



On the rank of general linear series on stable curves

Karl Christ^{1,2}

Received: 9 May 2022 / Revised: 11 November 2022 / Accepted: 23 January 2023
© The Author(s) 2023

Abstract

We study the dimension of loci of special line bundles on stable curves and for a fixed semistable multidegree. In case of total degree $d = g - 1$, we characterize when the effective locus gives a Theta divisor. In case of degree $g - 2$ and g , we show that the locus is either empty or has the expected dimension. This leads to a new characterization of semistability in these degrees. In the remaining cases, we show that the special locus has codimension at least 2. If the multidegree in addition is non-negative on each irreducible component of the curve, we show that the special locus contains an irreducible component of expected dimension.

Mathematics Subject Classification 14H51 · 14H40 · 14H20

1 Introduction

If X is a smooth curve of genus g , then a *general* line bundle L of degree d satisfies

$$h^0(X, L) = \max \{0, d - g + 1\}.$$

The locus of *special* line bundles in $\text{Pic}^d(X)$, those that have additional global sections, is empty for $d < 0$ and $d > 2g - 2$, and irreducible of dimension d if $0 \leq d \leq g - 1$ and irreducible of dimension $2g - 2 - d$ if $g - 1 \leq d \leq 2g - 2$.

The author was supported by the Israel Science Foundation (Grant No. 821/16) and by the Center for Advanced Studies at BGU.

✉ Karl Christ
kchrist@math.uni-hannover.de

¹ Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, 84105 Be'er Sheva, Israel

² Institute of Algebraic Geometry, Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany

In this paper, we are interested in how this picture changes if X is no longer assumed to be a smooth curve, but allowed to have nodal singularities. More precisely, we assume X to be a *stable* curve. Stable curves give a well-understood compactification of the moduli space of smooth curves. In this way they provide important tools for understanding the geometry of the moduli space, as well as studying smooth curves via degeneration techniques.

Two basic properties remain as in the smooth case: First, the Riemann Roch theorem still holds, and hence also the lower bound $h^0(X, L) \geq d - g + 1$. And second, loci of special line bundles (or, more generally, any Brill Noether locus) can be realized as a degeneracy locus of a map between vector bundles. Such a locus is either empty, or each of its irreducible components has at least the expected dimension.

If X is reducible, the similarities with the case of smooth curves do not go much further. More precisely, in this case also the degree d Picard scheme $\text{Pic}^d(X)$ is no longer irreducible. To be able to talk about generic behaviour, one thus needs to restrict to an irreducible component of $\text{Pic}^d(X)$. We denote such an irreducible component by $\text{Pic}^{\underline{d}}(X)$, parametrizing line bundles on X of fixed *multidegree* \underline{d} . A multidegree is a tuple of integers, one for each irreducible component X_v of X , that prescribes the degree of the restriction of a line bundle to X_v .

For almost all choices of multidegree \underline{d} of fixed total degree d none of the statements for smooth curves in the first paragraph remains true if X is reducible. In fact, for any integer r and fixed total degree d , there are only finitely many multidegrees \underline{d} of total degree d such that there exists a line bundle L of multidegree \underline{d} and with $h^0(X, L) \leq r$ – whereas there are infinitely many multidegrees of fixed total degree d .

To remedy this, we focus in this paper on *semistable multidegrees*, as introduced by Caporaso to construct a universal compactified Jacobian over the moduli space of stable curves [5]. See Definition 3.1. Line bundles with semistable multidegree are those that are slope semistable with respect to the dualizing sheaf ω_X of X by [3, §1].

Our choice of multidegrees is motivated by two previous results. First, as in the case of smooth curves, outside of the range $0 \leq d \leq 2g - 2$ every line bundle with semistable multidegree is non-special by [7, Theorem 2.3]. And second, the case of $d = g - 1$ allows for a theory of Theta divisors, as we explain below. Even more, one can characterize semistability for degree $g - 1$ in these terms. We note however, that in other regards there are better-behaved classes of multidegrees, and there seems to be no overall best-behaved choice. See for example [11, 12] for the question of an upper bound on $h^0(X, L)$ for all line bundles L of fixed multidegree.

1.1 Results

Suppose \underline{d} is a multidegree of total degree $d \leq g - 1$. Then the expected dimension of the effective locus

$$W_{\underline{d}}(X) := \{[L] \in \text{Pic}^{\underline{d}}(X) \mid h^0(X, L) \geq 1\}$$

is d . In particular, if $d = g - 1$, $W_{\underline{d}}(X)$ is expected to be a divisor in $\text{Pic}^{\underline{d}}(X)$, called the *Theta divisor*.

Since $W_{\underline{d}}(X)$ can be realized as a degeneracy locus, there are three options for the actual dimension of $W_{\underline{d}}(X)$ in degree $g - 1$: $W_{\underline{d}}(X)$ is either empty, a divisor in $\text{Pic}^{\underline{d}}(X)$, or all of $\text{Pic}^{\underline{d}}(X)$. Thus to show that $W_{\underline{d}}(X)$ is a divisor, it suffices to show that there exist both an effective and a non-effective line bundle of multidegree \underline{d} . For the latter, a very pleasing answer was found by Beauville [4, Lemma 2.1]. Namely, a multidegree \underline{d} of total degree $g - 1$ is semistable if and only if there exists a non-effective line bundle of multidegree \underline{d} .

In this paper, we first settle the remaining part of the question, namely when the effective locus $W_{\underline{d}}(X)$ is empty. It turns out, that $W_{\underline{d}}(X)$ indeed can be empty, even if \underline{d} is semistable of degree $g - 1$. Example 4.7 gives instances for arbitrary $g \geq 2$. Before we can characterize when this happens, we need to recall one particularly convenient combinatorial description of semistable multidegrees of total degree $g - 1$.

We denote by \mathbb{G}_X the dual graph of X . To an orientation O of the edges of \mathbb{G}_X , one can associate a multidegree \underline{d}_O . Namely, the value of \underline{d}_O on an irreducible component X_v of X is given by

$$\text{indeg}_O(v) - 1 + g_v.$$

Here v is the vertex of \mathbb{G}_X corresponding to X_v , $\text{indeg}_O(v)$ denotes the number of incoming edges at v in the orientation O , and g_v denotes the geometric genus of X_v . As observed in [3, 4], a multidegree \underline{d} is semistable of total degree $g - 1$ if and only if $\underline{d} = \underline{d}_O$ for some orientation O . As a final ingredient, recall that a directed cycle in an orientation O is a cycle in which every vertex is adjacent to one incoming and one outgoing edge of the cycle.

Theorem 1.1 *Let X be a stable curve and \underline{d} a multidegree of total degree $g - 1$. Then the effective locus $W_{\underline{d}}(X) \subset \text{Pic}^{\underline{d}}(X)$ has codimension 1 if and only if \underline{d} is semistable and either X contains an irreducible component that is not rational, or the orientation giving \underline{d} contains a directed cycle.*

See Theorem 4.8. One can view this as a small correction to [4, Proposition 2.2], in which it is claimed that \underline{d} is semistable if and only if a Theta divisor exists. The possibility of an empty effective locus however is not considered.

The existence of a Theta divisor is a foundational result in the theory of compactified Jacobians and generalized Prym varieties, since a Theta divisor allows to define a canonical polarization on these varieties. See, for example, [1, 3, 4, 6, 13, 15]. In these cases, the multidegrees \underline{d} do satisfy the second condition of Theorem 1.1 and no issue arises. See Remark 4.9 for a detailed discussion.

We use Theorem 1.1 and combinatorics of semistable multidegrees to obtain new results for semistable multidegrees of total degrees other than $g - 1$. Recall that a multidegree \underline{d} itself is called *effective*, if it is non-negative on each irreducible component of X .

Theorem 1.2 *Let X be a stable curve and \underline{d} a semistable multidegree of total degree $d \leq g - 2$. Then each irreducible component of the effective locus $W_{\underline{d}}(X)$ has dimension at most $g - 2$. If \underline{d} is in addition effective, then the effective locus $W_{\underline{d}}(X)$ contains an irreducible component of expected dimension, d .*

See Theorem 6.5. It is not hard to see, that a line bundle L is semistable or special, if and only if its residual $\omega_X \otimes L^{-1}$ is so (cf. Remark 3.6). Hence Theorem 1.2 also gives the analogous statements for special loci in case $d \geq g$.

The irreducible component of expected dimension in the last statement of Theorem 1.2 is the image $A_{\underline{d}}(X)$ of the rational Abel map $\alpha_{\underline{d}}$. This map is defined whenever \underline{d} is effective, as follows:

$$\alpha_{\underline{d}}: \prod_v (X_v^{\text{sm}})^{\underline{d}_v} \rightarrow \text{Pic}^{\underline{d}}(X), (p_1, \dots, p_d) \mapsto \mathcal{O}_X(p_1 + \dots + p_d).$$

Here X_v^{sm} denotes the intersection of X_v with the smooth locus of X and \underline{d}_v the value of the multidegree \underline{d} on X_v . In general, the dimension of the image $A_{\underline{d}}(X)$ can be smaller than d even if $d \leq g$, and $A_{\underline{d}}(X)$ need not be an irreducible component of the effective locus. See Remark 4.11. The claim of the theorem is, that both these properties however do hold when \underline{d} is semistable.

Notice next, that Theorem 1.2 shows that the effective locus is either empty or has the expected dimension, when \underline{d} is semistable of total degree $d = g - 2$. As opposed to the case $d = g - 1$, this no longer characterizes semistability. See Examples 5.5 and 5.6. On the other hand, we obtain a new characterization of semistability in degree $g - 2$ in Lemma 5.8. Since it allows for a more elegant formulation, we state here the residual version for total degree g , as in Theorem 5.10.

Theorem 1.3 *Let X be a stable curve and \underline{d} a multidegree of total degree g . Then \underline{d} is semistable if and only if \underline{d} is effective and the rational Abel map $\alpha_{\underline{d}}$ is dominant.*

As mentioned above, it is well-known that semistability in degree $g - 1$ shows many surprising interactions, among others with combinatorics and the existence of Theta divisors. In addition to [8, 14], Theorem 1.3 provides further evidence that semistability in degree g exhibits similarly special behaviour.

Finally, for $d < g - 2$ the effective locus can have larger than expected dimension, even if \underline{d} is semistable. This can be the case already for $d = g - 3$ as in Example 6.7.

1.2 Structure of the paper

In Sect. 2, we fix some notation. In Sect. 3, we recall basic properties of semistable and special line bundles. In Sect. 4, we discuss the case $d = g - 1$ and the existence of Theta divisors. To do so, we recall the description of semistability in terms of orientations in Sect. 4.1. After establishing existence of Theta divisors in Sect. 4.2, we recall a description of the irreducible components of Theta divisors due to Coelho and Esteves [9] in Sect. 4.3. In Sect. 5, we discuss the case $d = g - 2$ and its residual case $d = g$. We show that in these cases the special loci have expected dimension if \underline{d} is semistable in Sect. 5.1 and characterize semistability in Sect. 5.2. Finally, we discuss in Sect. 6 the general case and prove Theorem 1.2.

2 Notations and conventions

Throughout the paper, we work over an algebraically closed field k of characteristic 0. We consider a curve X over k , which we will always assume to be reduced with nodal singularities. If not specified otherwise, we will assume X to be connected.

We denote the *dual graph* of X by \mathbb{G}_X . That is, \mathbb{G}_X contains a vertex v for every irreducible component X_v of X ; an edge between vertices v and w for each node in $X_v \cap X_w$, possibly with $v = w$; and each vertex v is assigned the weight g_v given by the geometric genus of X_v . In particular, \mathbb{G}_X may contain multiple edges between the same two vertices, as well as loop edges. We denote by $V(\mathbb{G}_X)$ and $E(\mathbb{G}_X)$ the sets of vertices and edges of \mathbb{G}_X , respectively.

The *genus* of \mathbb{G}_X is defined as

$$g(\mathbb{G}_X) = 1 - \chi(\mathbb{G}_X) + \sum_{v \in V(\mathbb{G}_X)} g_v, \tag{1}$$

where $\chi(\mathbb{G}_X)$ is the Euler characteristic of \mathbb{G}_X . That is, $1 - \chi(\mathbb{G}_X) = 1 - |V(\mathbb{G}_X)| + |E(\mathbb{G}_X)|$. The genus of \mathbb{G}_X equals the arithmetic genus $g(X)$ of X . We write $g := g(X) = g(\mathbb{G}_X)$ if X is clear from the context. The *valence* $\text{val}(v)$ of a vertex v is the number of edges adjacent to v with loops counted twice.

For a subcurve $Y \subset X$ we write $Y^c = \overline{X \setminus Y}$ for the closure of the complement in X . In particular, $Y \cap Y^c$ is a finite union of nodes. Any subcurve $Y \subset X$ corresponds to an *induced subgraph* \mathbb{G}_Y of \mathbb{G}_X , that is, a subgraph that contains all edges of \mathbb{G}_X between vertices contained in \mathbb{G}_Y . We denote by \mathbb{G}_Y^c the dual graph of Y^c .

We denote by ω_X the *dualizing sheaf* of X . It has total degree $2g - 2$. The restriction of ω_X to a subcurve Y of X has total degree $2g(Y) - 2 + |Y \cap Y^c|$. In particular, the restriction of ω_X to an irreducible component X_v has degree $2g_v - 2 + \text{val}(v)$. A curve X is *stable* if it is connected, $g(X) \geq 2$ and whenever an irreducible component X_v is smooth and rational, then $|X_v \cap X_v^c| \geq 3$. Equivalently, X is stable if it is connected and ω_X is ample.

We write \underline{d} for a *multidegree*, that is, a formal linear combination of vertices of \mathbb{G}_X with integer coefficients. In particular, we can add and subtract multidegrees coefficient-wise. We denote by \underline{d}_v the coefficient at a vertex v of \mathbb{G}_X . We write $\underline{d} \pm v$ for the multidegree obtained by adding or subtracting 1 from the coefficient at v . Any line bundle L on X has an associated multidegree $\underline{\text{deg}}(L)$, defined by $\underline{\text{deg}}(L)_v = \text{deg}(L|_{X_v})$. The *total degree* of \underline{d} is $\sum_v \underline{d}_v$, which coincides with the total degree $\text{deg}(L)$ of L if \underline{d} is the multidegree of L . A multidegree is *effective*, if $\underline{d}_v \geq 0$ for all vertices v of \mathbb{G}_X .

3 Semistable multidegrees and Brill Noether loci

In this section, we continue the preliminaries by introducing the main objects of interest for this paper—semistable multidegrees in Sect. 3.1 and special loci in Sect. 3.2.

3.1 Semistable multidegrees

We begin with semistable line bundles and their multidegrees in the sense of [5].

Definition 3.1 Let X be a stable curve. A line bundle L on X of total degree d is called *semistable* if for any subcurve $Y \subset X$ we have

$$g(Y) - 1 + (d - g + 1) \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} \leq \deg(L|_Y). \quad (2)$$

The line bundle L is called *stable* if the inequality above is strict for every proper subcurve $Y \subsetneq X$.

Whether L is semistable or stable depends only on its multidegree \underline{d} , since

$$\deg(L|_Y) = \sum_{X_v \subset Y} \underline{d}_v.$$

We will say that a multidegree \underline{d} is semistable or stable, if the corresponding line bundles are so.

Lemma 3.2 *A line bundle L is semistable or stable if and only if its residual $\omega_X \otimes L^{-1}$ is so.*

Proof Rearranging terms turns Inequality (2) into

$$d \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} - \frac{|Y \cap Y^c|}{2} \leq \deg(L|_Y). \quad (3)$$

We want to apply Inequality (3) to the subcurve $Y^c = \overline{X \setminus Y}$. Before doing so, note that $\deg(L|_{Y^c}) = d - \deg(L|_Y)$ and recall that the degree of the restriction of the dualizing sheaf ω_X to Y is $2g(Y) - 2 + |Y \cap Y^c|$. In particular, $2g(Y^c) - 2 + |Y \cap Y^c| = 2g - 2 - (2g(Y) - 2 + |Y \cap Y^c|)$. We obtain from (3) applied to Y^c :

$$\begin{aligned} d \frac{2g(Y^c) - 2 + |Y \cap Y^c|}{2g - 2} - \frac{|Y \cap Y^c|}{2} &\leq \deg(L|_{Y^c}) \\ \Leftrightarrow d - d \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} - \frac{|Y \cap Y^c|}{2} &\leq d - \deg(L|_Y) \\ \Leftrightarrow (2g - 2 - d) \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} - (2g(Y) - 2 + |Y \cap Y^c|) - \frac{|Y \cap Y^c|}{2} & \\ &\leq -\deg(L|_Y) \\ \Leftrightarrow (2g - 2 - d) \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} - \frac{|Y \cap Y^c|}{2} & \\ &\leq 2g(Y) - 2 + |Y \cap Y^c| - \deg(L|_Y) \end{aligned}$$

The last inequality is Inequality (3) for $\omega_X \otimes L^{-1}$ and hence the claim follows. \square

3.2 Brill Noether loci

Given a multidegree \underline{d} on X we write $\text{Pic}(X)$ for the Picard scheme of X , and $\text{Pic}^{\underline{d}}(X)$ for the connected component of $\text{Pic}(X)$ that parametrizes line bundles of multidegree \underline{d} . For a line bundle L , we denote by $[L]$ the corresponding point of $\text{Pic}(X)$.

Definition 3.3 The Brill Noether locus $W_{\underline{d}}^r(X)$ is the subset of $\text{Pic}^{\underline{d}}(X)$ given by

$$W_{\underline{d}}^r(X) := \left\{ [L] \in \text{Pic}^{\underline{d}}(X) \mid h^0(X, L) \geq r + 1 \right\}.$$

We will write $W_{\underline{d}}(X) := W_{\underline{d}}^0(X)$ for the effective locus.

The next proposition is the standard observation that the Brill Noether locus $W_{\underline{d}}^r(X)$ can be realized as a degeneracy locus of a map between vector bundles. We sketch the argument for the convenience of the reader, following the presentation in the proof of [4, Proposition 2.2].

Proposition 3.4 Let X be a stable curve and \underline{d} a multidegree of total degree d . Then $W_{\underline{d}}^r(X)$ is a closed subset of $\text{Pic}^{\underline{d}}(X)$ with either $W_{\underline{d}}^r(X) = \emptyset$, or each irreducible component of $W_{\underline{d}}^r(X)$ has dimension at least $\min\{g, g - (r + 1)(g - d + r)\}$.

Note, in particular, that the effective locus $W_{\underline{d}}(X)$ is either empty, or each irreducible component has dimension at least $\min\{d, g\}$.

Proof Let $\mathcal{P} \rightarrow \text{Pic}^{\underline{d}}(X) \times X$ be a Poincaré bundle, that is, a line bundle that restricts to L over $[L] \times X$. Denote by pr_1 and pr_2 the projections from $\text{Pic}^{\underline{d}}(X) \times X$ to $\text{Pic}^{\underline{d}}(X)$ and X , respectively.

Recall that we denote by ω_X the dualizing sheaf of X . We choose $D = \sum_{i=1}^k p_i$ where the $p_i \in X$ are k smooth points, such that

$$h^0(X, \omega_X \otimes L^{-1} \otimes \mathcal{O}_X(D)) = \deg(\omega_X \otimes L^{-1} \otimes \mathcal{O}_X(D)) - g + 1 = g - 1 - d + k$$

for all $[L] \in \text{Pic}^{\underline{d}}(X)$. This is always possible, see [10, Lemma 2.1] or [7, Lemma 2.5]. We get a short exact sequence

$$0 \rightarrow \mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_X(-D) \rightarrow \mathcal{P} \rightarrow \mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D \rightarrow 0.$$

We apply $(\text{pr}_1)_*$ to this sequence and set $E_1 = (\text{pr}_1)_*(\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D)$, which is locally free of rank k , and $E_2 = R^1(\text{pr}_1)_*(\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_X(-D))$, which is locally free of rank $g - 1 - d + k$ by the choice of D . Since $R^1(\text{pr}_1)_*(\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D) = 0$, the higher direct image sequence induces an exact sequence

$$E_1 \xrightarrow{u} E_2 \rightarrow R^1(\text{pr}_1)_* \mathcal{P} \rightarrow 0.$$

Over $[L] \in \text{Pic}^{\underline{d}}(X)$ it restricts to

$$(E_1)_{[L]} \xrightarrow{u_{[L]}} (E_2)_{[L]} \rightarrow H^1(X, L) \rightarrow 0.$$

By the Riemann Roch theorem, $h^0(X, L) \geq r + 1$ if and only if $h^1(X, L) \geq g - d + r$. On the other hand, the exact sequence gives $h^1(X, L) = g - 1 - d + k - \dim(\operatorname{Im}(u_{[L]}))$. Thus $W_{\underline{d}}^r(X)$ is the locus where u has rank at most $k - 1 - r$. The expected dimension of this locus is the Brill Noether number

$$\rho = g - (k - k + 1 + r)(g - 1 - d + k - k + 1 + r) = g - (r + 1)(g - d + r).$$

The claim now follows by the observation that a degeneracy locus is either empty, the whole space, or each of its irreducible components has dimension at least the expected dimension (see, e.g., [2, Section 2.4]). \square

We will be interested in loci of special line bundles, which are the first of the Brill Noether loci defined above: We call a line bundle L of total degree d *special*, if $h^0(X, L) > \max\{0, d - g + 1\}$. The term ‘special’ is of course taken from the case of irreducible curves, where non-special line bundles are dense in $\operatorname{Pic}^d(X)$.

Lemma 3.5 *Let X be a stable curve with dualizing sheaf ω_X . Then a line bundle L on X is special, if and only if its residual $\omega_X \otimes L^{-1}$ is.*

Proof Suppose $d = \deg(L) \leq g - 1$, the case $d \geq g - 1$ is analogous. By the Riemann Roch theorem we have $h^0(X, L) = d - g + 1 + h^0(X, \omega_X \otimes L^{-1})$. Thus

$$h^0(X, L) > 0 \Leftrightarrow h^0(X, \omega_X \otimes L^{-1}) > g - d - 1 = (2g - 2 - d) - g + 1,$$

and the claim follows, since $\deg(\omega_X \otimes L^{-1}) = 2g - 2 - d$. \square

Remark 3.6 By Lemmas 3.2 and 3.5, it suffices to restrict to semistable multidegrees of total degree $d \leq g - 1$ and to consider effective loci $W_{\underline{d}}(X)$ to obtain the analogous information about special loci also for $d > g - 1$ by passing to residuals. More precisely, if \underline{d} is semistable of total degree d , then $\underline{d}' = \deg(\omega_X) - \underline{d}$ is semistable of degree $2g - 2 - d$ by Lemma 3.2 and the isomorphism $\operatorname{Pic}^d(X) \rightarrow \operatorname{Pic}^{\underline{d}'}(X)$, $[L] \mapsto [\omega_X \otimes L^{-1}]$ preserves special loci by Lemma 3.5.

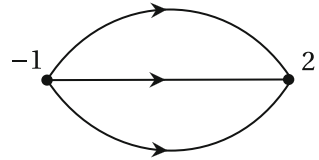
4 Theta divisors

In this section, we consider the case $d = g - 1$. Semistable multidegrees admit a very convenient description in this case, using orientations on the dual graph. We collect the necessary notions in Sect. 4.1. In Sect. 4.2, we discuss existence of Theta divisors, with the main results in Theorem 4.8. In Sect. 4.3, we recall a description of the irreducible components of Theta divisors due to Coelho and Esteves [9], that will be useful in the next section.

4.1 Graph orientations and their associated multidegrees

We begin by fixing some standard notions from graph theory. An *orientation* O on the graph \mathbb{G}_X is the assignment of a direction to each edge of \mathbb{G}_X . The multidegree

Fig. 1 An orientation and its associated multidegree. Both vertices are weightless and the three edges form a directed cut



\underline{d}_O associated to an orientation O has by definition value

$$\text{indeg}_O(v) - 1 + g_v$$

on a vertex $v \in V(\mathbb{G}_X)$, where $\text{indeg}_O(v)$ is the number of edges adjacent to v that are oriented towards v in O . A multidegree \underline{d} is called *orientable*, if there is an orientation O of the edges of \mathbb{G}_X such that $\underline{d} = \underline{d}_O$. Note that an orientation O giving \underline{d} need not be unique. An orientable multidegree has total degree

$$\begin{aligned} \sum_{v \in \mathbb{G}_X} \underline{d}_v &= \sum_{v \in V(\mathbb{G}_X)} (g_v - 1 + \text{indeg}_O(v)) \\ &= |E(\mathbb{G}_X)| - |V(\mathbb{G}_X)| + \sum_{v \in V(\mathbb{G}_X)} g_v = g - 1. \end{aligned}$$

Lemma 4.1 *Let L be a line bundle on X of multidegree \underline{d} . Assume $\underline{d} = \underline{d}_O$ and let O' be the orientation obtained from O by reversing the direction of each edge. Then*

$$\underline{\text{deg}}(\omega_X \otimes L^{-1}) = \underline{d}_{O'}.$$

Proof For any vertex $v \in V(\mathbb{G}_X)$, we have by definition $\underline{d}_v = g_v - 1 + \text{indeg}_O(v)$. On the other hand,

$$\underline{\text{deg}}(\omega_X \otimes L^{-1})_v = 2g_v - 2 + \text{val}(v) - \underline{d}_v = g_v - 1 + \text{val}(v) - \text{indeg}_O(v).$$

Any loop based at v contributes 2 to $\text{val}(v)$ and 1 to $\text{indeg}_O(v)$. Any other edge adjacent to v is either oriented towards v or away from it. Thus $\text{val}(v) - \text{indeg}_O(v)$ equals the number of loop edges based at v plus the number of edges adjacent to v and oriented away from v in O . This is by construction $\text{indeg}_{O'}(v)$ and hence the claim follows. \square

Recall that any subcurve $Y \subset X$ defines an induced subgraph $\mathbb{G}_Y \subset \mathbb{G}_X$. The subset of edges of \mathbb{G}_X that are adjacent to a vertex in \mathbb{G}_Y and a vertex not in \mathbb{G}_Y is called the *cut* defined by \mathbb{G}_Y . They correspond to the nodes in $Y \cap Y^c$. If O is an orientation on \mathbb{G}_X , a cut is called a *directed cut* if every edge in the cut is directed towards \mathbb{G}_Y .

A *cycle* in \mathbb{G}_X is a connected subgraph, in which each vertex has valence 2. A *directed cycle* in an orientation O is a cycle in \mathbb{G}_X , such that every vertex has exactly one edge directed towards it in the restriction of O to the cycle; in particular, a loop edge gives a directed cycle in any orientation.

Lemma 4.2 *For every orientation O on \mathbb{G}_X , any edge e of \mathbb{G}_X is contained in a directed cut or a directed cycle, but not both.*

Proof This is well-known and usually attributed to [18]. We sketch a proof for the convenience of the reader. Let e be directed in O from v to w .

Suppose e is contained in a directed cut and a directed cycle. Then the directed cut would restrict to a non-empty directed cut on the directed cycle, which is not possible. Thus e is not contained in a directed cut and a directed cycle.

To see that e is contained in a directed cut or a directed cycle, let $V \subset V(\mathbb{G}_X)$ denote the set of vertices v' for which there exists a directed path from w to v' in O . If $v \in V$, e completes the directed path from w to v to a directed cycle that contains e . If $v \notin V$, the induced subgraph with vertices V defines a directed cut containing e . \square

An orientation O is called *acyclic* if it contains no directed cycles, and hence by Lemma 4.2 every edge is contained in a directed cut. It is called *totally cyclic* if every edge is contained in a directed cycle, and hence by Lemma 4.2 it contains no directed cuts.

A *source* of an orientation O is a vertex v such that all edges adjacent to v are oriented away from v . The following lemma is another well-known fact from graph theory.

Lemma 4.3 *Let \mathbb{G}_X be a graph and O an orientation on \mathbb{G}_X .*

- (1) *The multidegree \underline{d}_O is not effective, if and only if O contains a source v with $g_v = 0$.*
- (2) *If O is an acyclic orientation, then it contains a source v .*

Proof The first claim follows immediately from the definition of \underline{d}_O .

For the second claim observe that if O is acyclic, then Lemma 4.2 implies that \mathbb{G}_X contains a directed cut oriented towards an induced subgraph \mathbb{G}_Y . The restriction of an acyclic orientation remains acyclic, and hence the restriction of O to a connected component of \mathbb{G}_Y^c is acyclic. We can repeat this argument, until \mathbb{G}_Y^c , the complement of \mathbb{G}_Y , contains a single vertex, which by construction will be a source. \square

The following proposition motivates our interest in orientations in the context of semistable multidegrees.

Proposition 4.4 [3, Proposition 3.6] *Let X be a stable curve and \underline{d} a multidegree of total degree $g - 1$. Then \underline{d} is semistable, if and only if \underline{d} is orientable. It is stable, if and only if it can be given by a totally cyclic orientation.*

Remark 4.5 Inserting $d = g - 1$ in Definition 3.1, a multidegree \underline{d} of total degree $g - 1$ is semistable, if and only if for every induced subgraph $\mathbb{G}_Y \subset \mathbb{G}_X$, the total degree of \underline{d} on \mathbb{G}_Y is at least $g(\mathbb{G}_Y) - 1$. In this formulation, Proposition 4.4 is a version of Hakimi's Theorem [16].

4.2 Existence of Theta divisors

Apart from the combinatorial characterization of semistability in terms of orientations, there is the following characterization in terms of the existence of non-special line bundles.

Proposition 4.6 [4, Lemma 2.1] *Let X be a stable curve and \underline{d} a multidegree of total degree $g - 1$. Then \underline{d} is semistable, if and only if there is a line bundle L of multidegree \underline{d} with $h^0(X, L) = 0$.*

Note that for $d = g - 1$, the expected dimension of the effective locus $W_{\underline{d}}(X)$ is $g - 1$. Thus combining with Proposition 3.4, it follows that if \underline{d} is semistable, $W_{\underline{d}}(X)$ is either empty or a divisor in $\text{Pic}^{\underline{d}}(X)$, the so called *Theta divisor*. The next example shows, that the effective locus can indeed be empty.

Example 4.7 Consider a curve X with dual graph \mathbb{G}_X that has two vertices v, w of weight 0 and k edges between v and w . Let the multidegree \underline{d} be given as $\underline{d}_v = -1$ and $\underline{d}_w = k - 1$ with total degree $g - 1 = k - 2$. Then $\underline{d} = \underline{d}_O$, where O is given by orienting all edges from v to w , and hence \underline{d} is semistable by Proposition 4.4. See Fig. 1 for the case $k = 3$. Let L be any line bundle of multidegree \underline{d} on X . Since L has negative degree on X_v , any global section s of L vanishes along all of X_v . In particular, the restriction of s to X_w also vanishes at the k points in $X_v \cap X_w$. Since $L|_{X_w}$ has degree $k - 1$, this is only possible for the zero section. Thus $h^0(X, L) = 0$ for all line bundles of multidegree \underline{d} and $W_{\underline{d}}(X) = \emptyset$.

Recall that a multidegree \underline{d} is called effective, if $\underline{d}_v \geq 0$ for all vertices v of \mathbb{G}_X .

Theorem 4.8 *Let X be a stable curve and \underline{d} a multidegree of total degree $g - 1$. Then we have for the effective locus $W_{\underline{d}}(X) \subset \text{Pic}^{\underline{d}}(X)$:*

- (1) $W_{\underline{d}}(X)$ is empty or has pure dimension $g - 1$ if and only if \underline{d} is semistable.
- (2) $W_{\underline{d}}(X)$ is empty if and only if all irreducible components of X are rational and $\underline{d} = \underline{d}_O$ with O acyclic. If \underline{d} is effective, $W_{\underline{d}}(X)$ is not empty.

Proof As mentioned above, the first claim follows by Propositions 3.4 and 4.6. For the second claim, we show more generally that for an orientable multidegree $\underline{d} = \underline{d}_O$ on a nodal curve X , which is not necessarily stable or connected, $W_{\underline{d}}(X) = \emptyset$ if and only if $g_v = 0$ for all $v \in V(\mathbb{G}_X)$ and O is acyclic. Note first, that to prove this stronger claim, we still may assume that X is connected, since the condition can be checked on each connected component.

We first prove the last claim: if \underline{d} is effective, then $W_{\underline{d}}(X) \neq \emptyset$. Indeed, in this case let p_i be a collection of smooth points of X , \underline{d}_v of them contained in X_v . Then $\mathcal{O}_{X_v}(\sum_{i=1}^{\underline{d}_v} p_i)$ is effective of multidegree \underline{d} . Furthermore, if \underline{d} is effective and orientable, either $g_v \neq 0$ for some v or O is not acyclic by Lemma 4.3. Thus we can assume from now on that \underline{d} is not effective.

We show the claim by induction on the number of non-loop edges of \mathbb{G}_X . The base case is if \mathbb{G}_X contains no non-loop edges, that is, X is irreducible and \mathbb{G}_X contains a single vertex v . Since \underline{d} is orientable and not effective, we need to have $g_v = 0$ and \mathbb{G}_X contains no edges, in which case $\underline{d}_v = -1$ and $W_{\underline{d}}(X) = \emptyset$ is immediate.

For the induction step, let v be a vertex such that $\underline{d}_v < 0$. By Lemma 4.3 (1), we need to have $g_v = 0$ and v is a source in O . Let X_v^c be the closure of the complement of X_v in X . For a line bundle L on X of multidegree \underline{d} , let \underline{d}' denote the multidegree of $L|_{X_v^c}(-X_v \cap X_v^c)$. Since all vertices are oriented away from v in O , \underline{d}' is orientable. Namely, it is given by the restriction of O to the dual graph of X_v^c . Since any global

section of L vanishes along all of X_v , we have

$$h^0(X, L) = h^0(X_v^c, L|_{X_v^c}(-X_v \cap X_v^c)). \quad (4)$$

Indeed, $H^0(X_v^c, L|_{X_v^c}(-X_v \cap X_v^c))$ is naturally identified with the kernel of the evaluation map $H^0(X, L) \rightarrow H^0(X_v, L|_{X_v})$. If all global sections of L vanish along X_v , then $H^0(X, L)$ coincides with this kernel and (4) follows.

Now suppose, that there is a line bundle L on X with multidegree \underline{d} and $h^0(X, L) \geq 1$. Then also $h^0(X_v^c, L|_{X_v^c}(-X_v \cap X_v^c)) \geq 0$ by (4). By induction it follows that there is either $w \in V(\mathbb{G}_X) \setminus \{v\}$ with $g_w \neq 0$ or $\underline{d}' = \underline{d}_{O'}$ with O' not acyclic. In the first case, X_w is a non-rational component also of X . In the second case, we may extend O' to an orientation O on \mathbb{G}_X by orienting the edges adjacent to v away from v . By construction, we have $\underline{d} = \underline{d}_O$. If C is an oriented cycle of O' , then it is also an oriented cycle of O and hence O is not acyclic.

Conversely, suppose that $W_{\underline{d}}(X)$ is empty. Then also $W_{\underline{d}'}(X_v^c)$ is empty by (4). So by induction, $g_w = 0$ for all $w \in V(\mathbb{G}_X) \setminus \{v\}$ and $\underline{d}' = \underline{d}_{O'}$ with O' acyclic. By assumption, we have also $g_v = 0$, and hence all irreducible components of X are rational. We can extend O' to an orientation O with $\underline{d} = \underline{d}_O$ as above by orienting the edges adjacent to v away from v . We are finished if we can show that O is acyclic. To this end, suppose to the contrary that C is an oriented cycle of O . By Lemma 4.2, C contains none of the edges adjacent to v since they are all contained in a directed cut. Thus C would restrict to an oriented cycle of O' , which is not possible. \square

Remark 4.9 Example 4.7 contradicts [4, Proposition 2.2], in which it is claimed that for total degree $d = g - 1$, the effective locus is a divisor in $\text{Pic}^d(X)$ if and only if \underline{d} is semistable. The possibility of an empty effective locus is not accounted for in the proof of *loc. cit.*, and Theorem 4.8 provides the necessary correction.

In [4] Theta divisors are used to define a polarization on generalized Prym varieties. The multidegree \underline{d} giving the Theta divisor is that of a Theta characteristic L_0 , that is, a line bundle such that $L_0^{\otimes 2}$ is isomorphic to the dualizing sheaf of X . Thus the multidegree \underline{d} of a Theta characteristic has value $\underline{d}_v = g(X_v) - 1 + \frac{|X_v \cap X_v^c|}{2}$ on X_v . In particular, \underline{d} is effective whenever X is not a disjoint union of smooth rational curves. The subcurves used in [4, Proposition 5.2 and Theorem 5.4] to define the generalized Prym varieties are not the disjoint union of smooth rational curves. Thus no issue arises, since a Theta divisor exists if \underline{d} is effective and semistable by Theorem 4.8.

The second main application of Theta divisors concerns the theory of compactified Jacobians. Recall that the compactified Jacobian admits a stratification with strata $P^{(\underline{d}, S)}$, parametrizing line bundles that have stable multidegree \underline{d} on the partial normalization of X at a subset of nodes S . By Proposition 4.4, a stable multidegree \underline{d} in degree $g - 1$ is given by a totally cyclic orientation on the dual graph of this partial normalization, i.e., a subgraph of \mathbb{G}_X . Since a totally cyclic orientation is acyclic only if the graph has no edges, Theorem 4.8 gives that the only stratum in which $W_{\underline{d}}(X)$ is not a divisor is the one in which S is the set of all nodes and then also only if $\bar{g}_v = 0$ for all irreducible components X_v of X . But in this case $P^{(\underline{d}, S)}$ has dimension 0 so no issue arises. In particular, the use of [4, Proposition 2.2] as [3, Lemma 3.8] and

[6, Proposition 1.3.7] to extend the Theta divisor to compactified Jacobians causes no issues.

4.3 Rational Abel maps and components of Theta divisors

We next recall a description of the irreducible components of a Theta divisor $W_{\underline{d}}(X)$ given in [9].

For an irreducible component X_v of X , denote by X_v^{sm} the locus of points in X_v that are smooth points of X . Following [6, Section 1.2.7], we define the *rational Abel map* associated to an effective multidegree \underline{d} of total degree d as the map

$$\alpha_{\underline{d}}: \prod_{v \in V(\mathbb{G}_X)} (X_v^{\text{sm}})^{d_v} \rightarrow \text{Pic}^d(X),$$

given by sending a d -tuple of points (p_1, \dots, p_d) to $\mathcal{O}_X\left(\sum_{i=1}^d p_i\right)$. We denote the image of $\alpha_{\underline{d}}$ by $A_{\underline{d}}(X)$; clearly, it is irreducible of dimension at most $\min\{d, g\}$.

Lemma 4.10 *Let \underline{d} be an effective multidegree of total degree $0 \leq d \leq g$. Then the dimension of $A_{\underline{d}}(X)$ is d if and only if a general $[L] \in A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$.*

Proof The domain of $\alpha_{\underline{d}}$ is irreducible of dimension d , thus $A_{\underline{d}}(X)$ has dimension d if and only if $\alpha_{\underline{d}}$ is generically finite.

The fiber of $\alpha_{\underline{d}}$ over a point $[L] = \left[\mathcal{O}_X\left(\sum_{i=1}^d p_i\right)\right]$ in the image of $\alpha_{\underline{d}}$ consists of tuples (p'_1, \dots, p'_d) such that $\mathcal{O}_X\left(\sum_{i=1}^d p_i\right) \simeq \mathcal{O}_X\left(\sum_{i=1}^d p'_i\right)$. In particular, if $h^0(X, L) = 1$ for a general L in $A_{\underline{d}}(X)$, then $\alpha_{\underline{d}}$ is generically finite.

Conversely, suppose a line bundle of the form $L = \mathcal{O}_X\left(\sum_{i=1}^d p_i\right)$ admits two linearly independent global sections s_1, s_2 , the first of them vanishing only at the p_i . Then the subspace of $H^0(X, L)$ spanned by the s_i consists of sections of the form $as_1 + bs_2$ for $a, b \in k$. For general choices of a, b , these global sections do not vanish along nodes or entire irreducible components of X , since this is true for s_1 . Hence their associated divisor is of the form $p'_1 + \dots + p'_d$ for smooth points p'_i and $\alpha_{\underline{d}}$ has fibers of dimension at least 1 over such points $[L]$. Thus if the general $[L] \in A_{\underline{d}}$ satisfies $h^0(X, L) \geq 2$, then $\alpha_{\underline{d}}$ is not generically finite, as claimed. \square

Remark 4.11 The assumption of Lemma 4.10, that a general $[L] \in A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$, is not automatically satisfied, even if $d \leq g$. For example, suppose X contains a smooth rational component X_v , and \underline{d} is the multidegree with value $d_v = |X_v \cap X_v^c| + 1$ on v and 0 on all other irreducible components of X . Then every line bundle L in $A_{\underline{d}}(X)$ satisfies $h^0(X, L) \geq 2$ even though \underline{d} is effective. In this case $\dim(A_{\underline{d}}(X)) < d$ and it follows from Proposition 3.4, that $A_{\underline{d}}(X)$ is not an irreducible component of the effective locus $W_{\underline{d}}(X)$.

Let L be a line bundle of multidegree \underline{d} and $Y \subset X$ a subcurve. Let $\nu: X^{\nu} \rightarrow X$ be the partial normalization of X at nodes in $Y \cap Y^c$. We obtain the pull-back map

$$\nu^*: \text{Pic}^d(X) \rightarrow \text{Pic}^{d_Y}(Y) \times \text{Pic}^{d_{Y^c}}(Y^c),$$

where \underline{d}_Y and \underline{d}_{Y^c} are the multidegrees of $L|_Y$ and $L|_{Y^c}$, respectively. Suppose now, that the multidegree \underline{d}' of $L|_Y(-Y \cap Y^c)$ is effective on Y . Then we can define

$$V := \left\{ [L'(Y \cap Y^c)] \mid [L'] \in A_{\underline{d}'}(Y) \right\} \subset \text{Pic}^{\underline{d}_Y}(Y)$$

and

$$W_{\underline{d},Y}(X) := (\nu^*)^{-1} \left(\text{Pic}^{\underline{d}_{Y^c}}(Y^c) \times V \right) \subset \text{Pic}^{\underline{d}}(X).$$

Thus, roughly speaking, $W_{\underline{d},Y}(X)$ is the locus of line bundles on X of multidegree \underline{d} , that are in the image of a rational Abel map on Y shifted by $Y \cap Y^c$, and arbitrary away from Y .

Recall that by Proposition 4.4 a multidegree \underline{d} of total degree $g - 1$ is semistable if and only if it is orientable, i.e., $\underline{d} = \underline{d}_O$ for some orientation O on \mathbb{G}_X .

Proposition 4.12 [9, Theorem 3.6] *Let X be a stable curve and $\underline{d} = \underline{d}_O$ a semistable multidegree of total degree $g - 1$. Then*

$$W_{\underline{d}}(X) = \bigcup_Y W_{\underline{d},Y}(X),$$

and the $W_{\underline{d},Y}(X)$ are irreducible divisors in $\text{Pic}^{\underline{d}}(X)$. The union is over connected subcurves $Y \subset X$, such that the multidegree associated to the restriction $O|_{\mathbb{G}_Y}$ of O is effective on the dual graph $\mathbb{G}_Y \subset \mathbb{G}_X$, and \mathbb{G}_Y defines a directed cut with edges oriented towards \mathbb{G}_Y in O .

Remark 4.13 In case that $\underline{d} = \underline{d}_O$ is a stable multidegree, O contains by Proposition 4.4 no directed cut except for the empty one given by $Y = X$. In this case, Proposition 4.12 implies that $W_{\underline{d}}(X)$ is an irreducible divisor, as was established in [6, Theorem 3.1.2]. Notice also, that Theorem 4.8 is consistent with Proposition 4.12 concerning emptiness of the effective locus: one can argue similar as in the proof of Theorem 4.8, to show that there exists no connected subcurve Y of X as required in Proposition 4.12 if and only if X has only rational components and O is acyclic.

Lemma 4.14 *Let $W_{\underline{d},Y}(X)$ be an irreducible component of $W_{\underline{d}}(X)$ as in Proposition 4.12. Let $p \in X_\nu$ be a smooth point with $X_\nu \subset Y$. Then a general line bundle $[L] \in W_{\underline{d},Y}(X)$ satisfies $h^0(X, L) = 1$ and p is not a base point of L .*

Proof The first claim, that $h^0(X, L) = 1$ for L general in $W_{\underline{d},Y}(X)$, is part of [9, Proposition 3.5]. The second claim follows, since global sections of L vanishing on Y^c can by construction of $W_{\underline{d},Y}(X)$ be identified with those of a sheaf of the form $\mathcal{O}_Y \left(\sum_{i=1}^k p_i \right)$ for some smooth points p_i . Thus for L such that $p \neq p_i$ for all i , p is not a base point of L . \square

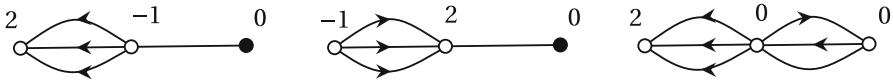


Fig. 2 Multidegrees given by partial orientations as in Remark 5.2, where bold vertices have weight 1 and circled ones weight 0. Only the multidegree in the middle picture is semistable

5 The cases $d = g - 2$ and $d = g$

In this section, we study effective loci for semistable multidegrees of total degree $d = g - 2$, and its residual case $d = g$. The main results are Proposition 5.4 and Theorem 5.10.

5.1 Dimension of special loci

In [8], an analogue in degree g of the description in terms of orientations in degree $g - 1$ of Proposition 4.4 was established. Recall that for an orientation O an induced subgraph $\mathbb{G}_Y \subset \mathbb{G}_X$ induces a directed cut, if all edges of \mathbb{G}_X adjacent to \mathbb{G}_Y but not contained in \mathbb{G}_Y are directed towards \mathbb{G}_Y in O .

Lemma 5.1 [8, Lemmas 1.7.4 and 3.3.2] *A multidegree \underline{d} of total degree g is semistable if and only if there is $v \in V(\mathbb{G}_X)$ and an orientation O on \mathbb{G}_X , such that $\underline{d} = \underline{d}_O + v$ and there is no directed cut oriented towards an induced subgraph \mathbb{G}_Y with $v \in V(\mathbb{G}_Y)$. If \underline{d} is semistable, such an orientation exists for every choice of vertex $v \in V(\mathbb{G}_X)$.*

Remark 5.2 Recall that if $\text{deg}(L) = \underline{d}_O$ is an orientable multidegree, then $\text{deg}(\omega_X \otimes L^{-1}) = \underline{d}_{O'}$, where O' is the orientation obtained from O by reversing the direction of all edges (see Lemma 4.1). Since semistability is preserved in passing to the residual by Lemma 3.2, Lemma 5.1 thus gives the following characterization in total degree $d = g - 2$. A multidegree \underline{d} of total degree $d = g - 2$ is semistable if and only if for every vertex $v \in V(\mathbb{G}_X)$, we have $\underline{d} = \underline{d}_O - v$ with O an orientation on \mathbb{G}_X such that there is no directed cut oriented away from an induced subgraph \mathbb{G}_Y with $v \in V(\mathbb{G}_Y)$. If \underline{d} is semistable, such an orientation exists for every choice of vertex $v \in V(\mathbb{G}_X)$.

The characterizations of semistability in Lemma 5.1 and Remark 5.2 can be encoded conveniently by biorienting one edge, respectively leaving one edge unoriented in the orientation. The condition then is that all directed cuts are oriented away from the bioriented edge, respectively towards the unoriented edge. See [8] for examples in degree g and Fig. 2 for an illustration in degree $g - 2$.

Lemma 5.3 *Let X be a stable curve and \underline{d} a multidegree of total degree d .*

- (1) *If $d = g - 2$, \underline{d} is semistable if and only if $\underline{d} + v$ is semistable for all vertices $v \in V(\mathbb{G}_X)$.*
- (2) *If $d = g$, \underline{d} is semistable if and only if $\underline{d} - v$ is semistable for all vertices $v \in V(\mathbb{G}_X)$.*

Proof We show the second claim and assume $d = g$. The first statement for $d = g - 2$ follows by Lemma 3.2 from the second statement via passing to residuals. So

suppose first, that \underline{d} is semistable of total degree g . Then Lemma 5.1 implies that $\underline{d} - v$ is orientable for every vertex $v \in V(\mathbb{G}_X)$, and hence $\underline{d} - v$ is semistable by Proposition 4.4.

Conversely, suppose $\underline{d} - v$ is semistable for every vertex $v \in V(\mathbb{G}_X)$. Again by Proposition 4.4, it follows that $\underline{d}_O = \underline{d} - v$ for some orientation O . To conclude that \underline{d} is semistable, it suffices by Lemma 5.1 to show, that for every directed cut that is oriented towards an induced subgraph \mathbb{G}_Y we have $v \notin V(\mathbb{G}_Y)$.

Suppose to the contrary, that $v \in V(\mathbb{G}_Y)$ and \mathbb{G}_Y induces a directed cut oriented towards \mathbb{G}_Y . Let \mathbb{G}_Y^c be the dual graph of $Y^c = X \setminus Y$. Since by assumption all edges that are directed towards a vertex in \mathbb{G}_Y^c are already contained in \mathbb{G}_Y^c , the restriction of \underline{d} to \mathbb{G}_Y^c is induced by the restriction of O to \mathbb{G}_Y^c . In particular, we have $\sum_{X_w \subset Y^c} \underline{d}_w = g(\mathbb{G}_Y^c) - 1$. Now fix a $w' \in V(\mathbb{G}_Y^c)$ and consider $\underline{d} - w'$. Then

$$\sum_{X_w \subset Y^c} (\underline{d} - w')_w = g(\mathbb{G}_Y^c) - 2 < g(\mathbb{G}_Y^c) - 1$$

and thus $\underline{d} - w'$ is not semistable by Remark 4.5. This contradicts the assumption. \square

Proposition 5.4 *Let X be a stable curve and \underline{d} a semistable multidegree of total degree d .*

- (1) *If $d = g - 2$, the effective locus $W_{\underline{d}}(X)$ is either empty, or of pure dimension $g - 2$.*
- (2) *If $d = g$, the special locus $W_{\underline{d}}^1(X)$ is either empty, or of pure dimension $g - 2$.*

Proof Suppose first $d = g - 2$. If $W_{\underline{d}}(X) = \emptyset$, there is nothing to show. Otherwise, let $W \subset W_{\underline{d}}(X)$ be an irreducible component. Let $v \in V(\mathbb{G}_X)$ be a vertex of the dual graph. By Proposition 4.4 and Lemma 5.3, we have $\underline{d} + v = \underline{d}_O$ for some orientation O of \mathbb{G}_X . Choose a smooth point $p \in X_v$ and consider the isomorphism

$$\varphi_p: \text{Pic}^{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}_O}(X), [L] \mapsto [L(p)].$$

If L is effective, then so is $L(p)$. Thus the image $\varphi_p(W)$ is contained in an irreducible component of the effective locus in $\text{Pic}^{\underline{d}_O}(X)$. Recall that Proposition 4.12 gives a description of such irreducible components $W_{\underline{d}_O, Y}(X)$. Here Y is a subcurve such that \mathbb{G}_Y induces a directed cut in O , oriented towards \mathbb{G}_Y . So we may assume $\varphi_p(W) \subset W_{\underline{d}_O, Y}(X)$.

By Remark 5.2, we need to have $X_v \subset Y$ since the cut induced by \mathbb{G}_Y is directed towards \mathbb{G}_Y in O and \underline{d} is semistable. Thus we can apply Lemma 4.14, and obtain that for a general line bundle $[L'] \in W_{\underline{d}_O, Y}(X)$, we have $h^0(X, L') = 1$ and p is not a base point of L' . This in turn implies that $\varphi_p(W) \neq W_{\underline{d}_O, Y}(X)$ since either p is a base point of $L(p)$ or $h^0(X, L(p)) = h^0(X, L) + 1 \geq 2$. By Theorem 4.8, we have $\dim(W_{\underline{d}_O, Y}(X)) = g - 1$ since \underline{d}_O is semistable. Since $\varphi_p(W)$ is a closed subset of $W_{\underline{d}_O, Y}(X)$, this implies $\dim(W) \leq g - 2$. The claim now follows from Proposition 3.4, since the expected dimension of W is $d = g - 2$.

For $d = g$ the claim follows from the case of $d = g - 2$ by passing to residuals (see Remark 3.6). \square

5.2 Characterizing semistability

Recall that Theorem 4.8 (1) states that in degree $g - 1$ the effective locus is empty or of expected dimension if and only if \underline{d} is semistable. Thus Proposition 5.4 generalizes the ‘if’ part of this statement to degree $g - 2$. The ‘only if’ part no longer holds, as the following examples show.

Example 5.5 Let \mathbb{G}_X be the graph with three vertices v_1, v_2, v_3 of weight $(0, 0, 1)$, a triple edge between v_1 and v_2 , and a single edge between v_2 and v_3 . Consider the multidegree $\underline{d} = (2, -1, 0)$. Its total degree is $1 = g - 2$ and one checks that \underline{d} is not semistable. See the left picture in Fig. 2 for an illustration. However, the effective locus $W_{\underline{d}}(X)$ is empty in this case. Notice also, that the multidegree $(-1, 2, 0)$ on the same graph, as in the middle picture in Fig. 2, is semistable with empty effective locus.

Example 5.6 Let \mathbb{G}_X be the graph with vertices v_1, v_2, v_3 , all of weight 0, and three edges between each of the pairs v_1, v_2 and v_2, v_3 . Denote by X_i the irreducible component of X corresponding to v_i . Consider the multidegree $\underline{d} = (2, 0, 0)$. Its total degree is $2 = g - 2$ and one checks that \underline{d} is not semistable. See the right picture in Fig. 2 for an illustration. Effective line bundles $[L] \in \text{Pic}^{\underline{d}}(X)$ are exactly those with $L|_{X_2 \cup X_3} = \mathcal{O}_{X_2 \cup X_3}$. Thus $W_{\underline{d}}(X)$ is irreducible and 2-dimensional, the two parameters being given by the gluing data along the three nodes in $X_1 \cap X_2$.

Note that for the dual graph in Example 5.5, the effective locus for a semistable multidegree of total degree $g - 1$ is never empty by Theorem 4.8 (2) since there is v with $g_v \geq 1$, whereas it is empty for the semistable multidegree $(-1, 2, 0)$ of total degree $g - 2$. In general, we do not know, how to generalize the second part of Theorem 4.8 to degree $g - 2$, that is:

Question 5.7 When is the effective locus for a semistable multidegree of total degree $g - 2$ empty?

Examples 5.5 and 5.6 show in particular, that semistability in degree $g - 2$ cannot be characterized by the existence of non-special line bundles as in Proposition 4.6 for degree $g - 1$. We next give a refinement that does allow for a similar characterization.

Lemma 5.8 *Let X be a stable curve and \underline{d} a multidegree of total degree $g - 2$. Then \underline{d} is semistable if and only if there is a line bundle L of multidegree \underline{d} such that $h^0(X, L) = 0$ and for every irreducible component X_v of X and a general point $p_v \in X_v$, we still have $h^0(X, L(p_v)) = 0$.*

Proof If there is a line bundle L of multidegree \underline{d} as described in the lemma, pick a smooth point $p_v \in X_v$ for every irreducible component X_v such that $h^0(X, L(p_v)) = 0$. Thus $\underline{d} + v$ is semistable for every $v \in V(\mathbb{G}_X)$ by Proposition 4.6 and \underline{d} is semistable by Lemma 5.3.

Conversely, suppose \underline{d} is semistable. Then $\underline{d} + v$ is semistable for all vertices $v \in V(\mathbb{G}_X)$ by Lemma 5.3. Choose a smooth point $p_v \in X_v$ on each irreducible component X_v . This gives the isomorphism

$$\varphi_{p_v} : \text{Pic}^{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}+v}(X), [L] \mapsto [L(p_v)].$$

Under this identification, there is a dense open set $U_v \subset \text{Pic}^{\underline{d}}(X)$ such that for all line bundles $[L] \in U_v$ we have $h^0(X, L(p_v)) = 0$ by Theorem 4.8, since $\underline{d} + v$ is semistable. Then an $[L]$ in the intersection of the finitely many U_v satisfies the conditions in the lemma by construction. \square

The residual statement of Lemma 5.8 gives the statement of Theorem 1.3. To prove it, we will need the following easy consequence of the Riemann Roch theorem:

Lemma 5.9 *Let X be a stable curve, $p \in X$ a smooth point and L a line bundle on X . Then p is a base point of L if and only if it is not a base point of $(\omega_X \otimes L^{-1})(p)$.*

Proof Applying the Riemann Roch theorem gives on the one hand

$$h^0(X, L) - h^0\left(X, \omega_X \otimes L^{-1}\right) = d - g + 1$$

and on the other

$$h^0(X, L(-p)) - h^0\left(X, (\omega_X \otimes L^{-1})(p)\right) = d - 1 - g + 1.$$

Hence $h^0(X, L) = h^0(X, L(-p))$ if and only if $h^0(X, \omega_X \otimes L^{-1}) = h^0(X, (\omega_X \otimes L^{-1})(p)) - 1$. \square

Recall that we defined the rational Abel map in Sect. 4.3.

Theorem 5.10 *Let X be a stable curve and \underline{d} a multidegree of total degree g . Then the following are equivalent:*

- (1) *The multidegree \underline{d} is semistable.*
- (2) *There is a line bundle L of multidegree \underline{d} such that $h^0(X, L) = 1$ and non-zero global sections of L do not vanish along a whole irreducible component X_v of X .*
- (3) *The multidegree \underline{d} is effective and the associated rational Abel map $\alpha_{\underline{d}}$ is dominant.*

Proof By Lemma 3.2 a line bundle L is semistable if and only if its residual $\omega_X \otimes L^{-1}$ is so. If L has total degree g , then $\omega_X \otimes L^{-1}$ has total degree $g - 2$. Thus, using Lemma 5.8, \underline{d} is semistable if and only if there is a line bundle L of multidegree \underline{d} , such that $h^0(X, \omega_X \otimes L^{-1}) = 0$ and for any irreducible component X_v of X and general point $p_v \in X_v$, p_v is a base point of $(\omega_X \otimes L^{-1})(p_v)$. By the Riemann Roch theorem and Lemma 5.9, these two conditions translate to condition (2) on L . Thus equivalence of (1) and (2) follows.

Next, we show that (3) implies (2). If the image $A_{\underline{d}}(X)$ of the rational Abel map $\alpha_{\underline{d}}$ has dimension g , then a general $[L]$ in $A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$ by Lemma 4.10. By definition, such an L is of the form $\mathcal{O}_X\left(\sum_{i=1}^d p_i\right)$, which satisfies also the second part of condition (2), since the global sections of \mathcal{O}_X do not vanish along whole irreducible components of X .

Finally, suppose that \underline{d} satisfies the equivalent conditions (1) and (2). If $\deg(L|_{X_v}) < 0$ for some irreducible component X_v , any global section of L vanishes along all of X_v . Thus (2) immediately implies that \underline{d} is effective and thus the rational Abel map

$\alpha_{\underline{d}}$ is defined. By Lemma 4.10, if a general $[L] \in A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$, then $\dim(A_{\underline{d}}(X)) = d = g$. Since this is also the dimension of $\text{Pic}^d(X)$, it suffices to show this in order to prove (3).

Since $d > 0$ and \underline{d} is effective, there exists a vertex v , such that $\underline{d} - v$ is still effective. Hence we can consider the corresponding rational Abel map and its image $A_{\underline{d}-v}(X) \subset \text{Pic}^{d-v}(X)$. Fix a smooth point $p_v \in X_v$ and consider once again the isomorphism

$$\varphi_{p_v} : \text{Pic}^{d-v}(X) \rightarrow \text{Pic}^d(X), [L] \mapsto [L(p_v)].$$

Then by construction $\varphi_{p_v}(A_{\underline{d}-v}(X)) \subset A_{\underline{d}}(X)$. Since $\underline{d} - v$ is semistable by Lemma 5.3, it follows that $A_{\underline{d}-v}(X)$ is an irreducible component of the effective locus in $\text{Pic}^{d-v}(X)$ by Proposition 4.12 and has dimension $g - 1$ by Theorem 4.8. Since φ_{p_v} is an isomorphism, we get that $A_{\underline{d}}(X)$ has dimension at least $g - 1$. In particular, $A_{\underline{d}}(X)$ cannot be contained in the special locus, which by Proposition 5.4 is either empty or has dimension $g - 2$. In other words, a general $[L] \in A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$, as claimed. \square

6 Effective loci for general semistable multidegrees

Finally, we consider in this section the dimension of effective loci for semistable multidegrees of total degree $d \leq g - 2$. In Sect. 6.1 we lay some combinatorial groundwork, and in Sect. 6.2 we prove Theorem 6.5, the main result of this section. We conclude with some counterexamples to possible stronger claims in Sect. 6.3.

6.1 Combinatorial considerations

Recall that a line bundle L of total degree d is by definition semistable if and only if

$$g(Y) - 1 + (d - g + 1) \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} \leq \deg(L|_Y),$$

for every subcurve Y of X . Recall furthermore, that if $d = g$ and X is stable, this is equivalent to requiring

$$g(Y) \leq \deg(L|_Y)$$

for every subcurve Y of X , since $0 < \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} < 1$ for proper subcurves Y .

Lemma 6.1 *Let X be a stable curve and L a line bundle of total degree $d > g$. Suppose $g(Y) \leq \deg(L|_Y)$ for all subcurves $Y \subset X$. Then there is a vertex $v \in V(\mathbb{G}_X)$ such that for all subcurves Y with $X_v \subset Y$ the strict inequality $g(Y) < \deg(L|_Y)$ holds.*

Proof We first claim, that if L is as in the assumptions of the lemma and Y is a subcurve such that $\deg(L|_Y) = g(Y)$, then Y needs to be connected. Indeed, if to the contrary

$Y = Y_1 \sqcup Y_2$, then $g(Y) = g(Y_1) + g(Y_2) - 1$. Since $\deg(L|_Y) = \deg(L|_{Y_1}) + \deg(L|_{Y_2})$ we thus need to have $\deg(L|_{Y_1}) < g(Y_1)$ or $\deg(L|_{Y_2}) < g(Y_2)$. But this contradicts the assumption on L .

Now let $Y, Z \subset X$ be two connected subcurves such that $\deg(L|_Y) = g(Y)$ and $\deg(L|_Z) = g(Z)$. Assume furthermore, that $Y \cup Z$ is connected, as well. Notice that both Y and Z need to be proper subcurves of X , since $d > g$. We claim, that we then need to also have $\deg(L|_{Y \cup Z}) = g(Y \cup Z)$. By assumption, we have $\deg(L|_{Y \cup Z}) \geq g(Y \cup Z)$ and we need to show the other inequality.

On the one hand, we have

$$\deg(L|_{Y \cup Z}) = g(Y) + g(Z) - \deg(L|_{Y \cap Z}), \quad (5)$$

where we set $\deg(L|_{Y \cap Z}) = 0$ if $Y \cap Z$ consists only of nodes of X . Write k for the number of edges in \mathbb{G}_X that are adjacent to both dual graphs \mathbb{G}_Y and \mathbb{G}_Z of Y and Z , but contained in neither. Then one checks using the definition of genus (1), that we have on the other hand

$$g(Y) + g(Z) = g(Y \cup Z) + 1 - k + g(Y \cap Z) - l, \quad (6)$$

where $l = 0$ if $Y \cap Z$ consists only of nodes, in which case we set $g(Y \cap Z) = 0$, and $l = 1$ otherwise. Substituting (6) in (5) gives

$$\begin{aligned} \deg(L|_{Y \cup Z}) &= g(Y \cup Z) + 1 - k + g(Y \cap Z) - l - \deg(L|_{Y \cap Z}) \\ &\leq g(Y \cup Z) + 1 - k - l, \end{aligned}$$

where we used that by assumption $\deg(L|_{Y \cap Z}) \geq g(Y \cap Z)$. Since $Y \cup Z$ is connected, not both k and l can be 0, and hence $\deg(L|_{Y \cup Z}) \leq g(Y \cup Z)$, as claimed.

To conclude assume to the contrary, that for every vertex v there is a subcurve $Y \subset X$ containing X_v and such that $\deg(L|_Y) = g(Y)$. Then the union of all such subcurves for varying v covers X . Applying what we showed above to this union implies $d = g$, a contradiction. \square

Recall that a multidegree \underline{e} is called effective, if it is non-negative on each irreducible component of X .

Lemma 6.2 *Let X be a stable curve and \underline{d} a semistable multidegree of total degree $d \geq g$. Then there are effective multidegrees \underline{e} and \underline{e}' such that $\underline{d} - \underline{e}$ and $\underline{d} - \underline{e}'$ are semistable of total degree g and $g - 1$, respectively.*

Proof We show the existence of \underline{e} , that is, an effective multidegree such that $\underline{d} - \underline{e}$ is semistable of total degree g . Then $\underline{d} - \underline{e} - v$ is semistable of degree $g - 1$ by Lemma 5.3 for any vertex v and thus we obtain also $\underline{e}' := \underline{e} + v$ as claimed. If $d = g$, there is nothing to show, so assume $d > g$ and let L be a line bundle of multidegree \underline{d} .

Since \underline{d} is semistable, we have by definition that for all subcurves $Y \subset X$

$$g(Y) - 1 + (d - g + 1) \frac{2g(Y) - 2 + |Y \cap Y^c|}{2g - 2} \leq \deg(L|_Y).$$

By assumption we have $d - g + 1 > 1$ and since X is stable, we have $0 < \frac{2g(Y)-2+|Y \cap Y^c|}{2g-2}$. In particular, we have for any proper subcurve $Y \subset X$

$$g(Y) \leq \deg(L|_Y).$$

Thus we can apply Lemma 6.1 and there is a vertex $v \in V(\mathbb{G}_X)$ such that $g(Y) < \deg(L|_Y)$ whenever $X_v \subset Y$. Thus for every proper subcurve Y of X and smooth point $p \in X_v$ we have

$$g(Y) \leq \deg(L|_Y(-p)). \tag{7}$$

If $d = g + 1$, then $\underline{d} - v$ has total degree g and Eq. (7) gives that $\underline{d} - v$ is semistable. Otherwise, (7) ensures that $\underline{d} - v$ again satisfies the assumption of Lemma 6.1. We obtain w such that $\underline{d} - v - w$ satisfies an inequality as in (7). Repeating this procedure $g - d$ times gives an effective multidegree $\underline{e} := v + w + \dots$ such that $\underline{d} - \underline{e}$ is semistable of total degree g , as claimed. \square

Since semistability is preserved in passing to residuals by Lemma 3.2, Lemma 6.2 gives an analogous statement in degrees $d \leq g - 2$. We will later use this version, and hence state it next.

Lemma 6.3 *Let X be a stable curve and \underline{d} a semistable multidegree of total degree $d \leq g - 2$. Then there are effective multidegrees \underline{e} and \underline{e}' such that $\underline{d} + \underline{e}$ and $\underline{d} + \underline{e}'$ are semistable of total degree $g - 2$ and $g - 1$, respectively.*

The following example shows, that a claim analogous to Lemma 6.3 does not hold for arbitrary total degrees. More precisely, it shows that for total degrees $d < d' < g - 2$ and a semistable multidegree \underline{d} of total degree d , there need not be an effective multidegree \underline{e} of total degree $d' - d$ such that $\underline{d} + \underline{e}$ is semistable.

Example 6.4 Let \mathbb{G}_X be the graph with three vertices v_1, v_2, v_3 with two edges between v_1 and v_2 and two edges between v_2 and v_3 . Let the weight of the vertices be $(2, 1, 2)$ and consider the multidegree $\underline{d} = (0, 3, 0)$. Then the total degree is $3 = g - 4$ and one checks that \underline{d} is semistable. But $\underline{d} + v$ is not semistable for any vertex v of \mathbb{G}_X . Indeed, a semistable multidegree on \mathbb{G}_X of total degree 4 needs to have value 1 or 2 on both v_1 and v_3 .

6.2 Effective loci for semistable multidegrees

We are now ready to prove the main statement of this section. Recall that for a multidegree \underline{d} we denote by $W_{\underline{d}}(X)$ the effective locus in $\text{Pic}^d(X)$, that is, the locus of line bundles L with multidegree \underline{d} and $h^0(X, L) \geq 1$. Its expected dimension is d , but it may very well be empty or of larger dimension. Recall furthermore, that the multidegree \underline{d} itself is called effective, if it is non-negative on each irreducible component.

Theorem 6.5 *Let X be a stable curve and \underline{d} a semistable multidegree of total degree $d \leq g - 2$. Then each irreducible component of $W_{\underline{d}}(X)$ has dimension at most $g - 2$.*

If \underline{d} is in addition effective, then the effective locus $W_{\underline{d}}(X)$ contains an irreducible component of dimension d .

Proof Let \underline{d} be a semistable multidegree of total degree $d \leq g - 2$. If $d < 0$, then the residual $\omega_X \otimes L^{-1}$ has total degree greater than $2g - 2$ and is still semistable by Lemma 3.2. Hence $\omega_X \otimes L^{-1}$ is non-special by [7, Theorem 2.3] and thus $h^0(X, L) = 0$ by Lemma 3.5. So in this case $W_{\underline{d}}(X) = \emptyset$. Since a multidegree of negative total degree cannot be effective, this gives the claim.

Assume now $d \geq 0$. By Lemma 6.3, there is a semistable multidegree \underline{d}' of total degree $g - 2$ such that $\underline{e} := \underline{d}' - \underline{d}$ is effective. Fix a line bundle $[L_{\underline{e}}] \in A_{\underline{e}}(X)$ of multidegree \underline{e} , where $A_{\underline{e}}(X)$ denotes the image of the rational Abel map. Recall from Sect. 4.3, that this means $L_{\underline{e}} = \mathcal{O}_X(p_1 + \cdots + p_{g-2-d})$ for a collection of smooth points p_i , with \underline{e}_v of them contained in an irreducible component X_v of X . Consider the isomorphism

$$\varphi: \text{Pic}^{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}'}(X), [L] \mapsto [L \otimes L_{\underline{e}}].$$

That is, φ maps L to $L(p_1 + \cdots + p_{g-2-d})$. In particular, $h^0(X, L) \leq h^0(X, L(p_1 + \cdots + p_{g-2-d}))$ and thus $\varphi(W_{\underline{d}}(X)) \subset W_{\underline{d}'}(X)$. Since \underline{d}' is semistable of total degree $g - 2$, $W_{\underline{d}'}(X)$ is either empty or of pure dimension $g - 2$ by Proposition 5.4. Hence the first claim follows.

For the second claim, we assume \underline{d} is effective and thus we can consider the rational Abel map for \underline{d} . Its image $A_{\underline{d}}(X) \subset \text{Pic}^{\underline{d}}(X)$ is irreducible and contained in an irreducible component W of $W_{\underline{d}}(X)$. By Lemma 6.3, there is a semistable multidegree \underline{d}'' , this time of total degree $g - 1$, such that $\underline{e}' := \underline{d}'' - \underline{d}$ is effective. Consider the map

$$\phi: W \times A_{\underline{e}'}(X) \rightarrow V, ([L], [L_{\underline{e}'}]) \mapsto [L \otimes L_{\underline{e}'}].$$

Here V is an irreducible component of the effective locus $W_{\underline{d}''}(X)$ containing the image of ϕ . Since $A_{\underline{d}}(X) \subset W$, we have $A_{\underline{d}''}(X) \subset V$ (note that \underline{d}'' is effective since \underline{d} is). Thus $A_{\underline{d}''}(X) = V$ and $\dim(V) = g - 1$ by Proposition 4.12, since \underline{d}'' is semistable. In particular, the restriction of ϕ to $A_{\underline{d}}(X) \times A_{\underline{e}'}(X)$ is surjective onto V .

We claim that $\dim(A_{\underline{e}'}(X)) = g - 1 - d$. Indeed, a general line bundle $[L''] \in A_{\underline{d}''}(X)$ satisfies $h^0(X, L'') = 1$ by Lemma 4.14. Since $h^0(X, L_{\underline{e}'}) \leq h^0(X, L_{\underline{e}'} \otimes L)$ for line bundles $[L_{\underline{e}'}] \in A_{\underline{e}'}(X)$ and $[L] \in A_{\underline{d}}(X)$, and the restriction of ϕ to $A_{\underline{d}}(X) \times A_{\underline{e}'}(X)$ is surjective onto $A_{\underline{d}''}(X)$, it follows that $h^0(X, L_{\underline{e}'}) = 1$ for a general line bundle $[L_{\underline{e}'}] \in A_{\underline{e}'}(X)$. Thus $\dim(A_{\underline{e}'}(X))$ equals the total degree of \underline{e}' , $g - 1 - d$, by Lemma 4.10.

To finish the argument, choose $[L] \in W$ and $[L_{\underline{e}'}] \in A_{\underline{e}'}(X)$. The restrictions of ϕ to $\{[L]\} \times A_{\underline{e}'}(X)$ and $W \times \{[L_{\underline{e}'}]\}$ are injective. Denote by W_L and $W_{L_{\underline{e}'}}$ the respective images. By injectivity, we have $W_L \cap W_{L_{\underline{e}'}} = \{[L \otimes L_{\underline{e}'}]\}$. Combining these observations, we can use the usual estimate for the dimension of the intersection (see, for example, [17, Proposition I.7.1]) to obtain

$$0 = \dim(W_L \cap W_{L_{\underline{e}'}}) \geq \dim(W) + \dim(A_{\underline{e}'}) - \dim(A_{\underline{d}''}) = \dim(W) - d.$$

Thus $\dim(W) \leq d$. On the other hand, W is not empty, and hence $\dim(W) \geq d$ by Proposition 3.4. \square

6.3 Further counterexamples

We conclude with two examples, that exclude some possible strengthenings of Theorem 6.5.

First, we give an example, where \underline{d} is semistable, but the effective locus is irreducible and of dimension greater than the expected dimension. In particular, requiring that \underline{d} is effective for the second claim in Theorem 6.5 is necessary.

Example 6.6 Let \mathbb{G}_X be the graph with two vertices v_1 and v_2 , of respective weights 1 and 5, and three edges between them. Consider the multidegree $\underline{d} = (-1, 3)$, which one checks to be semistable (in fact, stable). Denote by X_1 and X_2 the irreducible components of X , corresponding to v_1 and v_2 , respectively. The effective locus $W_{\underline{d}}(X)$ is then given as follows: any choice for $L|_{X_1}$, any choice of gluing data, and $L|_{X_2} = \mathcal{O}_{X_2}(X_1 \cap X_2)$. Thus $W_{\underline{d}}(X)$ is irreducible of dimension 3, whereas the total degree of \underline{d} is 2.

Finally, we give an example where the multidegree \underline{d} is semistable and effective, but $W_{\underline{d}}(X)$ has a component of dimension greater than the expected dimension. In particular, in the second claim of Theorem 6.5 not all components need to be of expected dimension.

Example 6.7 Let \mathbb{G}_X be the graph with two vertices v_1 and v_2 , of respective weight 3 and 4, and a single edge between them. Hence $g = 7$. Consider the multidegree $\underline{d} = (2, 2)$, which is effective and semistable (in fact, stable) of total degree $4 = g - 3$. Denote by X_1 and X_2 the irreducible components of X , corresponding to v_1 and v_2 , respectively. Set $p = X_1 \cap X_2$. The effective locus $W_{\underline{d}}(X)$ has three irreducible components in this case: the first is the closure of $A_{\underline{d}}(X)$, which has dimension 4, equal to the expected dimension. The second is $\text{Pic}^2(X_1) \times \{\mathcal{O}_{X_2}(p + q) | q \in X_2\}$, which again has dimension 4. Finally, the third is $\{\mathcal{O}_{X_1}(p + q) | q \in X_1\} \times \text{Pic}^2(X_2)$ which has dimension $5 = g - 2$.

Acknowledgements Many discussions with Lucia Caporaso and Ilya Tyomkin helped shape this paper, and I am very grateful for the insights and suggestions they provided. In addition, I would like to thank Sam Payne and an anonymous referee, whose comments on an earlier draft led to many improvements in presentation.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability statement No datasets were created or analysed during the current study.

Declarations

Conflict of interest The author states, that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence,

and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Alexeev, V., Birkenhake, C., Hulek, K.: Degenerations of Prym varieties. *J. Reine Angew. Math.* **553**, 73–116 (2002)
- Arbarello, E., Cornalba, M., Griffiths, P. A., Harris, J.: Geometry of algebraic curves. Volume I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267. Springer, New York (1985)
- Alexeev, V.: Compactified Jacobians and Torelli map. *Publ. Res. Inst. Math. Sci.* **40**(4), 1241–1265 (2004)
- Beauville, A.: Prym varieties and the Schottky problem. *Invent. Math.* **41**(2), 149–196 (1977)
- Caporaso, L.: A compactification of the universal Picard variety over the moduli space of stable curves. *J. Am. Math. Soc.* **7**(3), 589–660 (1994)
- Caporaso, L.: Geometry of the theta divisor of a compactified Jacobian. *J. Eur. Math. Soc. (JEMS)* **11**(6), 1385–1427 (2009)
- Caporaso, L.: Linear series on semistable curves. *Int. Math. Res. Not. IMRN* **13**, 2921–2969 (2011)
- Caporaso, L., Christ, K.: Combinatorics of compactified universal Jacobians. *Adv. Math.* **346**, 1091–1136 (2019)
- Coelho, J., Esteves, E.: Brill-Noether locus of rank 1 and degree $g - 1$ on a nodal curve. *Commun. Algebra* **43**(7), 2748–2762 (2015)
- Catanese, F., Franciosi, M.: Divisors of small genus on algebraic surfaces and projective embeddings, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, pp. 109–140. Bar-Ilan Univ., Ramat Gan (1996)
- Christ, K.: A Clifford inequality for semistable curves. [arXiv:2204.08234](https://arxiv.org/abs/2204.08234) (2022)
- Caporaso, L., Len, Y., Melo, M.: Algebraic and combinatorial rank of divisors on finite graphs. *J. Math. Pures Appl. (9)* **104**(2), 227–257 (2015)
- Casalaina-Martin, S., Grushevsky, S., Hulek, K., Laza, R.: Extending the Prym map to toroidal compactifications of the moduli space of abelian varieties. *J. Eur. Math. Soc. (JEMS)* **19**(3), 659–723 (2017). **(With an appendix by Mathieu Dutour Sikirić)**
- Christ, K., Payne, S., Shen, T.: Compactified Jacobians as Mumford models. [arXiv:1912.03653](https://arxiv.org/abs/1912.03653) (2019)
- Caporaso, L., Viviani, F.: Torelli theorem for stable curves. *J. Eur. Math. Soc. (JEMS)* **13**(5), 1289–1329 (2011)
- Hakimi, S.: On the degrees of the vertices of a directed graph. *J. Frankl. Inst.* **279**, 260–308 (1965)
- Hartshorne, R.: Algebraic Geometry, Graduate Texts in Mathematics. Springer, New York (1977)
- Minty, G.J.: On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network-programming. *J. Math. Mech.* **15**, 485–520 (1966)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.