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On $\mathcal{N}=2$ supersymmetric Ruijsenaars–Schneider models

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A R T I C L E   I N F O

A B S T R A C T

We construct an $\mathcal{N}=2$ supersymmetric extension of n-particle Ruijsenaars–Schneider models. The guiding feature is a deformation of the phase space. The supercharges have a "free" form linear in the fermions but produce an interacting four-fermion Hamiltonian. A field-dependent unitary transformation maps to standard fermions obeying conventional Poisson brackets. In this frame, the supercharges and Hamiltonian have long "fermionic tails". We also comment on previous attempts in this direction.

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1. Introduction

The Ruijsenaars–Schneider models [1] are known for more than three decades as a "relativistic" variant[1] of the well known Calogero–Moser–Sutherland models [3–5]. An $\mathcal{N}=2$ supersymmetric extension of the latter has been constructed many years ago [6,7]. Soon after, its symmetry algebra and eigenfunctions have been analyzed [8], introducing the Jack superpolynomials [9] as superspace analogs of the Jack polynomials. In contrast, its "relativistic" cousin – an $\mathcal{N}=2$ supersymmetric extension of Ruijsenaars–Schneider models – remained almost completely unexplored for a couple of decades. Presumably responsible for this relative silence is the unfamiliar structure of its action (the lack of a potential), which impedes trusted techniques of supersymmetric mechanics [10] in a relativistic setting.

The first (integrable) $\mathcal{N}=2$ supersymmetric generalization of the (quantum) trigonometric Ruijsenaars–Schneider model has been reported in [11]. Its construction is based on (a) the integrability of the bosonic system, (b) a modification of the anticommutation relations between fermions, and (c) a complicated definition of adjoints. We shall comment on it in the Conclusions.

A second example of an $\mathcal{N}=2$ supersymmetric Ruijsenaars–Schneider system was elaborated in [12] for the case of three particles. Its ansatz for the supercharges mimics those in the supersymmetric Calogero–Moser–Sutherland models [13–15]. Unfortunately, it is unclear how this permutation-symmetry breaking solution can be extended to an arbitrary number of particles.

Here, we succeed in constructing $\mathcal{N}=2$ supersymmetric n-particle Ruijsenaars–Schneider models. Our guidelines are:

- The supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are taken to be "free", i.e. only linear in the fermions,
- The "interactions" are entirely encoded in a highly non-trivial structure of the Poisson brackets.

With this prescription, $\mathcal{N}=2$ supercharges may easily be constructed. The $\mathcal{N}=2$ Poincaré superalgebra then produces an interacting Hamiltonian. With a (field-dependent) unitary transformation one comes back to the standard fermions with standard Poisson brackets. The ensuing supercharges appear as a natural generalization of the "non-relativistic" fermions to the "relativistic" case. When passing to the standard fermions, the Hamiltonian gets modified by a long "fermionic tail".

The paper is organized as follows. In Section 2 we briefly review the relevant properties of the bosonic Ruijsenaars–Schneider model, emphasizing on its free-Hamiltonian representation, whose price is deformed Poisson brackets. Section 3 extends the non-standard bosonic Poisson brackets to the $\mathcal{N}=2$ supersymmetric case, where the "free" supercharges produce a proper interacting $\mathcal{N}=2$ supersymmetric

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[1] For a justification of the term "relativistic", see e.g. the discussion in [2].

Hamiltonian. The precise relation with standard fermions (obeying standard Poisson brackets) is presented in Section 4, which also computes the Hamiltonian in this frame. In the Conclusions we compare with the previous efforts [11,12] and mention possible further developments.

2. Bosonic models

The Ruijsenaars–Schneider models are integrable many-body systems in one dimension which are described by the equations of motion [1]

$$\ddot{x}_i = 2 \sum_{j \neq i}^{n} \dot{x}_i \dot{x}_j W(x_i - x_j) , \tag{2.1}$$

where the function W is one of the following functions[2]

$$W(x) \in \left\{ 1/x, \ 1/\sin(x), \ 1/\sinh(x), \ 1/\tan(x), \ 1/\tanh(x) \right\} . \tag{2.2}$$

The standard description of such systems is based on the Hamiltonian [1]

$$H = \tfrac{1}{2} \sum_{i}^{n} e^{2\theta_i} \prod_{j(\neq i)}^{n} f(x_i - x_j) , \tag{2.3}$$

where the rapidities θ_i and the coordinates x_j obey standard Poisson brackets,

$$\left\{ x_i, \theta_j \right\} = \delta_{ij} \qquad \text{and} \qquad \left\{ x_i, x_j \right\} = \left\{ \theta_i, \theta_j \right\} = 0 . \tag{2.4}$$

The functions W in (2.1) and f in (2.3) are related (in this order):

$$f(z) \in \left\{ \frac{1}{z}, \frac{1}{\sinh(z)}, \frac{1}{\sin(z)}, \frac{1}{\tanh(\frac{z}{2})}, \frac{1}{\tan(\frac{z}{2})} \right\} \quad \Leftrightarrow \quad W(z) \in \left\{ \frac{1}{z}, \frac{1}{\tanh(z)}, \frac{1}{\tan(z)}, \frac{1}{\sinh(z)}, \frac{1}{\sin(z)} \right\} . \tag{2.5}$$

We prefer the following re-interpretation of the Ruijsenaars–Schneider systems. Let us cast the Hamiltonian H into a free form,[3]

$$H = \tfrac{1}{2} \sum_{i=1}^{n} p_i^2 , \tag{2.6}$$

with new momenta

$$p_i = e^{\theta_i} \prod_{j(\neq i)}^{n} \sqrt{f(x_i - x_j)} . \tag{2.7}$$

This redefinition ($\theta_i \to p_i$) clearly changes the Poisson brackets to[4]

$$\left\{ x_i, p_j \right\} = \delta_{ij} p_j \qquad \text{and} \qquad \left\{ p_i, p_j \right\} = p_i p_j W(x_i - x_j) . \tag{2.8}$$

One may check that the Hamiltonian H (2.6) and brackets (2.8) result in the equations of motion (2.1). Indeed, from (2.6) and (2.8) we have

$$\dot{x}_i \equiv \{ x_i, H \} = p_i^2 , \tag{2.9}$$

and, therefore,

$$\ddot{x}_i \equiv \{ \dot{x}_i, H \} = \left\{ p_i^2, H \right\} = 2 \sum_{j(\neq i)}^{n} p_i^2 p_j^2 W(x_i - x_j) = 2 \sum_{j(\neq i)}^{n} \dot{x}_i \dot{x}_j W(x_i - x_j) , \tag{2.10}$$

as it should be.

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[2] We will not consider the elliptic variant in this paper.
[3] This form of the Hamiltonians and Poisson brackets is explicitly written in [12] but may be older.
[4] We assume that $W(0) = 0$, which allows us to avoid writing $\{ p_i, p_j \} = (1 - \delta_{ij}) \, p_i p_j W(x_i - x_j)$.

3. $\mathcal{N}=2$ supersymmetric Ruijsenaars–Schneider models

We have seen that the bosonic Ruijsenaars–Schneider models can be described by a free Hamiltonian, while the interaction moved to the Poisson brackets. We shall use the same strategy to construct an $\mathcal{N}=2$ supersymmetric extension of the Ruijsenaars–Schneider models.

Such a model is equivalent to the existence of supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ forming an $\mathcal{N}=2$ Poincaré superalgebra

$$\left\{\mathcal{Q}, \overline{\mathcal{Q}}\right\} = -2i\mathcal{H} \qquad \text{and} \qquad \{\mathcal{Q}, \mathcal{Q}\} = \left\{\overline{\mathcal{Q}}, \overline{\mathcal{Q}}\right\} = 0 \tag{3.1}$$

together with the Hamiltonian $\mathcal{H}$ whose bosonic sector coincides with the Hamiltonian H (2.6).

To construct such supercharges we extend the $2n$ phase-space variables x_i and p_j, obeying the brackets (2.8), by $2n$ fermions ψ_i and $\bar{\psi}_j = (\psi_j)^\dagger$, subject to the brackets

$$\{\psi_i, \psi_j\} = -\psi_i \psi_j W(x_i - x_j), \quad \{\bar{\psi}_i, \bar{\psi}_j\} = -\bar{\psi}_i \bar{\psi}_j W(x_i - x_j), \quad \{\psi_i, \bar{\psi}_j\} = -i\delta_{ij} + \psi_i \bar{\psi}_j W(x_i - x_j),$$

$$\{p_i, \psi_j\} = \tfrac{i}{2}\delta_{ij} \, p_i \psi_i \sum_k \psi_k \bar{\psi}_k W'(x_i - x_k) - \tfrac{i}{2} p_i \psi_j \psi_i \bar{\psi}_i W'(x_i - x_j), \qquad \{x_i, \psi_j\} = 0, \tag{3.2}$$

$$\{p_i, \bar{\psi}_j\} = -\tfrac{i}{2}\delta_{ij} \, p_i \bar{\psi}_i \sum_k \psi_k \bar{\psi}_k W'(x_i - x_k) + \tfrac{i}{2} p_i \bar{\psi}_j \psi_i \bar{\psi}_i W'(x_i - x_j) \qquad \{x_i, \bar{\psi}_j\} = 0.$$

It is rather easy to check that the Jacobi identities are fulfilled.

Finally, we verify that the supercharges

$$\mathcal{Q} = \sum_i^n p_i \psi_i \qquad \text{and} \qquad \overline{\mathcal{Q}} = \sum_i^n p_i \bar{\psi}_i \tag{3.3}$$

form an $\mathcal{N}=2$ Poincaré superalgebra (3.1) with the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2} \sum_{i=1}^n p_i^2 + i \sum_{i,j}^n p_i p_j \psi_i \bar{\psi}_j W(x_i - x_j) + \tfrac{1}{2} \sum_{i,j}^n p_i^2 \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j W'(x_i - x_j). \tag{3.4}$$

Thus, the set $\{\mathcal{Q}, \overline{\mathcal{Q}}, \mathcal{H}\}$ describes an $\mathcal{N}=2$ supersymmetric extension of the Ruijsenaars–Schneider models.

It should be noted that everything works fine for *any* antisymmetric function $W(x)$. Thus, the explicit choices in (2.2) are dictated by integrability and not by $\mathcal{N}=2$ supersymmetry.

Finally, we comment on the rationale leading to the form of the brackets (3.2). We insist that the supercharges Q and $\overline{Q}$ obey the brackets $\{Q, Q\} = \left\{\overline{Q}, \overline{Q}\right\} = 0$ and have the structure (3.3). To achieve this, the brackets $\{\psi_i, \psi_j\} = -\psi_i \psi_j W(x_i - x_j)$ and $\{\bar{\psi}_i, \bar{\psi}_j\} = -\bar{\psi}_i \bar{\psi}_j W(x_i - x_j)$ compensate contributions coming from $\{p_i, p_j\}$. For a complete cancellation of $\{Q, Q\}$, we further make the ansatz $\{p_i, \psi_j\} = p_i \psi_i \mathcal{A}_{i,j}$. The functions $\mathcal{A}_{i,j}$ as well as the yet undetermined brackets $\{\psi_i, \bar{\psi}_j\}$ finally follow uniquely from the Jacobi identities.

4. Playing with the fermions

One-dimensional supersymmetric systems feature a rich possibility to define the fermions. For example, the quite nonlinear redefinition of the fermions in [16] brings the supercharges of the $\mathcal{N}$-extended supersymmetric A_n Calogero model introduced in [17] to the standard form maximally cubic in the fermions. The mystery with the brackets (3.2) has the same origin. Indeed, one may define alternative fermions

$$\xi_i = \exp\left\{\tfrac{i}{2} \sum_j^n \psi_j \bar{\psi}_j W(x_i - x_j)\right\} \psi_i \qquad \text{and} \qquad \bar{\xi}_i = \exp\left\{-\tfrac{i}{2} \sum_j^n \psi_j \bar{\psi}_j W(x_i - x_j)\right\} \bar{\psi}_i, \tag{4.1}$$

which are subject to the standard brackets

$$\{\xi_i, \xi_j\} = \{\bar{\xi}_i, \bar{\xi}_j\} = 0, \quad \{\xi_i, \bar{\xi}_j\} = -i\delta_{ij} \qquad \text{and} \qquad \{p_i, \xi_j\} = \{p_i, \bar{\xi}_j\} = 0. \tag{4.2}$$

This exponential redefinition is inspired by the relation (2.7) between momenta and rapidities. Observing that

$$\xi_i \bar{\xi}_i = \psi_i \bar{\psi}_i \qquad \forall i, \tag{4.3}$$

the inverse transformation reads

$$\psi_i = \exp\left\{-\tfrac{i}{2} \sum_j^n \xi_j \bar{\xi}_j W(x_i - x_j)\right\} \xi_i, \quad \bar{\psi}_i = \exp\left\{\tfrac{i}{2} \sum_j^n \xi_j \bar{\xi}_j W(x_i - x_j)\right\} \bar{\xi}_i. \tag{4.4}$$

Thus, the supercharges (3.3) and the Hamiltonian (3.4) can be represented as

$$\mathcal{Q} = \sum_i^n p_i \exp\left\{-\frac{i}{2}\sum_j^n \xi_j\bar{\xi}_j W(x_i-x_j)\right\}\xi_i\,, \qquad \overline{\mathcal{Q}} = \sum_i^n p_i \exp\left\{\frac{i}{2}\sum_j^n \xi_j\bar{\xi}_j W(x_i-x_j)\right\}\bar{\xi}_i\,, \tag{4.5}$$

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^n p_i^2 \,+\, i\sum_{i,j}^n p_i p_j \exp\left\{-\frac{i}{2}\sum_k \xi_k\bar{\xi}_k\big(W(x_i-x_k)-W(x_j-x_k)\big)\right\}\xi_i\bar{\xi}_j W(x_i-x_j)$$

$$+\,\frac{1}{2}\sum_{i,j}^n p_i^2 \xi_i\bar{\xi}_i \xi_j\bar{\xi}_j W'(x_i-x_j). \tag{4.6}$$

Thus, in this frame, the supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ and the Hamiltonian $\mathcal{H}$ acquire long "fermionic tails".

5. Conclusions

We have constructed an $\mathcal{N}=2$ supersymmetric extension of the Ruijsenaars–Schneider system, starting from the bosonic equations of motion (2.1). Our construction is valid for any antisymmetric function W. Only the demand of integrability will restrict this function to the choices known from Ruijsenaars–Schneider models. When re-expressed in terms of standard fermions, the Hamiltonian and the supercharges are natural "relativistic" generalizations of the "non-relativistic" ones.

Let us compare our results with previous attempts on this problem. The work of [11] contains fermionic brackets similar to (3.2) between ψ_i and $\bar{\psi}_j$. However, it does not feature deformed Poisson brackets between fermions and momenta like in (3.2), but rather introduces complicated conjugation properties. Of course, our construction is pure classical and, for the time being, ignores integrability, but we do not foresee obstacles for implementing the latter.

A three-particle Ruijsenaars–Schneider model was proposed in [12]. Its supercharges

$$Q = \sum_{i=1}^3 p_i\xi_i - i\,p_1\xi_1\xi_3\bar{\xi}_3 W(x_1-x_3) - i\,p_2\xi_2\xi_1\bar{\xi}_1 W(x_2-x_1) - i\,p_3\xi_3\xi_2\bar{\xi}_2 W(x_3-x_2)\,,$$

$$\overline{Q} = \sum_{i=1}^3 p_i\bar{\xi}_i + i\,p_1\bar{\xi}_1\xi_3\bar{\xi}_3 W(x_1-x_3) + i\,p_2\bar{\xi}_2\xi_1\bar{\xi}_1 W(x_2-x_1) + i\,p_3\bar{\xi}_3\xi_2\bar{\xi}_2 W(x_3-x_2) \tag{5.1}$$

indeed form an $\mathcal{N}=2$ Poincaré algebra (3.1), and the fermions ξ_i and $\bar{\xi}_j$ obey the standard brackets (4.2). To compare these supercharges to ours one should put them into the form

$$Q = \sum_i^3 p_i\rho_i \qquad \text{and} \qquad \overline{Q} = \sum_i^3 p_i\bar{\rho}_i \tag{5.2}$$

with new fermions

$$\rho_i = \xi_i\left(1 - i\xi_{i-1}\bar{\xi}_{i-1} W(x_i-x_{i-1})\right) \qquad \text{cyclicly in} \quad i=1,2,3\,. \tag{5.3}$$

While the brackets $\{\rho_i,\rho_j\}$, $\{\bar{\rho}_i,\bar{\rho}_j\}$ and $\{\rho_i,\bar{\rho}_j\}$ coincide with the corresponding ones for ψ_i and $\bar{\psi}_i$ from (3.2), the non-covariant structure of the ρ fermions results in completely different brackets $\{p_i,\rho_j\}$ and $\{p_i,\bar{\rho}_j\}$. For example, it follows from (5.3) that $\{p_1,\rho_3\}=0$ in explicit contradiction with $\{p_1,\psi_3\}\neq 0$ from (3.2), when restricted to the three-particle case. We conclude that our model for $n=3$ differs from the one in [12]. Another road to the same conclusion expands the supercharges (4.5) in powers of fermions. Even for three particles, (4.5) produces five-fermion terms, in the contradiction with the ansatz in [12].

There is a number of interesting open issues regarding these extended supersymmetric Ruijsenaars–Schneider models. The list contains prospective integrability, an off-shell superfield Lagrangian formulation, and $\mathcal{N}=4$ generalizations. We plan to clarify some of these points elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] S. Ruijsenaars, H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. 170 (1986) 370.
[2] H.W. Braden, R. Sasaki, The Ruijsenaars–Schneider model, Prog. Theor. Phys. 97 (1997) 1003.
[3] F. Calogero, Solution of a three-body problem in one dimension, J. Math. Phys. 10 (1969) 2191.
[4] F. Calogero, Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971) 419.

[5] A. Polychronakos, Physics and mathematics of Calogero particles, J. Phys. A 39 (2006) 12793, arXiv:hep-th/0607033.

[6] B.S. Shastry, B. Sutherland, Superlax pairs and infinite symmetries in the $1/r^2$ system, Phys. Rev. Lett. 70 (1993) 4029.

[7] L. Brink, A. Turbiner, N. Wyllard, Hidden algebras of the (super)Calogero and Sutherland models, J. Math. Phys. 39 (1998) 1285.

[8] P. Desrosiers, L. Lapointe, P. Mathieu, Supersymmetric Calogero–Moser–Sutherland models and Jack superpolynomials, Nucl. Phys. B 606 (2001) 547, arXiv:hep-th/0103178;
P. Desrosiers, L. Lapointe, P. Mathieu, Jack polynomials in superspace, Commun. Math. Phys. 242 (2003) 331, arXiv:hep-th/0209074.

[9] P. Desrosiers, L. Lapointe, P. Mathieu, Orthogonality of Jack polynomials in superspace, Adv. Math. 212 (2007) 361;
P. Desrosiers, L. Lapointe, P. Mathieu, Evaluation and normalization of Jack superpolynomials, Int. Math. Res. Not. 23 (2012) 5267.

[10] D. Freedman, P. Mende, An exactly solvable N particle system in supersymmetric quantum mechanics, Nucl. Phys. B 344 (1990) 317.

[11] O. Blondeau-Fournier, P. Desrosiers, P. Mathieu, Supersymmetric Ruijsenaars–Schneider model, Phys. Rev. Lett. 114 (2015) 121602.

[12] A. Galajinsky, Ruijsenaars–Schneider three-body models with N=2 supersymmetry, J. High Energy Phys. 1804 (2018) 079, arXiv:1802.08011 [hep-th].

[13] N. Wyllard, (Super)conformal many body quantum mechanics with extended supersymmetry, J. Math. Phys. 41 (2000) 2826, arXiv:hep-th/9910160.

[14] A. Galajinsky, O. Lechtenfeld, K. Polovnikov, $\mathcal{N}=4$ mechanics, WDVV equations and roots, J. High Energy Phys. 0903 (2009) 113, arXiv:0802.4386 [hep-th].

[15] S. Krivonos, O. Lechtenfeld, Many-particle mechanics with $D(2,1;\alpha)$ superconformal symmetry, J. High Energy Phys. 1102 (2011) 042, arXiv:1012.4639 [hep-th].

[16] S. Krivonos, O. Lechtenfeld, A. Provorov, A. Sutulin, Extended supersymmetric Calogero model, Phys. Lett. B 791 (2019) 385, arXiv:1812.10168 [hep-th].

[17] S. Krivonos, O. Lechtenfeld, A. Sutulin, $\mathcal{N}$-extended supersymmetric Calogero models, Phys. Lett. B 784 (2018) 137, arXiv:1804.10825 [hep-th].