

New $\mathcal{N} = 2$ superspace Calogero models

Sergey Krivonos,^a Olaf Lechtenfeld^b and Anton Sutulin^a

^a*Bogoliubov Laboratory of Theoretical Physics, JINR,
141980 Dubna, Russia*

^b*Institut für Theoretische Physik and Riemann Center for Geometry and Physics,
Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany*

E-mail: krivonos@theor.jinr.ru, olaf.lechtenfeld@itp.uni-hannover.de,
sutulin@theor.jinr.ru

ABSTRACT: Starting from the Hamiltonian formulation of $\mathcal{N} = 2$ supersymmetric Calogero models associated with the classical A_n, B_n, C_n and D_n series and their hyperbolic/trigonometric cousins, we provide their superspace description. The key ingredients include n bosonic and $2n(n-1)$ fermionic $\mathcal{N} = 2$ superfields, the latter being subject to a nonlinear chirality constraint. This constraint has a universal form valid for all Calogero models. With its help we find more general supercharges (and a superspace Lagrangian), which provide the $\mathcal{N} = 2$ supersymmetrization for bosonic potentials with arbitrary repulsive two-body interactions.

KEYWORDS: Extended Supersymmetry, Superspaces, Integrable Hierarchies

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1 Introduction

The rational Calogero model [1, 2] and its various generalizations (see [3] for a review) play a special role in mathematical physics. Being an integrable and solvable multi-particle one-dimensional system it appears in many different areas such as fluid mechanics, spin chains, gauge theory and string theory. The standard $\mathcal{N} = 2$ supersymmetric extension of the n -particle Calogero models is based on n scalar bosonic $\mathcal{N} = 2$ superfields with a component content $(1, 2, 1)$ i.e. one physical boson, 2 fermions and one auxiliary field for each particle [4–6]. Being quite satisfactory for $\mathcal{N} = 2$ supersymmetry, this construction fails to reproduce $\mathcal{N} = 4$ supersymmetric Calogero models beyond four particles [7].

Recent progress in the construction of supersymmetric extensions of Calogero models was achieved by adding to the system more fermions as compared to the standard supersymmetrization [4–6]. It was inspired by supersymmetric extensions of the matrix models which, upon reduction or gauge fixing, give rise to the familiar bosonic systems. The superfield approach developed in [8–10] for the rational spin-Calogero models with $\mathcal{N} = 2, 4$ supersymmetry was recently extended to $\mathcal{N} = 2, 4$ supersymmetric hyperbolic Calogero models [11, 12]. However, it suffers from an unclear structure of the bosonic matrix model one has to start from.

In the series of papers [13–15] we developed a different approach. Mainly working in the Hamiltonian formulation, we worked out an ansatz for the supercharges which accommodates all Calogero models associated with the classical A_n, B_n, C_n and D_n Lie algebras and their trigonometric/hyperbolic extensions [15]. Having at hands the Hamiltonian description of \mathcal{N} -extended supersymmetric Calogero models, it is of interest to gain also its superfield formulation, at least in the simplest case of $\mathcal{N} = 2$ supersymmetry.¹ The superspace picture, alternative to the one constructed in [8–12], will help in understanding

¹A superspace description of the \mathcal{N} -extended supersymmetric Euler-Calogero-Moser system has been provided in [16].

the general supersymmetry structure and clarify the role played by the additional matrix fermions. This is the main goal of this work.

The plan of the paper is as follows. In section 2 we review the Hamiltonian description of the supersymmetric Calogero models. Their superfield treatment is performed in section 3 ($A_1 \oplus A_{n-1}$ models) and in section 4 (B_n, C_n, D_n models). The most remarkable result here is a universal nonlinear fermionic chiral supermultiplet which collects all matrix fermions occurring in all super-extended Calogero models. In section 5 we present more general supercharges (and a superspace Lagrangian), which provide an $\mathcal{N} = 2$ supersymmetrization for a bosonic potential $\frac{g^2}{2} \sum_{i < j} f(x_i - x_j)^2$ with an arbitrary function f . We conclude with a short summary and some arguments in favor of the integrability of the constructed supersymmetric systems.

2 Hamiltonian description of $\mathcal{N} = 2$ supersymmetric Calogero models

In the Hamiltonian approach the n -particle supersymmetric Calogero model with $\mathcal{N} = 2$ extended supersymmetry [13–15] features the following degrees of freedom:

- n bosonic coordinates x_i and momenta p_i with $i = 1, \dots, n$,
- $2n$ fermions ψ_i and $\bar{\psi}_i$,
- $2n(n-1)$ fermions ξ_{ij} and $\bar{\xi}_{ij}$ with $\xi_{ii} = \bar{\xi}_{ii} = 0$.

Their non-vanishing Poisson brackets are

$$\{x_i, p_j\} = \delta_{ij}, \quad \{\psi_i, \bar{\psi}_j\} = -i\delta_{ij}, \quad \{\xi_{ij}, \bar{\xi}_{km}\} = -i(1-\delta_{ij})(1-\delta_{km})\delta_{im}\delta_{jk}. \quad (2.1)$$

A central role of our construction take the composite objects Π_{ij} and $\tilde{\Pi}_{ij}$ defined as

$$\Pi_{ij} = (\psi_i - \psi_j)\bar{\xi}_{ij} + (\bar{\psi}_i - \bar{\psi}_j)\xi_{ij} + \sum_{k=1}^n (\xi_{ik}\bar{\xi}_{kj} + \bar{\xi}_{ik}\xi_{kj}), \quad (2.2)$$

$$\tilde{\Pi}_{ij} = 2\delta_{ij}\psi_i\bar{\psi}_i + (\psi_i + \psi_j)\bar{\xi}_{ij} - (\bar{\psi}_i + \bar{\psi}_j)\xi_{ij} + \sum_{k=1}^n (\xi_{ik}\bar{\xi}_{kj} - \bar{\xi}_{ik}\xi_{kj}). \quad (2.3)$$

One may easily check that Π_{ij} and $\tilde{\Pi}_{ij}$ together form an $s(u(n) \oplus u(n))$ algebra,²

$$\begin{aligned} \{\Pi_{ij}, \Pi_{km}\} &= i(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}), & \{\Pi_{ij}, \tilde{\Pi}_{km}\} &= i(\delta_{im}\tilde{\Pi}_{kj} - \delta_{kj}\tilde{\Pi}_{im}), \\ \{\tilde{\Pi}_{ij}, \tilde{\Pi}_{km}\} &= i(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}). \end{aligned} \quad (2.4)$$

Our $\mathcal{N} = 2$ supersymmetric Calogero models of A -type [14, 15] are defined by a generic form of their supercharges,

$$\begin{aligned} Q &= \sum_{i=1}^n p_i \psi_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji}, \\ \bar{Q} &= \sum_{i=1}^n p_i \bar{\psi}_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji}, \end{aligned} \quad (2.5)$$

²We remind that $\sum_i \Pi_{ii} = 0$.

with some function f , to be specified in a moment. Note that $\tilde{\Pi}_{ij}$ does not appear here. These supercharges form an $\mathcal{N} = 2$ super-Poincaré algebra,

$$\{Q, \bar{Q}\} = -2i\mathcal{H} \quad \text{and} \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \quad (2.6)$$

together with the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] \\ & + \frac{\alpha}{2} \sum_{i,j}^n \Pi_{ij} \Pi_{ji}. \end{aligned} \quad (2.7)$$

Here, we abbreviated

$$z_{ij} = x_i - x_j, \quad (2.8)$$

and the constant parameter α and the function f are given as follows,

rational Calogero model	$\alpha = 0, \quad f(z_{ij}) = \frac{1}{z_{ij}} = \frac{1}{x_i - x_j},$
hyperbolic Calogero-Moser model	$\alpha = -1, \quad f(z_{ij}) = \frac{1}{\sinh(z_{ij})} = \frac{1}{\sinh(x_i - x_j)},$
trigonometric Calogero-Moser model	$\alpha = 1, \quad f(z_{ij}) = \frac{1}{\sin(z_{ij})} = \frac{1}{\sin(x_i - x_j)}.$

(2.9)

For the B , C and D -type models, the supercharges take a more complicated generic form (including $\tilde{\Pi}_{ij}$),

$$\begin{aligned} \mathcal{Q} = & \sum_{i=1}^n p_i \psi_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji} \\ & + i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ij} \right] \xi_{ji} \\ & + i \sum_i^n \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right] \psi_i, \\ \bar{\mathcal{Q}} = & \sum_{i=1}^n p_i \bar{\psi}_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji} \\ & - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ij} \right] \bar{\xi}_{ji} \\ & - i \sum_i^n \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right] \bar{\psi}_i. \end{aligned} \quad (2.10)$$

Here,

$$y_{ij} = x_i + x_j, \quad (2.11)$$

and the function f is the same as in (2.9). The supercharges (2.10) form the same $\mathcal{N} = 2$ super-Poincaré algebra (2.6) together with the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] \\ & + \frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ij} \right] \left[(g + \Pi_{ii}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ji} \right] \\ & + \frac{1}{2} \sum_i^n \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right] \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right]. \end{aligned} \quad (2.12)$$

Its bosonic sector reads

$$\mathcal{H}_{bos} = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{g^2}{2} \sum_{i \neq j}^n (f^2(z_{ij}) + f^2(y_{ij})) + \frac{g'^2}{2} \sum_i^n f^2(y_{ii}). \quad (2.13)$$

Due to the presence of only two coupling constants, g and g' , we may describe B , C and D -type models in the rational case and C and D (but not B)-type models in the hyperbolic/trigonometric case.

3 $\mathcal{N} = 2$ superspace $\mathbf{A}_1 \oplus \mathbf{A}_{n-1}$ Calogero models

To provide a superspace description of $\mathcal{N} = 2$ supersymmetric Calogero models one has, firstly, to assemble the physical components x_i , ψ_i , $\bar{\psi}_i$, ξ_{ij} and $\bar{\xi}_{ij}$ into $\mathcal{N} = 2$ superfields.

It immediately follows from the structure of the supercharges Q and \bar{Q} (2.5) that under $\mathcal{N} = 2$ supersymmetry the coordinates x_i transform into the fermions ψ_i and $\bar{\psi}_i$:

$$\delta x_i \equiv \{x_i, i\bar{\epsilon}Q + i\epsilon\bar{Q}\} = i\bar{\epsilon}\psi_i + i\epsilon\bar{\psi}_i. \quad (3.1)$$

Thus, one is left to n bosonic $\mathcal{N} = 2$ superfields \mathbf{x}_i with the components³

$$x_i = \mathbf{x}_i|, \quad \psi_i = -iD\mathbf{x}_i|, \quad \bar{\psi}_i = -i\bar{D}\mathbf{x}_i|, \quad A_i = \frac{1}{2} [\bar{D}, D] \mathbf{x}_i|. \quad (3.2)$$

Concerning the fermionic components ξ_{ij} , $\bar{\xi}_{ij}$, we have no other possibility than to put them into $2n(n-1)$ new fermionic superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ with vanishing diagonal parts, i.e.

$$\boldsymbol{\xi}_{ii} = \bar{\boldsymbol{\xi}}_{ii} = 0 \quad \forall i. \quad (3.3)$$

As $\mathcal{N} = 2$ superfields the $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ contain a lot of components. Hence, they have to be constrained somehow.⁴ The appropriate constraints derive from the explicit form of the

³We use the $\mathcal{N} = 2$ spinor covariant derivatives D and \bar{D} obeying $\{D, \bar{D}\} = 2i\partial_t$ and $\{D, D\} = \{\bar{D}, \bar{D}\} = 0$. We denote by $\mathcal{A}|$ the $\theta = \bar{\theta} = 0$ limit of a superspace expression \mathcal{A} .

⁴It was first mentioned in [17] that $2n(n-1)$ fermionic components can be placed into $n(n-1)$ supermultiplets of type $(0, 2, 2)$.

supercharges Q and \bar{Q} (2.5), which leads to the following supersymmetry transformations of the leading components ξ_{ij} and $\bar{\xi}_{ij}$ of these superfields,

$$\begin{aligned} \delta_Q \xi_{ij} &\sim i\bar{\epsilon} \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} + \xi_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \xi_{ik} - \sum_{k \neq j}^n f(z_{jk}) \xi_{jk} \right) \right. \\ &\quad \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} \right) \xi_{ik} \xi_{kj} \right], \\ \delta_{\bar{Q}} \bar{\xi}_{ij} &\sim i\epsilon \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} + \bar{\xi}_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \bar{\xi}_{ik} - \sum_{k \neq j}^n f(z_{jk}) \bar{\xi}_{jk} \right) \right. \\ &\quad \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} \right) \bar{\xi}_{ik} \bar{\xi}_{kj} \right]. \end{aligned} \quad (3.4)$$

To realize this transformation property we are forced to impose a nonlinear chirality condition on the superfields ξ_{ij} and $\bar{\xi}_{ij}$,

$$\begin{aligned} D\xi_{ij} &= i \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} + \xi_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \xi_{ik} - \sum_{k \neq j}^n f(z_{jk}) \xi_{jk} \right) \right. \\ &\quad \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} \right) \xi_{ik} \xi_{kj} \right], \\ \bar{D}\bar{\xi}_{ij} &= i \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} + \bar{\xi}_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \bar{\xi}_{ik} - \sum_{k \neq j}^n f(z_{jk}) \bar{\xi}_{jk} \right) \right. \\ &\quad \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} \right) \bar{\xi}_{ik} \bar{\xi}_{kj} \right]. \end{aligned} \quad (3.5)$$

This condition leaves in the superfields ξ_{ij} and $\bar{\xi}_{ij}$ only the components

$$\xi_{ij} = \xi_{ij}|, \quad B_{ij} = \bar{D}\xi_{ij}|, \quad \bar{\xi}_{ij} = \bar{\xi}_{ij}|, \quad \bar{B}_{ij} = D\bar{\xi}_{ij}|. \quad (3.6)$$

Finally, to obtain the correct brackets (2.1) for $(\psi_i, \bar{\psi}_i)$ and $(\xi_{ij}, \bar{\xi}_{ij})$ after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must have the form

$$\mathcal{L}_{kin}^\psi = \frac{i}{2} \sum_{i=1}^n (\psi_i \dot{\bar{\psi}}_i - \dot{\psi}_i \bar{\psi}_i) \quad \text{and} \quad \mathcal{L}_{kin}^\xi = \frac{i}{2} \sum_{i,j}^n (\dot{\xi}_{ij} \bar{\xi}_{ji} - \xi_{ij} \dot{\bar{\xi}}_{ji}). \quad (3.7)$$

In $\mathcal{N} = 2$ superspace, this amounts to the free action ($g = 0$)

$$S_0 = \int dt d^2\theta \left[-\frac{1}{2} \sum_{i=1}^n D\mathbf{x}_i \bar{D}\mathbf{x}_i + \frac{1}{2} \sum_{i,j}^n \xi_{ij} \bar{\xi}_{ji} \right] \quad \text{with} \quad d^2\theta \equiv D\bar{D}. \quad (3.8)$$

More interesting is the construction of the interaction terms. Again, some hints come from the transformation properties of the fermions ξ_{ij} and $\bar{\xi}_{ij}$ under \bar{Q} and Q supersymmetry, respectively,

$$\delta_{\bar{Q}} \xi_{ij} \sim i\epsilon g f(z_{ij}) + \dots, \quad \delta_Q \bar{\xi}_{ij} \sim i\bar{\epsilon} g f(z_{ij}) + \dots \quad (3.9)$$

To reproduce such terms in superspace, the unique possibility is to add to the action S_0 (3.8) a term

$$S_{\text{int}} = i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^n f(z_{ij}) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^n f(z_{ij}) \bar{\xi}_{ij}. \quad (3.10)$$

To be supersymmetrically invariant, the integrands in (3.10) must be chiral and antichiral, respectively. It is not too hard to check that this is indeed so: the nonlinear chirality constraint (3.5) implies that

$$D \left(\sum_{i \neq j}^n f(z_{ij}) \xi_{ij} \right) = 0 \quad \text{and} \quad \bar{D} \left(\sum_{i \neq j}^n f(z_{ij}) \bar{\xi}_{ij} \right) = 0. \quad (3.11)$$

Combining all these facts together, we conclude that the superfield action reads

$$S = \int dt d^2\theta \left[-\frac{1}{2} \sum_{i=1}^n D\mathbf{x}_i \bar{D}\mathbf{x}_i + \frac{1}{2} \sum_{i,j}^n \xi_{ij} \bar{\xi}_{ji} \right] + i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^n f(z_{ij}) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^n f(z_{ij}) \bar{\xi}_{ij}, \quad (3.12)$$

where the superfields ξ_{ij} and $\bar{\xi}_{ij}$ are subject to the nonlinear chirality constraint (3.5).

It is important to note that, after passing to new fermionic superfields

$$\lambda_{ij} \equiv f(z_{ij}) \xi_{ij} \quad \text{and} \quad \bar{\lambda}_{ij} \equiv f(z_{ij}) \bar{\xi}_{ij}, \quad (3.13)$$

the nonlinear constraint (3.5) is slightly simplified to

$$D\lambda_{ij} = i \left[\lambda_{ij} \sum_{k \neq i}^n \lambda_{ik} - \lambda_{ij} \sum_{k \neq j}^n \lambda_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j}^n \lambda_{ik} \lambda_{kj} \right], \quad (3.14)$$

$$\bar{D}\bar{\lambda}_{ij} = i \left[\bar{\lambda}_{ij} \sum_{k \neq i}^n \bar{\lambda}_{ik} - \bar{\lambda}_{ij} \sum_{k \neq j}^n \bar{\lambda}_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j}^n \bar{\lambda}_{ik} \bar{\lambda}_{kj} \right].$$

In this form, the constraint has lost any f -dependence, which however will reappear in the action,

$$S = \int dt d^2\theta \left[-\frac{1}{2} \sum_{i=1}^n D\mathbf{x}_i \bar{D}\mathbf{x}_i + \frac{1}{2} \sum_{i,j}^n \frac{\lambda_{ij} \bar{\lambda}_{ji}}{f(z_{ij}) f(z_{ji})} \right] + i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^n \lambda_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^n \bar{\lambda}_{ij}. \quad (3.15)$$

Also, the component Lagrangian, Hamiltonian and Poisson brackets will be more complicated in terms of the composite superfields λ_{ij} and $\bar{\lambda}_{ij}$.

Despite the extremely simple form of the superfield action (3.12), its component version looks quite complicated due to the constraint (3.5). Indeed, after integration over θ in (3.12) we get the off-shell Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{pot}}, \quad \text{where} \quad (3.16)$$

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2} \sum_i^n \left(\dot{x}_i \dot{x}_i + A_i A_i \right) + \frac{i}{2} \sum_i^n \left(\dot{\psi}_i \bar{\psi}_i - \psi_i \dot{\bar{\psi}}_i \right) + \frac{i}{2} \sum_{i,j}^n \left(\dot{\xi}_{ij} \bar{\xi}_{ji} - \xi_{ij} \dot{\bar{\xi}}_{ji} \right) \\ & + \frac{1}{2} \sum_{i,j}^n \left(\xi_{ij} D(\bar{D} \bar{\xi}_{ji}) - \bar{D}(D \xi_{ij}) \bar{\xi}_{ji} - D \xi_{ij} \bar{D} \bar{\xi}_{ji} + B_{ij} \bar{B}_{ji} \right), \end{aligned} \quad (3.17)$$

$$\mathcal{L}_{\text{pot}} = -\frac{g}{2} \sum_{i,j}^n f'(z_{ij}) \left((\psi_i - \psi_j) \bar{\xi}_{ij} + (\bar{\psi}_i - \bar{\psi}_j) \xi_{ij} \right) + i \frac{g}{2} \sum_{i,j}^n f(z_{ij}) (B_{ij} + \bar{B}_{ij}).$$

To eliminate the auxiliary fields A_i and B_{ij} one firstly has to evaluate the terms in the second line of (3.17) by using the constraint (3.5). This is a straightforward but rather tedious calculation. After employing the equations of motion for the auxiliary components we finally obtain the desired result,

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_{i=1}^n \dot{x}_i \dot{x}_i + \frac{i}{2} \sum_{i=1}^n \left(\dot{\psi}_i \bar{\psi}_i - \psi_i \dot{\bar{\psi}}_i \right) + \frac{i}{2} \sum_{i,j}^n \left(\dot{\xi}_{ij} \bar{\xi}_{ji} - \xi_{ij} \dot{\bar{\xi}}_{ji} \right) \\ & - \frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] \\ & - \frac{\alpha}{2} \sum_{i,j}^n \Pi_{ij} \Pi_{ji}. \end{aligned} \quad (3.18)$$

Thus, the superfield action (3.12) with the superfields ξ_{ij} and $\bar{\xi}_{ij}$ subject to the nonlinear chirality constraint (3.5) indeed describes all $\mathcal{N} = 2$ supersymmetric $A_1 \oplus A_{n-1}$ Calogero models.

4 $\mathcal{N} = 2$ superspace B_n , C_n and D_n Calogero models

The supercharges of the $\mathcal{N} = 2$ supersymmetric B , C and D -type Calogero models (2.10) have a more complicated structure than those in (2.5). Therefore, it is expected that the nonlinear chirality constraint for the superfields ξ_{ij} and $\bar{\xi}_{ij}$ are more intricate as well. Indeed, the explicit structure of the supercharges (2.10) uniquely fixes this constraint to be

$$\begin{aligned} D \xi_{ij} = & i \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} - \frac{f'(y_{ij})}{f(y_{ij})} (\psi_i + \psi_j) \xi_{ij} \right. \\ & + \left\{ \left(\frac{f'(y_{ii})}{f(y_{ii})} - f(y_{ii}) \right) \psi_i + \left(\frac{f'(y_{jj})}{f(y_{jj})} + f(y_{jj}) \right) \psi_j \right\} \xi_{ij} \\ & + \xi_{ij} \left(\sum_{k \neq i}^n (f(z_{ik}) + f(y_{ik})) \xi_{ik} - \sum_{k \neq j}^n (f(z_{jk}) + f(y_{jk})) \xi_{jk} \right) \\ & \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} - \frac{f'(y_{ik})}{f(y_{ik})} + \frac{f'(y_{kj})}{f(y_{kj})} \right) \xi_{ik} \xi_{kj} \right], \end{aligned}$$

$$\begin{aligned}
 \overline{D}\bar{\xi}_{ij} = & i \left[-\frac{f'(z_{ij})}{f(z_{ij})}(\bar{\psi}_i - \bar{\psi}_j)\bar{\xi}_{ij} - \frac{f'(y_{ij})}{f(y_{ij})}(\bar{\psi}_i + \bar{\psi}_j)\bar{\xi}_{ij} \right. \\
 & + \left\{ \left(\frac{f'(y_{ii})}{f(y_{ii})} + f(y_{ii}) \right) \bar{\psi}_i + \left(\frac{f'(y_{jj})}{f(y_{jj})} - f(y_{jj}) \right) \bar{\psi}_j \right\} \bar{\xi}_{ij} \\
 & + \bar{\xi}_{ij} \left(\sum_{k \neq i}^n (f(z_{ik}) - f(y_{ik})) \bar{\xi}_{ik} - \sum_{k \neq j}^n (f(z_{jk}) - f(y_{jk})) \bar{\xi}_{jk} \right) \\
 & \left. - \sum_{k \neq i,j}^n \left(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})} - \frac{f'(y_{ik})}{f(y_{ik})} + \frac{f'(y_{kj})}{f(y_{kj})} \right) \bar{\xi}_{ik} \bar{\xi}_{kj} \right]. \quad (4.1)
 \end{aligned}$$

The complicated form of this constraint disappears after passing to the composite superfields

$$\lambda_{ij} = (f(z_{ij}) + f(y_{ij}))\xi_{ij} \quad \text{and} \quad \bar{\lambda}_{ij} = (f(z_{ij}) - f(y_{ij}))\bar{\xi}_{ij}, \quad (4.2)$$

in which it acquires its familiar form (3.14),

$$\begin{aligned}
 D\lambda_{ij} &= i \left[\lambda_{ij} \sum_{k \neq i}^n \lambda_{ik} - \lambda_{ij} \sum_{k \neq j}^n \lambda_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j}^n \lambda_{ik} \lambda_{kj} \right], \\
 \overline{D}\bar{\lambda}_{ij} &= i \left[\bar{\lambda}_{ij} \sum_{k \neq i}^n \bar{\lambda}_{ik} - \bar{\lambda}_{ij} \sum_{k \neq j}^n \bar{\lambda}_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j}^n \bar{\lambda}_{ik} \bar{\lambda}_{kj} \right].
 \end{aligned}$$

Finally, the superfield action reads

$$\begin{aligned}
 S = & \int dt d^2\theta \left[-\frac{1}{2} \sum_{i=1}^n D\mathbf{x}_i \overline{D}\mathbf{x}_i + \frac{1}{2} \sum_{i,j}^n \xi_{ij} \bar{\xi}_{ji} - \frac{1}{2} g' h(y_{ii}) \right] \\
 & + i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^n (f(z_{ij}) + f(y_{ij})) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^n (f(z_{ij}) + f(y_{ij})) \bar{\xi}_{ij}, \quad (4.3)
 \end{aligned}$$

where

$$h'(y_{ii}) = f(y_{ii}). \quad (4.4)$$

Compared to the action of the $A_1 \oplus A_{n-1}$ Calogero models (3.15), only the term $\frac{1}{2} g' \int dt d^2\theta h(y_{ii})$ carrying the new coupling constant g' appears in the action (4.3). All other terms just mimic those in (3.15).

It is a matter of straightforward but tedious calculations to check that, after excluding the auxiliary fields by their equations of motion, the final Lagrangian acquires the expected form

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{2} \sum_{i=1}^n \dot{x}_i \dot{x}_i + \frac{i}{2} \sum_{i=1}^n (\dot{\psi}_i \bar{\psi}_i - \psi_i \dot{\bar{\psi}}_i) + \frac{i}{2} \sum_{i,j}^n (\dot{\xi}_{ij} \bar{\xi}_{ji} - \xi_{ij} \dot{\bar{\xi}}_{ji}) \\
 & - \frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right]
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ij} \right] \left[(g + \Pi_{ii}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \tilde{\Pi}_{ji} \right] \\
& -\frac{1}{2} \sum_i^n \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right] \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \tilde{\Pi}_{ii} \right]. \quad (4.5)
\end{aligned}$$

Thus, the superfield action (4.3) with the superfields ξ_{ij} and $\bar{\xi}_{ij}$ subject to the nonlinear chirality constraint (4.1) indeed describes the $\mathcal{N} = 2$ supersymmetric B , C and D -type Calogero models.⁵

5 New $\mathcal{N} = 2$ supersymmetric n -particle models

Our point of departure was the explicit form of the $\mathcal{N} = 2$ supercharges for the Calogero models (2.5) and (2.10) constructed in [14, 15]. In all cases we considered the function f to be not arbitrary but to be chosen from the list (2.9). Indeed, for generic f the supercharges (2.5) do not form the closed algebra (2.6). As a consequence, the nonlinear chirality condition (3.5) is not self-consistent for an arbitrary function f (i.e. D acting on the r.h.s. of (3.5) does not vanish). On the other hand, the universal chirality constraint (3.14) is self-consistent. This implies a weaker chirality condition on ξ_{ij} and $\bar{\xi}_{ij}$:⁶

$$\begin{aligned}
D\xi_{ij} &= i \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} + \xi_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \xi_{ik} - \sum_{k \neq j}^n f(z_{jk}) \xi_{jk} \right) \right. \\
&\quad \left. + \sum_{k \neq i, j}^n \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \xi_{ik} \xi_{kj} \right], \quad (5.1) \\
\bar{D}\bar{\xi}_{ij} &= i \left[-\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} + \bar{\xi}_{ij} \left(\sum_{k \neq i}^n f(z_{ik}) \bar{\xi}_{ik} - \sum_{k \neq j}^n f(z_{jk}) \bar{\xi}_{jk} \right) \right. \\
&\quad \left. + \sum_{k \neq i, j}^n \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \bar{\xi}_{ik} \bar{\xi}_{kj} \right].
\end{aligned}$$

One may check that this nonlinear chirality constraint is perfectly self-consistent for an arbitrary function f .

Now, let us again start from the superspace action (3.8), but where the fermionic superfields ξ_{ij} and $\bar{\xi}_{ij}$ must obey the constraint (5.1). Passing to the components and eliminating the auxiliary components one arrives at

$$\begin{aligned}
\mathcal{Q} &= \sum_{i=1}^n p_i \psi_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \bar{\xi}_{ij} \right. \\
&\quad \left. + \sum_{k \neq i, j}^n \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \xi_{ik} \bar{\xi}_{kj} \right] \xi_{ji},
\end{aligned}$$

⁵Except again for B -type models in the trigonometric/hyperbolic case.

⁶For the sake of simplicity we consider here the A -type case, in which the $(\lambda_{ij}, \bar{\lambda}_{ij})$ and $(\xi_{ij}, \bar{\xi}_{ij})$ are related as in (3.13).

$$\bar{\mathcal{Q}} = \sum_{i=1}^n p_i \bar{\psi}_i - i \sum_{i \neq j}^n \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \xi_{ij} + \sum_{k \neq i,j}^n \frac{f(z_{ik}) f(z_{ij})}{f(z_{kj})} \bar{\xi}_{ik} \xi_{kj} \right] \bar{\xi}_{ji}. \quad (5.2)$$

For the functions listed in (2.9) these supercharges coincide with the ones in (2.5). However, the supercharges (5.2) generate the $\mathcal{N} = 2$ super-Poincaré algebra (2.6) for an arbitrary function f . It is not too hard to find the bosonic part of the new Hamiltonian,

$$\mathcal{H}_{\text{bos}} = \frac{1}{2} \sum_i^n p_i^2 + \frac{g^2}{2} \sum_{i \neq j}^n f^2(z_{ij}). \quad (5.3)$$

Thus, the supercharges (5.2) provide an $\mathcal{N} = 2$ supersymmetrization of a wide class of multi-particle systems with a bosonic Hamiltonian of the type (5.3). A detailed analysis of such models will be given elsewhere.

6 Conclusions

In this paper we have provided a superspace description of the $\mathcal{N} = 2$ supersymmetric Calogero models, rational as well as trigonometric/hyperbolic, associated with the classical A_n , B_n , C_n and D_n Lie algebras. We presented a minimal superfield content accommodating the $2n^2$ fermions for the $\mathcal{N} = 2$ supersymmetric n -particle model. As $2n$ fermions accompany the n bosonic coordinates in general bosonic $\mathcal{N} = 2$ superfields, the remaining $2n(n-1)$ fermions must be put into additional fermionic $\mathcal{N} = 2$ superfields, which have to be constrained such as to describe those fermions alone. The nonlinear chirality condition (3.14) written in terms of composite superfields solves this task. These composite fermionic $\mathcal{N} = 2$ superfields make the constraint look simple and universal but complicate the Lagrangian. In terms of the fundamental fermions it is the other way around. We finally presented more general supercharges (and the superspace Lagrangian) which provides an $\mathcal{N} = 2$ supersymmetrization of bosonic n -particle systems with an arbitrary repulsive two-body interaction.

One might criticize that the approach presented here is unnecessarily complicated, because all $\mathcal{N} = 2$ supersymmetric Calogero models can be more or less straightforwardly formulated in the standard fashion employing the minimum of $2n$ fermions [4]. This, however, is no longer the case with $\mathcal{N} > 2$ supersymmetric Calogero models, where our treatment with additional fermions becomes essential. Hence, we consider our results here as a preparation for attacking Calogero systems with more supersymmetry in a superspace setting. The main excuse for presenting of our $\mathcal{N} = 2$ results is the universal form of the nonlinear chirality constraint together with the almost trivial generalization of the supercharges to $\mathcal{N} = 4$ supersymmetry [15] which will further the superspace construction of $\mathcal{N} \geq 4$ Calogero models.

There is no doubt that the importance of Calogero-type models is related with their integrability. In the standard case when $\mathcal{N} = 2$ supersymmetrization of n -particle models

involves $2n$ fermionic degrees of freedom the issue of integrability has been considered, for example, in [5, 6]. In our case we deal with many more fermionic degrees of freedom, namely $2n^2$ instead of $2n$. Nevertheless, the L - M pair, at least for rational Calogero models ($\alpha = 0, f(z_{ij}) = \frac{1}{x_i - x_j}$ in (2.9)), has the quite simple form

$$\begin{aligned} L_{ij} &= \delta_{ij} p_j - i(1 - \delta_{ij}) \frac{g + \Pi_{jj} - \Pi_{ij}}{x_i - x_j}, \\ M_{ij} &= -i \delta_{ij} \sum_{k \neq i} \frac{g + \Pi_{kk} - \Pi_{ik}}{(x_i - x_k)^2} + i(1 - \delta_{ij}) \frac{g + \Pi_{jj} - \Pi_{ij}}{(x_i - x_j)^2} \end{aligned} \quad (6.1)$$

such that

$$\frac{d}{dt} L_{ij} \equiv \{L_{ij}, H\} = \sum_k (L_{ik} M_{kj} - M_{ik} L_{kj}) \equiv [L, M]_{ij}. \quad (6.2)$$

This construction is quite similar to those considered in [5, 6] and [12].

Clearly, the independent conserved currents read

$$\mathcal{H}_k = \text{Tr} \left(L^k \right), \quad k = 1, 2, \dots, n. \quad (6.3)$$

Additional currents can be constructed if we note that

$$\frac{d}{dt} \Pi_{ij} \equiv \{\Pi_{ij}, H\} = [\Pi, M]_{ij}. \quad (6.4)$$

Therefore, any operator that can be written as the trace of a polynomial function \mathcal{F} depending only on the matrices L and Π is conserved,

$$\frac{d}{dt} \text{Tr} \mathcal{F}(L, \Pi) = 0. \quad (6.5)$$

Keeping in mind that our Hamiltonians (2.7) depend on fermionic degrees of freedom only through the composites Π_{ij} , one expects that the above set of conserved currents contains all currents in involution, as needed for integrability. This will be a strong indication in favor of complete integrability of the constructed systems. However, a detailed discussion of integrability is out of the scope of the present paper.

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A Self-consistency of the nonlinear chirality constraint

The nonlinear chirality constraint reads

$$\frac{1}{i} D \lambda_{ij} = \lambda_{ij} \sum_{k \neq i} \lambda_{ik} - \lambda_{ij} \sum_{k \neq j} \lambda_{jk} + (1 - \delta_{ij}) \sum_{k \neq i, j} \lambda_{ik} \lambda_{kj}, \quad \lambda_{ii} = 0, \quad (A.1)$$

$$\frac{1}{i} \bar{D} \bar{\lambda}_{ij} = \bar{\lambda}_{ij} \sum_{k \neq i} \bar{\lambda}_{ik} - \bar{\lambda}_{ij} \sum_{k \neq j} \bar{\lambda}_{jk} + (1 - \delta_{ij}) \sum_{k \neq i, j} \bar{\lambda}_{ik} \bar{\lambda}_{kj}, \quad \bar{\lambda}_{ii} = 0. \quad (A.2)$$

Let us prove that (A.1) is self-consistent. For this we act with D on it. The l.h.s. is identically zero (since $D^2 = 0$), while the r.h.s. should vanish by virtue of the constraint itself.

$$\begin{aligned}
 \text{r.h.s.} &= D\lambda_{ij} \sum_{k \neq i} \lambda_{ik} - \lambda_{ij} \sum_{k \neq i} D\lambda_{ik} - D\lambda_{ij} \sum_{k \neq j} \lambda_{jk} + \lambda_{ij} \sum_{k \neq j} D\lambda_{jk} \\
 &\quad + (1 - \delta_{ij}) \sum_{k \neq i, j} (D\lambda_{ik} \lambda_{kj} - \lambda_{ik} D\lambda_{kj}) \\
 &= \left(\lambda_{ij} \sum_{n \neq i} \lambda_{in} - \lambda_{ij} \sum_{n \neq j} \lambda_{jn} + (1 - \delta_{ij}) \sum_{n \neq i, j} \lambda_{in} \lambda_{nj} \right) \left[\sum_{k \neq i} \lambda_{ik} - \sum_{k \neq j} \lambda_{jk} \right] \\
 &\quad - \lambda_{ij} \sum_{k \neq i} \left(\lambda_{ik} \sum_{n \neq i} \lambda_{in} - \lambda_{ik} \sum_{n \neq k} \lambda_{kn} + (1 - \delta_{ik}) \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \right) \\
 &\quad + \lambda_{ij} \sum_{k \neq j} \left(\lambda_{jk} \sum_{n \neq j} \lambda_{jn} - \lambda_{jk} \sum_{n \neq k} \lambda_{kn} + (1 - \delta_{jk}) \sum_{n \neq j, k} \lambda_{jn} \lambda_{nk} \right) \\
 &\quad + (1 - \delta_{ij}) \sum_{k \neq i, j} \left(\lambda_{ik} \sum_{n \neq i} \lambda_{in} - \lambda_{ik} \sum_{n \neq k} \lambda_{kn} + (1 - \delta_{ik}) \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \right) \lambda_{kj} \\
 &\quad - (1 - \delta_{ij}) \sum_{k \neq i, j} \left(\lambda_{kj} \sum_{n \neq k} \lambda_{kn} - \lambda_{kj} \sum_{n \neq j} \lambda_{jn} + (1 - \delta_{kj}) \sum_{n \neq j, k} \lambda_{kn} \lambda_{nj} \right) \lambda_{ik}.
 \end{aligned} \tag{A.3}$$

Let us first collect all terms proportional to $(1 - \delta_{ij})$,

$$\begin{aligned}
 &\sum_{n \neq i, j} \lambda_{in} \lambda_{nj} \sum_{k \neq i} \lambda_{ik} - \sum_{n \neq i, j} \lambda_{in} \lambda_{nj} \sum_{k \neq j} \lambda_{jk} + \sum_{k \neq i, j} \lambda_{ik} \sum_{n \neq i} \lambda_{in} \lambda_{kj} \\
 &\quad - \sum_{k \neq i, j} \lambda_{ik} \sum_{n \neq k} \lambda_{kn} \lambda_{kj} + \sum_{k \neq i, j} \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \lambda_{kj} - \sum_{k \neq i, j} \lambda_{kj} \sum_{n \neq k} \lambda_{kn} \lambda_{ik} \\
 &\quad + \sum_{k \neq i, j} \lambda_{kj} \sum_{n \neq j} \lambda_{jn} \lambda_{ik} - \sum_{k \neq i, j} \sum_{n \neq k, j} \lambda_{kn} \lambda_{nj} \lambda_{ik}.
 \end{aligned} \tag{A.4}$$

The fourth and sixth terms cancel each other out. Then, replacing the indices in the first and second terms as $k \leftrightarrow n$ makes them cancel with the third and seventh terms. Thus, after replacing the indices in the first term as $k \leftrightarrow n$, we remain with

$$\sum_{n \neq i, j} \sum_{k \neq i, n} \lambda_{ik} \lambda_{kn} \lambda_{nj} - \sum_{k \neq i, j} \sum_{n \neq k, j} \lambda_{ik} \lambda_{kn} \lambda_{nj}. \tag{A.5}$$

A sum over k in the first term can be split into two pieces, corresponding to $k = j$ or $k \neq j$, yielding

$$\sum_{n \neq i, j} \sum_{k \neq i, n} \lambda_{ik} \lambda_{kn} \lambda_{nj} = \sum_{n \neq i, j} \lambda_{ij} \lambda_{jn} \lambda_{nj} + \sum_{n \neq i, j} \sum_{k \neq i, j, n} \lambda_{ik} \lambda_{kn} \lambda_{nj}. \tag{A.6}$$

Performing the analogous splitting for the index n in the second term of (A.5), $n = i$ or $n \neq i$, one finds

$$\sum_{k \neq i, j} \sum_{n \neq k, j} \lambda_{ik} \lambda_{kn} \lambda_{nj} = \sum_{k \neq i, j} \lambda_{ik} \lambda_{ki} \lambda_{ij} + \sum_{k \neq i, j} \sum_{n \neq i, j, k} \lambda_{ik} \lambda_{kn} \lambda_{nj}. \tag{A.7}$$

After relabeling $k \leftrightarrow n$, the second terms on the r.h.s. of (A.6) and (A.7) become identical. Hence, their contribution to (A.5) cancels,

$$\sum_{k \neq i, j} \sum_{n \neq i, j, k} \left(\lambda_{in} \lambda_{nk} \lambda_{kj} - \lambda_{ik} \lambda_{kn} \lambda_{nj} \right) = 0, \quad (\text{A.8})$$

so we are left with the first terms on the r.h.s. of (A.6) and (A.7),

$$\lambda_{ij} \sum_{n \neq i, j} \left(\lambda_{jn} \lambda_{nj} - \lambda_{in} \lambda_{ni} \right). \quad (\text{A.9})$$

Let us secondly analyze all terms in (A.3) multiplying λ_{ij} and add to them (A.9), which is of the same form,

$$\begin{aligned} & \sum_{n \neq i} \lambda_{in} \sum_{k \neq i} \lambda_{ik} - \sum_{n \neq i} \lambda_{in} \sum_{k \neq j} \lambda_{jk} - \sum_{n \neq j} \lambda_{jn} \sum_{k \neq i} \lambda_{ik} + \sum_{n \neq j} \lambda_{jn} \sum_{k \neq j} \lambda_{jk} \\ & - \sum_{k \neq i} \lambda_{ik} \sum_{n \neq i} \lambda_{in} + \sum_{k \neq i} \lambda_{ik} \sum_{n \neq k} \lambda_{kn} - \sum_{k \neq i} \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \\ & + \sum_{k \neq j} \lambda_{jk} \sum_{n \neq j} \lambda_{jn} - \sum_{k \neq j} \lambda_{jk} \sum_{n \neq k} \lambda_{kn} + \sum_{k \neq j} \sum_{n \neq j, k} \lambda_{jn} \lambda_{nk} \\ & + \sum_{k \neq i, j} \lambda_{jk} \lambda_{kj} - \sum_{k \neq i, j} \lambda_{ik} \lambda_{ki}. \end{aligned} \quad (\text{A.10})$$

It remains to show that these twelve terms combine to zero. The sum of the first five term cancels with the eighth term. Rewriting the sixth term as a sum of two pieces, separating $n = i$ and $n \neq i$,

$$\sum_{k \neq i} \lambda_{ik} \sum_{n \neq k} \lambda_{kn} = \sum_{k \neq i} \lambda_{ik} \lambda_{ki} + \sum_{k \neq i} \lambda_{ik} \sum_{n \neq k, i} \lambda_{kn}, \quad (\text{A.11})$$

one may observe it to cancel with the seventh and the twelfth term. Analogously, the ninth, tenth and eleventh term add to zero. Nothing remains. This completes the proof.

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