RESEARCH ARTICLE

Uncertainty relations for general phase spaces

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We describe a setup for obtaining uncertainty relations for arbitrary pairs of observables related by a Fourier transform. The physical examples discussed here are the standard position and momentum, number and angle, finite qudit systems, and strings of qubits for quantum information applications. The uncertainty relations allow for an arbitrary choice of metric for the outcome distance, and the choice of an exponent distinguishing, e.g., absolute and root mean square deviations. The emphasis of this article is on developing a unified treatment, in which one observable takes on values in an arbitrary locally compact Abelian group and the other in the dual group. In all cases, the phase space symmetry implies the equality of measurement and preparation uncertainty bounds. There is also a straightforward method for determining the optimal bounds.

1   Introduction

Uncertainty relations are quantitative expressions of two fundamental features of quantum mechanics. The first feature is the observation that there are *no dispersion-free states* [1, Sec. IV.1]. That is, we cannot find states that give fixed (non-statistical) results for all observables. This is already seen for many pairs of observables, most famously for position and momentum. The well-known Heisenberg-Kennard relation is a "preparation uncertainty relation", i.e., a quantitative expression of the observation that there is no quantum state for which both the position distribution and the momentum distribution are sharp. The second feature is loosely referred to as *complementarity*, or the existence of mutually exclusive experimental arrangements. More precisely, there are measurements that cannot be simulated as marginals of a joint measurement device. The word "simulated" here indicates that complementarity runs much deeper than the trivial impossibility of building two different experiments in the same space, like setting the same polarizer simultaneously at 0 and 45 degrees. The basic impossibility statement is in terms of observables (positive operator valued measures), which encode just the statistical "quantum input to classical output" behavior

of a device. Two measurement devices A and B may thus be incompatible or "not jointly measurable" in the sense that it is impossible to build a new device C with outcome pairs $c = (a, b)$, such that ignoring the b-outcomes leaves one with a device statistically equivalent to A, and similarly for a and B. There is a particular way of attempting such a joint measurement, namely by measuring first one observable, and then trying to retrieve the other from the post-measurement state. For complementary observables, this is bound to fail, which is another way of saying that the first measurement necessarily disturbs the system. A "measurement uncertainty relation" is a quantitative expression of the non-existence of joint measurements, and hence also encodes the *error-disturbance trade-off* associated with measurements.

The preparation and the measurement aspect of uncertainty are logically independent because they refer to quite different experimental scenarios. An experiment verifying a preparation uncertainty relation between observables A and B will determine separately the distributions of A and B, so that no individual particle is subject to both kinds of measurement. The minimum uncertainty objects in this case are states. In contrast, for a measurement uncertainty an A-value and a B-value is obtained for each particle, often in succession as in the error-disturbance scenario. The minimum uncertainty objects are approximate joint measurements of A and B. There is no direct operational link between these scenarios, and the quantitative bounds for preparation and mea-

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surement uncertainty are, in general, different. Indeed, for two projection-valued (standard) observables, which are for this purpose mainly specified by their eigenbases, preparation uncertainty is zero if the two bases share *one* eigenvector, whereas measurement uncertainty vanishes when the observables commute, i.e., when they share *all* eigenvectors. The distinction is also borne out by the detailed study of angular momentum uncertainty [2], where the preparation and measurement uncertainties require quite different methods. It is therefore somewhat surprising that for the case of position and momentum [3–5], the measurement uncertainty relations are quantitatively the same as the preparation uncertainty relations. The abstract reason for this is *phase space symmetry*.

However, this type of symmetry and the result mentioned are by no means restricted to the standard position/momentum case. The purpose of this paper is to review the application of these ideas to other phase spaces. The common features of the systems considered are the following: One has a pair of observables, which we will continue to call position and momentum, related to each other by a Fourier transform. Position will take values in a space X (generalizing $X = \mathbb{R}^n$), on which translations make sense. Hence, we view it as an Abelian group. The unitary operators implementing translations in position space will be functions of momentum. Symmetrically, there is a momentum space $\widehat{X}$ whose translations are generated by unitary operators, which are functions of position. Such pairs appear in many traditional systems in physics, e.g., number/phase, or lattice site/quasi-momentum. Quantum information has additionally generated a lot of interest in finite cases, like qudit systems or qubit strings. For qubit strings, the position and momentum observables are the readout of strings in Z and X basis, respectively. A typical uncertainty question would be how accurately an eavesdropper can possibly measure a string in one basis without disturbing the readouts in another basis, when both errors are ascertained, for example, from the Hamming distance.

Because we claim a quantitative agreement between measurement and preparation uncertainty bounds, we need to express the bounds by a uniform set of criteria. It turns out that all it takes is to fix, for each observable, a metric on the outcome space together with a certain exponent. This allows closely connected definitions of variances for the preparation uncertainties and the distance of probability distributions needed for the measurement uncertainty. To summarize, each case of the theory developed here involves the following choices:

- a *phase space* $\Xi = X \times \widehat{X}$, which is given in terms

of a locally compact Abelian group X and its dual $\widehat{X}$. We will refer to X and $\widehat{X}$ as the position and momentum space, respectively.
- a translation invariant *metric* on the space X, and another one on $\widehat{X}$.
- *error exponents* $1 \leqslant \alpha, \beta \leqslant \infty$, which determine whether the error measures emphasize small or large distances.

We will develop the theory in full generality for any such choice. This includes the equality of the measurement and preparation uncertainty bounds. The bounds are best expressed in terms of the set of achievable pairs $(\Delta P, \Delta Q)$ of uncertainties, especially in terms of the trade-off curve of pairs, along which neither uncertainty can be reduced without increasing the other. There is a concrete prescription on how to calculate this curve: Each point on the curve is determined by finding the ground state of a certain operator and this solution gives the corresponding minimum uncertainty state (resp. minimum uncertainty joint measurement). Sometimes the ground state problem is very simple. For example, the standard position/momentum case leads to the problem of finding the ground state of a harmonic oscillator, making the minimum uncertainty states pure Gaussians. This case also has an additional dilatation symmetry, so that with each achievable uncertainty pair also the hyperbola $(\lambda \Delta P, \lambda^{-1} \Delta Q)$ is achievable. Therefore, the uncertainty region is completely described by the lowest lying hyperbola, i.e., by the lowest product $\Delta P \Delta Q$. However, this is the only case in which an uncertainty product adequately describes the trade-off curve. The general ground state problem cannot be solved in such a simple form. Therefore, we look at concrete cases, listed in Table 1, selected in part for their physical interest and in part to illustrate some of the features that may occur.

The paper is organized as follows: We will review the basic theory of phase space quantum mechanics in the next section and the relevant notions of uncertainty in Section 3. This is followed in Section 5 by a discussion of the special instances, as summarized in Table 1.

Table 1 Phase spaces and parameters considered in this paper.

System	X	$\widehat{X}$	Metrics	α, β	Section	Ref.
Canonical pair	$\mathbb{R}$	$\mathbb{R}$	abs	all		[4]
Mechanics	$\mathbb{R}^n$	$\mathbb{R}^n$	Euclidean	1,2	5.1	
Angle/number	$\mathbb{T}$		arc or chordal	2		[6]
		$\mathbb{Z}$	discrete or abs	1,2	5.2	
Qudit	$\mathbb{Z}_d$	$\mathbb{Z}_d$	discrete	1	5.3	
Qubit string	$\mathbb{Z}_2^N$	$\mathbb{Z}_2^N$	Hamming	all	5.4	

2 Phase spaces

Here, we outline phase space quantum mechanics in the general setting outlined above. The generality forces us to use a relatively abstract (i.e., mathematical) language. Physicists not feeling comfortable with this level of abstraction should read this section and the next with two concrete examples from Table 1 in mind, one of which should be the "standard" case of the position/momentum pair.

The origin of the theory outlined here is in Ref. [7, 8], where it is carried out for the standard phase spaces $\mathbb{R}^n \times \mathbb{R}^n$. The generalization to general phase spaces is straightforward for the parts we need for the current context, and only needs some standard results of the harmonic analysis (Fourier theory) of locally compact Abelian groups [9, 10]. A detailed treatment, including the fine points, is in preparation together with Jussi Schultz.

From now on we assume that a group X of "position shifts" is given. Technically, any locally compact Abelian group is allowed, but in physical or quantum information applications, we will be talking about one of the groups from Table 1. Apart from one or more canonical degrees of freedom, like position/momentum of quantum optical field quadratures, we may also have angle or phase variables with an intrinsic periodicity given by the group $\mathbb{T}$ of phases (complex numbers with modulus 1 under multiplication) or, equivalently, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Furthermore, there may be discrete variables given by integers, either unbounded ($X = \mathbb{Z}$) or modulo some number d ($X = \mathbb{Z}_d$). Furthermore, arbitrary combinations of these choices are allowed.

We will denote integration with respect to the Haar measure on X by "$\int dx$". This measure is unique up to a constant and is characterized by its translation invariance, i.e., by the possibility to substitute a shifted variable without functional determinant factors. In the discrete cases, it is often natural to give each point unit measure. Integrals with respect to this "counting measure" are just sums over x. In the compact cases ($\mathbb{T}$ and $\mathbb{Z}_d$ and their products) the total Haar measure is finite, and it is often convenient to take it as a probability measure, i.e., normalized to 1. Note that these natural choices are in conflict for $\mathbb{Z}_d$, which means that we have to make decisions analogous to the choice about where to put the normalization factors 2π for the standard Fourier transform.

The basic Hilbert space of our systems will now be $\mathcal{H} = \mathcal{L}^2(X, dx)$, the square integrable functions on X. In this space, the projection-valued position observable acts by multiplication operators, i.e., the position probability density associated with a vector ψ will be $|\psi(x)|^2$. The unitary shift operators $(U_x \psi)(y) = \psi(y - x)$ are clearly not functions of position. However, because the underlying group X is Abelian, the operators commute and the group can therefore be diagonalized jointly, i.e., be represented as multiplication operators in another representation. Of course, this will be the momentum representation reached by the Fourier transform. The Fourier transform of a function $\psi : X \to \mathbb{C}$ will be a function $\mathcal{F}\psi : \widehat{X} \to \mathbb{C}$, where $\widehat{X}$ is the dual group of X. This is abstractly the set of continuous multiplicative functions from X to $\mathbb{T}$. If $p \in \widehat{X}$ labels such a function, we write it as $x \mapsto \lfloor p | x \rfloor$. By definition, $\lfloor p | x_1 + x_2 \rfloor = \lfloor p | x_1 \rfloor \lfloor p | x_2 \rfloor$. The sum in $\widehat{X}$ is defined by $\lfloor p_1 + p_2 | x \rfloor = \lfloor p_1 | x \rfloor \lfloor p_2 | x \rfloor$. Concretely, when $X = \mathbb{R}^n$, we also have $\widehat{X} = \mathbb{R}^n$ and $\lfloor p | x \rfloor = \exp(ip \cdot x)$, where the dot denotes the scalar product. Similarly, for the pair $\Xi = \mathbb{T} \times \mathbb{Z}$ we have $\lfloor \alpha | n \rfloor = \exp(i\alpha n)$. Note that changing α to $\alpha + 2\pi$ (which represents the same element in $\mathbb{T}$) does not change the value of $\lfloor \alpha | n \rfloor$ and this property is what forces $n \in \mathbb{Z}$. The same reasoning leads to the form $\lfloor p | x \rfloor = \exp(2\pi ipx/d)$ for $\Xi = \mathbb{Z}_d \times \mathbb{Z}_d$. Now the Fourier transform and its inverse are defined by

$$(\mathcal{F}\psi)(p) = \int dx \, \overline{\lfloor p | x \rfloor} \, \psi(x) \quad \text{and}$$

$$(\mathcal{F}^*\phi)(x) = \int dp \, \lfloor p | x \rfloor \, \phi(x). \tag{1}$$

Here the bar means complex conjugation. Note that each of these formulas fixes a normalization of the Haar measure on $\widehat{X}$ relative to that on X and a theorem says that these two, potentially distinct, conventions do coincide [10, Thm.4.4.14]. $\mathcal{F}$ is a unitary operator with inverse $\mathcal{F}^*$ and the usual formulas relating the product of functions to the convolution of their Fourier transforms hold, with the pertinent powers of 2π absorbed into the definition of the measures. The momentum observable acts by multiplication after the Fourier transform and the momentum probability density associated with a state vector ψ is just $|(\mathcal{F}\psi)(p)|^2$.

As an example, let us consider a qubit, which is usually not done in these terms. The group is now $X = \{0, 1\}$, i.e., a single bit with addition mod 2. The dual group $\widehat{X}$ is the same with $\lfloor p | x \rfloor = \exp(2\pi ipx/2) = (-1)^{px}$. The Fourier transform acts on $\mathcal{H} = \mathbb{C}^2$ by a Hadamard matrix. The position observable is given by the diagonal matrices (functions of σ_z) and the momentum observable is given by the functions of σ_x.

Momentum translations will act in the position representation by multiplication with $\lfloor p | x \rfloor$. Combining these with the position translations we get the phase space

translation operators, or *Weyl operators*,

$$(W(q,p)\psi)(x) = \lfloor p|x \rfloor \psi(x+q). \qquad (2)$$

These form a projective representation of the phase space translation group $\Xi = X \times \widehat{X}$:

$$W(q_1,p_1)W(q_2,p_2) = \lfloor p_1|q_2 \rfloor W(q_1+q_2, p_1+p_2). \qquad (3)$$

Sometimes it is customary to change each Weyl operator by a phase, particularly in the standard $\mathbb{R}^n \times \mathbb{R}^n$ case where this simplifies the relation for the adjoint to $W(\xi)^* = W(-\xi)$. With the choice (2) this reads instead

$$W(q,p)^* = \lfloor p|q \rfloor W(-q,-p). \qquad (4)$$

The factor in (3) depends on phase conventions, but the commutation phase

$$W(q_1,p_1)W(q_2,p_2) = \lfloor p_1|q_2 \rfloor \overline{\lfloor p_2|q_1 \rfloor} W(q_2,p_2)W(q_1,p_1) \qquad (5)$$

does not.

For many purposes, it is not necessary that the notation refers separately to position and momentum, so we will just write $\xi \in \Xi$ for the pair $\xi = (q,p) \in X \times \widehat{X}$ and "$d\xi$" for "$dq\,dp$". With the above conventions about normalizing the measures, this translates into $dp\,dq/(2\pi)$ for the standard phase space. Thus, the phase space volume is measured in units of Planck's constant $h = 2\pi\hbar = 2\pi$. It should be noted that while the normalizations of the individual measures dq and dp contain a conventional factor, the phase space measure is independent of such conventions. The phase space translations of quantum observables (operators $A \in \mathcal{B}(\mathcal{H})$) and classical observables (functions $f : \Xi \to \mathbb{C}$) are now given by

$$\alpha_\xi(A) = W(\xi)^* A W(\xi) \quad \text{and} \quad (\alpha_\xi f)(\eta) = f(\eta - \xi). \qquad (6)$$

Similarly, we can define the operation of phase space inversion by the parity operator $(\Pi\psi)(x) = \psi(-x)$ as

$$\beta(A) = \Pi A \Pi \quad \text{and} \quad (\beta f)(\eta) = f(-\eta). \qquad (7)$$

This notation is chosen to emphasize the quantum-classical analogy and helps generalize the *convolution* from phase space functions to operators [8]. Indeed, the convolution of functions can be alternatively written as

$$(f * g)(\xi) = \int d\eta\, f(\eta) g(\xi - \eta)$$
$$= \int d\eta\, f(\eta)(\alpha_\eta g)(\xi) = \int d\eta\, f(\eta)(\alpha_\xi \beta g)(\eta),$$
$$(f * g) = \int d\eta\, f(\eta)(\alpha_\eta g), \qquad (8)$$

where the last line is just another version of the second, read as an equation between functions. This version allows the definition of the convolutions between functions and operators (giving an operator). The second expression on the second line, with the trace of functions substituted for phase space integrals, suggests the convolution of two operators, which is then again a phase space function:

$$f * A = A * f = \int d\eta\, f(\eta)(\alpha_\eta A), \qquad (9)$$

$$(A * B)(\xi) = \text{tr}\big(A(\alpha_\xi \beta B)\big). \qquad (10)$$

We have not specified the analytic conditions for these integrals to exist. For functions on an infinite phase space (like $f = g = 1$) the integral may diverge. A crucial Lemma in this theory, based on the square integrability of matrix elements $\langle \phi, W(\xi)\psi \rangle$, is that if all factors involved are either integrable functions (i.e., $\|f\|_1 = \int d\xi\, |f(\xi)| < \infty$) or "trace class" operators (i.e., $\|A\|_1 = \text{tr}\,|A| < \infty$), then the same holds for their convolution. Convolution is then a commutative and associate product that determines a Banach algebra with the 1-norm. It also has the crucial property that the convolution of positive factors is positive. The convolution can also be extended to the case where one factor is just a bounded function or operator. However, in this case the result can only be guaranteed to be bounded, and in a product of several factors we can usually only allow one such factor.

The main upshot of this formalism for our purpose is the characterization of *covariant phase space observables*. By definition, these are normalized positive $\mathcal{B}(\mathcal{H})$-valued measures F that commute with phase space translations. We use the compact notation $F[f] = \int F(d\xi)\, f(\xi)$, i.e., F with parentheses is a function on the subsets of Ξ, and F with brackets is the linear operator $F[\cdot] : \mathcal{L}^\infty(\Xi, d\xi) \to \mathcal{B}(\mathcal{H})$ one gets from this by integration. Because, conversely, $F(\sigma) = F[\chi_\sigma]$, with χ_σ the indicator function of a measurable set $\sigma \subset \Xi$, we consider these two to be essentially the same object. Covariance then means that $\alpha_\xi F[f] = F[\alpha_\xi f]$. The basic theorem on the subject [8, Prop.3.3] then states that the covariant phase space observables are in one-to-one correspondence with the density operators ρ_F on $\mathcal{H}$, given by the formula

$$F[f] = \rho_F * f. \qquad (11)$$

That ρ_F is a density operator has a double meaning: On one hand, it describes the conditions for the correspondence $F \leftrightarrow \rho_F$, namely $\rho_F \geqslant 0$ and $\text{tr}\,\rho_F = 1$. Somewhat accidentally, these are the conditions for an operator describing a mixed quantum state. On the other hand, the

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measure $F(\cdot)$ has an operator-valued density with respect to $d\xi$, namely the translated $\alpha_\xi(\rho_F)$. This "accident" will be crucial later for establishing the equivalence between the measurement uncertainty relations for F and preparation uncertainty relations for a certain state, namely ρ_F.

For the measurement uncertainty, we need the position and momentum marginals of such observables, i.e., the expectations of functions of only position or only momentum. So let $f : X \to \mathbb{C}$ be some function on position space. We can consider it either as a classical function on phase space $f_q : \Xi \to \mathbb{C}$ by $f_q(x,p) = f(x)$, or as a quantum operator $f(Q) \in \mathcal{B}(\mathcal{H})$ as determined by functional calculus. This is the multiplication operator $(f(Q)\psi)(x) = f(x)\psi(x)$. Then expectations can alternatively be written as an integral over phase space (resp. a trace) or as an integral over just X with respect to a suitable "marginal." Thus, if ρ is a density operator and μ is a probability density on phase space, we define the marginals μ^q and ρ^Q by

$$\int \mathrm{d}x\, \mu^q(x) f(x) = \int \mathrm{d}\xi\, \mu(\xi) f_q(\xi), \qquad (12)$$

$$\int \mathrm{d}x\, \rho^Q(x) f(x) = \mathrm{tr}\, \rho f(Q). \qquad (13)$$

Thus, ρ^Q is just the position probability density associated with the quantum state ρ. Classically, μ^q arises from μ by integrating out the momenta. Similarly, in the quantum case, integrating over all momentum translates of ρ produces an operator, namely $\int dp\, \alpha_{(0,p)}(\rho) = (\rho^Q)(Q)$. Now suppose we have prepared a quantum state ρ, measured the covariant observable F, and evaluated the expectation of a function f_q depending on position only. Then the overall expectation is

$$\mathrm{tr}\, \rho(\rho_F * f_q) = \rho * \beta(\rho_F * f_q)(0) = \int \mathrm{d}x\, (\rho * \beta\rho_F)^q f(x)$$

$$= \int \mathrm{d}x\, \mu^q(x) f(x) \qquad (14)$$

with $\qquad \mu^q = \rho^Q * (\beta\rho_F)^Q. \qquad (15)$

This has a remarkable interpretation, which is the basis of the equivalence between measurement and preparation uncertainty in our setting: The probability density for the position marginal of a covariant observable F in the state ρ is the convolution of the density ρ^Q for the ideal position observable in the same state and the corresponding density of another state, $\beta\rho_F$. Because convolution is an operation representing the sum of independent random variables, we arrive at the following statement:

The position marginal of a covariant phase space observable can be simulated by first making an ideal position measurement and adding

to the outcome some random noise with a fixed distribution, independent of the input state. The distribution of the noise is the position distribution of another quantum state characterizing the observable.

Of course, the same holds mutatis mutandis for momentum (and letters p, P replacing q, Q), with *the same* state $\beta\rho_F$ characterizing the observable. Thus, the preparation uncertainty trade-off of having either $(\beta\rho_F)^Q$ or $(\beta\rho_F)^P$ sharp translates directly into the measurement uncertainty trade-off of measuring either position or momentum precisely, but never both.

3 Measurement and preparation uncertainty

The statement that measurement and preparation uncertainty bounds are quantitatively equal for phase space observables presupposes that the errors and variances are defined in a closely related way. This begins by choosing, for each observable a metric d on the set X of outcomes. This not only fixes the units in which all deviations are measured, but also is an adaptation to the concrete problem at hand. For example, for discrete outcomes we might just be interested in whether outcomes coincide, without assigning a numerical weight (other than a constant) to their distance in case they do not. This is then simply expressed by choosing the discrete metric $d(x,y) = 1 - \delta_{xy}$. For real valued observables like position and momentum we always take the standard distance $d(x,y) = |x - y|$. The only requirement on the metric will be that it is translation invariant, i.e., $d(x+z, y+z) = d(x,y)$.

In addition we will fix, for every observable, an error exponent α with $1 \leqslant \alpha \leqslant \infty$. Then if μ is a probability measure on X, we define its *deviation from a point* $x \in X$ as

$$d(\mu, x) = \left(\int \mu(\mathrm{d}y) \mathrm{d}(x,y)^\alpha \right)^{\frac{1}{\alpha}}. \qquad (16)$$

Thus, for $\alpha = 2$ we get the mean quadratic deviation, for $\alpha = 1$ the mean absolute deviation, and in the limit $\alpha \to \infty$ the maximal deviation (discounting sets of μ-measure zero). The *spread* of the probability measure, which we just denote by $d(\mu)$ is its smallest deviation from any point, i.e.,

$$d(\mu) = \min_x d(\mu, x). \qquad (17)$$

The notation (16) suggests that this expression somehow extends the original metric on X to one on the probability measures. This is intentional, and for the formulation

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of measurement uncertainty we actually also need the extension to the case where both arguments are probability measures, say ν and μ. In this case, we set

$$d(\nu,\mu) = \inf_\gamma \left(\int \gamma(\mathrm{d}x\,\mathrm{d}y) d(x,y)^\alpha \right)^{\frac{1}{\alpha}}, \qquad (18)$$

where the infimum is over all "couplings" of μ and ν, i.e., all joint distributions on $X \times X$ such that the first variable is distributed according to ν and the second according to μ. When ν is concentrated on the point x this expression reduces to (16). This metric is called the *transport metric* [11] associated with d and α. It expresses the minimal cost of converting ν into μ, when transferring one mass unit from x to y costs $d(x,y)^\alpha$. In particular, when $\mu = \nu$, the best coupling (= transport plan) is to leave everything as is. Therefore it corresponds to γ spread out on the diagonal of $X \times X$, giving $d(\mu,\mu) = 0$. Similarly, when μ arises from ν by a translation a of the variable, we have $d(\mu,\nu) = d(a,0)$. Finally, for a convolution of probability measures we get

$$d(\mu * \nu, \mu) \leqslant d(\nu, 0). \qquad (19)$$

Using this notation, we can say that the preparation uncertainty theory for the observables P and Q is the study of the set of pairs

$$\mathsf{PUR} = \left\{ \left(d(\rho^P), d(\rho^Q) \right) \,\middle|\, \rho \text{ a state} \right\}, \qquad (20)$$

where ρ^P, ρ^Q denote the position and momentum distributions of the state ρ. In particular, we want to show that this "uncertainty region" contains no points near the origin.

Measurement uncertainty is a property of any (approximate) joint measurement F of two observables. For each of them, i.e., in our case P and Q, we compare the output marginal distributions in a state ρ, denoted by ρF^Q and ρF^P, with what one would have got with the corresponding ideal measurement. We want the result to be uniformly good for all input states, i.e., we look at

$$d(F^Q, Q) = \sup_\rho d\left(\rho F^Q, \rho^Q \right) \qquad (21)$$

and the corresponding quantity for P. This vanishes if and only if the position distribution obtained by F is the same as the usual one for arbitrary input states ρ. In that case, Heisenberg [12] told us to expect that the corresponding quantity fail badly for momentum. The trade-off is thus given by the measurement uncertainty region

$$\mathsf{MUR} = \left\{ \left(d(F^P, P), d(F^Q, Q) \right) \,\middle|\, F \text{ a joint measure-} \right.$$

$$\left. \text{ment} \right\}. \qquad (22)$$

For variants and a discussion of these notions, see Ref. [3].

Now for a covariant measurement F it is easy to compute both measurement uncertainties. Combining (14) with (19) we get the bound in terms of the "noise generated by F", i.e.,

$$d\left(\rho F^Q, \rho^Q \right) \leqslant d\left((\beta \rho_F)^Q, 0 \right) \leqslant d\left((\beta \rho_F)^Q \right). \qquad (23)$$

Here the last inequality holds with equality iff the mean of the position distribution $(\beta \rho_F)^Q$ is zero, which can be achieved easily by just shifting all position outcomes. Any other choice of a constant offset would clearly be sub-optimal, so we have equality in the optimal case. Moreover, equality holds in (19) if μ is a point measure. Therefore, because of the supremum in (21), we have

$$d(F^P, P) = d\left((\beta \rho_F)^P \right) \quad \text{and}$$
$$d(F^Q, Q) = d\left((\beta \rho_F)^Q \right) \qquad (24)$$

for all covariant (and centered) F. Hence, measurement uncertainties for F are the same as the preparation uncertainties for the state $\beta \rho_F$. Ref. [4, 5] showed that the general case can be reduced to the covariant one by an averaging procedure. Hence, we have

$$\mathsf{MUR} = \mathsf{PUR}. \qquad (25)$$

4 How to compute the bounds from a ground state problem

General methods for efficiently computing measurement uncertainty relations are still scarce. We therefore use the known methods for preparation uncertainty. First, it is better to work with variances rather than with deviations, i.e., to omit the root in definition (16). For the purpose of drawing uncertainty diagrams this is just a rescaling, but the linearity in ρ makes estimates more straightforward. Second, we can reduce to the case of centered states, for which the minimum in (17) is attained at $x = 0$ (resp. $p = 0$). This can always be achieved by a translation. Hence, for the position variance we just have to compute the expectation of the function $x \mapsto d(x,0)^\alpha$, or, when written in functional calculus, the expectation of the unbounded operator $d(Q,0)^\alpha$. The trade-off is taken into account by considering linear combinations of variances with positive weights, and minimizing these over all states. That is, for $t > 0$:

$$d(\rho^P)^\beta + t d(\rho^Q)^\alpha = \operatorname{tr} \rho \left(d(P,0)^\beta + t\, d(Q,0)^\alpha \right)$$
$$\equiv \operatorname{tr} \rho H_{\alpha\beta}(t). \qquad (26)$$

It is clear that the operator $H_{\alpha\beta}(t)$ appearing here is usu-

ally unbounded, but positive, so technically speaking we mean its Friedrichs extension. In all cases it has a discrete spectrum, and minimizing the above expression over ρ is just finding its ground state "energy" $E_{\alpha\beta}(t)$. Essentially, this function is the Legendre transform of the trade-off curve we want to determine: For a fixed value of $d(\rho^Q)$ we find the best otherwise state-independent bound on $d(\rho^P)$ by treating t as a parameter to be optimized. This gives the state independent bound

$$d(\rho^P)^\beta \geqslant \sup_t \left\{ E_{\alpha\beta}(t) - t\,\Delta \right\} \quad \text{if } \Delta = d(\rho^Q)^\alpha. \quad (27)$$

This is the description of the trade-off curve (or rather: its best convex approximation) and the following examples will all be based on this method.

5 Examples

Here we will provide some concrete examples of the theory outlined in the previous sections.

5.1 The standard case: $\Xi = \mathbb{R}^n \times \mathbb{R}^n$ with Euclidean distance

Due to the dilation symmetry $(x, p) \mapsto (\lambda x, \lambda^{-1}p)$ the uncertainty region will be bounded by a hyperbola, and completely described by the best constant c in

$$d(\rho^Q)\, d(\rho^P) \geqslant c_{\alpha,\beta}(n)\hbar. \quad (28)$$

This scaling symmetry is what makes "dimensional analysis" work. Thus, in the above relation we brought in the dimensional constant $\hbar$ to make c dimensionless, but will take $\hbar = 1$ from now on. The textbook case is $c_{2,2}(1) = 1/2$. For $n = 1$ the constants (from Ref. [4]) are shown in Fig. 1.

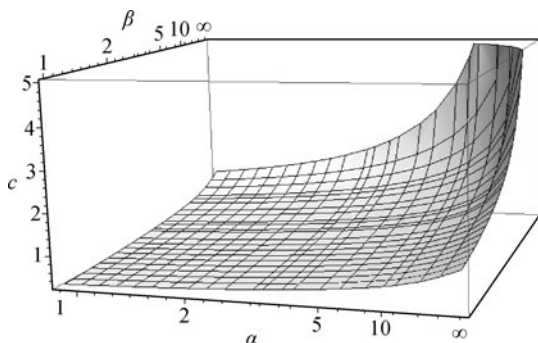Depending on the application there may be good reasons to explore exponents other than 2. For example,

Fig. 1 Uncertainty constants $c_{\alpha\beta}(1)$. The axes have been scaled non-linearly to represent the infinite range. We have $c_{\infty\infty}(1) = \infty$, because there are no states with strictly bounded support in both position and momentum.

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$\alpha = \infty$ corresponds to the case of strict spatial confinement, such as the lateral position when passing a slit. If we are interested in the root mean square momentum spread after the slit, the constant $c_{\infty,2}$ will give a much better bound than first converting the slit information to a constraint on the root mean square deviation in position and using $c_{2,2}$ instead.

To relate the constants $c_{\alpha,\beta}(n)$ to a ground state problem we consider the two-parameter family of Hamiltonians and ground states

$$H_{\alpha\beta}(a, b) = a \left(\sum_{i=1}^n Q_i^2 \right)^{\frac{\alpha}{2}} + b \left(\sum_{i=1}^n P_i^2 \right)^{\frac{\beta}{2}}$$
$$\geqslant E(a, b)\,\mathbb{1}. \quad (29)$$

Then E satisfies the identities $E(\mu a, \mu b) = \mu E(a, b)$ from homogeneity, and $E(\lambda^\alpha a, \lambda^{-\beta}b) = E(a, b)$ from dilation symmetry, so that

$$E(a, b) = a^{\frac{\beta}{\alpha+\beta}}\, b^{\frac{\alpha}{\alpha+\beta}}\, E, \quad (30)$$

with $E = E(1, 1)$ for short. We now optimize λ on the right hand side of the following inequality to get

$$E \leqslant \lambda^\alpha d(\rho^Q)^\alpha + \lambda^{-\beta} d(\rho^P)^\beta$$
$$= (\alpha + \beta)\, \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} \left(d(\rho^Q) d(\rho^P) \right)^{\frac{\alpha\beta}{\alpha+\beta}}, \quad (31)$$

which shows (28).

The dimension dependence is straightforward in the quadratic case, because variances just add up to give the Euclidean variance, i.e., $H_{\alpha\beta}$ separates in Cartesian coordinates. We get $E = n$, and hence

$$c_{2,2}(n) = \frac{n}{2}. \quad (32)$$

In general we can still use the rotation symmetry to simplify the problem, seeking joint eigenfunctions of $H_{\alpha\beta}$, and the angular Laplacian L^2. If the eigenvalue for the latter operator is λ, we have to find the smallest E for which we can solve the radial equation

$$r^\alpha \phi(r) + \left(-\frac{d^2}{dr^2} + \frac{4\lambda + (n-1)(n-3)}{4r^2} \right)^{\frac{\beta}{2}} \phi(r)$$
$$= E\, \phi(r). \quad (33)$$

We have chosen to include the weight coming from the integration in polar coordinates into the wave function, so the radial Laplacian, i.e., the operator in parentheses, contains no first derivatives. Of course, unless β is even, this is not a differential operator of finite order. No general solution is then available.

However, e.g., the case $\alpha = \infty$, $\beta = 2$ is tractable. For $\alpha = \infty$, $d(\rho^Q)$ is the radius of the smallest ball containing the support of the position distribution. We fix

this to be $r = 1$, and include other values by scaling symmetry. Then the "potential" term in (33) becomes zero inside the ball, but diverges outside. Because we are seeking wave functions with finite $d(\rho^P)$, we cannot have a jump at the boundary and must impose a zero boundary condition at $r = 1$. The bottom eigenvalue E of P^2 in (33) is then the lowest admissible value of $d(\rho^P)^2$ and we find $E = c_{\infty,2}(n)^2$, either directly from (28) or from (31). Clearly, E is the lowest for $\lambda = 0$, i.e., a purely radial function. At $r = 0$ the ground state wave function, written in Cartesian coordinates goes to a constant, so $\phi(r) \sim r^{(n-1)/2}$. This singles out the Bessel function

$$\phi(r) \propto \sqrt{r}\, J_{n/2-1}\big(\sqrt{E}\, r\big). \tag{34}$$

The scaling E in the argument then has to be chosen such that we have the first zero $z_1(n/2-1)$ of the Bessel function at $r = 1$, which determines the bottom eigenvalue as $E = z_1(n/2-1)^2$. Hence

$$c_{\infty,2}(n) = z_1\left(\frac{n}{2}-1\right) \approx \frac{n}{2} + 1.47292\, n^{1/3} - 1 + \mathbf{o}(1), \tag{35}$$

where the asymptotic expansion of z_1 is taken from [13, 9.5.14]. The n-dependence of this expression is clearly not as simple as (32), although it is asymptotically linear (see Fig. 2). A direct derivation of this observation will be given below (Section 5.4).

5.2 Number and angle

This case is treated in detail in Ref. [6]. For definiteness, we treat the discrete variable as a position Q, and the angle-valued one as a momentum P. Two metrics naturally suggest themselves for either side: For the discrete variable, say two numbers $x, y \in \mathbb{Z}$, we can look at $|x-y|$, but we may just as well be interested in the probability of two numbers coinciding, which is expressed by the discrete metric $1 - \delta_{xy}$. For two angles p, r we may either measure angle along the unit circle, i.e., $|p-r+2\pi n|$, with
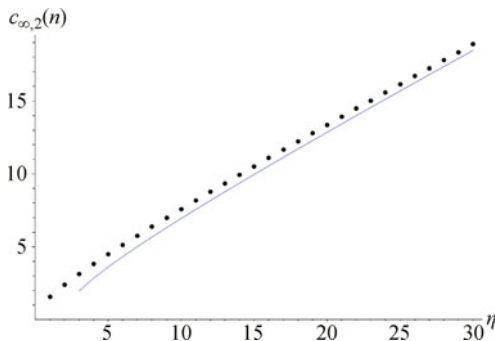

Fig. 2 The dots show the dimension dependence of the constant in (28), for $\alpha = \infty$, $\beta = 2$. The blue line is the asymptotic expansion (35).

n chosen to minimize this expression, or the length of the chord through the circle, $2|\sin(p-r)/2|$. The trade-off curves are readily computed numerically, but there are few analytic expressions. For example, for the discrete metric on $\mathbb{Z}$ ($\alpha = 1$), the chordal metric for angles ($\beta = 2$) we have

$$d(\rho^Q)^2 + d(\rho^P)^2\big(4 - d(\rho^P)^2\big) \geqslant 1. \tag{36}$$

5.3 Qudits: $\mathbb{Z}_n \times \mathbb{Z}_n$

In this case, the discrete metric is the natural one, especially when one is interested in quantum information coding problems. For the discrete metric $d(x, x')^\alpha = d(x, x')$. Therefore, changing the error exponent gives no new information and we take $\alpha = \beta = 1$. In this space of discrete distributions on n points,

$$\Delta = 1 - 1/n \tag{37}$$

is the "radius", i.e., the distance from the totally mixed state to a pure state and hence the largest possible variance. The "diameter", i.e., the largest distance between any distributions is 1, attained at a pair of distinct pure states. It is clear that when the position is sharp, the momentum has a flat distribution. Therefore, the points $(0, \Delta)$, $(\Delta, 0)$ will be in the uncertainty region.

Now $d(Q, 0) = \mathbb{1} - |0\rangle\langle 0|$ and $d(Q, 0) = \mathbb{1} - |\phi\rangle\langle\phi|$, with the zero-momentum eigenvector $\phi = n^{-1/2} \sum_j |j\rangle$. The ground state of (26) is to be found in the span of $|0\rangle$ and $|\phi\rangle$. Hence, the pairs of expectations $(\mathrm{tr}\,\rho d(P, 0), \mathrm{tr}\,\rho d(Q, 0))$ are an affine image of a qubit state space and thus lie on an ellipse, joined with the point $(1, 1)$ for states orthogonal to both $|0\rangle$ and $|\phi\rangle$. The ellipse fits exactly into the unit square and also contains the antipodal points $(1 - \Delta, 1)$ and $(1, 1 - \Delta)$. This fixes the trade-off curve (see Fig. 3). The resulting uncertainty relation is thus for all $d(\rho^P), d(\rho^Q) \leqslant \Delta$,

$$\left(d(\rho^P) - \Delta\right)^2 + \left(d(\rho^P) - \Delta\right)^2 + \left(2 - \frac{4}{n}\right)d(\rho^P)d(\rho^Q)$$
$$\leqslant \Delta^2. \tag{38}$$

In this form it is easy to see that if one uncertainty vanishes, the other has to be equal to Δ.

It is interesting to compare this relation, in its version as a measurement uncertainty relation, with a simple ansatz for a joint measurement using the idea of approximate cloning (cf. also Ref. [14]). To this end, we consider an asymmetric cloner, given by an isometry $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ of the form

$$V\phi = a\phi \otimes \Omega + b\Omega \otimes \phi, \quad \text{with}$$

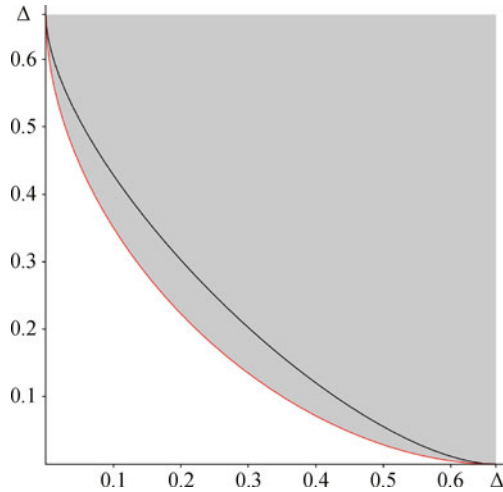$$1 = |a|^2 + |b|^2 + 2\Re e(\overline{a}b)/n,$$

Fig. 3 The uncertainty region (shaded) for a qudit as described by (38), drawn for $n = 3$. The second ellipse inside the region shows what can be achieved by universal cloning and measuring on the clones.

$$\Omega = n^{-1} \sum_j |jj\rangle. \qquad (39)$$

Thus, V maps systems to three copies of the systems, of which the middle one is then traced out as an ancilla. The parameters a, b determine which of the two output copies is more faithful: When $a = 1$, the first copy is just the original system, and the second is completely depolarized. This is reversed for $a = 0$. The distinguishing feature of this one-parameter family of cloning maps is the intertwining relation

$$VU = (U \otimes \overline{U} \otimes U)V \qquad (40)$$

for arbitrary n-dimensional unitary operators. It implies "universality" in the sense that no direction and no basis in Hilbert space is singled out.

When F, E are arbitrary full basis projective measurements, we get a joint measurement by

$$G_{x,y} = V^*(F_x \otimes \mathbb{1} \otimes E_y)V. \qquad (41)$$

Its marginals are readily computed to be

$$F'_x := \sum_y G_{x,y} = (1 - |b|^2)F_x + \frac{|b|^2}{n} \mathbb{1}. \qquad (42)$$

Thus, F' differs from F by the admixture of state independent noise with a flat distribution and "probability" $|b|^2$. Here the quotes are added as a warning that this may fail to be a proper probability: when $ab < 0$, we can have $|b|^2 > 1$. The coefficient of F_x may thus negative, but the coefficient of the noise term is always positive. The largest distance between the output distributions of F and F' is achieved at an eigenstate of F. This gives

$$d(F, F') = \Delta |b|^2. \qquad (43)$$

For position and momentum, we get a joint measurement, which is also covariant because of (40). It is therefore generated by a density operator, namely $\rho_F = nV^*(|0\rangle\langle 0| \otimes \mathbb{1} \otimes |\phi\rangle\langle\phi|)V$. One readily verifies that this is not pure and hence cannot be optimal. The comparison of the uncertainty pairs generated by cloning and the optimal bound is given in Fig. 3.

This suggests relaxing the intertwining (40) to only phase space shifts. In this way we arrive at a phase space covariant cloning device (not to be confused with a "phase covariant" cloner). Because the phase space structure is the main theme of this paper, we briefly describe how to obtain such maps. It turns out to be convenient to look, not at V, but at an operator $\widehat{V}$ with rearranged matrix elements, which takes $\mathbb{C}^n \otimes \mathbb{C}^n$ to itself, namely

$$\langle jkl|V|i\rangle = \langle jl|\widehat{V}|ki\rangle. \qquad (44)$$

One then verifies easily that (40) is equivalent to $[U \otimes U, \widehat{V}] = 0$ and that (41) becomes $G_{x,y} = \operatorname{tr}_1 \widehat{V}^*(F_x \otimes E_y)\widehat{V}$, where tr_1 denotes the partial trace over the first factor. These relations are only demanded for $U = W(q, p)$. Therefore, $\widehat{V}$ lies in the algebra spanned by the Weyl operators commuting with the group of operators $W(q, p) \otimes W(q, p)$. It is hence a linear combination

$$\widehat{V} = \sum_{q,p} u(q, p)W(q, p) \otimes W(-q, -p), \qquad (45)$$

where $u(q, p)$ are suitable complex coefficients. The normalization condition is $\operatorname{tr}_2 \widehat{V}^*\widehat{V} = \mathbb{1}$, which can by guaranteed by adjusting an overall scalar factor, because the left hand side commutes with all Weyl operators and is hence a multiple of the identity. For the same reason as in the case of the universal cloner, the phase space covariant cloner will give a covariant observable. To explore the possibilities, it suffices to determine the density operators $\rho_F = nG_{0,0}$ obtained by various choices of u. Direct computation gives (up to irrelevant constant factors)

$$\rho_F = \sum_{q,p,q',p'} u(q', p')\overline{u(q, p)}\, \delta_{qq'}\, |p\rangle\langle p'| = \sum_q |\psi_q\rangle\langle\psi_q|, \qquad (46)$$

where $\psi_q = \sum_p \overline{u(q, p)}|p\rangle$ and the kets $|p\rangle$ are eigenkets of momentum. Clearly, we can choose $u(q, p)$ so that the ψ_q are the eigenvectors (times the square root of the eigenvalue) of any density operator we choose. It follows that *every* covariant observable can be realized by phase space covariant cloning.

5.4 Qubit strings: $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$

In this case "position" corresponds to the readout in the computational basis, say the product of the Z eigenbases

for every qubit, and "momentum" is the readout in the product of some conjugate eigenbases, say X. As the distance function, we take the Hamming distance per qubit:

$$d(x, x') = \frac{1}{n} \sum_{i=1}^{n} |x_i - x_i'|. \qquad (47)$$

The Hamiltonian is now a many-body operator with non-commuting terms. However, for large n the two terms commute approximately, and the ground state problem is within the scope of mean-field theory, as laid out in Ref. [15]. The basic result is that the ground state energy is obtained asymptotically by minimizing instead a classical function on the one-particle state space. We associate a "classical Hamiltonian function" with the Hamiltonian $H_{\alpha\beta}(t)$ on the set of one-particle density matrices ρ_1, namely

$$\left(h_{\alpha\beta}(t) \right)(\rho_1) = \left(\operatorname{tr} \rho_1 d(P_1, 0) \right)^{\alpha} + t \left(\operatorname{tr} \rho_1 d(Q_1, 0) \right)^{\beta}, \qquad (48)$$

where P_1, Q_1 are the one-particle position and momentum, respectively. In the limit $n \to \infty$, the ground state energy converges to the minimum of this function. We do not have to compute this minimum explicitly, because we are only interested in the uncertainty region that it outlines. This is given directly by the two terms in (48), with ρ_1 ranging over the one-particle state space. Taking qubits with X and Z measurements, we parametrize ρ_1 by its Bloch sphere coordinates, and find that the boundary curve of the asymptotic uncertainty region PUR_{∞} is given by

$$t \mapsto \left(\left(\frac{1}{2}(1 + \cos t) \right)^{\alpha}, \left(\frac{1}{2}(1 + \sin t) \right)^{\beta} \right). \qquad (49)$$

This method works for all systems of a large number of equal copies. We can also use it to get a handle on the dimension dependence in Section 5.1. Let $E_n(a, b)$ be the constants in (29) and (30) with the n-dependence made explicit. If we set $a = n^{-\alpha/2}$ and $b = n^{-\beta/2}$, the resulting ground state problem is of the mean field type, and we get

$$\begin{aligned} \lim_n E_n(a, b) &= \min_{\rho_1} \left\{ \left(\operatorname{tr} \rho_1 Q_1^2 \right)^{\alpha/2} + \left(\operatorname{tr} \rho_1 P_1^2 \right)^{\beta/2} \right\} \\ &= \min_v \left(v^{\alpha/2} + (4v)^{-\beta/2} \right) \\ &= (\alpha + \beta)\, 2^{-\frac{\alpha\beta}{\alpha+\beta}} \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}}. \end{aligned} \qquad (50)$$

On the other hand, from (30) we get $E_n(a, b) = n^{-\frac{\alpha\beta}{\alpha+\beta}} E$

and hence from (31) we have

$$E_n(a, b) = n^{-\frac{\alpha\beta}{\alpha+\beta}} (\alpha + \beta)\alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} c_{\alpha\beta}(n)^{\frac{\alpha\beta}{\alpha+\beta}}. \qquad (51)$$

Combining this we get the remarkably simple result that

$$\lim_n c_{\alpha\beta}(n) \frac{2}{n} = 1, \quad \text{for all } \alpha, \beta \geqslant 1. \qquad (52)$$

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