$\mathcal{N} = 4$ supersymmetric Calogero-Sutherland models

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Starting from the Hamiltonian formulation of supersymmetric Calogero models associated with the classical A_n, B_n, C_n and D_n series we construct the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric extensions of their hyperbolic/trigonometric Calogero-Sutherland cousins. The bosonic core of these models are the standard Calogero-Sutherland hyperbolic/trigonometric systems.

I. INTRODUCTION

There is a lot of confirmation that $\mathcal{N} = 4$ supersymmetric extensions of Calogero-Moser systems must include a large number of fermions—far more than the $4n$ fermions expected within the standard (but not very successful) approach [1–6]. The source of these fermions is the supersymmetrization of the matrix models from which, in the purely bosonic cases, the Calogero-Moser systems can be obtained by a reduction (see, e.g., [7]).

A suitable approach to supersymmetric Calogero-like models has been proposed in [1–3]. Starting from a supersymmetrization of the Hermitian matrix model, the resulting matrix fermionic degrees of freedom are packaged in $\mathcal{N} = 4$ superfields. In a recent paper [2], $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric extensions of the multiparticle hyperbolic Calogero-Sutherland system were constructed by applying a gauging procedure [8] to one-dimensional matrix superfield systems. However, for $\mathcal{N} = 4$ their bosonic part does not reproduce the ordinary Calogero systems but only spin-Calogero ones.[1]

In a series of papers [4–6] we developed a different approach. Mainly working in the Hamiltonian formulation, we worked out an ansatz for the supercharges which accommodates all Calogero models associated with the classical A_n, B_n, C_n and D_n Lie algebras. Here, the supercharges contain the fermion-cube terms only through

the combination *fermion × fermion bilinear*, where the fermion bilinears span an $s(u(n) \oplus u(n))$ algebra.

In this paper we use this ansatz to construct $\mathcal{N} = 4$ supersymmetric extensions of the Calogero-Sutherland models associated with the classical A_n, B_n, C_n and D_n series (Sec. II). As a separate application, we also find the $\mathcal{N} = 4$ supersymmetric extensions of the trigonometric/hyperbolic cousins of the Euler-Calogero-Moser system (Sec. III).

II. CALOGERO-SUTHERLAND MODELS

A. Basic ingredients

The starting point of our construction is the same set of the fields as in the $\mathcal{N}$-extended supersymmetric Calogero-Moser model [4] which is nothing but a supersymmetric extension of the Hermitian matrix model [7,9,10]. This set of fields includes the following ones:

(a) n bosonic coordinates x_i, which come from the diagonal elements of the Hermitian matrix X, and the corresponding momenta p_i for $i, j = 1, \ldots, n$ which obey the standard brackets

$$\{x_i, p_j\} = \delta_{ij}. \tag{2.1}$$

(b) Fermionic matrices containing $\mathcal{N} n^2$ elements ξ_{ij}^a, $\bar{\xi}_{ija}$ for $a = 1, \ldots, \mathcal{N}/2$ with $(\xi_{ij}^a)^{\dagger} = \bar{\xi}_{jia}$ and brackets

$$\{\xi_{ij}^a, \bar{\xi}_{kmb}\} = -\mathrm{i}\delta_b^a \delta_{im}\delta_{jk}. \tag{2.2}$$

Using these ingredients one may construct the fermionic bilinears

$$\Pi_{ij} = \sum_{a=1}^{\mathcal{N}/2} \sum_{k=1}^{n} (\xi_{ik}^a \bar{\xi}_{kja} + \bar{\xi}_{ika} \xi_{kj}^a), \qquad \sum_i \Pi_{ii} = 0, \tag{2.3}$$

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[1]For a quantization of the $\mathcal{N} = 2$ case, see [3].

$$\tilde{\Pi}_{ij} = \sum_{a=1}^{\mathcal{N}/2} \sum_{k=1}^{n} (\xi_{ik}^a \bar{\xi}_{kja} - \bar{\xi}_{ika} \xi_{kj}^a), \qquad (2.4)$$

which form an $s(u(n) \oplus u(n))$ algebra,

$$\{\Pi_{ij}, \Pi_{km}\} = \{\tilde{\Pi}_{ij}, \tilde{\Pi}_{km}\} = i(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}) \quad \text{and}$$
$$\{\Pi_{ij}, \tilde{\Pi}_{km}\} = i(\delta_{im}\tilde{\Pi}_{kj} - \delta_{kj}\tilde{\Pi}_{im}). \qquad (2.5)$$

Using these ingredients in [6] the supercharges and Hamiltonian have been constructed for arbitrary even-$\mathcal{N}$ supersymmetric extensions of the A_n, B_n, C_n and D_n rational Calogero models. In what follows we will use the same ingredients to construct $\mathcal{N} = 2, 4$ trigonometric/hyperbolic Calogero-Sutherland models with the supercharges and the Hamiltonian obeying the $\mathcal{N} = 2$, 4-extended super-Poincaré algebra.

B. $\mathcal{N} = 2$ supersymmetric $A_{n-1} \oplus A_1$ Calogero-Sutherland models

In this simplest case the supercharges have a quite simple structure[2]

$$Q = \sum_{i=1}^{n} p_i \xi_{ii} - i \sum_{i \neq j}^{n} \left[f(z_{ij})(g + \Pi_{jj}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji},$$
$$\bar{Q} = \sum_{i=1}^{n} p_i \bar{\xi}_{ii} - i \sum_{i \neq j}^{n} \left[f(z_{ij})(g + \Pi_{jj}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji}.$$
$$(2.6)$$

Note that $\tilde{\Pi}_{ij}$ does not appear here and the function f will be specified in a moment.

These supercharges form an $\mathcal{N} = 2$ super-Poincaré algebra

$$\{Q, \bar{Q}\} = -2iH \quad \text{and} \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \qquad (2.7)$$

together with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right]$$
$$\times \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] - \frac{\beta^2}{2} \sum_{i,j}^{n} \Pi_{ij} \Pi_{ji}.$$
$$(2.8)$$

Here, we abbreviated

$$z_{ij} = x_i - x_j, \qquad (2.9)$$

and the constant parameter β and the function f are given as follows,

rational Calogero model $\quad \beta = 0$,

$$f(z_{ij}) = \frac{1}{z_{ij}} = \frac{1}{x_i - x_j},$$

hyperbolic Calogero-Sutherland model $\quad \beta = 1$,

$$f(z_{ij}) = \frac{1}{\sinh(z_{ij})} = \frac{1}{\sinh(x_i - x_j)},$$

trigonometric Calogero-Sutherland model $\quad \beta = i$,

$$f(z_{ij}) = \frac{1}{\sin(z_{ij})} = \frac{1}{\sin(x_i - x_j)}. \qquad (2.10)$$

Thus, the supercharges (2.6) and the Hamiltonian (2.8) describe an $\mathcal{N} = 2$-extended supersymmetric Calogero-Sutherland models of type $A_{n-1} \oplus A_1$.

It should be noted that when checking that the supercharges form the superalgebra (2.7) it is not enough to know the brackets between Π_{ij} and the fermions ξ_{ij}, $\bar{\xi}_{ij}$. Instead, the explicit expressions for Π_{ij} (2.3) have to be substituted into Eq. (2.6). This makes the calculations slightly more complicated as compared to those discussed in [6].

C. $\mathcal{N} = 4$ supersymmetric $A_{n-1} \oplus A_1$ Calogero-Sutherland models

Due to the absence of any guiding rules for construction of $\mathcal{N} = 4$ supercharges, the reasonable starting point is the straightforward generalization of the $\mathcal{N} = 2$ supercharges (2.6) to the $\mathcal{N} = 4$ supersymmetry reads

$$Q^a = \sum_{i=1}^{n} p_i \xi_{ii}^a - i \sum_{i \neq j}^{n} \left[f(z_{ij})(g + \Pi_{jj}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji}^a,$$
$$\bar{Q}_b = \sum_{i=1}^{n} p_i \bar{\xi}_{iib} - i \sum_{i \neq j}^{n} \left[f(z_{ij})(g + \Pi_{jj}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{jib},$$
$$a, b = 1, 2. \qquad (2.11)$$

Unfortunately, this guess is not correct and the supercharges (2.11) do not form the $\mathcal{N} = 4$ superalgebra

$$\{Q^a, \bar{Q}_b\} = -2i\delta_b^a H \quad \text{and} \quad \{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0, \qquad (2.12)$$

in contrast with their $\mathcal{N} = 2$ cousins (2.6). The possible modification of the supercharges looks as follows:

$$\mathcal{Q}^a = Q^a - i\beta \sum_{i,j}^{n} \xi_{ij}^a \tilde{\Pi}_{ji}, \qquad \bar{\mathcal{Q}}_a = \bar{Q}_a + i\beta \sum_{i,j}^{n} \bar{\xi}_{ija} \tilde{\Pi}_{ji}.$$
$$(2.13)$$

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[2]We omit the indices a, b which all are equal to 1 in this case.

The supercharges (2.13) form $\mathcal{N} = 4$ super Poincaré algebra (2.12) together with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]$$
$$\times \left[(g + \Pi_{ii})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ji}\right] - \frac{\beta^2}{2}\sum_{i,j}^{n}\Pi_{ij}\Pi_{ji},$$

(2.14)

which has the same form as $\mathcal{N} = 2$ Hamiltonian (2.8).

It should be mentioned that in the $\mathcal{N} = 2$ case the additional, β-dependent terms in the supercharges (2.13) are automatically nullified in virtue of the structure of $\tilde{\Pi}_{ij}$ (2.3)

$$\sum_{i,j}^{n}\xi_{ij}^{a}\tilde{\Pi}_{ji} = \sum_{i,j}^{n}\bar{\xi}^{ija}\tilde{\Pi}_{ji} = 0, \quad \text{if } a, \quad b = 1, \quad (2.15)$$

and thus, the supercharges (2.13) reduce to the supercharges (2.6) in the limit $a, b = 1$.

Finally, all we said above is valid only for the functions f from the list (2.10).

D. $\mathcal{N} = 4$ supersymmetric B_n, C_n and D_n Calogero-Sutherland models

It is strange but for the B-, C- and D-type models the $\mathcal{N} = 4$ supercharges take the same form as $\mathcal{N} = 2$ ones. Indeed, one may check that the following supercharges (including $\tilde{\Pi}_{ij}$),

$$\mathcal{Q}^a = \sum_{i=1}^{n} p_i\xi_{ii}^a - i\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\xi_{ji}^a + i\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\tilde{\Pi}_{ij}\right]\xi_{ji}^a$$
$$+ i\sum_{i}^{n}\left[(g' + \Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\tilde{\Pi}_{ii}\right]\xi_{ii}^a,$$

$$\bar{\mathcal{Q}}_a = \sum_{i=1}^{n} p_i\bar{\xi}_{iia} - i\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\bar{\xi}_{jia} - i\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\tilde{\Pi}_{ij}\right]\bar{\xi}_{jia}$$
$$- i\sum_{i}^{n}\left[(g' + \Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\tilde{\Pi}_{ii}\right]\bar{\xi}_{iia},$$

(2.16)

form $\mathcal{N} = 4$ super-Poincaré algebra (2.12) together with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\left[(g + \Pi_{ii})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ji}\right]$$
$$+ \frac{1}{2}\sum_{i\neq j}^{n}\left[(g + \Pi_{jj})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\tilde{\Pi}_{ij}\right]\left[(g + \Pi_{ii})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\tilde{\Pi}_{ji}\right]$$
$$+ \frac{1}{2}\sum_{i}^{n}\left[(g' + \Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\tilde{\Pi}_{ii}\right]\left[(g' + \Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\tilde{\Pi}_{ii}\right]$$
$$- \frac{\beta^2}{2}\sum_{i,j}^{n}(\Pi_{ij}\Pi_{ji} + \tilde{\Pi}_{ij}\tilde{\Pi}_{ji}).$$

(2.17)

Here,

$$y_{ij} = x_i + x_j, \quad (2.18)$$

and the function f is the same as in (2.10).

The bosonic sector of the Hamiltonian (2.17) reads

$$\mathcal{H}_{\text{bos}} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^{n}(f^2(z_{ij}) + f^2(y_{ij})) + \frac{g'^2}{2}\sum_{i}^{n} f^2(y_{ii}). \quad (2.19)$$

Due to the presence of only two coupling constants, g and g', we may describe B-, C- and D-type models in the rational case and C- and D- (but not B-) type models in the hyperbolic/trigonometric case.

Finally, let us note that in the $\mathcal{N} = 2$ supersymmetric case the last term in the Hamiltonian (2.17) nullified automatically due to the structure of the Π_{ij} and $\tilde{\Pi}_{ij}$ (2.3)

$$\sum_{i,j}^{n} (\Pi_{ij}\Pi_{ji} + \tilde{\Pi}_{ij}\tilde{\Pi}_{ji}) = 0 \quad \text{if } a, b = 1. \qquad (2.20)$$

E. Toward higher supersymmetries

It is interesting to note that the supercharges (2.13) obey the relations

$$\{\mathcal{Q}^a, \mathcal{Q}^b\} = 0 \quad \text{and} \quad \{\bar{\mathcal{Q}}_a, \bar{\mathcal{Q}}_b\} = 0 \qquad (2.21)$$

for arbitrary range of the indices a, b running from 1 to $\mathcal{N}/2$. However, the anticommutators between these supercharges have more complicated structure

$$\{\mathcal{Q}_a, \bar{\mathcal{Q}}_b\} = -2i(\delta_b^a \mathcal{H} - \beta^2 \mathcal{V}_b^a), \qquad (2.22)$$

where

$$\mathcal{V}_b^a = \sum_{i,j}^{n} [\Pi_{ij} W_{jib}^a + \tilde{\Pi}_{ij} \tilde{W}_{jib}^a], \qquad (2.23)$$

and

$$W_{ijb}^a = \sum_{k}^{n} [\xi_{ik}^a \bar{\xi}_{kjb} + \bar{\xi}_{ikb} \xi_{kj}^a] \quad \text{and}$$

$$\tilde{W}_{ijb}^a = \sum_{k}^{n} [\xi_{ik}^a \bar{\xi}_{kjb} - \bar{\xi}_{ikb} \xi_{kj}^a]. \qquad (2.24)$$

Thus, the algebra becomes a nonlinear one. Note that for the $\mathcal{N} = 2$ supersymmetry the unique term $\mathcal{V}_1^1 = 0$, while for the $\mathcal{N} = 4$ supersymmetry we have $\mathcal{V}_1^1 = \mathcal{V}_2^2$ and $\mathcal{V}_2^1 = \mathcal{V}_1^2 = 0$. Thus, in the $\mathcal{N} = 4$ case these additional terms just modified the Hamiltonian.

Finally, one should note that the purely fermionic objects $\mathcal{V}_b^a$ (2.23) are, essentially, constants, because

$$\{\mathcal{H}, \mathcal{V}_b^a\} = 0. \qquad (2.25)$$

III. EULER-CALOGERO-MOSER MODELS

A. Basic ingredients

The construction of the supersymmetric extension [5] of the Euler-Calogero-Moser systems [9] is more economical as compared to the supersymmetric Calogero-Sutherland

systems we considered in the previous sections. Indeed, to construct the corresponding $\mathcal{N}$ supercharges one needs to introduce "only" $\mathcal{N}\frac{1}{2}n(n+1)$ fermions $\rho_{ij}^a, \bar{\rho}_{ija}$ symmetric over indices i, j $\rho_{ij}^a = \rho_{ji}^a, \bar{\rho}_{aij} = \bar{\rho}_{aji}$ and obeying the brackets

$$\{\rho_{ij}^a, \bar{\rho}_{kmb}\} = -\frac{i}{2}\delta_b^a(\delta_{im}\delta_{jk} + \delta_{ik}\delta_{jm}). \qquad (3.1)$$

The internal degrees of freedom of Euler-Calogero-Moser (ECM) models are encoded in the angular momenta $\ell_{ij} = -\ell_{ji}$ with the Poisson brackets forming the $so(n)$ algebra

$$\{\ell_{ij}, \ell_{km}\} = \frac{1}{2}(\delta_{ik}\ell_{jm} + \delta_{jm}\ell_{ik} - \delta_{jk}\ell_{im} - \delta_{im}\ell_{jk}). \qquad (3.2)$$

Similar to the construction of the supersymmetric Calogero-Sutherland systems, our ansatz for the supercharges includes the following fermionic bilinears:[3]

$$\Pi_{ij}^\rho = -i\sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n} (\rho_{ik}^a \bar{\rho}_{kja} - \rho_{jk}^a \bar{\rho}_{kia}), \qquad (\Pi_{ij}^\rho)^\dagger = \Pi_{ij}^\rho,$$

$$\tilde{\Pi}_{ij}^\rho = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n} (\rho_{ik}^a \bar{\rho}_{kja} + \rho_{jk}^a \bar{\rho}_{kia}), \qquad (\tilde{\Pi}_{ij}^\rho)^\dagger = \tilde{\Pi}_{ij}^\rho.$$

$$(3.3)$$

Finally, one may check that the $\mathcal{N}$ supercharges $Q^a, \bar{Q}_b$,

$$Q^a = \sum_{i=1}^{n} p_i\rho_{ii}^a - \sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij}^\rho)\rho_{ji}^a}{x_i - x_j},$$

$$\bar{Q}_a = \sum_{i=1}^{n} p_i\bar{\rho}_{iia} - \sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij}^\rho)\bar{\rho}_{jia}}{x_i - x_j} \qquad (3.4)$$

form $\mathcal{N}$-extended super-Poincaré algebra (2.12) [5] together with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij}^\rho)^2}{(x_i - x_j)^2}. \qquad (3.5)$$

B. $\mathcal{N} = 4$ supersymmetry

The construction of the supersymmetric extension of the hyperbolic/trigonometric ECM model is very similar to the case of the Calogero model. Again, we succeeded in the construction of the $\mathcal{N} = 4$ supersymmetric extensions, only. The main idea of our construction is to maximally preserve the anzatz for the supercharges (3.4), i.e., we admit the appearance of the three-linear fermionic terms in

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[3]These bilinears form a $u(n)$ algebra [5].

the supercharges only through the bilinears (3.3). Thus, our ansatz for the supercharges reads

$$
\mathcal{Q}^a = \sum_{i=1}^{n} p_i \rho_{ii}^a - \sum_{i \neq j}^{n} \left(f(z_{ij}) \ell_{ij} - \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij}^\rho - i\beta \tilde{\Pi}_{ij}^\rho \right) \rho_{ji}^a,
$$

$$
\bar{\mathcal{Q}}_a = \sum_{i=1}^{n} p_i \bar{\rho}_{iia} - \sum_{i \neq j}^{n} \left(f(z_{ij}) \ell_{ij} - \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij}^\rho + i\beta \tilde{\Pi}_{ij}^\rho \right) \bar{\rho}_{jia}.
$$

$$(3.6)$$

Here, the function $f(z_{ij})$ and the parameter α are defined in the list (2.10).

In the rational case $f(z_{ij}) = 1/z_{ij}, \beta = 0$ the supercharges (3.6) form $\mathcal{N}$-extended superalgebra (2.12) for the indices $a, b = 1, \ldots \mathcal{N}/2$. However, one may easily check that for the two other choices of $f(z_{ij}), \beta$ in Eq. (2.10), the supercharges (2.11) form only the $\mathcal{N} = 4$ superalgebra (2.12) together with the Hamiltonian

$$
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \left[\left(f(z_{ij}) \ell_{ij} - \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij}^\rho \right)^2 + \beta^2 \Pi_{ij}^\rho \Pi_{ji}^\rho \right].
$$

$$(3.7)$$

IV. CONCLUSIONS

In this paper we constructed $\mathcal{N} = 4$ supersymmetric extensions of the Calogero-Sutherland models associated with the classical A_n, B_n, C_n and D_n series. The guiding principle was the structure of the supercharges in which the fermion-cube terms are all built from fermionic bilinears (2.3) spanning an $s(u(n) \oplus u(n))$ algebra. We also

described $\mathcal{N} = 4$ supersymmetric extensions of the trigonometric/hyperbolic cousins of the Euler-Calogero-Moser system.

In contrast with the rational Calogero-Moser and/or Euler-Calogero-Moser system admitting an arbitrary number of supersymmetries [4–6], their trigonometric/hyperbolic versions can so far be supersymmetrized up to the $\mathcal{N} = 4$ cases only. If we try to construct the additional supercharges, they will span a soft variant of the super-Poincaré algebra with purely fermionic conserved R-charges in the commutator of $\mathcal{Q}^a$ with $\bar{\mathcal{Q}}_b$ (2.22). It would be interesting to understand the nature of these R-charges and their algebra in more detail.

For a further development, one of the key questions is the possible integrability or even superintegrability of the constructed systems. It seems there is no serious problem with the $L - A$ pairs, which mostly mimic the pairs from the bosonic case. However, the unusually large number of fermions complicates the situation. It should be noted that the Hamiltonians contain the fermions only through the bilinears. Thus, the "efficient" number of the degrees of freedom seems to be smaller that the "naive" number of degrees of freedom. A qualitative example of such a situation is provided by the Euler-Calogero-Moser models [9], where the internal degrees of freedom are encoded in currents spanning an $so(n)$ algebra. We will consider the integrability properties of the constructed supersymmetric systems elsewhere.

[1] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Supersymmetric Calogero models by gauging, Phys. Rev. D **79**, 105015 (2009).

[2] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Supersymmetric hyperbolic Calogero-Sutherland models by gauging, Nucl. Phys. **B944**, 114633 (2019).

[3] S. Fedoruk, $\mathcal{N} = 2$ supersymmetric hyperbolic Calogero-Sutherland model, Nucl. Phys. **B953**, 114977 (2020).

[4] S. Krivonos, O. Lechtenfeld, and A. Sutulin, $\mathcal{N}$-extended supersymmetric Calogero models, Phys. Lett. B **784**, 137 (2018).

[5] S. Krivonos, O. Lechtenfeld, and A. Sutulin, Supersymmetric many-body Euler-Calogero-Moser model, Phys. Lett. B **790**, 191 (2019).

[6] S. Krivonos, O. Lechtenfeld, A. Provorov, and A. Sutulin, Extended supersymmetric Calogero model, Phys. Lett. B **791**, 385 (2019).

[7] A. Polychronakos, Physics and mathematics of Calogero particles, J. Phys. A **39**, 12793 (2006).

[8] F. Delduc and E. Ivanov, Gauging $\mathcal{N} = 4$ supersymmetric mechanics, Nucl. Phys. **B753**, 211 (2006).

[9] S. Wojciechowski, An integrable marriage of the Euler equations with the Calogero-Moser system, Phys. Lett. **111A**, 101 (1985).

[10] J. Gibbons and T. Hermsen, A generalization of the Calogero-Moser system, Physica (Amsterdam) **11D**, 337 (1984).