \tag{A.10}$$

where the covariant derivative of β is

$$\mathcal{D}\beta = \mathrm{d}\beta + \tfrac{1}{2}c_{iJ}\tilde{T}_{iI}{}^J C^I . \tag{A.11}$$

A.2 Finding the correct electric-magnetic duality frame

In order to read off the gauge couplings M_{MN} and η_{MN}, we can consider the action with all charges T_{iJ}^I and t^i set to zero, and bring this action into the correct electric-magnetic duality frame. When no fields are charged with respect to the vector fields, the dualizations are of course simpler, and we find that replacing the vector fields B_i by their duals $B^{\bar{\imath}}$ brings the couplings into their canonical form.

Setting charges to zero, the terms in the action containing the fields B_i are

$$\begin{aligned} S_{B_i} = -\tfrac{1}{4}\int_{M_{1,3}} \Big[&\mathrm{e}^{-2\phi+\eta}\tilde{g}^{ij}(F_i - b_{12}\epsilon_{ik}\mathrm{d}G^k)\wedge *(F_j - b_{12}\epsilon_{jl}\mathrm{d}G^l) \\ &- 2(F_i - b_{12}\epsilon_{ik}\mathrm{d}G^k)\wedge\epsilon^{ij}L_j\Big], \end{aligned} \tag{A.12}$$

where $F_i = \mathrm{d}B_i$ and we have introduced the shorthand notation

$$\begin{aligned} L_i =& (c_{iI} + a_i b_I)\mathcal{D}C^I + \left(\gamma_i - \tfrac{1}{2}a_i b_I b^I\right)\mathrm{d}A + a_i\mathcal{D}\tilde{C} \\ &- \left(\epsilon_{ij}\beta + a_i\gamma_j + \tfrac{1}{2}c_{iI}c_j^I + a_i b_I c_j^I\right)\mathcal{D}G^j . \end{aligned} \tag{A.13}$$

We now introduce the dual fields $B^{\bar{\imath}}$ by adding the following term to the action (A.12),

$$\delta S = -\tfrac{1}{2}\int_{M_{1,3}} \delta_{\bar{\imath}i}B^{\bar{\imath}}\wedge\epsilon^{ij}\mathrm{d}F_j . \tag{A.14}$$

Eliminating the two-forms F_i using its equations of motion, we arrive at the dual action

$$S_{B^{\bar{\imath}}} = -\tfrac{1}{4}\int_{M_{1,3}} \left[\mathrm{e}^{2\phi-\eta}\tilde{g}^{ij}(\delta_{i\bar{\imath}}\mathrm{d}B^{\bar{\imath}} + L_i)\wedge *(\delta_{j\bar{\jmath}}\mathrm{d}B^{\bar{\jmath}} + L_j) + 2b_{12}\delta_{\bar{\imath}i}\mathrm{d}B^{\bar{\imath}}\wedge\mathrm{d}G^i\right] . \tag{A.15}$$

Substituting these results into the complete vector action (4.17), we can see that the gauge kinetic couplings are now indeed of the canonical form presented in equation (4.3). This allows us to read off the matrices M_{MN} and η_{MN}.

B SO$(6,n)$ coset matrix

The entries of the matrix M_{MN} (with indices M, N taking the $6+n$ values i, $\bar\imath$, 5, 6, I) can be easily extracted from the kinetic terms for the vectors in eqs. (4.17) and (A.15), by comparison with the general form of this term for $\mathcal{N}=4$ supergravity in eq. (4.3). The result is

$$\begin{aligned}
M_{ij} &= \mathrm{e}^{-2\phi}\tilde g_{ij} + \mathrm{e}^{-\rho}a_i a_j + \mathrm{e}^{\rho}(\gamma_i + b_I c_i^I)(\gamma_j + b_J c_j^J) + H_{IJ}c_i^I c_j^J \\
&\quad + \mathrm{e}^{2\phi}\tilde g^{kl}\big(\epsilon_{ki}\beta + a_k\gamma_i + \tfrac12 c_{kI}c_i^I + a_k b_I c_i^I\big)\big(\epsilon_{lj}\beta + a_l\gamma_j + \tfrac12 c_{lI}c_j^I + a_l b_I c_j^I\big)\,, && \text{(B.1)}\\
M_{i\bar\jmath} &= \mathrm{e}^{2\phi}\tilde g^{jk}\delta_{\bar\jmath}^{j}\big(\epsilon_{ki}\beta + a_k\gamma_i + \tfrac12 c_{kI}c_i^I + a_k b_I c_i^I\big)\,, && \text{(B.2)}\\
M_{i5} &= -\mathrm{e}^{-\rho}a_i + \mathrm{e}^{\rho}b_I b^I(\gamma_i + b_J c_i^J) + H_{IJ}b^I c_i^J && \text{(B.3)}\\
&\quad + \mathrm{e}^{2\phi}\tilde g^{jk}\big(\gamma_j - \tfrac12 a_j b_I b^I\big)\big(\epsilon_{ki}\beta + a_k\gamma_i + \tfrac12 c_{kI}c_i^I + a_k b_I c_i^I\big)\,, && \text{(B.4)}\\
M_{i6} &= -\mathrm{e}^{\rho}(\gamma_i + b_I c_i^I) - \mathrm{e}^{2\phi}\tilde g^{jk}a_j\big(\epsilon_{ki}\beta + a_k\gamma_i + \tfrac12 c_{kI}c_i^I + a_k b_I c_i^I\big)\,, && \text{(B.5)}\\
M_{iI} &= -H_{IJ}c_i^J - \mathrm{e}^{\rho}b_I(\gamma_i + b_J c_i^J) && \text{(B.6)}\\
&\quad + \mathrm{e}^{2\phi}\tilde g^{jk}(c_{jI} + a_j b_I)\big(\epsilon_{ki}\beta + a_k\gamma_i + \tfrac12 c_{kI}c_i^I + a_k b_I c_i^I\big)\,, && \text{(B.7)}\\
M_{\bar\imath\bar\jmath} &= \mathrm{e}^{2\phi}\tilde g^{ij}\delta_{i\bar\imath}\delta_{j\bar\jmath}\,, && \text{(B.8)}\\
M_{\bar\imath 5} &= \mathrm{e}^{2\phi}\tilde g^{ij}\delta_{i\bar\imath}\big(\gamma_j - \tfrac12 a_j b_I b^I\big)\,, && \text{(B.9)}\\
M_{\bar\imath 6} &= -\mathrm{e}^{2\phi}\tilde g^{ij}\delta_{i\bar\imath}a_j\,, && \text{(B.10)}\\
M_{\bar\imath I} &= \mathrm{e}^{2\phi}\tilde g^{ij}\delta_{i\bar\imath}(c_{jI} + a_j b_I)\,, && \text{(B.11)}\\
M_{55} &= \mathrm{e}^{-\rho} + \tfrac14\mathrm{e}^{\rho}(b_I b^I)^2 + \mathrm{e}^{2\phi}\tilde g^{ij}\big(\gamma_i - \tfrac12 a_i b_I b^I\big)\big(\gamma_j - \tfrac12 a_j b_J b^J\big) + H_{IJ}b^I b^J\,, && \text{(B.12)}\\
M_{56} &= -\tfrac12\mathrm{e}^{\rho}b_I b^I - \mathrm{e}^{2\phi}\tilde g^{ij}a_i\big(\gamma_j - \tfrac12 a_j b_I b^I\big)\,, && \text{(B.13)}\\
M_{5I} &= -\tfrac12\mathrm{e}^{\rho}b_I b^I b_J - H_{IJ}b^J + \mathrm{e}^{2\phi}\tilde g_{ij}(c_{iI} + a_i b_I)\big(\gamma_j - \tfrac12 a_j b_J b^J\big)\,, && \text{(B.14)}\\
M_{66} &= \mathrm{e}^{\rho} + \mathrm{e}^{2\phi}\tilde g^{ij}a_i a_j\,, && \text{(B.15)}\\
M_{6I} &= \mathrm{e}^{\rho}b_I - \mathrm{e}^{2\phi}\tilde g^{ij}a_i(c_{jI} + a_j b_I)\,, && \text{(B.16)}\\
M_{IJ} &= H_{IJ} + \mathrm{e}^{\rho}b_I b_J + \mathrm{e}^{2\phi}\tilde g^{ij}(c_{iI} + a_i b_I)(c_{jJ} + a_j b_J)\,, && \text{(B.17)}
\end{aligned}$$

with the other entries determined by symmetry. It can be checked that this matrix indeed satisfies $M_{MN}\eta^{NP}M_{PQ} = \eta_{MQ}$, with the SO$(6,n)$ metric given in eq. (4.21).

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