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New $D(2,1;\alpha)$ mechanics with spin variables

S. Fedoruk,[a] **E. Ivanov**[a] **and O. Lechtenfeld**[b]

[a] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia*

[b] *Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, D-30167 Hannover, Germany*

E-mail: fedoruk@theor.jinr.ru, eivanov@theor.jinr.ru, lechtenf@itp.uni-hannover.de

Abstract: We elaborate on a novel superconformal mechanics model possessing $D(2,1;\alpha)$ symmetry and involving extra U(2) spin variables. It is the one-particle case of the $\mathcal{N}=4$ superconformal matrix model recently proposed in arXiv:0812.4276 [hep-th], and it generalizes to arbitrary $\alpha\neq 0$ the OSp(4|2) superconformal mechanics of arXiv:0905.4951 [hep-th]. As in the latter case, the U(2) spin variables are described by a Wess-Zumino action and define the first Hopf map $S^3 \to S^2$ in the target space. Upon quantization, they represent a fuzzy sphere. We find the classical and quantum generators of the $D(2,1;\alpha)$ superalgebra and their realization on the physical states. The super wave-function encompasses various multiplets of the $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ subgroups of $D(2,1;\alpha)$, with fixed isospins. The conformal potential is determined by the external magnetic field in the Wess-Zumino term, whose strength is quantized like in the OSp(4|2) case. As a byproduct, we reveal new invariant subspaces in the enveloping algebra of $D(2,1;\alpha)$ for our quantum realization.

Keywords: Extended Supersymmetry, Superspaces, Black Holes

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Contents

1 Introduction

The interest in various models of $\mathcal{N}$=4 superconformal mechanics is mainly caused by the possibility of using them for the description of supergravity black-hole solutions within the AdS/CFT correspondence, as was first suggested in [1].

In [2], we constructed a new $\mathcal{N}$=4 superconformal matrix model with U(n) gauge symmetry. This model is described by the following harmonic superspace action,

$$S = -\frac{1}{4(1+\alpha)}\int \mu_H \mathrm{Tr}\left(\mathcal{X}^{-1/\alpha}\right) + \frac{1}{2}\int \mu_A^{(-2)}\mathcal{V}_0\widetilde{\mathcal{Z}}^+\mathcal{Z}^+ + \frac{i}{2}\,c\int \mu_A^{(-2)}\,\mathrm{Tr}\,V^{++}\,, \qquad (1.1)$$

where α is a real parameter which can take any non-zero value. The first term in (1.1) is the gauged action of the (**1,4,3**) multiplets which are described by hermitian $(n{\times}n)$-matrix superfields $\mathcal{X} = (\mathcal{X}_a^b)$, $a,b = 1,\dots,n$. They are in the adjoint of U(n) and are subject to appropriate gauge-covariant constraints. These constraints involve the gauge connections which are expressed through the analytic harmonic gauge superfield $V^{++}(\zeta,u)$ [3]. The third term in (1.1) is a Fayet-Iliopoulos (FI) term for V^{++} and the real constant c is its strength. The second term in (1.1) is a Wess-Zumino (WZ) action describing n commuting analytic superfields $\mathcal{Z}_a^+$ which represent off-shell $\mathcal{N}$=4 multiplets of type (**4,4,0**) and are in the fundamental of U(n). The superfield $\mathcal{V}_0(\zeta,u)$ is a real analytic gauge prepotential for the U(n) singlet (**1,4,3**) superfield $\mathcal{X}_0 \equiv \mathrm{Tr}\,(\mathcal{X})$.

After passing to the WZ gauge, eliminating auxiliary degrees of freedom and fixing a gauge with respect to the residual gauge group, the model (1.1) involves n bosonic fields x_a which are the first components of the diagonal superfields $\mathcal{X}_a^a$ (no sum over a), n^2 fermionic fields ψ_a^b which are the second components in the θ expansion of $\mathcal{X}_a^b$, and the lowest commuting components of the superfields $\mathcal{Z}_a^+$. The latter variables are described by Wess-Zumino-type $d = 1$ actions and parametrize n independent target spheres S^2 . Thus, they may be interpreted as target harmonic variables. After quantization, they become a sort of non-dynamical spin variables representing n "fuzzy" spheres.

The model (1.1) is invariant under the most general $\mathcal{N}$=4 superconformal symmetry $D(2,1;\alpha)$ (with the more customary OSp(4|2) and SU(1,1|2) symmetries as particular cases). It contains two SU(2) R-symmetry subgroups one of which acts only on fermions. In the case of $D(2,1;\alpha{=}-\frac{1}{2}) \simeq$ OSp(4|2), this model yields a new $\mathcal{N}$=4 supersymmetric extension of the U(2) spin A_{n-1} Calogero system.

Note that for $\alpha{=}-1$ we have $D(2,1;\alpha{=}-1) \simeq$ SU(1,1|2)$\otimes$SU(2). It was argued in [4] that the large-n limit of the n-particle SU(1,1|2) superconformal Calogero model provides a microscopic description of the extreme Reissner-Nordström (RN) black hole in the near-horizon limit. This hypothesis is based on the assertion that for a large number of particles and in a limit when all coordinates of the Calogero model, except for one, are treated as "small", the Calogero model reduces to the conformal mechanics for this "allocated" coordinate.

For all values of $\alpha{\neq}-1/2$, the actions (1.1) yield non-trivial conformal sigma models in the bosonic limit. Therefore, the model (1.1) can hardly be utilized to describe a single black hole along the lines of [4]. Yet, it may be relevant to the multi-black-hole system, since the corresponding moduli spaces of n black holes in four- and five-dimensional supergravities are known to be described by sigma-model-type multi-black-hole quantum mechanics [5–9]. They become flat precisely in the case of OSp(4|2) superconformal symmetry, i.e. at $\alpha{=}-1/2$.

Note that the construction of a self-consistent n-body generalization of black-hole quantum mechanics is a rather complicated problem [5–9] beyond the one- and two-body cases. In order to have a normalizable ground state in the latter cases, one should apply a proper time redefinition, just as in conformal quantum mechanics [10]. If the general multi-black-hole quantum mechanics amounts to supersymmetric Calogero models, one can employ the powerful machinery developed for integrable super-Calogero systems (see e.g. [11–17]).

In the present paper we investigate the n=1 case of the model (1.1), which describes the center-of-mass motion in the general super-Calogero model and, therefore, corresponds to a single black hole. The special case of $\alpha{=}-1/2$, both on classical and quantum levels, was considered in detail in [18]. Here, we extend this consideration to all non-zero values of α.[1] We hope that an exhaustive understanding of the n=1 case will be helpful for attacking the quantum $D(2,1;\alpha)$ model for arbitrary values of n.

We use the standard notations of $\mathcal{N}$=4, d=1 supersymmetric theories, following [20, 21] and [18].

[1]Another view of the $D(2,1;\alpha)$ superconformal mechanics models with spin variables (based on an $su(2)$ Hamiltonian reduction at the classical component level) was presented in [19].

2 Superfield setup

The one-particle limit of the model (1.1) involves superfields corresponding to three off-shell $\mathcal{N}$=4 supermultiplets: **(i)** the "radial" multiplet (**1,4,3**); **(ii)** the Wess-Zumino ("isospin") multiplet (**4,4,0**); and **(iii)** the gauge ("topological") multiplet. The total action has the form

$$S = S_{\mathcal{X}} + S_{FI} + S_{WZ}\,. \tag{2.1}$$

The first term in (2.1) is the standard free action of the (**1,4,3**) multiplet ($\alpha \neq 0$)

$$S_{\mathcal{X}} = -\frac{1}{4(1+\alpha)}\int \mu_H\, \mathcal{X}^{-1/\alpha}\,, \tag{2.2}$$

where the even real superfield $\mathcal{X}$ is subjected to the constraints

$$D^{++}\,\mathcal{X} = 0\,, \tag{2.3}$$

$$D^+D^-\,\mathcal{X} = 0\,, \qquad \bar{D}^+\bar{D}^-\,\mathcal{X} = 0\,, \qquad (D^+\bar{D}^- + \bar{D}^+D^-)\,\mathcal{X} = 0\,. \tag{2.4}$$

The set of conditions (2.3) and (2.4) is equivalent to the standard constraints $D^iD_i\,\mathcal{X} = 0$, $\bar{D}_i\bar{D}^i\,\mathcal{X} = 0$, $[D^i,\bar{D}_i]\,\mathcal{X} = 0$ for the superfield $\mathcal{X}$ living in the "central basis $\mathcal{N}$=4 superspace" parametrized by the coordinates θ_i, $\bar{\theta}^i$ and t.

Note that the action (2.2) is in fact non-singular at $\alpha = -1$. Indeed, making use of the fact that $\int \mu_H\,\mathcal{X}$ is an integral of total derivative, we cast the action (2.2) in the equivalent form

$$S_{\mathcal{X}} = -\frac{1}{4(1+\alpha)}\int \mu_H\,\left(\mathcal{X}^{-1/\alpha} - \mathcal{X}\right)\,.$$

Thus in the limit $\alpha = -1$ we obtain the standard action

$$S_{\mathcal{X}}\Big|_{\alpha=-1} = -\frac{1}{4}\int \mu_H\, \mathcal{X}\,\ln\!\mathcal{X}\,, \tag{2.5}$$

The action (2.2) is not defined at α=0, and this special case needs a separate analysis (see section 5). In what follows we always assume that $\alpha \neq 0$.

The second term in (2.1) is FI term

$$S_{FI} = \frac{i}{2}\,c\int \mu_A^{(-2)}\,V^{++} \tag{2.6}$$

for the gauge supermultiplet. The even analytic gauge superfield $V^{++}(\zeta,u)$, $D^+V^{++} = 0$, $\bar{D}^+V^{++} = 0$, is subjected to the gauge transformations

$$V^{++\prime} = V^{++} - D^{++}\lambda, \quad \lambda = \lambda(\zeta,u)\,, \tag{2.7}$$

which are capable to gauge away, *locally*, all the components from V^{++}. However, the latter contains a component which cannot be gauged away *globally*. This is the reason why this $d = 1$ supermultiplet was called "topological" in [3].

Last term in (2.1) is Wess-Zumino (WZ) term

$$S_{WZ} = \frac{1}{2}\,b\int \mu_A^{(-2)}\,\mathcal{V}\,\widetilde{\mathcal{Z}}^+\,\mathcal{Z}^+\,. \tag{2.8}$$

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Here, the complex analytic superfield $\mathcal{Z}^+, \widetilde{\mathcal{Z}}^+$ $(D^+\mathcal{Z}^+ = \bar{D}^+\mathcal{Z}^+ = 0)$, is subjected to the harmonic constraints

$$\mathcal{D}^{++}\,\mathcal{Z}^+ \equiv (D^{++} + i\,V^{++})\,\mathcal{Z}^+ = 0\,, \qquad \mathcal{D}^{++}\,\widetilde{\mathcal{Z}}^+ \equiv (D^{++} - i\,V^{++})\,\widetilde{\mathcal{Z}}^+ = 0 \tag{2.9}$$

and describes a gauge-covariantized version of the $\mathcal{N}$=4 multiplet (**4,4,0**). The relevant gauge transformations are

$$\mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^+, \qquad \widetilde{\mathcal{Z}}^{+\prime} = e^{-i\lambda}\widetilde{\mathcal{Z}}^+\,. \tag{2.10}$$

We explicitly included a coupling constant b in (2.8) in order to track the contribution of WZ term to the full component action. Afterwards, this constant will be put equal to 1.

The superfield $\mathcal{V}(\zeta, u)$ in (2.8) is a real analytic gauge superfield $(D^+\,\mathcal{V} = \bar{D}^+\,\mathcal{V} = 0)$, which is a prepotential solving the constraints (2.3) and (2.4) for $\mathcal{X}$. It is related to the superfield $\mathcal{X}$ in the central basis by the harmonic integral transform [22]

$$\mathcal{X}(t, \theta_i, \bar{\theta}^i) = \int du\,\mathcal{V}\left(t_A, \theta^+, \bar{\theta}^+, u^\pm\right)\Big|_{\theta^\pm = \theta^i u_i^\pm,\ \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}\,. \tag{2.11}$$

The unconstrained analytic prepotential $\mathcal{V}$ possesses its own pregauge freedom

$$\delta\mathcal{V} = D^{++}\lambda^{--}\,, \quad \lambda^{--} = \lambda^{--}(\zeta, u)\,, \tag{2.12}$$

which can be exploited to show that $\mathcal{V}$ describes just the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ (after choosing the appropriate Wess-Zumino gauge) [22]. The coupling to the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ in (2.8) is introduced for ensuring superconformal invariance. We shall see that, upon passing to components, it gives rise to non-trivial interactions for the physical fields. The invariance of (2.8) under (2.12) is ensured by the constraints (2.9).

Besides the gauge U(1) symmetry (2.7), (2.10) and pregauge symmetry (2.12), the action (2.1) respects the rigid $\mathcal{N}$=4 superconformal symmetry $D(2,1;\alpha)$. All superconformal transformations are contained in the closure of the supertranslations and superconformal boosts.

Invariance of the action (2.1) under the supertranslations $(\bar{\varepsilon}^i = \overline{(\varepsilon_i)})$

$$\delta t = i(\theta_k\bar{\varepsilon}^k - \varepsilon_k\bar{\theta}^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar{\theta}^k = \bar{\varepsilon}^k$$

is automatic because we use the $\mathcal{N}$=4 superfield approach.

The coordinate realization of the superconformal boosts of $D(2,1;\alpha)$ [3, 21] is as follows $(\bar{\eta}^i = \overline{(\eta_i)})$:

$$\delta' t = it(\theta_k\bar{\eta}^k + \bar{\theta}^k\eta_k) + (1+\alpha)\,\theta_i\bar{\theta}^i(\theta_k\bar{\eta}^k + \bar{\theta}^k\eta_k)\,, \tag{2.13}$$

$$\delta'\theta_i = \eta_i t - 2i\alpha\,\theta_i(\theta_k\bar{\eta}^k) + 2i(1+\alpha)\,\theta_i(\bar{\theta}^k\eta_k) - i(1+2\alpha)\,\eta_i(\theta_k\bar{\theta}^k)\,, \tag{2.14}$$

$$\delta'\bar{\theta}^i = \bar{\eta}^i t - 2i\alpha\,\bar{\theta}^i(\bar{\theta}^k\eta_k) + 2i(1+\alpha)\,\bar{\theta}^i(\theta_k\bar{\eta}^k) + i(1+2\alpha)\,\bar{\eta}^i(\theta_k\bar{\theta}^k)\,, \tag{2.15}$$

$$\delta' t_A = \alpha^{-1}\Lambda t_A\,, \qquad \delta' u_i^+ = \Lambda^{++}u_i^-\,, \tag{2.16}$$

$$\delta'\theta^+ = \eta^+ t_A + 2i(1+\alpha)\eta^-\theta^+\bar{\theta}^+\,, \qquad \delta'\bar{\theta}^+ = \bar{\eta}^+ t_A + 2i(1+\alpha)\bar{\eta}^-\theta^+\bar{\theta}^+\,, \tag{2.17}$$

$$\delta'(dtd^4\theta) = -\alpha^{-1}\,(dtd^4\theta)\,\Lambda_0\,, \quad \delta'\mu_H = \mu_H\left(2\Lambda - \alpha^{-1}(1+\alpha)\Lambda_0\right)\,, \quad \delta'\mu_A^{(-2)} = 0\,, \tag{2.18}$$

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where

$$\Lambda = \tilde{\Lambda} = 2i\alpha(\bar{\eta}^-\theta^+ - \eta^-\bar{\theta}^+)\,,\ \Lambda^{++} = D^{++}\Lambda = 2i\alpha(\bar{\eta}^+\theta^+ - \eta^+\bar{\theta}^+)\,,\ D^{++}\Lambda^{++} = 0\,, \tag{2.19}$$

$$\Lambda_0 = 2\Lambda - D^{--}\Lambda^{++} = 2i\alpha(\theta_k\bar{\eta}^k + \bar{\theta}^k\eta_k)\,,\quad D^{++}\Lambda_0 = 0\,. \tag{2.20}$$

Taking the field transformations in the form (here we use the "passive" interpretation of them)

$$\delta'\mathcal{X} = -\Lambda_0\,\mathcal{X}\,,\qquad \delta'\mathcal{V} = -2\Lambda\,\mathcal{V}\,,\qquad \delta'\mathcal{Z}^+ = \Lambda\,\mathcal{Z}^+\,,\qquad \delta' V^{++} = 0\,, \tag{2.21}$$

it is easy to check the invariance of the action (2.1). Note that the constraints (2.3), (2.4) and (2.9) as well as the actions (2.2), (2.6) and (2.8), are invariant with respect to the $D(2,1;\alpha)$ transformations with an arbitrary $\alpha\neq 0$. It is worth pointing out that the action (2.8) is superconformally invariant just due to the presence of the analytic prepotential $\mathcal{V}$.

3 Component actions

3.1 Action for (1,4,3) supermultiplet

The solution of the constraint (2.3), (2.4) is as follows (in the analytic basis):

$$\begin{aligned}\mathcal{X} = {}& x + \theta^-\psi^+ + \bar{\theta}^-\bar{\psi}^+ - \theta^+\psi^- - \bar{\theta}^+\bar{\psi}^- + \theta^-\bar{\theta}^-N^{++} + \theta^+\bar{\theta}^+N^{--} + (\theta^-\bar{\theta}^+ + \theta^+\bar{\theta}^-)N \\ & + \theta^-\theta^+\bar{\theta}^-\Omega^+ + \bar{\theta}^-\bar{\theta}^+\theta^-\bar{\Omega}^+ + \theta^-\bar{\theta}^-\theta^+\bar{\theta}^+D\,.\end{aligned} \tag{3.1}$$

Here

$$N^{\pm\pm} = N^{ik}u^\pm_i u^\pm_k\,,\quad N = i\dot{x} - N^{ik}u^+_i u^-_k\,,\quad D = 2\ddot{x} + 2i\dot{N}^{ik}u^+_i u^-_k\,, \tag{3.2}$$

$$\psi^\pm = \psi^i u^\pm_i\,,\qquad \bar{\psi}^\pm = \bar{\psi}^i u^\pm_i\,,\qquad \Omega^+ = 2i\dot{\psi}^+\,,\qquad \bar{\Omega}^+ = -2i\dot{\bar{\psi}}^+ \tag{3.3}$$

and $x(t_A)$, $N^{ik} = N^{(ik)}(t_A)$, $\psi^i(t_A)$, $\bar{\psi}_i(t_A) = (\overline{\psi^i})$ are d=1 fields.

Inserting (3.1) in (2.2) we obtain

$$\begin{aligned} S_{\mathcal{X}} = {}& \frac{1}{4\alpha^2}\int dt\, x^{-\frac{1}{\alpha}-2}\left[\dot{x}\dot{x} - i\left(\bar{\psi}_k\dot{\psi}^k - \dot{\bar{\psi}}_k\psi^k\right) - \frac{1}{2}N^{ik}N_{ik}\right] \\ & - \frac{1}{4\alpha^2}\left(\frac{1}{\alpha}+2\right)\int dt\, x^{-\frac{1}{\alpha}-3}\,N^{ik}\psi_{(i}\bar{\psi}_{k)} \\ & - \frac{1}{12\alpha^2}\left(\frac{1}{\alpha}+2\right)\left(\frac{1}{\alpha}+3\right)\int dt\, x^{-\frac{1}{\alpha}-4}\,\psi^i\bar{\psi}^k\psi_{(i}\bar{\psi}_{k)}\,. \end{aligned} \tag{3.4}$$

In the central basis the θ expansion (3.1) takes the form:

$$\mathcal{X}(t,\theta_i,\bar{\theta}^i) = x + \theta_i\psi^i + \bar{\psi}_i\bar{\theta}^i + \theta^i\bar{\theta}^k N_{ik} + \frac{i}{2}(\theta)^2\dot{\psi}_i\bar{\theta}^i + \frac{i}{2}(\bar{\theta})^2\theta_i\dot{\bar{\psi}}^i + \frac{1}{4}(\theta)^2(\bar{\theta})^2\ddot{x}\,, \tag{3.5}$$

where $(\theta)^2 \equiv \theta_i\theta^i = -2\theta^+\theta^-$, $(\bar{\theta})^2 \equiv \bar{\theta}^i\bar{\theta}_i = 2\bar{\theta}^+\bar{\theta}^-$. Then, from (2.11) we can identify the fields appearing in the WZ gauge for $\mathcal{V}$ with the fields in (3.5)

$$\mathcal{V}(t_A,\theta^+,\bar{\theta}^+,u^\pm) = x(t_A) - 2\,\theta^+\psi^i(t_A)u^-_i - 2\,\bar{\theta}^+\bar{\psi}^i(t_A)u^-_i + 3\,\theta^+\bar{\theta}^+N^{ik}(t_A)u^-_i u^-_k\,. \tag{3.6}$$

This expansion will be used to express the action (2.8) in terms of the component fields.

3.2 FI and WZ actions

Using the U(1) gauge freedom (2.7), (2.10) we can choose WZ gauge

$$V^{++} = -2i\,\theta^+\bar\theta^+ A(t_A)\,. \tag{3.7}$$

Then

$$S_{FI} = c\int dt\, A\,. \tag{3.8}$$

The solution of the constraint (2.9) in WZ gauge (3.7) is

$$\mathcal{Z}^+ = z^i u^+_i + \theta^+\varphi + \bar\theta^+\phi + 2i\,\theta^+\bar\theta^+\nabla_{t_A} z^i u^-_i\,, \qquad \widetilde{\mathcal{Z}}^+ = \bar z_i u^{+i} + \theta^+\bar\phi - \bar\theta^+\bar\varphi + 2i\,\theta^+\bar\theta^+\nabla_{t_A}\bar z_i u^{-i}$$

where

$$\nabla z^k = \dot z^k + iA\,z^k\,, \qquad \nabla\bar z_k = \dot{\bar z}_k - iA\,\bar z_k\,. \tag{3.9}$$

In (3.9), $z^i(t_A)$ and $\varphi(t_A)$, $\phi(t_A)$ are $d{=}1$ fields, bosonic and fermionic, respectively. The fields z^i form a complex doublet of the R-symmetry SU(2) group, while the fermionic fields are singlets of the latter. Another ("mirror") R-symmetry SU(2) is not manifest in the present approach: the bosonic fields are its singlets, while the fermionic fields form a doublet with respect to it.

Inserting expressions (3.9) and (3.6) in the action (2.8) and performing integration over θ s and harmonics there, we obtain a component form of the WZ action

$$\begin{aligned} S_{WZ} =& \frac{i}{2}\,b\int dt\,\left(\bar z_k\nabla z^k - \nabla\bar z_k\, z^k\right)x - \frac{1}{2}\,b\int dt\, N^{ik}\bar z_i z_k \\ &+\frac{1}{2}\,b\int dt\left[\psi^k\left(\bar\varphi\, z_k + \bar z_k\phi\right) + \bar\psi^k\left(\bar\phi\, z_k - \bar z_k\varphi\right) - x\left(\bar\phi\,\phi + \bar\varphi\,\varphi\right)\right]. \end{aligned} \tag{3.10}$$

The fermionic fields ϕ, φ are auxiliary. The action is invariant under the residual local U(1) transformations

$$A' = A - \dot\lambda_0\,, \quad z^{i\prime} = e^{i\lambda_0} z^i\,, \;\bar z_i{}' = e^{-i\lambda_0}\bar z_i \tag{3.11}$$

(and similar phase transformations of the fermionic fields).

The total component action is a sum of (3.4), (3.8) and (3.10). Eliminating the auxiliary fields N^{ik}, ϕ, $\bar\phi$, φ, $\bar\varphi$, from this sum by their algebraic equations of motion,

$$N_{ik} = -2b\alpha^2 x^{\frac{1}{\alpha}+2} z_{(i}\bar z_{k)} - \left(\frac{1}{\alpha}+2\right) x^{-1}\,\psi_{(i}\bar\psi_{k)}\,, \tag{3.12}$$

$$\phi = -\frac{\bar\psi^k z_k}{x}\,, \quad \bar\phi = \frac{\psi^k\bar z_k}{x}\,, \quad \varphi = -\frac{\psi^k z_k}{x}\,, \quad \bar\varphi = -\frac{\bar\psi^k\bar z_k}{x}\,, \tag{3.13}$$

and making the redefinition

$$x' = x^{-\frac{1}{2\alpha}}\,, \qquad \psi'_k = -\frac{1}{2\alpha}\,x^{-\frac{1}{2\alpha}-1}\psi_k\,, \qquad z'^i = x^{1/2}\,z^i\,, \tag{3.14}$$

we obtain the on-shell form of the action (2.1) in WZ gauge (we omitted the primes on x, ψ and z)

$$S = S_b + S_f\,, \tag{3.15}$$

$$S_b = \int dt\left[\dot x\dot x + \frac{i}{2}\,b\left(\bar z_k\dot z^k - \dot{\bar z}_k z^k\right) - \frac{b^2\alpha^2(\bar z_k z^k)^2}{4x^2} - A\left(b\bar z_k z^k - c\right)\right], \tag{3.16}$$

$$S_f = -i\int dt\left(\bar\psi_k\dot\psi^k - \dot{\bar\psi}_k\psi^k\right) + 2b\alpha\int dt\,\frac{\psi^i\bar\psi^k z_{(i}\bar z_{k)}}{x^2} + \frac{2}{3}\,(1+2\alpha)\int dt\,\frac{\psi^i\bar\psi^k\psi_{(i}\bar\psi_{k)}}{x^2}. \tag{3.17}$$

It is still invariant under the gauge transformations (3.11). The $d{=}1$ gauge connection $A(t)$ in (3.16) is the Lagrange multiplier for the constraint

$$\bar{z}_k z^k = c\,. \tag{3.18}$$

This constraint implies $c > 0$. After varying with respect to A, the action (3.15) is gauge invariant only with taking into account the constraint (3.18) which is gauge invariant by itself. The constant b in (3.16), (3.17) marks the contributions of the superfield WZ term to the physical component action. It can be eliminated by a proper rescaling of the variables $z^i, \bar{z}_i$, so hereafter we choose $b = 1$.

It is convenient to fully fix the residual gauge freedom by choosing the phases of z^1 and z^2 opposite to each other. In this gauge, the constraint (3.18) is solved by

$$z^1 = \kappa \cos\frac{\gamma}{2}\, e^{i\beta/2}\,, \qquad z^2 = \kappa \sin\frac{\gamma}{2}\, e^{-i\beta/2}\,, \qquad \kappa^2 = c\,. \tag{3.19}$$

In terms of the newly introduced fields the bosonic action (3.16) takes the form[2]

$$S_b = \int dt \left[\dot{x}\dot{x} - \frac{\alpha^2 c^2}{4\,x^2} - \frac{c}{2}\cos\gamma\,\dot{\beta}\right]. \tag{3.20}$$

As argued in section 5, this action can be relevant to describing some particular orbits near horizon of the extreme $D{=}5$ black holes. The spinor z^k provides a parametrization of the angular part of the set of the horizon coordinates.

Unconstrained fields in the action (3.15), three bosons x, γ, β and four fermions ψ^k, $\bar{\psi}_k$, constitute some on-shell supermultiplet with three bosonic and four fermionic fields. As opposed to the off-shell (**3,4,1**) supermultiplet considered in [20, 23, 24] the action (3.16) contains "true" kinetic term only for one bosonic component x which also possesses the conformal potential, whereas two other fields parametrizing the coset $\mathrm{SU}(2)_R/\mathrm{U}(1)_R$ are described by a WZ term. Taken separately, the WZ term provides an example of Chern-Simons mechanics [25–31]. The variables $\gamma(t)$ and $\beta(t)$ (or z^k and $\bar{z}_k$ in the manifestly SU(2) covariant formulation) become (iso)spin degrees of freedom (target SU(2) harmonics) upon quantization. The realization of the $D(2,1;\alpha)$ superconformal transformations on these fields will be given in the next section.

It should be stressed that the considered model realizes a new mechanism of generating conformal potential $\sim 1/x^2$ for the field $x(t)$. Before eliminating auxiliary fields, the component action contains no explicit term of this kind. It arises as a result of varying with respect to the Lagrange multiplier $A(t)$ and making use of the arising constraint (3.18). As we shall see, in quantum theory this new mechanism entails a quantization of the constant c.

The naive inspection of the bosonic action (3.16) could lead to the conclusion that angular variables completely decouple from a radial variable, and, hence, are superfluous. Moreover, the classical dynamics associated with the WZ term in (3.16) is trivial. However, like in other Chern-Simons-type theories, this term has a non-trivial impact on the quantum properties of the model. Indeed, as we shall see in the quantum case, owing to the non-trivial geometry of the angular space the quantum state vectors necessarily

[2] The fermionic action (3.17) can also be rewritten in terms of β and γ, like its $\alpha{=}-1/2$ prototype [18].

carry quantum numbers of the SU(2) spin. Though in the bosonic limit this symmetry is purely internal (it commutes with the $d = 1$ conformal group SL$(2, R)$), the presence of angular variables leads to the property that the wave function encompasses non-trivial SU(2) multiplets.[3] In the supersymmetry case, when the full action (3.15) is considered, the situation becomes even more involved. Now this SU(2) symmetry in addition acts on fermions in parallel with the second SU(2) R-symmetry which from the very beginning is realized only on fermionic fields, and these either SU(2) are an essential part of the superconformal group. Examining the action (3.15), we were not able to find any change of variables which would decouple the angular variables from other ones. Actually, we already observed the same phenomenon in the particular OSp(4|2) case [18]. Now we see that it persists at any choice of the parameter α in $D(2, 1; \alpha)$. Even at the classical level, the WZ term yields, e.g., a non-trivial additional contribution to the fermionic equations of motion (coming from the term proportional to b in (3.17)). Although in the β, γ parametrization both γ and $\dot{\beta}$ can be expressed through fermions and some integration constant by their classical equations of motion, an essential trace of the WZ couplings still remains in the equations of motion for fermions, producing a mass term for them and modifying the coefficients before the third order terms.[4] The Hamiltonian, $\mathcal{N}$=4 supercharges and other $D(2, 1; \alpha)$ generators also involve important new pieces caused by the WZ term and additional fermionic couplings associated with it (see below).

3.3 $\mathcal{N}$=4 superconformal symmetry in WZ gauge

The transformations and their generators look most transparent in terms of the SU(2) doublet quantities z^k and $\bar{z}_k$.

To determine the superconformal transformations of component fields, we should know the appropriate compensating gauge transformations needed to preserve the WZ gauge (3.7). For supertranslations and superconformal boosts the parameter of the compensating gauge transformations is as follows

$$\lambda = 2i \left[(\theta^+ \bar{\varepsilon}^- - \bar{\theta}^+ \varepsilon^-) + t_A \, (\theta^+ \bar{\eta}^- - \bar{\theta}^+ \eta^-) \right] A \tag{3.21}$$

where

$$\varepsilon^- := \varepsilon^i u_i^- \,, \quad \eta^- := \eta^i u_i^- \,. \tag{3.22}$$

Taking this into account, we obtain the relevant infinitesimal $D(2, 1; \alpha)$ transformations which leave the action (3.15) invariant (as in (3.15) we omit 'primes' on the newly intro-

[3]In the bosonic case, in accord with the general concept of separating variables, one can postulate that the wave function is a product of the chargeless conformal mechanics wave function by the lowest Landau level wave function associated with the SU(2) WZ term. No such a separation is possible in the generic superconformal case due to the presence of fermions interacting with both types of bosonic variables.

[4]We thank S. Krivonos for a discussion on this issue.

duced variables):

$$\delta x = -\omega_i\psi^i + \bar\omega^i\bar\psi_i\,, \tag{3.23}$$

$$\delta\psi^i = i\bar\eta^i x - i\bar\omega^i\dot x - \alpha\frac{\bar\omega_k z^{(i}\bar z^{k)}}{x} - (1+2\alpha)\frac{\bar\omega_k\psi^k\bar\psi^i + \omega_k\psi^k\psi^i}{x}\,, \tag{3.24}$$

$$\delta\bar\psi_i = -i\eta_i x + i\omega_i\dot x - \alpha\frac{\omega^k z_{(i}\bar z_{k)}}{x} + (1+2\alpha)\frac{\omega^k\bar\psi_k\psi_i + \bar\omega^k\bar\psi_k\bar\psi_i}{x}\,, \tag{3.25}$$

$$\delta z^i = -2\alpha\,\frac{\omega^{(i}\psi^{k)} + \bar\omega^{(i}\bar\psi^{k)}}{x}\,z_k\,, \qquad \delta\bar z_i = 2\alpha\,\frac{\omega_{(i}\psi_{k)} + \bar\omega_{(i}\bar\psi_{k)}}{x}\,\bar z^k\,, \tag{3.26}$$

$$\delta A = 0\,, \tag{3.27}$$

where $\omega_i = \varepsilon_i + t\,\eta_i$ and $\bar\omega^i = \bar\varepsilon^i + t\,\bar\eta^i$. Note that the closure of d=1 Poincarè supersymmetry transformations is a sum of the time translations and residual U(1) gauge transformation with a field-dependent parameter. Such a sum turns out to vanish for the d=1 gauge field A. In appendix this specifically d=1 phenomenon is expounded on a simple example of toy $\mathcal{N}$=2 supersymmetric model.

Now, using the Nöther procedure, we can directly find the classical generators of the supertranslations

$$Q^i = p\,\psi^i + 2i\alpha\frac{z^{(i}\bar z^{k)}\psi_k}{x} + i(1+2\alpha)\frac{\psi_k\psi^k\bar\psi^i}{x}\,, \tag{3.28}$$

$$\bar Q_i = p\,\bar\psi_i - 2i\alpha\frac{z_{(i}\bar z_{k)}\bar\psi^k}{x} + i(1+2\alpha)\frac{\bar\psi^k\bar\psi_k\psi_i}{x}\,, \tag{3.29}$$

where $p \equiv 2\dot x$, as well as of the superconformal boosts:

$$S^i = -2\,x\psi^i + t\,Q^i, \qquad \bar S_i = -2\,x\bar\psi_i + t\,\bar Q_i\,. \tag{3.30}$$

The remaining (even) generators of the supergroup $D(2,1;\alpha)$ can be found by evaluating mutual anticommutators of the odd generators.

As follows from the action (3.15), the SU(2) spinor variables are canonically self-conjugate due to the presence of second-class constraints for their momenta. As a result, non-vanishing canonical Dirac brackets (at equal times) have the following form

$$[x,p]_D = 1, \quad [z^i,\bar z_j]_D = -i\delta^i_j, \quad \{\psi^{ii'},\psi^{kk'}\}_D = \frac{i}{2}\,\epsilon^{ik}\epsilon^{i'k'} \quad \left(\{\psi^i,\bar\psi_j\}_D = \frac{i}{2}\,\delta^i_j\right) \tag{3.31}$$

where we introduced the notations

$$\psi^{ii'} = (\psi^{i1'},\psi^{i2'}) = (\psi^i,\bar\psi^i), \qquad (\overline{\psi^{ii'}}) = \psi_{ii'} = \epsilon_{ik}\epsilon_{i'k'}\psi^{kk'}\,, \quad (\epsilon_{12} = \epsilon^{21} = 1). \tag{3.32}$$

Using Dirac brackets (3.31), we arrive at the following closed superalgebra:

$$\{Q^{ai'i}, Q^{bk'k}\}_D = 2i\left(\epsilon^{ik}\epsilon^{i'k'}T^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}J^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}I^{i'k'}\right), \tag{3.33}$$

$$[T^{ab},T^{cd}]_D = -\epsilon^{ac}T^{bd} - \epsilon^{bd}T^{ac}, \tag{3.34}$$

$$[J^{ij},J^{kl}]_D = -\epsilon^{ik}J^{jl} - \epsilon^{jl}J^{ik}, \quad [I^{i'j'},I^{k'l'}]_D = -\epsilon^{ik}I^{j'l'} - \epsilon^{j'l'}I^{i'k'}, \tag{3.35}$$

$$[T^{ab},Q^{ci'i}]_D = \epsilon^{c(a}Q^{b)i'i}, \quad [J^{ij},Q^{ai'k}]_D = \epsilon^{k(i}Q^{ai'j)}, \quad [J^{i'j'},Q^{ak'i}]_D = \epsilon^{k'(i'}Q^{aj')i}\,. \tag{3.36}$$

In (3.33)–(3.36) we use the notation

$$Q^{21'i} = -Q^i\,, \qquad Q^{22'i} = -\bar{Q}^i\,, \qquad Q^{11'i} = S^i\,, \qquad Q^{12'i} = \bar{S}^i\,, \tag{3.37}$$

$$T^{22} = H\,, \qquad T^{11} = K\,, \qquad T^{12} = -D\,. \tag{3.38}$$

The explicit expressions for the generators are

$$H = \frac{1}{4}\,p^2 + \alpha^2\,\frac{(\bar{z}_k z^k)^2}{4x^2} - 2\alpha\,\frac{\psi^i\bar{\psi}^k z_{(i}\bar{z}_{k)}}{x^2} - (1+2\alpha)\,\frac{\psi_i\psi^i\,\bar{\psi}^k\bar{\psi}_k}{2x^2}\,, \tag{3.39}$$

$$K = x^2 - t\,xp + t^2\,H\,, \tag{3.40}$$

$$D = -\frac{1}{2}\,xp + t\,H\,, \tag{3.41}$$

$$J^{ij} = i\left[z^{(i}\bar{z}^{j)} + \psi^{ik'}\psi^j{}_{k'}\right] = i\left[z^{(i}\bar{z}^{k)} + 2\psi^{(i}\bar{\psi}^{k)}\right]\,, \tag{3.42}$$

$$I^{i'j'} = i\psi^{ki'}\psi_k{}^{j'} \qquad \left(I^{1'1'} = -i\psi_k\psi^k\,, \quad I^{2'2'} = i\bar{\psi}^k\bar{\psi}_k\,, \quad I^{1'2'} = -i\psi_k\bar{\psi}^k\right)\,. \tag{3.43}$$

The relations (3.33)–(3.36) provide the standard form of the superalgebra $D(2,1;\alpha)$ (see, e.g., [23, 32, 33]). Bosonic generators $T^{ab} = T^{ba}$, $J^{ik} = J^{ki}$, $I^{i'k'} = I^{k'i'}$ form mutually commuting $su(1,1)$, $su(2)_R$ and $su(2)_L$ algebras, respectively.[5]

It is worth pointing out one important feature of the basic relation $\{Q^i, \bar{Q}_j\}_D = 2iH\delta^i_j$. Although Q and $\bar{Q}$ contain terms of the third order in ψ with the coefficients $(1+2\alpha)$, no quartic fermionic term $\sim (1+2\alpha)^2$ appears in the Hamiltonian. This is because of the vanishing Dirac bracket

$$\{\psi_k\psi^k\bar{\psi}^i, \bar{\psi}^l\bar{\psi}_l\psi_j\}_D = 0\,. \tag{3.44}$$

The expression (3.39) coincides with the canonical Hamiltonian associated with the action (3.15). Owing to the A-term in (3.15), there is also the first-class constraint

$$D^0 - c \equiv \bar{z}_k z^k - c \approx 0\,, \tag{3.45}$$

which should be imposed on the wave functions in quantum case.

Casimir operators (on classical level) of the $su(1,1)$, $su(2)_R$ and $su(2)_L$ algebras are

$$T^2 \equiv \frac{1}{2}\,T^{ab}T_{ab} = HK - D^2 = \frac{1}{4}\alpha^2(z^k\bar{z}_k)^2 - 2\alpha\,z^{(i}\bar{z}^{k)}\psi_{(i}\bar{\psi}_{k)} - \frac{1}{2}(1+2\alpha)\psi_i\psi^i\,\bar{\psi}^k\bar{\psi}_k, \tag{3.46}$$

$$J^2 \equiv \frac{1}{2}\,J^{ik}J_{ik} = \frac{1}{4}\,(z^k\bar{z}_k)^2 - 2z^{(i}\bar{z}^{k)}\psi_{(i}\bar{\psi}_{k)} - \frac{3}{2}\,\psi_i\psi^i\,\bar{\psi}^k\bar{\psi}_k\,, \tag{3.47}$$

$$I^2 \equiv \frac{1}{2}\,I^{i'k'}I_{i'k'} = I\bar{I} - (I_3)^2 = \frac{3}{2}\,\psi_i\psi^i\,\bar{\psi}^k\bar{\psi}_k\,. \tag{3.48}$$

Using these expressions and

$$\frac{i}{4}\,Q^{ai'i}Q_{ai'i} = \frac{i}{2}\,(Q^i\bar{S}_i - S^i\bar{Q}_i) = 4\alpha\,z^{(i}\bar{z}^{k)}\psi_{(i}\bar{\psi}_{k)} + 2(1+2\alpha)\,\psi_i\psi^i\,\bar{\psi}^k\bar{\psi}_k\,, \tag{3.49}$$

[5] It would be of interest to clarify the precise relation of our realization of $D(2,1;\alpha)$ derived from the concrete model to the realization found recently in [34] from a different reasoning.

we obtain that the second-order (classical) Casimir operator of $D(2,1;\alpha)$,

$$C_2 = T^2 + \alpha J^2 - (1+\alpha) I^2 + \frac{i}{4}\, Q^{ai'i} Q_{ai'i}\,, \tag{3.50}$$

takes the form

$$C_2 = \frac{1}{4}\,\alpha(\alpha+1)\,(z^k \bar{z}_k)^2 = \frac{1}{4}\,\alpha(\alpha+1)\,(D^0)^2\,. \tag{3.51}$$

It is important to note that the (iso)spin (angular) variables make significant contributions to $D(2,1;\alpha)$, $su(1,1)$ and $su(2)_R$ Casimirs (3.46), (3.47), (3.51). Additional terms in these operators are generated by the second and third terms in the Hamiltonian (3.39) and the first terms in the generators (3.42), all arising from the terms $\propto b$ in the actions (3.16) and (3.17).

By inspecting the expressions (3.46)–(3.49), we observe that the following quantity M vanishes identically for this particular realization of the $D(2,1;\alpha)$ superalgebra:

$$M \equiv T^2 - \alpha^2 J^2 - \frac{1}{3}\,(1-\alpha^2)\, I^2 + \frac{i}{8}\,(1-\alpha)\, Q^{ai'i} Q_{ai'i} = 0\,. \tag{3.52}$$

Using this identity together with the expression (3.50), we obtain the constraint

$$(\alpha+1)\Big[T^2 - \alpha J^2 - \frac{1}{3}\,(\alpha-1) I^2\Big] - (\alpha-1) C_2 = 0\,, \tag{3.53}$$

which relates the Casimir of $D(2,1;\alpha)$ to the Casimirs of the three mutually commuting bosonic subgroups SU(1,1), $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ in our model. Plugging the expression (3.51) for the $D(2,1;\alpha)$ Casimir in this constraint, we find that

$$(\alpha+1)\Big[T^2 - \alpha J^2 - \frac{1}{3}\,(\alpha-1) I^2 - \frac{1}{4}\,\alpha(\alpha-1)\,(D^0)^2\Big] = 0\,. \tag{3.54}$$

Using the expressions (3.46)–(3.48), we can check that the term in the square brackets is vanishing, that is the expression

$$T^2 = \alpha J^2 + \frac{1}{3}\,(\alpha-1) I^2 + \frac{1}{4}\,\alpha(\alpha-1)\,(D^0)^2 \tag{3.55}$$

is valid for all $\alpha \neq 0$, including $\alpha = -1$.

Note that the Hamiltonian (3.39) has the standard form of the Hamiltonian of (super)conformal mechanics[6]

$$H = \frac{1}{4}\, p^2 + \frac{T^2}{x^2}\,. \tag{3.56}$$

Using the expression (3.55), we can represent the Hamiltonian in the convenient equivalent form

$$H = \frac{1}{4}\, p^2 + \alpha(\alpha-1)\,\frac{(D^0)^2}{4x^2} + \alpha\,\frac{J^2}{x^2} + (\alpha-1)\,\frac{I^2}{3x^2}\,. \tag{3.57}$$

The last two terms involve the Casimirs of the groups $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$. The second term contains the quantity $D^0 = \bar{z}_k z^k$ which is the generator of some extra U(1) commuting with $D(2,1;\alpha)$.

[6]From $H = \frac{1}{4}(p^2 + \frac{g}{x^2})$ and the expressions (3.40), (3.41) we obtain $T^2 = g/4$.

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It is worth pointing out that at $\alpha = -1$, when $D(2,1;\alpha)$ degenerates into $\mathrm{SU}(1,1|2) \oplus\!\!\!\!\times \mathrm{SU}(2)_L$, the $\mathrm{SU}(2)_L$ Casimir I^2 drops out from the expression (3.50) for the Casimir C_2, as it should be. However, since in the model under consideration this $\mathrm{SU}(2)_L$ is realized only on fermions, the Casimir I^2 reappears in the subsequent formulas from the term $\frac{i}{4}\,Q^{ai'i}Q_{ai'i}$. Hence, even for a fixed $D(2,1;\alpha)$ Casimir (3.50), the term $\frac{i}{4}\,Q^{ai'i}Q_{ai'i}$ makes a contribution $\sim I^2$ to the $\mathrm{SU}(1,1)$ Casimir (3.55). As a result, the term with the $\mathrm{SU}(2)_L$ Casimir I^2 is retained in (3.57) even at $\alpha = -1$. Incidentally, the simplest form of the Hamiltonian is achieved at $\alpha = 1$.

In the next section we shall construct a quantum realization of the $D(2,1;\alpha)$ superalgebra.

4 $D(2,1;\alpha)$ quantum mechanics

4.1 Operator realization of $D(2,1;\alpha)$ superalgebra

Quantum operators of physical coordinates and momenta satisfy the quantum brackets, obtained in the standard way from (3.31)

$$[X,P] = i\,, \qquad [Z^i,\bar Z_j] = \delta^i_j\,, \qquad \{\Psi^i,\bar\Psi_j\} = -\frac{1}{2}\,\delta^i_j\,. \tag{4.1}$$

Quantum supertranslation and superconformal boost generators are defined by the classical expressions (3.28), (3.29), (3.30). We take Weyl ordering of the fermionic quantities in the last terms of (3.28) and (3.29):

$$\mathbf{Q}^i = P\Psi^i + 2i\alpha\frac{Z^{(i}\bar Z^{k)}\Psi_k}{X} + i(1+2\alpha)\frac{\langle\Psi_k\Psi^k\bar\Psi^i\rangle}{X}\,, \tag{4.2}$$

$$\bar{\mathbf{Q}}_i = P\bar\Psi_i - 2i\alpha\frac{Z_{(i}\bar Z_{k)}\bar\Psi^k}{X} + i(1+2\alpha)\frac{\langle\bar\Psi^k\bar\Psi_k\Psi_i\rangle}{X}\,, \tag{4.3}$$

$$\mathbf{S}^i = -2\,X\Psi^i + t\,\mathbf{Q}^i, \qquad \bar{\mathbf{S}}_i = -2\,X\bar\Psi_i + t\,\bar{\mathbf{Q}}_i\,. \tag{4.4}$$

The symbol $\langle ... \rangle$ denotes Weyl ordering. Note that

$$\langle\Psi_k\Psi^k\bar\Psi^i\rangle = \Psi_k\Psi^k\bar\Psi^i + \frac{1}{2}\,\Psi^i\,, \qquad \langle\bar\Psi^k\bar\Psi_k\Psi_i\rangle = \bar\Psi^k\bar\Psi_k\Psi_i + \frac{1}{2}\,\bar\Psi_i$$

and $\bar{\mathbf{Q}}_i = -\left(\mathbf{Q}^i\right)^+$, $\bar{\mathbf{S}}_i = -\left(\mathbf{S}^i\right)^+$.

Evaluating the anticommutators of the odd generators (4.2), (4.4), one determines uniquely the full set of quantum generators of superconformal algebra $D(2,1;\alpha)$. We obtain[7]

$$\begin{aligned}\mathbf{H} &= \frac{1}{4}P^2 + \alpha^2\frac{(\bar Z_k Z^k)^2 + 2\bar Z_k Z^k}{4X^2} - 2\alpha\frac{Z^{(i}\bar Z^{k)}\Psi_{(i}\bar\Psi_{k)}}{X^2} \\ &\quad - (1+2\alpha)\frac{\langle\Psi_i\Psi^i\,\bar\Psi^k\bar\Psi_k\rangle}{2X^2} + \frac{(1+2\alpha)^2}{16X^2}\,,\end{aligned} \tag{4.6}$$

$$\mathbf{K} = X^2 - t\,\frac{1}{2}\,\{X,P\} + t^2\,\mathbf{H}\,, \tag{4.7}$$

$$\mathbf{D} = -\frac{1}{4}\,\{X,P\} + t\,\mathbf{H}\,, \tag{4.8}$$

$$\mathbf{J}^{ik} = i\left[Z^{(i}\bar Z^{k)} + 2\Psi^{(i}\bar\Psi^{k)}\right]\,, \tag{4.9}$$

$$\mathbf{I}^{1'1'} = -i\Psi_k\Psi^k\,,\qquad \mathbf{I}^{2'2'} = i\bar\Psi^k\bar\Psi_k\,,\qquad \mathbf{I}^{1'2'} = -\frac{i}{2}\,[\Psi_k,\bar\Psi^k]\,. \tag{4.10}$$

Note that

$$\begin{aligned}\langle\Psi_i\Psi^i\,\bar\Psi^k\bar\Psi_k\rangle &= \frac{1}{2}\left\{\Psi_i\Psi^i,\bar\Psi^k\bar\Psi_k\right\} - \frac{1}{4} = \Psi_i\Psi^i\,\bar\Psi^k\bar\Psi_k - \Psi_i\bar\Psi^i + \frac{1}{4}\,,\\ \Psi^i\langle\Psi_l\Psi^l\,\bar\Psi^k\bar\Psi_k\rangle &= -\langle\Psi_l\Psi^l\,\bar\Psi^k\bar\Psi_k\rangle\Psi^i = \frac{1}{2}\,\langle\Psi_l\Psi^l\bar\Psi^i\rangle\,,\\ \bar\Psi_i\langle\Psi_l\Psi^l\,\bar\Psi^k\bar\Psi_k\rangle &= -\langle\Psi_l\Psi^l\,\bar\Psi^k\bar\Psi_k\rangle\bar\Psi_i = \frac{1}{2}\,\langle\bar\Psi^k\bar\Psi_k\Psi_i\rangle\,.\end{aligned}$$

It can be directly checked that the generators (4.2)–(4.10) indeed form the $D(2,1;\alpha)$ superalgebra which is obtained from the DB superalgebra (3.33)–(3.36) in the standard fashion (changing altogether DB by (anti)commutators and multiplying the right-hand sides by i):

$$\{\mathbf{Q}^{ai'i},\mathbf{Q}^{bk'k}\} = -2\left(\epsilon^{ik}\epsilon^{i'k'}\mathbf{T}^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}\mathbf{J}^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}\mathbf{I}^{i'k'}\right)\,, \tag{4.11}$$

$$[\mathbf{T}^{ab},\mathbf{T}^{cd}] = -i\left(\epsilon^{ac}\mathbf{T}^{bd} + \epsilon^{bd}\mathbf{T}^{ac}\right)\,, \tag{4.12}$$

$$[\mathbf{J}^{ij},\mathbf{J}^{kl}] = -i\left(\epsilon^{ik}\mathbf{J}^{jl} + \epsilon^{jl}\mathbf{J}^{ik}\right)\,,\quad [\mathbf{I}^{i'j'},\mathbf{I}^{k'l'}] = -i\big(\epsilon^{ik}\mathbf{I}^{j'l'} + \epsilon^{j'l'}\mathbf{I}^{i'k'}\big)\,, \tag{4.13}$$

$$[\mathbf{T}^{ab},\mathbf{Q}^{ci'i}] = i\epsilon^{c(a}\mathbf{Q}^{b)i'i},\ \ [\mathbf{J}^{ij},\mathbf{Q}^{ai'k}] = i\epsilon^{k(i}\mathbf{Q}^{ai'j)}, [\mathbf{J}^{i'j'},\ \ \mathbf{Q}^{ak'i}] = i\epsilon^{k'(i'}\mathbf{Q}^{aj')i}\,. \tag{4.14}$$

As in (3.33)–(3.36), in (4.11)–(4.14) we use the notation

$$\mathbf{Q}^{21'i} = -\mathbf{Q}^i\,,\quad \mathbf{Q}^{22'i} = -\bar{\mathbf{Q}}^i\,,\qquad \mathbf{Q}^{11'i} = \mathbf{S}^i\,,\quad \mathbf{Q}^{12'i} = \bar{\mathbf{S}}^i\,, \tag{4.15}$$

$$\mathbf{T}^{22} = \mathbf{H}\,,\quad \mathbf{T}^{11} = \mathbf{K}\,,\quad \mathbf{T}^{12} = -\mathbf{D}\,. \tag{4.16}$$

[7]It is worth making here an important clarifying remark which refers as well to our previous paper [18]. In (4.1) and below we assign to quantum operators the following Hermitian conjugation properties

$$X^+ = X,\quad P^+ = P\,,\qquad \bar Z_i = -\left(Z^i\right)^+\,,\qquad \bar\Psi_i = -\left(\Psi^i\right)^+\,, \tag{4.5}$$

whereas for classical quantities we still have $\bar z_i = \overline{(z^i)}$, $\bar\psi_i = \overline{(\psi^i)}$. This change of conventions in the quantum case is necessary for ensuring the standard Clifford algebra for quantum fermionic operators and standard quantum supersymmetry algebra with the positive-definte right-hand side of the basic anticommutator (see the comments after (4.11)–(4.16)). As we show in appendix B, the standard conjugation conventions can be restored by performing the time reversal $t \to -t$ in the initial model, thus bringing the opposite (standard) sign to kinetic terms of all involved $d{=}1$ spinor fields.

Note that due to (4.5) we have

$$\left(\mathbf{Q}^{ai'i}\right)^{+} = -\epsilon_{ik}\epsilon_{i'k'}\mathbf{Q}^{ak'k} \tag{4.17}$$

and, as a result, the basic anticommutator has the standard form $\{\mathbf{Q}, \mathbf{Q}^{+}\} = \mathbf{H}$.

In the quantum case, the classical relation (3.44) is replaced by

$$\{\langle \Psi_k \Psi^k \bar{\Psi}^i \rangle, \langle \bar{\Psi}^l \bar{\Psi}_l \Psi_j \rangle\} = \frac{1}{8}\,\delta^i_j\,. \tag{4.18}$$

and, due to (4.18), the term $\frac{(1+2\alpha)^2}{16X^2}$ appears in the quantum Hamiltonian (4.6). This term is necessary also for preserving the basic supersymmetry relations $[\mathbf{H}, \mathbf{Q}] = [\mathbf{H}, \bar{\mathbf{Q}}] = 0$. The appearance of such a "conformal" term when quantizing $\mathcal{N}$=4 superconformal systems was earlier observed in [15–17].

The quantization of the pure bosonic limit (3.16) of the classical system (3.15) does not lead to appearance of the additional term $\frac{(1+2\alpha)^2}{16X^2}$ in the corresponding quantum Hamiltonian which is thus a sum of only first two terms in (4.6). Using the same procedure as in [18] this Hamiltonian can be represented in the form

$$H = \frac{1}{4}\left[P^2 + 4\alpha^2 \frac{Y_a Y_a}{X^2}\right], \tag{4.19}$$

where

$$Y_a = \frac{1}{2}\,\bar{Z}_i (\sigma_a)^i{}_j Z^j \tag{4.20}$$

and σ_a, $a = 1, 2, 3$ are Pauli matrices. The quantities Y_a, obtained via the first Hopf map from the SU(2) spinors $Z^i, \bar{Z}_i$, generate $\mathrm{SU}(2)_R$ transformations in the bosonic sector of the model (the second $\mathrm{SU}(2)_L$ R-symmetry group of $D(2,1;\alpha)$ acts in the fermionic sector only). The operator Y_aY_a in the second term of (4.19) is the Casimir operator of the group $\mathrm{SU}(2)_R$ for its realization in the bosonic sector. Due to the constraint (3.45) (for definiteness, we adopt $\bar{Z}_k Z^k$-ordering in it; see also (4.58) and (4.65)), this Casimir takes the definite value $\frac{c}{2}\left(\frac{c}{2}+1\right)$. Thus, in the pure bosonic limit our model describes a conformal particle with the quantum potential $\alpha^2 \frac{c}{2}\left(\frac{c}{2}+1\right)/X^2$ which possesses the fixed $\mathrm{SU}(2)_R$ spin $\frac{c}{2}$. In the entire supersymmetric model, with all fermions taken into account, the generators of $\mathrm{SU}(2)_R$ contain additional fermionic parts (see (4.9)) and the corresponding full $\mathrm{SU}(2)_R$ Casimir operator proves not to be fixed. A thorough consideration of the pure bosonic case of the $\alpha = -1/2$ model can be found in our paper [18].

The second-order Casimir operator of the whole supergroup $D(2,1;\alpha)$ is given by the following expression [36]

$$\mathbf{C}_2 = \mathbf{T}^2 + \alpha\,\mathbf{J}^2 - (1+\alpha)\,\mathbf{I}^2 + \frac{i}{4}\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}\,. \tag{4.21}$$

Using the relations

$$
\begin{aligned}
\mathbf{T}^2 &\equiv \frac{1}{2}\,\mathbf{T}^{ab}\mathbf{T}_{ab} = \frac{1}{2}\,\{\mathbf{H},\mathbf{K}\} - \mathbf{D}^2 = \frac{1}{4}\,\alpha^2\left[(\bar{Z}_k Z^k)^2 + 2\bar{Z}_k Z^k\right] - 2\alpha Z^{(i}\bar{Z}^{k)}\Psi_{(i}\bar{\Psi}_{k)} \qquad (4.22)\\
&\quad -\frac{1}{2}\,(1+2\alpha)\langle\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k\rangle + \frac{1}{16}\,(1+2\alpha)^2 - \frac{3}{16}\,,\\
\mathbf{J}^2 &\equiv \frac{1}{2}\,\mathbf{J}^{ik}\mathbf{J}_{ik} = \frac{1}{4}\,\left[(\bar{Z}_k Z^k)^2 + 2\bar{Z}_k Z^k\right] - \frac{3}{2}\,\left(\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k - \Psi_i\bar{\Psi}^i\right) - 2Z^{(i}\bar{Z}^{k)}\Psi_{(i}\bar{\Psi}_{k)}, \qquad (4.23)\\
\mathbf{I}^2 &\equiv \frac{1}{2}\,\mathbf{I}^{i'k'}\mathbf{I}_{i'k'} = \frac{1}{2}\,\{\bar{\mathbf{I}},\mathbf{I}\} - (\mathbf{I}_3)^2 = \frac{3}{2}\,\left(\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k - \Psi_i\bar{\Psi}^i\right) + \frac{3}{4} \qquad (4.24)
\end{aligned}
$$

together with

$$
\begin{aligned}
\frac{i}{4}\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i} &= \frac{i}{4}\,[\mathbf{Q}^i,\bar{\mathbf{S}}_i] + \frac{i}{4}\,[\bar{\mathbf{Q}}_i,\mathbf{S}^i] \qquad (4.25)\\
&= 4\alpha Z^{(i}\bar{Z}^{k)}\Psi_{(i}\bar{\Psi}_{k)} + 2(1+2\alpha)\left(\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k - \Psi_i\bar{\Psi}^i\right) + (1+\alpha)\,,
\end{aligned}
$$

we finally cast $\mathbf{C}_2$ in the form

$$
\mathbf{C}_2 = \frac{1}{4}\,\alpha(1+\alpha)\left[(\bar{Z}_k Z^k)^2 + 2\bar{Z}_k Z^k + 1\right]. \qquad (4.26)
$$

4.2 Invariant spaces in the enveloping algebra of $D(2,1;\alpha)$

An important property is that the enveloping algebra of $D(2,1;\alpha)$ superalgebra has several subspaces which are closed under the action of $D(2,1;\alpha)$. The presence of such subspaces provides an explanation why some bilinear combinations of the $D(2,1;\alpha)$ generators in the considered realization identically vanish without conflict with the $D(2,1;\alpha)$ covariance. This phenomenon is encountered already at the classical level (see (3.52)). As we shall see, the realization of the $D(2,1;\alpha)$ generators in the considered model is such that the operators forming one of the invariant subspaces just mentioned are vanishing. As a result, the physical states form a module of such a restricted representation of $D(2,1;\alpha)$.

One invariant subspace is formed by the bilinear combinations

$$
\begin{aligned}
\mathbf{M} &\equiv \mathbf{T}^2 - \alpha^2\,\mathbf{J}^2 - \frac{1}{3}\,(1-\alpha^2)\,\mathbf{I}^2 + \frac{i}{8}\,(1-\alpha)\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}\,, \qquad (4.27)\\
\mathbf{M}^{ai'i} &\equiv \frac{i}{4}\,\left(\{\mathbf{T}^a_b,\mathbf{Q}^{bi'i}\} - \alpha\,\{\mathbf{J}^i_j,\mathbf{Q}^{ai'j}\} + \frac{1}{3}\,(1-\alpha)\,\{\mathbf{I}^{i'}_{j'},\mathbf{Q}^{aj'i}\}\right), \qquad (4.28)\\
\mathbf{M}^{ik,\,i'k'} &\equiv \alpha\,\{\mathbf{J}^{ik},\mathbf{I}^{i'k'}\} - \frac{i}{2}\,\mathbf{Q}^{b(i'(i}\mathbf{Q}_b{}^{k')k)}\,, \qquad (4.29)\\
\mathbf{M}^{ac,\,i'k'} &\equiv \{\mathbf{T}^{ac},\mathbf{I}^{i'k'}\} - \frac{i}{2}\,\mathbf{Q}^{(a(i'j}\mathbf{Q}^{c)k')}{}_j\,, \qquad (4.30)\\
\mathbf{M}^{ai,\,i'j'k'} &\equiv i\{\mathbf{I}^{(i'j'},\mathbf{Q}^{ak')k}\}\,, \qquad (4.31)\\
\mathbf{M}^{i'j'k'l'} &\equiv \{\mathbf{I}^{(i'j'},\mathbf{I}^{k'l')}\}\,. \qquad (4.32)
\end{aligned}
$$

On this set a linear finite-dimensional representation of $D(2,1;\alpha)$ is realized

$$[\mathbf{M},\mathbf{Q}^{ai'i}] = (1{+}\alpha)\,\mathbf{M}^{ai'i}\,, \tag{4.33}$$

$$\{\mathbf{M}^{ai'i},\mathbf{Q}^{ck'k}\} = -i\,\epsilon^{ac}\epsilon^{i'k'}\epsilon^{ik}\mathbf{M}+\frac{i}{3}\,(2{+}\alpha)\,\epsilon^{ac}\mathbf{M}^{ik,\,i'k'}-\frac{i}{3}\,(1{+}2\alpha)\,\epsilon^{ik}\mathbf{M}^{ac,\,i'k'}, \tag{4.34}$$

$$[\mathbf{M}^{ik,\,i'k'},\mathbf{Q}^{bj'j}] = 4\epsilon^{j(i}\epsilon^{j'(i'}\mathbf{M}^{bk')k)}+(1{+}2\alpha)\epsilon^{j(i}\mathbf{M}^{bk),\,i'j'k'}\,, \tag{4.35}$$

$$[\mathbf{M}^{ac,\,i'k'},\mathbf{Q}^{bj'j}] = -4\epsilon^{b(a}\epsilon^{j'(i'}\mathbf{M}^{c)k')j}+(2{+}\alpha)\epsilon^{b(a}\mathbf{M}^{c)j,\,i'j'k'}\,, \tag{4.36}$$

$$\{\mathbf{M}^{ai,\,i'j'k'},\mathbf{Q}^{bl'l}\}= -2i\epsilon^{ba}\epsilon^{l'(i'}\mathbf{M}^{il,j'k')}-2i\epsilon^{li}\epsilon^{l'(i'}\mathbf{M}^{ab,j'k')}+2i(1{+}\alpha)\epsilon^{ba}\epsilon^{li}\mathbf{M}^{i'j'k'l'}\,, \tag{4.37}$$

$$[\mathbf{M}^{i'j'k'l'},\mathbf{Q}^{bn'n}] = \epsilon^{n'(i'}\mathbf{M}^{bn,\,i'j'k')}\,. \tag{4.38}$$

The second invariant subspace is formed by the quantities

$$\mathbf{N} \equiv \mathbf{T}^2+\frac{1}{3}\,\alpha(2{+}\alpha)\,\mathbf{J}^2-(1{+}\alpha)^2\,\mathbf{I}^2+\frac{i}{8}\,(2{+}\alpha)\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}\,, \tag{4.39}$$

$$\mathbf{N}^{ai'i} \equiv \frac{i}{4}\left(\{\mathbf{T}^a_b,\mathbf{Q}^{bi'i}\}+\frac{1}{3}\,(2{+}\alpha)\,\{\mathbf{J}^i_j,\mathbf{Q}^{ai'j}\}+(1{+}\alpha)\,\{\mathbf{I}^{i'}_{j'},\mathbf{Q}^{aj'i}\}\right), \tag{4.40}$$

$$\mathbf{N}^{i'k',\,ik} \equiv -(1{+}\alpha)\,\{\mathbf{J}^{ik},\mathbf{I}^{i'k'}\}-\frac{i}{2}\,\mathbf{Q}^{b(i'(i}\mathbf{Q}_b{}^{k')k)}\,, \tag{4.41}$$

$$\mathbf{N}^{ac,\,ik} \equiv \{\mathbf{T}^{ac},\mathbf{J}^{ik}\}-\frac{i}{2}\,\mathbf{Q}^{(aj'(i}\mathbf{Q}^{c)}{}_{j'}{}^{k)}\,, \tag{4.42}$$

$$\mathbf{N}^{ai',\,ijk} \equiv i\{\mathbf{J}^{(ij},\mathbf{Q}^{ai'k)}\}\,, \tag{4.43}$$

$$\mathbf{N}^{ijkl} \equiv \{\mathbf{J}^{(ij},\mathbf{J}^{kl)}\}\,. \tag{4.44}$$

They can also be shown to constitute a basis of a linear finite-dimensional representation of $D(2,1;\alpha)$.

At last, the third invariant subspace is formed by the bilinear operators

$$\mathbf{L} \equiv \frac{1}{3}\,(1+2\alpha)\,\mathbf{T}^2+\alpha^2\,\mathbf{J}^2-(1+\alpha)^2\,\mathbf{I}^2+\frac{i}{8}\,(1+2\alpha)\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}\,, \tag{4.45}$$

$$\mathbf{L}^{ai'i} \equiv \frac{i}{4}\left(\frac{1}{3}\,(1+2\alpha)\,\{\mathbf{T}^a_b,\mathbf{Q}^{bi'i}\}+\alpha\,\{\mathbf{J}^i_j,\mathbf{Q}^{ai'j}\}+(1+\alpha)\,\{\mathbf{I}^{i'}_{j'},\mathbf{Q}^{aj'i}\}\right), \tag{4.46}$$

$$\mathbf{L}^{i'k',\,ac} \equiv -(1+\alpha)\,\{\mathbf{I}^{i'k'},\mathbf{T}^{ac},\}-\frac{i}{2}\,\mathbf{Q}^{(a(i'j}\mathbf{Q}^{c)k')}{}_j\,, \tag{4.47}$$

$$\mathbf{L}^{ik,\,ac} \equiv \alpha\{\mathbf{J}^{ik},\mathbf{T}^{ac}\}-\frac{i}{2}\,\mathbf{Q}^{(aj'(i}\mathbf{Q}^{c)}{}_{j'}{}^{k)}\,, \tag{4.48}$$

$$\mathbf{L}^{ii',\,abc} \equiv i\{\mathbf{T}^{(ab},\mathbf{Q}^{c)i'i}\}\,, \tag{4.49}$$

$$\mathbf{L}^{abcd} \equiv \{\mathbf{T}^{(ab},\mathbf{T}^{cd)}\}\,. \tag{4.50}$$

As for two previous invariant subspaces, these operators are closed under the action of $D(2,1;\alpha)$.

These three invariant subspaces in the enveloping algebra have the following properties.

First, these subspaces and one-dimensional space formed by the Casimir operator (4.21) exhaust all possible invariant subspaces in the enveloping algebra, such that they are bilinear in the $D(2,1;\alpha)$ generators and involve singlets of all three bosonic subgroup SL$(2,R)$, SU$(2)_R$ and SU$(2)_L$.

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Second, these subspaces are related to each other via some discrete transformations. Namely, the subspaces (4.27)–(4.32) and (4.39)–(4.44) are *dual* to each other. That is, the discrete transformation

$$\alpha \quad \leftrightarrow \quad -(1+\alpha)\,, \qquad\qquad \mathbf{J}^{ik} \quad \leftrightarrow \quad \mathbf{I}^{i'k'}\,, \tag{4.51}$$

which is an automorphism of the $D(2,1;\alpha)$ algebra (4.11)–(4.14), takes the space (4.27)–(4.32) into the space (4.39)–(4.44) and vice versa. The subspace (4.45)–(4.50) is a fixed point of the mapping (4.51).

The subspace (4.45)–(4.50) is related to the subspaces (4.27)–(4.32) and (4.39)–(4.44) via similar discrete transformations. E.g., under the transformation

$$\alpha \quad \to \quad \alpha^{-1}\,, \qquad \mathbf{T}^{ab} \quad \leftrightarrow \quad \mathbf{J}^{ik}\,, \qquad \mathbf{Q}^{ai'i} \quad \to \quad \alpha^{-1/2}\mathbf{Q}^{ai'i} \tag{4.52}$$

the space (4.39)–(4.44) goes over into the space (4.45)–(4.50). Note, however, that the change (4.52) (and its analog taking (4.27)–(4.32) into (4.45)–(4.50)) is ill defined for the real form of the superalgebra $D(2,1;\alpha)$ since it takes the $sl(2,R)$ generators into the $su(2)$ ones. These transformations present a true automorphism of the complexified $D(2,1;\alpha)$ algebra.

In the case of $\alpha = -1/2$ (when $1+2\alpha = 0$) the subspaces (4.27)–(4.32) and (4.39)–(4.44) coincide. Moreover, the subspace formed by

$$\mathbf{M}\,, \qquad \mathbf{M}^{ai'i}\,, \qquad \mathbf{M}^{ik,\,i'k'} \tag{4.53}$$

(or $\mathbf{N}$, $\mathbf{N}^{ai'i}$, $\mathbf{N}^{i'k',\,ik}$) is invariant under the $D(2,1;\alpha = -1/2)$. Just this subspace was exploited in [18].

In the case of $\alpha = -1$ (when $D(2,1;\alpha)$=SU$(1,1|2)\rtimes$SU$(2)_L$) the operator (4.27) coincides with the Casimir (4.21),

$$\mathbf{C_2} = \mathbf{M} = \mathbf{T}^2 - \mathbf{J}^2 + \frac{i}{4}\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}\,. \tag{4.54}$$

Thus in this special case the appropriate invariant subspaces degenerate into the singlets of the superconformal group SU$(1,1|2)\otimes$SU(2).[8]

Actually, in the case of generic α, for the particular representation of generators given by eqs. (4.22)–(4.24) all quantities (4.27)–(4.32) identically vanish:

$$\mathbf{M} = 0\,, \quad \mathbf{M}^{ai'i} = 0\,, \quad \mathbf{M}^{ik,\,i'k'} = 0\,, \quad \mathbf{M}^{ac,\,i'k'} = 0\,, \quad \mathbf{M}^{ai,\,i'j'k'} = 0\,, \quad \mathbf{M}^{i'j'k'l'} = 0\,. \tag{4.55}$$

As a consequence of these identities, there arises the relation

$$(1+\alpha)\mathbf{T}^2 - \alpha(1+\alpha)\mathbf{J}^2 + \frac{1}{3}\,(1-\alpha^2)\mathbf{I}^2 = -(1-\alpha)\mathbf{C}_2\,. \tag{4.56}$$

[8] Although our mechanical system is ill defined at α=0, the $D(2,1;\alpha)$ algebra (4.11)–(4.14) as it stands still admits such a choice, and it gives rise to the superalgebra $D(2,1;\alpha$=0)=SU$(1,1|2)\otimes$SU$(2)_R$. In this case the operator (4.39) coincides with the Casimir (4.21), $\mathbf{C_2} = \mathbf{N} = \mathbf{T}^2 - \mathbf{I}^2 + \frac{i}{4}\,\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}$.

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In the case of $\alpha = -1$ the constraint (4.56) leads to the condition $\mathbf{C}_2 = 0$ that agrees with eqs. (4.54) and 4.55, as well as with (4.26).

Using the expression (4.26) for the Casimir in r.h.s. of (4.56) we can represent the relation (4.56) in the form

$$\mathbf{T}^2 - \alpha\,\mathbf{J}^2 + \frac{1}{3}\,(1-\alpha)\mathbf{I}^2 = -\alpha(1-\alpha)\left[\frac{1}{2}\,D^0\left(\frac{1}{2}\,D^0+1\right)+\frac{1}{4}\right], \tag{4.57}$$

which is valid for any value of α. Thus, for an irreducible representation of $D(2,1;\alpha)$ with the fixed $\mathbf{C}_2$ (see (4.69) below), the values of the Casimir operators $\mathbf{T}^2$, $\mathbf{J}^2$, $\mathbf{I}^2$ of the three bosonic subgroups $sl(2,R)$, $su(2)_R$, $su(2)_L$ prove to be always related according to (4.57).

The operator

$$D^0 = \bar{Z}_k Z^k\,, \tag{4.58}$$

entering the right-hand side of (4.57) commutes with all generators of the superalgebra $D(2,1;\alpha)$ (as in the classical case).

4.3 Quantum spectrum

The Hamiltonian (4.6) and the SL$(2,R)$ Casimir operator (4.22) can be represented as

$$\mathbf{H} = \frac{1}{4}\left(P^2 + \frac{\hat{g}}{X^2}\right), \tag{4.59}$$

$$\mathbf{T}^2 = \frac{1}{4}\,\hat{g} - \frac{3}{16}\,, \tag{4.60}$$

where

$$\hat{g} \equiv 4\alpha^2\frac{1}{2}\,\bar{Z}_k Z^k\left(\frac{1}{2}\,\bar{Z}_k Z^k+1\right) - 8\alpha Z^{(i}\bar{Z}^{k)}\Psi_{(i}\bar{\Psi}_{k)} - 2(1+2\alpha)\langle\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k\rangle + \frac{1}{4}\,(1+2\alpha)^2\,. \tag{4.61}$$

The operators (4.59) and (4.60) formally look like those given in the model of [10]. However, there is an essential difference. Whereas the quantity $\hat{g}$ is a constant in the model of [10], in our case $\hat{g}$ is an operator which takes fixed, but different, constant values on different components of the full wave function.

To find the quantum spectrum of (4.59) and (4.60), we make use of the realization

$$\bar{Z}_i = v_i^+\,, \qquad Z^i = \partial/\partial v_i^+ \tag{4.62}$$

for the bosonic operators Z^k and $\bar{Z}_k$, as well as the following realization of the odd operators Ψ^i, $\bar{\Psi}_i$

$$\Psi^i = \psi^i\,, \qquad \bar{\Psi}_i = -\frac{1}{2}\,\partial/\partial\psi^i\,, \tag{4.63}$$

where ψ^i are complex Grassmann variables. Then, the wave function is defined as

$$\Phi = A_1 + \psi^i B_i + \psi^i\psi_i A_2\,. \tag{4.64}$$

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	$\mathbf{T}^2$	$\mathbf{J}^2$	$\mathbf{I}^2$	$\frac{i}{4}\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}$
$A^{(c)}_{k'}$	$\frac{\alpha^2(c+1)^2-1}{4}$	$\frac{(c+1)^2-1}{4}$	$\frac{3}{4}$	$1+\alpha$
$B'^{(c)}_{k}$	$\frac{\alpha^2(c+1)^2-2\alpha(c+1)}{4}$	$\frac{(c+1)^2-2(c+1)}{4}$	0	$\alpha(c+1)$
$B''^{(c)}_{k}$	$\frac{\alpha^2(c+1)^2+2\alpha(c+1)}{4}$	$\frac{(c+1)^2+2(c+1)}{4}$	0	$-\alpha(c+1)$

Table 1. The values of the Casimirs of the bosonic subgroups and $\frac{i}{4}\mathbf{Q}^{ai'i}\mathbf{Q}_{ai'i}$.

The full wave function is subjected to the same constraints (3.45) as in the bosonic limit (we use the normal ordering for the even SU(2)-spinor operators, with all operators Z^i standing on the right)

$$D^0\Phi = \bar{Z}_i Z^i \Phi = v^+_i \frac{\partial}{\partial v^+_i}\,\Phi = c\,\Phi. \tag{4.65}$$

Like in the bosonic limit, requiring the wave function $\Phi(v^+)$ to be single-valued gives rise to the condition that the constant c is integer, $c \in \mathbb{Z}$. We take c to be positive in order to have a correspondence with the bosonic limit where c becomes SU(2) spin. Then (4.65) implies that the wave function $\Phi(v^+)$ is a homogeneous polynomial in v^+_i of the degree c:

$$\Phi = A^{(c)}_1 + \psi^i B^{(c)}_i + \psi^i\psi_i A^{(c)}_2\,, \tag{4.66}$$

$$A^{(c)}_{i'} = A_{i',k_1\ldots k_c} v^{+k_1}\ldots v^{+k_c}\,, \tag{4.67}$$

$$B^{(c)}_i = B'^{(c)}_i + B''^{(c)}_i = v^+_i B'_{k_1\ldots k_{c-1}} v^{+k_1}\ldots v^{+k_{c-1}} + B''_{(ik_1\ldots k_c)} v^{+k_1}\ldots v^{+k_c}\,. \tag{4.68}$$

In (4.68) we extracted SU(2) irreducible parts $B'_{(k_1\ldots k_{c-1})}$ and $B''_{(ik_1\ldots k_c)}$ of the component wave functions, with the SU(2) spins $(c-1)/2$ and $(c+1)/2$, respectively.

On the physical states (4.65), (4.66) Casimir operator (4.26) takes the value

$$\mathbf{C}_2 = \alpha(1+\alpha)(c+1)^2/4\,. \tag{4.69}$$

On the same states, the Casimir operators (4.22)–(4.24) of the bosonic subgroups SU(1,1), SU(2)$_R$ and SU(2)$_L$ take the values given in the table 1.[9] For different component wave functions, the quantum numbers r_0, j and i, defined by

$$\mathbf{T}^2 = r_0(r_0-1)\,,\qquad \mathbf{J}^2 = j(j+1)\,,\qquad \mathbf{I}^2 = i(i+1)\,,$$

take the values listed in the table 2. The fields B'_i and B''_i form doublets of SU(2)$_R$

[9] Here we use that

$$\Psi_i\Psi^i\,\bar\Psi^k\bar\Psi_k - \Psi_i\bar\Psi^i = \frac{1}{4}\left(\psi^i\psi_i\,\frac{\partial}{\partial\psi_k}\frac{\partial}{\partial\psi^k} - 2\psi^i\,\frac{\partial}{\partial\psi^i}\right), \qquad Z^i\bar Z^k\Psi_{(i}\bar\Psi_{k)} = -\frac{1}{2}\left(v^{+i}\frac{\partial}{\partial v^+_j}\psi_{(i}\frac{\partial}{\partial\psi^{j)}}\right).$$

Therefore, we have

$$\left(\Psi_i\Psi^i\,\bar\Psi^k\bar\Psi_k - \Psi_i\bar\Psi^i\right)\Phi = -\frac{1}{2}\psi^i B_i\,, \qquad \left(Z^i\bar Z^k\Psi_{(i}\bar\Psi_{k)}\right)\Phi = -\frac{1}{2}v^{+i}\frac{\partial}{\partial v^+_j}\psi_{(i}B_{j)} = \frac{1}{4}\psi^i\left[(c+2)B'_i - c\,B''_i\right].$$

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	r_0	j	i
$A^{(c)}_{k'}(x,v^+)$	$\frac{\|\alpha\|(c+1)+1}{2}$	$\frac{c}{2}$	$\frac{1}{2}$
$B'^{(c)}_{k}(x,v^+)$	$\frac{\|\alpha\|(c+1)+1}{2} - \frac{1}{2}\,\mathrm{sign}(\alpha)$	$\frac{c}{2}-\frac{1}{2}$	0
$B''^{(c)}_{k}(x,v^+)$	$\frac{\|\alpha\|(c+1)+1}{2} + \frac{1}{2}\,\mathrm{sign}(\alpha)$	$\frac{c}{2}+\frac{1}{2}$	0

Table 2. The SU(1, 1), $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ quantum numbers.

	l
$A^{(c)}_{k'}(x,v^+)$	$\|\alpha\|(c+1)-\frac{1}{2}$
$B'^{(c)}_{k}(x,v^+)$	$\|\alpha\|(c+1)-\frac{1}{2}-\mathrm{sign}(\alpha)$
$B''^{(c)}_{k}(x,v^+)$	$\|\alpha\|(c+1)-\frac{1}{2}+\mathrm{sign}(\alpha)$

Table 3. Values of the constant l.

generated by $\mathbf{J}^{ik}$, whereas the component fields $A_{i'} = (A_1, A_2)$ form a doublet of $\mathrm{SU}(2)_L$ generated by $\mathbf{I}^{i'k'}$. If the super-wave function (4.64) is bosonic (fermionic), the fields $A_{i'}$ describe bosons (fermions), whereas the fields B'_i, B''_i present fermions (bosons). It is easy to check that the relation (4.56) is valid in all cases.

Each of the component wave functions $A_{i'}$, B'_i, B''_i carries an infinite-dimensional unitary representation of the discrete series of the universal covering group of the one-dimensional conformal group SU(1,1). Such representations are characterized by positive numbers r_0 [37, 38] (for the unitary representations of SU(1,1) the constant $r_0 > 0$ must be (half)integer). Basis functions of these representations are eigenvectors of the compact SU(1,1) generator

$$\mathbf{R} = \frac{1}{2}\left(a^{-1}\mathbf{K} + a\mathbf{H}\right),$$

where a is a constant of the length dimension. These eigenvalues are $r = r_0 + n$, $n \in \mathbb{N}$ [10, 37, 38].

Using the expressions (4.6), (4.22)–(4.24) and the values of Casimirs from the table 1, we can write the Hamiltonian in the unified form:

$$\mathbf{H} = \frac{1}{4}\left(P^2 + \frac{l(l+1)}{X^2}\right) \tag{4.70}$$

where the constant l takes, on the separate wave functions, the values listed in the table 3.

In the above quantization, we took into account all the conditions implied by the initial classical system. Due to the presence of additional invariant spaces in the enveloping algebra, we may try to impose additional conditions on the wave function, e.g.

$$\mathbf{L}\,\Phi = 0 \tag{4.71}$$

where $\mathbf{L}$ was defined in (4.45). As a result, we could expect to obtain more restricted spectrum at certain values of the parameters α and c. Regrettably, this conjecture fails: in order to preserve the superconformal $D(2,1;\alpha)$ covariance, we are led to assume that all operators from the set (4.45)–(4.50), on equal footing with $\mathbf{L}$, annihilate the physical states, and these restrictions prove to be too strong. It is an open question whether the constraints of this kind could have a non-trivial solution in some other $D(2,1;\alpha)$ invariant superconformal mechanics models.

Let us focus on some peculiar properties of the $D(2,1;\alpha)$ quantum mechanics constructed.

As opposed to the standard $SU(1,1|2)$ superconformal mechanics [12, 20, 39], the construction presented here essentially uses the variables z_i (or v_i^+) parametrizing the two-sphere S^2, in addition to the standard (dilatonic) coordinate x.

The presence of additional "(iso)spin" S^2 variables in our construction leads to a richer quantum spectrum. Besides, the relevant wave functions involve representations of the two independent SU(2) groups, in contrast to the $SU(1,1|2)$ models of [12, 15–17, 20, 39] where only the SU(2) realized on fermionic variables really matters.

Also, in a contradistinction to the previously considered models (and in the same way as in our previous paper [18] devoted to the particular $\alpha = -1/2$ case), there naturally appears a quantization of the conformal coupling constant which is expressed as a SU(2) Casimir operator, with both integer and half-integer eigenvalues. This happens already in the bosonic sector of the model, and is ensured by the S^2 variables.[10]

Note that the variables v_i^+ in the expansions (4.67) and (4.68) can be identified with a half of the target space harmonic-like variables $v_i^\pm$ (though without the standard constraint $v^{+i}v_i^- \sim const$). Within a different quantization scheme used e.g. in [40, 41], we would have even more literal harmonic interpretation of the bosonic isospinor variables. In both schemes, the S^2 constraint (3.18) is not explicitly solved before quantization, it is imposed on the wave functions as in (4.65). An alternative quantization scheme would be to deal with an explicit parametrization of the two-spere S^2, e.g. the stereographic projection parametrization [30, 31] or the parametrization by the Euler angles β and γ as in (3.20), and then to apply the canonical methods (Gupta-Bleuler quantization or Dirac procedure).[11] An important role in this case is played by the requirement of the square-integrability of the wave function on S^2, which substitutes the constraint (4.65) of the parametrization-independent quantization schemes. As follows from the consideration in [29–31], this demand ensures the wave function to contain unitary representations of SU(2). General issues of the canonical quantization of Chern-Simons mechanics were addressed in [27].

---

[10] Note that the strength of the conformal potential is related to the strength of the WZ term and so is quantized also in the $\mathcal{N}=4$ superconformal mechanics associated with the $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet (without non-dynamical S^2 variables) [23]. However, no direct relation between these parameters and SU(2) Casimirs appears in this case.

[11] One more approach is to quantize in the oscillator variables [25, 26, 29].

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4.4 Comment on the SU(1, 1|2) case

Let us here focus on some peculiarities of the case of SU(1, 1|2) superconformal symmetry.

In the case of $\alpha = -1$ one has $D(2,1;\alpha{=}-1) \simeq \mathrm{SU}(1,1|2) \otimes \mathrm{SU}(2)_L$, and thus our model is invariant under SU(1, 1|2) superconformal group and an outer automorphism group $\mathrm{SU}(2)_L$ acting only on the fermions. In general, the supergroup SU(1, 1|2) is known to admit a non-vanishing central charge which breaks this second R-symmetry SU(2) group down to U(1) [20][12]. Thus, if we require our model to be invariant under $\mathrm{SU}(2)_L$ (as in the case of generic α) the corresponding SU(1, 1|2) algebra cannot include a central charge.

There arises the question as to whether a different version of the $\mathcal{N}{=}4$ superconformal mechanics model with spin variables exists, such that it possesses SU(1, 1|2) symmetry with a non-vanishing central charge. The answer is affirmative, and it can be derived from the results of refs. [12, 15–17, 20].

When only SU(1, 1|2) symmetry is required, while $\mathrm{SU}(2)_L$ symmetry is allowed to be broken, the constraints (2.3) and (2.4) for the even real superfield $\mathcal{X}$ can be weakened [20] by adding nonzero constants in their right-hand sides. The simplest choice is the following set of the constraints

$$\text{(a)} \quad D^i D_i \,\mathcal{X} = 0\,, \quad \bar{D}_i \bar{D}^i \,\mathcal{X} = 0\,; \qquad \text{(b)} \quad [D^i, \bar{D}_i]\,\mathcal{X} = m \tag{4.72}$$

where m is a constant. The solution of the constraints (4.72a) is a sum of (3.5) and additional term $-\frac{1}{4}\theta\bar{\theta}A$, where A is some undefined constant. The constraint (4.72b) serves to fix this constant to be m. Then the action (2.2) (with $\alpha{=}-1$) will give rise to additional contributions to the physical component Lagrangian (3.15), such that they are proportional to m^2/x^2 and $m\psi\bar{\psi}/x^2$ [20]. These additional terms appear in the Hamiltonian, and they are induced by the appropriate new terms in the Noether supercharges. Comparing these modified SU(1, 1|2) generators with those given in [12, 15–17], one can see that they correspond just to the SU(1, 1|2) algebra with a central charge proportional to m.

More detailed analysis of the U(2) spin $\mathcal{N}{=}4$ superconformal mechanics in which the even real superfield $\mathcal{X}$ is subjected to the constraints (4.72) with $m \neq 0$ will be given elsewhere. An interesting new feature of such a model is the presence of two complementary mechanisms of generating the conformal potential $\sim x^{-2}$: the on-shell one via coupling to the auxiliary superfields $\mathcal{Z}^+$ as in the case of generic α, and the off-shell one based on the deformed constraints (4.72) and a non-zero central charge in the SU(1, 1|2) algebra. It should be stressed that such a modification of the constraints is admissible only in the case of $\alpha{=}-1$; at any other value of α (not belonging to the equivalence class of the choice $\alpha{=}-1$) the superconformal invariance requires the constants in the right-hand sides of the constraints to vanish.

5 Summary and outlook

In this paper we presented a new version of $\mathcal{N}{=}4$ mechanics with $D(2,1;\alpha)$ superconformal symmetry. It is obtained as the one-particle reduction of the many-particle Calogero-type

[12] The quotient of the general SU(1, 1|2) over the central charge generator is sometimes denoted as PSU(1, 1|2) .

systems proposed in [2]. This system generalizes the OSp(4|2) superconformal mechanics constructed in our previous work [18], and it shares many characteristic features of the latter. In the bosonic sector it involves two complex fields (world-line harmonics) parametrizing the first Hopf map $S^3 \to S^2$.

Due to the presence of spin variables in the superconformal mechanics, the quantum spectrum involves diverse $D(2,1;\alpha)$ representations characterized by the specific values of the Casimir operator (4.21), (4.26). In these representations, the particle states carry representations of the bosonic subgroups SU(1,1), $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$, the Casimirs of which are related to each other by the constraint (4.57). This constraint is identically satisfied for the particular realization of the $D(2,1;\alpha)$ generators pertinent to our model.

The appearance of this constraint is related to the existence of some invariant subspaces in the enveloping algebra of $D(2,1;\alpha)$. We found that at generic α there exist more invariant subspaces than for the degenerate case of $\alpha{=}-1/2$ corresponding to OSp(4|2) [18], where some invariant subspaces are identified.

The $D(2,1;\alpha)$ superconformal mechanics was considered here for $\alpha{\neq}0$. Formally, we can take the limit $\alpha \to 0$ in the final relations, and we observe that the target harmonic degrees of freedom decouple (see, e.g., (3.16), (3.17) and (3.23)–(3.27)). Nevertheless, the superconformal superfield action of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet is of a special form for $\alpha{=}0$, so this case requires a separate study. Here we give a brief comment on the construction of the superfield superconformal action at $\alpha{=}0$.

We note that $D(2,1;\alpha{\to}0)$ reduces to $\mathrm{SU}(1,1|2){\otimes}\mathrm{SU}(2)_R$. The "passive" superconformal variation (2.21) of $\mathcal{X}$ disappears in this case, while the integration measure μ_H is transformed as (see (2.18))

$$\delta^\prime \mu_H = -2i(\theta_k\bar\eta^k + \bar\theta^k\eta_k)\,\mu_H\,. \tag{5.1}$$

As suggested in [22, 23], in order to ensure the superconformal invariance, it is necessary to modify the transformation law of $\mathcal{X}$ and, therefore, of $\mathcal{V}$ in the following way,

$$\delta^\prime_{mod}\mathcal{X} = 2i(\theta_k\bar\eta^k + \bar\theta^k\eta_k)\,, \qquad \delta^\prime_{mod}\mathcal{V} = 4i(\bar\eta^-\theta^+ - \eta^-\bar\theta^+)\,. \tag{5.2}$$

Then the most general $D(2,1;\alpha{=}0)$ superconformal action for the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet reads [22]

$$S^{\mathcal{X}}_{\alpha=0} = -\frac{1}{4}\int \mu_H\, e^{\,\mathcal{X}} + \int \mu_A^{(-2)} c^{+2}\,\mathcal{V}\,, \tag{5.3}$$

where $c^{+2} = c^{ij}u^+_i u^+_j$, and c^{ij} are constant parameters. The second FI term in (5.3) is superconformal only at $\alpha{=}0$. It yields a conformal potential for the dilaton field with a strength $\sim c^{ik}c_{ik}$, breaks the decoupled $\mathrm{SU}(2)_R$ down to U(1) and induces a central charge $\sim c^{ik}$ in SU(1,1|2). Actually, this action is dual to the $\alpha{=}-1$ action for $\mathcal{X}$ with the modified constraints (4.72) [42]: the duality interchanges $\mathrm{SU}(2)_L$ with $\mathrm{SU}(2)_R$ and also α with $-(1{+}\alpha)$. However, the $D(2,1;\alpha{=}0)$ superconformal invariance is not compatible with the presence of $\mathcal{V}$ in the WZ term of the action (2.8), still implying the transformation laws (2.21) for $\mathcal{Z}^+$ and for V^{++}. As a consequence, the WZ term and the FI term of V^{++}

decouple from the $\mathcal{X}$ action:

$$S^{\mathcal{Z},V^{++}}_{\alpha=0} = \frac{1}{2}\int \mu_A^{(-2)}\, \widetilde{\mathcal{Z}}^{+}\mathcal{Z}^{+} + \frac{i}{2}\, c\int \mu_A^{(-2)}\, V^{++}\,. \tag{5.4}$$

i.e. we loose any interaction between the superfields $\mathcal{X}$ and $\mathcal{Z}^{+}$. This situation is quite analogous to what happens in the $\mathcal{N}$=1 and $\mathcal{N}$=2 super Calogero models considered in [2], where the center-of-mass supermultiplet $\mathcal{X}$ decouples from the WZ and gauge supermultiplets. Note that in the many-particle $\mathcal{N}$=4 super Calogero models the (matrix) $\mathcal{X}$ supermultiplet will still interact with the (column) $\mathcal{Z}$ supermultiplet via the gauge supermultiplet even in the α=0 case.

Based on the duality just mentioned between the cases of α=0 and α=−1, one may expect that in the α=0 case the interaction of the superfield $\mathcal{X}$ with the U(2) spin variables can still be gained by placing the latter into a "mirror" $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet, for which the $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ R-symmetry groups switch their roles. In this context, it is worth noting that the bi-harmonic $\mathcal{N}$=4 approach [35] achieves a unified description of systems with $D(2,1;\alpha)$ and $D(2,1;-1-\alpha)$ invariance. It allows one to naturally incorporate mirror counterparts for all $\mathcal{N}$=4 supermultiplets with four fermions. Hence, it may provide an extension of the $D(2,1;\alpha)$ superconformal models considered here, by adding such extra supermultiplets. Upon quantization, the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ auxiliary multiplets would produce a second family of target harmonic-like U(2) variables.

For the remainder of this outlook and as a continuation of the discussion in the Introduction, let us illustrate how the models considered in this paper and in [2, 18] could be inscribed into the context of D=5 extreme black-hole quantum mechanics.

The motion of a test particle with mass m near the horizon of an extremal Tangherlini black hole of charge Q (a straightforward D=5 generalization of the D=4 extremal Reissner-Nordström solution) is described by the simple action [5–9]

$$S = \frac{mQ^2}{2}\int dt\, |\dot{\vec{y}}|^2\,, \tag{5.5}$$

where $\vec{y}$ are the coordinates of Euclidean four-space which are related to the isotropic near-horizon black-hole coordinates $\vec{x}$ via $\vec{y} = \vec{x}/|\vec{x}|^2$.

Making a polar decompostiion of the 4-vector $\vec{y}$ into a radial part $\rho = |\vec{y}|$ and an S^3 angular part, we rewrite the action (5.5) in first-order form as

$$S = \int \left[p_\rho d\rho + \vec{J}{\cdot}\vec{\omega} - dt\, \frac{1}{2mQ^2}\left(p_\rho^2 + \frac{4\vec{J}{\cdot}\vec{J}}{\rho^2}\right)\right]. \tag{5.6}$$

Here, ω_i are the invariant one-forms on $S^3 \sim \mathrm{SU}(2)$, parametrized by the Euler angles ($0{\le}\gamma{\le}\pi$, $0{\le}\beta{\le}2\pi$, $0{\le}\phi{<}4\pi$):

$$\omega_1 = -\sin\phi\, d\gamma + \cos\phi\, \sin\gamma\, d\beta\,, \quad \omega_2 = \cos\phi\, d\gamma + \sin\phi\, \sin\gamma\, d\beta\,, \quad \omega_3 = d\phi + \cos\gamma\, d\beta\,. \tag{5.7}$$

In the Hamiltonian approach, the quantities $\vec{J}$ generate some SU(2) invariance [43]. It is easy to see that the action (5.5) is indeed reproduced by eliminating p_ρ and $\vec{J}$ in (5.6) by their algebraic equations of motion. Firstly, we obtain the action

$$S=\frac{mQ^2}{2}\int dt\left[\dot\rho\dot\rho+\rho^2\frac{1}{4}\vec\omega_t\cdot\vec\omega_t\right]\qquad\text{where}\qquad \vec\omega=\vec\omega_t dt\,. \tag{5.8}$$

However, $\frac{1}{4}\vec\omega\cdot\vec\omega$ is precisely the S^3 metric [43, 44]. Therefore secondly, the action takes the form

$$S=\frac{mQ^2}{2}\int dt\left[\dot\rho\dot\rho+\rho^2\dot{\vec n}\cdot\dot{\vec n}\right]\qquad\text{where}\qquad |\vec n|=1\,. \tag{5.9}$$

This is just (5.5) with $\vec y=\rho\,\vec n$.

Performing in (5.6) a reduction with respect to the variables $\vec J$ [45],

$$J_1=J_2=0\,,\qquad J_3=a=\text{const}\,, \tag{5.10}$$

and identifying $\rho=bx$, $p_\rho=b^{-1}p_x$, $a=-c/2$, where $b^2=\frac{2}{mQ^2}$, we obtain the one-particle bosonic limit (3.20) of the action (1.1) at $|\alpha|=1$.

The fact that just this particular value of α comes out is not surprising because the action (5.5) was obtained in [5–9] as the bosonic limit of the $SU(1,1|2)$ superconformal model. It is interesting that the action (3.20) at *arbitrary* non-zero value of α can still be reproduced by the same reduction (5.10) from a deformation of the action (5.5) (or, equivalently, of (5.6)).

This can be done in two different ways. One option is to substitute $4(J_1J_1+J_2J_2+\alpha^2J_3J_3)/\rho^2$ for $4\vec J\cdot\vec J/\rho^2$ in the last term of (5.6). The action (5.8) deformed in this way involves the metric $\frac{1}{4}(\omega_{t1}\omega_{t1}+\omega_{t2}\omega_{t2}+\alpha^{-2}\omega_{t3}\omega_{t3})$ instead of $\frac{1}{4}\vec\omega_t\cdot\vec\omega_t$. Such a system describes the particle motion on a squashed 3-sphere, with α^{-2} as the squashing parameter. This model may bear a tight relation to $D{=}5$ rotating black holes, whose horizon is known to be a squashed 3-sphere [44, 46–48]. The O(4) symmetry of (5.5) is broken to O(3) in this situation.

Another possibility is to replace $4\vec J\cdot\vec J/\rho^2$ in the last term of (5.6) by $4\alpha^2\vec J\cdot\vec J/\rho^2$. The Lagrangian in (5.8) is then deformed into $[\dot\rho\dot\rho+\alpha^{-2}\rho^2\frac{1}{4}\vec\omega_t\cdot\vec\omega_t]$. This system describes particle motion on a 4-dimensional cone $C(S^3)$ over the round sphere S^3 of radius α^{-2} as the base [23, 49]. This cone is conformally flat and exhibits O(4) isometry at any $\alpha{\neq}0$, including the values $\alpha{=}\pm1$ which correspond to the action (5.5).

In both cases, the reduction (5.10), performed in the relevant counterparts of the action (5.6), exactly yields our action (3.20). It is amusing that the parameter α acquires a nice geometric meaning within such a framework.

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A Toy model with $\mathcal{N}$=2 supersymmetry

Here we consider $\mathcal{N}$=2 supersymmetric model describing a "matter" supermultiplet coupled to U(1) gauge background. Matter is represented by two chiral superfields $Z^k(t_L, \theta)$, $\bar{Z}_k(t_R, \bar{\theta}) = (Z^k)^+$, $t_{L,R} = t \pm i\theta\bar{\theta}$, satisfying irreducible conditions $\bar{D}Z^k = 0$, $D\bar{Z}_k = 0$, $k = 1, 2$. Here, the covariant spinor derivatives are

$$D = \partial_\theta + i\bar{\theta}\partial_t\,, \quad \bar{D} = -\partial_{\bar{\theta}} - i\theta\partial_t\,, \quad \{D, \bar{D}\} = -2i\partial_t\,.$$

The gauge prepotential is a real superfield $V(t, \theta, \bar{\theta})$, $(V)^+ = V$. The action has the following form

$$S = \int dt d^2\theta \left[\bar{Z}_k\, e^{2V} Z^k + c\, V\right]. \tag{A.1}$$

It is invariant under the local U(1) transformations:

$$Z^k \to e^{-i\Lambda} Z^k\,, \qquad \bar{Z}_k \to e^{i\bar{\Lambda}} \bar{Z}_k\,, \qquad V \to V + \frac{i}{2}\left(\Lambda - \bar{\Lambda}\right) \tag{A.2}$$

where $\Lambda(t_L, \theta)$, $\bar{\Lambda}(t_R, \theta) = (\Lambda)^+$ are chiral and antichiral superfield gauge parameters.

Supersymmetry transformations of a general $\mathcal{N}$=2 superfield F are defined by

$$\delta F = -(\delta t \partial_t + \delta\theta\partial_\theta + \delta\bar{\theta}\partial_{\bar{\theta}})\, F = -(\varepsilon\, Q - \bar{\varepsilon}\, \bar{Q})\, F \tag{A.3}$$

where the generators of SUSY transformations are

$$Q = \partial_\theta - i\bar{\theta}\partial_t\,, \quad \bar{Q} = -\partial_{\bar{\theta}} + i\theta\partial_t\,.$$

Component contents of the superfields defined above are

$$Z^k = z^k + 2i\theta\phi^k + i\theta\bar{\theta}\dot{z}^k\,, \qquad \bar{Z}_k = \bar{z}_k + 2i\bar{\theta}\bar{\phi}_k - i\theta\bar{\theta}\dot{\bar{z}}_k\,, \qquad V = v + \theta\chi - \bar{\theta}\bar{\chi} + \theta\bar{\theta}A\,, \tag{A.4}$$

where ϕ^k, $\bar{\phi}_k = (\overline{\phi^k})$ and χ, $\bar{\chi} = (\overline{\chi})$ are fermionic fields. For the component fields the transformations (A.3) yield

$$\delta z^k = -2i\varepsilon\phi^k\,, \qquad \delta\bar{z}_k = -2i\bar{\varepsilon}\bar{\phi}_k\,, \qquad \delta\phi^k = -\bar{\varepsilon}\dot{z}^k\,, \qquad \delta\bar{\phi}_k = -\varepsilon\dot{\bar{z}}_k\,, \tag{A.5}$$

$$\delta v = -\varepsilon\chi + \bar{\varepsilon}\bar{\chi}\,, \quad \delta\chi = -\bar{\varepsilon}(A + i\dot{v})\,, \quad \delta\bar{\chi} = -\varepsilon(A - i\dot{v})\,, \quad \delta A = -i(\varepsilon\dot{\chi} + \bar{\varepsilon}\dot{\bar{\chi}})\,. \tag{A.6}$$

Let us consider the action (A.1) in the WZ gauge,

$$V(t, \theta, \bar{\theta}) = \theta\bar{\theta}A(t)\,, \quad e^{2V} = 1 + 2\theta\bar{\theta}A\,. \tag{A.7}$$

It takes the form ($\int d^2\theta\, (\theta\bar{\theta}) = 1$)

$$S^{WZ} = \int dt \left[i(\bar{z}_k \nabla z^k - \nabla\bar{z}_k z^k) + c\, A\, - 4\bar{\phi}_k\phi^k\right],$$

where ∇z and $\nabla\bar{z}$ are the gauge-covariant derivatives,

$$\nabla z^k = \dot{z}^k - iAz^k\,, \qquad \nabla\bar{z}_k = \dot{\bar{z}}_k + iA\bar{z}_k\,. \tag{A.8}$$

The action (A.8) is invariant under the residual local U(1) transformations

$$\delta z^k = -i\lambda z^k\,, \qquad \delta \bar z_k = i\lambda \bar z_k\,, \qquad \delta A = -\dot\lambda\,, \tag{A.9}$$

where $\lambda(t)$ is the $d{=}1$ gauge parameter.

Supersymmetry transformations (A.5)–(A.6) do not preserve the WZ gauge conditions $v = 0$, $\chi = 0$, $\bar\chi = 0$, and we are led to modify these transformations by a field-dependent compensating gauge transformation with the parameter

$$\Lambda = -2i\theta\bar\varepsilon A\,, \qquad \bar\Lambda = -2i\bar\theta\varepsilon A\,.$$

Then the supersymmetry transformations leaving invariant the action (A.8) are given by

$$\delta^{WZ} z^k = -2i\varepsilon\phi^k\,, \quad \delta^{WZ}\bar z_k = -2i\bar\varepsilon\bar\phi_k\,, \quad \delta^{WZ}\phi^k = -\bar\varepsilon\nabla z^k\,, \quad \delta^{WZ}\bar\phi_k = -\varepsilon\nabla\bar z_k\,, \tag{A.10}$$

$$\delta^{WZ} A = 0\,. \tag{A.11}$$

Let us study the closure of these transformations. On the fields z^k we have

$$(\delta_1^{WZ}\delta_2^{WZ} - \delta_2^{WZ}\delta_1^{WZ})\, z^k = 2i\,(\varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1)\,\nabla z^k = 2ia_{12}\dot z^k - i\lambda_{12}z^k\,, \tag{A.12}$$

where

$$a_{12} = \varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1\,, \qquad \lambda_{12} = 2i\,(\varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1)\,A\,. \tag{A.13}$$

Thus, the r.h.s. of (A.12) is the time translation with the parameter a_{12} accompanied by a residual gauge transformation with the parameter λ_{12}. Clearly, the closure on the gauge field $A(t)$ should be the same. We find

$$\delta_{12}^{WZ} A = 2ia_{12}\dot A - \dot\lambda_{12} \;= 0, \tag{A.14}$$

in agreement with (A.11).

On shell, after eliminating the auxiliary fields ϕ, $\bar\phi$ in the action (A.8),

$$\phi^k = 0\,, \qquad \bar\phi_k = 0\,, \tag{A.15}$$

the action (A.8) and the supersymmetry transformations (A.10), (A.11) become

$$S^{WZ} = \int dt \left[i(\bar z_k\nabla z^k - \nabla\bar z_k z^k) + c\,A \right], \tag{A.16}$$

$$\tilde\delta^{WZ} z^k = 0\,, \qquad \tilde\delta^{WZ}\bar z_k = 0\,, \qquad \tilde\delta^{WZ} A = 0\,. \tag{A.17}$$

Taking into account the equations of motion

$$\nabla z^k = \nabla\bar z_k = 0\,,$$

these on-shell transformations close on the time translations and gauge transformation like their off-shell counterparts (A.10), (A.11).

The structure of the component $\mathcal{N}{=}4$ supersymmetry transformations in the WZ gauge in our $D(2,1;\alpha)$ superconformal mechanics model is basically the same as in the toy model just considered.

B Time reversal in mechanics

Let us consider the simple mechanical model with the Lagrangian

$$L_1 = \dot{x}^2 - i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - i(\bar{z}\dot{z} - \dot{\bar{z}}z) - U(x, \psi, \bar{\psi}, z, \bar{z})\,. \tag{B.1}$$

The canonical momenta are[13]

$$p = 2\dot{x}, \qquad p_\psi = -i\bar{\psi}, \quad p_{\bar{\psi}} = -i\psi, \qquad p_z = -i\bar{z}, \quad p_{\bar{z}} = iz \tag{B.2}$$

with Poisson brackets

$$[x, p]_P = 1, \qquad [z, p_z]_P = [\bar{z}, p_{\bar{z}}]_P = 1, \qquad \{\psi, p_\psi\}_P = \{\bar{\psi}, p_{\bar{\psi}}\}_P = 1\,. \tag{B.3}$$

Therefore, the Hamiltonian is

$$H = p\dot{x} + p_\psi\dot{\psi} + p_{\bar{\psi}}\dot{\bar{\psi}} + p_z\dot{z} + p_{\bar{z}}\dot{\bar{z}} - L = \frac{1}{4}p^2 + U. \tag{B.4}$$

The definition (B.2) implies second-class constraints

$$\begin{gathered} G_\psi = p_\psi + i\bar{\psi} \approx 0, \quad G_{\bar{\psi}} = p_{\bar{\psi}} + i\psi \approx 0, \quad G_z = p_z + i\bar{z} \approx 0, \quad G_{\bar{z}} = p_{\bar{z}} - iz \approx 0 \\ \{G_\psi, G_{\bar{\psi}}\}_P = 2i\,, \quad [G_z, G_{\bar{z}}]_P = 2i\,. \end{gathered} \tag{B.5}$$

Introducing Dirac brackets

$$\begin{aligned} [A, B\}_D = [A, B\}_P &+ \frac{i}{2}[A, G_\psi\}_P[G_{\bar{\psi}}, B\}_P + \frac{i}{2}[A, G_{\bar{\psi}}\}_P[G_\psi, B\}_P; \\ &-\frac{i}{2}[A, G_z\}_P[G_{\bar{z}}, B\}_P + \frac{i}{2}[A, G_{\bar{z}}\}_P[G_z, B\}_P \end{aligned}$$

we obtain

$$[x, p]_D = 1, \qquad [z, \bar{z}]_D = \frac{i}{2}, \qquad \{\psi, \bar{\psi}\}_D = \frac{i}{2}\,. \tag{B.6}$$

Then, passing to quantum theory, we obtain the following operator algebra

$$[X, P] = i, \qquad [Z, \bar{Z}] = -\frac{1}{2}, \qquad \{\Psi, \bar{\Psi}\} = -\frac{1}{2}\,. \tag{B.7}$$

The time-reversed system is described by the Lagrangian[14]

$$L_2 = \dot{x}^2 + i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) + i(\bar{z}\dot{z} - \dot{\bar{z}}z) - U(x, \psi, \bar{\psi}, z, \bar{z})\,. \tag{B.8}$$

Performing the same procedure as above we obtain that the system (B.8) has the same Hamiltonian (B.4), but different Dirac brackets

$$[x, p]_D = 1, \qquad [z, \bar{z}]_D = -\frac{i}{2}, \qquad \{\psi, \bar{\psi}\}_D = -\frac{i}{2} \tag{B.9}$$

which yield

$$[X, P] = i, \qquad [Z, \bar{Z}] = \frac{1}{2}, \qquad \{\Psi, \bar{\Psi}\} = \frac{1}{2}\,. \tag{B.10}$$

Comparing (B.10) with (B.7), we observe that the former turns into the latter after redefining

$$\bar{Z} = -\,(Z)^+\,, \qquad \bar{\Psi} = -\,(\Psi)^+\,.$$

[13] We use the notations which are related to those in [50, 51] through a redefinition. In particular, we define the fermionic momenta as right derivatives of the Lagrangian.

[14] To be more precise, under the time reversal we also need to change the sign of the overall normalization constant before the invariant action since the integral $\int dt$ changes its sign.

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