Liouville description of conical defects in dS$_4$, Gibbons-Hawking entropy as modular entropy, and dS$_3$ holography

Cesar Arias,[a,b] **Felipe Diaz,**[c] **Rodrigo Olea**[c] **and Per Sundell**[c,d]

[a]*Department of Mathematics, University of California,*
Davis CA 95616, U.S.A.

[b]*Riemann Center for Geometry and Physics, Leibniz Universität Hannover,*
Appelstraße 2, 30167 Hannover, Germany

[c]*Departamento de Ciencias Físicas, Universidad Andres Bello,*
Sazié 2212, Piso 7, Santiago de Chile

[d]*Department of Physics, Nanjing University,*
22 Hankou Road, Nanjing, China

E-mail: carias@math.ucdavis.edu, f.diazmartinez@uandresbello.edu,
rodrigo.olea@unab.cl, per.anders.sundell@gmail.com

ABSTRACT: We model the back-reaction of a static observer in four-dimensional de Sitter spacetime by means of a singular $\mathbb{Z}_q$ quotient. The set of fixed points of the $\mathbb{Z}_q$ action consists of a pair of codimension two minimal surfaces given by 2-spheres in the Euclidean geometry. The introduction of an orbifold parameter $q > 1$ permits the construction of an effective action for the bulk gravity theory with support on each of these minimal surfaces. The effective action corresponds to that of Liouville field theory on a 2-sphere with a finite vacuum expectation value of the Liouville field. The intrinsic Liouville theory description yields a thermal Cardy entropy that we reintrepret as a modular free energy at temperature $T = q^{-1}$, whereupon the Gibbons-Hawking entropy arises as the corresponding modular entropy. We further observe that in the limit $q \to \infty$ the four-dimensional geometry reduces to that of global dS$_3$ spacetime, where the two original minimal surfaces can be mapped to the future and past infinities of dS$_3$ by means of a double Wick rotation. In this limit, the Liouville theories on the minimal surfaces become boundary theories at zero temperature whose total central charge equals that computed using the dS$_3$/CFT$_2$ correspondence.

Contents

1 Introduction

The non-trivial topology of de Sitter (dS) spacetime comprises two disconnected spacelike boundaries and causally disconnected interior regions. Recently, it has been argued [1] that the Gibbons-Hawking entropy of dS spacetime [2] arises from the entanglement between the past and future conformal infinities or, alternatively, from the entanglement between two antipodal and causally disconnected bulk observers located at opposites Rindler wedges of the dS interior.

One of the central ideas behind the above argument is that in order to measure any observer-dependent quantity, in particular the thermal properties of the dS cosmological horizon, one has to go beyond the standard probe approximation of a static observer. In other words, the observer back-reaction should be taken into account.

Following this idea and motivated by the conically singular geometries induced by point particles in three dimensions [3–5], we model the back-reaction of a static observer in dS$_4$ spacetime via the quotient dS$_4$/$\mathbb{Z}_q$. That is, we think of observers that back-react with the background geometry as inducing codimesion two defects that correspond to the fixed points of the $\mathbb{Z}_q$ action. In this sense, we treat the orbifold dS$_4$/$\mathbb{Z}_q$ as the fundamental manifold on which the gravity theory is formulated and think of the dS$_4$ spacetime only as

a smooth limit of it. When such back-reaction is taken into account, we shall refer to the static observer as a *massive observer*, and we shall think of the $q \to 1$ limit as its massless probe limit in which one recovers the original, non-singular dS$_4$ geometry.

The aim of this note is to show that massive observers in dS$_4$ admits an intrinsic description in terms of a two-dimensional conformal field theory. We shall argue that the introduction of an orbifold parameter $q > 1$ permits to build up a reduced two-dimensional action functional with support on the pair of codimension two minimal surfaces that define the set of fixed points of the $\mathbb{Z}_q$ action. Each of these minimal and tensionful surfaces have the topology of a 2-sphere in the Euclidean geometry and they can be formally thought of as the "worldvolume" of a massive observer, whose massless limit is equivalent to the tensionless limit $q \to 1$. As we shall argue, the resulting effective two-dimensional Euclidean action can be identified with a Liouville theory on a 2-sphere, in which the Liouville field acquires q-dependent vacuum expectation value.

The correspondence between the effective action of a massive observer and the Liouville theory action links the gravitational parameters, namely the dS$_4$ radius ℓ and the four-dimensional Newton's constant G_4, with the Liouville coupling constant $\gamma^2 \sim \hbar$. This relation results in a semiclassical central charge given by

$$c_q = \left(1 - \frac{1}{q}\right) \frac{3\ell^2}{G_4} \,. \tag{1.1}$$

This q-dependent central charge arguably encodes degrees of freedom associated to a massive observer which are not present in the massless limit $q \to 1$. Consequently and by means of the thermal Cardy formula, the central charge (1.1) predicts a Cardy entropy that equals a modular free energy whose corresponding modular entropy correctly reproduces the Gibbons-Hawking area law.

We conclude by observing that in the $q \to \infty$ limit of the quotient dS$_4/\mathbb{Z}_4$, the four-dimensional geometry reduces to the global geometry of dS$_3$, where the two minimal surfaces of the former can be mapped — via a double Wick rotation — to the two conformal boundaries of the latter. In this limit, the two Liouville theories on the bulk minimal surfaces become boundary theories (one for each boundary) at zero temperature. Moreover, upon taking the $q \to \infty$ limit, the two dS$_3$ boundaries inherit a central charge from the Liouville theory on the corresponding minimal surface in one higher dimension. As we shall see, the total central charge of the two boundaries reproduces exactly the central charge derived in the context of the dS$_3$/CFT$_2$ correspondence [6–12].

2 Static observers in dS$_4$

In four dimensions, de Sitter spacetime (dS$_4$) can be viewed as a four-dimensional timelike hypersurface embedded in five-dimensional Minkowski space $\mathcal{M}^{1,4}$. Taking the embedding coordinates to be $X^\mu \in \mathcal{M}^{1,4}$, $\mu = 0, \ldots, 4$, and considering the Minkowski metric

$$ds^2_{\mathcal{M}^{1,4}} = -(dX^0)^2 + \sum_{1 \leq i \leq 4} (dX^i)^2 \,, \tag{2.1}$$

the dS$_4$ hypersurface is defined by

$$X_\mu X^\mu = \ell^2 \,, \tag{2.2}$$

where ℓ^2 is the dS$_4$ radius. The hyperboloid (2.2) has the topology of $\mathbb{R} \times S^3$ with manifest $O(4,1)$ symmetries.

2.1 Massless probe observers

The standard description of a static observer in dS$_4$ is obtained by parametrizing the embedding coordinates as

$$X^0 = \sqrt{\ell^2 - \hat{r}^2}\sinh(\hat{t}/\ell)\,, \quad X^1 = \sqrt{\ell^2 - \hat{r}^2}\cosh(\hat{t}/\ell)\,, \quad X^i = \hat{r}\hat{y}_i\,, \;\; 2 \le i \le 4\,, \tag{2.3}$$

where the $\hat{y}_i$ denote the coordinates of the unit 2-sphere. The resulting line element

$$ds^2 = -\left(1 - \frac{\hat{r}^2}{\ell^2}\right)d\hat{t}^2 + \frac{d\hat{r}^2}{1 - \frac{\hat{r}^2}{\ell^2}} + \hat{r}^2 d\Omega_2^2\,, \tag{2.4}$$

where the radial coordinate runs from $0 \le \hat{r} < \ell$ and $d\Omega_2^2$ is the metric on the unit 2-sphere.

The time-independent metric (2.4) describes the worldline of a *single* static observer located at the origin $\hat{r} = 0$. The observer is causally connected with only part of the full spacetime. Such region is dubbed the Rindler wedge (or static patch) of the observer, and its boundary defines an observer-dependent cosmological horizon $\mathcal{H}$. This has the fix time topology of a 2-sphere and is located at $\hat{r} = \ell$.

In the Euclidean vacuum, a static observer detects a temperature and a corresponding Gibbons-Hawking entropy [2] given by

$$T_{\rm dS} = \frac{1}{2\pi\ell}\,, \quad \mathcal{S}_{\rm dS} = \frac{\pi\ell^2}{G_4}\,. \tag{2.5}$$
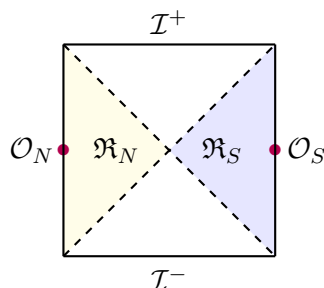
2.2 Massive observers and antipodal defects

The above characterization of a static observer in dS spacetime considers the observer as a massless probe object. Here, instead, we treat an observer as a massive object which modify the local geometry of the spacetime; we propose to model the back-reaction of such massive observer by means of a singular $\mathbb{Z}_q$ quotient. This construction, which we shall now briefly review, has been spelled out in full detail in [1].

To begin with, we note that the constraint (2.2) can be alternatively solved by parameterizing the embedding coordinates as

$$\begin{aligned} X_0 &= \sqrt{\ell^2 - \xi^2}\cos\theta\,\sinh(t/\ell)\,, & X_1 &= \sqrt{\ell^2 - \xi^2}\cos\theta\,\cosh(t/\ell)\,, \\ X_2 &= \xi\cos\theta\,, \quad X_3 = \ell\sin\theta\cos\phi\,, & X_4 &= \ell\sin\theta\sin\phi\,, \end{aligned} \tag{2.6}$$

where

$$-\infty < t < \infty\,, \quad -\ell < \xi < \ell\,, \quad 0 \le \theta \le \pi\,, \quad 0 \le \phi < 2\pi\,. \tag{2.7}$$

Figure 1. Penrose diagram of dS$_4$, with coordinates (τ, Θ). In the conformal time $\tau \in [-\pi/2, +\pi/2]$, the future and past infinities $\mathcal{I}^{\pm}$ are located at $\tau = \pm \pi/2$. The global polar coordinates $\Theta \in [0, \pi]$ defines the north and south poles by the points $\Theta = 0, \pi$, respectively. The metric (2.8) covers the two Rindler wedges of the dS$_4$ interior: $\mathfrak{R}_N = \{0 \leq \theta < \pi/2\}$ and $\mathfrak{R}_S = \{\pi/2 < \theta \leq \pi\}$. The location of the two antipodal observers $\mathcal{O}_N$ and $\mathcal{O}_S$ defined in (2.9) coincides with the global north and south poles $\Theta = \theta = 0$ and $\Theta = \theta = \pi$, respectively.

The resulting dS$_4$ line element, that we shall simply denote by g_4, is

$$g_4 = \ell^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + \cos^2 \theta \left[- \left(1 - \frac{\xi^2}{\ell^2} \right) dt^2 + \frac{d\xi^2}{1 - \frac{\xi^2}{\ell^2}} \right] . \tag{2.8}$$

The metric (2.8) has the warped product form $S^2 \times_w \mathrm{dS}_2^{\pm}$, where the 2-sphere has radius ℓ and dS$_2^{\pm}$ denotes the radially extended dS$_2$ space, with the extended radial coordinate $\xi \in (-\ell, \ell)$, as indicated in (2.7). This geometry describes the worldline of two antipodal static observers

$$\mathcal{O}_N := (\theta = 0, \xi = 0) \in \mathfrak{R}_N \,, \qquad \mathcal{O}_S := (\theta = \pi, \xi = 0) \in \mathfrak{R}_S \,, \tag{2.9}$$

which are causally disconnected (as any light ray can not be sent from one observer into the other). The foliation (2.8) covers the union $\mathfrak{R}_N \cup \mathfrak{R}_S$ of both northern and southern Rindler wedges, as depicted in figure 1.

In order to incorporate the observers back-reaction, one next deforms the S^2 sector in (2.8) by performing a $S^2/\mathbb{Z}_q$ orbifold. This is done via the discrete identification $\phi \sim \phi + \frac{2\pi}{q}$, with an orbifold parameter $q > 1$. The four-dimensional orbifold $\widehat{\mathrm{dS}}_4 := \mathrm{dS}_4/\mathbb{Z}_q$ is then endowed with the metric

$$\widehat{g}_4 = \ell^2 g_{\mathrm{spindle}} + w^2 g_2^{\pm} \,, \tag{2.10}$$

where the warp factor $w = \cos \theta$ satisfy the holonomy conditions $w|_{0,\pi} = 1$ and $w'|_{0,\pi} = 0$, and

$$g_{\mathrm{spindle}} = d\theta^2 + \frac{\sin^2 \theta}{q^2} \, d\phi^2 \,, \qquad g_2^{\pm} = - \left(1 - \frac{\xi^2}{\ell^2} \right) dt^2 + \frac{d\xi^2}{1 - \frac{\xi^2}{\ell^2}} \,. \tag{2.11}$$

The azimuthal identification deforms the S^2 geometry into that of a Thurston's *spindle* [13]. The latter geometry has two antipodal conical singularities at the points $\theta = 0, \pi$, which are precisely the locations of the two static observers (2.9). We interpret

these singularities as the response of the background geometry to the presence of a massive observer, with a mass proportional to $(q-1)$.

The set of fixed points under the $\mathbb{Z}_q$ action

$$\mathcal{F} = \Sigma_N \cup \Sigma_S\,, \qquad \Sigma_N := \widehat{\mathrm{dS}}_4\big|_{\theta=0}\,, \qquad \Sigma_S := \widehat{\mathrm{dS}}_4\big|_{\theta=\pi}\,, \tag{2.12}$$

defines two antipodal, codimension two surfaces Σ_N and Σ_S, both endowed with the induced metric

$$h = \widehat{g}_4\big|_{\theta=0,\pi} = g_2^{\pm}\,. \tag{2.13}$$

In what follows, we shall refer to the submanifolds (Σ_N, h) and (Σ_S, h) as *defects*.

In terms of the gravity action and in order to have a well defined variational principle, the two conical singularities are resolved by adding to the Einstein-Hilbert action a pair of Nambu-Goto terms with support on Σ_N and Σ_S [14]

$$I[\widehat{\mathrm{dS}}_4] = \frac{1}{16\pi G_4} \int\limits_{\widehat{\mathrm{dS}}_4\backslash(\Sigma_N\cup\Sigma_S)} d^4x\sqrt{-g}\left(R - \frac{6}{\ell^2}\right) - \mathcal{T}_q\int_{\Sigma_N}d^2y\sqrt{-h} - \mathcal{T}_q\int_{\Sigma_S}d^2y\sqrt{-h}\,. \tag{2.14}$$

In the above, the support of the first integral excludes the location of the defects Σ_N and Σ_S. The two Nambu-Goto terms are coupled through the tension

$$\mathcal{T}_q = \frac{1}{4G_4}\left(1 - \frac{1}{q}\right)\,, \tag{2.15}$$

where the limit $q \to 1$ corresponds to the tensionless limit in which one recovers the usual Einstein-Hilbert action on the smooth dS_4 geometry.

Hence, by construction, Σ_N and Σ_S are codimension two minimal surfaces with an induced stress-energy tensor given by

$$T_{ij} = \mathcal{T}_q\,h_{ij}\,. \tag{2.16}$$

The localized stress energy tensor (2.16) is a strong sign of the existence of an underlying field theory defined on the two minimal surfaces. As we shall next argue, this theory corresponds to an Eucliedan Liouville theory on a 2-sphere.

3 Liouville theory description of a massive observer

In this section, we construct an effective two-dimensional action with support on the codimension two minimal surfaces Σ_N and Σ_S. These surfaces are the set of fixed points of the $\mathbb{Z}_q$ action. Each of them contain the worldline of one of the massive observers $\mathcal{O}_N$ and $\mathcal{O}_S$, and they both have the topology of a 2-sphere in the Euclidean geometry, *viz.*

$$\Sigma_N^E \cong \Sigma_S^E \cong S^2\,, \tag{3.1}$$

with induced metric $d\Omega_2^2$ (which corresponds to the analytic continuation of (2.13)). In the above, the label "E" denotes Euclidean geometry. Hereafter, we shall drop this label when is clear from context.

3.1 Effective two-dimensional action

To begin with, we recall that the total Euclidean gravity action (2.14) on the conically singular manifold $\widehat{\mathrm{dS}}_4 := \mathrm{dS}_4/\mathbb{Z}_q$ consists of a bulk piece plus a pair of two-dimensional Nambu-Goto terms

$$I^E_{\mathrm{total}}[\widehat{\mathrm{dS}}_4] = I_{\mathrm{bulk}}[\widehat{\mathrm{dS}}_4] + I_{\mathrm{NG}}[\Sigma_N] + I_{\mathrm{NG}}[\Sigma_S]\,, \tag{3.2}$$

where the Euclidean integrals

$$I_{\mathrm{bulk}}[\widehat{\mathrm{dS}}_4] := -\frac{1}{16\pi G_4} \int\limits_{\widehat{dS}_4 \backslash (\Sigma_N \cup \Sigma_S)} d^4x \sqrt{g} \left(R - \frac{6}{\ell^2} \right), \tag{3.3}$$

$$I_{\mathrm{NG}}[\Sigma_N] := \mathcal{T}_q \int_{\Sigma_N} d^2 y \sqrt{h}\,, \qquad I_{\mathrm{NG}}[\Sigma_S] := \mathcal{T}_q \int_{\Sigma_S} d^2 y \sqrt{h}\,.$$

Although the support of the bulk integral above excludes the location of the defects, we can define a "free energy inflow" from the bulk to Σ_N and Σ_S by dimensional reducing I_{bulk} down to two dimensions

$$I_{\mathrm{bulk}}[\widehat{\mathrm{dS}}_4] \overset{\mathrm{dim\,red}}{\longrightarrow} I_{2d}[\Sigma_N] + I_{2d}[\Sigma_S]\,, \tag{3.4}$$

as to define an effective action on each of the defects, which comprises the inflow (3.4) and the corresponding Nambu-Goto term, *viz.*

$$I_{\mathrm{eff}}[\Sigma_N] = I_{2d}[\Sigma_N] + I_{\mathrm{NG}}[\Sigma_N]\,, \quad I_{\mathrm{eff}}[\Sigma_S] = I_{2d}[\Sigma_S] + I_{\mathrm{NG}}[\Sigma_S]\,, \tag{3.5}$$

and such that the total on-shell action (3.2)

$$I^E_{\mathrm{total}}[\widehat{\mathrm{dS}}_4] \approx I_{\mathrm{eff}}[\Sigma_N] + I_{\mathrm{eff}}[\Sigma_S]\,. \tag{3.6}$$

(From here and in what follows, we shall use the notation "$\approx$" to indicate on-shell equalities.)

The reduced Euclidean action I_{2d} in (3.4) can be computed using the line elements (2.11) and integrating out the spindle coordinates (θ, ϕ). This gives

$$I_{\mathrm{bulk}}[\widehat{\mathrm{dS}}_4] \approx -\frac{\ell^2}{4qG_4} \int d^2 y \sqrt{h}\, \mathcal{R}\,, \tag{3.7}$$

where the integral is over the two-dimensional submanifold coordinatized by $y = (\tau, \xi)$ (with τ denoting the Euclidean time), and $\mathcal{R} = \mathcal{R}[h]$ is the intrinsic two-dimensional scalar of curvature built up from the induced metric on the defects (2.13). This reduction holds upon imposing Einstein's equations $\ell^2 R_{\theta\theta} = 3g_{\theta\theta}$ (and likewise the $\phi\phi$-equation) and by making use of the codimension two identity $R_{ij} = 3\cos^2\theta\, \mathcal{R}_{ij}$.

Due to the antipodal symmetry relating Σ_N and Σ_S [1], we further assign to each of the defects half of the total inflow (3.7)

$$I_{2d}[\Sigma_N] := -\frac{\ell^2}{8qG_4} \int_{\Sigma_N} d^2 y \sqrt{h}\, \mathcal{R}\,, \quad I_{2d}[\Sigma_S] := -\frac{\ell^2}{8qG_4} \int_{\Sigma_S} d^2 y \sqrt{h}\, \mathcal{R}\,, \tag{3.8}$$

so that the effective action (3.5) on the northern defect is given by

$$I_{\text{eff}}[\Sigma_N] \approx -\frac{\ell^2}{8qG_4} \int_{\Sigma_N} d^2y\sqrt{h}\,\mathcal{R} + \frac{1}{4G_4}\left(1 - \frac{1}{q}\right)\int_{\Sigma_N} d^2y\sqrt{h}\,, \qquad (3.9)$$

idem for $I_{\text{eff}}[\Sigma_S]$.

3.2 On-shell correspondence with Liouville theory

We now observe that the structure of the reduced effective action (3.9) closely resembles the Liouville theory action [15]:

$$I_{\text{L}}[g,\Phi;\gamma] = -\frac{1}{2}\int_{M_2} d^2y\sqrt{g}\Big(g^{ij}\partial_i\Phi\partial_j\Phi + Q\mathcal{R}\Phi + 4\pi\mu e^{2\gamma\Phi}\Big)\,. \qquad (3.10)$$

In the above, (M_2,g) is a two-dimensional Euclidean manifold and $\gamma^2 \sim \hbar$ is the only coupling constant of the theory; its strength dictates the classical and quantum regimes and further defines the background charge to be $Q = \gamma^{-1} + \gamma$, as required for conformal invariance (for a brief review of Liouville theory see appendix A). It is important to point out that the action (3.10) differs from the one given in (A.1) by an overall factor of -2π; such a normalization is needed in order to uniformize the definition of the stress-energy tensor[1] while comparing (3.9) and (3.10).

Indeed, the reduced effective action (3.9) corresponds precisely to the Liouville action (3.10) on the 2-sphere $(M_2,g) = (\Sigma_N, h)$, upon giving to the Liouville field a fix expectation value $\langle\Phi\rangle = \Phi_0$. That is

$$I_{\text{eff}}[\Sigma_N] \approx I_{\text{L}}\big|_{\langle\Phi\rangle=\Phi_0}\,, \qquad (3.11)$$

and similarly for $I_{\text{eff}}[\Sigma_S]$. This on-shell relation permits to establish the existence of an effective field theory on each of the minimal surfaces, Σ_N and Σ_S, given by a broken phase of Liouville theory. As we shall now see, the effective field theoretic description encode a number of compatibility conditions that in turn yield a q-dependent central charge which we propose encode the degrees of freedom associated to a massive observer.[2]

The on-shell correspondence (3.11) holds provided

$$\frac{\ell^2}{8qG_4} = \frac{Q\Phi_0}{2}\,, \qquad \frac{1}{4G_4}\left(1 - \frac{1}{q}\right) = -2\pi\mu e^{2\gamma\Phi_0}\,, \qquad (3.12)$$

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[1]We recall that in our conventions the definition of the gravitational stress energy tensor (2.16) differs from the standard convention used in the CFT context:

$$T^{ij}_{\text{grav}} = \frac{2}{\sqrt{h}}\frac{\delta I}{\delta h_{ij}}\,, \qquad T^{ij}_{\text{CFT}} = \frac{-4\pi}{\sqrt{h}}\frac{\delta I}{\delta h_{ij}}\,.$$

The overall factor of -2π propagates when computing the operator product expansion $\langle TT\rangle$, which in turns produces a relative factor in the central charge. Comparison of the effective gravitational action (3.9) with the Liouville action (A.1) thus requires the normalization implemented in (3.10).

[2]A similar idea has been previously discussed in [16] where the dimensional reduction of Einstein gravity to two-dimensional Liouville theory is proposed to describe the underlying degrees of freedom of black hole horizons.

as follows from matching the terms of the same order in derivatives of the metric in (3.9) and (3.10). In addition, the expectation value Φ_0 must satisfy the Liouville equation of motion for a constant field, which is given by

$$Q\mathcal{R} + 8\pi\gamma\mu e^{2\gamma\Phi_0} = \frac{2Q}{\ell^2} + 8\pi\gamma\mu e^{2\gamma\Phi_0} = 0\,, \tag{3.13}$$

where the first equality made use of the constant positive curvature $\mathcal{R} = 2\ell^{-2}$ of Σ_N.

Compatibility of the equations (3.12) and (3.13) yields

$$\Phi_0 = \frac{1}{2\gamma(q-1)}\,, \qquad \mu = \frac{1}{8\pi G_4}\left(\frac{1-q}{q}\right)\exp\left(\frac{1}{1-q}\right)\,, \tag{3.14}$$

and

$$\frac{Q}{\gamma} = \left(1 - \frac{1}{q}\right)\frac{\ell^2}{2G_4}\,. \tag{3.15}$$

Observe that the bound $q > 1$ for the orbifold parameter can be understood as a consistency condition: one the one hand, from (3.13) it follows that positivity of $\mathcal{R}[h] = 2\ell^{-2} > 0$ is only possible if $\mu < 0$, which according to (3.14) requires q to be greater than one. On the other hand and remembering that $Q = \gamma + \gamma^{-1}$, the bound $q > 1$ ensures the reality of the couplings γ and ℓ in (3.15).

In what follows, we shall see that the semiclassical limit of (3.15) provides a nontrivial link between the Liouville coupling constant γ, in terms of which the central charge of the theory is defined, and the gravitational coupling ℓ^2/G_4 which in turns defines (up to a factor of π in dimension four) the entropy of the dS$_4$ space.

3.3 Central charge and Cardy formula

In the semiclassical regime $\gamma^2 \ll 1$, where thus $Q \sim \gamma^{-1}$, there exists a $\mathcal{O}(1/\gamma^2)$ contribution to the Liouville central charge [17]

$$c = 1 + 6Q^2 \approx \frac{6}{\gamma^2}\,, \tag{3.16}$$

whose value can be computed in terms of the gravity couplings and the orbifold parameter q. Indeed, from the semiclassical limit of (3.15), we straightforwardly find

$$c_q = \left(1 - \frac{1}{q}\right)\frac{3\ell^2}{G_4}\,. \tag{3.17}$$

This value of the central charge is consistent with the classical conformal anomaly equation $\langle T \rangle = \frac{c}{12}\mathcal{R}$, where $T = h^{ij}T_{ij}$ is the trace of stress-energy tensor (2.16) and $\mathcal{R} = 2\ell^{-2}$ is the curvature of the corresponding defect. Also, we note that since $q > 1$, then the central charge $c_q > 0$, which indicates unitarity of the theory.

Is it worth to notice that the central charge (3.17) is q-dependent and vanishes in the tensionless limit $q \to 1$. This is similar to what occurs in the AdS$_3$/CFT$_2$ context, where the bulk orbifold AdS$_3/\mathbb{Z}_q$ induces a q-dependence of the central charge of the boundary theory [18–22], with the Brown-Henneaux central charge being recovered by rescaling the Newton's constant as $G \sim qG$.

Having obtained the central charge (3.17) and by virtue of the thermal Cardy formula in the canonical ensemble [23, 24]

$$\mathcal{S}_q^{\text{Cardy}} = \frac{\pi^2}{3} c_{q,L} T_L + \frac{\pi^2}{3} c_{q,R} T_R \,, \tag{3.18}$$

a q-dependent entropy can be computed (as usual, L and R label left and right-movers central charge and temperature). Indeed, based on the arguments of [25], the Cardy formula holds in the extended range of large central charge and large gap in operator dimension above zero. Accordingly, in our case, since

$$c_q \sim \frac{\ell^2}{G_4} \gg 1 \,, \quad \Delta_0 \sim c_q \gg 1 \,, \tag{3.19}$$

where Δ_0 is the (semiclassical) conformal dimension of the bound state (see (A.13)), the Cardy formula (3.18) applies.

For a non-chiral Liouville theory, we have

$$c_{q,L} = c_{q,R} = c_q \,, \qquad T_L = T_R = \frac{1}{2\pi} \,, \tag{3.20}$$

where c_q is given in (3.17) and T_L and T_R correspond to the temperature of the generalized Hartle-Hawking vacuum of dS space. This is known to be equivalent to a thermal state $\rho = e^{-2\pi H_{\mathfrak{R}}}$ defined by the Rindler Hamiltonian $H_{\mathfrak{R}}$ [26, 27](see also [28]).

Hence, using (3.17) and (3.20) in the Cardy formula (3.18), we find the q-dependent Cardy entropy

$$\mathcal{S}_q^{\text{Cardy}} = \left(1 - \frac{1}{q}\right) \frac{\pi \ell^2}{G_4} \,. \tag{3.21}$$

Note that minus the derivative of the above entropy with respect to $1/q$ gives the Gibbons-Hawking entropy (2.5). Based on this simple observation, we shall next reinterpret the Cardy entropy (3.21) as *modular free energy.*

3.4 Modular free energy and Gibbons-Hawking entropy

The Cardy entropy (3.21) can be understood as the modular free energy F_q whose derivative with respect to the dimesionless temperature[3] $T = q^{-1}$ [29, 30] yields the Gibbons-Hawking area law. To this end, we define the modular Hamiltonian

$$H := -\log \rho \,, \qquad \rho^q = e^{-qH} \,. \tag{3.22}$$

Thus, we can write the modular partition function as

$$\mathcal{Z} = \operatorname{tr} \rho^q = \operatorname{tr} e^{-qH} \,, \tag{3.23}$$

in terms of which the modular free energy is given by

$$F_q = -\frac{1}{q} \log \mathcal{Z} = -\frac{1}{q} \log \operatorname{tr}\rho^q \,. \tag{3.24}$$

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[3]Note that, by identifying $T = q^{-1}$, the orbifold parameter q induces a Boltzmann factor $\exp(-qH)$ (with H denoting the modular Hamiltonian (3.22)) which makes the Liouville theory thermal and thus amounts to using the thermal Cardy formula (3.18).

Next, we compute the modular free energy (3.24) on the 4-sphere defined by

$$S^4 := (\mathcal{R}_S^E \cup \mathcal{R}_N^E)/\Pi \,. \tag{3.25}$$

Here, $\mathcal{R}_S^E$ and $\mathcal{R}_N^E$ denote the analytic continuation of the southern and northern Rindler wedges (both given by 4-spheres), and $\Pi : S^4 \to S^4$ is the antipodal map that sends every point in the southern Rindler wedge to the corresponding antipodal point in the northern Rindler wedge [1]. It is important to point out that the 4-sphere (3.25) is equivalent to the analytic continuation of a single Rindler wedge and also equivalent to the Euclidean continuation of global dS$_4$ spacetime. Moreover, it naturally admits a $\mathbb{Z}_q$ action (given by azimutal identifications), with a q-fold branched cover that we denote by S_q^4. Using the Calabrese-Cardy formula [31]

$$\operatorname{tr} \rho^q = \frac{\mathcal{Z}[S_q^4]}{(\mathcal{Z}[S^4])^q} \,, \tag{3.26}$$

it follows that

$$F_q[S^4] = -\frac{1}{q}\log \mathcal{Z}[S_q^4] + \log \mathcal{Z}[S^4] \approx I^E[S^4/\mathbb{Z}_q] - I^E[S^4] = 2\left(1 - \frac{1}{q}\right)\frac{\pi\ell^2}{G_4}\,. \tag{3.27}$$

In the above, we have used the semiclassical approximation $\mathcal{Z}[S^4] \approx \exp(-I^E[S^4])$ and the locality of the gravity action to write $I^E[S_q^4] = q I^E[S^4/\mathbb{Z}_q]$. The value of the latter is given by the on-shell value of (2.14) (properly Euclideanized) restricted to a single Rindler wedge (which we recall is given by a 4-sphere in the Euclidean geometry), *viz.*

$$I^E[S^4/\mathbb{Z}_q] \approx \left(1 - \frac{2}{q}\right)\frac{\pi\ell^2}{G_4}\,. \tag{3.28}$$

The value of the modular free energy (3.27) comprises the contribution form both, northern and southern defects. For a single defect (say the southern one), we thus have

$$F_q^{\Sigma_S} = \frac{1}{2}F_q[S^4] = \left(1 - \frac{1}{q}\right)\frac{\pi\ell^2}{G_4}\,, \tag{3.29}$$

which corresponds exactly to the value of the Cardy entropy (3.21).

Finally, we can compute the modular entropy

$$\widetilde{\mathcal{S}}_q = -\frac{\partial F_q}{\partial T} = (1 - q\partial_q)\log \mathcal{Z}\,, \tag{3.30}$$

which gives

$$\widetilde{\mathcal{S}}_q = q^2 \frac{\partial}{\partial q}\left(1 - \frac{1}{q}\right)\frac{\pi\ell^2}{G_4} = \frac{\pi\ell^2}{G_4} = \mathcal{S}_{\mathrm{dS}}\,. \tag{3.31}$$

This is precisely the Gibbons-Hawking entropy (2.5). Observe that although (3.31) has its origin in the modular free energy (3.24), its value is independent of the modular parameter q and hence this remains fix in the tensionless limit $q \to 1$, in which one recovers the standard description of the dS$_4$ spacetime.
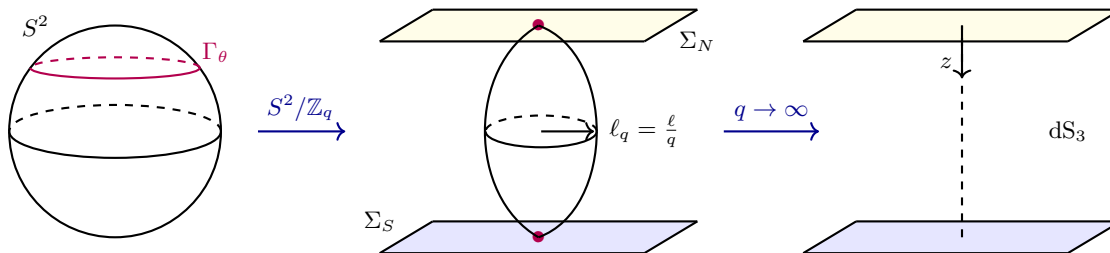
Figure 2. The large q limit of the spindle $S^2/\mathbb{Z}_q$. This corresponds to the zero radius limit $\ell_q \to 0$, where the two-dimensional geometry between the northern and southern defects Σ_N and Σ_S shrinks to a single transverse dimension. The resulting geometry is that of global dS$_3$ spacetime.

4 The large q limit and dS$_3$ holography

Here, we consider the $q \to \infty$ limit of the orbifold $\widehat{\mathrm{dS}}_4 := \mathrm{dS}_4/\mathbb{Z}_q$. We will first argue that this limit yields an alternative realization of the global dS$_3$ geometry. We will further propose that the large q limit provides a new mechanism to study dS$_3$/CFT$_2$ holography, whereby the dual field theory defined on the two conformal boundaries of dS$_3$ has a higher dimensional origin, namely, it is inherited from the Euclidean Liouville theory on the two minimal surfaces Σ_S and Σ_N (embedded in four dimensions).

4.1 3D conformal boundaries from codimension two defects in 4D

The limit $q \to \infty$ is equivalent to the zero radius limit of the $S^2/\mathbb{Z}_q$ spindle, *viz.* $\ell_q := q^{-1}\ell \to 0$. In this limit, the two-dimensional geometry between the northern and southern defects Σ_N and Σ_S collapses to a single transverse direction, say $z := \ell\theta$, with Σ_N located at $z = 0$ and Σ_S at $z = \pi\ell$. The situation is illustrated in figure 2.

In the above limit, the four-dimensional geometry of the manifold $(\widehat{\mathrm{dS}}_4, \widehat{g}_4)$ reduces to the three-dimensional geometry of global dS$_3$ spacetime with a radius equals to ℓ. This can be seen directly from the embedding coordinates (2.6) by first identifying $\phi \sim \phi + \frac{2\pi}{q}$ and then taking $q \to \infty$. This operation sets $X_4 = 0$. The remaining coordinates

$$X_0 = \sqrt{\ell^2 - \xi^2}\,\cos\theta\,\sinh(t/\ell)\,, \quad X_1 = \sqrt{\ell^2 - \xi^2}\,\cos\theta\,\cosh(t/\ell)\,, \tag{4.1}$$
$$X_2 = \xi\cos\theta\,, \qquad\qquad\qquad X_3 = \ell\sin\theta\,,$$

parametrize the embedding dS$_3 \hookrightarrow \mathcal{M}^{1,3}$ of the dS$_3$ hyperboloid, defined by the hypersurface equation $-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = \ell^2$, into four-dimensional Minkowski spacetime $\mathcal{M}^{1,3}$.

After taking the limit, the resulting geometry is

$$g_3 = dz^2 + \cos^2(z/\ell)\,h\,, \tag{4.2}$$

where h is the two-dimensional induced metric on the defects defined in (2.11) and (2.13). We futher observe that the line element (4.2) can be mapped to the global foliation of dS$_3$. This is done via analytical continuation of the transverse coordinate $z \in [0, \pi\ell]$ and the time $t \in (-\infty, \infty)$ (the latter being the time coordinate in h), that is

$$z \to iT\,, \qquad t \to i\tau\,. \tag{4.3}$$

As a result, the compact coordinate z becomes the global time $-\infty < T < \infty$ and the induced metric $h \to d\Omega_2^2$, where $d\Omega_2^2$ denotes the metric on the unit 2-sphere:

$$g_3 = -dT^2 + \cosh^2(T/\ell)d\Omega_2^2\,. \tag{4.4}$$

Clearly, this is the global foliation of dS$_3$ spacetime. Under (4.3), the original codimension two defects Σ_N and Σ_S are respectively sent to $T \to -\infty$ and $T \to \infty$. Hence, in the large q limit, they reincarnate as the past and future infinities of dS$_3$.

4.2 dS$_3$/CFT$_2$ central charge

The above maneuvers show that the global dS$_3$ geometry can be thought of as the limit

$$(\widehat{\mathrm{dS}}_4, \widehat{g}_4) \overset{q \to \infty}{\longrightarrow} (\mathrm{dS}_3, g_3) \tag{4.5}$$
$$(\Sigma_S, \Sigma_N) \longmapsto (\mathcal{I}^+, \mathcal{I}^-)\,,$$

where the minimal surfaces Σ_S and Σ_N are sent to the past and future infinities $\mathcal{I}^\pm$ of dS$_3$ (after the double analytical continuation (4.3)). Thus, recalling from section 3 that on Σ_N and Σ_S there exist an Euclidean Liouville theory, from the dS$_3$ perspective one should expects to have some Liouville-type theory on each of the boundaries $\mathcal{I}^\pm$. This is consistent with the known fact that the asymptotic dynamics of pure dS$_3$ gravity — when formulated as two copies Chern-Simons theory with gauge group SL$(2, \mathbb{C})$ — is described by an Euclidean Liouville theory on $\mathcal{I}^+ \cup \mathcal{I}^-$ [12]. Indeed, in the large q limit, the Liouville theory on each the minimal surfaces reaches its zero temperature limit becoming a non-thermal theory, in agreement with the results established in [12] which, in the context of the dS$_3$/CFT$_2$ correspondence, predicts a non-thermal dual theory.

Accordingly, the total central charge of the composite boundary $\mathcal{I}^+ \cup \mathcal{I}^-$

$$c = c(\mathcal{I}^+) + c(\mathcal{I}^-)\,, \tag{4.6}$$

can be computed by means of (4.5) as

$$c = c_\infty(\Sigma_N) + c_\infty(\Sigma_S) = \frac{6\ell^2}{G_4}\,, \tag{4.7}$$

where $c_\infty(\Sigma_N) = c_\infty(\Sigma_S) = \frac{3\ell^2}{G_4}$ denote the Liouville central charge (3.17) in the limit $q \to \infty$. Note that the four-dimensional Newton's constant can be expressed in terms of the three-dimensional one as

$$G_4 = \mathrm{Vol}(S^1)\,G_3\,, \tag{4.8}$$

where $\mathrm{Vol}(S^1)$ is defined as the average volume of a meridian Γ_θ located at a polar angle θ (see figure 2). This average is given by

$$\mathrm{Vol}(S^1) = \langle \Gamma_\theta \rangle = 2\pi\ell\langle \sin\theta \rangle = 4\ell\,. \tag{4.9}$$

(In the above, we have used that $\pi\langle \sin\theta \rangle = \int_0^\pi d\theta\,\sin\theta = 2$.) Then

$$G_4 = 4\ell G_3\,, \tag{4.10}$$

and therefore one finds that the total central charge (4.7) is

$$c = \frac{3\ell}{2G_3} \,, \tag{4.11}$$

in accordance with the result derived in the context of the dS$_3$/CFT$_2$ correspondence [6–12]. Note that this result is consistent with the fact that, in the large q limit, the Cardy entropy of the two defects

$$\mathcal{S}_q^{\Sigma_N \cup \Sigma_S} := \mathcal{S}_q^{\mathrm{Cardy}}[\Sigma_N] + \mathcal{S}_q^{\mathrm{Cardy}}[\Sigma_S] = 2\left(1 - \frac{1}{q}\right)\frac{\pi\ell^2}{G_4} \,, \tag{4.12}$$

correctly reproduces the thermodynamic entropy of three-dimensional dS spacetime [32] (upon using the dimensional reduction of the Newton constant (4.10)), *viz.*

$$\mathcal{S}_q^{\Sigma_N \cup \Sigma_S} \overset{q\to\infty}{\longrightarrow} \mathcal{S}_{\mathrm{dS}_3} = \frac{\pi\ell}{2G_3} \,. \tag{4.13}$$

The results (4.11) and (4.13) seem to indicate that dS$_3$ holography may emerge as the large q limit of the dS$_4$/$\mathbb{Z}_q$ orbifold geometry.

5   Conclusions

In this work, we have modeled the back-reaction of a static observer in four-dimensional de Sitter spacetime via the singular quotient dS$_4$/$\mathbb{Z}_q$. The latter geometry exhibits two antipodal conical singularities that we interpret as being created by a pair of massive observers, $\mathcal{O}_S$ and $\mathcal{O}_N$, defined in (2.9). The massless probe limit is defined by $q \to 1$ in which one recovers the smooth dS$_4$ spacetime.

The set of fixed points of the $\mathbb{Z}_q$ action defines a pair of codimension two surfaces, Σ_S and Σ_N, as indicated in (2.12). Each of these two surfaces contains the worldline of one static observer and they both have the topology of a 2-sphere in the Euclidean geometry. Moreover, they are by construction minimal surfaces in the sense that their area functional must be coupled to the Einstein-Hilbert action in order to have a well defined variational principle; cf. equation (2.14).

By introducing an orbifold parameter $q > 1$, we have proposed the existence of an intrinsic field theoretic description of each of the minimal surfaces in terms of a two-dimensional conformal field theory. To this end, we have built up an effective two-dimensional action functional with support on Σ_S and Σ_N, which comprises a free energy inflow coming from dimensionally reducing the four-dimensional Einstein-Hilbert action, plus the corresponding Nambu-Goto term of the surface. The resulting effective action, given in equation (3.9), corresponds to that of a Liouville theory on a 2-sphere with a fixed vacuum expectation value of the Liouville field.

The correspondence between the reduced action on the minimal surfaces and the Liouville theory action provides a non-trivial link between the couplings and parameters of both theories. These consistency conditions, displayed in (3.14) and (3.15), in particular lead to the q-dependent central charge (3.17). Making use of the thermal Cardy formula, we have
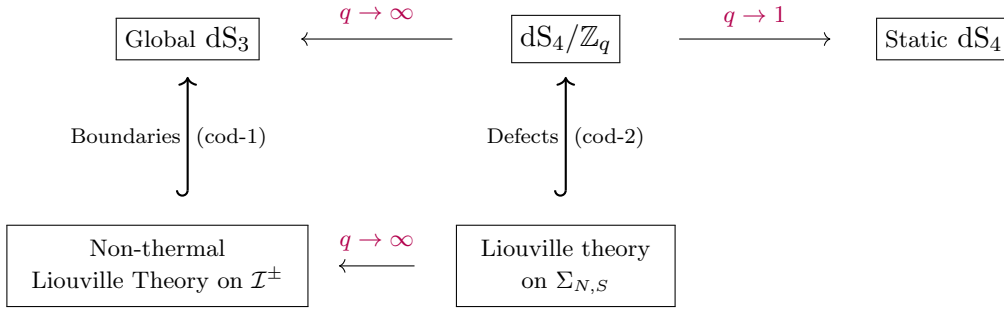
Figure 3. Different limits of the $dS_4/\mathbb{Z}_q$ geometry and its defect/boundary field theory description.

computed the Cardy entropy (3.21) which, upon identifying the modular parameter with the inverse of the (dimensionless) temperature $q = T^{-1}$, gives a modular free energy whose modular entropy equals the Gibbons-Hawking entropy.

The above construction permits the interpretation of the Gibbons-Hawking entropy as representing microscopic degrees of freedom of the massive observer: the back-reaction of such observer induces a conical defect which in turn is the locus of codimension two minimal surface. This two-dimensional surface encode their own field theoretic description in terms of Liouville theory, which yields the central charge (3.17) and that we propose captures the degrees of freedom of the observer (that are only visible when $q > 1$).

We finally studied the $q \to \infty$ limit of the quotient $dS_4/\mathbb{Z}_q$, which is equivalent to the zero radius limit of the $S^2/\mathbb{Z}_q$ spindle (see figure 2). In this limit, the four-dimensional geometry reduces to the global geometry of dS_3 spacetime where the two minimal surfaces Σ_S and Σ_N are mapped, upon double analytical continuation, to the future and past conformal boundaries $\mathcal{I}^+$ and $\mathcal{I}^-$ of dS_3, as indicated in (4.5).

From the relation between the modular parameter and the temperature $q = T^{-1}$, it follows that the limit $q \to \infty$ is also equivalent to zero temperature limit of the Liouville theory on the minimal surfaces. As a result, the future and past infinities of dS_3 inherit from the minimal surfaces a non-thermal Liouville theory. Schematically, our findings can be summarized in figure 3.

Accordingly, the total central charge of the composite dS_3 boundary $\mathcal{I}^+ \cup \mathcal{I}^-$ comprises two separate contributions, one from Σ_S and another one Σ_N, as displayed in (4.7). This can be directly computed by taking the large q limit of the Liouville central charge (3.17). The result correctly reproduces the value of the dS_3/CFT_2 central charge for the boundary field theory.

Regarding directions for future work, one may speculate that our construction belongs to a broader scheme whereby (higher spin) gravity theories are formulated as quasi-topological field theories of the AKSZ type [33]. These theories are naturally formulated on manifolds with multiple boundaries and they incorporate extended objects of various codimensions; Hilbert spaces are assigned to boundaries (encoding boundary states of the bulk theory) as well as to defects (encoding defect states labeled by the codimension number). In this moduli space, it is natural to expect that the Hilbert spaces associated to

boundaries and defects are related via a (co)dimensional ladder of dualities involving different limits of the moduli parameters. The case presented here would then be a concrete example of such a duality in which the Hilbert space of a codimension two defect in four dimension gives rise, in the large q limit, to the boundary Hilbert space of dS_3. We plan to refine and present these ideas in a separate work.

Acknowledgments

A Liouville theory

In this appendix we collect the most relevant results of Liouville field theory and its semiclassical limit. For a more detailed analysis see, for instance, [34–36] and references therein.

Quantum theory. Let (Σ, h) be a two-dimensional Riemann surface. Liouville theory is an exact two-dimensional conformal field theory on Σ, defined by the action

$$I_L = \frac{1}{4\pi} \int_\Sigma d^2y \sqrt{h} \Big(h^{ij} \partial_i \Phi \partial_j \Phi + Q\mathcal{R}\Phi + 4\pi\mu e^{2\gamma\Phi} \Big) , \tag{A.1}$$

where the interaction parameter μ depends on the curvature of Σ, and the coupling $\gamma^2 \sim \hbar$ controls the quantum effects. When considering the theory on a Lorentzian manifold, the action (A.1) acquires an extra overall minus sign.

Conformal invariance at the full quantum level sets the brackground charge

$$Q = \frac{1}{\gamma} + \gamma , \tag{A.2}$$

which is thus invariant under the shift $\gamma \to \gamma^{-1}$. In complex coordinates, the (holomorphic part of the) stress-enery tensor

$$T := T_{zz} = Q\partial^2\Phi - (\partial\Phi)^2 , \tag{A.3}$$

gives rise, via the operator product expansion

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2} + \cdots , \tag{A.4}$$

to the central charge of the theory

$$c = 1 + 6Q^2 . \tag{A.5}$$

The vertex operators

$$V_\alpha(z) = e^{2\alpha\Phi(z)}, \tag{A.6}$$

labeled by the quantum number α, create the spectrum of primary operators of the theory. It follows that the operator product

$$T(z_1)V(z_2) = \frac{\Delta\, V_\alpha(z_2)}{(z_1 - z_2)^2} + \frac{\partial V_\alpha(z_2)}{z_1 - z_2} + \cdots \tag{A.7}$$

determines the conformal dimension of primaries in terms of the background charge and the momentum

$$\Delta = \alpha(Q - \alpha)\,. \tag{A.8}$$

The momenta of normalizable states

$$\alpha = \frac{1}{2}Q + i\lambda\,, \quad \lambda \in \mathbb{R}\,, \tag{A.9}$$

in terms of which

$$\Delta = \frac{1}{4}Q^2 + \lambda^2 \geq \frac{1}{4}Q^2 =: \Delta_0\,. \tag{A.10}$$

Semiclassical limit. The semiclassical limit of the theory is taken through the double scaling

$$\Phi \to \gamma^{-1}\Phi\,, \quad \mu \to \gamma^{-2}\mu\,, \tag{A.11}$$

under which the quantum action (A.1) scales to

$$I_L = \frac{1}{4\pi\gamma^2} \int_\Sigma d^2y \sqrt{h} \left(h^{ij}\partial_i\Phi\partial_j\Phi + \mathcal{R}\Phi + 4\pi\mu e^{2\Phi} \right), \tag{A.12}$$

in the limit where $\gamma^2 \to 0$. In this regime, the central charge (A.5) and the bound state conformal weight in (A.10) is well approximated by

$$c \approx \frac{6}{\gamma^2}\,, \quad \Delta_0 \approx \frac{c}{24}\,. \tag{A.13}$$

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