

QUANTUM-CLASSICAL HYBRID SYSTEMS AND THEIR QUASIFREE TRANSFORMATIONS

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Summary

The focus of this work is the description of a framework for quantum-classical hybrid systems. The main emphasis lies on continuous variable systems described by canonical commutation relations and, more precisely, the quasifree case. Here, we are going to solve two main tasks: The first is to rigorously define spaces of states and observables, which are naturally connected within the general structure. Secondly, we want to describe quasifree channels for which both the Schrödinger picture and the Heisenberg picture are well defined.

We start with a general introduction to operator algebras and algebraic quantum theory. Thereby, we highlight some of the mathematical details that are often taken for granted while working with purely quantum systems. Consequently, we discuss several possibilities and their advantages respectively disadvantages in describing classical systems analogously to the quantum formalism. The key takeaway is that there is no candidate for a classical state space or observable algebra that can be put easily alongside a quantum system to form a hybrid and simultaneously fulfills all of our requirements for such a partially quantum and partially classical system. Although these straightforward hybrid systems are not sufficient enough to represent a general approach, we use one of the candidates to prove an intermediate result, which showcases the advantages of a consequent hybrid ansatz: We provide a hybrid generalization of classical diffusion generators where the exchange of information between the classical and the quantum side is controlled by the induced noise on the quantum system.

Then, we present solutions for our initial tasks. We start with a CCR-algebra where some variables may commute with all others and hence generate a classical subsystem. After clarifying the necessary representations, our hybrid states are given by continuous characteristic functions, and the according state space is equal to the state space of a non-unital C^* -algebra. While this C^* -algebra is not a suitable candidate for an observable algebra itself, we describe several possible subsets in its bidual which can serve this purpose. They can be more easily characterized and will also allow for a straightforward definition of a proper Heisenberg picture. The subsets are given by operator-valued functions on the classical phase space with varying degrees of regularity, such as universal measurability or strong*-continuity. We describe quasifree channels and their properties, including a state-channel correspondence, a factorization theorem, and some basic physical operations. All this works solely on the assumption of a quasifree system, but we also show that the more famous subclass of Gaussian systems fits well within this formulation and behaves as expected.

Keywords: quantum-classical hybrids, quasifree systems, CCR-algebra

Zusammenfassung

Der Schwerpunkt dieser Arbeit ist die Beschreibung eines mathematischen Rahmens für quanten-klassische Hybridsysteme. Der Fokus liegt dabei auf Systemen mit kontinuierlichen Variablen, welche durch die kanonischen Vertauschungsrelationen beschrieben werden, wobei wir uns im Speziellen auf den quasifreien Fall konzentrieren werden. Hierbei haben wir zwei große Aufgaben zu lösen: Die erste besteht darin sowohl den Zustandsraum, als auch die Observablenalgebra rigoros zu definieren und ihre natürliche Verbindung innerhalb des Formalismus zu präzisieren. Als zweites wollen wir auf dieser Struktur quasifreie Kanäle beschreiben, für die sowohl das Schrödinger-Bild, als auch das Heisenberg-Bild definiert sind.

Wir beginnen mit einer allgemeinen Einführung in Operatoralgebren und algebraische Quantentheorie. Besonderen Wert legen wir dabei auf einige der mathematischen Details, die bei der Arbeit mit reinen Quantensystemen oft als selbstverständlich angesehen werden. Ausgehend davon diskutieren wir verschiedene Möglichkeiten und deren Vor- bzw. Nachteile bei der Beschreibung von klassischen Systemen, analog zum üblichen Formalismus der Quantenmechanik. Eines der wichtigsten Ergebnisse dabei ist, dass es keinen Kandidaten für einen klassischen Zustandsraum bzw. eine klassische Observablenalgebra gibt, der direkt mit einem Quantensystem zu einem Hybrid zusammengefügt werden kann und gleichzeitig alle unsere Anforderungen an ein solch teilweises Quanten- und teilweises klassisches System erfüllt. Obwohl diese einfachen Hybride nicht für einen allgemeinen Ansatz ausreichen, verwenden wir einen dieser Kandidaten, um ein Zwischenergebnis für den konsequenten Hybridansatz zu beweisen: Wir zeigen eine Verallgemeinerung der klassischen Diffusionsgeneratoren für Hybride, bei welcher der Informationsaustausch zwischen der klassischen und der Quantenseite durch das induzierte Rauschen des Quantensystems limitiert ist.

Dann widmen wir uns den Lösungen für die oben formulierten Aufgaben: Wir beginnen mit der CCR-Algebra, in der einige Variablen untereinander kommutieren und damit ein klassisches Teilsystem beschreiben. Nachdem wir Darstellungstheorie dieser Algebra betrachtet haben, können wir hybride Zustände als kontinuierliche charakteristische Funktionen beschreiben. Der entsprechende Zustandsraum ist gleich dem Zustandsraum einer nicht-unitalen C^* -Algebra. Diese C^* -Algebra ist für sich zwar keine geeignete Observablenalgebra, aber wir untersuchen mehrere mögliche Untermengen in ihrem Bidual, welche hierfür benutzt werden können. Diese Untermengen besitzen sowohl eine praktische Charakterisierung, als auch eine direkte Möglichkeit zur Definition eines Heisenberg-Bildes. Konkret sind diese Untermengen als operatorwertige Funktionen auf dem klassischen Phasenraum gegeben, wobei sie sich durch unterschiedliche Voraussetzungen an die Eigenschaften der Funktionen unterscheiden, wie z.B. universelle Messbarkeit oder starke- $*$ -Stetigkeit. Weiter beschreiben wir quasifreie Kanäle und ihre Eigenschaften, einschließlich einer Zustands-Kanal-Korrespondenz, ein Faktorisierungstheorem und einige grundlegende physikalische Operationen. All dies funktioniert unter der Annahme eines quasifreien Systems, aber wir zeigen, dass auch die bekannte Untermenge der Gaußschen Systeme sich in diesen Formalismus eingliedert und wie erwartet verhält.

Schlagwörter: Quanten-klassische Hybride, Quasifreie Systeme, CCR-Algebra

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Chapter 1

Introduction

1.1 Motivation

Let us start by formulating the guiding question of this part:

Why study quantum-classical hybrids?

In fact, there are plenty of possible answers on several levels of detail, the most general being relatively straightforward: We, that is the author and most likely the reader, live our daily life in a *classical* world. Hence, anything that connects this with the quantum regime of photons, electrons, and atoms is likely a hybrid at some point. Here, the prime example is a partial or non-destructive measurement of a quantum system, which naturally ends up with a non-trivial hybrid system. The mathematical description of this operation is commonly known as a quantum instrument, see Fig. 1.1. This example is good to keep in mind because it nicely illustrates the archetype of a classical system that we want to join our quantum part to form our quantum-classical hybrid system.

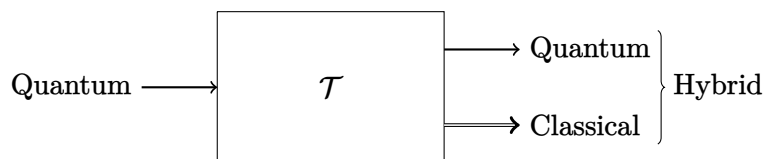


Figure 1.1: The instrument: A quantum system is measured and outputs a hybrid system consisting of a quantum part joined by a classical system that carries the measurement result.

Besides this rather general motivation, let us get more specific. In quantum theory, many of the most powerful theorems state something like:

There is no quantum state ... every observable ... all quantum operations ...

Even without writing these statements out, they all demand that we know what all states, observables, and possible operations precisely mean. Consequently, a work on quantum-classical hybrids should give a comprehensive answer to the following questions:

- What are the possible configurations of a quantum-classical hybrid, i.e., what is the state space?
- What are the possible measurements of a hybrid system, i.e., what is the observable algebra?
- How to characterize the admissible operations on a quantum-classical hybrid?

Here, the last item hardly depends on the first two, which for themselves are highly intertwined. Unlike purely quantum and classical systems, the precise structure of states and observables for hybrids needs to be better established.

Like many other works regarding hybrids, the early focus of this work was much more on a direct application of hybrids before shifting to these general questions above. In doing so, we had to go deeper into the functional analysis of such systems than we had anticipated. In return, we can state a precise description of the aforementioned spaces and operations with a practical calculus for the large subclass of quasifree quantum-classical hybrids.

Another point of motivation to study hybrids more deeply is their upcoming relevance. As real-world quantum computation is getting more advanced, the possibilities for implementing quantum information protocols increase just as well. A good example of these natural hybrid protocols is quantum teleportation [1], see Fig. 1.2. A detailed description of it can be found in nearly every textbook about quantum information, e.g., [2].

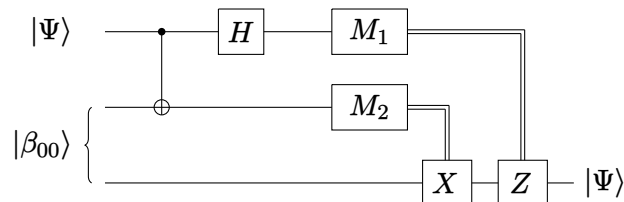


Figure 1.2: Quantum teleportation: A quantum state $|\Psi\rangle$ gets teleported using a shared entangled quantum state $|\beta_{00}\rangle$ and classical communication (double lines).

Indeed, this protocol highlights two hybrid aspects: Firstly, the measurements M_1 and M_2 transform our fully quantum system, consisting of three qubits (one as $|\Psi\rangle$ and two as $|\beta_{00}\rangle$), into one qubit and classical information, i.e. a proper hybrid system as described above. Secondly, the operations or gates X and Y depend on the classical outcome of M_1 and M_2 , i.e., they are truly hybrid operations.

Typically, the teleportation protocol only utilizes bits and qubits, i.e., the basic units of quantum, respectively, classical information. Hence, the necessary mathematics can be reduced to the direct sum of finite-dimensional matrices, which thoroughly describe the states, observables, and operations. For more complex hybrid scenarios, we need two generalizations: Infinite-dimensional quantum systems and continuous sets for our classical side. This is where things quickly become more challenging.

1.2 Structure

Chap. 2 is a review of algebraic quantum theory, including the necessary mathematical definitions and facts from functional analysis and operator algebras. Although we leave out most proofs, we give a sufficient and self-contained introduction to all the necessary tools and provide a detailed list of references in the last section.

Chap. 3 is a general introduction to quantum-classical hybrids and also contains a short review of commutative C^* - and W^* -algebras. In Sect. 3.1, we start with a discussion of hybrids that have a finite-dimensional quantum system and a discrete classical part. In preparation for the discussion of continuous classical systems, we review the basic definitions and results regarding commutative C^* - and W^* -algebras in Sect. 3.2. With this, we discuss possible approaches to building hybrid systems in Sect. 3.3. At times, these hybrid frameworks can provide a sufficient structure for some aspects, but each fails at a different point to constitute a basis for the purpose of this work. Then, in Sect. 3.4, we group and classify the previous approaches to the topic of quantum-classical hybrids and put them into context.

Chap. 4 is about generalizing diffusions to the hybrid setting and our first result about hybrids, showing the capabilities of this approach. For this, we start with a short introduction to Feller semigroups in Sect. 4.1. Just like quantum dynamical semigroups, described by the famous Lindblad theorem, Feller semigroups have a distinct generator form. So in Sect. 4.2, taking one of the hybrid algebras from the chapter before, we generalize this classical theorem to the hybrid case. Here, a positivity condition naturally bounds the information flow from the quantum to the classical systems.

Chap. 5 is the central chapter and contains a calculus and a description of the according algebras for quasifree hybrid systems. These results are published in [3]. We start in Sect. 5.1 with a review of continuous-variable quantum systems and their calculus, on which we will build our hybrid generalization. In Sect. 5.2, we introduce the hybrid framework, including representations of the hybrid CCR-algebra and the according state space. By hybrid CCR-algebra, we mean a CCR-algebra but with a degenerate symplectic form, i.e., with some variables that may commute with all others and hence generate a classical subsystem. Here, the *good* or standard states are those with a continuous characteristic function and are equal to the state space of a non-unital C^* -algebra $C^*(\Xi, \sigma)$. This algebra is isomorphic to the tensor product of the continuous functions vanishing at infinity on the classical part of the phase space, joined by the compact operators on the quantum part. Also we study the famous subclass of quasifree systems, the Gaussian ones, in Sect. 5.2.4. After we have discussed our states, we study the possible observable algebras in Sect. 5.3. Of course, we could state $C^*(\Xi, \sigma)^{**}$ as the observable algebra, but even without the quantum part, the bidual of the classical algebra is way too unpractical. Hence, we propose several subsets of it as candidates for hybrid observables algebras. With states and observables, we introduce quasifree channels and some of their properties in Sect. 5.4. This includes the characterization of the subclass of noiseless channels and a noise factorization theorem, which allows us to write every hybrid quasifree channel as enlarging the system followed by a noiseless operation. We end this work with two more practical sections. In Sect. 5.5, we describe basic physical operations like disturbance, observables, and instruments in terms of this hybrid framework.

Chapter 2

Review: Algebraic Quantum Theory

... what this chapter is and isn't

The following chapter is a short guide on how to get from the Hilbert space formulation of quantum mechanics to its algebraic form. Thereby, we mention the necessary facts but do not cover every detail, as there are many textbooks about different topics. We give a detailed list of references for the statements in this chapter in Sect. 2.5, such that the interested reader can find more details and proofs there. Nevertheless, we try to be as self-contained as possible in the sense that the following section has several summaries, hopefully improving the flow of reading.

2.1 Quantum Mechanics on Hilbert Space

The *Hilbert space formulation* of quantum mechanics is its most idealized description. While still providing the necessary framework for different research areas, we need a more general approach. Working on a new framework by starting with the special case of a special case is not advisable. To get there, we start with a brief review of this formulation to get a common ground and point towards the upcoming generalization.

Let \mathcal{H} be a complex **Hilbert space**, that is, \mathcal{H} is a vector space over the complex numbers \mathbb{C} , equipped with a scalar product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, which is linear in the second argument and anti-linear in the first. It is complete with respect to the norm induced by the scalar product: $\|\Psi\| = \sqrt{\langle \Psi, \Psi \rangle}$. One usually assumes the Hilbert space \mathcal{H} separable, i.e. \mathcal{H} has a countable basis.

The **states**, which characterize the possible configurations of a system, are given by unit vectors $|\Psi\rangle \in \mathcal{H}$. In quantum theory, there are two different ways of combining them. One is by superposition, and the second is a mixture. Mathematically, a superposition of states is their linear combination on the Hilbert space level. For example, the vectors $|\Psi_i\rangle \in \mathcal{H}$ can be superposed to the state

$$|\Phi\rangle = \sum_i^n c_i |\Psi_i\rangle, \quad c_i \in \mathbb{C}. \quad (2.1)$$

The naive translation of this concept to classical configurations, like the condition of a cat, underlines the differences between quantum and classical theories. In contrast to superposition, the notion of mixing is present in both classical and quantum systems: For this, we take our first step out of the Hilbert space and use the one-dimensional projectors $|\Psi_i\rangle\langle\Psi_i|$ as pure states instead of the Hilbert space vectors $\Psi_i \in \mathcal{H}$. We can mix them by forming non-trivial convex combinations

$$\rho = \sum_i^n c_i |\Psi_i\rangle\langle\Psi_i|, \quad (2.2)$$

with $c_i \geq 0$ and $\sum_i c_i = 1$. These ρ are called mixed states. The operational interpretation of these states is readily understood if we read the coefficients $\{c_i\}$ as a *classical* probability distribution. Then, the state ρ in Eq. (2.2) means that our system is with probability c_i in the state $|\Psi_i\rangle\langle\Psi_i|$. In general, this decomposition is not unique and should be read in the sense of ensembles.

The **measurements** or observables in quantum mechanics are often modeled by hermitian or self-adjoint operators $A^* = A$. Similar to the eigenvalue decomposition of hermitian matrices in the finite-dimensional case, self-adjoint or hermitian operators have a spectral decomposition

$$A = \int_{\mathbb{R}} x A(dx), \quad (2.3)$$

where $A(dx)$ is a projection-valued measure (PVM) on the possible measurement outcomes. Note that this is a very brief and simple description of the measurements in quantum theory. For example, the measure $A(dx)$ does not need to be projection-valued, leading to the more general positive operator-valued measure (POVM), see Sect. 2.3. For the moment, we leave it there and discuss another important aspect,

namely, what type of information we get out of a measurement. It is essential to highlight that, in comparison to most classical theories, quantum mechanics is probabilistic and not deterministic. So, in general, the figure of merit is not a statement about a single shot but a series of measurements characterized by a probability distribution and statistical quantities like the **expectation value**:

$$\langle x \rangle_{\Psi} = \langle \Psi | A | \Psi \rangle \quad \text{or} \quad \langle x \rangle_{\rho} = \text{tr}[\rho A]. \quad (2.4)$$

The **dynamics**, that is, the time evolution between the preparation of the state and the measurement, is typically described by the famous Schrödinger equation

$$-i\hbar\partial_t\Psi(x, t) = H\Psi(x, t). \quad (2.5)$$

Here H is the Hamiltonian, which is also a self-adjoint operator and, according to Eq. (2.5), determines the infinitesimal evolution of the state Ψ . For time-independent Hamiltonians, this leads to the time-evolution operator:

$$U_t = e^{-\frac{i}{\hbar}Ht} \quad \text{with} \quad \Psi(t) = U_t\Psi(0). \quad (2.6)$$

The operator U_t is unitary, so $U_t^* = U_{-t}$ is the inverse and describes the reverse dynamics. It is also possible to handle mixed states, where the infinitesimal behavior is also generated by the Hamiltonian

$$\partial_t\rho = i[H, \rho]. \quad (2.7)$$

The analog to the Schrödinger equation, Eq. (2.7) is sometimes called *Liouville- or von Neumann* equation. The long-term evolution for a mixed state is then described by

$$\rho(t) = U_t\rho(0)U_t^*. \quad (2.8)$$

With regards to the dynamics of the system, there is one point left to discuss, namely the choice of **picture** we are working in. For example, the evolution of an expectation value can be described in two equivalent ways:

$$\langle A \rangle_{\Psi_t} = \langle U_t\Psi | A | U_t\Psi \rangle = \langle \Psi | U_t^* A U_t | \Psi \rangle = \langle A(t) \rangle_{\Psi}. \quad (2.9)$$

In the left part of Eq. (2.9), we evolve the state, while on the right-hand side, the measurement operator A is the dynamical element. The first case, in which the dynamics are acting on the states, is called the **Schrödinger picture**, and the second, i.e., when we evolve the observables, the **Heisenberg picture**. Here, both pictures yield the same results, and based on the situation, one picture can be more beneficial than the other. Having this choice and switching between them is a valuable feature of quantum mechanics.

The three fundamental building blocks of preparation, dynamics, and measurement that can be used to break up any experiment in quantum theory and the two different pictures for the evolution are illustrated in Fig. 2.1.

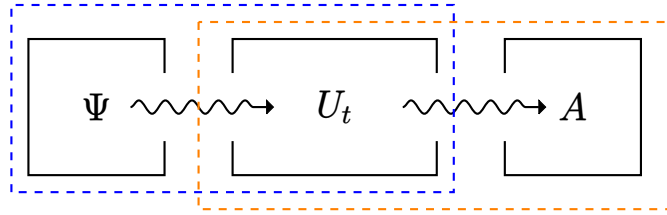


Figure 2.1: A general experiment in quantum theory with a state Ψ , the dynamics described by U_t and an observable A . The *Schrödinger picture* is marked in blue and the *Heisenberg picture* in orange.

Algebraic quantum theory Let us summarize the generalizations in the description of quantum theory we will need.

At first, our focus point will switch from the Hilbert space \mathcal{H} to the operator algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the bounded linear operators over \mathcal{H} , which is our observable algebra in quantum theory (see Sect. 2.2).

A subset of this algebra are the trace-class operators $\mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, which define quantum states by density operators ρ . As we will see, this does not exploit all possibilities for states in quantum theory. Indeed, it is one of the main advantages of algebraic quantum theory to handle the *full* state space of an observable algebra. It consists not only of density operators but positive linear functionals $\omega : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$. In general, states are best understood as *expectation value functionals*, including the previous definitions by the identifications

$$\omega_\Psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}, \quad \omega_\Psi(A) = \langle \Psi | A | \Psi \rangle \quad (2.10)$$

$$\omega_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}, \quad \omega_\rho(A) = \text{tr}[\rho A]. \quad (2.11)$$

Alongside this shift of focus from the states to the observables, the Heisenberg picture is more prominent in algebraic quantum theory than in the Hilbert space formulation. Nevertheless, both pictures are present and viable (see Sect. 2.3).

Another generalization we will need is the transition from closed system dynamics to open systems. This means that instead of reversible dynamics, we have to deal with irreversible operations. Mathematically, we have to move from unitary implemented groups to completely positive maps and the representations of semigroups (see Sect. 2.4).

2.2 Operator Algebras

In this section, we will recall some basic facts from the theory of operator algebras. Their development is closely connected to those of quantum theory. Witnesses for this are that W^* -algebras are also known as *von Neumann* algebras because von Neumann was one of the founding fathers of both fields and the nomenclature for positive functionals as *states*.

2.2.1 C^* -algebras

We start with the common approach showing that the space $\mathcal{B}(\mathcal{H})$ has the structure of what is called a **C^* -algebra** [4, 5] and some basic facts from functional analysis [6, 7]. For this, let us start with the formal definition of the elements in $\mathcal{B}(\mathcal{H})$:

Definition 1 (bounded linear operators). *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ a map or operator. We denote the set of all bounded linear operators by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and in the case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.*

Thereby bounded refers to the **operator norm**:

Definition 2 (operator norm). *For $A \in \mathcal{B}(\mathcal{H})$ we define the operator norm $\|A\|$ by*

$$\|A\| = \sup_{\Psi} \frac{\|A\Psi\|}{\|\Psi\|} = \sup_{\|\Psi\|=1} \|A\Psi\|. \quad (2.12)$$

In the following, we will drop the word *bounded* and assume every operator to be bounded else, we will especially state an operator to be an **unbounded operator**. Indeed, two of the most prominent operators in quantum theory, namely the position Q and momentum P , are prime examples of unbounded operators, and we will have a closer look at them in Chap. 5.

Note that a linear map between two normed vector spaces is bounded if and only if it is continuous (with respect to the norm topology). For $A, B \in \mathcal{B}(\mathcal{H})$ we define their sum by $(A + B)\Psi = A\Psi + B\Psi$ and scalar multiplication by $(\lambda A)\Psi = \lambda(A\Psi)$, $\lambda \in \mathbb{C}$. This turns $\mathcal{B}(\mathcal{H})$ into a **vector space**.

The vector space $\mathcal{B}(\mathcal{H})$ is complete with respect to the operator norm, so it becomes a complete normed vector space, which is also called a **Banach space**. From the definition of the operator norm, we have that any $A \in \mathcal{B}(\mathcal{H})$ and $\Psi \in \mathcal{H}$ satisfy

$$\|A\Psi\| \leq \|A\| \|\Psi\|. \quad (2.13)$$

Then for two operators $A, B \in \mathcal{B}(\mathcal{H})$ we get

$$\|AB\| = \sup_{\|\Psi\|=1} \|AB\Psi\| \leq \|A\| \|B\| \sup_{\|\Psi\|=1} \|\Psi\| \leq \|A\| \|B\|. \quad (2.14)$$

Hence, if A, B are bounded, so is their product $AB \in \mathcal{B}(\mathcal{H})$, which we read as applying them successively. This makes $\mathcal{B}(\mathcal{H})$ an **algebra**. An algebra, which is also a Banach space, is called a **Banach algebra**. Finally, we will see that $\mathcal{B}(\mathcal{H})$ is a Banach $*$ -algebra and especially a C^* -algebra. For this, we need an **involution** or $*$ -operation, which satisfies the following properties for $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$:

1. $(A^*)^* = A$
2. $(AB)^* = B^*A^*$
3. $(\lambda A + B)^* = \bar{\lambda}A^* + B^*$
4. $\|A^*A\| = \|A\|^2$.

The items 1-3. are the conditions for a ***-algebra** and 4. is called **C*-property**, which turns a *-algebra into a **C*-algebra**. The *-operation on $\mathcal{B}(\mathcal{H})$ is defined by the adjoint in the following way:

Definition 3 (adjoint operator). For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\Phi \in \mathcal{H}_1, \Psi \in \mathcal{H}_2$ we define its adjoint operator $A^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ by

$$\langle \Phi, A^*\Psi \rangle_{\mathcal{H}_1} = \langle A\Phi, \Psi \rangle_{\mathcal{H}_2}. \quad (2.15)$$

One can easily check that the adjoint satisfies the above requirements of a *-involution, which makes $\mathcal{B}(\mathcal{H})$ a C*-algebra.

Note that an abstract C*-algebra is not required to have a multiplicative unit like $\mathbb{1}$ in $\mathcal{B}(\mathcal{H})$. If it does, we call it a **unital C*-algebra** or C*-algebra with a unit. Nonetheless, one can show that every C*-algebra contains an approximate unit [8, Sect. I.7]:

Definition 4 (approximate unit). Let \mathcal{A} be a C*-algebra and $\{u_\lambda \mid \lambda \in \Lambda\}$ a net of positive elements with $\|u_\lambda\| \leq 1$ for all λ . If $\lambda < \mu$ implies $u_\lambda \leq u_\mu$ and

$$\lim_{\lambda \in \Lambda} \|A(1 - u_\lambda)\| = 0 \quad (2.16)$$

for all $A \in \mathcal{A}$, then this net is called an **approximate unit**.

If \mathcal{A} is separable, i.e., it contains a countable dense subset, the approximate unit is bounded and can be represented by a sequence u_i instead of a net u_λ . Another approach is **adjoining a unit** or the **unitization** of \mathcal{A} . If \mathcal{A} is a non-unital Banach algebra, we expand it to $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$, which does have a unit $\mathbb{1}_{\tilde{\mathcal{A}}}$ in the second summand [9, Sect. C.5]. The norm (as a Banach algebra) is given by

$$\|A + \lambda \mathbb{1}_{\tilde{\mathcal{A}}}\| = \|A\| + |\lambda|. \quad (2.17)$$

This can be extended to also work for C*-algebras, see [9, Prop. C.30].

2.2.1.1 Operators

In this part, we characterize the objects of a C*-algebra. Certain facts are distinct to the quantum case, i.e., $\mathcal{B}(\mathcal{H})$, but wherever possible, we state most facts on the level of a general unital C*-algebra \mathcal{A} . We start by generalizing the concept of eigenvalues from linear algebra to arbitrary elements in a C*-algebra:

Definition 5 (spectrum). For $A \in \mathcal{A}$ we define its **spectrum** $\sigma(A)$ by

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda \mathbb{1}) \text{ is not invertible}\}. \quad (2.18)$$

The complement of the spectrum is called the **resolvent**.

Next, we classify certain types of elements in \mathcal{A} and have a look at some of their properties:

Definition 6. We call an element $A \in \mathcal{A}$: **normal** if $AA^* = A^*A$, **self-adjoint** if $A = A^*$, **an isometry** if $A^*A = \mathbb{1}$, **unitary** if $A^*A = AA^* = \mathbb{1}$ and a **projection** if $A^2 = A = A^*$.

Obviously, every self-adjoint element is normal, and every unitary element is an isometry. One can also show that the spectrum of a self-adjoint element is real, and for unitary elements, the spectrum lies on the complex unit circle.

On a concrete Hilbert space, we can state some more properties: An operator $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, if and only if $\langle \Psi, A\Psi \rangle \in \mathbb{R}$ for all $\Psi \in \mathcal{H}$. An isometry, and especially every unitary, does not change the norm of vectors in \mathcal{H}

$$\|\Psi\|^2 = \langle \Psi, \Psi \rangle = \langle A^*A\Psi, \Psi \rangle = \langle A\Psi, A\Psi \rangle = \|A\Psi\|^2 \quad (2.19)$$

and is always injective.

An important class of objects are the **positive elements**. On Hilbert spaces, there is a straightforward definition for the positivity of an operator $A \in \mathcal{B}(\mathcal{H})$, which is that $\langle \Psi, A\Psi \rangle \geq 0$ for all $\Psi \in \mathcal{H}$. For the general case, we have the following:

Definition 7 (positive elements). *An element $A \in \mathcal{A}$ is positive, $A \geq 0$, if one of the equivalent conditions holds true:*

- A is self-adjoint and $\sigma(A) \in \mathbb{R}_+$
- there exists $B \in \mathcal{A}$ such that $A = B^*B$.

For two positive operators $A, B \geq 0$, we say $A \geq B$ if $A - B \geq 0$. This defines a partial order on the positive elements of \mathcal{A} , which we denote by \mathcal{A}_+ and, with the vector space structure, becomes a partially ordered vector space.

It is clear from the definition that every positive element $A \in \mathcal{A}_+$ is automatically self-adjoint. Moreover, every positive operator A has a unique **square root** \sqrt{A} , which satisfies $(\sqrt{A})^2 = A$. With this, we can define the **absolute value** for a self-adjoint operator $A \in \mathcal{A}$ via $|A| = \sqrt{A^2}$. Another important fact about the positive elements of a C*-algebra is that they form a cone:

Lemma 8. *Let \mathcal{A}_+ be the set of positive elements of \mathcal{A} . Then \mathcal{A}_+ is a norm closed convex cone.*

Recall that a subset X is called **convex**, if $\forall x, y \in X$ also $\lambda x + (1 - \lambda)y \in X$, $0 \leq \lambda \leq 1$ and a **cone** if for every $x \in X$ also $\lambda x \in X$ with $\lambda > 0$. The **extreme points** of a convex set are those on the boundary of X for which $\lambda x + (1 - \lambda)y = z$ and $0 < \lambda < 1$ it follows that $x = y = z$. Moreover we can split every self-adjoint A into $A_{\pm} = (|A| \pm A)/2$, where $A_{\pm} \in \mathcal{A}_+$ and $A = A_+ - A_-$.

Now, we can define the class of elements called **trace class operators**, which will be used in the description of normal states in quantum theory.

Definition 9 (trace class). *For $A \in \mathcal{B}(\mathcal{H})$ we define the trace of A by*

$$\text{tr}[A] = \sum_{i=1}^{\infty} \langle \phi_i, A\phi_i \rangle, \quad (2.20)$$

where ϕ_i is an orthonormal basis for \mathcal{H} and if $\text{tr}[\sqrt{A^*A}]$ is finite, we call A a trace class operator. We denote the set of trace class operators by $\mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

The trace is linear, i.e., for $A, B \in \mathcal{T}(\mathcal{H})$ we have $\text{tr}[\lambda A + B] = \lambda \text{tr}[A] + \text{tr}[B]$ with $\lambda \in \mathbb{C}$ and like the positive operators, the trace class operators form a vector space, which becomes a normed vector space with the following definition.

Definition 10 (trace norm). *Let $A \in \mathcal{T}(\mathcal{H})$ be a trace class operator, then*

$$\|A\|_{tr} = \text{tr}[\sqrt{A^*A}] \quad (2.21)$$

*defines a norm on $\mathcal{T}(\mathcal{H})$, which is called the **trace norm**.*

For the trace norm one can show that for $A \in \mathcal{T}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})$ we have $\|AB\| \leq \|A\|_{tr}\|B\|$, $\|A\| \leq \|A\|_{tr}$ and $\text{tr}[A^*] = \overline{\text{tr}[A]}$. Another important feature of the trace class operators is that their product with a bounded operator is again a trace class operator:

Lemma 11. *Let $A \in \mathcal{T}(\mathcal{H})$ be a trace class operator and $B \in \mathcal{B}(\mathcal{H})$, then AB and BA are both trace class operators, i.e. $\mathcal{T}(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$. For the value of the trace, we have*

$$\text{tr}[AB] = \text{tr}[BA]. \quad (2.22)$$

So, the trace is independent of the choice of the basis because it is invariant under the action of a unitary operator U , i.e., $\text{tr}[UAU^*] = \text{tr}[A]$. Next, we equip $\mathcal{T}(\mathcal{H})$ with an inner product.

Definition 12 (Hilbert-Schmidt product). *For $A, B \in \mathcal{T}(\mathcal{H})$ we define an inner product by*

$$\langle A, B \rangle_{HS} = \text{tr}[A^*B]. \quad (2.23)$$

Clearly it is conjugate symmetric, i.e. $\langle A, B \rangle_{HS} = \overline{\langle B, A \rangle_{HS}}$, and linear in the second argument, $\langle A, \lambda B + C \rangle_{HS} = \lambda \langle A, B \rangle_{HS} + \langle A, C \rangle_{HS}$. Also, it is positive $\langle A, A \rangle_{HS} \geq 0$, and one can show that it is non-degenerate, therefore an inner product.

Now we come to the class of **compact operators**, which does not share the same place in the spotlight as $\mathcal{B}(\mathcal{H})$ or $\mathcal{T}(\mathcal{H})$ when it comes to quantum theory, but plays an important part in the mathematical structure.

Definition 13 (compact operators). *Let $K \in \mathcal{B}(\mathcal{H})$ be an operator and U the unit ball of \mathcal{H} . Then K is compact if the closure of $K(U)$ is compact. We denote the set of compact operators by $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.*

Sometimes, it is useful to describe compact operators by **finite-rank operators** [9, B.131]. These are operators $F \in \mathcal{B}(\mathcal{H})$ with finite-dimensional span, i.e. $\{F\Psi \mid \Psi \in \mathcal{H}\}$ has finite dimension. Then $\mathcal{K}(\mathcal{H})$ is the norm-closure of all such operators, and an operator K is compact, if and only if it can be approximated by finite-rank operators K_n , i.e., $\|K - K_n\| \rightarrow 0$. Also, if K is compact, so is K^* and vice versa. The set of compact operators forms a C*-algebra, which, like the set of trace class operators, forms an ideal in $\mathcal{B}(\mathcal{H})$.

2.2.1.2 States and Dualities

We already mentioned that trace class or density operators are the normal states in quantum theory. For the general concept of a state on a C^* -algebra, we need the definition of the **dual space**.

Definition 14 (dual space). *Let \mathcal{A} be a (normed) vector space over the complex numbers \mathbb{C} . A functional ω is a (continuous) linear map from \mathcal{A} into \mathbb{C} . The space of functionals over \mathcal{A} is called the (topological) dual space \mathcal{A}^* .*

One can highlight the continuity assumption in the above definition and differ between the topological dual space and the algebraic dual space, where the functionals ω are not required to be continuous. The space \mathcal{A}^* is a vector space, where addition is defined point-wise, $(\omega + \mu)(x) = \omega(x) + \mu(x)$, and scalar multiplication in the usual way.

Before we can state the precise connection between $\mathcal{K}(\mathcal{H})$, $\mathcal{T}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$, we need the notion of duality or **dual pairs**, as neither of the aforementioned spaces naturally consists of functionals [10, 2.3.8].

Definition 15 (dual pair). *Let \mathcal{A} and \mathcal{B} be two vector spaces. They form a (algebraic) dual pair or are in (algebraic) duality if there exists a bilinear form*

$$\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{C}, \quad (2.24)$$

which separates points if seen as a functional $\langle \cdot, B \rangle : \mathcal{A} \rightarrow \mathbb{C}$, i.e.

$$\forall A \in \mathcal{A} \setminus \{0\} \quad \exists B \in \mathcal{B} \quad \text{such that} \quad \langle A, B \rangle \neq 0, \quad (2.25)$$

and the same for interchanged roles of \mathcal{A} and \mathcal{B} . If \mathcal{A} and \mathcal{B} have a norm and the functionals $\langle \cdot, B \rangle$, $\langle A, \cdot \rangle$ are continuous, we drop the word algebraic.

Crucial in this definition is the fact that functionals separate points. Otherwise, one can find trivial examples where the dual space has little or nothing to say about the underlying space, and many of the following applications would not work. This is sometimes highlighted by calling such a duality a separating duality [11, II. §6.1]. Now we can formulate the dualities mentioned above in a more precise way [10, Thm. 3.4.13]:

Theorem 16 (dualities). *The trace-class operators $\mathcal{T}(\mathcal{H})$ are the dual space of the compact operators $\mathcal{K}(\mathcal{H})$ and the bounded linear operators $\mathcal{B}(\mathcal{H})$ are the dual space of the trace-class operators, i.e.,*

$$\mathcal{K}(\mathcal{H})^* = \mathcal{T}(\mathcal{H}) \quad \text{and} \quad \mathcal{T}(\mathcal{H})^* = \mathcal{B}(\mathcal{H}). \quad (2.26)$$

The duality between the spaces is given by

$$\langle A, B \rangle = \text{tr}[AB]. \quad (2.27)$$

Another way of stating Eq. (2.26) is to say that the compact operators $\mathcal{K}(\mathcal{H})$ form the **predual** of the trace-class operators $\mathcal{T}(\mathcal{H})$, which are the predual of the bounded operators $\mathcal{B}(\mathcal{H})$. We write this as

$$\mathcal{K}(\mathcal{H}) = \mathcal{T}(\mathcal{H})_* \quad \text{and} \quad \mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_*. \quad (2.28)$$

Note that for a C^* -algebra, we can always define a dual space, but the predual does not necessarily exist. Indeed, the existence of a predual is the defining difference between C^* -algebras and W^* -algebras.

The dual space of a C^* -algebra \mathcal{A}^* naturally carries a **norm** which is defined as

$$\|\omega\| = \sup_{A \in \mathcal{A}, \|A\| \leq 1} |\omega(A)|. \quad (2.29)$$

With this, let us collect some basic definitions about the elements of \mathcal{A}^* :

Definition 17. *Let $\omega \in \mathcal{A}^*$ be a linear functional on a Banach $*$ -algebra \mathcal{A} .*

- We call ω **positive** if it maps positive elements to positive elements, i.e., $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. We denote the set of positive functionals on \mathcal{A} by \mathcal{A}_+^* .
- If ω is positive and has norm one, we call ω a **state** on \mathcal{A} . We denote the set of all states or the **state space** of \mathcal{A} by $\mathcal{S}(\mathcal{A})$.
- A functional ω is called **faithful** if $\omega(A^*A) \neq 0$ for all non-zero $A \in \mathcal{A}$.
- We define the **adjoint** of ω by complex conjugation, i.e., $\omega(A)^* = \overline{\omega(A)}$ and call ω **self-adjoint** if $\omega(A)^* = \omega(A)$.

In the same way, we defined a partial order on the positive elements of a C^* -algebra, we can define a partial order on the positive elements of \mathcal{A}^* by $\omega \geq \mu$ if $\omega - \mu \geq 0$.

The structure of quantum state spaces is an important point for the theory and, indeed will be likewise important for our hybrid framework. On this matter, let us discuss some consequences of the above definition. One can show that $\omega \in \mathcal{A}^*$ is positive, if

$$\lim \omega(u_\lambda) = \|\omega\| \quad (2.30)$$

for some approximate unit $\{u_\lambda\}$. If \mathcal{A} is unital, the condition simplifies to $\omega(\mathbb{1}) = \|\omega\|$ [12, Prop. 3.1.4]. Hence for unital C^* -algebras, we can characterize the state space by

$$\mathcal{S}(\mathcal{A}) = \{\omega \in \mathcal{A}^* \mid \|\omega\| \leq 1, \omega(\mathbb{1}) = 1\}. \quad (2.31)$$

If \mathcal{A} is unital, this set is convex and weak*-compact (see Sect. 2.2.1.4). For non-unital \mathcal{A} , the normalization functional is not continuous in the weak* topology, so the states are not a weak*-closed subset of the unit ball, hence not compact.

The extreme points of $\mathcal{S}(\mathcal{A})$ are called **pure states**, while the other states are called **mixed states**. Accordingly a state ω is pure if and only if $\eta \leq \omega$ implies $\eta = \lambda\omega$ for $0 \leq \lambda \leq 1$.

If the algebra \mathcal{A} is non-unital, we cannot use Eq. (2.31) so we need to use the **quasi-state space** [12, Sect. 3.2], which is the positive part of the unit ball in \mathcal{A}^* , i.e.

$$\mathcal{Q}(\mathcal{A}) = \{\omega \in \mathcal{A}^* \mid \omega \geq 0, \|\omega\| \leq 1\}. \quad (2.32)$$

This space is also convex and always weak*-compact. The main difference between $\mathcal{S}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$ is the existence of the zero-functional in the quasi-state space.

2.2.1.3 Representations and the GNS construction

Equipped with the definition of states for a general C^* -algebra, we can now state two important tools in the field of operator algebras, namely the Gelfand-Naimark theorem, which identifies any C^* -algebra as a subalgebra of $\mathcal{B}(\mathcal{H})$ and the famous GNS-construction. For this, let us recall the basic definitions and facts of the representation theory of C^* -algebras.

Definition 18. Let \mathcal{A} be a Banach $*$ -algebra. A $*$ -homomorphism π from \mathcal{A} to $\mathcal{B}(\mathcal{H})$, i.e.,

$$\pi(AB) = \pi(A)\pi(B) \quad \text{and} \quad \pi(A^*) = \pi(A)^*, \quad (2.33)$$

is called a **representation** of \mathcal{A} .

- The corresponding Hilbert space \mathcal{H} is called **representation space** of π and to highlight this connection we write \mathcal{H}_π or the pair as $\{\pi, \mathcal{H}\}$.
- Two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ are called **unitarily equivalent** if there exists an isometry U , such that for $A \in \mathcal{A}$ we have $U\pi_1(A)U^* = \pi_2(A)$.
- A representations for which $\pi(A) \neq 0$ for all non-zero elements is called **faithful**.
- If for any non-zero $\Psi \in \mathcal{H}_\omega$, there is an $A \in \mathcal{A}$ such that $\pi(A)\Psi \neq 0$, then π is called **non-degenerate**.
- The representation is called **cyclic** if there is a **cyclic vector** $\Omega \in \mathcal{H}_\omega$, such that the closure of $\pi(\mathcal{A})\Omega$ is \mathcal{H}_ω .

After this rather long collection of definitions, let us begin with some of the corresponding results. The first will allow us to restrict our work on cyclic representations [8, Thm. 9.17]:

Proposition 19. Any non-degenerate representation of a Banach $*$ -algebra is a direct sum of cyclic representations.

The next fact comes with no surprise to anybody familiar with Schur's lemma from representation theory. For this, we need two more definitions: If $\{\pi, \mathcal{H}\}$ is a representation and $\mathcal{N} \subset \mathcal{H}$ a closed subspace, we call \mathcal{N} an **invariant subspace** if $\pi(A)\mathcal{N} \subset \mathcal{N}$ for all $A \in \mathcal{A}$. If the only invariant subspaces are \mathcal{H} and $\{0\}$, we call a representation **irreducible**. Then, we have the following theorem [8, Thm. 9.20]:

Proposition 20. Let $\{\pi, \mathcal{H}\}$ be a representation of a Banach $*$ -algebra \mathcal{A} , then the following are equivalent:

- a) $\{\pi, \mathcal{H}\}$ is irreducible.
- b) Only scalar multiplication operators commute with $\pi(\mathcal{A})$.

The proof of the next theorem constitutes what is famously known as the Gelfand-Naimark-Segal or **GNS construction**. It can be found in any work of the referenced literature, see Sect. 2.5, although most authors strengthen the requirements of the theorem, i.e., assume \mathcal{A} to be unital C^* -algebra. This avoids some technicalities during the proof, but the basic structure remains the same [8, Thm. 9.14].

Theorem 21 (GNS construction). *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity. Then, for every positive linear functional $\omega \in \mathcal{A}_+^*$, there exists a cyclic representation $\{\pi_\omega, \mathcal{H}_\omega\}$ of \mathcal{A} , which is unique up to unitary equivalence. Furthermore, if $\Omega \in \mathcal{H}_\omega$ is the cyclic vector and $A \in \mathcal{A}$, we have*

$$\omega(A) = \langle \pi_\omega(A)\Omega, \Omega \rangle. \quad (2.34)$$

To highlight the cyclic vector, one often extends the representation $\{\pi_\omega, \mathcal{H}_\omega\}$ to the triple $\{\pi_\omega, \mathcal{H}_\omega, \Omega\}$. A version with less technicalities is given in [4, Thm. 2.5.3], whose proof may be better suited for a first round.

A rough sketch of the construction goes as follows: A positive linear functional ω defines a sesquilinear form by $\omega(AB)$, which can be completed into a Hilbert space on which the algebra can be represented. Note that for a separable algebra \mathcal{A} , the GNS Hilbert space can be chosen likewise separable [7, Thm. 5.17]. In Sect. 2.2.2, we will see that every C^* -algebra admits a faithful representation, and together with the above construction, one can prove the Gelfand-Naimark theorem, which states that any C^* -algebra is $*$ -isomorphic to a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} [8, Thm. 9.18].

The last result in this section gives rise to a connection between the purity of states and the associated representations in the GNS-construction [8, Thm. 9.22]:

Theorem 22. *Let $\omega \in \mathcal{A}_+^*$ be a positive linear functional on a C^* -algebra. Then the following are equivalent:*

- a) ω is pure.
- b) The cyclic representation $\{\pi_\omega, \mathcal{H}_\omega\}$ induced by ω is irreducible.

2.2.1.4 Topologies

In several instances, like the compactness of the unit ball in the dual space, we mentioned the dependence of the topology. The importance of a topology is due to the fact that continuity, convergence, compactness, denseness, and others are defined by using open sets, i.e., they depend on the particular choice of a topology. We will give a very brief general introduction to the topic [10, Sect. 1.2].

Topology basics Let X be a set and τ a collection of subsets of X . If τ contains X and the empty set \emptyset , is closed under an arbitrary union of elements and finite intersections, we call the pair (X, τ) a **topological space**, the elements of τ **open sets** and τ a **topology**. If X has the structure of a vector space and addition and scalar multiplication are continuous, we call it a **topological vector space**. We can compare two topologies τ_1 and τ_2 on X in the following way: we say τ_1 is **finer** than τ_2 or τ_2 is **coarser** than τ_1 , if $\tau_2 \subseteq \tau_1$, i.e. τ_1 has more open sets.

The typical way to define or construct a topology is by defining its open sets. One of the most commonly known and used is the **norm topology**: If X is a normed space, we can define a metric by $d(a, b) = \|a - b\|$ and get open sets by

$$B_\varepsilon(a) = \{b \in X \mid \|a - b\| < \varepsilon\}. \quad (2.35)$$

Then a sequence A_n converges to A in this topology if $\|A_n - A\| \rightarrow 0$. In many applications, this topology is too restrictive.

For example, we will see in Sect. 2.4, that if we demand a semigroup of operators to be norm convergent, it forces the generator of the semigroup to be bounded. This would exclude many applications, for example, in quantum optics, where most generators are unbounded. Therefore, we are going to introduce some more topologies.

Instead of using a norm, one can do a similar construction with a family of seminorms instead [7, IV.1.1f]. A function p is called **seminorm** if it fulfills the same requirements as a norm with the difference that it is allowed to vanish on non-zero elements, i.e., there may exist $0 \neq x \in X$, such that $p(x) = 0$. Hence, we need a whole family of seminorms instead of just one norm. Given a vector space X and a family of seminorms $\{p_i\}$, they define a topology by the sets

$$U_i(y) = \{x \in X \mid p_i(x - y) < \varepsilon\}. \quad (2.36)$$

A topological vector space with the topology defined by a family of seminorms is called **locally convex space**. All of the following examples for topologies on $\mathcal{B}(\mathcal{H})$ are generated by families of seminorms. In general, those topologies can be relatively wild, for example, they do not need to be Hausdorff. One can show that the generated topology is Hausdorff if and only if for every non-zero $x \in X$ there exists a seminorm p_i , such that $p_i(x) \neq 0$ [7, Def. 1.2f], which is always the case for the local convex topologies in this work.

A useful source for seminorms that define topologies is the dual space or a dual pair, leading to the **weak topology** [7, V.1]. Here we take the elements $\omega \in \mathcal{A}^*$ to define a family of seminorms on \mathcal{A} :

$$p_\omega(a) = \omega(a) = |\langle a, \omega \rangle|. \quad (2.37)$$

This topology is also denoted as the $\sigma(\mathcal{A}, \mathcal{A}^*)$ -topology and is the coarsest topology, that makes all $\omega \in \mathcal{A}^*$ continuous [13, II.5]. Further, we can switch the argument and parameter in the seminorm, i.e., use

$$p_a(\omega) = \omega(a) = |\langle \omega, a \rangle| \quad (2.38)$$

as seminorms on \mathcal{A}^* , which defines the **weak* topology**.

As an example, we can choose X as $\mathcal{T}(\mathcal{H})$, so X^* is $\mathcal{B}(\mathcal{H})$, and with the duality described in Thm. 16, we get the weak* topology on $\mathcal{B}(\mathcal{H})$, which is also called the σ -weak operator topology or ultraweak topology [12, 3.5.5]. Topologies that are defined on $\mathcal{B}(\mathcal{H})$ are generally called **operator topologies**. Other important operator topologies are the following [7, IX. Def. 1.2f]:

- The **strong operator topology** is the topology on $\mathcal{B}(\mathcal{H})$ generated by the family of seminorms

$$p_\Psi(A) = \|A\Psi\| \quad \Psi \in \mathcal{H}. \quad (2.39)$$

Accordingly a sequence A_n converges to A in this topology if $\|(A_n - A)\Psi\| \rightarrow 0$ for all $\Psi \in \mathcal{H}$.

- The **strong* operator topology** is an extension of the above, making the adjoint operator continuous. For this, we use the seminorms

$$p_\Psi(A) = \|A\Psi\| + \|A^*\Psi\| \quad \Psi \in \mathcal{H}. \quad (2.40)$$

- On the other hand, the **weak operator topology** is defined by the seminorms

$$p_{\Phi, \Psi}(A) = |\langle \Phi, A\Psi \rangle| \quad \Phi, \Psi \in \mathcal{H}. \quad (2.41)$$

Here A_n converges to A if and only if $|\langle \Phi, (A_n - A)\Psi \rangle| \rightarrow 0$ for all $\Psi, \Phi \in \mathcal{H}$.

In the following, the prefixes *weak* and *strong* will refer to the operator topologies.

Equipped with the proper definitions of the different topologies, we can now state some of the previously mentioned results. We begin with the compactness of the state spaces in Sect. 2.2.1.2, which is a direct implication of the Banach-Alaoglu theorem [10, Thm. 2.5.2]:

Theorem 23 (Banach-Alaoglu). *If X is a normed vector space, then the closed unit ball of X^* is weak*-compact.*

The quasi-state space $\mathcal{Q}(\mathcal{A})$, as a closed subspace of the weak*-compact unit ball in \mathcal{A}^* , is always weak*-compact. If the state space is also closed, we can use the same argumentation to conclude that $\mathcal{S}(\mathcal{A})$ is likewise weak*-compact [12, 3.2.1].

The compactness combined with the convexity allows us to characterize the state space as the closed convex hull of the pure states by the Krein-Milman theorem [10, Thm. 2.5.4]:

Theorem 24 (Krein-Milman). *Let X be a locally convex space and $K \subset X$ a compact, non-empty, and convex subset. Then K has extreme points, and the closure of the convex hull of the extreme points equals K .*

In short, this theorem guarantees us the existence of plenty extremal elements, which play a major role in the structure of our state spaces, respectively, quasi-state spaces.

2.2.1.5 The enveloping C*-algebra

Before we continue with even more structure by introducing von Neumann algebras, we will review the concept of the enveloping C*-algebra. With this construction, we can study Banach *-algebras by using the C*-algebraic toolkit. We will use this in Sect. 5.2.3 to define the twisted group algebra over our hybrid phase space, whose states correspond exactly with our definition of hybrid standard states.

The following results are taken from [14, Ch. 2.7], which also contains the according proofs and more details.

Proposition 25. *Let \mathcal{A} be a Banach *-algebra with approximate identity. Let \mathcal{R} be the set of representations of \mathcal{A} , $\mathcal{Q}(\mathcal{A})$ the quasi-state space, and $\mathcal{P}(\mathcal{A})$ the set of pure states of \mathcal{A} . For every $A \in \mathcal{A}$, we have*

$$\|A\|' = \sup_{\pi \in \mathcal{R}} \|\pi(A)\| = \sup_{\omega \in \mathcal{Q}(\mathcal{A})} \|\omega(A^*A)^{1/2}\| = \sup_{\omega \in \mathcal{P}(\mathcal{A})} \|\omega(A^*A)^{1/2}\| \leq \|A\|. \quad (2.42)$$

The map $A \mapsto \|A\|'$ is a seminorm on \mathcal{A} such that

$$\|AB\|' \leq \|A\|' \|B\|', \quad \|A^*\|' = \|A\|', \quad \|A^*A\|' = \|A\|'^2 \quad (2.43)$$

for any $A, B \in \mathcal{A}$.

Now $\|A\|'$ defines a norm on \mathcal{A}/I , where I is the closed self-adjoint two-sided ideal of elements $A \in \mathcal{A}$ with $\|A\|' = 0$. With this, we can now define:

Definition 26. *The completion of $\{\mathcal{A}/I, \|\cdot\|'\}$ is called the **enveloping C*-algebra** of \mathcal{A} and is denoted as $C^*(\mathcal{A})$.*

With the completeness and the properties inherited from \mathcal{A} , the space $C^*(\mathcal{A})$ is a C*-algebra. If \mathcal{A} is already a C*-algebra, then $\|A\| = \|A\|'$ and \mathcal{A} can directly be identified as its enveloping C*-algebra.

Further one can show that for a Banach *-algebra \mathcal{A} and its enveloping C*-algebra $C^*(\mathcal{A})$, with τ denoting the canonical map of \mathcal{A} into $C^*(\mathcal{A})$, there is exactly one representation ρ of $C^*(\mathcal{A})$, such that

$$\pi = \rho \circ \tau, \quad (2.44)$$

where π is a representation of \mathcal{A} . Also $\tau(C^*(\mathcal{A}))$ is the C*-algebra generated by $\pi(\mathcal{A})$ and the map $\pi \rightarrow \tau$ is a bijection of the set of representations of \mathcal{A} onto the set of representations of $C^*(\mathcal{A})$. The representation τ is non-degenerate, resp. irreducible if and only if π is non-degenerate, resp. irreducible.

These properties carry over from the representations to the states: If $\omega \in \mathcal{A}_+^*$, there is exactly one $\eta \in C^*(\mathcal{A})_+^*$, such that

$$\omega = \eta \circ \tau \quad \text{and} \quad \|\omega\| = \|\eta\|. \quad (2.45)$$

Also, the map $\omega \rightarrow \eta$ is a bijection of \mathcal{A}_+^* onto $C^*(\mathcal{A})_+^*$ and, when restricted to a bounded set, is bicontinuous in the corresponding weak* topologies.

2.2.2 Von Neumann algebras and the bidual

Next to C*-algebras, W*- or von Neumann algebras are central working spaces in the field of operator algebras. In contrast to C*-algebras, it is more subtle to define those. For a start, let us remark that W*-algebras and von Neumann algebras are, strictly speaking, not synonymous. Usually, different terminologies indicate which kind of definition is used by the author, i.e., an abstract one or one given a concrete Hilbert space. We begin historically in reverse with the abstract definition from Sakai [15]. Note that the equivalence of the two upcoming definitions (up to representation) is non-trivial and an important achievement in the field itself.

Definition 27 (W*-algebra). *A C*-algebra \mathcal{M} is called a **W*-algebra** if it has a predual, i.e., there exists a Banach space \mathcal{M}_* such that $(\mathcal{M}_*)^* = \mathcal{M}$.*

The next definition is from Murray and von Neumann himself [16]. They named the following object *ring of operators*, and while the field evolved, its name changed to the surname of one of the originators:

Definition 28 (von Neumann algebra). *A **von Neumann algebra** on \mathcal{H} is a *-subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ containing the unit, such that*

$$\mathcal{M} = \mathcal{M}'' \quad (2.46)$$

Here \mathcal{M}' denotes the **commutant**, i.e., the set

$$\mathcal{M}' = \{A \in \mathcal{B}(\mathcal{H}) \mid [A, B] = 0 \forall B \in \mathcal{M}\} \quad (2.47)$$

and \mathcal{M}'' is the **bicommutant** which is obtained by applying the above definition twice.

We can exchange the requirement in Eq. (2.46) with \mathcal{M} being closed in several operator topologies. This equivalence between the algebraic property using the commutant and the topological ones is famously known as von Neumann's **bicommutant theorem** [12, Thm. 2.2.2]:

Theorem 29 (bicommutant theorem). *Let \mathcal{M} be a von Neumann algebra. Then the following are equivalent:*

- a) \mathcal{M} is closed in the weak operator topology.
- b) \mathcal{M} is closed in the strong operator topology.
- c) $\mathcal{M} = \mathcal{M}''$

Clearly, $\mathcal{B}(\mathcal{H})$ satisfies both definitions, and every von Neumann algebra is a C*-algebra, so the results from the previous chapter can be carried over.

Now, we introduce the concept of a universal representation and the enveloping von Neumann algebra. First note that any representation $\{\pi, \mathcal{H}\}$ of a C*-algebra \mathcal{A} generates a von Neumann algebra \mathcal{M}_π by taking a closure or utilizing Thm. 29 via

$$\mathcal{M}_\pi = \pi(\mathcal{A})''. \quad (2.48)$$

This algebra depends on the representation chosen and hence comes with some ambiguity. We can eliminate this by using a specific representation, which is universal in the sense that any other representation of \mathcal{A} factors through it. This representation is called the **universal representation** [12, 3.7.6] and can be explicitly constructed: For a C*-algebra \mathcal{A} with state space $\mathcal{S}(\mathcal{A})$, we define it by taking the direct sum over all cyclic GNS-representations ω and the respective Hilbert spaces, i.e.

$$\pi = \bigoplus_{\omega} \pi_{\omega} \quad \text{and} \quad \mathcal{H} = \bigoplus_{\omega} \mathcal{H}_{\omega}. \quad (2.49)$$

The universal representation is faithful, which proves the aforementioned statement that every C*-algebra admits a faithful representation, leading to the Gelfand-Naimark theorem. The von Neumann algebra \mathcal{M}_π defined by Eq. (2.48) using the universal representation is called the **universal enveloping von Neumann algebra**. Obviously, this algebra can be quite large, but it always exists and is unique up to isomorphisms. Hence, it is often abbreviated as \mathcal{A}'' , leaving out the representation. Because the universal representation is faithful, we can identify \mathcal{A} as a C*-subalgebra of \mathcal{A}'' .

The universal enveloping von Neumann algebra can also be described in another way: For a normed space and hence any C*-algebra \mathcal{A} , we call the dual of the dual space \mathcal{A}^{**} the **bidual**. In general, this space can be much larger than \mathcal{A} , but by

$$i(A)\omega = \omega(A), \quad A \in \mathcal{A}, \omega \in \mathcal{A}^* \quad (2.50)$$

we get a isometric map from $i: \mathcal{A} \rightarrow \mathcal{A}^{**}$, called **evaluation map** [10, 2.3.7], which gives

$$\mathcal{A} \subset \mathcal{A}^{**}. \quad (2.51)$$

One can show that \mathcal{A}^{**} is again a C*-algebra, and as a bidual, it naturally has the predual \mathcal{A}^* , making it a W*- or von Neumann algebra. We already encountered an example for this with the compact operators, where $\mathcal{K}(\mathcal{H})^{**} = \mathcal{B}(\mathcal{H})$ and clearly $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. With the universal enveloping von Neumann algebra and the bidual, we have ways to define a von Neumann algebra based on a C*-algebra. While different at first glance, these concepts are closely connected [12, Prop. 3.7.8]:

Proposition 30. *The universal enveloping von Neumann algebra \mathcal{M} of a C*-algebra \mathcal{A} is isomorphic, as a Banach space, to the bidual \mathcal{A}^{**} .*

Hence, we will often denote the universal enveloping von Neumann algebra \mathcal{A}'' of a C*-algebra \mathcal{A} as \mathcal{A}^{**} . Before we come to the benefits of this construction, we give a more precise version of the loosely stated fact that any representation of \mathcal{A} factors through the universal representation. For this, we need the definition of normal maps between von Neumann algebras [12, 2.5.1]:

Definition 31. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras and $\mathcal{T}: \mathcal{M} \rightarrow \mathcal{N}$ a positive, linear map. It is called **normal** if for each bounded monotone increasing net $\{x_i\}$ in \mathcal{M}_{sa} with limit x , the net $\{\mathcal{T}(x_i)\}$ increases to $\mathcal{T}(x)$ in \mathcal{N}_{sa} .*

If we set $\mathcal{N} = \mathbb{C}$ in the above definition, we also get a definition of a normal functional ω , which will become important later on, but for now, we can state the following [12, Thm. 3.7.7]:

Proposition 32. *Let $\{\pi, \mathcal{H}\}$ be a non-degenerate representation of a C*-algebra \mathcal{A} . Then there exists a unique normal representation π'' of \mathcal{A}'' onto $\pi(\mathcal{A})$ that extends π , i.e. $\pi''(\mathcal{A}) = \pi(\mathcal{A})$ and $\pi''(\mathcal{A}'') = \pi(\mathcal{A})''$.*

As we now know how to construct von Neumann algebras out of a C*-algebra, it is certainly time to answer the question of what benefits we get by this, i.e., what structures a von Neumann algebra offers, that a C*-algebra misses. One of the benefits is that von Neumann algebras generally house many projections. Indeed, they contain enough projections, such that they generate the algebra [12, 2.2.6]: Let \mathcal{M} be a von Neumann algebra and $\mathcal{P}(\mathcal{M}) = \{m \in \mathcal{M} \mid m^2 = m = m^*\}$ be the set of projections in \mathcal{M} . Then \mathcal{M} is the norm-closure of the linear span of $\mathcal{P}(\mathcal{M})$:

$$\mathcal{M} = \mathcal{P}(\mathcal{M})''. \quad (2.52)$$

The projections in a von Neumann algebra also form a complete lattice, and according to Takesaki [8, p.290]:

It is not an overstatement to say that the study of the projection lattice of a von Neumann algebra is at the core of the whole theory.

For the *whole theory*, we refer to Takesaki's series [8] or the other references in Sect. 2.5 and continue with the application to quantum theory.

2.3 Quantum States and Measurements

Let us now take the tools mentioned above to work and update our description of quantum mechanics from the first chapter in a more precise way.

2.3.1 State spaces

Based on the context, the name *state space* can have two different meanings. One is the *mathematical* state space of a C*-algebra and, therefore the von Neumann algebra $\mathcal{B}(\mathcal{H})$ as described in Sect. 2.2.1.2. There, we have defined

$$\mathcal{S}_1 = \{\omega \in \mathcal{B}(\mathcal{H})^* \mid \omega \geq 0, \|\omega\| = 1\}. \quad (2.53)$$

Another reading is the space of **density matrices** ρ defined as

$$\mathcal{S}_2 = \{\rho \in \mathcal{T}(\mathcal{H}) \mid \rho \geq 0, \text{tr}[\rho] = 1\}. \quad (2.54)$$

Here, we have $\mathcal{S}_2 \subset \mathcal{S}_1$ in the sense of embedding a space into its bidual. Functionals that arise from the predual, i.e., the space \mathcal{S}_2 , rather than the full dual, are called **normal states**. Normal states on $\mathcal{B}(\mathcal{H})$ are exactly those elements of \mathcal{S}_1 , which are normal maps according to Def. 31 or weak*-continuous [12, Thm. 3.6.4]:

Theorem 33. *Let ω be a bounded functional on a von Neumann algebra \mathcal{M} in $\mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

- a) ω is normal
- b) ω is weak*-continuous
- c) There is a trace-class operator $\rho \in \mathcal{T}(\mathcal{H})$, such that $\omega = \text{tr}[\rho A]$ for all $A \in \mathcal{M}$.

In contrast, non-normal elements in \mathcal{S}_1 , also called **singular states**, can be relatively wild objects, definitely from a mathematical point of view. Nevertheless, in physics, singular states are commonly used, often as idealizations or *improper eigenstates* for the position and momentum operators [9, Sect. 4.2]. Although defining normal quantum states as general *quantum states* while discarding the rest is quite the standard. This is understandable, as the hassle that comes with the full dual space is often disproportionately compared to the benefits.

Another detail, which is often taken for granted, is the fact that the set of normal states in quantum mechanics contains such a large amount of pure states, which is not a priori guaranteed.

The pure states of $\mathcal{T}(\mathcal{H})$ are the one-dimensional projectors, and a typical representation for density operators $\rho \in \mathcal{S}_2$ is given by their **canonical convex decomposition**

$$\rho = \sum_i c_i P_i, \quad (2.55)$$

where c_i is a possibly infinite sequence of positive numbers that sum up to one [17, Thm. 2.5]. For the connection to the pure states $|\Psi\rangle \in \mathcal{H}$ from the Hilbert space formulation of quantum mechanics, one often writes $P_\Psi = |\Psi\rangle\langle\Psi|$, and accordingly those states are called **vector states**.

In view of our upcoming generalization for quantum-classical hybrids, let us rephrase this more abstractly: In quantum theory, the observable algebra is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ with the *well-behaved* normal states as elements of the predual. We call this allocation of states and observables the W^* - or *von Neumann view*. Whereas arbitrary C^* -algebras \mathcal{A} are not required to have a predual, their state space is naturally the full dual \mathcal{A}^* , which we call the C^* -*view*. An overview of the utilized spaces and dualities in quantum theory is depicted in Fig. 2.2.

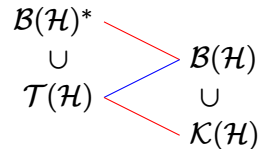


Figure 2.2: Dualities between operator spaces in quantum theory: The von Neumann-type duality with the states in the predual is marked in blue, while the dualities according to the C^* -view are drawn in red.

Note that it would be highly inadvisable to directly use $\mathcal{K}(\mathcal{H})$ as the observable algebra for quantum theory. The existence of an identity operator is elemental in the description of measurements. Clearly, $\mathbb{1} \notin \mathcal{K}(\mathcal{H})$, unless we take \mathcal{H} to be finite-dimensional, so in general the algebra of compact operators is too *small*.

2.3.2 Measures and measurements

Before we take a closer look at the mathematical description of quantum measurements, let us recall some terms of measure theory:

A σ -**algebra** over a set X is a collection of subsets \mathcal{F} , that

- contains the whole and the empty set: $\emptyset, X \in \mathcal{F}$
- is stable under the complement, i.e. for every $x \in \mathcal{F}$ we have $X \setminus x \in \mathcal{F}$
- is closed under countable unions, i.e., for every countable collection of sets $x_1, x_2, \dots \in \mathcal{F}$, we have $\bigcup_i x_i \in \mathcal{F}$.

The combination of the space and the σ -algebra (X, \mathcal{F}) is also called a **measurable space** and the elements $x \in \mathcal{F}$ are sometimes called **events**, based on the use in probability theory. An obvious choice for \mathcal{F} is the power set $\mathcal{P}(X)$, although it is too large for many applications. Most commonly used is the **Borel σ -algebra**, which is the smallest σ -algebra that contains all open sets of X and is usually denoted as $\mathcal{B}(X)$ ¹. The elements of $\mathcal{B}(X)$ are called **Borel sets**. With these definitions, we can define the normalized positive measures over X , better known as **probability measures**:

¹Not to be confused with the bounded linear operators

Definition 34 (probability measure). *A probability measure on the space (X, \mathcal{F}) is a map $\mu : \mathcal{F} \rightarrow [0, 1]$, which satisfies the following:*

- i) It is normalized in the sense that $\mu(\emptyset) = 0$ or $\mu(X) = 1$.*
- ii) The map μ is countably or σ -additive, that is*

$$\mu\left(\bigcup_i x_i\right) = \sum_i \mu(x_i) \quad (2.56)$$

for any countable, disjoint sets $x_i \in \mathcal{F}$.

We denote the set of all probability measures over X as $\mathcal{W}(X)$.

We will see in Sect. 3.2 that the space $\mathcal{W}(X)$ arises as the state space of a commutative C^* -algebra. A measurable set (X, \mathcal{F}) with a probability measure μ is called a **probability space**.

The overall procedure of a quantum measurement is typically connected to the word **observable**. Unfortunately, the precise object behind the word is different across the literature. It is used for a self-adjoint operator, a POVM, or a measurement in general. For this work, especially for the upcoming hybrid part, we choose the most general meaning, which is indeed quite simple: An observable is an element in the observable algebra. Now, let us collect some of the more precise definitions for this matter.

The quantum analog of probability measures needs the notation of **effects**:

$$\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq \mathbb{1}\}. \quad (2.57)$$

They model the basic building blocks of a quantum measurement and can be interpreted as *yes/no* measurements. The set $\mathcal{E}(\mathcal{H})$ is convex with the projections as extremal elements. Indeed, the states and effects are dual objects. That is, together, they define the statistics. A worthwhile and more in-depth description of this can be found in the section *2.1 Duality of states and effects* in [17]. Here, we will just reduce some of the key points of the theory [17, Sect. 3].

Definition 35 (positive operator-valued measure). *Let (X, \mathcal{F}) be a measurable space. A **positive operator-valued measure** or **POVM** is a map $M : \mathcal{F} \rightarrow \mathcal{E}(\mathcal{H})$, which satisfies the following:*

- i) M is normalized in the sense that $M(\emptyset) = 0$ or $M(X) = \mathbb{1}$.*
- ii) For any countable collection of disjoint elements $x_i \in \mathcal{F}$, we have*

$$M\left(\bigcup_i x_i\right) = \sum_i M(x_i), \quad (2.58)$$

which has to be read in weak operator topology.

Comparing this definition with the above, we see that a POVM is a probability measure when evaluated on a state:

$$\mu(dx) = \text{tr}[\rho M(dx)]. \quad (2.59)$$

Hence, POVMs are readily understood as the non-commutative analog to probability measures. Because of the normalization condition $M(X) = \mathbb{1}$, they are also known as **positive resolutions of the identity** [18]. The special case of **projection valued measures** or **PVMs** from the introduction are easily obtained by restricting POVMs to the subset of projections instead of all effects $\mathcal{E}(\mathcal{H})$. Note that the above is a more *measure theoretic* definition of POVMs. Often, it is suitable to reduce the definition to the discrete case, where $X = \{x_1, x_2, \dots\}$ are the possible measurement outcomes and the associate POVM is the collection of positive operators

$$M_x \geq 0 \quad \text{with} \quad \sum_x M_x = \mathbb{1}. \quad (2.60)$$

Accordingly, the operators M_x are also called POVM elements. Historically, the projection-valued measures predate the positive operator-valued measures. It was Davies [19] who successfully introduced this more general definition of a quantum measurement. The charm of the reduction of observables to PVMs and self-adjoint operators lies in their convenient connection by the spectral theorem. It states that for every self-adjoint operator A , there exists a unique PVM $A(dx)$, such that

$$A = \int_{\mathbb{R}} x A(dx). \quad (2.61)$$

Conversely, every PVM on the real line defines a self-adjoint operator by Eq. (2.61).

Now, calculating statistical quantities easily carries over from classical statistics. The **expectation value** for a preparation in the state ρ and the POVM M is just

$$\langle M \rangle_{\rho} = \int_X x \operatorname{tr}[\rho M(dx)], \quad (2.62)$$

which for the projection valued case can be written as $\operatorname{tr}[\rho A]$ or $\langle \Psi | A | \Psi \rangle$ for pure states. Closely related is the **variance**, which becomes

$$\Delta(M)_{\rho} = \int_X (x - \langle M \rangle_{\rho})^2 \operatorname{tr}[\rho M(dx)]. \quad (2.63)$$

We are going to finish this part with some remarks: In general, Eq. (2.61) has to be read with caution as A is not necessarily a bounded operator, which comes with all the difficulties, like domain considerations, that unbounded operators always include. As we already mentioned, the operators for position and momentum are the two prime examples for this and motivate the phase space formulation of quantum mechanics, see Chap. 5, where instead of the unbounded operators, we introduce their exponentials, called Weyl operators, which are bounded and even unitary.

2.4 Quantum Dynamics

The keyword quantum dynamics affects many different aspects, so let us start with a short overview. For us, **dynamics** will denote everything between the preparation and measurement of a given system, i.e., a possibly time-dependent map between the states or observables. For example, the order of a concatenation of maps, where time-dependence has to be understood as defining the order in which to apply them, can be called dynamics. Nonetheless, it often means continuous time-evolution, where the map has some continuous dependence upon a parameter t . The most commonly known example for the time-continuous case is the unitary evolution already introduced in Eq. (2.6). The mathematical concept behind admissible maps from one quantum system to another is *complete positivity*, which we will have a closer look at in the first part of this section.

Further, we can separate between **closed system** dynamics, which can be represented by the unitary evolution, or **open system** dynamics. Open systems enable us to describe irreversible transformations, for example, the system under consideration is allowed to be in exchange with its environment, or we describe measurement processes, i.e., we extract information out of a quantum system.

2.4.1 Complete positivity

The mathematical structure behind quantum dynamics is described by completely positive maps. Based on the previous sections, it seems reasonable to demand quantum dynamics \mathcal{T} to be just **positive maps**, that is, for $A \geq 0$, one would demand $\mathcal{T}(A) \geq 0$.

Indeed, this was the point of view before the concept of complete positivity was moved from a mathematical property to the general framework of quantum theory. While many aspects of the basic mathematical formulation of quantum theory were clear early on, the change towards complete positivity happened relatively late in the 1970s. A good witness of this process is found in the famous book *Quantum Theory of Open System* [19, Sect. 9.2, p. 136] by Davies, who wrote:

We remark that the condition of complete positivity could have been imposed on operations, instruments, and all subsequent results in this volume. We have refrained from doing so because the slight but wearisome extra details necessitated would have been completely unrewarded until this chapter.

Today, it is a common understanding that quantum dynamics are to be demanded as completely positive maps [17, Sect. 4].

Definition 36 (completely positive map). *Let \mathcal{A} be a C^* -algebra, $\mathcal{M}_n(\mathcal{A})$ the set of $n \times n$ matrices with entries in \mathcal{A} and $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$ a linear map. Then \mathcal{T} is called **n-positive**, if $\mathcal{T}_n = \mathcal{T} \otimes \mathbb{1}_n$ is a positive map on $\mathcal{M}_n(\mathcal{A})$ and **completely positive**, if \mathcal{T}_n is a positive map for every $n \in \mathbb{N}$.*

Obviously, every completely positive map is also positive, but the converse statement is false. Here, the most prominent example is the partial transposition on the finite-dimensional C^* -algebra of n -dimensional square matrices $\mathcal{A} = M_n(\mathbb{C})$, which is easily verified as a positive but not completely positive map. Given its more

restrictive nature than just positivity, completely positive maps have powerful properties. Before we discuss those, let us look at the physical motivation for Def. 36: The evolution of a quantum system should also yield an admissible quantum state if we extend the state by an ancilla or environment that has no interaction with the system itself. An illustration of this is depicted in Fig. 2.3.

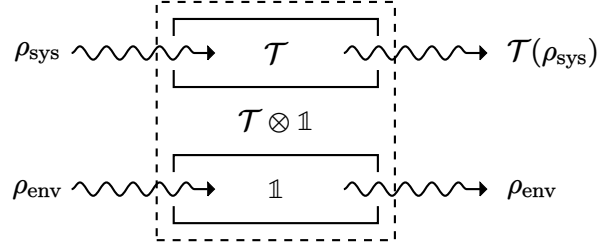


Figure 2.3: Physical motivation for complete positivity: A state ρ_{sys} , that evolves under the map \mathcal{T} , is extended by an arbitrary state ρ_{env} with no interaction in between. If \mathcal{T} is completely positive, the composite system is again a valid quantum state.

We continue with a simple lemma, which makes the above definition more practical [8, Cor. 3.4]:

Lemma 37. *Let \mathcal{A}, \mathcal{B} be C^* -algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ a linear map. Then ϕ is n -positive if and only if*

$$\sum_{i,j}^n y_i^* \phi(x_i^* x_j) y_j \geq 0 \quad \forall x_a \in \mathcal{A}, y_b \in \mathcal{B}, \quad (2.64)$$

and completely positive if the above holds for all n .

In order to check the above positivity condition, the following lemma provides a useful, necessary, and sufficient condition [8, Lem. 3.2]:

Lemma 38. *Let \mathcal{A} be a C^* -algebra. An element of $\mathcal{M}_n(\mathcal{A})$ is positive, if and only if it is a sum of matrices of the form $[a_i^* a_j]$ with $a_1, \dots, a_n \in \mathcal{A}$.*

In light of the topic of this work, a result worth mentioning is the necessity of non-commutativity to make a difference compared to positivity. On the classical side of our hybrid system, which for the moment we just read as *commutative*, these notions coincide [20, Thm. 1.2.4f]:

Theorem 39. *Let \mathcal{A} and \mathcal{B} be C^* -algebras and $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ a positive linear map. If \mathcal{A} or \mathcal{B} is commutative, then \mathcal{T} is completely positive.*

In general, it can be hard to check whether a map is completely or just n -positive. This changes if we assume a finite-dimensional quantum system, i.e., $\mathcal{A} = \mathcal{B}(\mathcal{H}) = \mathcal{M}_d(\mathbb{C})$, where the dimension bounds the number n that we have to test [20, Cor. 4.1.9]:

Lemma 40. *Let $\mathcal{T} : \mathcal{M}_a(\mathbb{C}) \rightarrow \mathcal{M}_b(\mathbb{C})$ be a positive linear map and $k = \min(a, b)$. Then \mathcal{T} is completely positive if and only if \mathcal{T} is k -positive.*

2.4.2 Dilations and the operator-sum representation

... from Stinespring to Kraus

One of the most powerful tools regarding completely positive maps is the Stinespring dilation theorem. While it is useful on its own, it also leads to the Kraus decomposition, which helps identify and construct completely positive maps in many applications. The original version by Stinespring is the following [21]:

Theorem 41 (Stinespring). *Let \mathcal{A} be a C^* -algebra with unit, let \mathcal{H} be a Hilbert space, and let \mathcal{T} be a linear map from \mathcal{A} to operators on \mathcal{H} . Then, a necessary and sufficient condition that \mathcal{T} has the form*

$$\mathcal{T}(A) = V^* \pi(A) V \quad \text{for all } A \in \mathcal{A}, \quad (2.65)$$

where V is a bounded linear transformation from \mathcal{H} to a Hilbert space \mathcal{K} and π is a $*$ -representation of \mathcal{A} into operators on \mathcal{K} , is that \mathcal{T} be completely positive.

Here, the Hilbert space \mathcal{K} is also called **dilation space**, the triple (π, V, \mathcal{K}) a **Stinespring representation** for \mathcal{T} , and if $\mathcal{A} = \mathcal{B}(\mathcal{H})$, we shorten this to the pair (V, \mathcal{K}) . Note that the space \mathcal{K} may be chosen larger, and the requirement that \mathcal{K} is spanned by the closure of $\pi(\mathcal{A})V\mathcal{H}$ defines a **minimal Stinespring dilation**, which is unique up to unitary equivalence. If \mathcal{T} is unital, i.e. $\mathcal{T}(\mathbb{1}) = \mathbb{1}$, then V is an isometry.

If we set the algebras as $\mathcal{A} = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{T} : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$, one can expand the theorem: If \mathcal{T} is normal, we can replace π with an amplification, a projection, and an isometry [8, Thm. 5.5]. In that case one can rewrite \mathcal{T} as [17, 4.18f]:

$$\mathcal{T}(A) = \tilde{V}^*(A \otimes \mathbb{1})\tilde{V}. \quad (2.66)$$

This gives rise to an interpretation of quantum operations, which is popular in quantum information theory. Here, one often works in finite dimensions, and many subtleties from the infinite case become much more manageable. It states that a completely positive map on the states can be understood as an expansion to an ancilla system or environment, followed by a unitary interaction and a reduction [22, Cor. 5.6]:

$$\mathcal{T}(\rho) = \text{tr}_{\mathcal{K}}[U(\rho \otimes \rho_{\text{env}})U^*]. \quad (2.67)$$

Here $\text{tr}_{\mathcal{K}}$ denotes the partial trace and $\rho_{\text{env}} \in \mathcal{T}(\mathcal{H}_{\text{env}})$ is a state of the environment. This is also called the **ancilla form** of the channel \mathcal{T} .

Now we come to the **Kraus decomposition**. Here, we introduce a basis on \mathcal{K} , and with a short calculation, we get a practical characterization of arbitrary completely positive maps [17, Prop. 4.21]:

Proposition 42 (Kraus decomposition). *A linear map $\mathcal{T} : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ is completely positive if and only if there exist $K_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, such that*

$$\mathcal{T}(A) = \sum_i K_i^* A K_i. \quad (2.68)$$

This decomposition is also known as the **operator-sum representation**. The operators K_i are usually called **Kraus operators** and if \mathcal{T} is **unital** they satisfy

$$\sum_i K_i^* K_i = \mathbb{1}. \quad (2.69)$$

Stineprings theorem also allows us to formulate a quantum analog to the classical Radon-Nikodym from measure theory [22, Thm. 5.8]:

Theorem 43. *Let $\mathcal{T}_x : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a family of completely positive maps and $\mathcal{T} = \sum_x \mathcal{T}_x$. Further let (V, \mathcal{K}) be a Stinespring representation for \mathcal{T} , then there exists positive operators $F_x \in \mathcal{B}(\mathcal{K})$, with $\sum_x F_x = \mathbb{1}$ and*

$$\mathcal{T}_x(A) = V^*(A \otimes F_x)V. \quad (2.70)$$

If the Stinespring representation is minimal, then the F_x are unique.

2.4.3 Quantum channels and dynamics

In physics, the mathematical property of complete positivity is translated into the concept of **quantum channels**, where we set the algebra as $\mathcal{B}(\mathcal{H})$ and add a normalization:

Definition 44 (quantum channel). *A normalized, completely positive map*

$$\mathcal{T} : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2) \quad (2.71)$$

is called a quantum channel.

The first thing we need to discuss are the different pictures for quantum dynamics, i.e., Schrödinger and Heisenberg picture. Let us start with the Schrödinger picture, where we evolve the states, or more precisely, the normal states $\rho \in \mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Because \mathcal{T} is completely positive, it has a Kraus decomposition, and as the trace class operators form a two-sided ideal in $\mathcal{B}(\mathcal{H})$, we know that \mathcal{T} leaves the trace-class invariant. The normalization condition, which we translate as *probability preserving*, means that \mathcal{T} has to be trace-preserving.

If we now switch to the Heisenberg picture, we need the dual or **adjoint map**, which is similar to the adjoint for operators on Hilbert spaces. For this we use the duality between $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$, described in Thm. 16, and define \mathcal{T}^* by

$$\mathrm{tr}[\mathcal{T}(\rho)A] = \mathrm{tr}[\rho\mathcal{T}^*(A)]. \quad (2.72)$$

Then \mathcal{T}^* is likewise completely positive, and normalization translates into \mathcal{T}^* being unital. If we relax the condition and demand \mathcal{T} just to be *probability non-increasing*, we get a sub-normalized quantum channel or **quantum operation**.

Note that the above description has a slight asymmetry: If we interchange the roles of \mathcal{T} and \mathcal{T}^* , i.e., start with \mathcal{T} in the Heisenberg picture as suggested by Eq. (2.71) and ask for \mathcal{T}^* in Schrödinger picture, we further need to assume that \mathcal{T} is a normal map. This ensures that Eq. (2.72) is still valid and $\mathcal{T}^*(\rho)$ is a trace-class operator. Alternatively, we could use the full dual instead of Eq. (2.72), which would include non-normal states and also yield a Schrödinger picture [17, Sect. 4.1.2].

The first example of a quantum channel we encountered is the unitary time evolution in Eq. (2.8)

$$\mathcal{T}_t(\rho) = U_t \rho U_t^*, \quad (2.73)$$

which is trace-preserving as U_t are unitary operators and completely positive because the channel is already in a Kraus decomposition with one Kraus operator.

A more profound example of a quantum channel is the **quantum instrument** [17, Sect. 5.1.2].

Example 45 (quantum instruments). A channel that describes a quantum measurement and yields not only the measurement result but also the post-measurement state is called a quantum instrument. Note that this is more the motivation than the precise definition of a quantum instrument (e.g. [17, Def. 5.4]), but it is easier applicable to the hybrid framework. The most basic example for constructing a quantum instrument from a POVM is the following. Let M_x be a collection of POVM elements with a discrete outcome set X . As each M_x is positive, we use Def. 7 to decompose each POVM element into $M_x = K_x^* K_x$ and immediately get the Kraus operators for our channel:

$$\mathcal{T}(\rho) = \sum_x \mathcal{T}_x(\rho) = \sum_x K_x^* \rho K_x. \quad (2.74)$$

The normalization of the POVM elements directly translates into the normalization of \mathcal{T} , and each \mathcal{T}_x is a quantum operation. The interpretation is straightforward: The overall channel \mathcal{T} describes the *unconditional* case, where we *ignore* the outcome, while each \mathcal{T}_x describes the *conditional* evolution, that is, the (sub-normalized) state after we looked at our measurement device and observed the outcome $x \in X$. For a proper post-measurement state we need to renormalize $\mathcal{T}_x(\rho)$ by p_x^{-1} , where $p_x = \text{tr}[\rho M_x]$ is the probability for the outcome $x \in X$. Also note that this construction is non-unique, as, for example, we could easily add a unitary operator, which leaves the probabilities invariant but changes the post-measurement state.

This construction also works the other way round, i.e., use the Kraus decomposition of a quantum channel and define the associated POVM.

In this basic description, the measurement output, or let us say the *classical* part of the system, has gotten a raw deal, as it is merely described by *words* rather than being a proper part of the output system. Indeed, more complex descriptions of instruments introduce a pointer system as a register for the measurement result, and the quantum instrument can be seen as the first operation that turns a pure quantum system into an effective hybrid [23, 4.6.8].

Before introducing the continuous-time dependency, let us draw attention to one last fact about quantum channels, namely the state-channel correspondence, which has been a handy tool in quantum information theory. It originated in Choi's thesis [24], which is often cited together with Jamiolkowski [25]. The most prominent result is today known as the **Choi-Jamiolkowski isomorphism** [17, 4.4.3].

For this part, we will reduce technicalities by only looking at finite-dimensions, i.e. $\mathcal{B}(\mathcal{H}) \cong \mathcal{M}_n(\mathbb{C})$. Let

$$\mathcal{T} : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_{d'}(\mathbb{C}) \quad (2.75)$$

be a positive linear map. Then, one can show that \mathcal{T} is completely positive if and only if the matrix

$$\Theta_{\mathcal{T}} = \begin{pmatrix} \mathcal{T}(|\phi_1\rangle\langle\phi_1|) & \dots & \mathcal{T}(|\phi_1\rangle\langle\phi_d|) \\ \vdots & \ddots & \vdots \\ \mathcal{T}(|\phi_d\rangle\langle\phi_1|) & \dots & \mathcal{T}(|\phi_d\rangle\langle\phi_d|) \end{pmatrix} \quad (2.76)$$

is positive, where $|\phi_i\rangle$ is an orthonormal basis for the d -dimensional Hilbert space $\mathcal{H}_d \cong \mathbb{C}^d$ and the matrix $\Theta_{\mathcal{T}}$ is also called **Choi-matrix** [17, 4.46]. For the formulation of the isomorphism, let

$$\Phi = \frac{1}{d} \sum_{i,j} |\phi_i \otimes \phi_j\rangle\langle\phi_i \otimes \phi_j| \quad (2.77)$$

be the projector on the maximally entangled state on $\mathcal{H}_d \otimes \mathcal{H}_d$. Then

$$\mathcal{J} : \mathcal{T} \mapsto \Omega_{\mathcal{T}} = (\mathcal{T} \otimes \mathbb{1})(\Phi) \quad (2.78)$$

is the **Choi-Jamiolkowski isomorphism**, that allows us to study maps \mathcal{T} by the means of states. Its inverse is given by

$$\mathcal{J} : \Omega \mapsto \mathcal{T}_{\Omega} : \mathcal{T}(A) = d \operatorname{tr}_2[(\mathbb{1} \otimes A^T)\Phi], \quad (2.79)$$

where tr_2 means the partial trace over the second tensor factor and shows that channels are isomorphic to a subset of the state space [17, 4.48].

Time-continuous processes

... from Stone to Lindblad

Now we have a closer look at the time-dependency, which we already used for the family of unitary operators $\{U_t\}$. Such a family is called a strongly continuous **one-parameter unitary group**, i.e. $U_0 = \mathbb{1}$, $U_s U_t = U_{s+t}$ for all $t, s \in \mathbb{R}$ and U_t is strongly continuous. We already mentioned that we could generate those by exponentiating the Hamiltonian via $U_t = \exp(itH)$. Indeed, the converse statement is also true and known as **Stone's theorem** [7, X. Thm. 5.6]:

Theorem 46 (Stone). *Let U_t be a one-parameter unitary group on a Hilbert space \mathcal{H} . Then U_t is generated by a self-adjoint operator H .*

These generators are not necessarily bounded, so in general, they are only defined on a **domain** $\mathcal{D} \subset \mathcal{H}$. For the transition to open systems, we can use the ancilla form of dilations described in Eq. (2.67), where we explicitly model the dynamics, including the environment, stay in the unitary description, and reduce to the desired systems afterwards. Usually, this direct approach is not practical, as the environment may be very large and hard to impossible to model. Instead, we introduce dynamics directly on the reduced system at the cost of introducing irreversibility.

Let us briefly recap this derivation, which is commonly done in the quantum physics literature [26]: We take a system \mathcal{S} that interacts with its environment \mathcal{E} .

For example, \mathcal{S} may be a trapped atom that interacts with a light-field \mathcal{E} . One assumes that the dynamics for \mathcal{S} and \mathcal{E} are determined by a Hamiltonian of the form $H = H_{\mathcal{S}} + H_{\mathcal{E}} + H_{\text{int}}$, where the first two terms describe the dynamics of the system and the environment and H_{int} their interaction. The dynamics for the state ρ_{sys} are then obtained by taking the partial trace over the environment, which yields the following infinitesimal behavior:

$$\dot{\rho}_{\text{sys}} = \mathcal{L}(\rho_{\text{sys}}) = i[H, \rho_{\text{sys}}] + \sum_{\alpha} L_{\alpha} L_{\alpha}^* \rho_{\text{sys}} + \rho_{\text{sys}} L_{\alpha} L_{\alpha}^* - L_{\alpha} \rho_{\text{sys}} L_{\alpha}^*. \quad (2.80)$$

Clearly, this procedure involves some approximations. Else we would have to deal with the same problems when explicitly maintaining the environment system: The first is that our system and the environment are weakly coupled, which is called the *Born-approximation* and allows us to neglect some higher-order terms. The second is that the environment is large compared to the system, so the environment undergoes only negligible changes, and the system does not depend on its interaction in the past. This is called *Markov-approximation*, as it ensures the markovianity of the process.

Note that this open system approach is more than an auxiliary construction but a powerful tool indeed. As we will see next, every bounded generator of a completely positive semigroup is of the form of Eq. (2.80). For finite-dimensional systems, this was shown by Gorini, Kossakowski, and Sudarshan [27] and for the infinite case, but with a bounded generator by Lindblad [28]. Therefore, it is sometimes called Gorini-Kossakowski-Sudarshan-Lindblad or **GKSL-equation**.

Now, we turn our focus to the according mathematics. Let \mathcal{A} be a Banach space and \mathcal{T}_t a family of bounded linear maps on \mathcal{A} . Similar to unitary groups, we call \mathcal{T}_t a **one-parameter semigroup**, if $\mathcal{T}_0 = \mathbb{1}$ and $\mathcal{T}_s \mathcal{T}_t = \mathcal{T}_{s+t}$ for all $t, s \in \mathbb{R}_+$. As we already mentioned, an important property of semigroups is their continuity. For norm continuous semigroups, one can show the following [29, Sect.3]:

Lemma 47. *Let \mathcal{A} be a Banach space and \mathcal{T}_t a norm continuous semigroup on \mathcal{A} , that is $\lim_{t \rightarrow 0} \|\mathcal{T}_t - \mathbb{1}\| = 0$. Then there is a bounded generator \mathcal{L} of \mathcal{T}_t , such that*

$$\mathcal{T}_t = \exp(t\mathcal{L}) \quad \text{and} \quad \left. \frac{d\mathcal{T}_t}{dt} \right|_{t=0} = \mathcal{L}. \quad (2.81)$$

Conversely, every bounded generator generates a norm continuous semigroup.

If \mathcal{T}_t is only strongly continuous, i.e., $\lim_{t \rightarrow 0} \|\mathcal{T}_t A - A\| = 0$, the generator is not bounded but closed and densely defined. In the quantum case, i.e., $\mathcal{A} = \mathcal{B}(\mathcal{H})$, we call a strongly continuous, completely positive one-parameter semigroup a **quantum dynamical semigroup**. A norm-preserving semigroup is called conservative, while a sub-normalized semigroup is called contractive. For the subclass of norm continuous quantum dynamical semigroups, we can state the complete characterization of generators by the Lindblad theorem [28]:

Theorem 48 (Lindblad). *Let $\mathcal{T}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a norm continuous completely positive semigroup. Then \mathcal{T}_t has a bounded generator \mathcal{L} in Lindblad form:*

$$\mathcal{L}(A) = K^* A + A K + \sum_{\alpha} L_{\alpha}^* A L_{\alpha} \quad \text{and} \quad \mathcal{L}(\mathbb{1}) = 0. \quad (2.82)$$

Conversely every operator satisfying Eq. (2.82) generates a norm continuous completely positive semigroup.

Note that in contrast to Eq. (2.80), the above is formulated in the Heisenberg picture, and the terms linear in the argument are summarized in the operators K .

Let us introduce some further nomenclature in the spirit of [30]:

The linear part of Eq. (2.82) generated by K and K^* is called the **no-event generator** and generates reversible dynamics by $T_t^0(\rho) = c_t \rho c_t^*$ with $c_t = \exp(tK)$. It is supplemented by a completely positive perturbation $\mathcal{P}(\rho) = \sum_\alpha L_\alpha \rho L_\alpha^*$ described by the **jump operators** L_α . For the semigroup, we then define the following:

Definition 49. A semigroup is called a **no-event semigroup** if it maps pure states to pure states.

The connection between the no-event definition for the generator and semigroup will be made more clear in the following two statements:

Theorem 50. Let \mathcal{L} be a Lindblad generator based on the contraction semigroup $c_t = \exp(tK)$, with jump operators L_α . Let $\phi \in \mathcal{H}$. Then the following are equivalent:

- (1) $e^{t\mathcal{L}}(|\phi\rangle\langle\phi|)$ is pure for all t .
- (2) For all $t \geq 0$ and $\alpha \in \mathbb{C}$, $c_t \phi$ is an eigenvector of L_α .

Proof. The minimal solution² is given by iteration via the series [31, Thm. 4.1]:

$$\begin{aligned} T_t \rho &= T_t^0 \rho + \int_0^t T_{t-s} \mathcal{P} T_s^0 \rho \, ds \\ &= T_t^0 \rho + \int_0^t (T_{t-s}^0 \mathcal{P} T_s^0 \rho + \int_0^{t-s} T_{t-s-r} \mathcal{P} T_r^0 \mathcal{P} T_s^0 \rho \, dr) \, ds, \end{aligned} \quad (2.83)$$

which alternatively can be expressed by the resolvent equation [30]. With this, we can write down the expression in (1) as

$$\begin{aligned} e^{t\mathcal{L}}(|\phi\rangle\langle\phi|) &= |c_t \phi\rangle\langle c_t \phi| + \sum_\alpha \int_0^t ds \, c_{t-s} L_\alpha c_s |\phi\rangle\langle\phi| c_s^* L_\alpha^* c_{t-s}^* \\ &\quad + \sum_\alpha \int_0^t ds \int_0^{t-s} dr \, e^{(t-s-r)\mathcal{L}} L_\alpha c_r L_\alpha c_s |\phi\rangle\langle\phi| c_s^* L_\alpha^* c_r^* L_\alpha^* \end{aligned} \quad (2.84)$$

and corresponding terms for each iteration. Now take $|\psi\rangle\langle\psi| \perp e^{t\mathcal{L}}(|\phi\rangle\langle\phi|)$. Then we get for the expectation value of this equation with respect to ψ

$$0 = \|\langle\psi, c_t \phi\rangle\|^2 + \sum_\alpha \int_0^t ds \|\langle\psi, c_{t-s} L_\alpha c_s \phi\rangle\|^2 + \dots \quad (2.85)$$

Since all summands in this equation are positive, they have to be zero. Therefore the $c_t \phi \perp \psi$. Since the integral needs to be zero and the integrand is positive, the integrand is zero almost everywhere. We also know that the integrand is continuous.

²Here minimal means the smallest possible solution in the completely positive order, i.e., for all other completely positive maps $\tilde{\mathcal{T}}_t$, we still have $\tilde{\mathcal{T}}_t - \mathcal{T}_t \geq 0$.

If it were non-zero for any $0 \leq r \leq t$, there would be an interval of measure $\mu > 0$ where the integrand is also non-zero, and the integral would have a positive value. Hence, we get $c_{t-s}L_\alpha c_s \phi \perp \psi$ for all $\psi \perp c_t \phi$, i.e. $c_{t-s}L_\alpha c_s \phi \in \text{span}(c_t \phi)$, therefore we can write

$$c_{t-s}L_\alpha c_s \phi = c_t \lambda_\alpha(s, t) \phi, \quad \text{for all } \alpha, 0 \leq s \leq t \quad (2.86)$$

and some $\lambda_\alpha(t, s) \in \mathbb{C}$. Since this is also true for $t = s$, we get the eigenvalue equation

$$L_\alpha c_s \phi = c_s \lambda_\alpha(s) \phi, \quad \text{for all } \alpha, 0 \leq s. \quad (2.87)$$

However, if the above eigenvalue equation is satisfied, the terms of all higher iterations which involve expressions $|\psi\rangle\langle\psi|$ with $\psi = c_{s_0}L_{\alpha_1}c_{s_1} \cdots L_{\alpha_n}c_{s_n} \phi$ will be proportional to $|c_t \phi\rangle\langle c_t \phi|$ with $t = \sum_{i=0}^n s_i$. Hence, the entire iteration series is pure, which proves the stated equivalence. \square

Based on this observation, we can state a more rigorous version of the characterization of the no-event generators:

Corollary 51. *A semigroup with a Lindblad generator takes all pure states to pure states if and only if the L_α are multiples of the identity, i.e., one can choose $L_\alpha = 0$ for all α .*

Proof. If the Lindblad semigroup takes all pure states to pure states, then by the above lemma, every $\phi = c_t \psi$ is an eigenvector of L_α , so $L_\alpha c_t \phi = \lambda_\alpha(t) c_t \phi$. But also, any other vector has to be an eigenvector of L_α , since $\psi = c_0 \psi$, so $L_\alpha \phi = \lambda_\alpha(0) \phi$ which implies that the eigenvalues are all the same, i.e., $L_\alpha = \lambda_\alpha \mathbb{1}$. As the semigroup is gauge-invariant under adding a scalar-multiple of the identity [30, 2.4], we can choose $L_\alpha = 0$. \square

Note that there is an important difference between the above lemma and its corollary: the lemma only handles one fixed but arbitrary state. In contrast, the corollary makes a statement if the semigroup maps all pure states to pure states. It is easy to construct examples where the semigroup takes some pure states to pure states but not all, hence not being a no-event semigroup:

Example 52 (coherent states). Let a and a^* be the annihilation and creation operator of the quantum harmonic oscillator and $n = a^*a$ the number operator with the eigenstates $|n\rangle$. Assume the semigroup with no-event generator $K = -i\omega a^*a - \frac{1}{2}\eta a^*a = \gamma a^*a$ and jump operator $L = \sqrt{\eta}a$. The no-event semigroup $c_t = \exp tK$ maps coherent states $e^\phi = \sum_n \frac{\phi^n}{\sqrt{n!}}|n\rangle$ to coherent states

$$\exp tK e^\phi = \sum_n \frac{e^{t\gamma n} \phi^n}{\sqrt{n!}}|n\rangle = \sum_n \frac{(e^{t\gamma} \phi)^n}{\sqrt{n!}}|n\rangle = e^{\tilde{\phi}(t)}. \quad (2.88)$$

Because coherent states are all eigenvectors of the annihilation operator a , we have that $c_t e^\phi$ is an eigenvector of L . Then Lem. 50 tells us that the semigroup generated by $\mathcal{L}\rho = K\rho + \rho K^* + L\rho L^*$ maps the pure coherent state e^ϕ to a multiple of the pure state $e^{\tilde{\phi}(t)}$. This can be verified by looking at the terms, which occur in Eq. (2.83)

$$T_t \rho \propto T_t^0 \rho + T_{t-s}^0 \mathcal{P} T_s \rho + \cdots \quad (2.89)$$

From Eq. (2.88), we already know that $T_t^0(|e^\phi\rangle\langle e^\phi|) = |e^{t\gamma}\phi\rangle\langle e^{t\gamma}\phi|$ and all other terms are proportional to this, for example

$$\begin{aligned} T_{t-s}^0 \mathcal{P}T_s(|e^\phi\rangle\langle e^\phi|) &= T_{t-s}^0 \mathcal{P}(|e^{s\gamma}\phi\rangle\langle e^{s\gamma}\phi|) = (e^{s\gamma}\phi)T_{t-s}^0(|e^{s\gamma}\phi\rangle\langle e^{s\gamma}\phi|) \\ &= (e^{s\gamma}\phi) |e^{e^{(t-s)\gamma}e^{s\gamma}\phi}\rangle\langle e^{e^{(t-s)\gamma}e^{s\gamma}\phi}| \\ &= (e^{s\gamma}\phi) |e^{t\gamma}\phi\rangle\langle e^{t\gamma}\phi|. \end{aligned} \quad (2.90)$$

Of course this semigroup is not a no-event semigroup, as states such as $|n\rangle\langle n|$ are getting mixed by T_t .

In physics, open quantum systems are often employed in quantum optics, where the operators K and L usually consist of canonical operators like position and momentum or annihilation and creation operators. As those are unbounded, so is the generator \mathcal{L} . Hence, the according semigroups cannot be norm but at most strongly continuous. Here, the Lindblad theorem is no longer valid in its full generality. However, the only counterexamples to the converse part, i.e., completely positive strongly continuous semigroups, which do not have a generator of Lindblad form, are more mathematically motivated than physically. The first counterexample was already published in 1996 [32] and was corrected and extended in the last years [33, 30]. Nevertheless, in physics, this fact is often ignored, which may be best summarized by the following quote from [34, Sect. 3.2, p.48]:

Conversely every semigroup a physicist was ever interested in, seems to have a Lindblad-type generator.

Before we finally turn our attention to hybrid systems, let us have a look at another expression of complete positivity, which will do most of the work in Sect. 4. It is known as **conditionally complete positivity**:

Definition 53 (conditionally complete positivity). *Let \mathcal{A} be a C^* -subalgebra of a C^* -algebra \mathcal{B} and $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{B}$ a self-adjoint bounded linear map. We call \mathcal{L} conditionally completely positive if for all $n \in \mathbb{N}$ and for all $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ with $\sum_{i=1}^n a_i b_i = 0$ we have:*

$$\sum_{i,j=1}^n b_i^* \mathcal{L}(a_i^* a_j) b_j \geq 0. \quad (2.91)$$

There are several equivalent definitions of conditionally complete positivity that can be found in [35, Lem. 14.5], and we choose the one above as most suitable for our application. With this, we can describe the fact that if a completely positive semigroup \mathcal{T} reaches the *minimum*, i.e., the border to negative elements, it can not decrease further in the next time-step, which is described by the following theorem [35, Thm. 14.7]:

Theorem 54. *Let \mathcal{L} be a self-adjoint bounded linear map on a C^* -algebra \mathcal{A} . Then \mathcal{L} is conditionally completely positive if and only if $e^{t\mathcal{L}}$ is completely positive for all $t \geq 0$.*

2.5 References and Literature

This section is devoted to the list of references we used in this chapter. Additionally, we add some personal comments as a guide for interested readers who want more information on one of the many areas we touched in this chapter. We do not claim the list to be complete but only reflect the author's subjective perception. Additionally, we give some freely available alternatives, that is, works published on the arXiv or the respective author's website.

2.2 Operator algebras Let us start with a short overview of some of the standard references in this field: For pure mathematics, common references are the series *Theory of Operator Algebras* by Takesaki [8, 36, 37], *Fundamentals of the Theory of Operator Algebras* by Kadison and Ringrose [38, 39], the book *Operator algebras: theory of C^* -algebras and von Neumann algebras* by Blackadder [40], the book *C^* -Algebras and W^* -algebras* by Sakai [15] and the works *C^* -algebras* [14] and *Von Neumann algebras* [41] by Dixmier. A common reference in mathematical physics are the two volumes of *Operator algebras and Quantum Statical Mechanics* by Bratteli and Robinson [5, 42]. We finish the list with two works by Pedersen *C^* -algebras and their Automorphism Groups* [12], and *Analysis NOW* [10], which, according to the author, can be read as *Analysis based on Norms, Operators and Weak topologies*. The use of the latter two as primary references for this work is based on Pedersen's influence on the field of multipliers and universally measurable elements, which will play a key role in our hybrid framework in Chap. 5.

Besides these books, which need to be bought or borrowed, we want to add some freely available alternatives: The *Lecture Notes on C^* -algebras, Hilbert C^* -modules, and Quantum Mechanics* by Landsman [43] and a preprint of the book *Quantum Spin Systems on Infinite Lattices* by Najjkens [4] are available on the arXiv. Both have a different main topic but include a good introduction to the general theory. Finally, the main matter as well as the extensive appendix of *Foundations of quantum theory: From Classical Concepts to Operator Algebras* by Landsman [9] are extremely useful, especially given the subtitle and the topic of this work. This work is not on the arXiv but is marked as open access and is available on the publisher's website.

A more thorough introduction to the wide field of topology can be found in books explicitly about that topic, for example, *Topology* by Waldmann [44] or *Topological vector spaces* by Schaefer and Wolf [13]. Because topology plays an essential role in operator theory, every of the above-defined standard references contains a section about the operator topologies and also books about functional analysis, like *Functional Analysis* by Rudin [6] or *A course in functional analysis* by Conway [7], from which the reader can choose his favorite. One has to keep in mind that the notation and nomenclature can vary, especially in this field.

2.3 Quantum states and measurements For the standard references, we have to mention the books *States, Effects, and Operations: Fundamental Notions of Quantum Theory* by Kraus [45], and *Quantum theory of open systems* by Davies [19], which are at the same time witnesses to the history of the field. If one wants to go even further back in time, one of the very first books in the field *Mathematische*

Grundlagen der Quantenmechanik by von Neumann himself [46] may not be a good recommendation for a textbook nowadays, but is interesting to read from a historical standpoint. A modern summary of the quantum mechanics framework can be found in the book *The mathematical language of Quantum Theory* by Heinosaari and Ziman [17], which is our primary reference for this matter. Other alternatives are *Probabilistic and statistical aspects of quantum theory* by Holevo [18], which at some points uses different nomenclature and the work *The quantum theory of measurement* by Busch, Lahti, Mittelstaedt, and Ylinen [47], which is very detailed and rigorously formulated.

Again, we want to highlight some freely available alternatives: A worked-out version of the lecture notes, on which our primary reference [17] is based, is available on the arXiv under the title *Guide to Mathematical Concepts of Quantum Theory* [48]. Another title with a good balance between mathematical rigor, readability, and physical interpretation is a series of notes under the title *Lectures in Quantum Noise Theory* by Attal [49].

2.4 Quantum dynamics Consistently, we use *The mathematical language of Quantum Theory* by Heinosaari and Ziman [17] as the main reference for this part. The mathematical details about the concept of completely positive maps are often part of standard works like *Theory of Operator Algebras* by Takesaki [8]. Other references more specific to this topic are the books *Completely bounded maps and operators algebras* by Paulsen [50], and *Positive linear maps of operator algebras* by Stømer [20]. The concept of conditional complete positivity can be found in *Dilations of Irreversible Evolutions in Algebraic Quantum Theory* by Evans and Lewis [35], which also offers many details about completely positive maps and semigroups.

The concept of quantum channels and the fundamental concepts of quantum theory, in general, are closely connected to the field of quantum information theory. The most common reference and one of the first textbooks about this topic is *Quantum Computation and Quantum Information* by Nielsen and Chuang [2]. Note that while the field evolved, the terminology changed slightly so this reference might require some adaptation. More modern alternatives are *Lectures on Quantum Information* edited by Bruß and Leuchs [22] or *Quantum Information Theory* by Wilde [23]. A detailed introduction to the mathematics behind one-parameter semigroups is *One-parameter semigroups for linear evolution systems* by Engel and Nagel [29] and especially for the quantum application *Quantum dynamical semigroups and applications* by Alicki, and Lendi [51]. More information about the application of open system techniques in quantum optics can be found in various books, for example, *An open systems approach to quantum optics* by Carmichael [26] or *Quantum Measurement and Control* by Wiseman and Milburn [52].

Chapter 3

Hybrid Algebras

In this chapter, we will discuss what possibilities we have for the classical systems that should go alongside our quantum system to form a hybrid. More precisely, we have to discuss the possible classical frameworks that describe the three building blocks of an experiment, i.e., preparations, dynamics, and measurement under the constraint, that they *harmonize* with our quantum system.

The first class of quantum-classical hybrids that we will study in this work has the structure of a tensor product. Before we go into more details, let us start with some preliminary remarks: For reasons we will discuss later, this ad hoc approach of a tensor product structure will pass over several details important to the phase space formulation for quasifree and Gaussian systems. The ansatz we will use will be more sophisticated and tailored towards this specific scenario. Nevertheless, they are good for building intuition and motivating the upcoming formulation for the phase space setting.

3.1 Hybrids with Discrete Classical Systems

We begin the discussion of quantum-classical hybrid systems with the simplest case: finite and discrete systems. Note that these systems are much more than a toy model. Finite-dimensional quantum systems, especially the two-dimensional case, also known as the qubit and its classical analog, the bit, constitute the fundamental quantities in information, respectively quantum information theory. Accordingly, it is also in this field where quantum-classical systems occur naturally and are routinely treated as what they are, namely *hybrids*.

Let us start with the basic framework. The restriction to finite dimensions on the quantum side simplifies matters quite a lot. Our Hilbert space becomes $\mathcal{H} = \mathbb{C}^n$, and for the observable algebra, we have $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$, which in large parts reduces the required mathematics from functional analysis to linear algebra. For example, all of our states are normal states, i.e., the description by density matrices ρ , with $\rho \geq 0$ and $\text{tr}[\rho] = 1$ is sufficient.

For the classical side, our first *classical phase space*, or better *outcome set*, is

$$X = \{x_1, x_2, \dots, x_s\}, \quad (3.1)$$

where x_i labels the possible configurations of our classical systems. A probability distribution on the set X is a function $p : X \rightarrow [0, 1]$ that is normalized

$$\sum_x p(x) = 1. \quad (3.2)$$

We call such a function **classical state**, and the collection of all such discrete probability distributions the classical state space.

With this, we can define our first prototype of a quantum-classical hybrid state. For example, in the teleportation protocol, the classical system carries the possible measurement outcomes of a quantum system, here a qubit, which means

$$X = \{\uparrow, \downarrow\} \cong \{0, 1\} \cong \{x_1, x_2\} \quad (3.3)$$

and a classical state is the probability distribution according to the quantum measurement.

Let us translate this idea into some more general formulas. Let $\rho_x \in \mathcal{T}(\mathcal{H})$ be a family of density operators and $p(x)$ a probability distribution on X . We define a **hybrid state** $\hat{\rho}$ as

$$\hat{\rho} = \bigoplus_x p(x) \rho_x = \begin{pmatrix} p(x_1) \rho_1 & 0 & 0 \\ 0 & p(x_2) \rho_2 & 0 \\ 0 & 0 & \dots \end{pmatrix}. \quad (3.4)$$

Here, each individual block $p(x)\rho_x$ is sub-normalized, but overall, it is a density operator in a block-diagonal form on the larger Hilbert space $\hat{\mathcal{H}} = \bigoplus_x \mathcal{H}$, as normalization and positivity are carried through.

The according hybrid observable algebra can be readily described by the block-diagonal, hermitian matrices $A \in \mathcal{A} = \bigoplus_x \mathcal{B}(\mathcal{H})$. Indeed, it is common for this type

of hybrid to see it embedded into a larger fully quantum setting $\mathcal{B}(\widehat{\mathcal{H}})$. From this, we can characterize \mathcal{A} by using so called Lüders or central projections:

$$P_x = |x\rangle\langle x| \otimes \mathbb{1}. \quad (3.5)$$

Here, $|x\rangle\langle x|$ is the identity on the block belonging to the label x and vanishing on all others, i.e., P_x projects to the subsystems belonging to the classical value x . With those, we can define **hybrid observables** \mathcal{A} , starting from the larger quantum systems $\mathcal{B}(\widehat{\mathcal{H}})$, as elements of the commutant

$$\mathcal{A} = \{P_x \mid x \in X\}' = \bigoplus_x \mathcal{B}(\mathcal{H}). \quad (3.6)$$

Also, we can reduce every observable $A \in \mathcal{B}(\widehat{\mathcal{H}})$ to an observable on the hybrid subsystem by $\mathcal{P}A = \sum_x P_x A P_x = \widehat{A}$. In both cases, we get the hybrid observables as block-diagonal hermitian matrices

$$\widehat{A} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & \dots \end{pmatrix}, \quad (3.7)$$

where every block $A_x \in \mathcal{B}(\mathcal{H})$ is a quantum observable. Here, we could add an additional factor $\lambda_x \in \mathbb{R}$ in each block from the classical observable algebra, which are the $s \times s$ -diagonal matrices. Indeed, every finite-dimensional unital commutative C^* -algebra is unitarily equivalent to this [9, Thm. A.21], and it is a common reading to translate *classical* with *diagonal* for these finite and discrete cases. The average or **expectation value** is readily defined by the trace on the larger system $\mathcal{B}(\widehat{\mathcal{H}})$:

$$\langle A \rangle_\rho = \text{tr} [\widehat{\rho} \widehat{A}] = \sum_x p(x) \text{tr} [\rho_x A_x]. \quad (3.8)$$

Note that finite hybrids are indeed well known. For example, the above characterization can be found in [53, Sect. 3.5] and [54, Sect. 6.2.1]. For a discussion of the general idea that classical systems originate from quantum systems as a limiting case with vanishing interference, i.e., off-diagonal terms, see, for example, [22, Sect. 5.5]

Summing up, states and observables for hybrids in the discrete case are conceptually direct accessible, and the structure, given by the direct sum, typically behaves quite well. The hybrid states, observables, and their combination to expectation values described in Eq. (3.4), Eq. (3.7), and Eq. (3.8) give a good impression on how a hybrid framework works. Because continuous-variable quantum systems naturally need an equally continuous classical part, our next step is the transition from a finite to a continuous classical phase space X .

Remarks on discrete dynamics

Before we start with this generalization, let us finish this section with an observation of the finite case dynamics. For this, let \mathcal{T} be a completely positive map on the larger observable algebra $\mathcal{B}(\widehat{\mathcal{H}})$, that leaves the hybrid algebra invariant, i.e.,

$$\mathcal{T} : \mathcal{B}(\widehat{\mathcal{H}}_{out}) \rightarrow \mathcal{B}(\widehat{\mathcal{H}}_{in}), \quad \text{with } \mathcal{T}(\mathcal{A}) \subset \mathcal{B}. \quad (3.9)$$

Here we denote the Lüders projectors on the respective spaces by $\{P_x\}_{x \in X}$ in $\mathcal{B}(\widehat{\mathcal{H}}_{out})$, respectively $\{Q_y\}_{y \in Y}$ for $\mathcal{B}(\widehat{\mathcal{H}}_{in})$ and the hybrid algebras by $\mathcal{A} = \{P_x\}'_{x \in X}$, respectively $\mathcal{B} = \{Q_y\}'_{y \in Y}$. Of course, not every completely positive map will obey the block-diagonal structure of the hybrid algebra, so the natural question arises: Which maps do and how to characterize them? For this, we introduce the notion of adapted maps:

Definition 55 (adapted Kraus representation). *Let $\mathcal{T} : \mathcal{B}(\widehat{\mathcal{H}}_{out}) \rightarrow \mathcal{B}(\widehat{\mathcal{H}}_{in})$ be a completely positive map. We call the Kraus decomposition $\mathcal{T}(A) = \sum_{\alpha} K_{\alpha}^* A K_{\alpha}$ x -adapted, if*

$$\forall \alpha \quad P_x K_{\alpha} Q_y \neq 0 \quad \text{for at most one } x. \quad (3.10)$$

We call it **adapted**, if

$$\forall \alpha \quad P_x K_{\alpha} Q_y \neq 0 \quad \text{for at most one pair } (x, y). \quad (3.11)$$

This definition establishes a connection between the Kraus operators of \mathcal{T} and the transition of information in our hybrid structure. Using the quantum Radon-Nikodym theorem (Thm. 43), this notion allows us to characterize the completely positive maps, which leaves the hybrid structure invariant:

Theorem 56. *Let $\mathcal{T} : \mathcal{B}(\widehat{\mathcal{H}}_{out}) \rightarrow \mathcal{B}(\widehat{\mathcal{H}}_{in})$ be a completely positive map. The map \mathcal{T} leaves the hybrid algebra invariant $\mathcal{T}(\mathcal{A}) \subset \mathcal{B}$ if and only if for every $x \in X$ there is an x -adapted Kraus representation.*

Proof. Let K_{α} be the Kraus operators for an x -adapted Kraus representation and choose an observable A_x on the x 'th-block, i.e., $[A_x, P_x] = 0$. Then

$$\begin{aligned} \mathcal{T}(A_x) &= \sum_{\alpha} K_{\alpha}^* A_x K_{\alpha} \\ &= \sum_{\alpha, y, y'} Q_{y'} K_{\alpha}^* P_x A_x P_x K_{\alpha} Q_y \\ &= \sum_{\alpha} Q_{y(\alpha)} K_{\alpha}^* P_x A_x P_x K_{\alpha} Q_{y(\alpha)} \\ &\in \bigoplus_y \mathcal{B}(Q_y \widehat{\mathcal{H}}_{in}). \end{aligned} \quad (3.12)$$

Here $y(\alpha)$ is the one $y \in Y$ for which $P_x K_{\alpha} Q_y \neq 0$. If no such $y(\alpha)$ exists, we already have $\mathcal{T}(A_x) = 0 \in \bigoplus_y \mathcal{B}(Q_y \widehat{\mathcal{H}}_{in})$.

For the converse choose an arbitrary but fixed $x \in X$ and $[\mathcal{T}(A_x), Q_y] = 0 \forall y \in Y$. Because \mathcal{T} is completely positive, there exists a minimal Stinespring dilation

$$\mathcal{T}(A) = V^*(A \otimes \mathbb{1}_{\mathcal{K}})V, \quad A \in \mathcal{B}(\widehat{\mathcal{H}}_{out}). \quad (3.13)$$

For the restriction of \mathcal{T} onto block-diagonal elements $A_x = P_x A P_x$, the map $\mathcal{T}(A_x) = V^*(A_x \otimes \mathbb{1})V$ is not necessarily a minimal Stinespring dilation, but we can find a subspace $\mathcal{K}_x \subset \mathcal{K}$ such that $\mathcal{K}_x = S_x \mathcal{K}$, and the dilation space is then given by $\text{lin}\{(A_x \otimes \mathbb{1})V\phi\} = (P_x \otimes S_x)(\widehat{\mathcal{H}} \otimes \mathcal{K})$. Since $[\mathcal{T}(A_x), Q_y] = 0 \forall y \in Y$, we have $\mathcal{T}(A_x)Q_y = Q_y \mathcal{T}(A_x)Q_y$ which is again completely positive and

$$\sum_y Q_y \mathcal{T}(A_x) Q_y = \mathcal{T}(A_x). \quad (3.14)$$

Hence we can use the quantum Radon-Nykodym theorem (Thm. 43), i.e., there exists an $F_y \geq 0$ with $\sum_y F_y = \mathbb{1}$ in $\mathcal{B}(\mathcal{K}_x)$, such that

$$\mathcal{T}(A_x)Q_y = V^*(A_x \otimes F_y)V. \quad (3.15)$$

Note that $\mathbb{1} \otimes F_y V = VQ_y$ because $A_x^* \otimes \mathbb{1}V\psi$ generates $\mathcal{H}_x \otimes \mathcal{K}_x$ and we have

$$\langle A_x^* \otimes \mathbb{1}V\psi, \mathbb{1} \otimes F_y V\phi - VQ_y\phi \rangle = \langle \psi, \mathcal{T}(A_x)Q_y - \mathcal{T}(A_x)Q_y\phi \rangle = 0. \quad (3.16)$$

With the above, we get

$$\begin{aligned} \mathbb{1} \otimes F_y V &= VQ_y^2 \\ &= \mathbb{1} \otimes F_y VQ_y \\ &= \mathbb{1} \otimes F_y^2 V. \end{aligned} \quad (3.17)$$

Since the Radon-Nykodym Theorem gives a unique decomposition, we have $F_y^2 = F_y$. We can therefore dissect \mathcal{K}_x into orthogonal subspaces and get

$$\mathcal{K} = \left(\bigoplus_y F_y \mathcal{K} \right) \otimes (1 - S_x)\mathcal{K}. \quad (3.18)$$

Now we choose a basis $\{e_\alpha\} \subset \mathcal{K}$ adapted to this dissection, i.e., we use $e_\alpha \in F_y \mathcal{K}$ such that $K_\alpha^y = (\mathbb{1} \otimes \langle e_\alpha |) V$ and get

$$\begin{aligned} P_x K_\alpha Q_{y'} &= (P_x \otimes \langle e_\alpha |) VQ_{y'} \\ &= P_x \otimes (\langle e_\alpha | F_{y'}) V \\ &= 0 \quad \text{if } y \neq y', \end{aligned} \quad (3.19)$$

and for $e_\alpha \in (1 - S_x)\mathcal{K}$ this is automatically zero. \square

Corollary 57. *Let \mathcal{T} be as above. Given a restriction $\mathcal{T}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{B}$, there is exactly one $\widehat{\mathcal{T}}$ ($= \mathcal{P}\mathcal{T}\mathcal{P}$), which has an adapted Kraus representation.*

Proof. The existence of an adapted Kraus representation means that

$$\forall \alpha \exists! (x, y) \quad \text{such that} \quad P_x K_\alpha Q_y \neq 0. \quad (3.20)$$

So we get that

$$\mathcal{T}_{xy} : \mathcal{B}(\mathcal{H}_x) \rightarrow \mathcal{B}(\mathcal{H}_y), \quad \mathcal{T}_{xy}(A_x) = \sum_\alpha (Q_y K_\alpha^* P_x) A_x (P_x K_\alpha Q_y) \quad (3.21)$$

is completely positive and uniquely defined by the restriction. Like this, \mathcal{T} can be written as

$$\sum_{x,y} \mathcal{T}_{xy} = \mathcal{P}_{in} \mathcal{T} \mathcal{P}_{out}, \quad (3.22)$$

where \mathcal{P}_{in} is defined by the projections $\{Q_y\}$ on the hybrid algebra \mathcal{B} and \mathcal{P}_{out} by $\{P_x\}$ on \mathcal{A} . \square

3.2 Review: Classical Systems

In the finite case, our classical system is more a pointer system rather than a system with its own structure, so let us take one step back and specify what precisely *classical* in our context means, as this choice will naturally determine our subsequent work. The common definition of classical as not quantum is too coarse since this would imply a theory of everything. The same problem arises with the identification of quantum by non-commutative, which determines classical as commutative.

To narrow down our classical part of the hybrid, we have to decide what our hybrid should describe. One possible way of constructing a hybrid is utilizing *hybrid mechanics*. Here, the typical approach is to unify the commutator on the quantum side with the Poisson bracket from classical mechanics. There are previous works on this attempt, which we will further discuss in Sect. 3.4. After this, the reader hopefully agrees that this approach has some issues and that the next scenario is more promising.

Our approach, which coincides with the previous construction for finite systems, is to see hybrids from an information-theoretic point of view. Again, the typical operation is the quantum instrument, where a pure quantum system gets (partially) measured and becomes an effective hybrid. This scenario is based on the exchange of information rather than quantities like energy, and we need to implement classical probability theory on our classical part of the hybrid.

Accordingly, our first goal is the definition of classical correspondents to the probabilistic quantities used in quantum theory, like the expectation value of a state ρ and observables A :

$$\langle A \rangle = \text{tr} [\rho A]. \quad (3.23)$$

Now, classical observables are random variables f , and the states are probability measures μ , so the duality that yields the expectation values is typically given by an integral of the form

$$\langle f \rangle = \int_X f d\mu. \quad (3.24)$$

Following this idea, we have two possible candidates for our algebraic formulation of classical probability theory, which we can classify according to the discussion in Sect. 2.3.1 for the quantum case:

C*-view This view is characterized by the fact that the observables form a C*-algebra \mathcal{A} , and our state space is part of the dual \mathcal{A}^* .

The specific choice is to take the observables as a sub-algebra of the continuous functions over the set X , i.e., $\mathcal{C}(X)$, and the states become probability measures by the Riesz representation theorem.

W*-view Here our observable algebra is a W*-algebra \mathcal{M} , which accordingly has predual \mathcal{M}_* , that houses the normal states.

The realization is the algebra of essentially bounded functions on X , denoted as $L^\infty(X, \mu)$, with the integrable functions $L^1(X, \mu)$ as its predual, where probability densities characterize the states.

Both settings are mathematically well studied, and our above choices for the classical part are indeed *the* commutative algebras for the respective setting. Also, they un-

derline the general philosophy behind C*- and W*-algebras, where the first are seen as *non-commutative topological spaces* and the latter as *non-commutative measure spaces* [40, Chap. III].

In order to discuss which approach is more suitable, we are going to recall some facts about these commutative algebras, as especially quantum physicists are often more familiar with the non-commutative versions. Commutative algebras are important in the general theory of operator algebras, so they are, to some extent, part of all standard works we described in Sect. 2.5. Because of the similar application of these algebras, our primary reference for the upcoming two sections will be the appendix of [9], which contains most of the necessary tools.

Lastly, we need to discuss the space X , which is the outcome space or **classical phase space**. This space is now considered to be continuous, and for our application, we basically want $X = \mathbb{R}^s$. As we will see, this *locally compact case* comes with some problems, and an alternative choice is the *compact case*, where we choose a compact subset or a compactification of \mathbb{R}^s . Both are mathematically *harmless* cases compared to arbitrary sets, yet their difference in compactness comes with some significant consequences for our hybrid algebra.

3.2.1 Commutative C*-algebras

We start with the definition and properties of an essential class of abelian C*-algebras, that is, different subsets of the continuous functions.

Let X be a locally compact Hausdorff space and $\mathcal{C}(X)$ the set of complex-valued continuous functions on X . A subspace of $\mathcal{C}(X)$ is the space of **bounded continuous functions**, denoted by

$$\mathcal{C}_b(X) = \left\{ f \in \mathcal{C}(X) \mid \sup_{x \in X} |f(x)| < \infty \right\}. \quad (3.25)$$

Because continuity, as well as boundedness, are preserved in sums and products, we can turn $\mathcal{C}_b(X)$ into an algebra with point-wise addition $(f + g)(x) = f(x) + g(x)$, multiplication $(fg)(x) = f(x)g(x)$ and scalar multiplication $(\lambda f)(x) = \lambda f(x)$. The space $\mathcal{C}_b(X)$ is complete with respect to the **supremum norm**

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \quad (3.26)$$

and as the multiplication is commutative, it is a commutative Banach algebra. We can introduce a *-operation on $\mathcal{C}_b(X)$ by complex conjugation $f(x)^* = \overline{f(x)}$, which turns $\mathcal{C}_b(X)$ into a commutative C*-algebra. Note that the constant 1-function is clearly bounded and is the multiplicative unit of this algebra, i.e., $\mathcal{C}_b(X)$ is unital.

Another subspace is $\mathcal{C}_0(X)$, describing the **continuous functions vanishing at infinity**. Its precise definition is

$$\mathcal{C}_0(X) = \left\{ f \in \mathcal{C}(X) \mid \forall \epsilon \geq 0 \{x \in X \text{ s.t. } |f(x)| \geq \epsilon\} \text{ is compact} \right\} \quad (3.27)$$

and one can show that $\mathcal{C}_0(X)$ is a closed subspace of $\mathcal{C}_b(X)$, so it is likewise a Banach space and a commutative C*-algebra. A significant difference is that the

unit of $\mathcal{C}_b(X)$ is not included in $\mathcal{C}_0(X)$, i.e., $\mathcal{C}_0(X)$ is in general not unital. The exception is when X is compact, but in this case we already have

$$\mathcal{C}(X) = \mathcal{C}_0(X) = \mathcal{C}_b(X). \quad (3.28)$$

The importance of the algebra \mathcal{C}_0 is easily seen by the next fact [9, Thm. C.8f]:

Theorem 58. *Every commutative C^* -algebra \mathcal{A} is isomorphic to $\mathcal{C}_0(X)$ for some locally compact Hausdorff space X , which is unique up to homeomorphism.*

For a deeper look at the space X and the isomorphism in the above theorem, we need some more definitions. For an arbitrary abelian C^* -algebra \mathcal{A} , we define its **Gelfand spectrum** $\Sigma(\mathcal{A})$ as the set of non-zero linear maps

$$\omega : \mathcal{A} \rightarrow \mathbb{C}, \quad \omega(AB) = \omega(A)\omega(B). \quad (3.29)$$

The elements of $\Sigma(\mathcal{A})$, i.e., non-zero algebra homomorphisms, are called **characters** and are part of \mathcal{A}^* . On the Gelfand spectrum we can define the **Gelfand transform** \widehat{A} , which maps every $A \in \mathcal{A}$ to a function

$$\widehat{A} : \Sigma(\mathcal{A}) \rightarrow \mathbb{C}, \quad \widehat{A}(\omega) = \omega(A). \quad (3.30)$$

This describes the stated isomorphism $\mathcal{A} \rightarrow \mathcal{C}_0(\Sigma(\mathcal{A}))$ of the above theorem. The **Gelfand topology** is defined as the weakest topology on $\Sigma(\mathcal{A})$, such that all \widehat{A} are continuous and coincides with the weak* topology. In this topology, the space $\Sigma(\mathcal{A})$ is locally compact and compact if \mathcal{A} is unital. In this case, the Gelfand spectrum coincides with the pure states of \mathcal{A} [9, C.14], and the evaluation map establishes a homeomorphism such that [9, Prop. C.19]

$$X \cong \Sigma(\mathcal{C}(X)), \quad (3.31)$$

which also justifies the commonly used identification of points $x \in X$ with the pure states.

Our next step is to bring in the probability measures, respectively, the dual space of $\mathcal{C}_0(X)$. A straightforward way to construct functionals on $\mathcal{C}_0(X)$ is by integration:

$$\omega(f) = \langle \omega, f \rangle = \int_X f(x) d\mu \quad f \in \mathcal{C}_0(X). \quad (3.32)$$

Indeed, this description of functionals is sufficient, which is known as the **Riesz representation theorem**.

For its statement, we need some extra definitions from measure theory, see [9, Sect. B.4f], in addition to the ones from Sect. 2.3.2:

Let (X, \mathcal{F}, μ) be a measure space. Then μ is called **complete** if $\mu(x_2) = 0$ and $x_1 \subset x_2$ for $x_1 \in \mathcal{P}(X)$ implies $x_1 \in \mathcal{F}$. Further, let $\mathcal{O}(X)$ denote the set of open subsets of X and $\mathcal{K}(X)$ the set of compact subsets of X . Now, the **outer measure** μ^* is defined as $\mu^*(x) = \inf\{\mu(u) \mid u \supseteq x, u \in \mathcal{O}(X)\}$ and the **inner measure** μ_* as $\mu_*(x) = \sup\{\mu(k) \mid k \subseteq x, k \in \mathcal{K}(X)\}$. A measure is called **regular** if

$$\mu(x) = \mu_*(x) = \mu^*(x), \quad \forall x \in \mathcal{F}. \quad (3.33)$$

Note that the Lebesgue measure on \mathbb{R}^s [55, Prop.1.4.1], every finite Borel measure on \mathbb{R}^s [55, Prop.1.5.6] and generally every finite Borel measure on a second countable locally compact Hausdorff space are regular [55, 7.2.3], i.e., for our use cases this property is not really an extra requirement.

Equipped with these definitions, we can finally state the following [9, Cor. B.21]:

Theorem 59. *Let X be locally compact. Then, there is a bijective correspondence between the positive linear functionals on $\mathcal{C}_0(X)$ and the complete regular finite measures on X . In particular, the states on $\mathcal{C}_0(X)$ correspond to regular probability measures on X .*

This result is indeed very practical, and our first option for a classical state space is exactly the set of probability measures over X , i.e., $\mathcal{W}(X)$. With the state space, we have a description of a subset of the dual, yet we still owe the reader a complete characterization of $\mathcal{C}_0(X)^*$.

Therefore, we have to drop the positivity and normalization condition, which leads to complex measures. As the name suggests, those are σ -additive maps from the σ -algebra \mathcal{F} to the complex numbers, see also [55, Sect. 4.1]. We denote the set of all complex measures over X as $\mathcal{M}(X)$.

Next we introduce the **variation** of a set $F \in \mathcal{F}$ and a measure $\mu \in \mathcal{M}(X)$ as

$$|\mu|(F) = \sup \left\{ \sum_n |\mu(F_n)| \right\}, \quad (3.34)$$

where the supremum is taken over all measurable partitions $F = \bigcup_n F_n$. The variation is a finite measure on (X, \mathcal{F}) and also the smallest positive measure, such that $|\mu|(F) \leq \nu(F)$ for all $F \in \mathcal{F}$ and positive measures ν [55, Prop.4.1.7f]. For $F = X$ we call $|\mu|(X) = \|\mu\|$ the **total variation** of the measure μ , which defines a norm on the space $\mathcal{M}(X)$. One can show, that $\mathcal{M}(X)$ is complete with respect to the total variation [55, Prop. 4.1.8] and describes the full dual space of $\mathcal{C}_0(X)$ [9, Thm. B.24]:

Theorem 60. *Let X be a locally compact Hausdorff space. Then, the dual space $\mathcal{C}_0(X)^*$ is isometrically isomorphic to the space $\mathcal{M}(X)$ of all complete regular complex measures μ on X , with the norm given by total variation.*

So by the above theorem, Eq. (3.32) defines a duality between the spaces $\mathcal{C}_0(X)$ and its dual space $\mathcal{C}_0(X)^* = \mathcal{M}(X)$.

3.2.2 Commutative W^* -algebras

The analogous concept to $\mathcal{C}(X)$ in the case of W^* - or von Neumann algebras are the $L^p(X, \mathcal{F}, \mu)$ function spaces. Let (X, \mathcal{F}, μ) be a σ -finite measure space and $1 < p < \infty$. A measure space is called σ -finite if for any countable union $X = \bigcup_n X_n$ we have $\mu(X_n) < \infty^1$. To define $L^p(X, \mathcal{F}, \mu)$ we first need the spaces

$$\mathcal{L}^p(X, \mathcal{F}, \mu) = \left\{ f \text{ measurable} \mid \|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} < \infty \right\}. \quad (3.35)$$

¹This requirement excludes pathological cases that hardly occur in practice [9, p.523]. For example, without this property, we would only get an isometry for $q = 1$ and $p = \infty$ in Thm. 61 [55, Prop. 3.5.5].

Note that the above-defined spaces are not normed spaces, and especially for $p = 2$, it is not the often used Hilbert space in quantum mechanics. The reason for $\|\cdot\|_p$ not being a norm on \mathcal{L}^p is the fact that $\|f\|_p = 0$ is not equivalent to $f = 0$. Take $f \in \mathcal{L}^p$ and define \tilde{f} by changing f on a subset of X with measure zero. Then $\|\tilde{f} - f\|_p = 0$, because $f = \tilde{f}$ μ -almost everywhere, but as a function they do not coincide. This is solved by introducing $L^p(X, \mathcal{B}, \mu)$, which is defined as the quotient space

$$L^p(X, \mathcal{B}, \mu) = \mathcal{L}^p(X, \mathcal{B}, \mu) / \sim. \quad (3.36)$$

Here, two functions $f, g \in \mathcal{L}^p(X, \mathcal{B}, \mu)$ are considered equivalent, i.e., $f \sim g$, if f and g only differ on a set of measure zero. On the space $L^p(X, \mathcal{B}, \mu)$, $\|\cdot\|_p$ defines a proper norm at the cost of changing the elements from functions to representatives of equivalence classes. So, in contrast to the previous section, $f \in L^p$ or $f \in \mathcal{C}_0(X)$ describe different objects f . For the special case $p = \infty$, i.e., the spaces \mathcal{L}^∞ and L^∞ , one uses the essential supremum

$$\|f\|_\infty = \inf \{t \in [0, \infty] \mid \mu(\{x \in X, |f(x)| > t\}) = 0\} \quad (3.37)$$

instead of $\|\cdot\|_p$. In applications where the measure is evident from the context, most often the Lebesgue measure dx , one typically abbreviates $L^p(X, \mathcal{B}, \mu)$ by $L^p(X)$, neglecting the fact that the L^p spaces depend on the measure chosen.

The L^p spaces, including $p = \infty$, are complete, i.e., they are Banach spaces and also become vector spaces with the point-wise operations [9, B.29]. Further, we can equip $L^\infty(X, \mathcal{F}, \mu)$ with a *-operation by the complex conjugation, making it a C*-algebra. Now the characterization of the dual, respectively the predual, of an L^p space is done by the following theorem [10, Thm. 5.5f]:

Theorem 61. *Let (X, \mathcal{F}, μ) be a measure space and $1 < p < \infty$. For $1/p + 1/q = 1$ and $g \in L^q(X, \mathcal{F}, \mu)$, the map $\omega_g(f) : L^p(X, \mathcal{F}, \mu) \rightarrow \mathbb{C}$*

$$\omega_g(f) = \langle f, g \rangle = \int_X f g d\mu \quad (3.38)$$

defines an isometric isomorphism of $L^q(X, \mathcal{F}, \mu)$ onto $L^p(X, \mathcal{F}, \mu)^$. If (X, \mathcal{F}, μ) is σ -finite, the same holds true for $q = 1$ and $p = \infty$.*

In conclusion, the C*-algebra $L^\infty(X, \mathcal{F}, \mu)$ has the predual $L^1(X, \mathcal{F}, \mu)$ and is therefor indeed a W*-algebra. Furthermore, like $\mathcal{C}_0(X)$ is the commutative C*-algebra, the same is true for L^∞ and commutative von Neumann algebras [9, C.140]:

Theorem 62. *Let $M \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra, then*

$$\mathcal{M} \cong L^\infty(X, \mu), \quad (3.39)$$

for some locally compact space X and probability measure μ on X .

Let us now focus on the states for this algebra, and some measure theoretic basic results. Up to now, we have only described the normal states of $L^\infty(X, \mu)$, which are elements of the predual $L^1(X, \mu)$, described by the (L^1, L^∞) duality in Eq. (3.38). For the characterization of the full dual space $L^\infty(X, \mu)^*$ and upcoming calculations, we again need some more definitions and theorems.

We call a measure μ **absolutely continuous** with respect to another measure ν , denoted as $\mu \ll \nu$ if every μ -null set is also a ν -null set. This property allows us to express one measure in terms of another, which is known as the Radon-Nikodym Theorem [55, Thm. 4.2.2]:

Theorem 63 (Radon-Nikodym). *Let (X, \mathcal{F}) be a measurable space and let μ and ν be σ -finite positive measures on (X, \mathcal{F}) . If μ is absolutely continuous with respect to ν , then there is an \mathcal{F} -measurable function $g : X \rightarrow [0, \infty)$ such that*

$$\mu(F) = \int_F g \, d\nu \quad (3.40)$$

holds for each $F \in \mathcal{F}$. The function g is unique up to ν -almost everywhere equality.

The function g in the above theorem is also called the **Radon-Nikodym derivative** and is accordingly sometimes denoted as $\frac{d\mu}{d\nu}$. More details about the required properties and possible extensions can be found in [55, Sect. 4.2]. The full state space of L^∞ is now characterized by the following theorem [9, Thm. B.31]:

Theorem 64. *Let (X, \mathcal{F}, μ) be a measure space. Then there is a bijective correspondence between the states of $L^\infty(X, \mu)$ and the finitely additive probability measures ν on (X, \mathcal{F}) that are absolutely continuous with respect to μ .*

Finitely additive measures are measures with the requirement of σ - or countable additivity reduced to only a finite collection of sets.

In conclusion, the whole state space of $L^\infty(X, \mu)$ consists of all finitely additive probability measures on (X, \mathcal{F}) that are absolutely continuous with respect to μ . The subset of normal states are those who are not only finitely but σ -additive, i.e., *standard* probability measures and can be described by L^1 -density using the Radon-Nykodym Theorem.

Representations on Hilbert spaces

Every C^* -algebra and hence every W^* -algebra can be represented as a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. For commutative C^* -algebras, we know that they are all isomorphic to $\mathcal{C}_0(X)$, where X is the Gelfand spectrum. To represent this algebra one can simply pick a state $\omega \in \mathcal{C}_0(X)^*$, respectively a measure μ , and do the GNS-construction, which yield [9, p.696]:

$$\mathcal{H}_\omega = L^2(X, \mu) \quad (3.41)$$

$$\pi_\omega(f) = m_f \quad (3.42)$$

$$\Omega_\omega = \mathbb{1}_X. \quad (3.43)$$

Here m_f is the multiplication operator, i.e., for $\Psi \in L^2(X, \mu)$ we have $m_f \Psi = f \Psi$.

For von Neumann algebras, where we identified $L^\infty(X, \mu)$ as the prototype of this class, we can likewise realize them on the well-known Hilbert space $L^2(X, \mu)$ as multiplication operators [9, B.108].

3.2.3 Tensor products

The focus of this work are quantum-classical hybrids. Therefore, we have to pay attention to bringing the quantum and the classical side together. A typical reflex of a physicist is most probable to take the tensor product. Unfortunately, there is no such thing as *the* tensor product for C*-algebras. In fact, there are many, and this part is considered [8, p.181]

... as one of the tricky parts of the theory.

Fortunately, our application is one of the special cases where things get much more straightforward. Therefore, we will not go into much detail about the general case, but we will not simply omit that the tensor product on the level of C*-algebras might cause ambiguities.

Tensor Product of Hilbert Spaces We start with a familiar example: The construction of those is something nearly every physicist has seen at least once in various levels of detail during his quantum mechanics lecture. As the basic idea is the same for C*-algebras, we will recall a brief sketch of this procedure:

Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. We define the **algebraic tensor product** as $\mathcal{H}_{\text{alg}} = \mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$. It is spanned by finite linear combinations of the form

$$\sum_i \Psi_i^1 \otimes_{\text{vec}} \Psi_i^2 \quad \Psi_i^1 \in \mathcal{H}_1, \Psi_i^2 \in \mathcal{H}_2, \quad (3.44)$$

where the symbol \otimes_{vec} denotes the tensor product of vector spaces. The algebraic tensor product is basically the basis for any of the upcoming constructions of tensor products. For the Hilbert space construction, we define for $\Psi, \Phi \in \mathcal{H}_{\text{alg}}$ the sesquilinear form

$$\langle \Psi, \Phi \rangle = \sum_i^n \sum_j^m \langle \Psi_i^1, \Phi_j^1 \rangle \langle \Psi_i^2, \Phi_j^2 \rangle, \quad (3.45)$$

which is positive-definite, hence \mathcal{H}_{alg} becomes an inner product space. As \mathcal{H}_1 and \mathcal{H}_2 are Hilbert spaces, we want $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ to be a Hilbert space too. To this end, we take the above scalar product and do the completion of \mathcal{H}_{alg} , which then yields the desired Hilbert space. The completion of the algebraic tensor product is then what is commonly known as the **Hilbert space tensor product**, which is unique up to unitary equivalence and is denoted by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

In this construction, the completion step might create ambiguities when applying this scheme to more general spaces. For Hilbert spaces, the completeness with respect to the norm induced by the scalar product makes the structure more *rigid* and demands a specific connection between a norm and the space. For Banach spaces, we could choose different norms in the completion, which yield different Banach spaces as tensor products.

Tensor Products of Banach Spaces and C*-Algebras We start with the case of Banach spaces [8, IV.Sect. 2]. So let \mathcal{B}_1 and \mathcal{B}_2 be Banach spaces and $\mathcal{B}_1 \otimes_{\text{alg}} \mathcal{B}_2$ the algebraic tensor product. In this case, the norm which is used in the completion of the algebraic tensor product is called a **cross-norm** $\|\cdot\|_\beta$ and satisfies

$$\|B_1 \otimes_{\text{alg}} B_2\|_\beta = \|B_1\| \|B_2\|, \quad B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2. \quad (3.46)$$

Unlike for Hilbert spaces, we can choose different norms $\|\cdot\|_\beta$, so one denotes the norm used in the construction of the tensor product as $\mathcal{B}_1 \otimes_\beta \mathcal{B}_2$. The greatest cross-norm on Banach spaces is called projective cross-norm

$$\|b\|_\gamma = \inf \left\{ \sum_i \|b_i^1\| \|b_i^2\| \mid b = \sum_i b_i^1 \otimes_{\text{alg}} b_i^2 \right\}, \quad (3.47)$$

leading to the **projective tensor product** \otimes_γ . It is the greatest possible cross-norm because it arises as the lower bound to the triangle inequality. Another one is called injective cross-norm

$$\|b\|_\lambda = \sup \left\{ \left| \sum_i f(b_i^1) g(b_i^2) \right| \mid f \in \mathcal{B}_1^*, \|f\| \leq 1, g \in \mathcal{B}_2^*, \|g\| \leq 1 \right\}, \quad (3.48)$$

which yield the **injective tensor product** \otimes_λ .

For C*-algebras [8, IV.Sect. 4] we have to take additional care for the involution to make sure that the resulting tensor product is again a C*-algebra. Let \mathcal{A}_1 and \mathcal{A}_2 be C*-algebras. The multiplication and the *-operation carry over to the algebraic tensor product that forms a *-algebra. For a C*-algebra one now needs a **C*-norm** $\|\cdot\|_\beta$ in the completion, i.e., it has to satisfy

$$\|ab\|_\beta \leq \|a\|_\beta \|b\|_\beta \quad (3.49)$$

$$\|a^*a\|_\beta = \|a\|_\beta^2, \quad a, b \in \mathcal{A}_1 \otimes_{\text{alg}} \mathcal{A}_2. \quad (3.50)$$

Examples of C*-norms are the projective C*-cross-norm

$$\|a\|_{\max} = \sup\{\|\pi(a)\| \mid \pi \text{ is a representation of } \mathcal{A}_1 \otimes_{\text{alg}} \mathcal{A}_2\}, \quad (3.51)$$

leading to the **projective C*-tensor product** denoted $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$. Another example is the injective C*-cross-norm

$$\|a\|_{\min} = \sup\{\|(\pi_1 \otimes \pi_2)(a)\| \mid \pi_i \text{ is a representation of } \mathcal{A}_i\}, \quad (3.52)$$

resulting in the **injective C*-tensor product** $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. As indicated by the subscript, another name for the projective C*-tensor product is **maximal tensor product**, while the injective C*-tensor product is often called **minimal tensor product**.

This notation is based on the fact that they give upper, respective lower bounds to all possible C*-norms, which also provides proof that any C*-norm is indeed a cross-norm [8, Thm. 4.19f].

From the representations in the above definitions, the existence of units or the calculations required for the above statements all require some more rigor in the

mathematical elaboration. Nevertheless, we hope that the reader who has never met anything but the tensor product for Hilbert spaces becomes a little sympathetic to the technicalities involved when dealing with tensor product of C^* -algebras and understands why we did not simply rush over these subtleties.

Although the classical side of our hybrid will make a deeper look into this topic redundant:

Definition 65. A C^* -algebra \mathcal{A} is called **nuclear**, if for any C^* -algebra \mathcal{B} , the projective and the injective C^* -norm coincide, i.e.,

$$\|\cdot\|_{\min} = \|\cdot\|_{\max}. \quad (3.53)$$

In other words, if one side of the tensor product is nuclear, the tensor product is uniquely defined. This is exactly the case for our tensor hybrids because commutative C^* -algebras are nuclear [39, Thm. 11.3.13], so we can write these hybrids with the tensor product \otimes without any risk of confusion.

Let us finish with some more practical observations. For this, let X be a locally compact Hausdorff space and \mathcal{A} a C^* -algebra. We denote by $\mathcal{C}_0(X, \mathcal{A})$ the set of \mathcal{A} -valued continuous functions $f : X \rightarrow \mathcal{A}$ that vanish at infinity, i.e., $x \mapsto \|f(x)\|_{\mathcal{A}} \in \mathcal{C}_0(X)$. This space can be equipped with the supremum norm

$$\|f\| = \sup \{\|f(x)\|_{\mathcal{A}} \mid x \in X\} \quad (3.54)$$

and together with point-wise operations, becomes a C^* -algebra. Indeed this is isomorphic to the tensor product of $\mathcal{C}_0(X)$ and \mathcal{A} [9, Prop. C.100]:

Proposition 66. Let X be locally compact and \mathcal{A} a C^* -algebra. Then

$$\mathcal{C}_0(X) \otimes \mathcal{A} \cong \mathcal{C}_0(X, \mathcal{A}). \quad (3.55)$$

The continuous extension of the map from $\mathcal{C}_0(X) \otimes \mathcal{A}$ to $\mathcal{C}_0(X, \mathcal{A})$ is defined by

$$f \otimes A \mapsto (fA : x \mapsto f(x)A). \quad (3.56)$$

Alongside the uniqueness, the characterization of states for tensor products with abelian tensor factors is much easier than the general theory. The following theorem tells us that pure states on a tensor hybrid are product states that are pure on the classical as well as on the quantum side [8, Thm. 4.14].

Theorem 67. Let $\mathcal{A}_1, \mathcal{A}_2$ be C^* -algebras. Then the following statements are equivalent:

- Either \mathcal{A}_1 or \mathcal{A}_2 is abelian.
- Every pure state ω of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is of the form $\omega = \omega_1 \otimes \omega_2$ for some pure states ω_1 of \mathcal{A}_1 and ω_2 of \mathcal{A}_2 .

Tensor Products of W^* - and von Neumann Algebras Like the basic definition of the space itself, the tensor product for W^* -algebras is a little more cumbersome. In return, there is basically only one tensor product for W^* - and von Neumann algebras (unless one dives really deep into the theory, e.g. [56]).

We give a (very) brief description of the construction [8, Sect. IV.5].

Let \mathcal{M}_1 and \mathcal{M}_2 be W^* -algebras and let $\mathcal{M}_0 = \mathcal{M}_1 \otimes_{\min} \mathcal{M}_2$ be their injective tensor product. The main idea is to define the W^* -tensor product as a suitable W^* -subalgebra of the universal enveloping von Neumann algebra \mathcal{M}_0^{**} of \mathcal{M}_0 . Now let

$$\overline{\mathcal{M}}^* = \mathcal{M}_1^* \otimes \mathcal{M}_2^* \quad (3.57)$$

be the closure with respect to the dual cross-norm², which is a closed subspace of \mathcal{M}_0^* . Because \mathcal{M}_i are W^* -algebras, they come with preduals $(\mathcal{M}_i)_*$ which are subsets of their biduals \mathcal{M}_i^* , so we can define

$$\overline{\mathcal{M}}_* = (\mathcal{M}_1)_* \otimes (\mathcal{M}_2)_*, \quad (3.58)$$

which is an invariant subspace of \mathcal{M}_0^* . Then there exists a central projection p in the universal enveloping von Neumann algebra \mathcal{M}_0^{**} such that $\overline{\mathcal{M}}_* = \mathcal{M}_0^* p$ [8, III. Thm. 2.7]. One can show that \mathcal{M}_0 is isometrically embedded in $((\mathcal{M}_1)_* \otimes_{\text{alg}} (\mathcal{M}_2)_*)^*$, which is isomorphic to $\mathcal{M}_0^{**} p$. Then $\mathcal{M}_0^{**} p$ is a W^* -algebra, called the **W^* -tensor product** of \mathcal{M}_1 and \mathcal{M}_2 , denoted by

$$\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2. \quad (3.59)$$

Its predual is the space $\overline{\mathcal{M}}_*$ and we have $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_* = (\mathcal{M}_1)_* \otimes_{\text{alg}} (\mathcal{M}_2)_*$. For the tensor product of von Neumann algebras, choose two faithful representations $\{\pi_i, \mathcal{H}_i\}$ of \mathcal{M}_i . Then the product representation $\pi_1 \otimes \pi_2$ of \mathcal{M}_0 can be uniquely extended to a faithful normal representation π of $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$, whose range is called the **von Neumann tensor product** [8, IV. Thm. 5.2].

²Every cross-norm also defines an adjoint or dual cross-norm on the algebraic tensor product of the respective dual spaces [8, Prop. 2.2] and the adjoint cross-norm of the injective and projective C^* -cross-norm coincide [8, Prop. 4.10].

3.3 Building Hybrids

3.3.1 Discussion of the different approaches

... what you get and give up

We now discuss the different possibilities to construct a continuous variable hybrid, i.e., choose the classical algebra with a state space that we put alongside the quantum pair

$$(\mathcal{T}(\mathcal{H}), \mathcal{B}(\mathcal{H})). \quad (3.60)$$

Besides the essential requirement that our algebra represents a non-trivial classical system, we want our hybrid structure to be able to describe non-trivial quantum-classical interactions like an instrument. This excludes systems like the combination of a continuous variable quantum system with a classical bit. We collect a more detailed list of criteria that our classical system and, thereby, our hybrid should satisfy.

In quantum mechanics and quantum information theory, the tensor product of observable algebras or states is usually considered the basic construction for combining two systems. In the case at hand, this would be a classical part, described by some algebra of functions on X , and a quantum part with the observable algebra $\mathcal{B}(\mathcal{H})$. It turns out that it is not so obvious how such a tensor product should be defined for hybrids or whether it is even a good way to describe this kind of composition. Nevertheless, for now, let us start with this ansatz.

Requirements

In quantum theory, the physical system determines the typical choice of a Hilbert space and, thereby, the states and observables. For example, if one wants to describe a qubit, this typically translates into the choice $\mathcal{H} = \mathbb{C}^2$ as the Hilbert space or $\mathcal{H} = L^2(\mathbb{R}^s, dx)$ for a quantum continuous variable system with n -degrees of freedom. In general, the Hilbert space determines the set of possible configurations.

In this spirit, the classical analog for the Hilbert space is the set X , the classical phase space, which at the beginning is just an infinite set of possible configurations. Additionally, we want that the set X and the framework on top, take the behavior at different scales into account: On a small scale, i.e., if two configurations are physically similar, as well as on a large scale, that is, during limit processes and studying the behavior when the system *escapes to infinity*. Mathematically, this requires the introduction of topology and a discussion about the compactness, respectively, the possible compactifications.

In summary, the formalism should be able to distinguish the different configurations and their behavior, which we entitle under the requirement that the classical algebra with the space X should allow for an **operational interpretation**.

Closely related is the requirement of a proper **state space**. For example, we want our classical side to have enough pure states, which is essential in many different aspects. At first, a key feature of the classical system is that no uncertainty principle forbids states with arbitrary sharp expectation values, i.e., point measures. Of course, these are the prime examples of classical pure states. Furthermore, pure

states are often the solution to physical problems like finding a ground state. Also, in general, it is a solid strategy to solve problems first on pure states and extend the solution afterwards [10, 2.5.5]. By Thm. 67, we know that for a tensor hybrid, the pure states are pure on each of the subsystems. Again, our classical system must have enough of them. Also, a hybrid framework is certainly well-advised to provide a *practical* description of the states, although this is less a requirement than a desired feature.

Besides the states, a **classical observable algebra** definitely has to have enough elements to describe measurements. Remember that one of the main reasons why the quantum observable algebra is $\mathcal{B}(\mathcal{H})$, and not the C*-algebra of compact operators $\mathcal{K}(\mathcal{H})$, is that for a measurement implementation, the identity plays a crucial role, see Sect. 2.3.2. Furthermore, we want the existence of unitary elements in our algebra, which also clearly requires the algebra to have an identity, i.e., our classical observable algebra best be unital.

Besides states and observables, we also want to describe **dynamics** on our hybrid. In the classical case, a standard requirement for admissible maps is positivity, and by Thm. 39, we know that for abelian algebras, this coincides with complete positivity. Hence, it is straightforward to demand hybrid dynamics to be represented by completely positive maps. Apart from the mathematical properties, our hybrid should also allow for time-continuous processes. This means that a classical system, which already comes with a suitable class of dynamics that can be generalized to the hybrid setting, is favorable.

We compare the different possible algebras for our classical and, thereby, for our hybrid system with this list of requirements.

Comparison

W*-view Let us start with the discussion with the von Neumann- or W*-approach, where we choose for the classical side

$$(L^1(X, \mu), L^\infty(X, \mu)). \quad (3.61)$$

This unified von Neumann approach certainly looks tempting: Both parts of our hybrid, i.e., the quantum and the classical one, are von Neumann algebras. The tensor product is clearly defined, and we have the respective normal states in the predual. Nevertheless, it comes with some noteworthy drawbacks.

Starting with the formal definition, we not only need to specify our space X but additionally need to choose a measure μ , which does have a significant impact on the properties of our algebra. For example, with our favorite classical phase space $X = \mathbb{R}^s$ and $\mu = dx$ as the Lebesgue measure, we lose all normal pure states:

For a von Neumann algebra \mathcal{M} , the normal pure states correspond to the non-zero minimal projections in \mathcal{M} (see Lem. 103), and for a non-atomic measure like dx , the algebra \mathcal{M} has no non-zero minimal projections. To put this less abstractly, the point measures do not have an L^1 -density. Of course, one could choose different measures and get some of the pure states back, but in this case, we always have to include the extra step of choosing a measure. Here, finding the *right* measure is another non-trivial part, so a description without this detour is clearly favorable.

Underlining that these problems are connected with the general choice of a von Neumann algebra and not just X or μ , let us look at \mathcal{M} as C^* -algebra and see that even if we drop the requirement for normal states, this does not help us on. By the Gelfand isomorphism, we know that any commutative von Neumann algebra \mathcal{M} is isomorphic to $\mathcal{C}(X)$, wherein this case X is an extremely disconnected compact Hausdorff space [38, Thm. 5.2.1]. These are defined by the fact that the closure of every open set is open. Examples are discrete sets, but any continuum with this property is abstract and beyond the scope of our application. In particular, when we think about X as a register for measurement outcome of the quantum part.

Lastly, the measure-theoretic approach makes it hard to define dynamics, i.e., semigroups with generators like in the quantum case. The description of such time-continuous processes is a well-known problem [57]. A solution is due to Feller, who gave a special role to the smaller algebra of continuous functions and defined what is nowadays known as Feller semigroups (see Chap. 4). We can summarize this as a hint to go for another approach and follow Fellers's choice of continuous functions for our classical part.

In conclusion, the W^* -view offers a unital observable algebra and the existence of normal states, which is a clear upside. The downside is that we need to choose a measure for the definition, and based on this choice, we lose some, if not all, normal pure states. Additionally, the dependence on X and μ makes this choice more difficult, and we already know that even without the quantum part around, defining dynamics analogous to the quantum case has some particular challenges in this setting.

C^* -view First, before going into details regarding the specific choice of the algebra or space X , note the following:

This approach forcibly leads to an asymmetry between the quantum and the classical part in a hybrid scenario. While we would like to exclude non-normal states on the quantum side and therefore only work with states in the predual, the states of the classical part are necessarily found in its full dual. This creates some extra challenges in the description of states and makes it harder if we want to define a proper Heisenberg and Schrödinger picture for our hybrid operations. Despite this fact, the choice of a C^* -setting for a classical analog to the quantum mechanical framework is quite common, see for example, [4, Example 2.4.2], [43, Sect. 4.1] or [9, Chap. I.1, Chap. I.3]. This hurdle aside, we will see that the C^* -approach checks a lot of the other requirements.

Clearly, the choice of our space X and the algebra depend on each other. The easiest way would be to start with a compact X , which directly implies

$$\mathcal{A} = \mathcal{C}_0(X) = \mathcal{C}_b(X) = \mathcal{C}(X). \quad (3.62)$$

This algebra is unital, and the state space of $\mathcal{C}(X)$ consists of all probability measures $\mathcal{W}(X)$, which is a convex set and has the point measures as extremal points [10, Prop. 2.5.7]. This exactly meets our requirements except for the fact that

$$X = \mathbb{R}^s \quad (3.63)$$

is not a valid choice. With our application of continuous-variable systems in mind, a framework that excludes this phase space from the get-go does simply not seem to be a good candidate. Translation-invariance and the possibility to define Gaussian functions should be included in our hybrid system, so we have to fix X as just locally compact and not compact.

A commutative C^* -algebra and in the spirit of Thm. 58 *the* commutative C^* -algebra, is of course $\mathcal{C}_0(X)$. With our requirement to include the case $X = \mathbb{R}^s$, this algebra disqualifies itself as an observable algebra because it is non-unital and, therefore, too small to describe measurements or unitary elements. An ansatz to solve this issue is to start with a locally compact X and choose a suitable compactification.

Here the first choice is the **one-point compactification**, where one adds the point ∞ and which we denote by

$$\dot{X} = X \cup \infty. \quad (3.64)$$

Now, the topology on \dot{X} is given by the open sets of X and the subsets of \dot{X} with a compact complement in X . The canonical injection $i : X \hookrightarrow \dot{X}$ is continuous and embeds X as an open subset into \dot{X} . Also every $f \in \mathcal{C}_0(X)$ is uniquely extended to a function $\dot{f} \in C(\dot{X})$ with $\dot{f}(\infty) = 0$. [9, Sect. C.6]. For the algebra, the one-point compactification is equivalent to the unitization $\dot{\mathcal{C}}_0(X)$ [9, Lem. C.38]:

$$\dot{\mathcal{C}}_0(X) \cong C(\dot{X}). \quad (3.65)$$

This is equivalent to saying that we do not consider arbitrary bounded continuous functions on X but only those converging to a constant at infinity. The value of this constant is then, by definition, the value of the function at the point ∞ . The one-point compactification or unitization largely maintains the structure of our set X , gives us an approximate unit, but fades out the behavior at the boundary. Despite the problems described above, the fact that $\mathcal{C}_0(X)$ also has a *decent* state space with plenty of pure states (see Thm. 59), this version of a classical algebra may be seen as a first working model.

Going forward, another candidate are the bounded continuous functions $\mathcal{C}_b(X)$. Also, with this choice of $X = \mathbb{R}^s$, this algebra is unital. The issue with this choice is the state space. The Gelfand spectrum of $\mathcal{C}_b(X)$ is more complicated and we will meet another compactification [9, Tab. B.1.2]:

$$\mathcal{C}_b(X) \cong C(\beta X). \quad (3.66)$$

Here βX is the **Stone-Ćech compactification** of the set X and every $f \in \mathcal{C}_b(X)$ extends uniquely to a function $\tilde{f} \in C(\beta X)$.

Alternative to being the Gelfand spectrum of $\mathcal{C}_b(X)$, the Stone-Ćech compactification can be constructed using the set of all ultrafilters on X . Either way, as the set determining our states, this compactification is way too large. Even in very simple cases, it is hard to describe βX in a more practical way [7, Chap. V.6].

Besides the occurrence of the Stone-Ćech compactification, there is another problem with $\mathcal{C}_b(X)$ as the classical part of hybrid algebra. Our general ansatz for this part is that our hybrid is built by a tensor product with our quantum algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$. For $\mathcal{C}_0(X)$ this is equivalent to looking at the \mathcal{A} -valued functions $f \in \mathcal{C}_0(X, \mathcal{A}) \cong \mathcal{C}_0(X) \otimes \mathcal{A}$ by Prop. 66.

If we now want to include observables that do not decay at infinity, the question arises whether this sort of isomorphism also holds for the bounded \mathcal{A} -valued norm continuous functions $\mathcal{C}_b(X, \mathcal{A})$ with the C^* -tensor product $\mathcal{A} \otimes \mathcal{C}_b(X)$. Here the candidate for this isomorphism is also

$$\iota : \mathcal{A} \otimes \mathcal{C}_b(X) \rightarrow \mathcal{C}_b(X, \mathcal{A}), \quad \iota(A \otimes f)(x) = f(x)A. \quad (3.67)$$

However, for our case, i.e., a locally compact space and especially $X = \mathbb{R}^s$, the embedding ι is not surjective [58]. Here we have

$$\iota(\mathcal{A} \otimes \mathcal{C}_b(X)) = \mathcal{C}(\beta X, \mathcal{A}), \quad (3.68)$$

but continuity of a bounded function $F : X \rightarrow \mathcal{A}$ does not necessarily imply the existence of a norm-continuous extension to βX . In fact, if such an extension exists, the range

$$F(X) = \{F(x) \mid x \in X\} \quad (3.69)$$

must have norm compact closure $F(\beta X) \subset \mathcal{A}$. As shown by Williams [58], this is precisely what could go wrong, i.e., we can turn this into a criterion describing the range of ι :

For example, take $F(x) = \sum_i P_i f_i(x)$, where P_i is a family of orthogonal projections in $\mathcal{A} = \mathcal{K}(\mathcal{H})$. The f_i are chosen to have disjoint supports in the elements of some countable partition of X , are positive, take the value 1 somewhere, and $\sum_i f_i = \mathbb{1}$. Then for a fixed function $f_0 \in \mathcal{C}_0(\mathbb{R})$, such that $f_0(0) = 1$ and $f_0(x) = 0$ for $|x| > 1/3$, we can set $f_i(x) = f_0(x - i)$. So $\{P_i\} \subset F(X)$ does not have norm compact closure. Note that the sum defining F cannot be obtained as a supremum-norm limit of finite partial sums, as would be required for $F \in \mathcal{A} \otimes \mathcal{C}_b(X)$, i.e., we have an element $F \in \mathcal{C}_b(X, \mathcal{A})$ with $F \notin \iota(\mathcal{A} \otimes \mathcal{C}_b(X))$.

Consequently, it is preferable to consider the larger algebra $\mathcal{C}_b(X, \mathcal{A}_1)$, rather than the tensor product, as a basic hybrid algebra. Note, however, that we can also change the topology of \mathcal{A} for which we demand continuity:

Consider the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, taken with the weak* topology, and hence the space $\mathcal{C}_w(X, \mathcal{B}(\mathcal{H}))$ of norm bounded, weak*-continuous functions $X \rightarrow \mathcal{B}(\mathcal{H})$. Since the unit ball of $\mathcal{B}(\mathcal{H})$ is now compact, the above argument of Williams no longer applies. Indeed if $F \in \mathcal{C}_w(X, \mathcal{B}(\mathcal{H}))$, and $\rho \in \mathcal{B}(\mathcal{H})_*$, the function $x \mapsto \text{tr } \rho F(x)$ is bounded and continuous, and hence extends to βX . The value $F_\rho(\hat{x})$ of this function at a point $\hat{x} \in \beta X$ is a bounded linear functional with respect to ρ , and there is an operator $F(\hat{x})$ representing this functional. In other words

$$\mathcal{C}_w(X, \mathcal{B}(\mathcal{H})) = \mathcal{C}_w(\beta X, \mathcal{B}(\mathcal{H})). \quad (3.70)$$

It is not obvious that this is even an algebra because the operator product is not continuous in the weak* topology.

In conclusion, without giving up our set $X = \mathbb{R}^s$, there is no suitable option that satisfies all of our demands for a tensor hybrid in this setting. Still, by using the one-point compactification and the algebra $\mathcal{C}(\dot{X})$, we have an unital algebra, with the probability measures $\mathcal{W}(\dot{X})$ as an accessible state space, plenty of pure states, and a well-known class of possible dynamics. The drawbacks are the asymmetry between normal and non-normal states for the quantum and the classical side and the fact that by using a compactification, we give up some information at the boundary, so we only get a part of our hybrid system.

Summary

We can summarize our discussion as follows: We have two different candidates for our classical side, see Fig. 3.1, namely the C*- and the W*-view, and none of the two checks all the boxes for a complete tensor hybrid. Both choices have their advantages and disadvantages, i.e., they work in different scenarios but not as a general starting point. For now, if we had to choose, the C*-approach seems to be the better working model for a tensor hybrid. This leaves us with two possibilities: Either use a compact X , accept the stated drawbacks for a general framework, and choose the pair

$$(\mathcal{C}(X)^*, \mathcal{C}(X))$$

as our classical algebra or look for another suitable hybrid algebra.

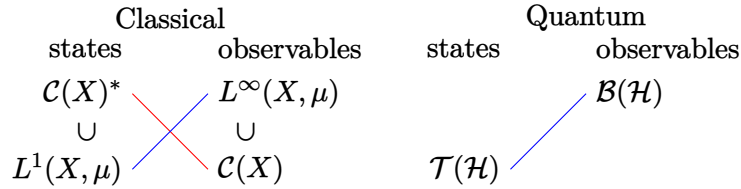


Figure 3.1: Dualities of spaces of states and observables, where a line indicates a dual pairing. The W*-approach (blue line) starts from the states and allows the full dual space as observables. The C*-approach (red line) makes the opposite choice. Traditional quantum mechanics has only the W*-approach.

The rest of this chapter and Chap. 4 is devoted to the first approach, while in Chap. 5 we take a second route: The algebra $\mathcal{C}_0(X)$ has precisely the type of state space we want, besides being too small to house our observables. This suggests a solution similar to the quantum case, where the algebra of compact operators is likewise too small but has a suitable state space with the trace class operators, whose dual $\mathcal{B}(\mathcal{H})$ is then the observable algebra, see Fig. 3.2. Note that this approach is

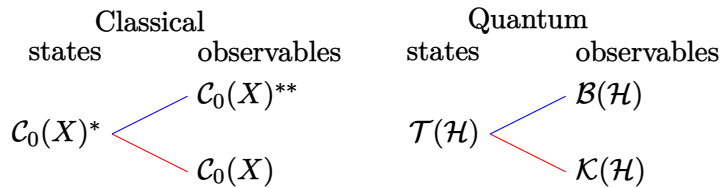


Figure 3.2: Extended dualities suitable for a joint generalization of the classical and the quantum case to hybrids. The states are here functionals on an underlying non-unital algebra \mathcal{A} , namely $\mathcal{C}_0(X)$ resp., $\mathcal{K}(\mathcal{H})$ in the classical resp. quantum case.

by no means free of complications: For example, the bidual $\mathcal{K}(\mathcal{H})^{**} = \mathcal{B}(\mathcal{H})$ as the bounded linear operators on a Hilbert space has a pretty characterization of its own, the space $\mathcal{C}_0(X)^{**}$ is by far less accessible. Finding more suitable subspaces is the subject of Sect. 5.3.

3.3.2 States on a tensor hybrid

Before we continue with the dynamics of a tensor hybrid, let us focus on the states. Consider now a general hybrid system of the kind

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{C}(X), \quad (3.71)$$

where X is a metrizable compact space and \mathcal{A}_1 is an arbitrary unital C^* -algebra, which of course later should become $\mathcal{B}(\mathcal{H})$ for a standard quantum system. Our aim is now to find a usable characterization of the states on such a system. Intuitively, one would write something like

$$\omega(A \otimes f) = \int \mu(dx) \omega_x(A) f(x), \quad (3.72)$$

where μ is a measure on X and $(\omega_x)_{x \in X}$ a family of states on \mathcal{A}_1 . Indeed, this form does not come automatically but requires some *lifting*, which we will describe in the following section.

Starting with a state ω and $A = \mathbb{1}$, we get that μ is the measure associated with the restriction of ω to the subalgebra $\mathcal{C}(X)$ by virtue of the Riesz representation theorem (Thm. 59). Extending this for $A \in \mathcal{A}_1$ we get the form

$$\omega(A \otimes f) = \int \mu_A(dx) f(x). \quad (3.73)$$

Further, we assume that $0 \leq A \leq \mathbb{1}$, which implies that $0 \leq \mu_A \leq \mu$ and in particular, μ -null sets are μ_A -null sets, so μ_A is absolutely continuous with respect to μ . Hence by the Radon-Nikodym Theorem (Thm. 63), there is a measurable function $w_A \geq 0$ such that $\mu_A = w_A \mu$. With the same argument applied to $\mu - \mu_A$, we also get $w_A \leq 1$. At this point we can write the state ω as

$$\omega(A \otimes f) = \int \mu(dx) w_A(x) f(x). \quad (3.74)$$

This already looks quite similar to Eq. (3.72), except that we use an A -dependent Radon-Nikodym derivative w_A , which is only defined μ -almost everywhere. Therefore, identities like

$$w_A(x) + w_B(x) = w_{A+B}(x) \quad (3.75)$$

only hold almost everywhere. As there are uncountably many such identities, the exceptional sets cannot be condensed into one, and it is not obvious how to ensure that for every individual x , we can choose ω_x to be a linear functional on \mathcal{A}_1 .

This is a well-known problem, even for abelian \mathcal{A}_1 , which goes under the keyword of *disintegration of measures*. The general solution in our context is given by the Lifting Theorem [59], extending the work of von Neumann [60]. It asserts that there is a positive linear operator

$$\Lambda : L^\infty(X, \mu) \rightarrow \mathcal{L}^\infty(X), \quad (3.76)$$

called lifting, assigning to each equivalence class $f \in L^\infty(X, \mu)$ of bounded measurable functions (with respect to almost everywhere equality) an individual function in that class such that positivity and identity are preserved. With this, we can write

$$\omega_x(A) = (\Lambda w_A)(x) \quad (3.77)$$

and get the characterization of ω in Eq. (3.72).

The existence of liftings is based on an Axiom of Choice argument, so it is very non-constructive. However, there are cases, like the Gaussian one, where the states ω_x are explicitly given. The above consideration also highlights some of the problems discussed in the previous section:

If we take $\mathcal{A}_1 = \mathcal{B}(\mathcal{H})$ and X a compactification of \mathbb{R}^s , the measure μ may live on the compactification points rather than properly on \mathbb{R}^s . Of course, the issues that arise this way hardly depend on the compactification chosen: The one-point compactification behaves much more tamely compared to the Stone-Čech-compactification, in which case the integral formula Eq. (3.72) is of little value.

The last issue that needs to be addressed is the case of singular states on $\mathcal{B}(\mathcal{H})$, which is typical for infinite-dimensional Hilbert spaces. As we have said, almost all of physics safely ignores this possibility, as did we. An arbitrary state $\omega \in \mathcal{A}^*$, of course, may have singular parts ω_x in Eq. (3.72). We would like to have a simple criterion excluding this possibility for our hybrid framework.

Indeed, with our outlined second approach to hybrids and the construction in Chap. 5 we will get a characterization of hybrid states, which solves both of the above issues.

3.3.3 Reversible dynamics and a no-go theorem

The typical benchmark for any attempt to build a quantum-classical hybrid is that both subsystems or marginals are correctly reproduced. At this level, the quantum part, with all its unique characteristics, certainly sets the agenda. One of these characteristics is the fundamental **information-disturbance tradeoff** [61]. The most general formulation of this principle is quite simple:

There is no information gain without disturbance.

In particular, any non-trivial quantum measurement involves some kind of irreversible disturbance. The formulation and quantification of these tradeoffs are part of the broad field of uncertainty relations [62]. For us, this principle can be rephrased and constitutes one of our basic guidelines for building hybrid systems:

Reversible dynamics on a hybrid system cannot transfer information from the quantum to the classical system.

The argument for this is straightforward: If such dynamics would exist, we could simply *read out*, respectively copy the classical system, learn something about the quantum side, and apply the inverse to hybrid, which instantly violates the above principle. As this principle does not rely on any specific form or construction of the hybrid, it should be valid in any realization and, therefore, constitutes a robust benchmark.

Note that this heavily depends on the classical part being classical: Swapping the classical part with another quantum system leads to the interpretation of the Stinespring dilation described in Eq. (2.67) and bypassing the information-disturbance tradeoff by copying the system is forbidden by the no-cloning theorem [63]. As we will see in Sect. 3.4, this can be seen as the core of many problems arising in different hybrid constructions.

3.4 Previous Works on Hybrids

The interaction and combination of quantum and classical systems is such a fundamental topic that it has been worked on several times. On some occasions, these works are explicitly talking about *hybrids*, but there are also many results not explicitly mentioning the word, yet they have useful results in the area. This especially applies to purely mathematical works but also happens in physics.

Also, the motivations to work on hybrids are quite diverse. Many works have specific focuses and, in consequence, devote a different amount of work to the various aspects of a general hybrid quantum-classical theory. In conclusion, this summary of previous works is almost surely not exhaustive, so what we describe is a classification of the most common approaches and key points to this topic.

At least it is safe to say that there is not *the* commonly accepted and tested framework for treating quantum-classical hybrids. Of course, this is also based on the simple fact that *classical physics* in general unites so many different areas and fields that a fully general quantum-classical framework is too ambitious.

At last, it should be noted that when it comes to comparing different approaches for hybrids, all reasonable work on the topic is bound to have some overlap when the two marginals *quantum theory* and *classical theory* are out of the discussion.

The mechanics approach A widespread and tempting approach to hybrids originates from *similar* structures when it comes to mechanics. Quantum systems, as well as classical systems, know *Hamiltonians* to generate the dynamics. Then, the basic idea is to combine quantum Hamiltonians and the commutator with classical Hamiltonians and the Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \quad (3.78)$$

into a hybrid or generalized version. One of the earliest works in this direction is the paper *Hamiltonian Systems and Transformation in Hilbert space* from 1931 by Koopmann [64]. Accordingly, the authors of [65], and others classify quantum-classical hybrids of this kind as *Koopman* type. A more in-depth discussion and further literature about this can be found in the review paper *Quantum-classical hybrid dynamics – a summary* by Elze [66] or in the respective section of work *On two recent proposals for witnessing nonclassical gravity* by Hall, and Reginatto [65], where the authors discuss several hybrid attempts in terms of their application to a recent proposal concerning the quantum nature of gravity.

In general, this approach often encounters problems like negative probabilities, which are then fixed by non-linear corrections. In short, taking this as a starting point for a hybrid construction immediately entails another chain of conceptual problems. The problems with this ansatz were noted several times [67, 68, 65].

For example, in [67], the authors take the *correspondence principle* as a benchmark for a hybrid formalism: A hybridization of a pair of two oscillators, i.e., quadratic Hamiltonians, coupled by a bilinear term, should reproduce the corresponding equations of motion on each side. However, they come to the conclusion that while there is no direct inconsistency in their hybrid formalism:

...the result violates the correspondence principle, which we would expect to hold exactly for a pair of oscillators with bilinear coupling. Therefore, such a theory appears quite abnormal from the point of view of physics.

Problems like these are not surprising if one remembers the observations from Sect. 3.3.3. Approaches to hybrid dynamics that involve Hamiltonian, i.e., reversible dynamics and non-trivial interactions between the quantum and the classical system, have to fail at some point, or else they would violate the *information-disturbance tradeoff*.

We see the described problems as a manifestation of principle and not for the respective hybrid formalism to be unfinished. Logically, we will not follow this approach, and in Chap. 5 we will start with a *blank* classical system, which is just the real vector space \mathbb{R}^s as phase space without any further structure.

Classical limits A common view on the interplay between the quantum and the classical world is that the classical arises out of the quantum in the classical or thermodynamic limit, and most physics students will have seen some examples where for $n \rightarrow \infty$, a quantum expression turns classical.

Following this concept, we should end up with a hybrid system if we do this limit only partially on a subsystem of a quantum one, and there is no problem writing down Hamiltonian interactions between the almost classical and the quantum part.

However, this procedure must uphold the aforementioned no-go theorem that forbids hybrid dynamics, which circumvent the information-disturbance tradeoff of quantum mechanics.

The classical variables in such a system will generally evolve into some combination involving their conjugates, or as Sherry and Sudarshan phrase this in the series of works called *Interaction between classical and quantum systems: A new approach to quantum measurement* [69], the classical variables lose their *integrity*.

While this technique is, in principle, a viable approach to the topic, it has some significant challenges: The required physical discussion on how a good approximation works depends highly on the chosen system. Also, without the requirement to end up with a *working* hybrid that does not explicitly or implicitly violates some basic physical principles, the thermodynamic limit constructions can be challenging on their own. Also, for small systems, for example, with one degree of freedom, such discussion may not be possible.

A more modern representative of this technique is, for example, the work *Quantum approach to coupling classical and quantum dynamics* by Diosi et al. [70], which contains more references regarding this technique, a discussion of typical problems, and possible solutions to some of the problems discussed.

It should be noted that the classical limit is very closely related to the mean-field limit, and the latter has indeed been proposed as a model for measurement processes involving large quantum systems, see the work *Quantum Theory of Measurement and Macroscopic Observables* by Hepp [71].

For us, the many-body aspect of the classical system will not come into play at all or even enter the formalism. Conceptually, this is because we consider that limit already being done, and we work with a much-reduced set of classical variables, a finite set of reals, such as a measurement record in a continuing observation process.

Embedding the classical system Besides limit constructions on a subsystem, one can also embed a classical into a quantum system and subsequently combine this with other quantum systems. For example, in the quantum information community, many researchers think of the observables of a classical system as the diagonal matrices embedded into a larger full matrix algebra, as we described in Sect. 3.1.

Similarly, for a classical particle described by position variables in \mathbb{R}^s , one can get a quantum extension by including the generators of the spatial shifts, i.e., conjugate momenta, in a crossed-product construction [8]. This construction can be done at the von Neumann algebra level so that the enlarged quantum system has the full algebra of bounded operators over $L^2(\mathbb{R}^s, dx)$ as observables. This is the approach to hybrids chosen, for example, in the work *A Note on a Formula of the Lévy-Khinchin Type in Quantum Probability* by Barchielli and Paganoni [72]. In this setting, the distribution of the classical variables in a normal state always has an L^1 -density, which excludes the pure states according to our discussion in Sect. 3.3.1. We will see later that the pure states of a modified hybrid also correspond to extremal quantum channels, so this approach excludes the optimal, e.g., minimal noise channels for some tasks.

In our approach, pure states are included from the outset, and the von Neumann algebraic crossed-product embedding is characterized as a special case for which states are norm continuous under translations, see Sect. 5.3.3.

Hybrids for specific systems In many works, a theory of quantum-classical hybrids is derived or used for a specific use case. Of course, there is nothing wrong with this approach, but it surely requires a careful reflection on the limits and the nature of the derived statements.

A positive example is the work [73] from Diosi, where he suggests a *fairly general form of hybrid master equations* and derives a *heuristic master equation*. This clear communication is especially important if one wants to deduce further general statements about hybrid systems.

In a series of works about the interaction of quantum and classical systems [74, 75] Blanchard and Jadczyk developed a theory of *Event-enhanced quantum theory*. A summary of their work can be found in [76]. One of their *essential ingredients* is the replacement of continuous time evolution with piecewise deterministic processes. With this, they can include *events* and get a *minimal extension of the standard quantum theory* that can treat SQUID-tank and a cloud chamber model with GRW spontaneous localization [77]. Overall, their focus lies more on the discussion and interpretation of quantum phenomena using classical degrees of freedom rather than a complete hybrid framework.

As the no-go theorem from the previous section strongly suggests the use of dissipative time evolutions to express the measurement interaction, there are several works towards this direction [73, 72, 76, 78]. For example, in [76], the author Olkiewicz discusses a mathematical framework for the coupling between classical and quantum systems. According to the author, his work is based on a *phenomenological assumption*, and he is using a typical von Neumann setting for his hybrid that comes with all the challenges described in Sect. 3.3. Within this framework, he also discusses two applications: a *semiclassical description of gravity* and a *quantum*

system coupled to all one-dimensional projectors considered as a classical device.

A common characteristic of these works is their focus on the dynamics instead of the underlying state spaces and observable algebras. Another example of this is the work [78] by Oppenheim et al., which we will discuss in more detail in Sect. 4.2.1. This work was also motivated by the application of hybrids to questions about the quantum nature of gravity in the next paragraph.

Clearly, this work also falls under this category, yet we put much effort into justifying all of the choices made from a broad point of view.

Hybrids for gravity A recent discussion of hybrids for quantum fields coupled to gravity illustrates several of the options mentioned above. In [79], we find an approach making the dissipative nature of the interaction implicit. In [80, 81], it is argued that gravitationally induced entanglement would serve as proof of the non-classical nature of gravity. This is contradicted by [65], where the authors emphasize that this will depend on the notion of hybrids and that the non-linear variant, in particular, would allow for entanglement via a classical intermediary.

Our motivation for hybrid structures is practical and comes from continuous observation and other measurement processes. Whether the resulting structures are also helpful for some fundamental theory is far beyond the scope of this work. However, we hope that a sharper understanding of the mathematical structures will also be helpful in such projects and underlines the necessity of a better-developed and tested hybrid formalism.

Hidden hybrids Getting an overview of the existing literature in a research field is always a tedious but necessary task. One can argue that for *hybrids*, this is more on the upper end of the difficulty scale. As we have mentioned at the beginning of the section, given the fixed marginals *quantum* and *classical*, many works look similar and are bound to have some overlap. Additionally, there is a potential lack of useful keywords: The moment an author does not explicitly use the word *hybrid*, the remaining indicators, *quantum*, *classical*, *interaction*, *states*, *Weyl operators*, *etc.*, are extremely widespread. For example, the upcoming hybrid Bochner theorem (Thm. 89) can also be found in the books *Photons in Fock space* [82] by Honegger and Rieckers. Here, the word hybrid is mentioned one time, in another context, on over two thousand pages. Also, it highlights another challenge in finding previous works on hybrids: One of the subtitles of [82] is *From classical to quantized radiation systems*. Using this transition to highlight certain quantum phenomena is indeed quite common and is often not connected to proper hybrids. Also, in Sect. 5.2, we found out afterwards that the idea of using a twisted version of the group algebra for a C*-description of continuous representations was already worked out by Grundling in [83, 84].

All the different approaches to hybrids over several fields and from a long period underline the relevance of the topic. At the same time, this shows that we are still missing *the* quantum-classical hybrid framework, and there is much work to be done. As a consequence, the focus of this work switched from a more applied focus to the very foundations, hopefully finding use in further applications.

Chapter 4

Hybrid Diffusions

In this part, we will take inspiration from the classical theory of diffusion generators and generalize this to our hybrid setting. Well-known characterization theorems describe these as the generators of Markov semigroups with a particular continuity condition, which excludes jumps. Typical assumptions of this type are the continuity of sample paths and the locality of the generator in the sense that in order to compute $\mathcal{L}F(x)$, one only needs to know F in a neighborhood of x . This involves the topology of X , and a key role is played by Feller's condition demanding that the evolution operators map $\mathcal{C}(X)$ into itself. Accordingly, we choose the previously described C^* -approach for our hybrid algebra.

The main result of this chapter will be Thm. 76, which is a generalization of the classical characterization theorems, as well as the Lindblad characterization of generators in the purely quantum case. The main ingredient will be the property of conditional complete positivity, which we introduced in Sect. 2.4.3 and is a direct generalization of the *maximum principle*. Also, we see the first advantages that a comprehensive hybrid treatment gives: Thm. 76 comes with an *uncertainty-like* positivity constraint for the generator that bounds the information flow from the quantum to the classical part.

4.1 Review: Feller Semigroups

The power of Feller semigroups is that they allow us to describe the dynamics of probabilistic Markov processes in terms of semigroup theory on Banach spaces, i.e., transition kernels become operators \mathcal{T}_t on a function space. This goes well with the description of quantum dynamical semigroups that we introduced in Sect. 2.4.3 as both are special applications of the general theory of semigroups on Banach spaces.

The primary reference for this short review will be the standard work of Kallenberg's *Foundations of Modern Probability* [57], especially Chap. 17 *Feller Processes and Semigroups*.

Let us start with the basic idea: If μ is a probability kernel on a locally compact metric space X and $f : X \rightarrow \mathbb{R}^+$ measurable, then a **transition operator** \mathcal{T} acts like

$$(\mathcal{T}f)(x) = \int \mu(x, dy) f(y). \quad (4.1)$$

For the stochastic interpretation, we want that for $0 \leq f \leq 1$, we also get $0 \leq \mathcal{T}f \leq 1$, so \mathcal{T} has to be a positive contraction operator. A short calculation shows that if we add a time dependency, the Chapman-Kolmogorov equation for the probability kernels μ_t of the underlying stochastic process is equivalent to the semigroup property of \mathcal{T}_t [57, Lem. 17.1]. Further we assume that $\mathcal{T}_t f(x_0)$ converges point-wise to $f(x_0)$ as $t \rightarrow 0$. Now a positive contraction semigroup \mathcal{T} is called **Feller semigroup** if it leaves $\mathcal{C}_0(X)$ invariant, i.e.,

$$\mathcal{T}_t \mathcal{C}_0(X) \subset \mathcal{C}_0(X). \quad (4.2)$$

Accordingly, Eq. (4.2) is also known as the **Feller condition**. Like quantum dynamical semigroups, Feller semigroups have a generator \mathcal{L} , which uniquely determines \mathcal{T}_t [57, Lem. 17.5]. These are generally unbounded, so they come with a domain $\text{dom}(\mathcal{L})$, which is dense in $\mathcal{C}_0(X)$ [57, Thm. 17.4]. One can show that the point-wise convergence together with the Feller property implies the strong-continuity of \mathcal{T}_t [57, Thm. 17.6]. Also the typical expression of $\mathcal{T}_t = e^{t\mathcal{L}}$ is justified, because for $f \in \text{dom}(\mathcal{L})$ we have

$$\frac{d}{dt}(\mathcal{T}_t f) = \mathcal{T}_t \mathcal{L}f = \mathcal{L}\mathcal{T}_t f, \quad t \geq 0. \quad (4.3)$$

When dealing with unbounded generators in general, it is common to use a **core** instead of the full domain, which is often challenging to work with.

For this, recall that the **graph** of an operator \mathcal{L} with domain $\text{dom}(\mathcal{L})$, acting on a Banach space \mathcal{A} , is defined as the subset

$$\{(f, \mathcal{L}f) \mid f \in \text{dom}(\mathcal{L})\}. \quad (4.4)$$

The operator \mathcal{L} is **closed** if its graph is a closed subset, and we call \mathcal{L} **closable** if there exists an operator $\bar{\mathcal{L}}$, such that the closure of the graph is the graph of $\bar{\mathcal{L}}$, which is then called the **closure** of \mathcal{L} .

Now the **core** of a closable operator \mathcal{L} is a linear subspace $\mathcal{D} \subset \text{dom}(\mathcal{L})$, such that the restriction of \mathcal{L} onto \mathcal{D} has the closure \mathcal{L} . One can show that the generator of a Feller semigroup \mathcal{L} is closed [57, Lem. 17.8] and any dense invariant subspace $\mathcal{D} \subset \text{dom}(\mathcal{L})$ is a core for \mathcal{L} [57, Prop. 17.9].

With this, we can now state the conditions under which an operator \mathcal{L} generates a Feller semigroup [57, Thm. 17.11]:

Theorem 68. *Let \mathcal{L} be a linear operator on $\mathcal{C}_0(X)$ with domain \mathcal{D} . Then \mathcal{L} is closable, and the closure is the generator of a Feller semigroup on $\mathcal{C}_0(X)$ if and only if the following conditions hold:*

- i) \mathcal{D} is dense in $\mathcal{C}_0(X)$.*
- ii) The range of $\lambda_0 - \mathcal{L}$ is dense in \mathcal{C}_0 for some $\lambda_0 \geq 0$.*
- iii) If $f \vee 0 \leq f(x)$ for some $f \in \mathcal{D}$ and $x \in X$, then $\mathcal{L}f(x) \leq 0$.*

The condition *iii)* is also known as the **positive maximum principle**, which states that if a function reaches its positive maximum, it can not increase in the next infinitesimal time-step.

Up to now, we have assumed that \mathcal{T}_t is only contractive, which can cause issues if one wants to connect a Feller semigroup to a Markov process, where probability has to be conserved, i.e., \mathcal{T}_t has to be conservative. The solution for this, like in our discussion of classical observable algebras, is utilizing a compactification: Adding *infinity* to the locally compact set X allows any Feller semigroup to be extended to a conservative one on the now compact space \dot{X} [57, Lem. 17.13]. This allows to connect any Feller semigroup with a unique semigroup of Markov transition kernels μ_t like in Eq. (4.1) [57, Prop. 17.14]. For our scenario, which is diffusions on $X = \mathbb{R}^n$, we need some definitions.

Let $\mathcal{C}_K^\infty(\mathbb{R}^n) \subset \mathcal{C}_0(\mathbb{R}^n)$ be the dense subset of smooth functions with bounded support. We call an operator \mathcal{L} with $\mathcal{C}_K^\infty(\mathbb{R}^n) \subset \text{dom}(\mathcal{L})$ a **local** operator if we have $\mathcal{L}f(x_0) = 0$ whenever f vanishes in some neighbourhood of x_0 . For diffusions, the requirement of \mathcal{L} to be a local operator can be substituted by asking the sample paths to be almost surely continuous [57, Thm. 17.24]. Now if a local operator \mathcal{L} acts as a generator, the positive maximum principle can be reduced to a **local positive maximum principle**, i.e., if $f \in \mathcal{C}_K^\infty(\mathbb{R}^n)$ has a positive local maximum at a point x_0 , then $\mathcal{L}f(x_0) \leq 0$.

With this, we can state the classical result [57, Thm. 17.24] that we want to translate into the hybrid setting:

Theorem 69 (Feller diffusions and elliptic operators). *Let \mathcal{L} be the generator of a Feller process X on \mathbb{R}^s , and assume that $\mathcal{C}_K^\infty(\mathbb{R}^n) \subset \text{dom}(\mathcal{L})$. If \mathcal{L} is local on $\mathcal{C}_K^\infty(\mathbb{R}^n)$, then there exists some functions a_{ij} , b_i , $c \in \mathcal{C}(\mathbb{R}^n)$, where $c \geq 0$ and the matrix a_{ij} is symmetric, nonnegative definite, such that for any $f \in \mathcal{C}_K^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}_+^n$*

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) f''_{ij}(x) + \sum_i b_i(x) f'_i(x) - c(x) f(x). \quad (4.5)$$

4.2 Hybrid Diffusion Generators

Before we begin to extend Thm. 69 to a hybrid setting, we must set our algebra. According to the previous discussions, we will now take the classical system described by continuous functions on the compactification $X = \dot{\mathbb{R}}^s$. Furthermore, we will assume the quantum part to be described in a finite-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^d$, so the hybrid algebra will be

$$\mathcal{A} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{C}(\dot{\mathbb{R}}^s) \cong \mathcal{C}(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H})). \quad (4.6)$$

We need to carry over the classical discussion of Feller processes into our hybrid framework. The first criterion for diffusions is, of course, Feller's condition: It is incorporated here by looking for a dynamical semigroup on $\mathcal{C}(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H}))$ in the first place. Differentiability would be too much imposed since we do not want to impose any continuing smoothness in the course of the evolution. However, we demand that the differentiable functions are, in fact, in the domain of the generator. Finally, the second condition is the locality of the generator in the sense of differential operators. That is, in order to compute $\mathcal{L}F$ at a point x , we only need to know F in an arbitrarily small neighborhood of x . Note that it is easy to build examples of processes with added jumps where this condition is not respected.

Summarizing the above requirements, we get the following definition:

Definition 70. *Let $s \in \mathbb{N}$ and \mathcal{H} a finite-dimensional Hilbert space. We denote by $\mathcal{C}_K^k(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H}))$ the set of k times continuously differentiable functions, whose support is compact and does not contain the compactification point ∞ . Then a **hybrid diffusion generator** is the generator \mathcal{L} of a completely positive semigroup on $\mathcal{C}(\dot{\mathbb{R}}^s; \mathcal{B}(\mathcal{H}))$ with $\mathcal{L}\mathbb{1} \leq 0$ and the following additional properties:*

- (1) $\text{dom } \mathcal{L} \supset \mathcal{C}_K^3(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H}))$ and
- (2) *the operator \mathcal{L} is local in the sense, that when $F \in \mathcal{C}_K^3(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H}))$, and F vanishes in some neighbourhood of $x \in \mathbb{R}^s$, then $(\mathcal{L}F)(x) = 0$.*

A general hybrid semigroup, of course, allows jumps, even jumps, depending on the quantum system. We exclude these in the spirit of diffusion theory, but they can be added at a later stage. We allow our dynamical semigroups to be subnormalized, i.e., map states to multiples of states with a factor ≤ 1 . The theory of positive perturbations [85, 30] then allows filling the normalization gap with jump probabilities. This was done in a diffusion context by [72]. The setting in this work is, on the one hand, simpler than ours by requiring translation invariance in \mathbb{R}^s . On the other hand, it is more general by allowing infinite-dimensional Hilbert spaces and a jump contribution. Together, this gives a quantum generalization of the Lévy-Khinchin formula. In the intersection of the two approaches, i.e., for the jumpless part of the formula in [72], the positivity condition agrees with ours, although it is written somewhat differently, see Sect. 4.2.1 for a discussion.

The principal tool for the following chapter is the *conditional complete positivity condition*, which we already introduced in Def. 53. In the classical case, the *complete* in conditional positivity can be omitted and translates into the positive maximum

principle from above. We will see in this section that, likewise, plain conditional positivity goes a long way for the characterization of hybrid diffusion generators. The first two steps, cutting \mathcal{L} to a differential operator of second-order, and the form of the second-order term, do not need the *complete* requirement. On the other hand, it is clear that it must enter somewhere since the characterization of quantum Lindblad generators is a special case.

We now translate these concepts into a hybrid one:

Definition 71. *Let \mathcal{L} be a hybrid generator on \mathcal{A} . Let $\sigma = \delta_{x_0} \otimes |\phi\rangle\langle\phi|$ be a pure state, i.e., $\sigma(F(x_0)) = \langle\phi|F(x_0)|\phi\rangle$, and let $F \in \text{dom}(\mathcal{L}) \subset \mathcal{A}$. Then the principle of conditional positivity (CP) states that the conditions*

$$\text{(CP1)} \quad F \geq 0 \text{ and}$$

$$\text{(CP2)} \quad \sigma(F) = \langle\phi|F(x_0)|\phi\rangle = 0$$

imply $\sigma(\mathcal{L}F(x)) = \langle\phi|\mathcal{L}F(x_0)|\phi\rangle \geq 0$. The principle of conditional complete positivity (CCP) states that this is also true if the semigroup is extended by the identity on another copy of \mathcal{H} .

A brief look at the proof of Thm. 69 in [57, Thm. 17.24] shows that the classical side merely takes one and a half pages. While we can work closely alongside this result, the additional quantum system, even in finite dimensions, lengthens the work quite a bit. For a better overview, we will separate the statement into several smaller pieces. The first property to establish is that a diffusion generator must be a second-order differential operator, possibly with operator coefficients:

Lemma 72. *Every hybrid diffusion generator is a differential operator of at most second order.*

Proof. Let $F \in \mathcal{C}^3(X, \mathcal{B}(\mathcal{H}))$ be a hermitian valued function, and let $x_0 \in X$. Denote by \tilde{F} the second order Taylor approximation at x_0 , and let g be a scalar quadratic polynomial with $g(x_0) = 0$ and $g(x) > 0$ for $x \neq x_0$. Pick $\varepsilon > 0$. Moreover, let h be a smooth, positive scalar function, which is constant = 1 in a neighborhood of x_0 to be determined later. Then, consider the two functions

$$G_{\pm}(x) = h(x) \left(\varepsilon g(x) \mathbb{1} \pm (F(x) - \tilde{F}(x)) \right). \quad (4.7)$$

Note that $G_{\pm}(x_0) = 0$ because $g(x_0)$ vanishes by construction, and we took the Taylor approximation of F in x_0 . Beyond x_0 , the functions $G_{\pm}(x)$ behave as $\varepsilon g \mathbb{1}$ up to second order, and hence are strictly positive definite in a neighborhood around x_0 . We now choose $h(x)$ to be zero outside such neighborhood, which gives $G_{\pm}(x) \geq 0$, $G_{\pm} \in \mathcal{C}^3$ and bounded. By the assumptions on hybrid generators, we have $G_{\pm} \in \text{dom } \mathcal{L}$ and $(\mathcal{L}G_{\pm})(x_0)$ does not depend on the explicit choice of h . So G_{\pm} fulfills the requirements for conditional complete positivity together with any pure quantum state $|\phi\rangle\langle\phi|$. The positivity conclusion then also holds for an arbitrary pure state and can be summarized in the operator inequality $\mathcal{L}G_{\pm} \geq 0$, or

$$\varepsilon(\mathcal{L}hg\mathbb{1})(x_0) \pm ((\mathcal{L}hF)(x_0) - (\mathcal{L}h\tilde{F})(x_0)) \geq 0. \quad (4.8)$$

Although h depends on ε through the cutoff condition, the terms containing h in this expression do not depend on it by locality, hence we can let $\varepsilon \rightarrow 0$. Because Eq. (4.8) holds true for G_+ and G_- we can conclude that

$$(\mathcal{L}hF)(x_0) - (\mathcal{L}h\tilde{F})(x_0) = 0. \quad (4.9)$$

Note that in the above argumentation, it is crucial that G_{\pm} meets the requirements for conditional complete positivity, i.e., $G_{\pm}(x) \geq 0$, therefore a linear g would not suffice, and a quadratic g is the best we can choose, i.e. all higher orders are included. Finally, the equality in Eq. (4.9) only depends on the second order Taylor expansion \tilde{F} of F , so any higher orders are not taken into account by \mathcal{L} . In conclusion, \mathcal{L} is a differential operator of at most second order. \square

The locality condition enters in this proof to simplify the CCP condition (Def. 71) in a characteristic way: In general, CCP requires a positive element F of the hybrid algebra (CP1), which has zero expectation at a pure state $\sigma = |\phi\rangle\langle\phi| \otimes \delta_{x_0}$ (CP2). With locality, it suffices that F is positive *in a neighborhood* of x_0 . Indeed, any such F can be multiplied with a cutoff function h , such that h is equal to 1 in a neighborhood of x_0 , and $h(x) = 0$, where $F(x) \not\geq 0$. Then hF satisfies (CP1) and (CP2) with σ , and, because $\mathcal{L}(hF) = \mathcal{L}F$ by locality, application of the CCP principle to hF gives $\langle\phi|\mathcal{L}(hF)(x_0)|\phi\rangle \geq 0$. This also holds for the ‘completed’ version of conditional positivity and will be used without further comment below. Throughout, the pure state in the CP condition will be fixed as $|\phi\rangle\langle\phi| \otimes \delta_{x_0}$ with $\phi \in \mathcal{H}$, and $P = (\mathbb{1} - |\phi\rangle\langle\phi|)$ as the projection onto the complement. For the CCP condition, we will denote the vectors in $\mathcal{H} \otimes \mathcal{H}$ by capital greek letters.

From the previous lemma, we now know that $\mathcal{L}F(x_0)$ depends only on the collection of partial derivatives of F up to second order. The indices for vectors in \mathbb{R}^s will be denoted by $\alpha, \beta \in \{1, \dots, s\}$, so we can write

$$\begin{aligned} (\mathcal{L}F)(x_0) &= \mathcal{L}_2(F''(x_0)) + \mathcal{L}_{1,\alpha}(F'(x_0)) + \mathcal{L}_0(F(x_0)) \\ &= \sum_{\alpha\beta} \mathcal{L}_{2,\alpha\beta} \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}(x_0) \right) + \sum_{\alpha} \mathcal{L}_{1,\alpha} \left(\frac{\partial F}{\partial x_\alpha}(x_0) \right) + \mathcal{L}_0(F(x_0)), \end{aligned} \quad (4.10)$$

where $\mathcal{L}_{2,\alpha\beta} = \mathcal{L}_{2,\beta\alpha}$, $\mathcal{L}_{1,\alpha}$, and \mathcal{L}_0 are linear operators on $\mathcal{B}(\mathcal{H})$ depending on x_0 . Our aim is now to study the consequences of the CCP condition for the form of these operators. We will proceed by decreasing order. The function F will be chosen judiciously to bring out the properties of various orders. The point x_0 will usually omitted in the notation and taken as $x_0 = 0$ when designing suitable functions $F(x)$.

For the second-order terms, let us start with a small technical lemma:

Lemma 73. *Let $\mathcal{L} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map, such that there exists $L(\phi)$ with*

$$\langle\phi|\mathcal{L}(X)\phi\rangle = L(\phi)\langle\phi|X\phi\rangle \quad \forall X. \quad (4.11)$$

Then $L(\phi) = L = \text{constant}$.

Proof. Instead of \mathcal{L} we will use the dual operator \mathcal{L}^* which is given by the duality between Heisenberg and Schrödinger picture via $\text{tr}(\mathcal{L}^*(\rho)X) = \text{tr}(\rho\mathcal{L}(X))$, so

$$\mathcal{L}^*(|\phi\rangle\langle\phi|) = L(\phi)|\phi\rangle\langle\phi|. \quad (4.12)$$

We take two orthogonal unit vectors $|\phi_1\rangle, |\phi_2\rangle$ in \mathbb{C}^n and define $|\phi_3\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$ and $|\phi_4\rangle = \bar{\alpha}|\phi_1\rangle - \bar{\beta}|\phi_2\rangle$, with $\alpha \neq 0, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$. Now the rank-one operators $|\phi_i\rangle\langle\phi_i|$, $i = 1, 2, 3$ are linear independent and

$$|\phi_4\rangle\langle\phi_4| = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| - |\phi_3\rangle\langle\phi_3|. \quad (4.13)$$

We evaluate \mathcal{L} on $|\phi_4\rangle\langle\phi_4|$

$$\mathcal{L}(|\phi_4\rangle\langle\phi_4|) = \sum_{i=1}^3 c_i L(\phi_i)|\phi_i\rangle\langle\phi_i| = L(\phi_4) \sum_{i=1}^3 c_i |\phi_i\rangle\langle\phi_i|, \quad (4.14)$$

and get

$$\sum_{i=1}^3 c_i (L(\phi_i) - L(\phi_4))|\phi_i\rangle\langle\phi_i| = 0. \quad (4.15)$$

As for $i = 1, 2, 3$ the operators $|\phi_i\rangle\langle\phi_i|$ are linear independent we get $L(\phi_i) - L(\phi_4) = 0$, hence $L(\phi_i) = L(\phi_4) = L$. So we have that for any two orthogonal vectors with a third at an angle, the statement is true, and by iteration, we then get the whole of $M_n(\mathbb{C})$. \square

After this, let us begin with the second-order terms:

Lemma 74. *Let \mathcal{L} be a hybrid diffusion generator, then the second order terms in Eq. (4.10) are*

$$\mathcal{L}_{2,\alpha\beta}(A) = \frac{1}{2}D_{\alpha\beta}A, \quad (4.16)$$

where $D_{\alpha\beta} = D_{\beta\alpha} \in \mathbb{C}$.

Proof. Consider first a function of the form

$$F(x) = P + x_\alpha x_\beta A, \quad (4.17)$$

where $A = A^*$ has support in $P\mathcal{H}$, i.e., $A = PAP$, and $\alpha, \beta \in \{1, \dots, s\}$ are any two indices. The choice of A guarantees that $F(x) \geq 0$ for *sufficiently small* x , so according to the remarks above, (CP1) is satisfied. Moreover, (CP2) holds because $\sigma(F) = \langle\phi|F(0)|\phi\rangle = \langle\phi|P|\phi\rangle = 0$. The CP principle then gives that the ϕ -expectation of (4.10) is positive.

Obviously, $F(0) = P$, $F'(0) = 0$, and in F'' only the (α, β) -derivative is non-zero and equal to A . Hence

$$0 \leq \langle\phi|(\mathcal{L}F)(x_0)|\phi\rangle = 2\langle\phi|\mathcal{L}_{2,\alpha\beta}(A)|\phi\rangle + \langle\phi|\mathcal{L}_0(P)|\phi\rangle. \quad (4.18)$$

Since A allows an arbitrary real scaling factor, this implies $\langle\phi|\mathcal{L}_{2,\alpha\beta}(A)|\phi\rangle = 0$. The operator $\mathcal{L}_{2,\alpha\beta}$ is complex linear, so this will also hold for non-hermitian A . It follows that $\langle\phi|\mathcal{L}_2(F'')|\phi\rangle$ is independent of $PF''P$.

Next we address the dependence of \mathcal{L}_2 on matrix elements $\langle \phi | \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} | \psi \rangle$ of second order derivatives between ϕ and a vector $\psi \perp \phi$. To this end, we introduce the function

$$F(x) = |\Psi(x)\rangle\langle\Psi(x)| \quad \text{with} \quad \Psi(x) = \psi + x_\alpha x_\beta \phi. \quad (4.19)$$

Then $F(0)\phi = 0$, $F'(0) = 0$, and in F'' only the α, β -derivative is non-zero and equal to $|\phi\rangle\langle\psi| + |\psi\rangle\langle\phi|$. Using the same argument as above, and complex linearity in ψ , we conclude that $\langle \phi | \mathcal{L}_2(|\phi\rangle\langle\psi|) | \phi \rangle = 0$ for $\psi \perp \phi$ and similarly for the adjoint $|\psi\rangle\langle\phi|$. Summarizing these last two steps, we can say that $\langle \phi | \mathcal{L}_2(F'') | \phi \rangle$ depends only on $\langle \phi | F'' | \phi \rangle$. Since the dependence is linear, we have

$$\langle \phi | \mathcal{L}_2(F'') | \phi \rangle = \sum_{\alpha\beta} \langle \phi | \mathcal{L}_{2,\alpha\beta} \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \right) | \phi \rangle = \frac{1}{2} \sum_{\alpha\beta} D_{\alpha\beta}(\phi) \langle \phi | \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} | \phi \rangle. \quad (4.20)$$

Here, the argument of D emphasizes that the argument was done entirely with a fixed ϕ , so, in principle, the matrix D could well depend on ϕ . However, this possibility can be negated by using Lem. 73. Because the partial derivatives commute, the matrix $D_{\alpha\beta}$ can be taken to be symmetric. As a result, the matrix $D_{\alpha\beta}$ does not depend on ϕ and we can write the (α, β) part of \mathcal{L}_2 as

$$\mathcal{L}_{2,\alpha\beta} \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \right) = \frac{1}{2} D_{\alpha\beta} \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \right). \quad (4.21)$$

□

As we have seen, the second order is merely a standard diffusion on the classical part of the hybrid algebra. Next, we look at the first-order terms, which are more interesting. As it will turn out, it is here where our classical system will be able to *learn* from the quantum part.

Lemma 75. *Let \mathcal{L} be a hybrid diffusion generator, then the first-order terms in Eq. (4.8) are of the form*

$$\mathcal{L}_{1,\alpha}(A) = A J_\alpha + J_\alpha^* A \quad (4.22)$$

where $J_\alpha \in \mathcal{B}(\mathcal{H})$ and $\text{tr } J_\alpha \in \mathbb{R}$.

Proof. Contrary to the second order, conditional positivity will no longer suffice. Therefore, we have to go from \mathcal{H} to $\mathcal{H} \otimes \mathcal{H}$ and use the CCP-conditions for $\mathcal{L} \otimes \mathbb{1}$ on the bigger space. In order to do so, we change our *evaluation* state from ϕ to the maximally entangled state

$$\Phi = \sum_i c_i |ii\rangle \quad (4.23)$$

on $\mathcal{H} \otimes \mathcal{H}$. Accordingly, we have to change the vector ψ to

$$\Psi = \sum_{ij} \psi_{ij} |ij\rangle \quad \text{with} \quad \sum_i c_i \psi_{ii} = 0, \quad (4.24)$$

where the last condition is as before $\Psi \perp \Phi$. The first choice for $F(x)$ is

$$F(x) = |\Psi + \lambda x_\alpha \Omega\rangle\langle\Psi + \lambda x_\alpha \Omega|, \quad (4.25)$$

where the direction α is arbitrary but fixed and $\lambda \in \mathbb{R}$. The CCP-conditions for $F(x)$ are easily checked, (CCP1) is true because $F(x)$ is a projector, and (CCP2) follows directly from the orthogonality condition of Ψ and Φ . So we get

$$\langle \Phi | (\mathcal{L} \otimes \mathbb{1}) (|\Psi + \lambda x_\alpha \Omega\rangle \langle \Psi + \lambda x_\alpha \Omega|) | \Phi \rangle \geq 0. \quad (4.26)$$

We decompose the above equation into separate orders and get

$$\begin{aligned} & \langle \Phi | (\mathcal{L}_0 \otimes \mathbb{1}) (|\Psi\rangle \langle \Psi|) | \Phi \rangle \\ & + \langle \Phi | (\mathcal{L}_{1,\alpha} \otimes \mathbb{1}) (\lambda |\Psi\rangle \langle \Omega| + \lambda |\Omega\rangle \langle \Psi|) | \Phi \rangle \\ & + 2 \langle \Phi | (\mathcal{L}_{2,\alpha\alpha} \otimes \mathbb{1}) (\lambda^2 |\Omega\rangle \langle \Omega|) | \Phi \rangle \geq 0. \end{aligned} \quad (4.27)$$

The first term containing \mathcal{L}_0 is positive, which follows if we choose $|\Psi\rangle \langle \Psi|$ as an input. The third term with $\mathcal{L}_{2,\alpha\alpha}$ yield $D_{\alpha\alpha}$, a scalar-valued function, so we demand $\Omega \perp \Phi$ in our ansatz for $F(x)$ and this term is zero. As Eq. (4.27) has to be positive for every choice of λ , the term with $\mathcal{L}_{1,\alpha}$ has to vanish, i.e.,

$$\langle \Phi | (\mathcal{L}_{1,\alpha} \otimes \mathbb{1}) (\lambda |\Psi\rangle \langle \Omega| + \lambda |\Omega\rangle \langle \Psi|) | \Phi \rangle = 0. \quad (4.28)$$

Notice that the above is complex linear, so we can use $i\Psi$ instead of Ψ , which does not change the \mathcal{L}_0 and $\mathcal{L}_{2,\alpha\alpha}$ term, but $\mathcal{L}_{1,\alpha}$ gets $i\lambda |\Psi\rangle \langle \Omega| - i\lambda |\Omega\rangle \langle \Psi|$ as an argument. The result is the same as Eq. (4.28) but with a minus sign. Together this yield

$$\langle \Phi | (\mathcal{L}_{1,\alpha} \otimes \mathbb{1}) (\lambda |\Psi\rangle \langle \Omega|) | \Phi \rangle = 0. \quad (4.29)$$

Let us introduce a basis and write out the above:

$$\begin{aligned} \langle \Phi | (\mathcal{L}_{1,\alpha} \otimes \mathbb{1}) (\lambda |\Psi\rangle \langle \Omega|) | \Phi \rangle &= \lambda \sum_{\dots} c_i c_j \bar{\Omega}_{nk} \Psi_{ml} \langle ii | (\mathcal{L}_{1,\alpha} \otimes \mathbb{1}) (|ml\rangle \langle nk|) | jj \rangle \\ &= \lambda \sum_{\dots} c_i c_j \Psi_{ml} \bar{\Omega}_{nk} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle \langle i | l \rangle \langle k | j \rangle \\ &= \lambda \sum_{ijmn} c_i c_j \Psi_{mi} \bar{\Omega}_{nj} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle \\ &= \lambda \sum_{mi} \Psi_{mi} \sum_{nj} c_i c_j \bar{\Omega}_{nj} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle \\ &=: \lambda \sum_{mi} \Psi_{mi} \Phi_{mi}^b = 0 \quad \forall \Psi \perp \Phi, \Omega \perp \Phi. \end{aligned} \quad (4.30)$$

Here, Φ^b is an abbreviation for the corresponding terms in the line above. By the above equation, it is characterized by being orthogonal to all Ψ , which are orthogonal to Φ . That is, $\Phi^b = \mu(\Omega)\Phi$, for some scalar $\mu(\Omega)$, possibly depending on Ω . This means

$$c_i \sum_{nj} c_j \bar{\Omega}_{nj} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle = \mu(\Omega) c_i \delta_{mi} \quad \forall m, i. \quad (4.31)$$

Assuming that $i \neq m$ and with the same argument as above we get

$$\sum_{nj} c_j \bar{\Omega}_{nj} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle = 0 \quad \Rightarrow \langle i | \mathcal{L}_{1,\alpha} (|m\rangle \langle n|) | j \rangle = \delta_{nj} \Lambda'_{im}. \quad (4.32)$$

Going back to (4.30) and doing the same argument with interchanged roles of the vectors Ω and Ψ while assuming $n \neq j$ then yields

$$\sum_{mi} c_i \Psi_{mi} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = 0 \quad \Rightarrow \langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = \delta_{im} \Lambda_{nj}. \quad (4.33)$$

We are left with three cases, which give non-zero matrix elements, namely ($i \neq m, n = j$), ($i = m, n \neq j$), and ($i = m, n = j$). By the convention $\Lambda'_{im} = 0$ for $i = m$ and $\Lambda_{nj} = 0$ for $n = j$ we can separate these cases into the three terms of the following form:

$$\langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = \delta_{im} \Lambda_{nj} + \delta_{nj} \Lambda'_{im} + \delta_{im} \delta_{nj} \Lambda''_{ni}. \quad (4.34)$$

Because \mathcal{L} is hermiticity preserving, we know that $\mathcal{L}_{1,\alpha}$ also preserves hermiticity. As a result, we have

$$\langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = \overline{\langle j | \mathcal{L}_{1,\alpha} (|n\rangle\langle m|) |i\rangle} = \delta_{jn} \bar{\Lambda}_{mi} + \delta_{mi} \bar{\Lambda}'_{jn} + \delta_{im} \delta_{nj} \bar{\Lambda}''_{in} \quad (4.35)$$

and by comparison with (4.34) we get for the matrices

$$\delta_{im} \delta_{nj} \Lambda''_{ni} = \delta_{im} \delta_{nj} \bar{\Lambda}''_{in} \quad \Rightarrow (\Lambda'')^* = \Lambda'' \quad (4.36)$$

$$\delta_{nj} \Lambda'_{im} = \delta_{jn} \bar{\Lambda}_{mi} \quad \Rightarrow \Lambda^* = \Lambda'. \quad (4.37)$$

Now we insert Eq. (4.34) into Eq. (4.30) and get

$$\begin{aligned} 0 &= \sum_{ijmn} c_i c_j \Psi_{mi} \bar{\Omega}_{nj} \langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle \\ &= \sum_{ijmn} c_i c_j \Psi_{mi} \bar{\Omega}_{nj} (\delta_{im} \Lambda_{nj} + \delta_{nj} \Lambda'_{im} + \delta_{im} \delta_{nj} \Lambda''_{ni}) \\ &= \sum_{ijn} c_i c_j \Psi_{ii} \bar{\Omega}_{nj} \Lambda_{nj} + \sum_{ijm} c_i c_j \Psi_{mi} \bar{\Omega}_{jj} \Lambda'_{im} + \sum_{ij} c_i c_j \Psi_{ii} \bar{\Omega}_{jj} \Lambda''_{ji}. \end{aligned} \quad (4.38)$$

The terms in (4.38) vanish because of the orthogonality of Ψ and Ω with Φ and the equation reduces to

$$\sum_{ij} c_j \bar{\Omega}_{jj} \Lambda''_{ji} \Psi_{ii} c_i = 0. \quad (4.39)$$

Now $a_i := \Psi_{ii} c_i$ and $b_j := c_j \bar{\Omega}_{jj}$ are arbitrary apart from the constraints

$$\sum_i a_i = \sum_j b_j = 0. \quad (4.40)$$

That means $\langle a | \Lambda'' | b \rangle = 0$ whenever $\langle a | e \rangle = \langle b | e \rangle = 0$, where $e = (1, 1, \dots, 1)$ in the chosen basis. Then Eq. (4.39) states that Λ'' only has non vanishing entries along the vector e , i.e. $\Lambda'' = |e\rangle\langle \lambda_1| + |\lambda_2\rangle\langle e|$. As we already know that Λ'' is hermitian, we further have $\lambda_1 = \lambda_2 = \lambda$ and $\langle \lambda | e \rangle \in \mathbb{R}$. We insert Eq. (4.37) and the above in Eq. (4.34), which now reads

$$\langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = \delta_{im} \Lambda_{nj} + \delta_{nj} \Lambda_{im}^* + \delta_{nj} \delta_{im} (\bar{\lambda}_i + \lambda_j) \text{ with } \sum_k \lambda_k \in \mathbb{R}. \quad (4.41)$$

Finally we summarize Λ and λ into the matrix $J_{nj} = \Lambda_{nj} + \delta_{nj}\lambda_j$, so we can write

$$\langle i | \mathcal{L}_{1,\alpha} (|m\rangle\langle n|) |j\rangle = \langle i | (|m\rangle\langle n| J_\alpha) |j\rangle + \langle i | (J_\alpha^* |m\rangle\langle n|) |j\rangle, \quad (4.42)$$

where we added the index α for the according direction. So we arrive at the stated form of $\mathcal{L}_{1,\alpha}$:

$$\mathcal{L}_{1,\alpha}(A) = AJ_\alpha + J_\alpha^*A. \quad (4.43)$$

Notice that Λ has a vanishing trace, so $\text{tr } J_\alpha$ is determined by the sum over the components of λ in Eq. (4.28), hence $\text{tr } J_\alpha \in \mathbb{R}$. \square

Note that $\text{tr } J_\alpha$ being real readily removes an ambiguity in Eq. (4.22): in that expression, we could add any purely imaginary multiple of the identity to the matrix J_α without changing the generator.

We continue with the zeroth or constant order. Here, the specific form is well known, as this is the standard Lindblad generator. However, we will not use that for the moment since a new evaluation of the positivity conditions for the whole generator is needed anyhow. Therefore, we will take \mathcal{L}_0 as the most general linear map from $\mathcal{B}(\mathcal{H})$ to itself. We parameterize this with respect to a linear basis $\{E_i\}_{i=1}^{d^2} \subset \mathcal{B}(\mathcal{H})$. Then

$$\mathcal{L}_0(F) = \sum_{i,j=1}^{d^2} M_{ij} E_i^* F E_j, \quad (4.44)$$

with a $d^2 \times d^2$ coefficient matrix M . This form is valid for any basis, if it is valid for one, by simple basis transformation. For the special basis of matrix units, i.e., $i = (a, b)$ and $E_i = |a\rangle\langle b|$, the sufficiency of this form is evident because the matrix elements of F are transformed to those of $\mathcal{L}_0 F$ by a general matrix.

The expansion in such a basis turns out to be useful also for the linear term, for which we can set

$$J_\alpha = \sum_i J_{\alpha i} E_i, \quad \text{and } J_\alpha^* = \sum_i J_{i\alpha} E_i^* := \sum_i J_{i\alpha}^* E_i, \quad (4.45)$$

where $J_{i\alpha} = \overline{J_{\alpha i}} := J_{i\alpha}^*$. If we choose the basis so that in addition $\text{tr } E_i = 0$ for $i \neq 1$, we get from Lem. 4.22 that $J_{\alpha 1} = (1/d) \text{tr } J_\alpha \in \mathbb{R}$.

In the following theorem, the main result of this section, we join the dimensions $i = 1, \dots, d^2$ and $\alpha = 1, \dots, s$ together so we can form a square block matrix of dimension $d^2 + s$ out of all the coefficients.

Theorem 76. *Let \mathcal{L} be a hybrid diffusion generator and E_i an operator basis for $\mathcal{B}(\mathcal{H})$ with $E_1 = \mathbb{1}$, and $\text{tr } E_i = 0$ for $i \neq 1$. Then \mathcal{L} is of the form*

$$\mathcal{L}F = \frac{1}{2} \sum_{\alpha\beta} D_{\alpha\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha i} J_{\alpha i} \frac{\partial F}{\partial x_\alpha} E_i + \sum_{\beta j} J_{j\beta}^* E_j^* \frac{\partial F}{\partial x_\beta} + \sum_{ij} M_{ij} E_i^* F E_j. \quad (4.46)$$

The coefficients $D_{\alpha\beta}$, $J_{\alpha i}$ and M_{ij} are continuous, and $D_{\alpha\beta}$ and $J_{\alpha 1}$ are real for all x . With these conditions the coefficients are uniquely determined by \mathcal{L} , and the square matrix \mathcal{D} of dimension $(-1 + d^2 + s)$, defined by

$$\mathcal{D} = \begin{pmatrix} M & J^* \\ J & D \end{pmatrix}_{i,j>1} \geq 0 \quad (4.47)$$

is positive semi-definite.

Note that this contains the characterization [57] of diffusion generators with $d = 1$ and the Lindblad-Gorini-Kossakovski-Sudarshan form for $s = 0$. This form is perhaps better recognized when one separates the terms containing M into those with $i = 1$ and those with $i > 1$, and diagonalizing $(M_{ij})_{i,j>1}$. This shows that we must use the full CCP condition since there are semigroups of positive but not completely positive operators.

Proof. The stated form of \mathcal{L} , is a consequence of Lem. 74, Lem. 75 and the form chosen in Eq. 4.44. By inserting particular choices of F , we can show that the coefficients are uniquely defined. Moreover, we keep track of the x -dependence, which we had ignored for a while by working at a specific x_0 . We use the basis of matrix units, but the results follow for an arbitrary basis by transformation.

For the zeroth order we insert constant functions $F(x) = |i\rangle\langle j|$, and get

$$\begin{aligned} \langle k | (\mathcal{L}F)(x) | l \rangle &= \sum_{abcd} \langle k | (M_{abcd}(x) | b \rangle \langle a | (|i\rangle\langle j|) | c \rangle \langle d |) | l \rangle \\ &= \sum_{abcd} \delta_{kb} \delta_{ai} \delta_{jc} \delta_{dl} M_{abcd}(x) = M_{ikjl}(x). \end{aligned} \quad (4.48)$$

Here i, j, a, b are arbitrary, therefore, we can extract every single matrix element of M . Moreover, because $(\mathcal{L}F) \in \mathcal{C}(\dot{\mathbb{R}}^s, \mathcal{B}(\mathcal{H}))$, we conclude that the matrix elements are all continuous functions.

To retrieve the first-order coefficients, we choose $F(x) = x_\alpha |i\rangle\langle j|$ in some neighborhood, where α is an arbitrary but fixed direction in \mathbb{R}^s , and F is modified outside the neighborhood so that it becomes bounded and remains suitably differentiable. Then

$$\begin{aligned} \langle k | \mathcal{L}F(x) | l \rangle - x_\alpha M_{ikjl}(x) &= \sum_{ab} \langle k | i \rangle \langle j | J_{\alpha,ab} | a \rangle \langle b | l \rangle + \sum_{cd} \langle k | J_{\alpha,cd}^* | d \rangle \langle c | i \rangle \langle j | l \rangle \\ &= \sum_{ab} \delta_{ki} \delta_{ja} \delta_{bl} J_{\alpha,ab} + \sum_{cd} \delta_{kd} \delta_{ci} \delta_{jl} J_{\alpha,cd}^* \\ &= \delta_{ki} J_{\alpha,jl}(x) + \delta_{jl} J_{\alpha,ik}^*(x). \end{aligned} \quad (4.49)$$

The left-hand side is continuous and determined by \mathcal{L} because we have already shown this for M . For $j \neq l$, the equation gives $J_{\alpha,jl}(x)$, so we retrieve the off-diagonal elements of J_α and conclude that they are continuous functions. Setting $k = i$ and $j = l$, we determine

$$J_{\alpha,kk}^*(x) + J_{\alpha,ll}(x) = \langle k | \mathcal{L}F(x) | l \rangle - x_\alpha M_{kkl}(x) \in \mathcal{C}(\mathbb{R}^s). \quad (4.50)$$

Summing this over both k and l gives $2\Re \operatorname{tr} J_\alpha(x) = 2 \operatorname{tr} J_\alpha(x) \in \mathcal{C}(\mathbb{R}^s)$. Here we could omit the real part, because $\Im \operatorname{tr} J_\alpha = 0$ by the assumption $\operatorname{tr} J_\alpha = J_{\alpha 1} \in \mathbb{R}$ (see Lem. 75). Thus, we can also extract the individual diagonal elements $J_{\alpha,ll}(x)$ by summing (4.50) over k .

For the second order, we choose $F(x) = x_\alpha x_\beta \mathbb{1}$, suitably modified in the large. Then with the same technique we get $D_{\alpha\beta}(x) + D_{\beta\alpha}(x) \in \mathcal{C}(\dot{\mathbb{R}}^s)$. Since we assumed D symmetric, this uniquely fixes $D_{\alpha\beta}(x)$ and shows it is continuous.

For positivity condition Eq. (4.47), we again need the technique of conditional complete positivity. We begin by using only conditional positivity, but it is clear

from the outset that this will not be enough since for $s = 0$, we must just come back to the Lindblad form. However, we can apply the resulting condition to $\mathcal{L} \otimes \mathbb{1}$. It will be convenient from now on to use a basis E_i with the stated properties.

We choose $\phi \in \mathcal{H}$ and $B_0, B_1, \dots, B_s \in \mathcal{B}(\mathcal{H})$ subject only to the condition $B_0\phi = 0$. Now

$$F(x) = \left| B_0 + \sum_{\alpha=1}^s x_\alpha B_\alpha \right|^2 \quad (4.51)$$

fulfils (CCP1) and (CCP2), so we get $\langle \phi | (\mathcal{L}F)(x_0) | \phi \rangle \geq 0$. The first and second derivatives are

$$\frac{\partial F}{\partial x_\alpha} \Big|_{x=0} = B_0^* B_\alpha + B_\alpha^* B_0 \quad \text{and} \quad \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \Big|_{x=0} = B_\alpha^* B_\beta + B_\beta^* B_\alpha. \quad (4.52)$$

We write out the generator and get

$$\begin{aligned} & \sum_{ij} M_{ij} \langle B_0 E_i \phi | B_0 E_j \phi \rangle + \sum_{\alpha i} J_{\alpha i} \langle B_\alpha \phi | B_0 E_i \phi \rangle \\ & + \sum_{\beta j} J_{j\beta}^* \langle B_0 E_j \phi | B_\beta \phi \rangle + \sum_{\alpha\beta} D_{\alpha\beta} \langle B_\alpha \phi | B_\beta \phi \rangle \geq 0, \end{aligned} \quad (4.53)$$

where we used that D is symmetric and the fact that B_0 vanishes on ϕ . We define the $(d^2 + s)$ square matrix G as $G_{ij} = \langle B_0 E_i \phi | B_0 E_j \phi \rangle$, $G_{i\alpha} = \langle B_0 E_i \phi | B_\alpha \phi \rangle$ and $G_{\alpha\beta} = \langle B_\alpha \phi | B_\beta \phi \rangle$. That is, G is the Gram matrix $G_{ij} = \langle v_i, v_j \rangle$ of the list of vectors v_i , given by $v_i = |B_0 E_i \phi\rangle$ for $i = 1, \dots, d^2$ and $v_{\alpha+d^2} = |B_\alpha \phi\rangle$ for $\alpha = 1, \dots, s$. With this definition we can rewrite Eq. (4.53) as

$$\text{tr} \left[\begin{pmatrix} M & J^* \\ J & D \end{pmatrix} G^T \right] =: \text{tr}[\tilde{\mathcal{D}}G^T] \geq 0. \quad (4.54)$$

We would like to conclude from this the positivity of $\tilde{\mathcal{D}}$. For this, we need to generate sufficiently many matrices G^T , resp. G , in the way described. How many do we need? The cone generated by the G -matrices has to be the full positive cone. This is generated by the rank-one elements $|\lambda\rangle\langle\lambda|$. Conversely, if a neighborhood of such an element would be missing from the cone, it would be too small, and thus, the condition (4.54) would allow some non-positive solutions. In other words, the rank-one Gram matrices are the key.

Now the Gram matrix $G_{ij} = \langle v_i, v_j \rangle$ of a set of vectors is rank-one if and only if the v_i span a 1-dimensional space, i.e., if and only if they are all proportional. One way to achieve this is to set $B_\alpha = b_\alpha |\phi\rangle\langle\phi|$ and $B_0 = |\phi\rangle\langle\psi|$ with $b_\alpha \in \mathbb{C}$ and $\psi \perp \phi$. Then, the Gram matrix G has rank one, as every vector is proportional to ϕ . In this case we can write $G = |v\rangle\langle v|$, where

$$v = (\langle\psi|E_1|\phi\rangle, \dots, \langle\psi|E_{d^2}|\phi\rangle, b_1, \dots, b_s). \quad (4.55)$$

Using the b_α we can get s linear independent vectors, but $\langle\psi|E_i|\phi\rangle$ will never yield more than d . That is, by using this method, we cannot conclude the positivity of $\tilde{\mathcal{D}}$.

This is not surprising, as we already said at the beginning: the Lindblad form is a special case, so we were bound to need the complete version of conditional positivity.

Therefore, we apply the ideas just developed to the generator $\mathcal{L} \otimes \mathbb{1}$ on the Hilbert space $\mathcal{H}_2 = \mathcal{H} \otimes \mathcal{H}$. This gives us vectors from \mathcal{H}_2 to choose from for building Gram matrices. Instead of ϕ , we use the entangled state $\Phi = \sum_i |ii\rangle$, $B_0, B_\alpha \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ and $E_i \mapsto E_i \otimes \mathbb{1}_{\mathcal{H}}$. Accordingly we choose $\Phi \perp \Psi$ so $B_0 = |\Phi\rangle\langle\Psi|$ and $B_\alpha = b_\alpha |\Phi\rangle\langle\Phi|$. Now the argument goes exactly as before, but instead of Eq. (4.55), we get the vectors

$$(\langle\Psi|E_1 \otimes \mathbb{1}|\Phi\rangle, \dots, \langle\Psi|E_{d^2} \otimes \mathbb{1}|\Phi\rangle, b_1, \dots, b_s) =: (c_1, \dots, c_{d^2}, b_1, \dots, b_s) = \lambda. \quad (4.56)$$

Note that the first component of all these vectors is 0 because $E_1 = \mathbb{1}$, and $\Psi \perp \Phi$. However, one easily checks that the vectors $E_i \otimes \mathbb{1}|\Phi\rangle$ span $\mathcal{H} \otimes \mathcal{H}$, as E_i runs over any operator basis. Therefore, the components c_2, \dots, c_{d^2} can be chosen arbitrarily. We can hence not conclude the positivity of $\tilde{\mathcal{D}}$, but the positivity of this matrix with the first row and column omitted, which is called \mathcal{D} in the theorem. \square

The theorem leaves the first row and column of $\tilde{\mathcal{D}}$ unconstrained. This can be understood as follows: The contribution of the terms with $i = 1$ or $j = 1$ to the generator is

$$\tilde{\mathcal{L}}(F) = \sum_{\alpha} J_{\alpha 1} \frac{\partial F}{\partial x_{\alpha}} + K^* F + F K, \quad (4.57)$$

where $K = \sum_j M_{1j} E_j - \frac{1}{2} M_{11} \mathbb{1}$. The first part here generates a classical drift, the second a no-event part, i.e., a semigroup of the form $F \mapsto \exp(tK^*) F \exp(tK)$. Since the sum of allowed generators is again a generator by the Trotter formula, such terms are always allowed, and the complete positivity condition cannot constrain them in magnitude.

We have only evaluated the requirements of complete positivity. However, the definition of hybrid diffusions also demands that the generated semigroup be subnormalized, i.e., $\exp(t\mathcal{L})(\mathbb{1}) \leq \mathbb{1}$, or in generator terms $\mathcal{L}\mathbb{1} \leq 0$. This translates directly to

$$\sum_{ij=1}^{d^2} M_{ij} E_i^* E_j = K + K^* + \sum_{ij=2}^{d^2} M_{ij} E_i^* E_j \leq 0. \quad (4.58)$$

Since the second term here is positive, $K + K^* \leq 0$, i.e., K must generate a contraction semigroup. This is just the normalization condition for Lindblad generators. In this case, the drift and diffusion parts are not constrained, reflecting the possibility of increasing the diffusion by adding an arbitrary positive matrix to D .

Besides the ambiguities for $\tilde{\mathcal{D}}$, the positivity condition for \mathcal{D} enables another interesting discussion. While the diagonal blocks D and M determine the dynamics on the purely quantum, respective classical side, the off-diagonal matrix block J specifies their interaction. This allows for an interpretation, which we will encounter similarly for the quasifree dynamics in the following chapter and is likewise an expression of the general information-disturbance tradeoff in quantum theory. Increasing the interaction and thereby the possible flow of information from our quantum to our classical system must necessarily include the addition of noise into our system, i.e., we can not *increase* J to an arbitrary amount without increasing the diffusion, such that \mathcal{D} stays positive.

4.2.1 Comparison with other works

Quantum Lévy-Khinchin type generators

Here, we verify that the positivity conditions derived in [72, Theorem 3] agree with Thm. 76. The cited theorem characterizes generators with a *Lévy-Khinchin representation*, which, according to [72, Theorem 1] is equivalent to the operator being the infinitesimal generator of a *semi-uniformly continuous semigroup of probability operators* (SCSPO). Here, the *semi-uniform continuity* refers to the quantum part (item f. of [72, Definition 1]), which is automatic in our case since we have assumed \mathcal{H} to be finite-dimensional. Of course, the restriction to the classical subalgebra cannot be uniformly continuous because it is a (necessarily unbounded) diffusion operator. On the other hand, an additional condition is imposed, namely covariance with respect to translations (item b.). This will have the effect that the matrix \mathcal{D} from our theorem does not depend on x . The generator is written out for product arguments $a \otimes f$ with $a \in \mathcal{B}(\mathcal{H})$, which in our notation would correspond to a function of the form $F(x) = af(x)$. We copy the generator [72, Eq. (3.5)] in (4.59) with minor adaptations of notation in addition to leaving out the terms describing jumps, which are K_2 and \mathcal{L}_2 . So we have

$$\mathcal{L}(a \otimes f) = f(x)\mathcal{L}_0(a) + f(x)\mathcal{L}_1(a) + \sum_{\alpha=1}^s b_{\alpha} \frac{\partial f}{\partial x_{\alpha}} a + K_1(a \otimes f) \quad (4.59)$$

$$K_1(a \otimes f) = \frac{1}{2} \sum_{\alpha\beta} D_{\alpha\beta} \frac{\partial^2 f(x)}{\partial x_{\alpha} \partial x_{\beta}} a + \sum_{\alpha} \sum_{k=1}^r \frac{\partial f(x)}{\partial x_{\alpha}} N_{\alpha k} (L_k^* a + a L_k), \quad (4.60)$$

with $\tilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_1$, which are characterized as follows:

- \mathcal{L}_0 is the generator of a norm-continuous quantum dynamical semi-group on $\mathcal{B}(\mathcal{H})$.
- $\mathcal{L}_1(a) = (1/2) \sum_{k=1}^r ([L_k^*, a]L_k + L_k^*[a, L_k])$ with $L_k \in \mathcal{B}(\mathcal{H})$.
- $b_{\alpha} \in \mathbb{R}$, $\alpha = 1, \dots, s$.
- D and N are real matrices of the appropriate dimension with $D = NN^T$.

To make contact with Thm. 76, let us first diagonalize M or, equivalently, write $L_k = \sum_i R_{ki} E_i$ for some coefficient matrix R . Also, remember that the positivity condition in our theorem is only stated for \mathcal{D} instead of $\tilde{\mathcal{D}}$, i.e., we left out the matrix elements corresponding to the basis element $E_1 = \mathbb{1}$. These terms remain unconstrained, which fits with the fact that these yield generators for no-event semigroups and classical drifts. For the comparison, we can ignore these types of contributions. With this we express the terms under consideration of $\mathcal{L}_1(a)$ as $\hat{\mathcal{L}}_1(a)$ which are:

$$\hat{\mathcal{L}}_1(a) = \sum_{k=1}^r L_k^* a L_k = \sum_{i,j=2}^{d^2} \sum_{k=1}^r \bar{R}_{ki} R_{kj} E_i^* a E_j. \quad (4.61)$$

Furthermore, we leave out the no-event contribution of \mathcal{L}_0 and note that any jump contribution of \mathcal{L}_0 may be written in the same way as in Eq. (4.61) and can therefore

be added by the coefficients R . Next, we diagonalize the first-order terms in K_1 , which read

$$\sum_{\alpha=1}^s \sum_{i=2}^{d^2} \sum_{k=1}^r \frac{\partial f}{\partial x_\alpha} N_{\alpha k} (\bar{R}_{ki} E_i^* a + R_{ki} a E_i). \quad (4.62)$$

Let us now write Eq. (4.59) as $\tilde{\mathcal{L}}(a \otimes f)$, where we leave out the classical drifts, i.e. $\sum_{\alpha} b_{\alpha} \frac{\partial f}{\partial x_{\alpha}} a$, and the no-event parts:

$$\begin{aligned} \tilde{\mathcal{L}}(a \otimes f) &= \frac{1}{2} \sum_{\alpha, \beta=1}^s D_{\alpha\beta} \frac{\partial^2 f(x)}{\partial x_{\alpha} \partial x_{\beta}} a + \sum_{\alpha=1}^s \sum_{i=2}^{d^2} \sum_{k=1}^r \frac{\partial f}{\partial x_{\alpha}} N_{\alpha k} (\bar{R}_{ki} E_i^* a + R_{ki} a E_i) \\ &\quad + \sum_{i,j=2}^{d^2} \sum_{k=1}^r \bar{R}_{ki} R_{kj} E_i^* a E_j. \end{aligned} \quad (4.63)$$

The connection with Thm. 76 is now obvious by defining

$$J_{\alpha i} = \sum_{k=1}^r N_{\alpha k} R_{ki} \quad \text{and} \quad M_{ij} = \sum_{k=1}^r \bar{R}_{ik} R_{kj}. \quad (4.64)$$

Remember that N was supposed to be real, hence $N^T = N^*$, so we can write $D = NN^*$. Now, the positivity condition is easily verified:

$$0 \leq \mathcal{D} = \begin{pmatrix} M & J^* \\ J & D \end{pmatrix} = \begin{pmatrix} R^* R & R^* N^* \\ NR & NN^* \end{pmatrix} = \begin{pmatrix} R^* \\ N \end{pmatrix} \cdot \begin{pmatrix} R & N^* \end{pmatrix}. \quad (4.65)$$

Besides this, there is a recent publication [86] in which [72] and the upcoming section (also in [3]) is developed further, see Sect. 5.5.4.

Generalized Pawula theorem

Closely connected to our and the above work is the recent article *The two classes of hybrid classical-quantum dynamics* by Oppenheim, Sparaciari, Šoda, and Weller-Davies [78], which is still extended [87, 88] and applied in the field of quantum gravity [89, 90].

Given that the hybrid dynamics are memoryless, the authors describe that there are only two possible options for the according evolution: One with finite-sized jumps in the classical phase space and one continuous. They describe its most general form of a hybrid master equation for the latter. They perform a short-time expansion and find that either the Kramer-Moyal expansion must have infinite many moments or has to be of the following form:

$$\frac{\partial \rho(z, t)}{\partial t} = \sum_{n=1}^{n=2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}} \right) \left(D_{n, i_1, \dots, i_n}^{00} \rho(z, t) \right) \quad (4.66)$$

$$+ \frac{\partial}{\partial z_i} \left(D_{1,i}^{0\alpha} \rho(z, t) L_{\alpha}^{\dagger} \right) + \frac{\partial}{\partial z_i} \left(D_{1,i}^{\alpha 0} L_{\alpha} \rho(z, t) \right) \quad (4.67)$$

$$- i[H(z), \rho(z, t)] + D_0^{\alpha\beta}(z) L_{\alpha} \rho(z) L_{\beta}^{\dagger} - \frac{1}{2} D_0^{\alpha\beta} \{ L_{\beta}^{\dagger} L_{\alpha}, \rho(z) \}_+. \quad (4.68)$$

Although the techniques used are different, they very well coincide with our findings in Thm. 76. The similarities extend to the positivity condition of the coefficients described in our Eq. (4.47), the construction described in the above paragraphs, and [78, Eq. (36)].

The interpretation of the individual terms also matches: Besides the usual quantum dynamics and a classical zeroth order term in Eq. (4.66), the quantum-classical coupling in Eq. (4.67) with $D_{1,i}^{0\alpha}$ is described as encoding *the strength of the CQ back-reaction* and the classical diffusion in Eq. (4.68) with $D_{2,ij}^{00}$ as representing *the necessity of diffusion in the classical phase space* due to the information-disturbance tradeoff.

It should be noted that the similarities also extend to the shortcomings of the results: As in our work, the results in [78] are based on finite-dimensional Hilbert spaces even though the authors expect they can be *extended to any bounded trace-class operation*. Furthermore, their focus lies specifically on the dynamics, so they discuss *CQ states* rather briefly and do not specify the according algebras.

Chapter 5

Hybrids on Phase Space

In this chapter, we will study hybrids that allow for a more complex quantum side to include the two most prominent observables of quantum mechanics: position and momentum. As before, we begin with the purely quantum case, as this calculus for continuous-variable quantum systems will be the basis for our hybrid generalization. Adding the classical system is notational-wise straightforward. The purely quantum and our hybrid formulation indeed look very similar from the outside. Our main task will be the description of the framework behind it.

We will keep the purely quantum review relatively short because of the upcoming hybrid versions of the following statements. These will naturally include the quantum versions by ignoring the classical part. Hence, we will not give a self-contained introduction to the topic, so for proofs, details, and an in-depth discussion of the quantum case, we refer the reader to the reference subsection at the end of this part (see Sect. 5.1.5).

5.1 Review: Quantum Mechanics on Phase Space

5.1.1 The CCR algebra

In mechanics, whether quantum or classical, two of the essential observables are, of course, position Q and momentum P . For a quantum mechanical system with n -degrees of freedom, they are typically represented on the well-known Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, dx)$ as multiplication operators and derivatives, that is, for each degree of freedom $i \in \{1, 2, 3, \dots, n\}$ we have

$$Q_i \Psi(x) = x_i \Psi(x), \quad P_i \Psi(x) = -i\hbar \partial_{x_i} \Psi(x), \quad \Psi \in \mathcal{H}. \quad (5.1)$$

It is part of nearly every quantum mechanics lecture to show that these operators satisfy the famous **canonical commutation relation** or **CCR**:

$$[X_i, P_j] = i\hbar \delta_{ij} \mathbb{1}. \quad (5.2)$$

Named after the originator, the representation of the CCR in Eq. (5.1) is known as the Schrödinger representation. Note that from now on and throughout this work, we will use the convention $\hbar = 1$, as there will be enough signs and factors to keep track of.

Now we introduce the quantum mechanical phase space: A quantum system with n canonical degrees of freedom has a position variable $q \in \mathbb{R}^n$, and its momentum counterpart p in the dual space, which is likewise \mathbb{R}^n . This means we have a scalar product $q \cdot p$, and the phase space of the system is described by the set of vectors $\xi = (q, p) \in \mathbb{R}^{2n}$. This space carries a natural **symplectic form** given by

$$\sigma((q, p), (q', p')) = q \cdot p' - p \cdot q' = \sum_{i,j} \xi_i \sigma_{ij} \xi'_j. \quad (5.3)$$

The $2n$ -dimensional square matrix σ that belongs to this standard form is called **standard symplectic matrix** and is given by

$$\sigma = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}. \quad (5.4)$$

More general, a **symplectic form** over a vector space Ξ is a bilinear map, typically denoted as $\sigma : \Xi \times \Xi \rightarrow \mathbb{R}$, that is further

- i) antisymmetric, i.e., $\sigma(\xi, \eta) = -\sigma(\eta, \xi)$, and
- ii) non-degenerate, i.e., the only vector ξ such that $\sigma(\xi, \xi') = 0$ for all ξ' is $\xi = 0$.

The pair of vector space and symplectic form (Ξ, σ) is also called a **symplectic vector space**. In finite dimensions, there always exists a basis, called Darboux coordinates [91, Sect. 1.1], such that σ will take the form as in Eq. (5.4). Especially for $\Xi = \mathbb{R}^{2n}$, the geometry in this space resembles the Euclidean one, yet there are noteworthy differences, meaning that some Euclidean relations can be easily transferred, while others fail.

For example, instead of the Euclidean scalar product, symplectic geometry uses the symplectic form or **symplectic scalar product** $\sigma(\xi, \eta)$, which consequently determines properties like the symplectic complement. So for a linear subspace $V \subset \Xi$ we have

$$V^\perp = \{\xi \in \Xi \mid \sigma(\xi, \eta) = 0 \forall \eta \in V\}, \quad (5.5)$$

which again is a linear subspace of Ξ and relations like $(V^\perp)^\perp = V$ still hold, but properties like Ξ being the sum of V and V^\perp or $V \cap V^\perp = \emptyset$ are no longer true in general [92].

Let us now link the symplectic structure on our phase space with the quantum operators from above. This will allow us to switch our primary workplace from infinite-dimensional spaces and unbounded operators like in Eq. (5.1) to linear algebra on \mathbb{R}^{2n} .

We remind the reader of the discussion in Sect. 3.4: While the symplectic formulations of quantum and classical mechanics look very similar, and especially the quantum version is often discussed as a quantization of the classical one, we will not try to unify these two. Instead, we will use the quantum version and add a classical system explicitly without any symplectic structure.

Definition 77. We call a pair (Ξ, σ) a **quantum phase space** if Ξ is a finite-dimensional symplectic vector space with the symplectic form σ .

While it is possible to work in a coordinate-free way, it is typically more convenient to think of a quantum phase space as $\Xi \cong \mathbb{R}^{2n}$ and σ in standard form. We label the phase space coordinates $(q, p) \in \mathbb{R}^{2n}$ as vectors $\xi \in \Xi \cong \mathbb{R}^{2n}$ and summarize the position and momentum operators into an operator-valued vector $R = (Q_1, \dots, Q_n, P_1, \dots, P_n)$, so that the CCR in Eq. (5.2) can be rewritten as

$$[R_i, R_j] = i\sigma_{ij}\mathbb{1}. \quad (5.6)$$

Accordingly, σ may also be referred to as the commutation form, and based on the application in field theory, the vector R or, respectively, its components are often called field operators.

A fundamental symmetry of the theory are the **phase space translations**, which add a constant, i.e., a multiple of the identity to each R_j . We denote this transformation by

$$\alpha_\xi(R_j) = R_j + \xi_j\mathbb{1}, \quad (5.7)$$

where ξ_j are the components of ξ . Clearly, α preserves the commutation relations in Eq. (5.6) and will always be a homomorphism, i.e., it preserves operator products.

There are many subtleties and technicalities in the task of finding all operators satisfying Eq. (5.6), which are mainly related to domain questions of these unbounded operators [93]. The main regularity condition singling out the Schrödinger representation in Eq. (5.1) is that the operators are essentially self-adjoint on their common domain so that they generate unitary groups, which then should satisfy an integrated version of Eq. (5.6). Indeed, it can be argued that this is historically more adequate, as the upcoming integrated version in Eq. (5.9) is older [94, Sect. 4.1], and due to Weyl, who proposed it to Max Born, even before the latter published Eq. (5.6).

We will take this as a starting point and pass to the operators

$$W(\xi) = \exp(i\xi \cdot R), \quad (5.8)$$

which are accordingly called **Weyl operators**. Here, the expression $\xi \cdot R := \sum_j \xi_j R_j$ means a mixed vector/operator scalar product and the canonical commutation relations become

$$W(\xi)W(\eta) = e^{-\frac{i}{2}\sigma(\xi, \eta)}W(\xi + \eta) \quad (5.9)$$

$$= e^{-i\sigma(\xi, \eta)}W(\eta)W(\xi). \quad (5.10)$$

We will refer to Eq. (5.9) as the **Weyl relation**, while Eq. (5.10) is called the **canonical commutation relation** in Weyl form. As the Weyl operators induce the phase space translations, they are also known as displacement operators, especially in the field of quantum optics. Now, the aforementioned Schrödinger representation of the CCR in Eq. (5.1) generates a representation of the CCR in Weyl form, which we likewise call Schrödinger representation. These unitary operators act as follows:

$$(W(a, b)\psi)(r) = e^{\frac{ia \cdot b}{2} + ia \cdot r}\psi(r + b), \quad \text{for } (a, b) \in \Xi, \psi \in \mathcal{H} = L^2(\mathbb{R}^n, dr). \quad (5.11)$$

But as the word representation suggests, one can define this more generally:

Definition 78 (CCR-algebra). *Let (Ξ, σ) be quantum phase space. We call the C^* -algebra generated by the elements $W(\xi)$ that satisfy $W(-\xi) = \overline{W(\xi)}$ and*

$$W(\xi)W(\eta) = e^{-\frac{i}{2}\sigma(\xi, \eta)}W(\xi + \eta) \quad (5.12)$$

*the **CCR-algebra** over (Ξ, σ) and denote it as $\text{CCR}(\Xi, \sigma)$.*

The algebra $\text{CCR}(\Xi, \sigma)$ is unital, it always exists, and is unique up to isomorphisms [95, Thm. 2.1]. The difference between the abstract $W(\xi)$ and their representation as unitary operators on a Hilbert space is often highlighted by calling the first Weyl elements or symbols and the second Weyl operators, see [95, Chap. 2].

Note that we do not consider the CAR-algebra [96, 35] in this work since the commutation of classical variables form a much less happy combination with the anticommutation of fermionic degrees of freedom.

Let us now come back to the question of how many possible representations of the canonical commutation relations there are. Indeed, this question was one of the very early discussions in the development of quantum mechanics and is answered by the **von Neumann Uniqueness Theorem** [97]. Before we can state the theorem, we need a closer look at the regularity condition that tames our possible algebra representation.

Prior to the Stone Theorem (Thm. 46), we defined one-parameter unitary groups as strongly continuous. Similarly, we call a Weyl operator $W(\xi)$ **regular**, if

$$t \mapsto W(t\xi) \quad (5.13)$$

is strongly continuous, which guarantees the existence of a generator as in Eq. (5.8). With this, we can formulate von Neumann's famous theorem, which can be found in various formulations: [92, Thm. 2.1], [98, Sect. 3.2, Cor. 1.], [95, Thm. 1.2].

Theorem 79 (von Neumann Uniqueness Theorem). *Any two irreducible and regular representations of the canonical commutator relations in Weyl form over a finite-dimensional symplectic vector space are unitary equivalent.*

As the Schrödinger representation is a regular and irreducible representation of the CCR-algebra [95, Prop. 1.1], we already have a complete characterization of all regular representations.

5.1.2 States

After discussing the observable algebra, let us turn to states and dynamics in this formulation. One of the advantages of using Weyl operators instead of the unbounded operators P and Q is a new way to describe the possible configurations of our system. In place of using normed trace-class operators ρ , the phase space formulation of quantum mechanics offers another description in terms of characteristic functions:

Definition 80. *Let $\rho \in \mathcal{T}(\mathcal{H})$ be a quantum state and $W(\xi)$ a Weyl operator. We call the function*

$$\chi(\xi) = \text{tr}[\rho W(\xi)] \quad (5.14)$$

*the **characteristic function** of the state ρ .*

The mapping $\rho \mapsto \chi(\xi)$ is also known as **non-commutative Fourier transformation** and its inverse is called **Weyl transformation**, which maps a complex-valued integrable function $f(\xi)$ to an operator

$$f \mapsto \hat{f} = \frac{1}{(2\pi)^n} \int f(\xi) W(-\xi) d\xi, \quad (5.15)$$

where the above integral has to be read in the weak-sense and converges as a Bochner integral with values in $\mathcal{B}(\mathcal{H})$ [18, Sect. 5.3]. The mathematical reason that we can use characteristic functions instead of trace-class operators is mainly due to the fact that the non-commutative Fourier transformation $\rho \mapsto \chi(\xi)$ extends uniquely to an isometric map from the class of Hilbert-Schmidt operators, of which the trace-class operators are a dense subset, to the space $L^2(\Xi)$. This fact is also known as the **non-commutative Parseval relation**, see [18, Thm. 5.3.3].

For the main result regarding the description of states in this framework, we need the notion of **σ -twisted positive definiteness**, which coincides with the common definition of a positive definite function, but *twisted* with a factor:

Definition 81. *Let (Ξ, σ) be a symplectic vector space. A function $f(\xi)$ is σ -twisted positive definite if for any choice $\xi_1, \dots, \xi_m \in \Xi$, the $m \times m$ -matrix*

$$M_{kl} = \chi(\xi_k - \xi_l) e^{\frac{i}{2}\sigma(\xi_k, \xi_l)} \quad (5.16)$$

is positive semi-definite.

Recall that a quantum state, which we translate as a normal state on $\mathcal{B}(\mathcal{H})$, is described by a positive trace-class operator with $\text{tr } \rho = 1$. Of course, for singular states ω we can also define a characteristic function via

$$\chi(\xi) = \omega(W(\xi)). \quad (5.17)$$

To distinguish between these two types of characteristic functions, belonging to different types of states, and arbitrary functions $\chi(\xi)$, the next result is key. The **Quantum Bochner Theorem** characterizes exactly those characteristic functions, which belong to quantum states [18, Thm. 5.4.1]. It was apparently first formulated by Araki [99], with further relevant work by [100, 101, 102, 103, 35].

Theorem 82 (Quantum Bochner Theorem). *A function $\chi : \Xi \rightarrow \mathbb{C}$ is the characteristic function of a quantum state if and only if it is*

- (1) *continuous,*
- (2) *normalized, i.e., $\chi(0) = 1$, and*
- (3) *σ -twisted positive definite.*

Here, the *if*-direction is quite straightforward: As $W(\xi)$ is a regular representation, the according characteristic function $\chi(\xi)$ is continuous for every quantum state ρ . Because $W(0) = \mathbb{1}$, the normalization condition is a direct consequence of $\text{tr } \rho = 1$, and the σ -twisted positive definiteness also follows from $\rho \geq 0$. The more

complicated direction is to prove the converse, i.e., that conditions (1)–(3) are sufficient for $\chi(\xi)$ being the characteristic function of a quantum state. For the proof, we refer to the above reference or the upcoming hybrid analog, see Thm. 89, which works in a similar manner.

Next, we need to say how to get the statistical quantities like mean and higher moments from this parametrization. For a state ρ , with characteristic function $\chi(\xi)$, we get the **mean value** for R_j by taking the according derivative at $\xi = 0$:

$$m_j = \text{tr}[\rho R_j] = \frac{1}{i} \frac{\partial}{\partial \xi_j} \chi(\xi) \Big|_{\xi=0}. \quad (5.18)$$

The vector of all $2n$ mean values of R can then be combined into a **mean vector** $m \in \Xi$. For quantities depending on higher orders, like the **covariance matrix** γ , we need to increase the order of derivatives:

$$\gamma_{jk} = 2\Re \text{tr} \left[\rho (R_j - m_j \mathbb{1})(R_k - m_k \mathbb{1}) \right] = -\frac{\partial^2}{\partial \xi_k \partial \xi_j} \chi(\xi) \Big|_{\xi=0} - m_j m_k. \quad (5.19)$$

Just as in classical statistics, the covariance matrix γ is real, symmetric, and positive semi-definite [92].

Lastly, a highly useful and easy-to-handle subset of the state space consists of the **Gaussian states**. This class of states plays a vital role in the description of quantum optical experiments and, at the same time, is especially easy to use. Like the classical Gaussian or normal distribution, they are completely specified by their first and second moments. Analogically, we call a state Gaussian if its characteristic functions is given by

$$\chi(\xi) = \exp \left(-\frac{1}{2} \xi \cdot \gamma \xi + i \xi \cdot m \right). \quad (5.20)$$

This characteristic function is clearly continuous, normalized and the σ -twisted positive definiteness is equivalent to the condition [18, Thm. 5.5.1]

$$\gamma + i\sigma \geq 0. \quad (5.21)$$

Note that Eq. (5.21) is also denoted as *Heisenberg's uncertainty principle* and is satisfied by the covariance matrix γ of any quantum state [104].

5.1.3 Dynamics

The transformations that preserve the CCR in Eq. (5.6) are called **symplectic transformations**. This set of real, linear transformations which leaves the symplectic form invariant, i.e., $\sigma(S\xi, S\eta) = \sigma(\xi, \eta)$, is called the **symplectic group** and is typically denoted as

$$Sp(2n, \mathbb{R}) = \{S \in \mathcal{M}_{2n}(\mathbb{R}) \mid S\sigma S^T = \sigma\}. \quad (5.22)$$

The characterization is equivalent to the requirement that $S^T \sigma S = \sigma$ and in general for $S \in Sp(2n, \mathbb{R})$ we also have $-S, S^T, S^{-1} \in Sp(2n, \mathbb{R})$ [92].

The elements of the symplectic group $S \in Sp(2n, \mathbb{R})$ acting on a phase space are easily connected to unitary transformations on the Hilbert space, which is an elegant application of the von Neumann Uniqueness Theorem:

If $W(\xi)$ is a regular irreducible Weyl operator, so is $W(S\xi)$. By the von Neumanns Uniqueness Theorem $W(S\xi)$ is unitarily equivalent to $W(\xi)$, i.e., there exist unitary operators U_S such that

$$W(S\xi) = U_S W(\xi) U_S^* \quad \forall \xi \in \Xi, \quad (5.23)$$

which are unique up to a phase because the Weyl operators are irreducible. This representation is also called the **metaplectic representation**, see [92, Sect. 2.6] and [105]. One can show that the generators of these unitary operators are the Hamiltonians, which are quadratic expressions of the field operators. These can be further classified and give rise to a comprehensive study of quantum optics in terms of these operations, see [106]. So, with these unitary operations, we are already getting a considerable set of possible dynamics. However, given the unitary nature, we are still missing out on the large class of irreversible operations. For this, we will introduce quasifree dynamics.

Quasifree and Gaussian channels

The notion of quasifree operations (and states) arose in field theory and statistical mechanics [107, 108, 101]. In statistical mechanics, free time evolution is the non-interacting time evolution of a many-particle system. Indeed, in the absence of interaction, the time evolution on the one-particle Hilbert space should be automatically lifted to an evolution for the entire system.

We keep as the hallmark of quasifree evolutions that they can be characterized completely by linear operators at the phase space level. In contrast to the typical applications to unitary dynamics, we moreover include irreversible (completely positive) operations and later on, of course, hybrids (see [35, 109] for some early extensions in the irreversible direction and Sect. 5.4 for the hybrid scenario). For now, let us continue with the purely quantum case.

Let \mathcal{T} be a quantum channel, with $(\Xi_{\text{in}}, \sigma_{\text{in}})$ and $(\Xi_{\text{out}}, \sigma_{\text{out}})$ denoting the according quantum phase spaces.

Definition 83. *We call a quantum channel \mathcal{T} **quasifree** if its action on Weyl operators is given by*

$$\mathcal{T}^*(W(\xi)_{\text{out}}) = f(\xi) W(S\xi)_{\text{in}} \quad \forall \xi \in \Xi_{\text{out}}, \quad (5.24)$$

where $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$ is a linear map and $f : \Xi_{\text{out}} \rightarrow \mathbb{C}$ a function, which we call the **noise function** of the channel \mathcal{T} . If this function is a Gaussian, i.e.

$$f(\xi) = \exp\left(-\frac{1}{2}\xi \cdot N\xi + id \cdot \xi\right), \quad (5.25)$$

then \mathcal{T} is called a **Gaussian channel**.

Here, we started with the requirement that \mathcal{T} already is a quantum channel, and Eq. (5.24) alone is clearly not sufficient for \mathcal{T} being one. Let us now formulate the converse, i.e., which S and f in Eq. (5.24) lead to a normalized completely positive map.

Normalization in the Heisenberg picture means $\mathcal{T}^*(\mathbb{1}) = \mathbb{1}$, which demands $f(0) = 1$ for the noise function. Then for a linear map \mathcal{T}^* , that acts according to Eq. (5.24), complete positivity means that f is $\Delta\sigma$ -twisted positive definite in the sense of Def. 81 [110, 107], where $\Delta\sigma$ is defined as

$$\Delta\sigma = \sigma_{\text{out}} - S^\top \sigma_{\text{in}} S. \quad (5.26)$$

So together, quasifree channels are completely parametrized by pairs consisting of a linear map S and a noise function f that simultaneously fulfill these properties.

Now let us discuss why f is called noise function and the connection between general quantum channels and the reversible, respectively, unitary subclass:

If the transformation matrix S is symplectic, Eq. (5.26) becomes trivial, that is $\Delta\sigma = 0$, which widens our choices for an admissible f to general positive definite functions. Especially we can choose $f = 1$, and the action of our quasifree channel can be implemented unitarily like in Eq. (5.23). In that case, its inverse is given by

$$S^{-1} = \sigma S^\top \sigma^{-1}. \quad (5.27)$$

This is what motivates the nomenclature of f as noise function: Unitary channels allow for the least amount of noise added, i.e., $f = 1$, while any S that differs from the symplectic case necessarily comes with additional noise being added to the system. This connection can be seen as just another point of view on the information-disturbance tradeoff discussed in Sect. 3.3.3.

As with the states, the subclass of Gaussian dynamics is notable. One can quickly check that Gaussian channels map Gaussian states to Gaussian states, which is sometimes used as a defining feature of these channels. Note that the terms quasifree and Gaussian are sometimes used interchangeably, see [22, Sect. 3.3.2]. The advantage of limiting to this particular case is straightforward: A lot of real-world experiments can be well approximated as Gaussian while, at the same time, the formalism gets even more straightforward. A Gaussian state described by the characteristic function in Eq. (5.20) that undergoes a Gaussian channel with the noise function in Eq. (5.25) changes its mean and covariance by the following simple transformation rule:

$$m \mapsto Sm + d, \quad \gamma \mapsto S^\top \gamma S + N. \quad (5.28)$$

Additionally, one can simplify the $\Delta\sigma$ -twisted positive definiteness of a Gaussian channel into the short expression

$$N + i\Delta\sigma \geq 0. \quad (5.29)$$

Note that this condition does not include the shift parameter d , so like most of the works on this topic, we will use centered states and channels where possible and assume $d = m = 0$ from the beginning. Comparing Eq. (5.29) to Eq. (5.21) underlines a general connection between states and channels in quantum theory: A quantum channel with a trivial input system is just another way of preparing a quantum state.

We close this review with a discussion of time-continuous dynamics. First note that concatenating two Gaussian channels described by (S_1, N_1) and (S_2, N_2) is again a Gaussian channel with

$$(S_1 S_2, N_2 + S_2^T N_1 S_2). \quad (5.30)$$

Going further by introducing the structure of a semigroup, this is likewise carried through to the matrices S_t and N_t , describing **one-parameter Gaussian semigroups** [111, 112]. The conditions for S_t and N_t defining such a semigroup are that they depend continuously on $t \in \mathbb{R}^+$, satisfy the semigroup rules

$$S_t + S_{t'} = S_{t+t'} \quad \text{with} \quad N_{t+t'} = N_t + S_t^T N_{t'} S_t \quad (5.31)$$

and fulfil the boundary conditions $S_0 = \mathbb{1}$ and $N_0 = 0$. These matrix semigroups have the generators \dot{S} and \dot{N} , and we have for the individual matrices

$$N = t\dot{N} + \mathbf{o}(t) \quad \text{and} \quad S = \mathbb{1} + t\dot{S} + \mathbf{o}(t), \quad (5.32)$$

respectively $S_t = \exp(t\dot{S})$ and $N_t = \int_0^t ds S_s^T \dot{N} S_s$. In the next section, we will connect these to the more commonly known description by a Lindblad generator on the operator level.

5.1.4 Generators of Gaussian quantum semigroups

Before we start the discussion of quantum-classical hybrids, we provide a helpful result regarding the dynamics of Gaussian quantum systems. As discussed earlier, a Gaussian semigroup on phase space is characterized by its action on Weyl operators

$$\mathcal{T}_t^*(W(\xi)) = \exp\left(-\frac{1}{2}\xi \cdot N_t \xi\right) W(S_t \xi) \quad (5.33)$$

and ignoring possible shifts, it is fully characterized by the matrices S_t and N_t .

While this description is elegant in several ways, the more commonly known and used one is the description by a Lindblad or master equation:

$$\begin{aligned} \mathcal{L}(A) &= K^* A + A K + \sum_i L_i^* A L_i \\ &= i[\hat{H}, A] + \sum_i L_i^* A L_i - \frac{1}{2}(L_i^* L_i A + A L_i^* L_i). \end{aligned} \quad (5.34)$$

Our aim is now to state the connection between the Lindblad generators for Gaussian semigroups and the representation on phase space, i.e., a way to translate between the equations Eq. (5.33) or Eq. (5.32) and the more commonly known Eq. (5.34).

The standing claim is that we can write Eq. (5.34) with Lindblad jump operators, which are linear in the field operators R , and a contraction generator or Hamiltonian, which is at most quadratic. While one finds this as the definition or remarks about the equivalence [111, 112], we were not able to find the connection between the master equation and the dynamics formulated on phase space in a straightforward fashion. The best we could find is written down in [113], where this kind of connection is made on the level of covariance matrices and is commented by:

... From the equations of motion of the canonical coordinates or the Majorana operators in the Heisenberg picture, one finds after a tedious but straightforward computation that the covariance matrix satisfies a closed set of equations of motion

...

Knowing that the calculation on the generator level has most likely been done before, we think that its importance as the link between these two different ways of describing the dynamics justifies a closer look. At first, we rephrase Eq. (5.34) more practically.

Let $\alpha, \beta \in \{1, 2, 3, \dots, d\}$ denote the spatial degrees of freedom, and for the number of jump operators L_i we take $i \in \{1, 2, 3, \dots, n\}$. We introduce the coefficient matrices H and L by the following definition:

$$\hat{H} = (1/2) \sum_{\alpha, \beta} H_{\alpha\beta} R_\alpha R_\beta \quad \text{and} \quad L_i = \sum_{\alpha} L_{i\alpha} R_\alpha. \quad (5.35)$$

Note that we can take the matrix H to be symmetric (hence real) since the antisymmetric part would only add commutators of R 's, hence a multiple of the identity to the Hamiltonian, which is irrelevant in the generator. We insert Eq. (5.35) into the generator in Eq. (5.34), which now reads

$$\begin{aligned} \mathcal{L}(A) &= \sum_{\alpha, \beta} \frac{i}{2} [H_{\alpha\beta} R_\alpha R_\beta, A] + \sum_{\alpha, \beta} \sum_i L_{i\alpha}^* R_\alpha A L_{i\beta} R_\beta \\ &\quad - \frac{1}{2} (L_{i\alpha}^* R_\alpha L_{i\beta} R_\beta A + A L_{i\alpha}^* R_\alpha L_{i\beta} R_\beta) \\ &= \sum_{\alpha, \beta} \frac{i}{2} H_{\alpha\beta} [R_\alpha R_\beta, A] + \frac{1}{2} \sum_{\alpha, \beta} \sum_i L_{i\alpha}^* L_{i\beta} (R_\alpha [A, R_\beta] + [R_\alpha, A] R_\beta). \end{aligned} \quad (5.36)$$

Next, we define the manifestly positive semi-definite d -dimensional square matrix M by

$$M_{\alpha\beta} = \sum_i L_{i\alpha}^* L_{i\beta}. \quad (5.37)$$

The Gaussian Lindblad generator is now characterized by the two matrices M and H

$$\mathcal{L}(A) = \frac{1}{2} \sum_{\alpha\beta} M_{\alpha\beta} (R_\alpha [A, R_\beta] + [R_\alpha, A] R_\beta) + i H_{\alpha\beta} [R_\alpha R_\beta, A]. \quad (5.38)$$

We will now connect the generators \dot{N}, \dot{S} of the matrix semigroups N_t and S_t from Eq. (5.32) to the matrices M and H from above.

Proposition 84 (Gaussian generators). *There is bijective correspondence between pairs of (M, H) and (\dot{S}, \dot{N}) of $2d \times 2d$ -matrices such that*

- M is complex valued with $M \geq 0$, and H real with $H^T = H$.
- \dot{S}, \dot{N} are real with $\dot{N} + \frac{i}{2} (\dot{S}^T \sigma + \sigma \dot{S}) \geq 0$,

given by

$$\begin{aligned} \dot{N} &= \sigma (\Re M) \sigma^T, & \dot{S} &= (-\Im M + H) \sigma, \\ H &= -\frac{1}{2} (\dot{S} \sigma - \sigma \dot{S}^T), & M &= \sigma \dot{N} \sigma^T + \frac{i}{2} (\dot{S} \sigma + \sigma \dot{S}^T). \end{aligned} \quad (5.39)$$

In this case, both sets of matrices determine the same quasifree dynamical semigroup, namely via Eq. (5.33) and the Lindblad generator in Eq. (5.38).

Proof. To prove the claimed connection in Eq. (5.39), we compare the action of the two different formulations of generators on Weyl operators. Let us start with the generator on phase space:

$$\mathcal{L}(W(\xi)) = \frac{d}{dt} \mathcal{T}^*(W(\xi)) \Big|_{t=0} = \left(-\frac{1}{2} \xi \cdot \dot{N} \xi \right) W(\xi) + \frac{d}{dt} W(\xi + t \dot{S} \xi) \Big|_{t=0}. \quad (5.40)$$

We define the vector $\eta = \dot{S} \xi$ and calculate the derivative in the second term. Here, we use the Weyl relation in two different ways:

$$\begin{aligned} \frac{d}{dt} W(\xi + t\eta) \Big|_{t=0} &= \frac{d}{dt} \exp\left(-\frac{it}{2} \sigma(\xi, \eta)\right) W(\xi) W(t\eta) \Big|_{t=0} \\ &= W(\xi) \left(-\frac{i}{2} \sigma(\xi, \eta) + i\eta \cdot R \right) \end{aligned} \quad (5.41)$$

$$\begin{aligned} &= \frac{d}{dt} \exp\left(\frac{it}{2} \sigma(\xi, \eta)\right) W(t\eta) W(\xi) \Big|_{t=0} \\ &= \left(\frac{i}{2} \sigma(\xi, \eta) + i\eta \cdot R \right) W(\xi). \end{aligned} \quad (5.42)$$

For later use we calculate the commutator between the Weyl operators and the field operators R by taking Eq. (5.41) – Eq. (5.42):

$$\begin{aligned} W(\xi) \left(-\frac{i}{2} \sigma(\xi, \eta) + i\eta \cdot R \right) - \left(\frac{i}{2} \sigma(\xi, \eta) + i\eta \cdot R \right) W(\xi) &= 0 \\ W(\xi) i\eta \cdot R - W(\xi) \frac{i}{2} \sigma(\xi, \eta) - i\eta \cdot R W(\xi) - \frac{i}{2} \sigma(\xi, \eta) W(\xi) &= 0 \\ W(\xi) i\eta \cdot R - i\eta \cdot R W(\xi) &= i\sigma(\xi, \eta) W(\xi) \\ [W(\xi), \eta \cdot R] &= \sigma(\xi, \eta) W(\xi). \end{aligned} \quad (5.43)$$

We insert the derivative calculated in Eq. (5.42) into Eq. (5.40) and arrive at the following term for the action of the semigroup on the phase space:

$$\mathcal{L}(W(\xi)) = \left(-\frac{1}{2} \xi \cdot \dot{N} \xi + \frac{i}{2} \sigma(\xi, \dot{S} \xi) \right) W(\xi) + i(\dot{S} \xi) \cdot R W(\xi). \quad (5.44)$$

In the second step, we calculate the action of the generator on the operator level given by the Lindblad equation in Eq. (5.38). We start by using the derivation property for commutators, that is $[R_\alpha R_\beta, A] = [R_\alpha, A] R_\beta - R_\alpha [A, R_\beta]$, and get the following form:

$$\mathcal{L}(A) = \frac{1}{2} \sum_{\alpha\beta} (M + iH)_{\alpha\beta} [R_\alpha, A] R_\beta + (M - iH)_{\alpha\beta} R_\alpha [A, R_\beta]. \quad (5.45)$$

Before we apply the generator to the Weyl operator $A = W(\xi)$, we use the commutator calculated in Eq. (5.43) and multiply it with the field operator R_α from the left

$$R_\alpha [W(\xi), R_\beta] = \sigma(\xi, \beta) R_\alpha W(\xi). \quad (5.46)$$

For the second commutator in Eq. (5.45), we multiply it from the right and use the known commutator:

$$[R_\alpha, W(\xi)] R_\beta = -[W(\xi), R_\alpha] R_\beta = -\sigma(\xi, \alpha) W(\xi) R_\beta. \quad (5.47)$$

We sort the Weyl operators to the right,

$$\begin{aligned} [R_\alpha, W(\xi)]R_\beta &= -\sigma(\xi, \alpha) (R_\beta W(\xi) + [W(\xi), R_\beta]) \\ &= -\sigma(\xi, \alpha) (R_\beta W(\xi) + \sigma(\xi, \beta) W(\xi)) \\ &= -\sigma(\xi, \alpha) R_\beta W(\xi) - \sigma(\xi, \alpha)\sigma(\xi, \beta)W(\xi), \end{aligned} \quad (5.48)$$

and insert the two commutators in Eq. (5.45), applying the generator to a Weyl operator:

$$\begin{aligned} \mathcal{L}(W(\xi)) &= \frac{1}{2} \sum_{\alpha\beta} (M + iH)_{\alpha\beta} (-\sigma(\xi, \alpha) R_\beta W(\xi) - \sigma(\xi, \alpha)\sigma(\xi, \beta)W(\xi)) \\ &\quad + (M - iH)_{\alpha\beta} (\sigma(\xi, \beta) R_\alpha W(\xi)). \end{aligned} \quad (5.49)$$

Before we can compare the action of the two generators, we need some further calculations. We start by relabelling the indices in the first term and ordering the whole expression by $W(\xi)$ and $R_\alpha W(\xi)$:

$$\begin{aligned} \mathcal{L}(W(\xi)) &= \frac{1}{2} \sum_{\alpha\beta} \left((M + iH)_{\alpha\beta} (-\sigma(\xi, \alpha)\sigma(\xi, \beta)) \right) W(\xi) \\ &\quad + \left((M + iH)_{\beta\alpha} (-\sigma(\xi, \beta)) + (M - iH)_{\alpha\beta} \sigma(\xi, \beta) \right) R_\alpha W(\xi). \end{aligned} \quad (5.50)$$

We simplify the expression by using that M is positive, so $M_{\alpha\beta} = \overline{M}_{\beta\alpha}$ and the fact that H is symmetric, so we arrive at

$$\begin{aligned} \mathcal{L}(W(\xi)) &= \frac{1}{2} \sum_{\alpha\beta} \left(- (M + iH)_{\alpha\beta} \sigma(\xi, \alpha)\sigma(\xi, \beta) \right) W(\xi) \\ &\quad + \left(((\overline{M} - iH - M - iH)_{\beta\alpha}) \sigma(\xi, \beta) \right) R_\alpha W(\xi). \end{aligned} \quad (5.51)$$

Getting closer to the expression in Eq. (5.44), we write out the symplectic form with the symplectic matrix σ , i.e., $\sigma(\xi, \eta) = \sum_{\alpha\beta} \xi_\alpha \sigma_{\alpha\beta} \eta_\beta = \xi \cdot \sigma \eta$ and $\sigma(\xi, e_\alpha) = \sum_\gamma \xi_\gamma \sigma_{\gamma\alpha}$. With this, reordering and the fact that $-M + \overline{M} = -2i\Im m M$, we get the following intermediate result for the generator from the Lindblad equation:

$$\begin{aligned} \mathcal{L}(W(\xi)) &= \sum_{\alpha\beta\gamma\delta} \left(\xi_\gamma \sigma_{\gamma\alpha} \frac{1}{2} (M + iH)_{\alpha\beta} \sigma_{\beta\delta} \xi_\delta \right) W(\xi) \\ &\quad + \left(\xi_\gamma \sigma_{\gamma\beta} (-i\Im m M - iH)_{\beta\alpha} \right) R_\alpha W(\xi). \end{aligned} \quad (5.52)$$

Similar to the equation above, we rewrite the generator on phase space, that is Eq. (5.44), in components. Here we have to take care of two different types of \dot{S} , one occurring from a symplectic form and one from a scalar product, where we get an \dot{S}^T . Now the analog to Eq. (5.52) is

$$\mathcal{L}(W(\xi)) = \sum_{\alpha\gamma\delta} \left(-\frac{1}{2} \xi_\gamma \dot{N}_{\gamma\delta} \xi_\delta + \frac{i}{2} \xi_\gamma \sigma_{\gamma\alpha} \dot{S}_{\alpha\delta} \xi_\delta \right) W(\xi) + i(\xi_\gamma \dot{S}_{\gamma\alpha}^T) R_\alpha W(\xi). \quad (5.53)$$

Finally, we can do the coefficient comparison, which yields

$$\sum_{\alpha\gamma\delta} i(\xi_\gamma \dot{S}_{\gamma\alpha}^T) R_\alpha W(\xi) = \sum_{\alpha\gamma\delta} \left(\xi_\gamma \sigma_{\gamma\beta} (-i\Im m M - iH)_{\beta\alpha} \right) R_\alpha W(\xi) \quad (5.54)$$

and

$$\sum_{\alpha\gamma\delta} \left(-\frac{1}{2} \xi_\gamma \dot{N}_{\gamma\delta} \xi_\delta + \frac{i}{2} \xi_\gamma \sigma_{\gamma\alpha} \dot{S}_{\alpha\delta} \xi_\delta \right) W(\xi) = \sum_{\alpha\beta\gamma\delta} \left(\xi_\gamma \sigma_{\gamma\alpha} \frac{1}{2} (M + iH)_{\alpha\beta} \sigma_{\beta\delta} \xi_\delta \right) W(\xi). \quad (5.55)$$

From Eq. (5.54), we directly get the connection between \dot{S} and M, H :

$$\dot{S} = (-\Im m M + H)\sigma. \quad (5.56)$$

Note that Eq. (5.55) is of the form $\xi^T(\dots)\xi$, so for the coefficient comparison, we can only evaluate the symmetric part, i.e.,

$$\text{sym}(-\dot{N} + i\sigma\dot{S}) = \text{sym}(\sigma(M + iH)\sigma). \quad (5.57)$$

From this we can read of the relation between \dot{N} and M, H :

$$\frac{1}{2}(-\dot{N} + i\sigma\dot{S}) + \frac{1}{2}(-\dot{N} + i\sigma\dot{S})^T = \frac{1}{2}(\sigma(M + iH)\sigma) + \frac{1}{2}(\sigma(M + iH)\sigma)^T \quad (5.58)$$

$$-\dot{N} - \frac{i}{2}(\dot{S}^T\sigma - \sigma\dot{S}) = \frac{1}{2}\sigma(M + M^T)\sigma + i\sigma H\sigma \quad (5.59)$$

Using that $M^T = \overline{M}$, so $M + \overline{M} = 2\Re e M$ and the fact that \dot{S}, \dot{N}, H and σ are real matrices, we can compare the real part

$$\dot{N} = \sigma \Re e M \sigma^T \quad (5.60)$$

and the imaginary part

$$H = -\frac{1}{2}(\dot{S}\sigma - \sigma\dot{S}^T). \quad (5.61)$$

At last we express M by \dot{S} and \dot{N} . From Eq. (5.60) we get $\Re e M = -\sigma\dot{N}\sigma$ and from Eq. (5.56) we have $\Im m M = \dot{S}\sigma + H$. So we arrive at the stated form for M :

$$M = \sigma\dot{N}\sigma^T + i(\dot{S}\sigma + H) = \sigma\dot{N}\sigma^T + \frac{i}{2}(\dot{S}\sigma + \sigma\dot{S}^T). \quad (5.62)$$

□

This correspondence nicely illustrates the advantages as well as disadvantages of the formalism. Being able to describe the dynamics purely on the level of matrices heavily simplifies the mathematical toolbox but might lose some of the intuition connected to Lindblad generators. In this spirit, the above proposition hopefully helps any interested reader as a starting point for a transition to the quasifree world.

5.1.5 Notes and references

While the canonical commutation relation is closely tied to the development of quantum mechanics, the C*-algebraic aspects and the quasifree formulation had a surge of interest in the second half of the twentieth-century [114, 112, 107, 115, 108, 101, 102]. For a mathematical textbook introduction to the CCR-algebra and quasifree systems, we recommend the work *An Invitation to the Algebra of Canonical Commutation Relations* [95] by Petz. A textbook with more focus on physics is

Symplectic geometry and quantum mechanics [105] by de Gosson or Chap. 5 in *Probabilistic and statistical aspects of quantum theory* [18] by Holevo, which focuses especially on the Gaussian subset of the quasifree world.

Around the turn of the millennium, this area had another spike in interest from the quantum information community, for example [110, 104, 111, 113, 116, 117, 118], some of which should have congregated in a textbook [92]. Nevertheless, there are several reviews about Gaussian quantum information [119, 120], that also contain more details, especially about the practical application.

Let us end this section with some comments about the notational choices made in this work. The strength of this formulation is the transition from dealing with unbounded operators to linear algebra on a finite-dimensional phase space. On the one hand, problems become mathematically much easier to handle. On the other hand, one may lose some of the physical intuition if not used to this formulation. At the same time, there is no standard agreement on a general convention about the factors and the placement of σ through the formalism, making using results from different works harder than necessary. A reason for this may be that there is no commonly accepted *standard* textbook on this topic that could have delivered such a convention.

This being said, one common convention is to include σ in Eq. (5.8), i.e., one uses $\xi \cdot \sigma R$ in the exponent of a Weyl operator. In phase spaces with a proper symplectic form, this form is often used to identify the space with its dual (see [121] or [105, Sect. 1.1.1]). Unfortunately, this is the one convention we can not follow, as should be clear after the next section.

5.2 The Hybrid Phase and State Space

5.2.1 Adding the classical system

... the same, only different

Let us now introduce the classical side of our hybrid systems. The basic principle is remarkably simple and appeared in several levels of detail thorough the literature [114, 82, 95]. We eliminate the assumption that σ in the canonical commutation relation should be symplectic, i.e., non-degenerate. With the suitable choice of canonical coordinates, this process can be easily thought of as follows:

$$\sigma_{\text{quantum}} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 0 & \mathbb{1}_n & 0 \\ -\mathbb{1}_n & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) = \sigma_{\text{hybrid}}. \quad (5.63)$$

Now, the degenerate part of σ houses our classical system. Here, we only added one classical degree of freedom, i.e., $s = 1$, but while the dimension of the quantum part is always $2n$ and thus even, the dimension of the degenerate or classical part of σ can be extended to any integer $s \in \mathbb{N}$. Also, we can include the limiting cases $n = 0$ or $s = 0$, which eliminates the quantum or classical part of our hybrid.

The hybrid phase space

We take a system of n quantum canonical degrees of freedom and s classical ones. Here, the first advantage of this approach is that the presence of the classical subsystem requires only little to no changes in the notation at all. Dropping the assumption of non-degeneracy will allow non-zero null vectors for σ . In the standard basis, this means that the $2n$ variables p and q are now augmented by $0 \leq s < \infty$ unpaired classical variables $x \in \mathbb{R}^s$. These classical phase space vectors can be considered position variables without corresponding momenta or vice versa.

In this extended phase space $\Xi = \mathbb{R}^{2n+s}$ we now have the phase space vectors as sets of triples $\xi = (q, p, x)$, where an *extended symplectic form* can be defined as

$$\sigma((q, p, x), (q', p', x')) = p \cdot q' - q \cdot p' = \sum_{ij} \xi_i \sigma_{ij} \xi'_j. \quad (5.64)$$

This extended version remains antisymmetric bilinear, but as we explicitly introduced null vectors, it is no longer non-degenerate and hence not symplectic. If we want to emphasize the generalization, we call the pair (Ξ, σ) a **hybrid phase space**. As this will be the standard case from now on, we often drop the word hybrid and instead denote the respective special cases explicitly as quantum or classical. Also, we will consider arbitrary linear maps on these (hybrid) phase spaces so we can adopt a basis-free view, where a phase space is just a real vector Ξ space with antisymmetric bilinear form σ . In general, the classical part is always singled out as the space of null vectors:

$$\Xi_0 = \{\xi \in \Xi \mid \forall \eta : \sigma(\xi, \eta) = 0\}. \quad (5.65)$$

We can split $\Xi = \Xi_1 \oplus \Xi_0$, where Ξ_1 is a suitable subspace on which σ is non-degenerate as in Eq. (5.4), thus a standard quantum system.

The direct sum symbol indicates a unique decomposition $\xi = \xi_1 + \xi_0$ with $\xi_i \in \Xi_i$ for any vector ξ , and that the form σ also has a block structure, as in the coordinatization of Eq. (5.63). However, other than an orthogonal complement, the quantum part Ξ_1 is not uniquely defined, i.e., there are σ -preserving linear maps changing the decomposition. Some of our constructions depend on the decomposition $\Xi = \Xi_1 \oplus \Xi_0$, but we usually do not show explicitly that this dependence is harmless, and the necessary isomorphisms will be noiseless in the terminology of Sect. 5.4. Nevertheless, we will specify how the σ -preserving linear maps can be characterized in the hybrid scenario.

Automorphisms

As we have stated, the above definitions seem to depend on the precise splitting of $\Xi = \Xi_1 \oplus \Xi_0$. Nevertheless, it is often useful to think of Ξ and the bilinear antisymmetric form σ as a structure given in a coordinate-free way, so the splitting introduces an arbitrary choice. In this decomposition, Ξ_0 is the null space of σ , which is uniquely defined. However, since Euclidean orthogonality has no meaning in this structure, the complement Ξ_1 involves the choice of a linear projection $\pi : \Xi \rightarrow \Xi_0$. So

$$\pi' := \pi + \pi a(\mathbb{1} - \pi) \quad (5.66)$$

with an arbitrary linear operator a should do just as well. The two definitions of complement are connected by a shear transformation. This turns out to be the basic kind of automorphism of (Ξ, σ) , analogous to the symplectic transformations when σ is non-degenerate:

Lemma 85. *Given a hybrid phase space (Ξ, σ) as in Eq. (5.63). Let A be a linear map on Ξ with block decomposition*

$$A = \left(\begin{array}{cc|c} A_{11} & A_{12} & A_{10} \\ A_{21} & A_{22} & A_{20} \\ \hline A_{01} & A_{02} & A_{00} \end{array} \right). \quad (5.67)$$

Then A preserves σ , in the sense that $A^\top \sigma A = \sigma$, if and only if $A_{10} = A_{20} = 0$, and $A_Q = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is symplectic on (Ξ_1, σ_1) .

Proof. We write down the matrix product:

$$A^\top \sigma A = \begin{pmatrix} A_{11}^\top A_{21} - A_{21}^\top A_{11} & A_{11}^\top A_{22} - A_{21}^\top A_{12} & A_{11}^\top A_{20} - A_{21}^\top A_{10} \\ A_{12}^\top A_{21} - A_{22}^\top A_{11} & A_{12}^\top A_{22} - A_{22}^\top A_{12} & A_{12}^\top A_{20} - A_{22}^\top A_{10} \\ A_{10}^\top A_{21} - A_{20}^\top A_{11} & A_{10}^\top A_{22} - A_{20}^\top A_{12} & A_{10}^\top A_{20} - A_{20}^\top A_{10} \end{pmatrix} \stackrel{!}{=} \sigma. \quad (5.68)$$

The upper left 2×2 -submatrix, containing the blocks $A_{11}, A_{12}, A_{21}, A_{22}$, just tells us that this part of A has to be symplectic, just as if the classical block is zero-dimensional. Note that none of the blocks A_{0i} for $i = 1, 2, 0$ is constrained, so these will remain arbitrary.

Of the remaining five equations, two are redundant because the overall expression is antisymmetric. What remains are the equations $A_{11}^\top A_{20} - A_{21}^\top A_{10} = 0$, $A_{12}^\top A_{20} - A_{22}^\top A_{10} = 0$ and $A_{10}^\top A_{20} - A_{20}^\top A_{10} = 0$.

We can rewrite the first two as a matrix equation, i.e.,

$$\begin{pmatrix} A_{22}^\top & -A_{12}^\top \\ -A_{21}^\top & A_{11}^\top \end{pmatrix} \begin{pmatrix} A_{10} \\ A_{20} \end{pmatrix} = 0. \quad (5.69)$$

Because A_Q is symplectic, it is invertible, and it turns out that the coefficient matrix in Eq. (5.69) is exactly A_Q^{-1} . Hence, by multiplying Eq. (5.69) with A_Q , we find that $A_{10} = A_{20} = 0$. The final equation from the lower right matrix element is then automatically satisfied. \square

Note that in contrast to the symplectic case, we cannot conclude that A is invertible, which in this context is equivalent to the invertibility of A_{00} .

The case $A_Q = \mathbb{1}$, $A_{00} = \mathbb{1}$ corresponds exactly to the shear transformations mentioned above, with $a = (A_{01}, A_{02}) : \Xi_1 \rightarrow \Xi_0$.

Another consequence of this lemma is the confirmation that in contrast to quantum canonical variables, the classical variables can be uniformly stretched $A_{00} = \lambda \mathbb{1}$ without changing anything. This is in sharp contrast to approaches (see Sect. 3.4), in which the classical variables have their own symplectic form, which is used for the generation of classical Hamiltonian dynamics.

The hybrid commutation relations

After the phase space, we need to introduce our hybrid field operators. Here we extend the quantum set of operators $(Q_1, \dots, Q_n, P_1, \dots, P_n)$ by s classical operators X_i . Again, we refer to the classical X_i as operators out of convenience, although *random variables* might be more appropriate. These s classical components R_i with $i \in \{2n+1, \dots, 2n+s\}$ commute with all others, and besides their addition, we can keep the standard notation for the field operators

$$R = (R_1, \dots, R_{2n+s}) = (Q_1, \dots, Q_n, P_1, \dots, P_n, X_1, \dots, X_s) \quad (5.70)$$

and the commutation relations remain

$$[R_j, R_k] = i\sigma_{jk}\mathbb{1}. \quad (5.71)$$

Alongside, we can keep the notation for the phase space translations from Eq. (5.7) as $\alpha_\xi(R_j) = R_j + \xi_j\mathbb{1}$. Also, for the exponentiated versions of our operators and relations, we need to change very little. The Weyl operators, even though with classical arguments they may be more like functions, are kept as

$$W(\xi) = \exp(i\xi \cdot R) \quad (5.72)$$

and the Weyl relations and canonical commutation relations stay

$$W(\xi)W(\eta) = e^{-\frac{i}{2}\sigma(\xi,\eta)}W(\xi + \eta) \quad (5.73)$$

$$= e^{-i\sigma(\xi,\eta)}W(\eta)W(\xi). \quad (5.74)$$

Finally the CCR-algebra, i.e., the universal C*-algebra of these generators and relations, remains $\text{CCR}(\Xi, \sigma)$.

While the idea and the adaptation of the notation are pretty straightforward, the challenges arise if we follow this through. More specifically, we need to check what consequences this change brings to the triad of preparation, dynamics, and measurement and their mathematical description.

Notation

Let us close this section with a discussion about the notational conventions that we began at the end of Sect. 5.1.5, i.e., the absence of σ in Eq. (5.72). In our hybrid version, this would set the classical contribution to zero, so any constructions using this will not work. This means that in a coordinate-free spirit, the variable ξ in Eq. (5.72) does not lie in the phase space Ξ but in its dual $\widehat{\Xi}$. We will keep the notation simple by nevertheless identifying both spaces with \mathbb{R}^{2n+s} and using a dot for the standard scalar product. This convention will suffice for almost all of our purposes, that is unless we explicitly distinguish some components as position-like and others as momentum-like. In those rare cases, mainly the instrument in Sect. 5.5.7, we try to help readers keeping track by using corresponding letters:

For the phase space Ξ , and therefore also for the arguments of α_ξ we already introduced in the ordering

$$(q, p, x) \tag{5.75}$$

for the groups of $n + n + s$ variables. For the dual space $\widehat{\Xi}$, i.e., in the arguments of Weyl operators and characteristic functions, it is then suggestive to use the ordering

$$(\hat{p}, \hat{q}, k). \tag{5.76}$$

Here, we take into account that position space and momentum space are dual vector spaces, and k is the wave-number variable dual to classical shifts, as customary in $e^{ik \cdot x}$. In order to keep all appearances of the symplectic matrix explicit, we do not change one of the signs for elements in $\widehat{\Xi}$.

5.2.2 Standard representations

Defining the CCR-algebra $\text{CCR}(\Xi, \sigma)$ with an extended symplectic form is one thing, describing its representations is another. For this, we will start with establishing a hybrid analog of von Neumann's result (see Thm. 79).

The first step is to fix the analog of the Schrödinger representation from the quantum case: An explicit choice of Weyl operators satisfying the relation Eq. (5.73), initially without the claim that *all* good representations look like that. From the outset, it is clear that there will not be a unique standard representation in contrast to the quantum case due to the classical part of our hybrid.

The Hilbert space for an irreducible representation of a commutative C^* -algebra is one-dimensional. So already, there is no such uniqueness for a purely classical system, basically because there are uncountably many inequivalent irreducible representations of the classical observable algebra (labeled by the points of Ξ_0).

This non-uniqueness forces the choice of a measure μ on the classical subspace Ξ_0 , but in the end, that is all. Choosing μ will also be enough for the hybrid case.

Definition 86. Let $\Xi = \Xi_1 \oplus \Xi_0$ be a hybrid phase space with antisymmetric form $\sigma = \sigma_1 \oplus 0$. Then a **standard representation** is a representation of the Weyl relations in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes L^2(\Xi_0, \mu)$, where μ is some regular Borel measure on Ξ_0 , and \mathcal{H}_1 is the Hilbert space of the Schrödinger representation $W_1 : \Xi_1 \rightarrow \mathcal{B}(\mathcal{H}_1)$ for (Ξ_1, σ_1) . The Weyl operators are given by

$$W(\xi_1 \oplus \xi_0) = W_1(\xi_1) \otimes W_0(\xi_0), \quad (5.77)$$

where $W_0(\xi_0)$ is the multiplication operator

$$(W_0(\xi_0)\phi)(x) = e^{i\xi_0 \cdot x} \phi(x) \quad (5.78)$$

for $\phi \in L^2(\Xi_0, \mu)$ and $x \in \Xi_0$. A state on the CCR-algebra is called **standard** if it is given by a density operator on \mathcal{H} in a standard representation.

We remark that the standard representation depends on μ only up to equivalence: If two measures μ and μ' have the same null sets, the according Hilbert spaces $L^2(\Xi_0, \mu)$ and $L^2(\Xi_0, \mu')$ are connected by a unitary transformation that acts by multiplication with $\sqrt{d\mu/d\mu'}$ and in particular, intertwines the multiplication operators in Eq. (5.78) [8, Cor. III.1.5]. Therefore, we can always choose μ to be a probability measure, typically the classical marginal of a state under consideration. Another typical choice is the Lebesgue measure, which we will have a closer look at in Sect. 5.3.3. Finally, we will strive to get rid of the μ -dependence in the definition of observable algebras completely in Sect. 5.3.

The von Neumann algebra generated by a standard representation is

$$\mathcal{M}_\mu = \mathcal{B}(\mathcal{H}_1) \bar{\otimes} L^\infty(\Xi_0, \mu), \quad (5.79)$$

where $\bar{\otimes}$ denotes the tensor product of von Neumann algebras, see Sect. 3.2.3. Indeed, since the $W_1(\xi_1)$ are irreducible on \mathcal{H}_1 , they generate $\mathcal{B}(\mathcal{H}_1)$ as a von Neumann algebra, and similarly the Weyl multiplication operators generate the maximal abelian algebra of all multiplication operators M_f with $f \in L^\infty(\Xi_0, \mu)$, which is isomorphic to $L^\infty(\Xi_0, \mu)$ [8, Thm. III.1.2]. Putting this together using the commutation theorem for tensor products [8, Thm. IV.5.9] gives Eq. (5.79). Note that this algebra still depends on μ because in $L^\infty(\Xi_0, \mu)$ functions, which only agree μ -almost everywhere, are identified by definition. By identifying $A \otimes f$ with the function $x \mapsto f(x)A$ we can think of the elements of \mathcal{M}_μ as measurable $\mathcal{B}(\mathcal{H}_1)$ -valued functions on Ξ_0 .

Standard states

By the above definition, standard states are normal states on \mathcal{M}_μ , hence elements of the predual $\mathcal{T}^1(\mathcal{H}_1) \otimes L^1(\Xi_0, \mu)$, where $\mathcal{T}^1(\mathcal{H}_1)$ denotes the trace class [8, Thm. IV.7.17]. They can hence be decomposed as

$$\langle \omega, A \otimes f \rangle = \int \mu(dx) c(x) f(x) \text{tr}(\rho_x A), \quad (5.80)$$

where $c\mu$ is the probability measure determining the classical marginal, i.e., the expectations of multiplication operators, and $x \mapsto \rho_x$ is a measurable family of density operators. The factor $c(x)$ is introduced to allow that $\text{tr} \rho_x = 1$ for all x .

When we consider a particular state and its GNS-representation, we usually take μ directly as the marginal of that state, i.e., set $c(x) \equiv 1$. The required measurability conditions for the family of states ρ_x are described in [8, Sect. IV.7].

The definition of standard states brings in dependence on μ so that it is not a priori clear that convex combinations of standard states are standard. However, the integral decomposition in Eq. (5.80) makes clear that for a countable convex combination $\rho = \sum_j \lambda_j \rho_j$ we can take $\mu = \sum_j \lambda_j c_j \mu_j$, and set h_j to be the Radon-Nikodym derivative of $\lambda_j c_j \mu_j$ with respect to μ . Note that $0 \leq h_j(x) \leq 1$, and $\sum_j h_j = 1$. Then $\rho_0 = 1$ and $\rho_x = \sum_j h_j \rho_{x,j}$. In particular, a normal state in a direct sum of standard representations can be rewritten as a state using just a single summand, i.e., it is also standard in the sense of the above definition.

This argument also shows that the von Neumann algebra approach to hybrids can be made to work on larger and larger sets of states: If needed, one can consider any countable (and thereby any norm separable) family of states as absolutely continuous with respect to a common reference measure.

However, the set Ξ_0 has uncountably many points and hence point measures, which all have norm distance 2, so the set of measures on Ξ_0 is not norm separable. This means there is no single standard representation that can be used for *all* practical purposes. One could represent a single observable F by a net of functions

$$F_\mu \in \mathcal{B}(\mathcal{H}) \otimes L^\infty(\Xi_0, \mu), \quad (5.81)$$

each defined up to μ -a.e. equality. Then indices are ordered by absolute continuity $\mu \ll \nu$, i.e., ν has fewer null sets than μ , and in this case, F_ν is more sharply defined than F_μ . There is no natural limit to such nets because we cannot include all the uncountably many point measures. However, the notion of universally measurable sets and functions (see Sect. 5.3) does allow us to get rid of this.

Since standard states thus form a convex set, it makes sense to ask for the extreme points, i.e., the **pure states**. These are readily characterized:

Lemma 87. *A standard state ω on the CCR-algebra is extremal if and only if there is a point $x \in \Xi_0$ and a unit vector $\phi \in \mathcal{H}_1$ such that in the decomposition Eq. (5.80) $\mu = \delta_x$ is a point measure and $\rho_x = |\phi\rangle\langle\phi|$.*

Proof. Suppose that ω is extremal. Then let $f \in L^\infty(\Xi_0)$ with $\varepsilon < f < 1 - \varepsilon$ for some $\varepsilon > 0$. The state ω is then decomposed into the sum of two positive functionals

$$\omega(X) = \omega(f) \frac{\omega(fX)}{\omega(f)} + \omega(1-f) \frac{\omega((1-f)X)}{\omega(1-f)}. \quad (5.82)$$

This is a convex combination of states, so by extremality, the two states have to be proportional, i.e., $\omega(fX) = \lambda\omega(X)$ for all X . This forces $\lambda = \omega(f)$, by putting $X = \mathbb{1}$, and hence we conclude that $f = \omega(f)\mathbb{1}$ almost everywhere with respect to μ . Hence, μ is a point measure at some point x . The choice of ρ_y for $y \neq x$ is irrelevant because the whole complement of $\{x\}$ has measure zero. The state ρ_x is now given by a density operator, which has to be extremal as well, so $\rho_x = |\phi\rangle\langle\phi|$. \square

Note that a state ω may have no extremal components, i.e., no extreme points ω' such that $\omega \geq \lambda\omega'$ with $\lambda > 0$. Indeed, this will be the case whenever the measure

μ has no atoms (points of non-zero measure). Therefore, it is not a priori clear in which sense standard states can be decomposed into extreme points. This will be clarified in Sect. 5.2.3, where it will be seen that the standard states are the state space of a certain C*-algebra, so the convex combinations of extreme points are dense in a suitable weak* topology.

The Hybrid Uniqueness Theorem

It is straightforward to check that in the standard representation, $\xi \mapsto W(\xi)$ is continuous with respect to the strong operator topology. It turns out that this characterizes standard representations. This is the main content of the following Theorem, which is very close in its formulation and its proof to von Neumann's famous result [97].

Theorem 88 (Hybrid Uniqueness Theorem). *Every representation of the Weyl relations on a Hilbert space, for which the $\xi \mapsto W(\xi)$ is continuous in the strong operator topology, is unitarily equivalent to a direct sum of standard representations.*

In the literature, it is traditional [122] to use a weaker continuity condition, which does not demand joint continuity of W in all $2n + s$ variables in $\xi \in \Xi$, but only along one-dimensional subspaces. This is the minimum required to get self-adjoint canonical operators and is usually called *regularity* [42, 82]. This weaker version avoids some of the topological subtleties of infinite-dimensional Ξ , but for the finite-dimensional case of this work, it is still sufficient, and there is no difference.

Proof. Consider a strongly continuous representation W on a Hilbert space \mathcal{H} . For the most part, we will only need to use the representation $\xi_1 \mapsto W(\xi_1 \oplus 0)$ of the subgroup Ξ_1 . Following von Neumann, and even his notation up to a factor 2π , we introduce a Gaussian function $a : \Xi_1 \rightarrow \mathbb{C}$ and the operator

$$A = \int d\xi_1 a(\xi_1) W(\xi_1 \oplus 0). \quad (5.83)$$

The integral exists as a strong integral because W is continuous. Because a is integrable, A is a bounded operator. With von Neumann's choice, it is even a projection, and in the Schrödinger representation, it is just the one-dimensional projection $|\Omega\rangle\langle\Omega|$ onto the harmonic oscillator ground state vector $\Omega \in \mathcal{H}_1$. Since algebraic relations between A and anything in $\text{CCR}(\Xi_1, \sigma_1)$ are the same in any representation, it is hardly a surprise that we have

$$AW(\xi_1 \oplus 0)A = \langle\Omega|W_1(\xi_1)|\Omega\rangle A =: \chi(\xi_1)A, \quad (5.84)$$

where W_1 is the Schrödinger representation and $\chi(\xi_1) = \exp(-1/4 \xi_1^2)$ is a Gaussian, the characteristic function of the oscillator ground state. But, of course, one can also show this (as von Neumann does) by explicit computation based on the Weyl relations.

It is a key part of von Neumann's argument that A cannot vanish for any continuous representation of Ξ_1 . Indeed, in such a representation also $W(\eta \oplus 0)AW(\eta \oplus 0)^*$

would vanish for all η , which is exactly of the form in Eq. (5.83), with a kernel function

$$a_\eta(\xi_1) = \exp(i\eta \cdot \sigma \xi_1) a(\xi_1). \quad (5.85)$$

As a function of η , the integral of the modified Eq. (5.83) is thus the Fourier transform of an operator-valued L^1 -function, hence vanishes only if a does, which is false.

Consider now the subspace $\mathcal{H}_0 := A\mathcal{H}$ and the set M of vectors of the form $W(\xi_1 \oplus 0)\psi_0$ for $\xi_1 \in \Xi_1$ and $\psi_0 = A\psi_0 \in \mathcal{H}_0$. We claim that its linear span is dense. For if $\Psi \in \mathcal{H}$ were orthogonal to M , we would have that $\langle AW(\xi_1 \oplus 0)\Psi, A\psi_0 \rangle = 0$ for all ψ_0 , so A would vanish on the cyclic sub-representation space of Ξ_1 generated by Ψ , contradicting von Neumann's result $A \neq 0$.

We define a function $U : M \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_0$ by

$$UW(\xi_1 \oplus 0)\psi_0 = W_1(\xi_1)\Omega \otimes \psi_0. \quad (5.86)$$

Scalar products between different vectors on M are preserved, so, in particular, it sends a linear combination representing the null vector again to a linear combination with vanishing norm. That is, it extends to a linear operator on the algebraic linear span of M . Clearly, this extension is isometric as well, so extends by continuity to \mathcal{H} . Hence, Eq. (5.86) defines an isometry $U : \mathcal{H} \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_0$. It is also onto because the vectors $W_1(\xi_1)\Omega$ span \mathcal{H}_1 . To summarize: U is unitary. Now, since A commutes with $W(0 \oplus \eta_0)$ by virtue of the hybrid commutation relations, we can replace ψ_0 in Eq. (5.86) by $W(0 \oplus \eta_0)\psi_0$, and upgrade that equation to a full intertwining relation on M :

$$\begin{aligned} UW(\eta_1 \oplus \eta_0)W(\xi_1 \oplus 0)\psi_0 &= UW(\eta_1 \oplus 0)W(\xi_1 \oplus 0)(W(0 \oplus \eta_0)\psi_0) \\ &= W_1(\eta_1)W_1(\xi_1)\Omega \otimes W(0 \oplus \eta_0)\psi_0 \\ &= W_1(\eta_1) \otimes W(0 \oplus \xi_0)UW(\xi_1 \oplus 0)\psi_0. \end{aligned} \quad (5.87)$$

Hence $W(\eta_1 \oplus \eta_0) = U^*W_1(\eta_1) \otimes W(0 \oplus \eta_0)U$.

To complete the proof, it suffices to observe that the strongly continuous representation $\xi_0 \mapsto W(0 \oplus \xi_0)$ of the group $\Xi_0 \cong \mathbb{R}^s$ on \mathcal{H}_0 can be decomposed into a direct sum of cyclic ones, and the cyclic representations are of the form given in Def. 86. To see that this decomposition works together correctly with von Neumann's construction for Ξ_1 was the main reason to include an abridged version of his argument. \square

5.2.3 The hybrid state space

Given that our basic definitions remain basically the same, a state on the CCR-algebra is still completely determined by its expectations on Weyl operators, i.e., its characteristic function

$$\chi(\xi) = \omega(W(\xi)), \quad (5.88)$$

which has already been introduced for purely quantum systems in Sect. 5.1.2. Again, the natural question arises: Which are valid characteristic functions in our now extended hybrid scenario? This demands unifying the quantum version (Thm. 82) with its well-known classical analog: In the purely classical case, this is known as Bochner's Theorem (sometimes: Bochner-Khintchine Theorem [18]). Its hybrid version (also in [82], see Sect. 3.4) perfectly links these two.

Theorem 89 (Hybrid Bochner Theorem). *Let Ξ be a vector space with antisymmetric form σ . Then a function $\chi : \Xi \rightarrow \mathbb{C}$ is the characteristic function of a standard state on $\text{CCR}(\Xi, \sigma)$ if and only if it is*

- (1) *continuous,*
- (2) *normalized, $\chi(0) = 1$, and*
- (3) *σ -twisted positive definite, which means that, for any choice ξ_1, \dots, ξ_N , the $N \times N$ -matrix*

$$M_{kl} = \chi(-\xi_k + \xi_l) e^{-\frac{i}{2}\sigma(\xi_k, \xi_l)} \quad (5.89)$$

is positive semi-definite.

Proof. Just conditions (2) and (3) are equivalent to ω being a state on the CCR-algebra. The positive definiteness condition is precisely equivalent to $\omega(A^*A) \geq 0$, where $A = \sum_i c_i W(\xi_i)$, and the Weyl relations are used. By the GNS-construction, every positive linear functional comes from a Hilbert space representation, and by definition of the CCR algebra as the universal C*-algebra of the Weyl relations, the state thus extends to the whole algebra.

Continuity of χ for a standard ω is obvious because a standard representation is strongly continuous. Conversely, suppose that χ is continuous, and let $\Omega \in \mathcal{H}_\omega$ denote the cyclic vector of the GNS-representation of ω . Then

$$\xi \rightarrow \langle \pi_\omega(W(\eta_1))\Omega, \pi_\omega(W(\xi)) \pi_\omega(W(\eta_2))\Omega \rangle = \langle \Omega, \pi_\omega(W(-\eta_1)W(\xi)W(\eta_2)) \Omega \rangle \quad (5.90)$$

$$= \exp\left(\frac{i}{2}(\sigma(\xi, \eta_2) + \sigma(-\eta_1, \xi + \eta_2))\right) \langle \Omega, \pi_\omega(W(\xi - \eta_1 + \eta_2)) \Omega \rangle \quad (5.91)$$

$$= \chi(\xi - \eta_1 + \eta_2) \exp\left(\frac{i}{2}(\sigma(\xi, \eta_2) - \sigma(\eta_1, \xi) - \sigma(\eta_1, \eta_2))\right) \quad (5.92)$$

is continuous. Since the Weyl operators are bounded, this extends to the norm limits of linear combinations of $\pi_\omega(W(\eta_2))\Omega$, which is, by definition, all of \mathcal{H}_ω . Hence, $\pi_\omega(W(\cdot))$ is weakly continuous, but for unitary operators, this is the same as strong continuity. Hence, ω is normal in a strongly continuous representation.

By Thm. 88, this is a direct sum of standard representations, and by the argument preceding it, we conclude that ω itself is standard. \square

To see the power of the continuity condition, it may be helpful to point out some rather wild states of the CCR-algebra. Indeed, this algebra is just the hybrid version of the almost periodic functions, in the precise sense of Prop. 115. Pure states on the almost periodic functions form the Bohr compactification of Ξ_0 [123, Sect. 4.7], among which the points of Ξ_0 (i.e., their point evaluations) are just a tiny part and not even an open subset. This expresses the observation that almost periodic functions cannot distinguish a point from many others that are arbitrarily far away, so the finite and the infinite are intertwined more intimately than *observables* would ever distinguish. An algebra whose states are better behaved may be more adequate for physics.

This suggests finding a C*-algebra whose states are just the *good* ones described by Bochner's Theorem and will be done in the next paragraph.

The standard states as a C*-state space

The CCR-algebra is constructed so that its representations exactly correspond to the representations of the Weyl relations. In this correspondence, the topology of Ξ plays no role at all. The way to set up a similar correspondence for just the *continuous* unitary representations is well known from the theory of locally compact groups: One goes to the convolution algebra over the group. In fact, the term *group algebra* of a group is usually reserved for the C*-envelope of the convolution algebra $L^1(G)$, and not for the topology-free analog of the CCR algebra [14, Ch.13]. Von Neumann's proof uses the same idea by introducing the operator A in Eq. (5.83) as an integral. In this section, we follow this lead.

This will require a twisted version of the group algebra construction [124]. An alternative construction would be via the group C*-algebra of a related non-abelian group, a central extension [125, Ch.VII] of the additive group Ξ . The approach used below is a bit more direct. It avoids introducing the central phase parameter, which is integrated out anyhow in the end.

After we finished this work, we realized that the idea had already been followed through by Grundling [83, 84], with much the same motivation of getting a C*-description of continuous representations, and even extended to more general groups, also beyond locally compact ones. To keep this work self-contained, we nevertheless include our version.

For $h \in L^1(\Xi, d\xi)$ and any given measurable representation W of the Weyl relations we write the Bochner integral

$$W[h] = \int d\xi h(\xi)W(\xi). \quad (5.93)$$

The bracket notation indicates that $h \mapsto W[h]$ is closely related to the representation W . The multiplication rule and adjoints for such operators follow directly from the Weyl relations, namely $W[h]W[g] = W[h*_\sigma g]$ and $W[h]^* = W[h^*]$ with

$$(h*_\sigma g)(\xi) = \int d\eta h(\xi - \eta)g(\eta)e^{-\frac{i}{2}\xi \cdot \sigma \eta}, \quad (5.94)$$

$$h^*(\xi) = \overline{h(-\xi)}. \quad (5.95)$$

These operations turn $L^1(\Xi)$ into a Banach *-algebra, which we call the **σ -twisted convolution algebra** of Ξ . Any set of elements h_ε such that $h_\varepsilon \geq 0$, $\int d\xi h_\varepsilon(\xi) = 1$, and $h_\varepsilon(\xi) = 0$ for ξ outside a ball of radius ε around the origin is an approximate unit. As in the untwisted case, this follows from the strong continuity of translations on $L_1(\Xi)$. Utilizing the aforementioned C*-envelope that we already introduced in Sect. 2.2.1.5, we can now define the following:

Definition 90. *The enveloping C*-algebra of the convolution algebra is called the twisted group algebra of (Ξ, σ) and will be denoted by $C^*(\Xi, \sigma)$.*

By slight abuse of notation, we denote the element in $C^*(\Xi, \sigma)$ associated with $h \in L^1(\Xi)$ by the completion process again by h . This is justified by the observation that the canonical embedding $L^1(\Xi, \sigma) \hookrightarrow C^*(\Xi, \sigma)$ is injective.

The following proposition gives us a third way of looking at Weyl elements $W(\xi)$. Once they appeared as the abstract generators of a CCR-algebra (Def. 78). Then,

they were defined as explicit operators in any standard representation (Def. 86). These two views are equivalent by virtue of Bochner's Theorem, which identifies standard states with linear functionals ω' on $\text{CCR}(\Xi, \sigma)$ (Thm. 89). The next proposition allows us to further introduce, for each ξ , the linear functional

$$\omega \mapsto \omega'(W(\xi)). \quad (5.96)$$

This element of the bidual $C^*(\Xi, \sigma)^{**}$ is yet another version of the Weyl element, which we will also denote by $W(\xi)$. We cannot realize such an element in $C^*(\Xi, \sigma)$ since this algebra has no unit and hence contains no unitary elements. However, we can get close in the same sense that the expression Eq. (5.93) can be close to $W(\xi)$ if h is concentrated near ξ . The idea of the following proof is to do this limit in the GNS representation of ω .

Proposition 91. *Let (Ξ, σ) be a hybrid phase space. Then every state ω on $C^*(\Xi, \sigma)$ is given by a unique standard state ω' on $\text{CCR}(\Xi, \sigma)$ and conversely, such that*

$$\omega(h) = \int d\xi h(\xi) \omega'(W(\xi)). \quad (5.97)$$

Proof. It is a general feature of the enveloping C^* -algebra construction that the states ω on $C^*(\Xi, \sigma)$ are in bijective correspondence to the positive linear functionals $\tilde{\omega}$ on the convolution algebra with norm 1 (see Sect. 2.2.1.5 and [14, Prop. 2.7.5]). Here, the norm is taken as a linear functional on the Banach space $L^1(\Xi)$. That is, there is a function $\chi \in L^\infty(\Xi)$ with $\|\chi\|_\infty = 1$ such that

$$\tilde{\omega}(h) = \int d\xi h(\xi) \chi(\xi). \quad (5.98)$$

The main task of the proof is to show that the functions χ arising in this way are precisely the characteristic functions characterized by the Bochner Theorem, and in particular *continuous*. The uniqueness of the correspondence is clear from this equation since, on the one hand, it gives an explicit formula for ω (resp. $\tilde{\omega}$) in terms of ω' , and, on the other to states ω', ω'' satisfying it for the same $\tilde{\omega}$ would have to be equal as elements of $L^\infty(\Xi)$, hence equal almost everywhere, and hence equal by continuity.

We begin by defining a version of the Weyl operators acting on $L^1(\Xi)$, namely

$$(\widetilde{W}(\xi)h)(\eta) := e^{-\frac{i}{2}\xi \cdot \sigma \eta} h(\eta - \xi). \quad (5.99)$$

It is constructed so that

$$h *_\sigma g = \int d\xi h(\xi) \widetilde{W}(\xi)g. \quad (5.100)$$

Intuitively, we can think of $\widetilde{W}(\xi)$ as the operator of convolution with δ_ξ , the limit of probability densities concentrated near the point ξ . While this is not an element of the algebra, its operation is defined analogously to the approximate unit δ_0 . It is easy to check that the operators $\widetilde{W}(\xi)$ satisfy the Weyl relations in Eq. (5.73). However, unitarity does not make sense since $L^1(\Xi)$ is not a Hilbert space. The crucial observation is that $\xi \mapsto \widetilde{W}(\xi)g$ is continuous in the norm of $L^1(\Xi)$. Indeed, it is a product of a translation and a multiplication operator, which are both strongly

continuous on L^1 . Consider now a positive linear functional $\tilde{\omega}$ on $L^1(\Xi)$. Its GNS representation space $\tilde{\mathcal{H}}$ is the unique Hilbert space generated by vectors $v(h)$, $h \in L^1(\Xi)$, with the scalar product $\langle v(h)|v(k)\rangle = \tilde{\omega}(h^* *_{\sigma} k)$. On these the representation $W : L^1(\Xi) \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$ acts by left multiplication in the convolution algebra, i.e., according to the formula

$$W[h]v(g) = v(h *_{\sigma} g). \quad (5.101)$$

According to Thm. 21 the GNS space has a cyclic vector Ω and a representation $W : L^1(\Xi) \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$ such that $v(h) = W[h]\Omega$, and $\langle \Omega, W[h]\Omega \rangle = \tilde{\omega}(h)$. Indeed, one has $\Omega = \lim_{\varepsilon \rightarrow 0} v(h_{\varepsilon})$, where h_{ε} is a bounded approximate unit.

Our next aim is to show that W arises exactly as in Eq. (5.93) from the integration of a representation W of the Weyl relations (recall that the two functions will be typographically distinguished by their argument brackets). The obvious candidates for the Weyl operators $W(\cdot)$ are the operators $\tilde{W}(\xi)$ from Eq. (5.99), represented on $\tilde{\mathcal{H}}$ in GNS style. That is, in analogy to Eq. (5.101), we set

$$W(\xi)v(g) = v(\tilde{W}(\xi)g). \quad (5.102)$$

Then, it is elementary to check that $W(\xi)$ is unitary, and these operators satisfy the Weyl relations. Moreover, the L^1 -norm continuity of $\xi \mapsto \tilde{W}(\xi)g$ established earlier implies that $\xi \mapsto W(\xi)$ is continuous in the strong operator topology. Finally, Eq. (5.100) implies

$$W[h]v(g) = v(h *_{\sigma} g) = \int d\xi h(\xi)v(\tilde{W}(\xi)g) = \int d\xi h(\xi)W(\xi)v(g), \quad (5.103)$$

i.e., the GNS-representation $W[h]$ is related to the continuous representation W by Eq. (5.93). In particular,

$$\tilde{\omega}(h) = \int d\xi h(\xi)\langle \Omega, W(\xi)\Omega \rangle, \quad (5.104)$$

so Eq. (5.98) holds with $\chi(\xi) = \langle \Omega, W(\xi)\Omega \rangle$, which is clearly a normalized twisted positive definite function, and continuous because $W(\xi)$ is strongly continuous.

Conversely, given a state on the CCR-algebra, we can define its characteristic function $\chi(\xi) = \omega'(W(\xi))$. In general, that might fail to be even measurable, so the formula Eq. (5.98) might make no sense. However, for standard states, χ is continuous, and the integral is well defined. \square

We now determine the algebras $C^*(\Xi, \sigma)$ concretely. It turns out that this is best done by splitting into a purely classical and a purely quantum part.

Proposition 92. *Let (Ξ, σ) be a hybrid phase space, split as $\Xi = \Xi_1 \oplus \Xi_0$ with $\sigma = \sigma_1 \oplus 0$. Then*

$$C^*(\Xi, \sigma) = C^*(\Xi_1, \sigma_1) \otimes C^*(\Xi_0, 0) \cong \mathcal{K}(\mathcal{H}_1) \otimes \mathcal{C}_0(\Xi_0), \quad (5.105)$$

where $\mathcal{K}(\mathcal{H}_1)$ denotes the compact operators on the representation space \mathcal{H}_1 of the irreducible quantum system (Ξ_1, σ_1) , and $\mathcal{C}_0(\Xi_0)$ denotes the continuous functions on Ξ_0 vanishing at infinity.

Proof. We first observe that for the underlying L^1 -spaces, the direct sum naturally coincides with the projective product, which is predual to the tensor product of von Neumann algebras. That is:

$$L^1(\Xi_1 \oplus \Xi_0) = L^1(\Xi_1) \otimes L^1(\Xi_0). \quad (5.106)$$

Indeed, the tensor products $f \otimes g$ on the right-hand side can be identified with the product functions $fg(\xi_0 \oplus \xi_1) = f(\xi_1)g(\xi_0)$, and this embedding is clearly isometric on step functions. Since the measurable structure of $\Xi_1 \oplus \Xi_0$, which is $\Xi_1 \times \Xi_0$ as a set, is defined as generated by rectangles, the product functions span a dense subspace of $L^1(\Xi_1 \oplus \Xi_0)$. One can verify that the isomorphism Eq. (5.106) is also consistent with the definitions of adjoint operation and convolution product.

The completion in the construction of the enveloping C^* -algebra also works out: As one side of the tensor product is abelian, and the maximal and the minimal tensor product coincide, the tensor product is uniquely determined, as is the algebra in tensor product form (see Sect. 3.2.3).

In the second step, we need to show the claimed isomorphisms: Beginning with the classical case, $L^1(\Xi_0)$ is the convolution algebra of $\Xi_0 \cong \mathbb{R}^s$. The Gelfand isomorphism describes its irreducible representations for abelian Banach algebras: In this case, they are given by the point evaluations of the Fourier transform. Hence, the C^* -norm of the enveloping algebra, $\|h\| = \sup_{\pi} \|\pi[h]\|$, is equal to the supremum norm of the Fourier transform of h . Now, by the Riemann-Lebesgue Lemma [6, Sect. 7.5], the Fourier transforms of L^1 -functions are continuous and go to zero at infinity. On the other hand, by the Stone-Weierstraß Theorem [6, Thm. 5.7], these Fourier transforms separate points and are hence uniformly dense. Together we have

$$C^*(\Xi_0) = C_0(\Xi_0). \quad (5.107)$$

For quantum systems, note that every continuous representation of the Weyl relations is isomorphic to the Schrödinger representation on \mathcal{H}_1 by the von Neumann Uniqueness Theorem (see Thm. 79). Hence, we only need to show that in that representation, the operators of the form $W[h]$ with $h \in L^1(\Xi_1)$ are compact, and these operators form a dense subalgebra of $\mathcal{K}(\mathcal{H}_1)$. This follows immediately by the correspondence theory [121, Cor. 5.1.(4)].

An alternative approach using better-known facts goes via first showing that operators $h \mapsto W[h]$ are not only continuous from L^1 to $\mathcal{B}(\mathcal{H}_1)$ but also isometries for the 2-norms, i.e., $\|W[h]\|_2 = \|h\|_2$, for $h \in L^1(\Xi) \cap L^2(\Xi) \equiv \mathcal{A}$, and the Schatten 2-norm (Hilbert-Schmidt norm) on the operator side. Continuous extension via these 2-norms is even unitary. Hence, $W[\mathcal{A}]$ consists of Hilbert-Schmidt operators, which are compact, and by taking limits in 2-norm, we find that $W[\mathcal{A}]$ is operator norm dense in the Hilbert-Schmidt class, hence in $\mathcal{K}(\mathcal{H}_1)$. \square

We note that as a consequence of this characterization, we find that there are many extremal standard states, since the state space of $C^*(\Xi, \sigma)$ is the weak*-closed convex hull of its extreme points, by the Banach-Alaoglu and Krein-Milman Theorems (see Sect. 2.2.1.4). Of course, these were already identified in Lem. 87.

Restoring translation symmetry

Translations were part of our basic setup from the outset since the phase space Ξ is a vector space. The notion of standard representations (Def. 86) breaks the translation symmetry. However, it is restored in the twisted convolution construction. Indeed, combining the hybrid version of the shift (Eq. (5.7)) and the hybrid Weyl operator (Eq. (5.72)) we get

$$\alpha_\eta(W(\xi)) = e^{i\eta \cdot \xi} W(\xi). \quad (5.108)$$

Although Weyl operators are not themselves in $C^*(\Xi, \sigma)$, we think of this algebra as generated by integrated Weyl operators Eq. (5.93), and so we define

$$(\alpha_\eta h)(\xi) = e^{i\eta \cdot \xi} h(\xi), \quad (5.109)$$

which extends to the enveloping algebra $C^*(\Xi, \sigma)$. In the tensor product structure of Prop. 92, we can apply this separately to the quantum and classical parts so

$$\alpha_{\eta_1 \oplus \eta_0} = \alpha_{\eta_1} \alpha_{\eta_0}. \quad (5.110)$$

On the classical part $\mathcal{C}_0(\Xi_0)$ the action becomes the shift $(\alpha_{\eta_0} f)(\xi_0) = f(\xi_0 + \eta_0)$. Similarly, we can compute the action on the quantum part, finding

$$\alpha_\eta(X) = W(\sigma\eta)^* X W(\sigma\eta). \quad (5.111)$$

In this expression, we use σ as a matrix acting on the vector $\eta \in \Xi$, which is possible because we choose a fixed basis in $\Xi \cong \mathbb{R}^{2n+s}$. Since σ vanishes on the classical part, the component η_0 of the translation argument automatically drops out, and only the quantum Weyl operators are used.

The tensor product $\mathcal{K}(\mathcal{H}_1) \otimes \mathcal{C}_0(\Xi_0)$ can be considered as the algebra of norm continuous functions $F : \Xi_0 \rightarrow \mathcal{K}(\mathcal{H}_1)$ vanishing at infinity, by identifying $K \otimes f$ with the function $F(\xi) = f(\xi)K$. In this function form, which will later extend to certain subspaces of $C^*(\Xi, \sigma)^{**}$, the action of translations becomes, for $\eta = \eta_1 \oplus \eta_0$,

$$(\alpha_\eta(F))(\xi_0) = W(\sigma\eta)^* F(\xi_0 + \eta_0) W(\sigma\eta). \quad (5.112)$$

Continuity of state translations

We note that α_η^* is not strongly continuous on the Banach space of states, i.e., the function $\xi \mapsto \alpha_\eta^* \rho$ is not continuous in norm. Indeed, an arbitrarily small shift applied to a point measure moves it as far away as possible in the natural norm on states. Since translations are strongly continuous on $L^1(\mathbb{R}^n, dx)$, this is different for probability measures with absolutely continuous densities. We can use this to single out one particular standard representation, namely, using the Lebesgue measure for μ .

Proposition 93. *Let $\omega \in C^*(\Xi, \sigma)^*$ be a state with characteristic function χ , and let μ be its marginal probability measure on Ξ_0 . Then the following are equivalent:*

- (1) ω is norm continuous under phase space translations, i.e.,

$$\lim_{\eta \rightarrow 0} \|\omega - \alpha_\eta^*(\omega)\| = 0. \quad (5.113)$$

(2) μ is absolutely continuous with respect to Lebesgue measure.

(3) ω is the restriction of a standard state $\widehat{\omega}$ on a purely quantum system, in which the classical variables in Ξ_0 also have conjugate momenta.

In this case $\chi \in \mathcal{C}_0(\Xi)$. As a partial converse, if $\chi \in L^p(\Xi, d\xi)$ for some $p \in [1, 2]$ then the above conditions hold.

Proof. (1) \Rightarrow (2): (2) only depends on the restriction of ω to the classical algebra. So, this is a purely classical observation, which is valid for any locally compact group. Let α_x^* , $x \in \mathbb{R}^n$ denote the action of translations on measures over \mathbb{R}^n . Then (1) says $\lim_{x \rightarrow 0} \|\mu - \alpha_x^*(\mu)\| = 0$, where the norm is the dual norm of the supremum norm on $\mathcal{C}_0(\mathbb{R}^n)$, i.e., the total variation norm on measures. For any $\varepsilon > 0$, there is thus a neighborhood $U_\varepsilon \ni 0$ such that $\|\mu - \alpha_x^*(\mu)\| \leq \varepsilon$ for $x \in U_\varepsilon$. We pick a positive measurable function $h \in L^1(\mathbb{R}^n, dx)$ with integral 1 and support in U_ε , and set

$$\alpha_h^*(\mu) = \int dx h(x) \alpha_x^*(\mu), \quad (5.114)$$

which is to be read as a weak* integral. Using the triangle inequality, we get that $\|\mu - \alpha_h^*(\mu)\| \leq \varepsilon$. On the other hand, $\alpha_h^*(\mu)$ is absolutely continuous because Eq. (5.114) is the convolution of the two measures μ and $h dx$. Explicitly, for an arbitrary $f \in \mathcal{C}_0(\mathbb{R}^n)$, we get

$$(\alpha_h^*\mu)(f) = \int dx h(x) \int \mu(dy) f(y-x) = \int \mu(dy) \int dx h(y-x) f(x), \quad (5.115)$$

which is to say that $\alpha_h^*\mu$ has density $\widetilde{h}(x) = \int \mu(dy) h(y-x)$ with respect to Lebesgue measure. We also recall that the variation norm of absolutely continuous measures is just the L^1 norm of their densities. Now since $\alpha_h^*(\mu)$ converges in norm to μ as $\varepsilon \rightarrow 0$, the corresponding densities \widetilde{h} form a Cauchy net in $L^1(\mathbb{R}^n, dx)$. Its limit \widetilde{h}_0 must be in L^1 by completeness of L^1 . This is then a density for μ , so μ is itself absolutely continuous.

(2) \Rightarrow (3): Consider the standard representation for μ in the Hilbert space $\mathcal{H}_1 \otimes L^2(\Xi_0, \mu)$. We can consider this as a subspace of $\mathcal{H}_1 \otimes L^2(\Xi_0, dx)$ with the Lebesgue measure. By Def. 86, ω , as a state on the CCR algebra, can be represented as a normal state on this space. Apart from the canonical position operators in $L^2(\Xi_0, dx)$, which are already part of the hybrid setup, we can take the shift generators in this tensor factor as further canonical momentum operators. The full set of canonical operators is then clearly irreducible, so the extended system is purely quantum.

(3) \Rightarrow (1): The Weyl translations in a standard representation are strongly continuous, which implies that the action $\alpha_\xi^* \rho = W(\sigma\xi) \rho W(\sigma\xi)$ is norm continuous for every $\rho \in \mathcal{T}(\mathcal{H})$ so we can restrict the continuity condition for the extended system to those translations η which make sense in the original hybrid.

The final remark in the proposition is clear in one direction from the Riemann-Lebesgue Lemma and its quantum version [121]. In the converse direction, it follows that the Fourier transform of χ is the density with respect to the Lebesgue measure, which is then even continuous and goes to zero. \square

L^p -spaces

If one is not interested in pure states, a good setting for hybrids is to restrict consideration to the norm continuous states characterized by Prop. 93. This leads to a purely von Neumann algebraic picture: Since all probability measures μ are then absolutely continuous with respect to Lebesgue measure dx , we can represent all states in $\mathcal{H} = \mathcal{H}_1 \otimes L^2(\Xi_0, dx)$, which leads to the spaces

$$L^1(\Xi, \sigma) := \mathcal{T}^1(\mathcal{H}_1) \otimes L^1(\Xi_0, dx) \quad \text{and} \quad L^\infty(\Xi, \sigma) := \mathcal{B}(\mathcal{H}_1) \bar{\otimes} L^\infty(\Xi_0, dx). \quad (5.116)$$

Here, the tensor product on the right uses the 1-norm completion, and \mathcal{T}^p denotes the Schatten classes for $1 \leq p < \infty$, so \mathcal{T}^1 is the trace class. \mathcal{T}^∞ would be ambiguous, meaning either all bounded operators $\mathcal{B}(\mathcal{H})$ or just the compact operators $\mathcal{K}(\mathcal{H}_1)$, so we prefer to specify explicitly. The tensor product on the right is then the von Neumann algebra version, constructed as the completion of the product operators in the weak (or similar) operator topology. This choice was adopted, for example, in [72, 76, 126], see Sect. 3.4. Note that the von Neumann algebra $L^\infty(\Xi, \sigma)$ is not a subalgebra of $\mathcal{U}(\Xi, \sigma)$ but a quotient, because the (L^1, L^∞) -duality is defined by selecting a subspace of states (cf. Fig. 5.2). In fact, it does not make sense to evaluate an element $F \in L^\infty(\Xi, \sigma)$ on a pure state because functions differing at one point, e.g., the point $x \in \Xi_0$ on which the pure state lives, are identified.

The combined Schatten/Lebesgue spaces $L^p(\Xi, \sigma)$ can be obtained by a purely von Neumann algebraic construction using the semifinite trace

$$\widehat{\text{tr}}(f \otimes A) = \int dx f(x) \text{tr} A \quad (5.117)$$

on $L^\infty(\Xi, \sigma)$. Then $L^p(\Xi, \sigma)$ comes out as the p -norm completion of the elements $F \in L^\infty$ so that $\|F\|_p^p := \widehat{\text{tr}}|F|^p < \infty$. These spaces are also connected by interpolation [127]. The only case of interest to us, however, is $p = 2$ because of a fact which is well known in both the classical and the quantum case, so its generalization to hybrids is unsurprising: The **Fourier-Weyl transform** for hybrids

$$(\mathcal{F}F)(\xi) = \widehat{\text{tr}}(FW(\xi)) \quad (5.118)$$

is a unitary isomorphism from $L^2(\Xi, \sigma)$ onto $L^2(\Xi, 0)$. This definition differs from that in [121] by a symplectic matrix in the argument, which would not make sense in the classical case. So for elements $F \in L^p(\Xi, \sigma) \cap L^2(\Xi, \sigma)$ and $G \in L^q(\Xi, \sigma) \cap L^2(\Xi, \sigma)$ with $p^{-1} + q^{-1} = 1$, $1 \leq p \leq \infty$ we have

$$\widehat{\text{tr}}F^*G = \int dx \text{tr}(F(x)^*G(x)) = (2\pi)^{-(n+s)} \int d\xi \overline{\text{tr}(FW(\xi))} \text{tr}(GW(\xi)), \quad (5.119)$$

where s and n are the numbers of degrees of classical and quantum freedom, respectively (see Sect. 5.2.1). This extends by continuity to all of $L^2(\Xi, \sigma)$, and the map is clearly onto. Its inverse is an instance of Weyl quantization, in our case, a partial one acting only on the quantum part of the system. A similar useful formula concerns an integral over translates:

For $F, G \in L^1(\Xi, \sigma) \cap L^\infty(\Xi, \sigma)$ we have

$$\int d\xi \widehat{\text{tr}}(F\alpha_\xi(G)) = (2\pi)^n \widehat{\text{tr}}(F)\widehat{\text{tr}}(G). \quad (5.120)$$

The proof is immediate for the classical part, and the quantum part is essentially the square integrability of the quantum Weyl operators [121, Lem. 3.1.].

5.2.4 Gaussian hybrids

In our review of purely quantum continuous-variable systems in Sect. 5.1 we have already briefly discussed Gaussian systems. The hallmark of those systems, whether it be quantum or classical, is their simple description combined with a wide field of possible applications. Also, whether it be states or channels, typically the *Gaussian* subset is more commonly known and used than its *quasifree* superset. This clearly suggests having a closer in the hybrid case.

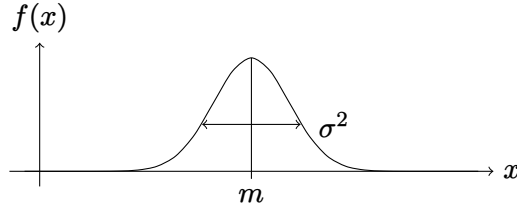


Figure 5.1: A one-dimensional Gaussian function.

The basic positivity condition

Our first step is to show that the common description of Gaussian states fits into our framework, i.e., we need to specialize to the case where all characteristic functions have a Gaussian form. Like in the purely quantum or classical setting, this allows a complete reduction to the finite-dimensional analysis of covariance matrices and means.

Definition & Lemma 94. *Consider a continuous variable system with hybrid commutation relations given by an antisymmetric matrix σ , as in Eq. 5.63. Then*

- (1) *For every state ω with finite second moments and characteristic function χ , we can define the **mean vector** $m \in \Xi$ and the **covariance matrix** γ by*

$$\begin{aligned} m_j &= \omega(R_j) &= \frac{1}{i} \frac{\partial}{\partial \xi_k} \chi(\xi) \Big|_{\xi=0} \\ \gamma_{jk} &= 2\Re \omega\left((R_j - m_j \mathbb{1})(R_k - m_k \mathbb{1})\right) &= -\frac{\partial^2}{\partial \xi_k \partial \xi_l} \chi(\xi) \Big|_{\xi=0} - m_j m_k. \end{aligned} \quad (5.121)$$

The covariance matrix γ is real, symmetric, and positive semi-definite.

- (2) *When (1) holds, $\gamma + i\sigma$, and hence its complex conjugate $\gamma - i\sigma$, are positive semi-definite.*

- (3) *When (2) holds,*

$$\chi(\xi) = \exp\left(-\frac{1}{4}\xi \cdot \gamma \xi + i\xi \cdot m\right) \quad (5.122)$$

*is the characteristic function of a state, called the **Gaussian state** with mean m and covariance γ .*

Proof. (1) is a definition.

(2) Let $\xi \in \mathbb{C}^{2n+s} = \Xi + i\Xi$, and define the operator $X = \sum_i \xi_i (R_i - m_i \mathbb{1})$. Then

$$\begin{aligned} \sum_{k,\ell} \bar{\xi}_k (\gamma_{k\ell} + i\sigma_{k\ell}) \xi_\ell &= 2 \sum_{k,\ell} \bar{\xi}_k \left(\Re \omega((R_k - m_k \mathbb{1})(R_\ell - m_\ell \mathbb{1})) \right. \\ &\quad \left. + i \Im \omega((R_k - m_k \mathbb{1})(R_\ell - m_\ell \mathbb{1})) \right) \xi_\ell \\ &= 2 \sum_{k,\ell} \bar{\xi}_k \omega((R_k - m_k \mathbb{1})(R_\ell - m_\ell \mathbb{1})) \xi_\ell \\ &= \omega(X^* X) \geq 0. \end{aligned} \tag{5.123}$$

(3) Here, we just need to verify Bochner's Theorem (Thm. 89), which is trivial for normalization and continuity. The proof of twisted positive definiteness is based on the well-known fact that the entrywise product, also called the Hadamard product or the Schur product, of positive semi-definite matrices is positive semi-definite. By summation this also applies to the *Hadamard exponential* [128] of a matrix M defined as $(\text{Hexp}M)_{kl} = \exp(M_{kl})$.

We now write

$$\chi(\xi_k - \xi_\ell) e^{i\xi_k \cdot \sigma \xi_\ell} = \left(\chi(\xi_k) \overline{\chi(\xi_\ell)} \right) \left(e^{\xi_k \cdot (\gamma + i\sigma) \xi_\ell / 2} \right) \tag{5.124}$$

This is the Hadamard product of two matrices, where the first is obviously positive definite, and the second is the Hadamard exponential of the Gram matrix

$$G_{k\ell} = \xi_k \cdot (\gamma + i\sigma) \xi_\ell, \tag{5.125}$$

which is positive semi-definite by assumption. \square

Minimality and purity

We have defined Gaussians in terms of their characteristic functions rather than by the form of their probability density (Def. 94). This has the consequence that we include singular Gaussians, for which γ has null directions. The extreme case of this is a classical system with $\gamma = 0$, and hence

$$\chi(\xi) = \exp(i\xi \cdot m), \tag{5.126}$$

i.e., the characteristic function of the point measure at m with *density* $\delta(\xi - m)$. More generally, we include cases where the classical measure is supported on a plane, whose direction is characterized by the kernel of γ and whose offset is given by m .

In the quantum case, this cannot happen: No normal state can have sharp position distribution because the position probability density is a sum of L^1 -functions $|\Psi(x)|^2$. Therefore, it is an interesting question what the pertinent conditions are in the hybrid case.

The next step in our characterization of quantum-classical hybrid states is the classification into pure and mixed ones. On the quantum side, this is well known. Here, the rank-one projectors or vectors give the pure states $|\Psi\rangle\langle\Psi|$. For classical systems, i.e., probability measures, the pure states are point measures. While certainly not every pure quantum state is a Gaussian one, all pure classical states are captured by our definition of Gaussian states:

They are of the form described in Eq. (5.122) for $\gamma = 0$. With these pure classical states, it is easy to construct hybrid ones with the non-vanishing quantum part. Take a Gaussian quantum state, with γ_Q as covariance and m_Q as mean. Now the pair $(\gamma_Q \oplus 0, m \oplus \xi_0)$ describes hybrid Gaussian state for $\sigma = \sigma_1 \oplus 0$.

In the subset of Gaussian quantum states, we also have a characterization of pure states. For n degrees of freedom, a Gaussian quantum state ω_Q with covariance γ_1 is pure if one of the equivalent statements holds true [92, 116]:

$$(a) \det \gamma_1 = 1, (b) \text{rank}(\gamma_1 + i\sigma_1) = n, (c) \gamma_1 \in \text{Sp}(2n, \mathbb{R}), (d) (\sigma_1^{-1} \gamma_1)^{-1} = -\mathbb{1}_{2n}.$$

Given that a hybrid must include the point measures on the classical part and the degeneracy of σ , we can not simply transfer this statement. Instead, we need to introduce minimal γ , which describes the covariance matrices of pure Gaussian hybrid states.

The idea of minimality is similar to [116], where the focus lies on the description of Gaussian quantum channels: There, the degenerate *symplectic* form comes into play because, as described in Sect. 5.1.3, a Gaussian quantum channel is fully described by the matrices N and S that fulfill $N + i\Delta\sigma \geq 0$ with $\Delta\sigma = \sigma_{\text{out}} - S^\top \sigma_{\text{in}} S$. Dependent on the choice of S , the antisymmetric form $\Delta\sigma$ can be degenerate, although this setting deals with the purely quantum case.

Definition & Lemma 95. *Let (Ξ, σ) and γ, m with $\gamma + i\sigma \geq 0$. Then the following are equivalent:*

- (1) γ is **minimal** in the sense that for any symmetric real matrix γ' , with $\gamma' \leq \gamma$ and $\gamma' + i\sigma \geq 0$, we have $\gamma' = \gamma$.
- (2) The Gaussian state with covariance γ and mean m is pure.
- (3) There is only one state with finite second moments, covariance γ and mean m .
- (4) There is a coordinate system for Ξ in which we have

$$\gamma + i\sigma = \left(\begin{array}{cc|c} \mathbb{1} & i\mathbb{1} & 0 \\ -i\mathbb{1} & \mathbb{1} & 0 \\ \hline 0 & 0 & 0 \end{array} \right). \quad (5.127)$$

Proof. Note that (1) and (4) do not contain m . Moreover, by a translation, we can change m , so if condition (2) or (3) holds for some m , it holds for all. Hence, we may assume $m = 0$ throughout.

(3) \Rightarrow (2): This direction is trivial: When there is only one state with the data (γ, m) and we know that there is always a Gaussian one, that unique state has to be the Gaussian.

(2) \Rightarrow (1): Suppose that γ is not minimal. Then we can find $\gamma' \leq \gamma$, but $\gamma' \neq \gamma$, such that γ' is still the covariance of an admissible state, i.e., $\gamma' + i\sigma \geq 0$. Then average over translations with a classical weight with covariance γ' to get the

Gaussian state back:

$$\begin{aligned}
\omega_\gamma(W(\xi)) &= \omega_{\gamma'}(W(\xi)) \exp\left(-\frac{1}{4}\xi(\gamma - \gamma')\xi\right) \\
&= \omega_{\gamma'}(W(\xi)) \int \mu_{\gamma-\gamma'}(d\eta) e^{i\xi\cdot\sigma\eta} \\
&= \int \mu_{\gamma-\gamma'}(d\eta) \omega_{\gamma'}(W(\sigma\eta)W(\xi)W(\sigma\eta)^*) \quad (5.128)
\end{aligned}$$

$$\omega_\gamma = \int \mu(d\eta) \alpha_\eta(\omega_{\tilde{\gamma}}). \quad (5.129)$$

Here $\mu_{\gamma-\tilde{\gamma}}$ describes a Gaussian measure with covariance $\gamma - \gamma'$. It is not pure since this is written as an explicit mixture of distinct states.

(1) \Rightarrow (4): Consider some admissible covariance matrix γ , so $\gamma + i\sigma \geq 0$, and denote by N_\pm the kernel of $\gamma \pm i\sigma$ in the complex Hilbert space $\Xi + i\Xi = \mathbb{C}^{2n+s}$. We will make γ smaller by subtracting rank-one operators. We claim that this is possible as long as $N_+^\perp \cap N_-^\perp$ contains non-zero vectors. Indeed, if $0 \neq \phi \in N_+^\perp \cap N_-^\perp$, the same holds for the complex conjugate, so either the real part or the imaginary part is non-zero, and we can assume ϕ to be real and normalized. Because $\phi \in N_+^\perp$ it lies in the span of those eigenvectors of the positive semi-definite matrix $\gamma + i\sigma$ with strictly positive eigenvalues. The smallest eigenvalue will be denoted by λ . Then $(\gamma + i\sigma) \geq \lambda|\phi\rangle\langle\phi|$, and $\gamma' = \gamma - \lambda|\phi\rangle\langle\phi|$ is still an admissible covariance matrix, and $\phi \in N_\pm'$, so the dimension of N_\pm has increased by 1. This implies that after finitely many steps, we reach a situation where $N_+^\perp \cap N_-^\perp = \{0\}$, and no more subtractions are possible. Covariance matrices with this property are our candidates for the minimal ones.

Since $N_+^\perp \cap N_-^\perp = (N_+ + N_-)^\perp$ this condition is equivalent to the spaces N_\pm together spanning $\Xi + i\Xi$. In particular, every real vector ξ can be written as $\xi = \xi_+ + \xi_-$ with $\xi_\pm \in N_\pm$. Since the complex conjugate vectors $\eta_\pm = \overline{\xi_\mp} \in N_\mp$ form an equally valid decomposition, so does $\xi'_\pm = (\xi_\pm + \overline{\xi_\mp})/2$, so we can choose ξ_\pm to be complex conjugates of each other. In other words, for every $\xi \in \Xi$ there is a vector $\eta \in \Xi$ such that $\xi + i\eta \in N_+$, which means that

$$0 = (\gamma + i\sigma)(\xi + i\eta) = (\gamma\xi - \sigma\eta) + i(\sigma\xi + \gamma\eta). \quad (5.130)$$

Of course, the two parentheses have to vanish separately. Now, if a vector is in the range of γ , i.e., of the form $\gamma\xi$, it is also of the form $\sigma\eta$, so in the range of σ , and the second parenthesis similarly shows the reverse inclusion. Hence, the ranges of σ and η are the same.

(4) \Rightarrow (3): For classical variables $\langle Q_j^2 \rangle = 0$ for $j > 2n$ implies that the state is concentrated at $\xi_0 = 0$. For quantum: $\omega(A^*A) = 0$ with $A = (Q + iP)$ it is the unique oscillator ground state ω_{osc} , so $\omega = \omega_{\text{osc}} \otimes \delta_0$. \square

Note that one characterization of purity, which works in the classical case, does not when quantum systems are around: Classically, the characteristic function of a point measure has constant modulus 1, and conversely. In the quantum case, the uncertainty relation prevents that.

Conditional states

The description of conditional states for hybrid systems is especially relevant because we are free to read out and copy the classical part thoroughly compared to the quantum systems. Also, it is a benchmark for our generalization of Gaussian systems as it would be highly beneficial if a Gaussian hybrid state would have the Gaussian versions on the respective subsystems. Indeed, for classical systems, i.e., multivariate normal distributions, this fact is well known. So the question for this chapter is the following: Given a Gaussian hybrid state ω with covariance γ , what do the measure $\mu(dx)$ and ρ_x in Eq. (5.80) look like? The basic answer to this is: Gaussian. For μ , this includes the possibility of point measures, which we consider as Gaussian as well, as described at the beginning. Mixed combinations, i.e., measures living on a hyperplane with a Gaussian density on that plane, are also possible.

It turns out that the conditioning works exactly as in classical probability. This is summarized in the following lemma, slightly adapted for our purposes.

Lemma 96. *Let γ be a positive semi-definite operator on a complex phase space $\Xi = \Xi_1 \oplus \Xi_0$, with the corresponding block matrix decomposition*

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{10} \\ \gamma_{01} & \gamma_{00} \end{pmatrix}. \quad (5.131)$$

(1) *Then there is an operator $A : \Xi_1 \rightarrow \Xi_0$ such $\gamma_{01} = \gamma_{00}A$, and we have the decomposition*

$$\langle \phi_1 \oplus \phi_0 \mid \gamma \mid \phi_1 \oplus \phi_0 \rangle = \langle \phi_1 \mid \gamma^S \mid \phi_1 \rangle + \langle \phi_0 + A\phi_1 \mid \gamma_{00} \mid \phi_0 + A\phi_1 \rangle, \quad (5.132)$$

where $\gamma^S = \gamma_{11} - A^*\gamma_{00}A \geq 0$ is called the **Schur complement** of γ_{00} in γ .

(2) *When γ_{00} is singular, A is not uniquely defined, but the above decomposition does not depend on the choice of A .*

(3) *When Ξ has a complex conjugation so that γ_{01} and γ_{00} are real, then A can also be chosen to be real.*

Proof. We start by choosing a basis for Ξ_0 , such that the non-singular part of γ_{00} is given by the upper left part, i.e.

$$\gamma = \begin{pmatrix} \gamma_{11} & \tilde{\gamma}_{10} & 0 \\ \tilde{\gamma}_{01} & \gamma_{00}^+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.133)$$

Note that because $\gamma \geq 0$, the submatrices γ_{01} and γ_{10} also vanish next to the singular part of γ_{00} and reduce to $\tilde{\gamma}_{01}$, resp. $\tilde{\gamma}_{10}$. The equation for the operator A now reads

$$\gamma_{01} = \begin{pmatrix} \tilde{\gamma}_{01} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_{00}^+ & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \quad (5.134)$$

A solution to Eq. (5.134) is $A_1 = (\gamma_{00}^+)^{-1}\tilde{\gamma}_{01}$ and $A_2 = 0$. With this, Eq. (5.132) is merely an algebraic transformation, so we proved (1). For (2), note that the solution

to Eq. (5.134) is unique, only if $A\Xi_0 \subset \gamma_{00}\Xi_0$, so it is exactly the singular part of γ_{00} , which allows arbitrary A_2 . Finally, (3) follows by taking the real part of Eq. (5.134). Because γ is a real matrix, the real part \tilde{A} of a complex solution A still satisfies $\gamma_{01} = \gamma_{00}\tilde{A}$. \square

Item (1) makes clear that we can define the Schur complement as

$$\langle \phi_1 | \gamma^S | \phi_1 \rangle = \min_{\phi_0} \langle \phi_1 \oplus \phi_0 | \gamma | \phi_1 \oplus \phi_0 \rangle. \quad (5.135)$$

Of course, when γ_{00} is invertible, we have $A = \gamma_{00}^{-1}\gamma_{01}$ and $\gamma^S = \gamma_{11} - \gamma_{10}\gamma_{00}^{-1}\gamma_{01}$. In the literature, the general definition is often given by replacing the inverse with some pseudo-inverse (cf. [129, 130]). Our above approach via A clarifies why the pseudo-inverse's exact nature is irrelevant. A also turns out to be useful in the computation of conditionals.

Proposition 97. *Let γ and A be as in Lem. 96.*

- (1) *When G is the Gaussian joint distribution of x_1 and x_0 with covariance γ and mean $m = m_1 \oplus m_0$, then the conditional distribution of x_1 given x_0 is Gaussian with covariance γ^S and mean $m_1 + A^\top(x_0 - m_0)$.*
- (2) *Let $\chi(\xi) = \exp(-\xi \cdot \gamma \xi / 4 + i(m_1 \oplus m_0) \cdot \xi)$ be the characteristic function of a hybrid system. Then in the decomposition in Eq. (5.80) the measure μ is Gaussian with covariance γ_{00} and mean m_0 and the states ω_{x_0} are Gaussian with characteristic function*

$$\chi_{x_0}(\xi_1) = \exp(-\xi_1 \cdot \gamma^S \xi_1 / 4 + i(m_1 + A^\top(x_0 - m_0)) \cdot \xi_1). \quad (5.136)$$

Proof. We prove the two parts together by only insisting on a vanishing commutator form σ_0 on Ξ_0 , but allowing the commutator form σ_1 to be degenerate as well. Then (2) is the version for non-degenerate σ_1 , and (1) is the version for $\sigma_1 = 0$. In the proof, this hardly makes a difference. Since covariance matrices are real symmetric, we can take Ξ_0 and Ξ_1 to be real spaces. Complex matrices will arise only at the end when we have to verify the positive conditions for the claimed conditional characteristic function.

Consider the Gaussian measure G_0 given by the Ξ_0 -marginal, i.e., the Gaussian with covariance γ_{00} and mean m_0 , assume ω_{x_0} to be hybrid Gaussian with some covariance $\tilde{\gamma}(x_0)$ and mean $\tilde{m}(x_0)$, both depending on x_0 in some way. We will see that for the overall state being Gaussian, these dependencies must be rather special. The characteristic function is now

$$\begin{aligned} \chi(\xi_1 \oplus \xi_0) &= \int G_0(dx_0) e^{i\xi_0 \cdot x_0} e^{-\xi_1 \cdot \tilde{\gamma}(x_0) \xi_1 / 4 + i\xi_1 \cdot \tilde{m}(x_0)} \\ &\stackrel{!}{=} e^{-(\xi_1 \oplus \xi_0) \cdot \gamma (\xi_1 \oplus \xi_0) / 4 + i\xi_1 \cdot m_1 + i\xi_0 \cdot m_0}. \end{aligned} \quad (5.137)$$

The integral should be expressed in terms of the characteristic function of G . For this, the exponent must be imaginary and affine in x_0 . Since $\tilde{\gamma}(x_0)$ must be positive for all x_0 , it must actually be independent of x_0 . We set $\tilde{m}(x_0) = \tilde{m}_1 + Bx_0$ for

some $B : \Xi_0 \rightarrow \Xi_1$. Extracting terms in the exponent, which do not depend on the integration variable, we get

$$\begin{aligned}\chi(\xi_1 \oplus \xi_0) &= e^{-\xi_1 \cdot \tilde{\gamma} \xi_1 / 4 + i \xi_1 \cdot \tilde{m}_1} \int G_0(dx_0) e^{i \xi_0 \cdot x_0 + i \xi_1 \cdot B x_0} \\ &= e^{-\xi_1 \cdot \tilde{\gamma} \xi_1 / 4 + i \xi_1 \cdot \tilde{m}_1} \chi_0(\xi_0 + B^\top \xi_1).\end{aligned}\quad (5.138)$$

Equating exponents in Eq. (5.137) and Eq. (5.138) order by order, we find

$$\xi_1 \cdot \gamma_{11} \xi_1 = \xi_1 \cdot \tilde{\gamma} \xi_1 + \xi_1 B \gamma_{00} B^\top \xi_1, \quad (5.139)$$

$$\xi_0 \cdot \gamma_{00} \xi_0 = \xi_0 \cdot \gamma_{00} \xi_0, \quad (5.140)$$

$$\xi_0 \cdot \gamma_{01} \xi_1 = \xi_0 \cdot \gamma_{00} B^\top \xi_1, \quad (5.141)$$

$$\xi_1 \cdot m_1 = \xi_1 \cdot (\tilde{m}_1 + B m_0), \quad (5.142)$$

$$\xi_0 \cdot m_0 = \xi_0 \cdot m_0. \quad (5.143)$$

Apart from tautologies in Eq. (5.140) and Eq. (5.143) which arise from taking G_0 as the marginal, we get $\gamma_{01} = \gamma_{00} B^\top$, which is solved by $B = A^\top$. Thus by Eq. (5.139) $\tilde{\gamma} = \gamma^S$ is the Schur complement. Finally, from Eq. (5.142) we get $\tilde{m}_1 = m_1 - A^\top m_0$, and hence the mean of the conditional state $\tilde{m}(x_0) = m_1 + A^\top(x_0 - m_0)$.

This proves the claims of (2), which immediately specializes to (1) in the purely classical case. The difference between these two is only the positivity condition on the covariance of the conditional state. Thus, it remains to be shown that χ_{x_0} in (2) indeed defines a quantum state, i.e., $\gamma^S + i\sigma_1 \geq 0$. To this end consider the Schur complement of γ_{00} in the complex positive definite matrix $\gamma + i(\sigma_1 \oplus \sigma_0)$ with $\sigma_0 = 0$. Then the same real operator A satisfies the required equation for this case as for the complement in γ . Therefore, the Schur complement in $\gamma + i\sigma$ is equal to $\gamma^S + i\sigma_1$, which is therefore positive semi-definite. \square

One subtlety has to be addressed: We allow singular Gaussians, which are supported on a hyperplane, and hence γ_{00} with a non-trivial null space. In that case, we saw that A is not uniquely determined. But the conditional states χ_{x_0} do depend on this choice. Let us take the most extreme example of this, namely $\gamma_{00} = 0$. Then we also have $\gamma_{01} = 0$ by positive semi-definiteness, i.e., *any* operator A satisfies the required equation. In this case, however, the support of the Ξ_0 -marginal is a point, which is then also the mean m_0 . For x_0 in the support of this measure, we have $A^\top(x_0 - m_0) = 0$, and also for any A , i.e., the mean is fixed independently of the choice of A . For x_0 not in the support, ω_{x_0} is anyhow irrelevant since it does not contribute to the representation integral in Eq. (5.80). More generally, for merely partially singular γ_{00} , the support of a singular Gaussian measure consists of those x_0 for which $(x_0 - m_0) = \gamma_{00}y$ for some y . But for these, $A^\top(x_0 - m_0) = A^\top \gamma_{00}y = (\gamma_{00}A)^\top y = \gamma_{10}y$ depends only on the given data, not on a choice of A .

In summary, we can state that our hybrid ansatz works well with the Gaussian world. Gaussian states are described exactly as one would have expected, and also looking at the marginals of a hybrid Gaussian recreates our standard quantum and classical analogues.

Squeezed and singular states

We finish this section with the description of the two-mode squeezed state [131], which is a Gaussian entangled state, and illustrates the limit towards singular states.

Let us start with the rough idea behind squeezing: We have already mentioned that quantum mechanics forbids sharply concentrated states in phase space, which is probably most famously captured by the Heisenberg Uncertainty Principle. While this uncertainty relation forbids the preparation of a sharply concentrated in both position and momentum, it only bounds the product of its variances. Hence, we can sharpen our distribution for one of the two observables at the cost of the other.

The limit of this process, loosely speaking, is then a sharply concentrated distribution in one observable, while the other is completely smeared out. Such states are the typical archetypes for singular states. Doing this limit more carefully and going to really sharp distributions is an interesting exercise in singular states [132]. It also highlights the importance of the continuity condition in Bochner's Theorem (Thm. 89), which excludes these states.

The topic is also closely connected to the famous classic EPR-paper [133] by Einstein, Podolsky, and Rosen: Think of a standard quantum system with phase space Ξ and non-degenerate σ . We then consider an *opposite* system over $(\Xi, -\sigma)$, which can be realized in the same Hilbert space, and the Weyl system \overline{W} with $\overline{W}(a, b) = W(-a, b)$. That is in the position representation Eq. (5.72) we can write

$$\overline{W}(\xi) = \Theta W(\xi) \Theta \quad (5.144)$$

with Θ the complex conjugation in that representation. Then the combined operators $\widetilde{W}(\xi) = W(\xi) \otimes \overline{W}(\xi)$ commute, and so the according characteristic function $\chi_2(\xi) = \omega_2(\widetilde{W}(\xi))$ is not subject to uncertainty constraints. That is, it may correspond to a probability distribution sharply concentrated at the origin. In the EPR paper, the state is to have sharp distribution for the canonical operators $Q_1 - Q_2$ and $P_1 + P_2$. What they actually write down is an unnormalizable wave function, but the intention is clearly to have *pretty sharp* distributions for these operators, of course, at the expense of their canonical conjugates $P_1 - P_2$ and $Q_1 + Q_2$.

First of all, by weak*-compactness, the limit of states with sharper and sharper distributions for $Q_1 - Q_2$ and $P_1 + P_2$ exists (at least along a subnet) as a state on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Since we have not specified any details of the sequence, there are many states that might arise as a weak* cluster point of such a sequence, and since we are implicitly invoking the axiom of choice, there is no way to give a finite specification ensuring convergence on *all* observables.

However, for some simple operators, like the Weyl operators themselves, the brief characterization is sufficient to determine the limit. Thus we get a *characteristic function*, namely

$$\langle \omega_{\text{EPR}}, W(\xi) \otimes \overline{W}(\eta) \rangle = \begin{cases} 1 & \xi = \eta \\ 0 & \text{otherwise.} \end{cases} \quad (5.145)$$

This is clearly discontinuous and so corresponds to no density operator. It is also insufficient to specify the state on observables not expressed as linear combinations of Weyl operators. However, some things can be read off easily. For example, the

marginal for the first party ($\eta = 0$ in the above) will have the *characteristic function*, which vanishes everywhere except for $\xi = 0$. This means that the probability for finding a value in some finite interval $[a, b]$ for the spectral resolution of any canonical operator is zero. Although the joint probability for Q_1 and Q_2 may be in some sense concentrated on $x_1 = x_2$, the values themselves are infinite with probability one and can only be specified as infinite points in some (non-constructive) compactification of position space. This is hardly what the authors intended, so it is much more useful to stick with *pretty sharp* distributions.

This is also a practical issue for quantum optics, where squeezed states play an important role. Here, the **two-mode squeezed state** is the typical prototype:

Consider a standard Gaussian product state of the doubled system that is

$$\chi_0(\xi_1 \oplus \xi_2) = \exp\left(-\frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)\right). \quad (5.146)$$

Now we apply the transformation $(p_1, p_2) \mapsto (cp_1 + sp_2, sp_1 + cp_2)$ with $c = \cosh \lambda$, $s = \sinh \lambda$ (a hyperbolic rotation), and the inverse to the positions, so that the symplectic form $\xi \cdot \sigma \xi' = p \cdot q' - q \cdot p'$ remains invariant. Then, taking into account the inversion of one of the arguments in \overline{W} , we get a state ω_λ with

$$\langle \omega_\lambda, W(\xi) \otimes \overline{W}(\eta) \rangle = \exp\left(-\frac{1}{4}(e^{2\lambda}(\xi - \eta)^2 + e^{-2\lambda}(\xi + \eta)^2)\right), \quad (5.147)$$

where $\xi^2 = (p, q)^2 = p^2 + q^2$. When $\eta = \xi$, this converges pointwise to Eq. (5.145), but since the limit is not continuous, Lévy's convergence theorem [3] does not apply, which would otherwise yield the existence of a continuous characteristic function, i.e., an according normal state.

Taking just the case $\xi = \eta$, we see that Eq. (5.147) converges pointwise to 1 as $\lambda \rightarrow \infty$. This property will be useful in the proof of Cor. 120. We therefore generalize it to arbitrary hybrids:

Lemma 98. *For any hybrid system (Ξ, σ) there is a family of states ω^ε for the system $(\Xi \oplus \Xi, \sigma \oplus (-\sigma))$ such that*

$$\lim_{\varepsilon \rightarrow 0} \langle \omega^\varepsilon, W(\xi) \otimes \overline{W}(\xi) \rangle \geq \exp\left(-\frac{\varepsilon}{2} \xi \cdot A \xi\right) \quad \text{for all } \xi \in \Xi, \quad (5.148)$$

where A is the covariance matrix of a quantum state on (Ξ, σ) , i.e., $A + i\sigma \geq 0$. In particular, the left-hand side goes to 1 as $\varepsilon \rightarrow 0$.

Proof. When (Ξ, σ) is a direct sum of other spaces, we can reduce the proof to each summand by taking tensor products. We need only two types of summands: single degree of freedom quantum systems for which the two-mode squeezed state provides just the required family of states with $\varepsilon = e^{-2\lambda}$, $\lambda \rightarrow \infty$, and one-dimensional classical systems. For these, we can take any probability measure on the doubled system, which is concentrated on the diagonal. This will satisfy the condition even with $\varepsilon = 0$. \square

5.3 Hybrid Observable Algebras

The settings

This section aims to complement the description of hybrid states by spaces of observables. Any standard representation in the sense of Def. 86 gives us a natural observable algebra, the von Neumann algebra

$$\mathcal{B}(\mathcal{H}_1) \otimes L^\infty(\Xi_0, \mu), \quad (5.149)$$

in which observables are $\mathcal{B}(\mathcal{H}_1)$ -valued functions on Ξ_0 and we have basically all the problems of the W^* -view described in Sect. 3.3.1.

This algebra depends on μ , and any choice of μ excludes some states. With the Lebesgue measure for μ , we exclude all pure states, and while we can add countably many point measures to μ , we would still miss uncountably many others.

When it comes to quasifree channels, a μ -dependent description brings in additional assumptions, so it becomes much more cumbersome to formulate results that hold for all quasifree channels. In contrast, our description of states on $C^*(\Xi, \sigma)$ is already free of such constraints, and we will now develop a matching description of observables and, later, of channels. So we have two complementary points of view:

The setting with fixed μ Here, we have a natural Hilbert space

$$\mathcal{H}_\mu = \mathcal{H}_1 \otimes L^2(\Xi_0, \mu), \quad (5.150)$$

where \mathcal{H}_1 is the Hilbert space of the Schrödinger representation of the quantum part Ξ_1 . It carries a standard representation (in the sense of Def. 86) of the Weyl operators and the basic C^* -algebra $C^*(\Xi, \sigma) = \mathcal{K}(\mathcal{H}_1) \otimes \mathcal{C}_0(\Xi_0)$. The density operators on \mathcal{H}_μ precisely give those states whose distribution for Ξ_0 is absolutely continuous with respect to μ , i.e., states in

$$\mathcal{T}^1(\mathcal{H}_1) \otimes L^1(\Xi_0, \mu). \quad (5.151)$$

Every state $\omega \in C^*(\Xi, \sigma)^*$ is represented in such a structure, but unless we want to go to non-separable Hilbert spaces, like the direct sum of *all* such \mathcal{H}_μ , every standard representation misses many states.

The μ -free setting This is the point of view based on the C^* -algebra

$$\mathcal{A} = C^*(\Xi, \sigma), \quad (5.152)$$

allowing all states of \mathcal{A} . The maximal space of observables, for which these states provide probability distributions, is, by definition, the bidual \mathcal{A}^{**} . To get the connection with the μ -dependent view, consider the map

$$i_\mu : \mathcal{T}^1(\mathcal{H}_1) \otimes L^1(\Xi_0, \mu) \rightarrow C^*(\Xi, \sigma)^*, \quad (5.153)$$

which identifies the states in the μ dependent view with the states in the μ -free setting. The adjoint of the embedding is then the restriction map, the representation

$$i_\mu^* : C^*(\Xi, \sigma)^{**} \rightarrow \mathcal{B}(\mathcal{H}_1) \bar{\otimes} L^\infty(\Xi_0, \mu) \subset \mathcal{B}(\mathcal{H}_1 \otimes L^2(\Xi_0, \mu)). \quad (5.154)$$

This representation destroys all information in $A \in C^*(\Xi, \sigma)^{**}$, which is irrelevant for states absolutely continuous with respect to μ , i.e., identifies functions coinciding μ -almost everywhere.

The requirements

Neither of the two obvious choices for observable spaces in the μ -free setting is feasible. The largest choice mentioned above is the second dual $C^*(\Xi, \sigma)^{**}$, which is in many ways *too large* so that individual elements often have no explicit description. The difficulty here arises mostly from the classical part: While the second dual of the compact operators is just the space of all bounded operators, the second dual $\mathcal{C}_0(X)^{**}$ is a rather complex object [134]. Elements of this space are not functions on X , but on a related, much larger Stonean topological space \widehat{X} . Its points are the extreme points of the normalized positive elements in the triple dual $C^*(\Xi, \sigma)^{***}$, another highly non-constructive object. Hence, while the elements in $C^*(\Xi, \sigma)^{**}$ admit a function representation, it is impossible to explicitly describe even a single point of the classical variables space on which they are supposed to be *functions*.

At the other end, small algebras, we have the algebra $C^*(\Xi, \sigma)$ itself. This is in many ways *too small*. Indeed, $C^*(\Xi, \sigma)$ does not have an identity, which is needed for the physical interpretation as an observable algebra. Also, it does not allow quantum operators with a continuous spectrum, barring the Weyl operators themselves. Therefore, we will have to choose some intermediate algebra \mathcal{M} with

$$C^*(\Xi, \sigma) \subset \mathcal{M} \subset C^*(\Xi, \sigma)^{**}. \quad (5.155)$$

The criteria for this choice are simple:

- \mathcal{M} should be constructed in a way that makes sense for every hybrid system.
- Applying a quasifree channel \mathcal{T} an observable in \mathcal{M}_{out} for the output system should give an observable in \mathcal{M}_{in} .

Roughly speaking, \mathcal{M} will describe a degree of regularity for observables, preserved by all quasifree channels, leading to an automatic Heisenberg picture between the corresponding observable algebras. This could be expressed as regularity properties of operator-valued functions on the classical phase space. However, such an approach introduces many case distinctions for proofs of the second of the above criteria: It depends on how inputs and outputs are split into classical and quantum parts and how a quasifree channel reshuffles these splits. It turns out to be much more efficient to work with constructions that apply to classical, quantum, and hybrid systems alike.

Here we follow the path of a functional analyst doing analysis, namely Gert Pedersen, who is the author of the textbook *Analysis now* [10], which we consequently already used as one of the primary references in the introduction (Sect. 2). The following section is based on his seminal early work and, despite its relative abstractness, leads painlessly to just the Heisenberg picture characterizations we wanted to see.

5.3.1 Review: Semicontinuity in C*-algebras

The constructions in this section are inspired by the commutative case [134] but have been generalized to arbitrary C*-algebras in [135, 136, 137] (see also [8, III,§6] and [12] for textbook versions).

The commutation relations are not needed for this, so we will consider first a general C*-algebra \mathcal{A} , typically without a unit, and only in the next section specialize to hybrids, i.e., $\mathcal{A} = \mathcal{C}_0(X, \mathcal{K}(\mathcal{H}))$. The standard states are then elements of the dual, and we are interested in well-behaved subalgebras of the bidual. A special role will be played by the pure states of \mathcal{A} .

We start with some more definitions and facts that were not already introduced in Sect. 2.2. From Sect. 2.2.2 we know that \mathcal{A}^{**} can be identified with its enveloping von Neumann algebra, which is the von Neumann algebra generated by \mathcal{A} in its universal representation. This is simply the direct sum of all GNS-representations of \mathcal{A} . The **center** $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$ of a von Neumann algebra \mathcal{M} is likewise a von Neumann algebra, and here it is the natural arena for the representation theory of \mathcal{A} .

We call \mathcal{M} a **factor** if the only elements in \mathcal{Z} are scalar multiples of the identity. Based on the factors, each von Neumann algebra can be decomposed and classified into three different types, called type I, II, and III. The self-adjoint elements of the center, denoted by \mathcal{Z}_{sa} , also form a complete real vector lattice [12, 2.6.1].

Definition 99. For an element $A \in \mathcal{M}_{\text{sa}}$, we define its **central cover** z_A as the infimum of all $Z \in \mathcal{Z}_{\text{sa}}$ with $A \leq Z$.

If A is a projection, so is z_A [12, 2.6.2]. Next, we connect the above definition of a central cover from elements in the universal enveloping von Neumann algebra \mathcal{A}^{**} to representations of the underlying C*-algebra \mathcal{A} .

Let $\{\pi, \mathcal{H}\}$ be a non-degenerate representation of \mathcal{A} . By Prop. 32, there exists a normal representation $\pi'' : \mathcal{A}'' \rightarrow \pi(\mathcal{A})''$ that extends π . One can show that the image of π'' is isomorphic to \mathcal{A}^{**} for some central projection $p \in \mathcal{Z}(\mathcal{A}^{**})$ [12, Cor. 2.5.5].

Definition 100. We call the central cover of the projection p , such that $p\mathcal{A}^{**}$ is isomorphic to $\pi(\mathcal{A})''$, the **central cover** of the representation $\{\pi, \mathcal{H}\}$ and denote it by z_π .

Before we can state the next result, we need another notion of equivalence for representations [12, 3.3.6].

Definition 101. We call two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ of a C*-algebra \mathcal{A} **quasi-equivalent** if there is an isomorphism h of $\pi_1(\mathcal{A})''$ to $\pi_2(\mathcal{A})''$ such that

$$h(\pi_1(A)) = \pi_2(A) \quad \text{for all } A \in \mathcal{A}. \quad (5.156)$$

Now, we can state the following theorem that will allow us to classify the representations of \mathcal{A} , up to quasi-equivalence, by the central projections in \mathcal{A}^{**} [12, Thm. 3.8.2].

Theorem 102. *Two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ of a C^* -algebra \mathcal{A} are quasi-equivalent if and only if $z_{\pi_1} = z_{\pi_2}$, and the map $(\pi, \mathcal{H}) \rightarrow z_\pi$ gives a bijective correspondence between equivalence classes of representations of \mathcal{A} and non-zero central projections in \mathcal{A}^{**} .*

Consider now a pure state $\omega \in \mathcal{A}^*$. Its GNS-representation is irreducible, so the corresponding central projection z_π is minimal. Here **minimal** means that there is no projection p in the center of \mathcal{A}^{**} , other than 0 and z_π , such that $0 \leq p \leq z_\pi$. Minimal projections are also called the **atoms** of the projection lattice, and the minimal projections of \mathcal{A}^{**} correspond precisely to the pure states on \mathcal{A} . We record this observation as a statement for arbitrary von Neumann algebras (see [138]).

Lemma 103. *Let \mathcal{M} be a von Neumann algebra. Then there is a one-to-one correspondence between minimal projections $p \in \mathcal{M}$ and extremal normal states $\omega \in \mathcal{M}_*$, given by*

$$pxp = \omega(x)p, \quad \text{for all } x \in \mathcal{M}. \quad (5.157)$$

Proof. For any projection p , we consider the von Neumann subalgebra $\widetilde{\mathcal{M}} = p\mathcal{M}p$. The crucial issue is whether this algebra is one-dimensional. When that is the case, we must have $pxp = \omega(x)p$ for some functional ω , which is necessarily a normal state. We will proceed by showing the implications “ p minimal” \Leftrightarrow “ $\dim \widetilde{\mathcal{M}} = 1$ ” \Rightarrow “ ω extremal” \Rightarrow “the support projection of ω satisfies $\dim \widetilde{\mathcal{M}} = 1$ ”.

Indeed, p is minimal iff, for any projection q , $0 \leq q \leq p$ implies $q = 0$ or $q = p$. This is equivalent to 0 and p being the only projections in $\widetilde{\mathcal{M}}$, i.e., to $\dim \widetilde{\mathcal{M}} = 1$.

In this case, consider the state ω . Then from $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ we conclude $\omega_1 \leq \lambda^{-1}\omega$, hence

$$|\langle \omega_1, x(1 - p) \rangle|^2 \leq \langle \omega_1, (1 - p)x^*x(1 - p) \rangle \leq \lambda^{-1} \langle \omega, (1 - p)x^*x(1 - p) \rangle = 0. \quad (5.158)$$

But then $\langle \omega_1, x \rangle = \langle \omega_1, pxp \rangle = \langle \omega, x \rangle \langle \omega_1, p \rangle$ and $\langle \omega_1, p \rangle = 1$ by choosing $x = \mathbb{1}$ in this equation. Hence $\omega_1 = \omega$, so ω is extremal.

Now let ω be extremal and normal, and let p be its support, i.e., the smallest projection such that $\langle \omega, p \rangle = 1$. Then ω restricted to $\widetilde{\mathcal{M}}$ is also pure and, in addition, faithful, i.e., $x \in \widetilde{\mathcal{M}}$ with $\langle \omega, x^*x \rangle = 0$ implies $x = 0$. Indeed, the eigenprojection $q \in \widetilde{\mathcal{M}}$ of x^*x for the spectral set $\{0\}$ must then satisfy $\langle \omega, q \rangle = 1$. Hence on the one hand $q \leq p$, because $q \in p\mathcal{M}p$, and $q \geq p$ by minimality of the support projection p . Therefore $x^*x = 0$.

So consider the GNS-representation π_ω of a faithful normal pure state. Faithfulness implies that the representation is injective, and general representation theorems [15, 1.16.2] imply that the image $\pi_\omega(\widetilde{\mathcal{M}})$ is a von Neumann algebra. By purity, $\pi_\omega(\widetilde{\mathcal{M}})$ is irreducible, so $\pi_\omega(\widetilde{\mathcal{M}}) = \mathcal{B}(\mathcal{H}_\omega)$. On the other hand, ω is given by a vector $\Omega \in \mathcal{H}_\omega$. So, unless $\dim \mathcal{H}_\omega = 1$, there is a vector orthogonal to it and hence a non-zero element x with $\pi_\omega(x)\Omega = 0$, and hence $\langle \omega, x^*x \rangle = 0$, contradicting faithfulness. Hence $\dim \mathcal{H}_\omega = 1$, and $\dim \widetilde{\mathcal{M}} = 1$. \square

For many von Neumann algebras, this is a statement about the empty set, namely when \mathcal{M} is of type II or III, or $L^\infty(X, \mu)$ when there are no points with positive μ -measure. However, for a second dual, there are many extreme points. Their central cover is the smallest projection z_a so that $p \leq z_a$ for all minimal projections and

$z_\alpha \mathcal{A}^{**}$ is called the **atomic representation** of \mathcal{A}^{**} . Similar to the construction of the universal representation in Eq. (2.49), we can replace the direct sum over all cyclic representation to those of pure states and get the **universal atomic representation** [8, Def. III.6.35]. This part will determine our subalgebras and be useful for a function representation.

Indeed, consider for a moment the classical case $\mathcal{A} = \mathcal{C}_0(X)$. Then there is a simple way to associate with an element $A \in \mathcal{A}^{**}$ a function \check{A} on X , namely to evaluate A in the pure state δ_x , the point measure at $x \in X$, setting $\check{A}(x) = \delta_x(A)$. However, when A has support in the complement of the atomic subspace, also called the **diffuse subspace**, we get

$$\delta_x(A)p = pAp = 0, \quad (5.159)$$

where p is the projection associated with δ_x via Eq. (5.157). Hence, $\check{A} = 0$, so the function \check{A} has nothing to say about A .

Nevertheless, for suitable subalgebras of \mathcal{A}^{**} , the atomic representation, and hence the function representation \check{A} , contains complete information. The idea of [135] is to use monotone limits to construct useful algebras with this property. Since these constructions work in the same way in arbitrary C^* -algebras, they also provide a Heisenberg picture for general dual channels.

In \mathcal{A}^{**} bounded, increasing nets are automatically weak*-convergent, and if this algebra is represented on a Hilbert space, the limits exist in the strong operator topology. This makes the most sense in the hermitian part \mathcal{A}_h^{**} of \mathcal{A}^{**} . For any subset $M \subset \mathcal{A}_h^{**}$, we denote by M^m the set of limit points of such nets from M . Similarly, $M_m = -(-M)^m$ represents the limits of decreasing nets from M [8, III.6].

Definition 104. *Let \mathcal{A} be a C^* -algebra. Then*

$$\mathcal{A}_\uparrow := (\mathcal{A}_h + \mathbb{R}\mathbb{1})^m$$

*is called the **lower semicontinuous cone** of \mathcal{A}_h^{**} . Accordingly the **upper semicontinuous cone** is defined as $\mathcal{A}_\downarrow = (\mathcal{A}_h + \mathbb{R}\mathbb{1})_m$.*

We remark that there are some subtle distinctions in defining the semicontinuous cone, depending on whether the unit is adjoined first (as above) and on whether a norm closure of the cone is taken. These are discussed carefully in [136, 137] and are beyond the scope of our application. Our focus lies in their connection to the multiplier algebra:

Definition 105. *The **multiplier algebra** of a C^* -algebra \mathcal{A} , denoted by $M(\mathcal{A})$, is the set of elements $m \in \mathcal{A}^{**}$ such that, for all $a \in \mathcal{A}$, $ma \in \mathcal{A}$ and $am \in \mathcal{A}$.*

Here the main observation for us is the following [8, Thm. III.6.24].

Theorem 106. *Let \mathcal{A} be a C^* -algebra and $M(\mathcal{A})$ its multiplier algebra. Then we have*

$$\mathcal{A}_\uparrow \cap \mathcal{A}_\downarrow = M(\mathcal{A})_h. \quad (5.160)$$

The multiplier algebra is the first candidate for our well-behaved subalgebras of the bidual. Another one is the following [8, Def. III.6.31f].

Definition 107. Let \mathcal{A} be a C^* -algebra. An element $a \in \mathcal{A}_h^{**}$ is called **universally measurable** if, for every state $\omega \in \mathcal{A}^*$, and every $\varepsilon > 0$, there are $x \in \mathcal{A}_\uparrow$, $y \in \mathcal{A}_\downarrow$ such that

$$x \leq a \leq y \quad \text{and} \quad \omega(y - x) < \varepsilon. \quad (5.161)$$

The real vector space of universally measurable elements of \mathcal{A} is denoted by $\mathcal{U}(\mathcal{A})$.

A useful property of this space is that it works perfectly with the above introduced atomic representation of the bidual [8, Thm. III.6.37]:

Theorem 108. The universal atomic representation of \mathcal{A} is isometric on the class $\mathcal{U}(\mathcal{A})$ of all universally measurable elements in \mathcal{A}_h^{**} .

In the classical case, $\mathcal{A} = \mathcal{C}_0(X)$ with X locally compact, the lower semicontinuous cone consists just of the bounded lower semicontinuous functions f in the sense of point-set topology (lower level sets $\{x | f(x) \leq a\}$ are closed). The multipliers are all bounded continuous functions [139], i.e.,

$$M(\mathcal{A}) = \mathcal{C}_b(X). \quad (5.162)$$

For the universally measurable functions, note that, for a fixed measure $\mu \in \mathcal{C}_0(X)^*$, by definition, all bounded Borel measurable functions can be integrated. However, one usually completes the Borel algebra by including all μ -null sets. The completion can be understood by adding all sets which can be approximated from above and below by Borel measurable sets, whose μ -volume differs by arbitrarily little. The completion construction depends on μ , but some sets will be added for all μ , and these are called universally measurable [55, 8.4].

The functions that are measurable for the completed σ -algebra are called μ -measurable, and their classes up to μ -a.e. equality form $L^\infty(X, \mu)$. The approximation from above and below for defining μ -measurable sets has its counterpart for functions in the definition given above, with fixed $\mu = \omega$. Hence, the universally measurable functions are those that are μ -measurable for all μ .

Heisenberg pictures

After we have now certainly fulfilled the relative abstractness promised at the end of the previous section, we still have to show that these subsets of \mathcal{A}^{**} allow for a proper definition of dynamics. Indeed, for both of these subsets of observables, there is an automatic Heisenberg picture for channels initially defined on states:

Lemma 109. Let \mathcal{A} and \mathcal{B} be C^* -algebras and $T : \mathcal{A}^* \rightarrow \mathcal{B}^*$ a linear map taking states to states and $T^* : \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ the dual map. Then the inclusions $T^*M(\mathcal{B}) \subset M(\mathcal{A})$, $T^*\mathcal{B}_\uparrow \subset \mathcal{A}_\uparrow$ and $T^*\mathcal{U}(\mathcal{B}) \subset \mathcal{U}(\mathcal{A})$ hold.

Proof. Dual channels $T^* : \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ preserve positivity and normalization. The latter condition can be written as $T^*\mathbb{1} = \mathbb{1}$. They map increasing nets to increasing nets and are continuous for the respective limits. Hence $T^*\mathcal{B}_\uparrow \subset \mathcal{A}_\uparrow$. Then, the characterization of multipliers as both upper and lower continuous (Thm. 106) shows $T^*M(\mathcal{B}) \subset M(\mathcal{A})$, which is actually not obvious from just the definition of multipliers.

For the universally measurable class we proceed directly: Fix $b \in \mathcal{U}(\mathcal{B})$ and $\varepsilon > 0$. Then by definition we can find $b_i \in \mathcal{B}_i$ for $i = \uparrow, \downarrow$ such that $b_\uparrow \leq b \leq b_\downarrow$ and $(T\omega)(b_\downarrow - b_\uparrow) \leq \varepsilon$ dualizing T in the last inequality and applying T^* to the inequality for b gives the required upper and lower bounds T^*b_i for T^*b . \square

We note that such inclusions are always equivalent to a continuity condition for T . If we chose subspaces $\tilde{\mathcal{A}} \subset \mathcal{A}^{**}$ and $\tilde{\mathcal{B}} \subset \mathcal{B}^{**}$, the inclusion $T^*(\tilde{\mathcal{B}}) \subset \tilde{\mathcal{A}}$ is equivalent [13, IV.2.1] to the continuity with respect to the weak topologies $\sigma(\mathcal{A}^*, \tilde{\mathcal{A}})$ and $\sigma(\mathcal{B}^*, \tilde{\mathcal{A}})$, which are defined to make just those linear functionals $\mathcal{A} \rightarrow \mathbb{C}$ continuous, which are given by elements of $\tilde{\mathcal{A}}$ (and similarly for \mathcal{B}).

5.3.2 Hybrid observables as functions

Let us now apply the ideas of the previous section to our hybrid systems. The symplectic form and the group theoretical structure of the phase space do not play a key role here, so we are looking at a slightly more general case, where the underlying C*-algebra is $\mathcal{A} = \mathcal{K}(\mathcal{H}) \otimes \mathcal{C}_0(X)$, where $\mathcal{K}(\mathcal{H})$ denotes the compact operators on a separable Hilbert space \mathcal{H} , and X is a locally compact metrizable space generalizing Ξ_0 . By Thm. 66, we can identify \mathcal{A} with $\mathcal{C}_0(X, \mathcal{K})$, the functions $X \mapsto \mathcal{K}$ which are continuous in norm and vanishing in norm at infinity. Under this identification, $A = f \otimes K$ becomes the function $A(x) = f(x)K$. In the introduction of [137], this algebra is actually suggested as the intuition-building model case for the theory we outlined in the previous section.

Going to the first dual \mathcal{A}^* , we find the states, and their disintegration as in Eq. (5.80): We can write each state ω as

$$\omega(f \otimes A) =: \langle \omega, f \otimes A \rangle = \int \mu(dx) m(x) f(x) \operatorname{tr}(\rho_x A), \quad (5.163)$$

where μ is a (not necessarily finite) measure on X , $m \in L^1(X, \mu)$ is a probability density, so that μm is an arbitrary probability measure. This splitting of the classical marginal merely emphasized that we may realize states absolutely continuous with respect to the same μ in the same Hilbert space. In Eq. (5.163) it is clear that the integrand at each point x is a linear functional in $A \in \mathcal{K}$, hence given by a density operator ρ_x . By taking linear combinations and norm limits, we can write this in terms of the operator-valued function $A(x) = \sum_j f_j(x) A_j$ as

$$\omega(A) =: \langle \omega, A \rangle = \int \mu(dx) m(x) \operatorname{tr}(\rho_x A(x)), \quad (5.164)$$

or, in a useful shorthand notation,

$$\omega = \int^{\oplus} \mu(dx) m(x) \rho_x. \quad (5.165)$$

Now Eq. (5.164) is the relation we want to extend to a much larger class of operator-valued functions $x \mapsto A(x)$.

It is worth noting that ρ_x could be modified on μ -null sets without change, and for the sake of this integral expression, $A(x)$ might be similarly modified. This is the hallmark of the μ -dependent approach. The states obtained with fixed μ are in the tensor product $\mathcal{T}(\mathcal{H}) \otimes L^1(X, \mu)$, the norm completion of the span of the product states $\rho \otimes m$ for which $\rho_x \equiv \rho$ is constant. We can also express this by the map

$$i_\mu : \mathcal{T}(\mathcal{H}) \otimes L^1(X, \mu) \rightarrow \mathcal{A}^*, \quad \langle i_\mu(\rho \otimes m), A \rangle = \int \mu(dx) m(x) \operatorname{tr}(\rho A(x)). \quad (5.166)$$

This provides the first (“ μ -dependent”) way of associating operator-valued functions to elements of \mathcal{A}^{**} :

For every $A \in \mathcal{A}^{**}$, we can consider $i_\mu^*(A) \in \mathcal{B}(\mathcal{H}) \bar{\otimes} L^\infty(X, \mu)$, the von Neumann algebra tensor product [8, Ch. IV.5], which is the dual of $\mathcal{T}(\mathcal{H}) \otimes L^1(X, \mu)$, and also the von Neumann algebra generated by \mathcal{A} in its representation on $\mathcal{H} \otimes L^2(X, \mu)$.

Note that while i_μ^* is onto, it is not injective: all details of A related to μ -null sets are obliterated.

The μ -free alternative is given by the formula

$$\langle \rho \otimes \delta_x, A \rangle = \text{tr } \rho \check{A}(x) \quad \text{for } A \in \mathcal{A}^{**}. \quad (5.167)$$

This is based on the observation that for fixed A , the evaluation on the left-hand side is a bounded linear functional with respect to ρ , and thus of the given form with a unique $\check{A}(x) \in \mathcal{B}(\mathcal{H})$. For fixed x this extends the point evaluation at x , i.e., $(\mathbb{1} \otimes \delta_x) : \mathcal{K} \otimes \mathcal{C}_0(X) \rightarrow \mathcal{K}$, to functions in \mathcal{A} . It is an extension by continuity for the weak topology induced by \mathcal{A}^* so $\check{A}(x) = (\mathbb{1} \otimes \delta_x)^{**}(A)$.

To contrast these approaches, take μ to be Lebesgue measure or any other *diffuse* measure assigning $\mu(\{x\}) = 0$ to every singleton. Let z_μ be the common support projection of the states $i_\mu(\rho \otimes m)$, i.e., the smallest projection in the W^* -algebra \mathcal{A}^{**} which gives probability 1 to all such states. Then, since δ_x and μ are disjoint, their support projections in $\mathcal{C}_0(X)^{**}$ are orthogonal, so $\check{z}_\mu = 0$, although $i_\mu^*(z_\mu) = 1$. Similarly, we can consider z_a , the smallest projection in \mathcal{A}^{**} giving 1 on all pure states $|\psi\rangle\langle\psi| \otimes \delta_x$. In this case $i_\mu^*(z_a) = 0$, although $\check{z}_a(x) = 1$ for all x . So, the two ways of assigning a function can be diametrically opposite. On the other hand, for $A \in \mathcal{A}$, both approaches give the continuous function representation that we started from, although $i_\mu^*(A)$ is strictly speaking an equivalence class up to μ -a.e. equality. The question is then how far we can extend this agreement if we avoid *wild* elements like z_μ and z_a . In the following proposition, this is answered by the notion of universal measurability and illustrated in Fig. 5.2.

Note that in Fig. 5.2 and in general, we will often abbreviate the algebra of certain spaces by the according arguments, i.e., write $M(X, \mathcal{K})$ instead of $M(\mathcal{C}_0(X, \mathcal{K}))$ or $\mathcal{U}(\Xi, \sigma)$ instead of $\mathcal{U}(C^*(\Xi, \sigma))$, when there is no risk of confusion.

Proposition 110. *Let $A \in \mathcal{U}(\mathcal{A})$, and let $\omega = \int^\oplus \mu(dx) m(x) \rho_x \in \mathcal{A}^*$ be a state. Then the function $x \mapsto \text{tr } \rho_x \check{A}(x)$ is μ -measurable, and μ -almost everywhere equal to $i_\mu^*(A)$. Moreover,*

$$\langle \omega, A \rangle = \int \mu(dx) m(x) \text{tr}(\rho_x \check{A}(x)). \quad (5.168)$$

and $\|A\| = \sup_x \|\check{A}(x)\|$.

Proof. Fix a state ω , and consider an increasing net $A_i \in \mathcal{A}_h + \mathbb{R}\mathbb{1}$ with limit $A \in \mathcal{A}_\uparrow$. Then from Eq. (5.167) we find $\check{A}(x) = \sup_i A_i(x)$. Also, in Eq. (5.164), the limit exists, and by the monotone convergence theorem, the integrand is indeed given by the pointwise supremum, i.e., $\text{tr}(\rho_x \check{A}(x))$. This shows the claim for $A \in \mathcal{A}_\uparrow$, and, of course, for $A \in \mathcal{A}_\downarrow$.

Now suppose $A \in \mathcal{U}(\mathcal{A})$, and $\varepsilon > 0$. Then we can find $X \in \mathcal{A}_\uparrow, Y \in \mathcal{A}_\downarrow$ such that $X \leq A \leq Y$ and $\langle \omega, Y - X \rangle \leq \varepsilon$. Let us first consider the case $\omega = \rho \otimes m$ with fixed ρ . Then the function $a_\rho(x) = \text{tr}(\rho \check{A}(x))$, and the similarly defined functions for X, Y (which are lower/upper semicontinuous) satisfy $x_\rho \leq a_\rho \leq y_\rho$, and their integrals with $m\mu$ differ by less than ε . By Def. 105, we conclude that a_ρ is universally measurable. This establishes the μ -measurability of the integrand in Eq. (5.168) for constant ρ and the formula itself. Hence $i_\mu^*(A)(x) = \check{A}(x)$, μ -almost everywhere.

Clearly, this extends to linear combinations $\sum_i \rho_i \otimes m_i$. These are norm dense in $\mathcal{T}(\mathcal{H}) \otimes L^1(X, \mu)$. Thus approximating a general state as in Eq. (5.165) by such step functions, we find that the functions $x \mapsto m(x) \operatorname{tr}(\rho_x \check{A}(x))$ converges in $L^1(X, \mu)$. So, the limit is μ -measurable, and the formula holds in general. We note that the function $x \mapsto \operatorname{tr}(\rho_x \check{A}(x))$ is *not universally* measurable, since the ρ_x depend on μ .

For the norm equality, we use Thm. 108, i.e., the fact that the atomic representation is isometric on $\mathcal{U}(\mathcal{A})$. Thus, we only have to show that

$$\sup_x \|\check{A}(x)\| = \sup_\omega \|\pi_\omega(A)\|, \tag{5.169}$$

where the supremum is over all pure states ω , and π_ω denotes the associated GNS representation. Now the pure states are of the form $\omega = |\psi\rangle\langle\psi| \otimes \delta_x$ with $\psi \in \mathcal{H}$, so we only have to show that $\|\check{A}(x)\| = \|\pi_\omega(A)\|$ for any pure state of this form, i.e., the right-hand side does not depend on ψ . Indeed, we will show that even $\pi_\omega(A) = \check{A}(x)$ up to the usual isomorphism around the GNS representation: By the discussion after the definition in Eq. (5.167) of $\check{A}(x)$, $(1 \otimes \delta_x)^{**}$ is a normal representation of \mathcal{A}^{**} on $\mathcal{B}(\mathcal{H})$, which is obviously irreducible, hence cyclic for any vector. Hence, if we identify ψ as the GNS vector, it exactly meets the description of π_ω . \square

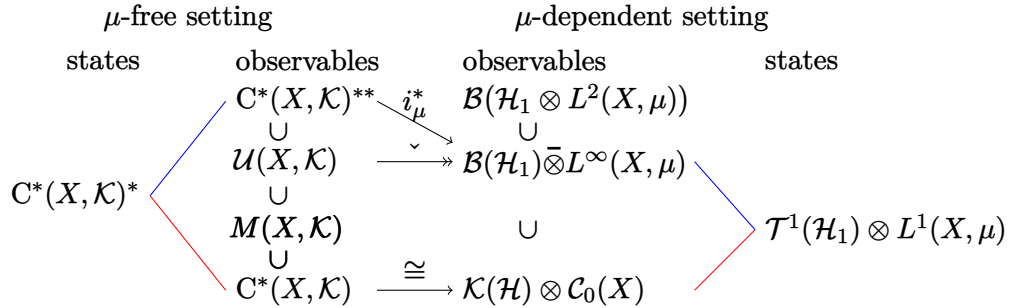


Figure 5.2: The two hybrid settings: The μ -free setting is based only on the C^* -algebra $C^*(X, \mathcal{K})$. In the μ -dependent setting, this algebra is represented on the Hilbert space $\mathcal{H}_\mu = \mathcal{H}_1 \otimes L^2(X, \mu)$. On the level of states, the connection is given by the map i_μ taking the state space on the far right to the one on the left (not shown). Its adjoint i_μ^* maps all the μ -free observable spaces to their μ -dependent counterparts. i_μ^* is surjective from $\mathcal{U}(X, \mathcal{K})$, indicated by a double arrow tip. This map is also realized by the function $A \mapsto \check{A}$ from Eq. (5.167) (cf. Prop. 110). It is injective on $M(X, \mathcal{K})$ if μ has full support.

Proposition 111. *Let X be locally compact, \mathcal{K} the algebra of compact operators on a separable Hilbert space \mathcal{H}_1 , and set $\mathcal{A} = \mathcal{C}_0(X, \mathcal{K})$. Then for $A \in M(\mathcal{A})$ the function \check{A} is strong*-continuous, i.e., $x \mapsto \check{A}(x)\psi$ and $x \mapsto \check{A}(x)^*\psi$ are both continuous for all $\psi \in \mathcal{H}$. Conversely, every uniformly bounded function \check{A} with this property defines a multiplier $A \in M(\mathcal{A})$.*

Sketch of proof: This is found in [139, Cor 3.4]. We nevertheless sketch the basic ideas of an approach not using the full-fledged theory.

According to Def. 105, multipliers can be thought of in terms of the operators

$$L_m, R_m : \mathcal{A} \rightarrow \mathcal{A}, \quad L_m(A) = mA, \quad R_m(A) = Am. \quad (5.170)$$

Their characteristic feature is $L_m(AB) = L_m(A)B$, and similarly for R_m . We note that the whole concept is redundant for C^* -algebras with a unit since then we just get multiplication with $L_m(\mathbb{1}) = m \in \mathcal{A}$. This suggests that the application of $L_m(U_\lambda)$ for a bounded approximate unit U_λ may help to characterize the action of L_m .

The first observation in the hybrid context is that these operators act pointwise, i.e., $(L_m(A))(x)$ depends only on $A(x)$, or, equivalently: $A(x) = 0 \Rightarrow (L_m A)(x) = 0$.

To show this, assume $A(x) = 0$, and take a bounded approximate unit $U_\lambda \in \mathcal{A}$ with $\|U_\lambda A - A\| \rightarrow 0$ for all $A \in \mathcal{A}$. Then, because L_m and U_λ are bounded, and the product in \mathcal{A} is defined pointwise:

$$\begin{aligned} \|(L_m A)(x)\| &\leq \|(L_m(A - U_\lambda A))(x)\| + \|L_m(U_\lambda)(x)A(x)\| \\ &\leq \|A - U_\lambda A\| + \|(L_m(U_\lambda))(x)\| \|A(x)\|. \end{aligned} \quad (5.171)$$

Then, the second term is equal to zero, and the first goes to zero as $\lambda \rightarrow 0$.

It follows that $L_m(A)(x) = L_m^x(A(x))$, where L_m^x is a multiplier of \mathcal{K} in the sense that it satisfies the basic relation $L_m^x(AB) = L_m^x(A)B$ for $A, B \in \mathcal{K}$. Now, the multiplier algebra of \mathcal{K} is known to be $M(\mathcal{K}) = \mathcal{B}(\mathcal{H})$. Indeed, for $A = |\Phi\rangle\langle\Psi|$ with a unit vector Ψ , we get

$$L_m^x(A) = L_m^x(A|\Psi\rangle\langle\Psi|) = L_m^x(A)|\Psi\rangle\langle\Psi| = |\tilde{\Phi}\rangle\langle\Psi| \quad (5.172)$$

for a suitable vector $\tilde{\Phi}$. Clearly, the map $\Phi \rightarrow \tilde{\Phi}$ is linear so that we can set $\tilde{\Phi} = M(x)\Phi$ for an operator $M(x)$, which is easily checked to be bounded. Since the operators $|\Phi\rangle\langle\Psi|$ span a dense subspace of the compact operators, $L_m^x(A) = M(x)A$.

On the right side, we get

$$R_m(A)(x) = (L_m^*(A^*)(x))^* = (M(x)^*(A^*(x)))^* = A(x)M(x). \quad (5.173)$$

It remains to check the continuity of M . To this end, we choose Φ, Ψ to be constant in a neighborhood of x . Then $M(x)|\Phi\rangle\langle\Psi|$ has to be a norm continuous function, i.e., $M(x)\Phi$ is continuous in norm. Using the right multiplier instead, we find that $M(x)^*\Psi$ likewise has to be continuous.

This concludes the argument starting from the multiplier condition. For the converse assume that, $x \mapsto A(x)$ is strong*-continuous, and $K \in \mathcal{C}_0(X, \mathcal{K})$. We have to show that $AK, KA \in \mathcal{C}_0(X, \mathcal{K})$. For this, we can assume that K is in a set whose linear hull is norm dense, namely the functions $K = |\psi\rangle\langle\psi| \otimes f$ with $\psi \in \mathcal{H}_1$ and $f \in \mathcal{C}_0(X)$ with compact support. By assumption, $A\psi$ is then norm continuous in \mathcal{H}_1 , and so AK is norm continuous. Since f has compact support, so does AK . The argument for $KA \in \mathcal{C}_0(X, \mathcal{K})$ is analogous.

We remark that in [139, Corr 3.4], the compacts are replaced by a more general algebra. The continuity is then in the natural topology for multipliers, the so-called strict topology, which is given by the seminorms

$$\|m\|_A = \|L_m A\| + \|R_m A\|. \quad (5.174)$$

For the compact operators, this coincides with the s^* -topology [40, I.8.6.3].

□

We do not have a similar characterization of $\mathcal{U}(\mathcal{A})$. In that case, it is clear from the proof of Prop. 110 that $\psi \mapsto \langle \psi, \check{A}(x)\psi \rangle$ must be universally measurable on X , and bounded by $\|A\|\|\psi\|^2$. This is enough to get the function representation in Eq. (5.168) of some element in $A \in \mathcal{A}^{**}$, but it is unclear (to us) whether this suffices to conclude $A \in \mathcal{U}(\mathcal{A})$. However, in the direction of stronger continuity conditions, we record the following for later use:

Corollary 112. *Assume that $X = \mathbb{R}^n$, and denote by $\alpha_x : \mathcal{A} \rightarrow \mathcal{A}$ the translation of functions, i.e., $(\alpha_x A)(y) = A(y + x)$. Then Eq. (5.167) provides a bijective correspondence between*

- (1) *elements $A \in \mathcal{U}(\mathcal{A})$, which are strongly continuous for α^{**} that is $\lim_{x \rightarrow 0} \|A - \alpha_x^{**}(A)\| = 0$, and*
- (2) *functions $\check{A} : X \rightarrow \mathcal{B}(\mathcal{H}_1)$, which are uniformly continuous in the sense that $\|\check{A}(x) - \check{A}(y)\| \leq \varepsilon$ if $|x - y| \leq \delta$.*

Such elements automatically are in $M(\mathcal{A})$.

Proof. Starting from (1), let \check{A} be the function defined by Eq. (5.167). Then the action of translations on the functions in $\mathcal{A} = \mathcal{K} \otimes \mathcal{C}_0(X)$ is lifted to the functions \check{A} . With the sign conventions analogous to Eq. (5.112), we get $\text{tr } \rho(\alpha_x \check{A})(y) = \langle \rho \otimes \delta_y, \alpha_x(A) \rangle = \langle \rho \otimes \delta_{x+y}, A \rangle = \text{tr } \rho \check{A}(x + y)$, and therefore $(\alpha_x^{**} A)(y) = \check{A}(x + y)$. By Prop. 110,

$$\|A - \alpha_x^{**}(A)\| = \sup_y \|\check{A}(y) - \check{A}(y + x)\|. \quad (5.175)$$

By assumption, this goes to zero as $x \rightarrow 0$, which is the stated uniform continuity with a coordinate change.

For the converse, uniform continuity of \check{A} implies the strong*-continuity of Prop. 111, so \check{A} defines a multiplier $A \in M(\mathcal{A})$, and the strong continuity estimate is again Eq. (5.175). \square

5.3.3 Translations and convolutions

We will later define quasifree channels by their covariance with respect to phase space translations. This hinges on the characterization of the joint eigenvectors of the translations, which is the topic of this section. Translations were part of our basic setup from the outset since the phase space Ξ is a vector space. The definition of twisted convolutions in Sect. 5.2.3 used this in an essential way. However, we need to introduce a notation for the translation maps themselves. In terms of canonical operators in a standard representation, a phase space shift acts as

$$R'_j = R_j + \eta_j \mathbb{1}, \quad (5.176)$$

where $\mathbb{1}$ stands for the identity operator or the constant 1-function. The size of the shift depends on j , and together, these parameters form the components of a vector $\eta \in \Xi$. In terms of Weyl operators, this means

$$W'(\xi) = \exp(i\xi \cdot R') = \exp(i\xi \cdot \eta) W(\xi) =: \alpha_\eta(W(\xi)). \quad (5.177)$$

Here, we have introduced the automorphism α_η , which expresses this symmetry as an automorphism on observables. If we think of $W(\xi)$ as an operator-valued function on Ξ_0 , we have to define it on more general operator-valued functions as

$$\alpha_\eta(F)(x) = W(\sigma\eta)F(x + \eta_0)W(\sigma\eta)^*, \quad (5.178)$$

where $\eta = \eta_1 \oplus \eta_0$. This formula makes sense for any of the observable algebras that are built from $\mathcal{B}(\mathcal{H}_1)$ -valued bounded functions on Ξ_0 , including the CCR-algebra in any standard representation, hence also $C^*(\Xi, \sigma)$. On standard states it is equivalent to

$$\alpha_\eta^*(\delta_x \otimes \rho) = \delta_{x-\eta_0} \otimes W(\sigma\eta)^* \rho W(\sigma\eta). \quad (5.179)$$

If we define α_η as an automorphism group on $C^*(\Xi, \sigma)$, in Eq. (5.177), we should really write α_η^{**} , but we will continue to use the same symbol also for this map. Then Eq. (5.177) just says that the Weyl operators $W(\xi) \in C^*(\Xi, \sigma)^{**}$ are joint eigenvectors of all translations. It will be crucial later on to turn this around:

Lemma 113. *Suppose that for some $F \in \mathcal{U}(\Xi, \sigma)$ we have $\alpha_\eta(F) = \lambda(\eta)F$, for some function $\lambda : \Xi \rightarrow \mathbb{C}$. Then there is $\xi \in \Xi$ and $c \in \mathbb{C}$, such that $F = cW(\xi)$, and $\lambda(\eta) = \exp(i\xi \cdot \eta)$.*

Proof. Note first that we must have $\lambda(\eta + \eta') = \lambda(\eta)\lambda(\eta')$. Moreover, λ must be a universally measurable function on Ξ [12, Prop. 7.4.5]. Since the only measurable characters on \mathbb{R}^{2s+n} are exponentials, we conclude that $\lambda(\eta) = \exp(i\xi \cdot \eta)$.

Consider $F' = FW(\xi)^*$. Then because α is a group of automorphisms, and $W(\xi)$ satisfies the required eigenvalue equation Eq. (5.177), we get that $\alpha_\eta(F') = F'$. It remains to prove that this implies that $F' = c\mathbb{1}$, since then $F = F'W(\xi) = cW(\xi)$.

It suffices to prove this in every standard representation, where Eq. (5.178) has a direct interpretation. Then, for all η_0, η_1 , $W(\eta_1)F'(x + \eta_0)W(\eta_1)^* = F'(x)$, where the operators $W(\eta_1)$ are Weyl operators in the Schrödinger representation. Setting first $\eta_0 = 0$, we thus conclude that $F'(x) = f(x)\mathbb{1}$ by irreducibility of the standard quantum Weyl operators. By setting $\eta_1 = 0$, we get that this f must be constant. Hence $F'(x) = c\mathbb{1}$. \square

Note that the measurability of $F \in \mathcal{U}(\Xi, \sigma)$ is required. For simplicity, we will construct an example in the classical case ($\sigma = 0$). Let $\xi \rightarrow \lambda(\xi)$ be an arbitrary homomorphism $\mathbb{R}^n \rightarrow \mathbb{C}$ into the unit circle, of which we do not require any continuity or measurability. It is well known that there are many discontinuous λ , which are then necessarily non-measurable (see [140, Ex. 3.2.4] or the review [141]). A simple construction uses a Hamel basis of \mathbb{R}^n as a vector space over \mathbb{Q} , i.e., a set of elements e_j , $j \in J$ such that every $\eta \in \mathbb{R}^n$ can be written uniquely as a finite linear combination $\eta = \sum_j \eta_j e_j$. Then we just set $\lambda(\eta) = \exp i \sum_j a_j \eta_j$, for arbitrary constants a_j . It is easily arranged that such a function is not continuous.

Now consider the set

$$\mathcal{M} = \left\{ F \in C^*(\Xi, \sigma)^{**} \mid \|F\| \leq 1, \text{ and, for all } \xi: \delta_\xi(F) = \overline{\lambda(\xi)} \right\}. \quad (5.180)$$

As a weak*-closed subset of the unit sphere, it is compact and nonempty because we can define F as a functional on the linear combinations of point measures by the condition in \mathcal{M} and then choose a Hahn-Banach extension. Now define the transformations $\beta_\eta = \overline{\lambda(\eta)} \alpha_\eta^{**}$. Because λ is a character, these maps leave \mathcal{M} invariant. They are also continuous and commute. Hence, by the Markov-Kakutani Fixed Point Theorem, they have a common fixed point F . F must be non-zero because it is in \mathcal{M} , and as a fixed point of the β_η it satisfies the equation $\alpha_\eta^{**} F = \lambda(\eta) F$. However, since λ is not continuous, it cannot be of the form given in the lemma.

Convolutions

In the previous section, we saw that hybrid observables in $M(X, \mathcal{K})$ are given by strong*-continuous operator-valued functions on X . Here, we will study a class with stronger continuity properties: On one hand, we demand the continuity to be in operator norm, and on the other, that it be uniform in X . This combination gives the continuity of $\xi \mapsto \alpha_\xi^{**}(A)$ in the norm of the hybrid bidual. In other words, F is strongly continuous for the translations. Since this property can be stated without explicitly observing the classical-quantum split, it will be easy to establish an automatic Heisenberg picture for quasifree channels (Prop. 121), even if quantum and classical degrees are strongly coupled.

For defining a suitable space of strongly continuous observables, we will make sure that the observable A has a good function representation in the first place, i.e., $A \in \mathcal{U}(\Xi, \sigma)$. This excludes unwanted elements like $\mathbb{1} - z_a$, which is even invariant under all α_ξ^{**} but has a vanishing function representation, as noted above. That is, we define

$$\mathcal{C}_u(\Xi, \sigma) = \{A \in \mathcal{U}(\Xi, \sigma) \mid \lim_{\eta \rightarrow 0} \|F - \alpha_\eta^{**}(F)\| = 0\}. \quad (5.181)$$

In this definition, we do not distinguish between quantum and classical translation directions. Restricting just to the classical part, Cor. 112 shows that $\mathcal{C}_u(\Xi, \sigma) \subset M(\Xi, \sigma)$. In addition, the argument $\check{A}(\xi_0) \in \mathcal{B}(\mathcal{H}_1)$ has to be strongly continuous for the quantum translations, and uniformly so with respect to ξ_0 .

A basic example is also given by the Weyl operators: Since we have $\alpha_\xi^{**} W(\eta) = \exp(i\xi \cdot \eta) W(\eta)$, the required continuity is immediate from the continuity of the phase factor. This shows that $\text{CCR}(\Xi, \sigma) \subset \mathcal{C}_u(\Xi, \sigma)$.

The algebra $\mathcal{C}_u(\Xi, \sigma)$ is still rather large, for example, not separable. In the context of Ludwig's axiomatic approach [142, 143], it seemed natural to single out a norm separable subspace $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ as a space of physical observables. One role of the space \mathcal{D} would be to determine a more realistic assessment of the distinguishability of states compared to norm or weak topologies. It turned out [121, 144] that in systems with canonical variables, the choices for \mathcal{D} on the quantum side are in one-to-one correspondence with choices on the classical side, which, in turn, can often be understood in terms of compactifications of phase space. For example, the CCR-algebra corresponds to the almost periodic functions and the Bohr compactification, whereas the compact operators correspond to $\mathcal{C}_0(\Xi)$ and adjoining the identity to the one-point compactification of X . In this section, we will show that the correspondence naturally also covers the hybrids between the fully quantum and the fully classical case. That is, the lattice of translation invariant closed subspaces of $\mathcal{C}_u(\Xi, \sigma)$ does not depend on σ .

This correspondence is best expressed in terms of the following notion of convolution. We denote by β_- the automorphism of phase space inversion, satisfying $\beta_-(W(\xi)) = W(-\xi)$, which is given by a coordinate change $\xi_0 \mapsto -\xi_0$ on the classical part and is implemented by the parity operator on the quantum part. The sign freedom in the following definition is due to the fact that $\mathcal{C}^*(\Xi, \sigma)^* = \mathcal{C}^*(\Xi, -\sigma)^*$: The twisted positive definiteness conditions in Eq. (5.89) for σ and $-\sigma$ both imply hermiticity ($\chi(-\xi) = \overline{\chi(\xi)}$), and with $\xi_k \mapsto -\xi_k$ and complex conjugation they become equivalent.

Definition 114. *Let Ξ be a real vector space with antisymmetric forms σ_1 and σ_2 and fix some signs $s_i = \pm 1$ for $i = 1, 2$. Then, for states $\omega_i \in \mathcal{C}^*(\Xi, \sigma_i)^*$ with characteristic functions χ_i , we define their **convolution**, denoted by $\omega_1 * \omega_2 \in \mathcal{C}^*(\Xi, s_1\sigma_1 + s_2\sigma_2)^*$ by its characteristic function $\chi(\xi) = \chi_1(\xi)\chi_2(\xi)$. For $\omega \in \mathcal{C}^*(\Xi, \sigma_1)^*$ and $F \in \mathcal{C}^*(\Xi, \sigma_2)^{**}$, we define $\omega * F = F * \omega \in \mathcal{C}^*(\Xi, s_1\sigma_1 + s_2\sigma_2)^{**}$ by evaluating it on an arbitrary $\omega' \in \mathcal{C}^*(\Xi, s_1\sigma_1 + s_2\sigma_2)^*$:*

$$\langle \omega', \omega * F \rangle = \langle \omega' * (\beta_- \omega), F \rangle. \quad (5.182)$$

Convolution is a bilinear operation $\mathcal{C}^*(\Xi, \sigma_1)^* \times \mathcal{C}^*(\Xi, \sigma_2)^* \rightarrow \mathcal{C}^*(\Xi, s_1\sigma_1 + s_2\sigma_2)^*$, which is obviously commutative, associative, and bi-positive. It is also translation invariant in the sense that $\alpha_\xi(\omega * F) = (\alpha_\xi \omega) * F = \omega * (\alpha_\xi F)$, which also shows why β_- is needed in Eq. (5.182). The freedom of the sign in the definition is used to get a classical state or observable function as the convolution of two quantum objects.

For pointwise estimates, it is useful to have a direct formula for the convolution, which bypasses the Fourier transform. When one factor is $\omega \in \mathcal{C}^*(\Xi, 0)^*$, e.g., a classical probability measure on Ξ , this is the usual average over translates of the other factor:

$$\omega * F = \int \omega(d\xi) \alpha_\xi(F), \quad (5.183)$$

where F could be an observable or another state, and the symplectic form is the same for F and $\omega * F$. For a state $\omega \in \mathcal{C}^*(\Xi, \sigma)^*$ and an observable $F \in \mathcal{C}_u(\Xi, \sigma)$ on the same hybrid, we get a uniformly continuous function $\omega * F \in \mathcal{C}_u(\Xi, 0)$:

$$(\omega * F)(\xi) = \omega(\alpha_\xi \beta_-(F)). \quad (5.184)$$

The hybrid generalization of correspondence theory [121, 145] is given in the following proposition. A state $\omega \in C^*(\Xi, \sigma_1 - \sigma_2)^*$ is called **regular** if it is norm continuous under translations (cf. Prop. 93) and its characteristic function vanishes nowhere.

Proposition 115. *Let Ξ be a vector space. Then the lattice of α -invariant closed subspaces of $\mathcal{C}_u(\Xi, \sigma)$ does not depend on σ . More precisely, let $\mathcal{D}_i \subset \mathcal{C}_u(\Xi, \sigma_i)$ be α -invariant closed subspaces, and $\omega_0 \in C^*(\Xi, \sigma_1 - \sigma_2)^*$ regular. Then the following are equivalent:*

- (1) $\omega * \mathcal{D}_1 \subset \mathcal{D}_2$ and $\omega * \mathcal{D}_2 \subset \mathcal{D}_1$ for all $\omega \in C^*(\Xi, \sigma_1 - \sigma_2)^*$.
- (2) The inclusions (1) hold for $\omega = \omega_0$.
- (3) \mathcal{D}_2 is the closure of $\omega_0 * \mathcal{D}_1$.
- (4) $\mathcal{D}_2 = \{A \in \mathcal{C}_u(\Xi, \sigma_1) \mid \omega_0 * A \in \mathcal{D}_1\}$.

Note that because (1) does not depend on ω_0 , each of the following items holds for all regular ω_0 if it holds for any one, and by the same token, is also equivalent to the same condition with \mathcal{D}_1 and \mathcal{D}_2 exchanged.

Proof. (See [121, Thm. 4.1] for more details.) The crucial fact here is Wiener's approximation theorem, which states that the translates of $\rho \in L^1(\Xi, d\xi)$ span a norm-dense subspace iff the Fourier transform vanishes nowhere. These are precisely the regular elements of $C^*(\Xi, 0)^*$. The proof uses the following arguments:

Lemma 116. *Let $\mathcal{D} \subset \mathcal{C}_u(\Xi, \sigma)$ be an α -invariant closed subspace and $\rho \in C^*(\Xi, 0)^*$. Then*

- (1) $\rho * \mathcal{D} \subset \mathcal{D}$,
- (2) when ρ is regular, this inclusion is norm dense.

Proof. (1) Now, ρ is a classical standard state, i.e., a probability measure on Ξ . The convolution integral $\rho * A = \int \rho(d\xi) \alpha_\xi \beta_-(A)$ can be approximated for strongly continuous A by partitioning the integration domain into regions, over which either $\alpha_\xi(A)$ changes little, or which have small total weight with respect to ρ . We may then replace $\alpha_\xi(A)$ by a constant in each region, thus approximating the convolution uniformly by a linear combination of translates $\alpha_\xi(A)$.

(2) For $A \in \mathcal{C}_u(\Xi, \sigma)$ we can find $\rho' \in L^1$ with sufficiently small support around the origin so that $\|\rho' * A - A\|$ is small. Approximating ρ' by a linear combination of translates $\alpha_\xi \rho$, we find that A itself lies in the closure of the translation-invariant subspace generated by $\rho * A$. \square

Coming back to the proof of the proposition, note that (1) \Rightarrow (2) is trivial. Given (2) we get $\omega_0 * \omega_0 * \mathcal{D}_1 \subset \omega_0 * \mathcal{D}_2 \subset \mathcal{D}_1$. But since $\omega_0 * \omega_0$ is regular, this inclusion is dense, which proves (3).

Next, we verify (3) \Leftrightarrow (4) by showing that the spaces defined by these conditions, which we temporarily call $\mathcal{D}_2^{(3)}$ and $\mathcal{D}_2^{(4)}$, are equivalent. Suppose that $A \in \mathcal{D}_2^{(4)}$. Then because $A \in \mathcal{C}_u(\Xi, \sigma_2)$, A lies in the closed translation invariant subspace

generated by $\omega_0 * \omega_0 * A \in \omega_0 * \mathcal{D}_1$, which is $\mathcal{D}_2^{(3)}$. Conversely, if $A \in \mathcal{D}_2^{(3)}$, it can be approximated by elements of the form $\omega_0 * A_1$, so $\omega_0 * A \approx \omega_0 * \omega_0 * A_1 \in \mathcal{D}_1$, which means that $A \in \mathcal{D}_2^{(4)}$.

It remains to show that (3) \Rightarrow (1). Indeed $\omega * \mathcal{D}_2 \subset \overline{\omega * \omega_0 * \mathcal{D}_1} \subset \mathcal{D}_1$. On the other hand, since $\omega_0 * \omega_0 * \mathcal{D}_1 \subset \mathcal{D}_1$ is dense, we find for arbitrary ω : $\omega * \mathcal{D}_1 \subset \overline{\omega * \omega_0 * \omega_0 * \mathcal{D}_1} \subset \overline{\omega * \omega_0 * \mathcal{D}_2} \subset \mathcal{D}_2$. \square

5.4 Hybrid Dynamics: Quasifree Channels

In Sect. 5.1.3, we introduced quasifree channels for quantum systems, which are completely characterized by linear operators at the phase space level. This linearity can be expressed as a *covariance condition* with respect to phase space translations. For general covariant channels, one has to fix representations of the symmetry group under consideration in the input system as well as in the output system, with the desired operations intertwining these two representations. In the case at hand, these will be two representations of the group of phase translations, and the difference between the representations is parametrized by a linear operator

$$S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}. \quad (5.185)$$

Our first step will be to characterize all channels satisfying such a covariance condition plus a regularity condition, which ensures that standard states in the sense of the previous sections are mapped to standard states. The action of these channels on states, i.e., the Schrödinger picture, will then be obvious. This was, in fact, the starting point of the present study. However, the corresponding Heisenberg pictures seemed initially rather unclear. Having clarified the necessary spaces in the previous section, we can now go on to apply these ideas and get Heisenberg picture channels for all quasifree channels without the need for any extra assumptions.

5.4.1 Definition

In the Schrödinger picture, a channel is a completely positive, normalization preserving, linear map

$$\mathcal{T} : C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^* \rightarrow C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^*. \quad (5.186)$$

It thus takes the input states to a device to the output states. Such channels include *measurements* when $(\Xi_{\text{out}}, \sigma_{\text{out}})$ is classical (i.e., $\sigma_{\text{out}} = 0$), *preparations* ($\Xi_{\text{in}} = \{0\}$), and all kinds of combinations in which, in addition to an operation on the quantum subsystem, classical information is used as an input, or is read out in the process (see Sect. 5.5).

The Heisenberg picture is always denoted by \mathcal{T}^* , and $\mathcal{T}^*(A)$ for an observable A of the output system is interpreted as that observable on the input system, which is obtained by first operating with the quantum device and then measuring A . The two pictures are thus related as two ways of viewing the same experiment. Since all observables have expectations in the standard hybrid state, they can be considered as elements of the dual, i.e., $C^*(\Xi, \sigma)^{**}$, and from interpretation, it is clear that \mathcal{T}^* must indeed be the Banach space adjoint of \mathcal{T} . In the definition, we use the notation $S^\top : \Xi_{\text{in}} \rightarrow \Xi_{\text{out}}$ for the linear algebra transpose (or adjoint) of the linear map $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$.

Definition 117. *Let $(\Xi_{\text{in}}, \sigma_{\text{in}})$ and $(\Xi_{\text{out}}, \sigma_{\text{out}})$ be hybrid phase spaces and take a linear map $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$. Then an **S -covariant channel** is a completely positive, normalisation preserving linear operator $\mathcal{T} : C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^* \rightarrow C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^*$ such that, for all $\xi \in \Xi_{\text{in}}$,*

$$\mathcal{T} \circ (\alpha_\xi^{\text{in}})^* = (\alpha_{S^\top \xi}^{\text{out}})^* \circ \mathcal{T}. \quad (5.187)$$

A **quasifree channel** is a channel, which is S -covariant for some S .

There is an alternative characterization in terms of \mathcal{T}^* , which also clarifies the data needed to specify an S -covariant channel.

Proposition 118. *Let \mathcal{T} be an S -covariant channel. Then there is a unique continuous and normalized function $f : \Xi_{\text{out}} \rightarrow \mathbb{C}$, which is twisted positive definite with respect to the antisymmetric form*

$$\Delta\sigma = \sigma_{\text{out}} - S^\top \sigma_{\text{in}} S, \quad (5.188)$$

such that, for all $\xi \in \Xi_{\text{out}}$,

$$\mathcal{T}^*(W_{\text{out}}(\xi)) = f(\xi)W_{\text{in}}(S\xi). \quad (5.189)$$

Conversely, every function f with this property defines an S -covariant channel.

We will again refer to f as the **noise function** of the channel \mathcal{T} , and to the hybrid state on $(\Xi_{\text{out}}, \Delta\sigma)$ with characteristic function f as its **noise state**, and denote it typically by τ .

Before going into the proof, let us explain why the form in Eq. (5.189) determines a unique state. At first glance, it defines the action of the channel only on the Weyl operators, hence by norm limits on the CCR-algebra, but no further. The point is that the formula really defines a transformation $\mathcal{T} : C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^* \rightarrow C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^*$ on *states*: By taking expectations of Eq. (5.189) with ω_{in} we get the characteristic function of $\omega_{\text{out}} = \mathcal{T}\omega_{\text{in}}$ as

$$\chi_{\text{out}}(\xi) = f(\xi)\chi_{\text{in}}(S\xi). \quad (5.190)$$

This shows that the channel is indeed specified completely by S and f . Another way to put this is to note that since the expectations of Weyl operators specify the state, the linear hull of these operators is weak*-dense in the bidual $C^*(\Xi, \sigma)^{**}$, and correspondingly in all the observable spaces. Therefore, Eq. (5.189) suffices to define the Heisenberg picture channel by first a linear extension and then an extension by weak*-continuity.

Proof of Prop. 118. Applying α_ξ^{in} to Eq. (5.189) and using the eigenvalue equation in Eq. (5.177) for $W_{\text{out}}(\xi)$ we find that

$$\alpha_\xi^{\text{in}} \circ \mathcal{T}^*(W(\eta)) = \mathcal{T}^*(\alpha_{S^\top \xi}^{\text{out}}(W(\eta))) = \mathcal{T}^*(e^{iS^\top \xi \cdot \eta} W(\eta)) = e^{i\xi \cdot S\eta} \mathcal{T}^*(W(\eta)). \quad (5.191)$$

That is, $\mathcal{T}^*(W(\eta))$ is a joint eigenvector of the translations and hence, by Lem. 113, must be proportional to $W(S\eta)$. We denote the proportionality factor by $f(\eta)$, see Sect. 5.3.3. This immediately implies Eq. (5.189). Now, we can choose a state ω_{in} such that χ_{in} vanishes nowhere, for example, a Gaussian. Since χ_{in} is continuous by Bochner's Theorem, and the channel maps standard states to standard states, so χ_{out} is also continuous, we conclude that f is continuous.

We now have to analyze the condition for complete positivity. Here, one should remember that the channel \mathcal{T} is primarily defined on $C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^*$ and complete positivity just means that $\mathcal{T} \otimes \mathbb{1}_n$ preserves positivity (i.e., positive semi-definiteness) for all n (see. Sect. 2.4.1). In order to give an equivalent formulation in the Heisenberg

picture, one can check complete positivity on any subalgebra $\mathcal{A} \subset C^*(\Xi_{\text{out}}, \sigma_{\text{out}})$ so that the positivity of any element of $\omega_n \in C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^* \otimes \mathcal{M}_n$ can be expressed as the positivity of expectation values of positive elements in $\mathcal{A} \otimes \mathcal{M}_n$, i.e., the positive cones are dual to each other. For this, any weak*-dense subalgebra \mathcal{A} will do, and we take here the linear span of the Weyl operators for \mathcal{A} .

We now assume that \mathcal{T}^* is completely positive and aim at deriving the stated twisted definiteness condition for f . To this end, we use that for a completely positive operator \mathcal{T}^* , and any choice of finitely many a_j, b_j , $X = \sum_{jk} a_j \mathcal{T}^*(b_j^* b_k) a_k^* \geq 0$. Here we choose $a_j = c_j W(S\xi_j)$ and $b_j = W(\xi_j)$ for an arbitrary choice of $\xi_1, \dots, \xi_n \in \Xi_{\text{out}}$, and $c_j \in \mathbb{C}$. The idea is that then X becomes a multiple of the identity, namely

$$\begin{aligned}
0 \leq X &= \sum_{jk} \bar{c}_j W(S\xi_j) \mathcal{T}^*(W(\xi_j)^* W(\xi_k)) c_k W(S\xi_k)^* \\
&= \sum_{jk} \bar{c}_j c_k e^{-i\xi_j \cdot \sigma_{\text{out}} \xi_k / 2} W(S\xi_j) \mathcal{T}^*(W(-\xi_j + \xi_k)) W(S\xi_k)^* \\
&= \sum_{jk} \bar{c}_j c_k e^{-i\xi_j \cdot \sigma_{\text{out}} \xi_k / 2} f(-\xi_j + \xi_k) W(S\xi_j) W(-S\xi_j + S\xi_k) W(S\xi_k)^* \\
&= \sum_{jk} \bar{c}_j c_k e^{-i\xi_j \cdot \sigma_{\text{out}} \xi_k / 2} f(-\xi_j + \xi_k) e^{-i(S\xi_j) \cdot \sigma_{\text{in}}(S\xi_k) / 2} \\
&\quad W(S\xi_j) W(S\xi_j)^* W(S\xi_k) W(S\xi_k)^* \\
&= \sum_{jk} \bar{c}_j f(-\xi_j + \xi_k) e^{i\xi_j \cdot \Delta \sigma \xi_k / 2} c_k \mathbb{1}. \tag{5.192}
\end{aligned}$$

The positivity of this expression for arbitrary c_j and ξ_j is exactly the stated twisted definiteness condition in Def. 81 but with a degenerate or hybrid form $\Delta\sigma$.

Conversely, when f satisfies the conditions, we can define \mathcal{T} acting on $C^*(\Xi_{\text{in}}, \sigma_{\text{in}})$ by Eq. (5.190), using Bochner's Theorem, and Prop. 91. Continuity and normalization of the output characteristic function are then guaranteed by the corresponding properties of f . Positivity will be addressed together with complete positivity.

We have to extend Bochner's Theorem to a version involving an additional tensor factor \mathcal{M}_n . So let $\omega^{\text{in}} \in C^*(\Xi_{\text{in}}, \sigma_{\text{in}}) \otimes \mathcal{M}_n$ be positive. The matrix elements $\omega_{\alpha\beta}^{\text{in}}$ then have characteristic functions $\chi_{\alpha\beta}^{\text{in}}(\eta) = \omega_{\alpha\beta}^{\text{in}}(W(\eta))$, and the positivity condition for ω^{in} is the positivity of the matrix

$$\chi_{\alpha\beta}^{\text{in}}(-\eta_j + \eta_k) e^{i\eta_j \cdot \sigma_{\text{in}} \eta_k / 2}, \tag{5.193}$$

for arbitrary η_1, \dots, η_N , where the indices of this matrix are considered to be the pairs (α, j) and (β, k) . Applying the channel $\mathcal{T} \otimes \mathbb{1}_n$ to ω^{in} means the application of Eq. (5.190) to each matrix element, resulting in a similar matrix for $\omega^{\text{out}} = (\mathcal{T} \otimes \mathbb{1}_n) \omega^{\text{in}}$, namely

$$\chi_{\alpha\beta}^{\text{out}}(\xi_i - \xi_j) e^{\frac{i}{2} \xi_i \cdot \sigma_{\text{out}} \xi_j} = \left(f(\xi_i - \xi_j) e^{\frac{i}{2} \xi_i \cdot \Delta \sigma \xi_j} \right) \left(\chi_{\alpha\beta}^{\text{in}}(S\xi_i - S\xi_j) e^{\frac{i}{2} S\xi_i \cdot \sigma_{\text{in}} S\xi_j} \right). \tag{5.194}$$

Here, we used the definition of $\Delta\sigma$. By assumption, the matrix in the first factor is positive definite. The second factor is positive because the input state in Eq. (5.193) is positive with the substitution $\eta_j = S\xi_j$. Hence, the left-hand side is also positive definite as the Hadamard product of two positive definite matrices. \square

5.4.2 State-channel correspondence

In this section, we will describe in more detail the geometry of the correspondence between an S -covariant channel \mathcal{T} and its noise state τ , which was set up in Prop. 118. The operator S will be fixed, and this is necessary if we want to consider a correspondence of convex sets: The convex combination of quasifree channels with different S is simply not quasifree. However, convex combinations do not exhaust the design possibilities for channels by engineering τ . Since arbitrary states are allowed, superpositions work just as well (see, e.g., [146]).

We begin with a condensed version of state-channel correspondence and cones in quantum theory: If we restrict to finite-dimensional systems, we can summarize this by saying that in quantum theory, there is only one isomorphism type of positive cone for the basic objects: For *observables*, the elements of the form A^*A , for *states*, the dual of the observable cone, and for *channels*, the completely positive cone.

The inclusion of direct sums of positive semi-definite cones extends this statement to quantum-classical hybrid systems. As an immediate consequence, we find that there is only one kind of order interval, which is an ordered vector space, i.e. a set of the kind

$$[x_1, x_2] = \{x | x_1 \leq x \leq x_2\}, \quad (5.195)$$

which is obviously determined by just the order relations. In particular, the possible decompositions $\rho = \rho_1 + \rho_2$ of a fixed state ρ into a sum of positive ρ_i are isomorphic to the corresponding interval $[0, \mathbb{1}]$ in which decompositions are just two-valued observables, and decompositions $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ of a channel into completely positive terms, i.e., an instrument with overall state change \mathcal{T} . This correspondence of order intervals is, in a sense, more robust than the correspondence of cones: It persists in infinite-dimensional systems while the isomorphism of cones breaks down. For example, $\mathcal{B}(\mathcal{H})$ has an order unit, whereas the trace class has none.

From the finite-dimensional case, it is clear that the difference between the spaces of states, observables, and channels lies in the respective normalization conditions. This is also reflected in the different structures of the convex sets of normalized elements: The extreme points are the projective Hilbert space for states, the projection lattice for observables, and something more complicated for channels. Moreover, we get different natural norms: The trace norm (Def. 10), the operator norm (Def. 2), and the **norm of complete boundedness**:

Definition 119. *Let \mathcal{A} be a unital C^* -algebra, $\mathcal{M}_n(\mathcal{A})$ the set of $n \times n$ matrices with entries in \mathcal{A} , $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$ a linear map and $\mathcal{T}_n = \mathcal{T} \otimes \mathbb{1}_n$. Then \mathcal{T} is completely bounded if $\sup_n \|\mathcal{T}_n\|$ is finite and we set*

$$\|\mathcal{T}\|_{\text{cb}} = \sup_n \|\mathcal{T}_n\|. \quad (5.196)$$

Then $\|\cdot\|_{\text{cb}}$ defines a norm on the space of completely bounded channels [50]. This norm is also (sometimes only in the Schrödinger picture) referred to as the diamond norm [147]. For completely positive maps, its value is determined by evaluating it on the unit [50, Prop. 3.6]:

$$\|\mathcal{T}\|_{\text{cb}} = \|\mathcal{T}(\mathbb{1})\|. \quad (5.197)$$

When using $\|\cdot\|_{\text{cb}}$ to quantify the distance between channels, their difference is of course no longer completely positive, and the cb-norm has a reputation of being not easy to compute [50, 148].

One surprising fact about the isomorphism $\mathcal{T} \leftrightarrow \tau$ is that it connects the normalized subsets of different categories: states on the one hand and quasifree channels on the other. In light of the above explanations, this is readily traced to the normalization conditions: For a channel, the normalization condition is $\mathcal{T}^*(\mathbb{1}) = \mathbb{1}$, and for general completely positive maps we have Eq. (5.197). This is not a linear function of \mathcal{T} . However, for a general bounded covariant map \mathcal{T} , we have shown (see Eq. (5.189) with $\xi = 0$) that $\mathcal{T}^*(\mathbb{1}) = f(0)\mathbb{1}$, so

$$\|\mathcal{T}\|_{\text{cb}} = f(0) = \|\tau\|. \quad (5.198)$$

Then, for positive elements, both norms depend only on one number, and this dependence is linear, i.e., the norm is additive on the positive cone. This is precisely what makes a complete channel-state correspondence possible here. Other classes of covariant channels share this feature, i.e., channels that intertwine automorphic actions of a group G , i.e., $\alpha_g^{\text{out}}\mathcal{T} = \mathcal{T}\alpha_g^{\text{in}}$ for $g \in G$. When the α_g^{in} are implemented by an irreducible set of unitaries, a projective representation of G , then, once again, $\mathcal{T}^*\mathbb{1}$ is a multiple of the identity, and the class of covariant channels is affinely isomorphic to a state space of a quantum system that can be computed from the representations involved [149]. We see here that the irreducibility of the implementing unitaries is not the key condition since, on the classical subsystem, no such unitaries exist. Instead, the decisive condition is that the representation on the input side has only the multiples of $\mathbb{1}$ as invariant elements, in the hybrid case, a special case of Lem. 113.

The following corollary summarizes the above discussion and lists some transfers-of-properties for the correspondence.

Corollary 120. *Fix hybrid systems with phase spaces $(\Xi_{\text{in}}, \sigma_{\text{in}})$ and $(\Xi_{\text{out}}, \sigma_{\text{out}})$, and a linear map $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$. Then there is a bijective correspondence between S -covariant channels \mathcal{T} in the sense of Def. 117, and noise states τ on the hybrid system $(\Xi_{\text{out}}, \Delta\sigma)$ as stated in Prop. 118. Then if $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ correspond to τ, τ_1, τ_2 , respectively, and $\lambda \in \mathbb{R}$, then*

- (1) $\mathcal{T} = \lambda\mathcal{T}_1 + (1 - \lambda)\mathcal{T}_2$ iff $\tau = \lambda\tau_1 + (1 - \lambda)\tau_2$,
- (2) $\lambda\mathcal{T}_2 - \mathcal{T}_1$ is completely positive iff $\lambda\tau_2 - \tau_1 \geq 0$,
- (3) $\|\mathcal{T}_1 - \mathcal{T}_2\|_{\text{cb}} = \|\tau_1 - \tau_2\|$,
- (4) for $\xi \in \Xi_{\text{out}}$, $\mathcal{T}_1 = \alpha_\xi^* \circ \mathcal{T}$ iff $\tau_1 = \alpha_\xi^*(\tau)$,
- (5) τ is extremal (= pure) iff \mathcal{T} is noiseless in the sense of Sect. 5.4.5,
- (6) τ is norm continuous under translations, iff \mathcal{T} is smoothing in the sense of Sect. 5.4.3.

Proof. The bijective correspondence is directly from Prop. 118. (1) and (2) are obvious, and (4) follows by noting that under the translations by ξ stated in that

item, the noise function $f(\eta) = \langle \tau, W(\eta) \rangle$ changes by a factor $\exp(i\xi \cdot \eta)$. (5) is non-trivial and will be shown in the section mentioned. (6) is trivial from the combination of (4) and (3), noting that smoothing means that $\|\alpha_\xi^* \circ \mathcal{T} - \mathcal{T}\|_{\text{cb}} \rightarrow 0$ for $\xi \rightarrow 0$. This proves all items except (3).

(3) Both norms are additive on the positive cone and coincide there. There is then a largest norm on the real linear span of the positive elements with this property, called the base norm [150]. The norm on states is of this type, which implies the inequality. A bit more explicitly, the base norm has the smallest unit ball of all the norms with the given restriction, just the convex hull of the positive and the negative elements of norm one.

$$\begin{aligned} \|\mathcal{T}_1 - \mathcal{T}_2\|_{\text{cb}} &\leq \inf\{p_+ + p_- \mid \mathcal{T}_\pm \text{ channels, } p_\pm \geq 0, (\mathcal{T}_1 - \mathcal{T}_2) = p_+\mathcal{T}_+ - p_-\mathcal{T}_-\} \\ &= \inf\{p_+ + p_- \mid \tau_\pm \text{ states, } p_\pm \geq 0, (\tau_1 - \tau_2) = p_+\tau_+ - p_-\tau_-\} \\ &= \|\tau_1 - \tau_2\|. \end{aligned} \quad (5.199)$$

This proves the inequality “ \leq ” in (3).

For the reverse inequality, consider the Weyl operators

$$\widetilde{W}(\xi) = W_{\text{out}}(\xi) \otimes \overline{W}_{\text{in}}(S\xi) \quad (\xi \in \Xi_{\text{out}}) \quad (5.200)$$

for an extended system $(\Xi_{\text{out}}, \sigma_{\text{out}}) \oplus (\Xi_{\text{in}}, \sigma_{\text{in}})$, where, as in the last part of Sect. 5.2.4, the overline is a complex conjugation inverting the symplectic form, see Eq. (5.144). Then \widetilde{W} is a strongly continuous representation of the relations for $(\xi_{\text{out}}, \Delta\sigma)$. Thus, using the notation in Eq. (5.93), for any $h \in L^1(\Xi_{\text{out}})$, we have $\|\widetilde{W}[h]\| \leq \|\widetilde{W}_\Delta[h]\|$, because the right-hand side is the supremum over all such representations. In the sequel, $h \in L^1(\Xi_{\text{out}})$ will be chosen with the only constraint that this norm is ≤ 1 . Then

$$\begin{aligned} (\mathcal{T}^* \otimes \mathbb{1})(\widetilde{W}[h]) &= \int d\xi h(\xi) \mathcal{T}^*(W_{\text{out}}(\xi)) \otimes \overline{W}_{\text{in}}(S\xi) \\ &= \int d\xi h(\xi) f(\xi) (W_{\text{in}}(S\xi)) \otimes \overline{W}_{\text{in}}(S\xi) \end{aligned} \quad (5.201)$$

We now apply the squeezed state ω^ε from Lem. 98 for $(\Xi_{\text{in}}, \sigma_{\text{in}})$:

$$\langle \omega^\varepsilon, (\mathcal{T}^* \otimes \mathbb{1})(\widetilde{W}[h]) \rangle = \int d\xi h(\xi) f(\xi) \langle \omega^\varepsilon, (W_{\text{in}}(S\xi)) \otimes \overline{W}_{\text{in}}(S\xi) \rangle. \quad (5.202)$$

Then, as $\varepsilon \rightarrow 0$, the expectation under the integral goes pointwise to 1, so by dominated convergence

$$\lim_{\varepsilon \rightarrow 0} \langle \omega^\varepsilon, (\mathcal{T}^* \otimes \mathbb{1})(\widetilde{W}[h]) \rangle = \int d\xi h(\xi) f(\xi) = \langle \tau, W_\Delta[h] \rangle. \quad (5.203)$$

Now, the left-hand side of this equation is linear in \mathcal{T}^* , and the right-hand side is linear in τ . Plugging in a difference, and observing $\|\widetilde{W}[h]\| \leq 1$ and $\|\omega^\varepsilon\| \leq 1$, we get

$$|\langle \tau_1 - \tau_2, W_\Delta[h] \rangle| \leq \|\mathcal{T}_1 - \mathcal{T}_2\|_{\text{cb}}. \quad (5.204)$$

The result then follows because $\|\tau_1 - \tau_2\|$ is the supremum over all h with the required norm bound. \square

5.4.3 Heisenberg pictures for S-covariant channels

The Heisenberg picture \mathcal{T}^* of a quasifree channel is initially defined on the bidual $C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^{**}$. However, it also maps better-behaved algebras into each other so that one can settle for one of these algebras as the basic observables in some contexts. Since the definitions given in Sect. 5.3 work for arbitrary C^* -algebras, commutative, quantum, or hybrid, the analytic properties defining these more special algebras are automatically preserved for all quasifree channels. As remarked already after Lem. 109, each inclusion $\mathcal{T}^* \mathcal{A} \subset \mathcal{B}$ can also be read as a continuity condition for \mathcal{T} .

Proposition 121. *For $(\Xi, \sigma) = (\Xi_{\text{in}}, \sigma_{\text{in}})$ or $(\Xi, \sigma) = (\Xi_{\text{out}}, \sigma_{\text{out}})$ consider the algebras*

$$\text{CCR}(\Xi, \sigma) \subset \mathcal{C}_{\text{u}}(\Xi, \sigma) \subset M(\Xi, \sigma) \subset \mathcal{U}(\Xi, \sigma) \subset C^*(\Xi, \sigma)^{**}. \quad (5.205)$$

Let \mathcal{T}^* be the Banach space adjoint of a quasifree channel

$$\mathcal{T} : C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^* \rightarrow C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^*.$$

Then \mathcal{T}^* maps the “out” version of an algebra in this inclusion chain to the corresponding “in” version.

Proof. \mathcal{T}^* is initially defined on the bidual, i.e., the largest element in the chain, so for this one, there is nothing to prove. For $\mathcal{U}(\Xi, \sigma)$ and $M(\Xi, \sigma)$ we have shown the claim in Lem. 109. The other cases use the quasifree structure. For $\text{CCR}(\Xi, \sigma)$ it is obvious from Prop. 118. For $\mathcal{C}_{\text{u}}(\Xi, \sigma)$, as defined in Eq. (5.181), it follows from the observation that if $\xi \mapsto \alpha_{\xi}^{\text{out}}(F)$ is norm continuous for some $F \in \mathcal{U}(\Xi_{\text{out}}, \sigma_{\text{out}})$, then

$$\xi \mapsto S^{\top} \xi \mapsto \alpha_{S^{\top} \xi}^{\text{out}}(F) \mapsto \mathcal{T}^* \circ \alpha_{S^{\top} \xi}^{\text{out}}(F) = \alpha_{\xi}^{\text{in}} \circ \mathcal{T}^*(F) \quad (5.206)$$

is also continuous. \square

The algebra $C^*(\Xi, \sigma)$ is conspicuously absent from the proposition’s list of algebras with an automatic Heisenberg picture. Indeed, it does not belong to that list. A simple counterexample is a depolarizing channel, for which $S = 0$, and $f = \chi_0$ is the characteristic function of some output state ω_0 . Then, after Eq. (5.190), $\chi_{\text{out}} = \chi_0$ for all input states. This translates to the Heisenberg picture as $\mathcal{T}(A) = \omega_0(A)1$. So even if $A \in C^*(\Xi_{\text{out}}, \sigma_{\text{out}})$, its image under the Heisenberg picture channel map is a multiple of the identity that is not in $C^*(\Xi_{\text{in}}, \sigma_{\text{in}})$. Nevertheless, there is an easily checkable condition that will ensure the Heisenberg picture in this case:

Lemma 122. *Let \mathcal{T} be an S-covariant channel. Then either*

- (1) $S\Xi_{\text{out}} = \Xi_{\text{in}}$ and $\mathcal{T}^* C^*(\Xi_{\text{out}}, \sigma_{\text{out}}) \subset C^*(\Xi_{\text{in}}, \sigma_{\text{in}})$, or
- (2) $S\Xi_{\text{out}} \neq \Xi_{\text{in}}$ and $(\mathcal{T}^* C^*(\Xi_{\text{out}}, \sigma_{\text{out}})) \cap C^*(\Xi_{\text{in}}, \sigma_{\text{in}}) = \{0\}$.

Proof. (1) Take an element $W[h] = \int d\xi h(\xi)W(\xi) \in C^*(\Xi_{\text{out}}, \sigma_{\text{out}})$. By definition of the algebra $C^*(\Xi_{\text{out}}, \sigma_{\text{out}})$ as the C^* -envelope of the twisted convolution algebra, such elements are dense. It therefore suffices to show that the image under the channel is again given by such an integral. Applying the channel gives

$$\mathcal{T}^* W[h] = \int d\xi h(\xi) f(\xi) W(S\xi). \quad (5.207)$$

We can split the integration variables into $\xi = (\xi_\perp, \xi_\parallel)$ with $S\xi_\perp = 0$ and a variable ξ_\parallel in a suitable linear complement of the kernel. Then ξ_\parallel uniquely specifies a point $S\xi_\parallel = \eta \in \Xi_{\text{in}}$. Carrying out the integral over ξ_\perp leaves $\mathcal{T}^*W[h] = W[h']$ with a function $h'(\eta) = \int d\xi_\perp h(\xi_\perp, \xi_\parallel) f(\xi_\perp, \xi_\parallel)$ which clearly lies in $L^1(\Xi_{\text{in}}, d\eta)$.

(2) When S is not surjective, there is a non-zero vector η orthogonal to $S\Xi_{\text{out}}$. Then we have $\alpha_\eta(W(S\xi)) = \exp(i\eta \cdot S\xi)W(S\xi) = W(S\xi)$ for all ξ . Integrating with an arbitrary $h \in L^1(\Xi_{\text{out}})$, it follows that

$$\alpha_\eta \circ \mathcal{T}^*W[h] = \int d\xi h(\xi) f(\xi) \alpha_\eta(W(S\xi)) = \mathcal{T}^*W[h]. \quad (5.208)$$

This transfers to $C^*(\Xi_{\text{out}}, \sigma_{\text{out}})$ by continuity. The image therefore consists of $F \in \mathcal{C}_u(\Xi_{\text{in}}, \sigma_{\text{in}})$, satisfying $\alpha_\eta F = F$. We will show that together with $F \in C^*(\Xi_{\text{in}}, \sigma_{\text{in}})$, this implies $F = 0$. With Eq. (5.178), the action of translations on functions $F \in \mathcal{C}_u(\Xi_{\text{in}}, \sigma_{\text{in}})$ is given by

$$(\alpha_\eta F)(\xi_0) = W(\sigma\eta)^* F(\xi_0 + \eta_0) W(\sigma\eta) = F(\xi_0), \quad (5.209)$$

where the last equality expresses our first conclusion. We take the norm on both sides so that the non-zero vector η enters only through its classical part η_0 . We claim that this classical part must vanish. Indeed, the sequence $n \mapsto \xi_0 + n\eta_0$ goes to infinity for all ξ_0 , and since $F \in C^*(\Xi_{\text{in}}, \sigma_{\text{in}}) \cong \mathcal{K}(\mathcal{H}_1) \otimes \mathcal{C}_0(\Xi_0)$ by Prop. 92, we have $\lim_n \|F(\xi_0 + n\eta_0)\| = 0$. But then $F(\xi_0) = 0$ for all ξ_0 , and $F = 0$. Hence $\eta_0 = 0$.

Now Eq. (5.209) says that the (supposedly) compact operator $F(\xi_0)$ commutes with a one-parameter subgroup of Weyl operators. Then the finite-dimensional eigenspaces of $F(\xi_0) + F(\xi_0)^*$ would have to be invariant under such a group. But since the generators in the Schrödinger representation have a continuous spectrum, this is impossible. So, the eigenspaces for non-zero eigenvalues have to be empty, which implies $F(\xi_0) + F(\xi_0)^* = 0$. Repeating this argument for $i(F(\xi_0) - F(\xi_0)^*)$ we get $F(\xi_0) = 0$ for all ξ_0 , hence $F = 0$. \square

Smoothing channels

The μ -dependent setting, with the particular choice $\mu = dx$ as the Lebesgue measure, has been singled out by the norm continuity of states under translations. The resulting structure also supports other L^p spaces and corresponding Schatten classes. A natural question is then whether a given quasifree channel preserves the continuity of states and therefore can be seen as a normal map between the corresponding hybrid von Neumann algebras $L^\infty(\Xi, \sigma)$ as defined in Eq. (5.116). The identity channel obviously has this property, but, for example, a depolarizing channel with a pure output state does not. The following lemma gives a positive answer for general non-singular S and arbitrary noise functions. Because all S_t in a matrix semigroup are non-singular, we conclude that this is sufficient for semigroup theory. It shows that the von Neumann algebra $L^\infty(\Xi, \sigma)$, as used in [72] (see also Sect. 4.2.1), is a sufficient arena for quasifree semigroups.

Lemma 123. *Let \mathcal{T} be the quasifree channel given by $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$ and $f : \Xi_{\text{out}} \rightarrow \mathbb{C}$. Suppose that S is injective. Then \mathcal{T} maps norm continuous states to norm continuous states.*

Proof. When S is injective, $S^\top \Xi_{\text{in}} \rightarrow \Xi_{\text{out}}$ is surjective. So let $\xi \in \Xi_{\text{out}}$, which we can consequently write as $\xi = S^\top \eta$. Suppose that ρ_{in} is norm continuous under translations, and consider $\rho_{\text{out}} = \mathcal{T} \rho_{\text{in}}$. Then by Eq. (5.187) the function

$$t \mapsto \alpha_{t\xi}^* \rho_{\text{out}} = \alpha_{tS^\top \eta}^* \mathcal{T} \rho_{\text{in}} = \mathcal{T}(\alpha_{t\eta}^* \rho_{\text{in}})$$

is continuous in norm. Since this holds for all $\xi \in \Xi_{\text{out}}$ and the translations commute, $\xi \mapsto \alpha_\xi^* \rho_{\text{out}}$ is also norm continuous. \square

Another criterion implying continuity of states is sufficient noise, no matter what S may be. In that case, all states, not just the continuous input states, become continuous. This is a smoothing property, and the following proposition collects some basic observations.

Proposition 124. *Let \mathcal{T} be a quasifree channel with noise state τ and noise function f . Consider the following statements:*

- (1) $f \in L^p(\Xi, d\xi)$ for some $p \in [1, 2]$.
- (2) τ is norm continuous under translations.
- (3) $\lim_{\xi \rightarrow 0} \|\alpha_\xi^* \circ \mathcal{T} - \mathcal{T}\|_{\text{cb}} = 0$.
- (4) For all $\omega \in C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^*$, $\mathcal{T}\omega$ is norm continuous under translations.
- (5) For all $A \in M(\Xi_{\text{out}}, \sigma_{\text{out}})$, $\mathcal{T}^* A \in \mathcal{C}_u(\Xi_{\text{in}}, \sigma_{\text{in}})$.

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \wedge (5). A channel \mathcal{T} with the property (3) will be called **smoothing**.

Proof. (1) \Rightarrow (2) is a part of Prop. 93 applied to the noise state. (2) \Rightarrow (3) is an immediate consequence of items (3) and (4) of Cor. 120, which also proves item (6) of that corollary. The remaining items follow from (3) by applying \mathcal{T} or \mathcal{T}^* to the respective arguments. For (3) \Rightarrow (5) note Prop. 121 and Cor. 112

For a converse at this point, one would need a uniformity condition on the modulus of continuity: For example, demanding the existence of an ω -independent function $\varepsilon(\xi)$, which goes to zero as $\xi \rightarrow 0$, and satisfies

$$\|\omega - (\alpha_\xi^* \circ \mathcal{T}) \otimes \mathbb{1}_n \omega\| \leq \varepsilon(\xi) \tag{5.210}$$

for all states on $C^*(\Xi_{\text{in}}, \sigma_{\text{in}}) \otimes \mathcal{M}_n$, is equivalent to (3). \square

5.4.4 Composition, concatenation, convolution

This section briefly considers three ways of combining channels or states. They correspond roughly to the parallel and serial execution of operations and to the addition of phase space variables. A case distinction for different configurations of classical and quantum variables is needed for none of these.

Composing subsystems with indices 1, 2 just gives the hybrid phase space

$$(\Xi_{12}, \sigma_{12}) = (\Xi_1 \oplus \Xi_2, \sigma_1 \oplus \sigma_2). \quad (5.211)$$

This corresponds to C^* -tensor products at the level of the non-unital algebras $C^*(\Xi, \sigma)$. The C^* -tensor product $C^*(\Xi_{12}, \sigma_{12}) = C^*(\Xi_1, \sigma_1) \otimes C^*(\Xi_2, \sigma_2)$ is uniquely defined, because the algebras involved are nuclear, so maximal and minimal norm on the algebraic tensor product coincide. This entails that states in $C^*(\Xi_{12}, \sigma_{12})^*$ can be weakly approximated by product elements, but the resulting *tensor product of state spaces* requires more than norm limits of product elements. For observable algebras, a simple approach using norm limits also fails (compare Sect. 3.3.1). It is clear that the compactification of a product is usually not the product of the compactifications. Even for the one-point compactification, corresponding to the observable algebras $\mathcal{A}_i = C^*(\Xi_i, \sigma_i) \oplus \mathbb{C}\mathbb{1}$, we get additional components, like $\mathbb{1} \otimes C^*(\Xi_2, \sigma_2) \not\subset C^*(\Xi_{12}, \sigma_{12}) \oplus \mathbb{C}\mathbb{1}$.

In spite of these subtleties, quasifree channels allow a straightforward composition operation $\mathcal{T}_1 \otimes \mathcal{T}_2$. When $S_i : \Xi_{i,\text{out}} \rightarrow \Xi_{i,\text{in}}$ and $f_i : \Xi_{i,\text{out}} \rightarrow \mathbb{C}$ are the data defining \mathcal{T}_i , the tensor product has

$$\begin{aligned} S(\xi_1 \oplus \xi_2) &= (S_1 \xi_1) \oplus (S_2 \xi_2) \\ f(\xi_1 \oplus \xi_2) &= f_1(\xi_1) f_2(\xi_2). \end{aligned} \quad (5.212)$$

Of course, with the composition of quantum systems comes **entanglement**. It is the observation that, while general states on a composite can be approximated by product elements, these product elements cannot be taken to be positive. Indeed, non-entangled or *separable* states are nowadays *defined* by the existence of a positive product approximation [151, 152]. Entanglement in Gaussian states is well understood [92, 104, 153, 154, 117], but the hybrid scenario creates no new interesting possibilities: The classical part of a composite hybrid is just the product of the classical parts. The pure classical states are point measures on a cartesian product, and hence product states. This is just saying that classical systems cannot be entangled. In the integral decomposition in Eq. (5.80) of an arbitrary hybrid state, all entanglement is therefore in the states ρ_x , where $x = (x_1, x_2)$ is a point in the cartesian product.

Executing one operation after the other is called, in different communities or contexts, **concatenation**, or composition (too unspecific, see the previous paragraph), or multiplication (too overloaded). Clearly, when $\mathcal{T}_1, \mathcal{T}_2$ are quasifree channels, so is concatenation $\mathcal{T} = \mathcal{T}_1 \mathcal{T}_2$. When we take S_1, S_2, S and f_1, f_2, f as the defining parameters of these channels then

$$\begin{aligned} S &= S_2 S_1, \\ f(\xi) &= f_1(\xi) f_2(S_1 \xi). \end{aligned} \quad (5.213)$$

We thus get a category whose objects are the hybrid systems and whose morphisms are the quasifree channels. Objects in a category are *the same* if they are connected by a morphism and its inverse morphism. An introduction to the topic of categories can be found in [155].

Isomorphism classes in our setting are labeled by the pairs $(n, s) \in \mathbb{N} \times \mathbb{N}$, where n is the number of quantum degrees of freedom, so that $\Xi_1 = \mathbb{R}^{2n}$ as a vector space on which σ is non-degenerate, and s is the number of classical dimensions, i.e., $\Xi_0 = \ker \sigma = \mathbb{R}^s$. Note that, in particular, our theory depends only on (Ξ, σ) and not on a particular splitting $\Xi = \Xi_1 \oplus \Xi_0$. We have used such splittings above, although only Ξ_0 , the null space of σ , is intrinsically defined by the structure (Ξ, σ) , and different complements Ξ_1 could be chosen. Changing this splitting is an isomorphism, leaving Ξ_0 fixed. It acts by an ξ_0 -dependent phase space translation of the quantum part, which is clearly quasifree and invertible as such.

Other categorical features (monomorphism, epimorphisms, etc.) can be worked out. An essential result of this kind is a characterization of channels with a one-sided inverse, see Prop. 129.

A trivial but frequently used concatenation is the formation of **marginals** of a channel, i.e., considering only one of the outputs and discarding the other (see Sect. 5.5 below). The discarding operation is itself a channel, the noiseless one with

$$S : \Xi_1 \rightarrow \Xi_1 \oplus \Xi_2, \quad S\xi_1 = \xi_1 \oplus 0, \quad (5.214)$$

or equivalently, is the tensor product of the identity on Ξ_1 with the destructive channel $\Xi_{\text{out}} = \{0\}$ on Ξ_2 .

We have already met the **convolution** of states in Def. 114. As in all group representation theory, one should think of convolutions as a contravariant encoding of the group multiplication. Consider two systems with the same set $\Xi = \Xi_1 = \Xi_2$, so the addition of phase space elements makes sense. For the moment, we do not care whether they are classical or quantum. Can we add signals of these types? The model for this is the addition of random variables. It corresponds to setting the Fourier arguments dual to the random variables x_1 and x_2 equal: The characteristic function for a sum is the expectation of $\exp(ik(x_1 + x_2))$, which we obtain from that of the joint distribution. So convolution, in general, corresponds to the linear map $S\xi = \xi \oplus \xi \in \Xi_1 \oplus \Xi_2$. This would suggest a channel acting as $\mathcal{T}(\rho_1 \otimes \rho_2) = \rho_1 * \rho_2$. This works as a noiseless channel when one of the factors is classical. In the quantum case, however, although the convolution of arbitrary states is well defined, the map \mathcal{T} with this property would not extend as a channel to entangled states. Thus one could either add noise or modify the definition by inverting the symplectic form in one factor, i.e., setting $\mathcal{T}(\rho_1 \otimes \rho_2^\top) = \rho_1 * \rho_2$, where ρ^\top denotes transposition, i.e., the inversion of momenta.

5.4.5 Noiseless operations

Every quasifree channel can be modified by multiplying f with an arbitrary (untwisted) positive definite function g . This corresponds to adding classical noise or averaging the output over translations α_ξ with a noise probability measure whose characteristic function is g . Since $|g(\xi)| \leq 1$ this always decreases $|f|$. In fact, unless the noise measure is concentrated on a single point, and we thus have a simple translation, we have $|g(\xi)| < 1$ for some ξ and the decrease of $|f(\xi)|$ is strict, at least for some ξ . Accordingly, we define:

Definition 125. *A quasifree channel is called a **minimal noise channel** if it cannot be constructed in the above way with $|g| \neq 1$.*

Those channels will be characterized below as the extremal S -covariant channels. In the same spirit, we define a noiseless channel:

Definition 126. *A quasifree channel is **noiseless**, if $|f(\xi)| = 1$, for all ξ , i.e., it is as large as consistent with any kind of twisted positive definiteness.*

These are characterized in the following proposition.

Proposition 127. *For a quasifree channel \mathcal{T} , specified by $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$ and $f : \Xi_{\text{out}} \rightarrow \mathbb{C}$, the following conditions are equivalent:*

- (1) $|f(\xi)| = 1$ for all ξ , i.e., \mathcal{T} is noiseless.
- (2) $\Delta\sigma = 0$, and there is some η such that $f(\xi) = \exp i\xi \cdot \eta$ for all ξ .
- (3) $\mathcal{T}^* : \text{CCR}(\Xi_{\text{out}}, \sigma_{\text{out}}) \rightarrow \text{CCR}(\Xi_{\text{in}}, \sigma_{\text{in}})$ is a homomorphism.
- (4) $\mathcal{T}^* : \text{C}^*(\Xi_{\text{out}}, \sigma_{\text{out}})^{**} \rightarrow \text{C}^*(\Xi_{\text{in}}, \sigma_{\text{in}})^{**}$ is a homomorphism.

Proof. Let us begin by establishing an equivalent condition for (3) in terms of S and f . Clearly, using the norm continuity of \mathcal{T}^* , it suffices to establish that

$$\mathcal{T}^*(W(\xi)W(\eta)) = \mathcal{T}^*(W(\xi))\mathcal{T}^*(W(\eta)). \quad (5.215)$$

Using the Weyl relations and our definitions in Eq. (5.188) and Eq. (5.189) we find that $\mathcal{T}^*(W(\xi)W(\eta)) = e^{-i\xi \cdot (\Delta\sigma)\eta/2} f(\xi + \eta)W(S\xi)W(S\eta)$. So, for our noise function, we get

$$f(\xi + \eta) = e^{i\xi \cdot (\Delta\sigma)\eta/2} f(\xi)f(\eta). \quad (5.216)$$

Clearly, this is satisfied when (2) holds, proving (2) \Rightarrow (3). Moreover, Eq. (5.216) implies that $\xi \mapsto |f(\xi)|$ is a homomorphism. Since f is twisted positive definite, we must have $|f(\xi)| \leq 1$, and by the homomorphism property $1 = |f(\xi)||f(-\xi)|$ so $|f(\xi)| \geq 1$, i.e., $|f(\xi)| = 1$. This shows that (3) \Rightarrow (1). The direction (1) \Rightarrow (2) follows immediately from Lem. 128 below (see also the remark following the proof).

It remains to verify (3) \Leftrightarrow (4). Here the direction (4) \Rightarrow (3) is trivial because $\text{CCR}(\Xi, \sigma) \subset \text{C}^*(\Xi, \sigma)^{**}$ and Weyl operators go to Weyl operators (see also the discussion in Sect. 5.4.3). For the converse direction, note that in a von Neumann algebra, $x \rightarrow xy$ is weak*-continuous. So the relation $\mathcal{T}^*(AB) = \mathcal{T}^*(A)\mathcal{T}^*(B)$, which is assumed to hold for $A, B \in \text{CCR}$ transfers to arbitrary $B \in \text{C}^*(\Xi_{\text{out}}, \sigma_{\text{out}})^{**}$ by weak*-continuity, and because CCR is weak*-dense. Repeating this argument for the first factor extends the relation to all A, B . \square

The following Lemma was needed in the proof above, and we separated it because it is of independent interest.

Lemma 128. *Let (Ξ, σ) be a vector space with antisymmetric form, and suppose that χ is a normalized σ -twisted positive definite function on Ξ . Suppose that $|\chi(\eta)| = 1$ for some $\eta \neq 0$. Then $\sigma\eta = 0$ and, for all $\xi \in \Xi$,*

$$\chi(\xi + \eta) = \chi(\xi)\chi(\eta). \quad (5.217)$$

Proof. Consider a 3×3 -matrix M of the form Eq. (5.89). Abbreviating the matrix entries as $M_{k\ell} = \chi(\xi_k - \xi_\ell) \exp(\frac{i}{2}\sigma(\xi_k, \xi_\ell))$, it is of the form

$$M = \begin{pmatrix} 1 & M_{12} & \overline{M_{31}} \\ \overline{M_{12}} & 1 & M_{23} \\ M_{31} & \overline{M_{13}} & 1 \end{pmatrix}. \quad (5.218)$$

Its determinant is

$$0 \leq \det M = 1 + M_{12}M_{23}M_{31} + \overline{M_{12}M_{23}M_{31}} - |M_{12}|^2 - |M_{23}|^2 - |M_{31}|^2. \quad (5.219)$$

Now take the triple of vectors as $(-\eta, 0, \xi)$, where $\xi \in \Xi$ is arbitrary. Then $M_{12} = \chi(-\eta) = \chi(\eta)$, $M_{23} = \chi(\xi)$, $M_{31} = \chi(\xi + \eta) \exp(-\frac{i}{2}\xi \cdot \sigma\eta)$. In particular, $|M_{12}| = |\chi(\eta)| = 1$, so this expression simplifies to

$$0 \leq \det M = -\left| M_{31} - \overline{M_{12} M_{23}} \right|^2. \quad (5.220)$$

This can only be positive if the absolute value vanishes, which means that

$$\chi(\xi + \eta) = \chi(\xi)\chi(\eta)e^{\frac{i}{2}\xi \cdot \sigma\eta}. \quad (5.221)$$

Changing $\xi \mapsto -\xi$ and $\eta \mapsto -\eta$, which also satisfies the assumption of the lemma, every characteristic function in the last expressions changes to its complex conjugate, while the exponent does not. Hence, the exponential factor has to be 1. Since $\xi \mapsto \lambda\xi$ is also allowed, the exponent has to be zero. \square

This shows that the maximal absolute value of χ can only be reached on the classical subsystem. We have not assumed that $|\chi(\lambda\eta)| = 1$ also holds for all scalar multiples $\lambda\eta$ as well. If that is the case, and χ is continuous, then $\chi(\eta) = \exp(i\mu \cdot \eta)$ for some $\mu \in \Xi$. We remark that this assumption may fail, and so, even in 1 dimension, we cannot conclude from the assumptions of the lemma that χ is the characteristic function of a point measure. For example, the classical characteristic function of a measure supported by the integers is 2π -periodic, so $\chi(2\pi) = 1$, but except for a point measure we have $|\chi(\eta)| < 1$ for $0 < \eta < 2\pi$.

Let us recapitulate which of the basic operations are noiseless: The *states* with the homomorphism property, i.e., $\omega(AB) = \omega(A)\omega(B)$, are only the pure states of classical systems, corresponding to point measures on $\Xi = \Xi_0$. Noiseless quantum states do not exist, which also excludes such states on hybrids with non-vanishing σ .

Noiseless *observables* are the projection-valued ones: The homomorphism property implies

$$F(M)^2 = \mathcal{T}^*(\chi_M)^2 = \mathcal{T}^*(\chi_M^2) = \mathcal{T}^*(\chi_M). \quad (5.222)$$

When the observable is considered as acting on a function algebra $\mathcal{C}_b(\Xi_{\text{out}})$, this is the property of having a von Neumann-style functional calculus, $\mathcal{T}^*(\Phi(A)) = \Phi(\mathcal{T}^*(A))$ for $\Phi : \mathbb{R} \rightarrow \mathbb{R}$. That is, postprocessing of outcomes with a function Φ is the same as applying this function to the operator in the functional calculus.

Noiseless *channels* from an irreducible quantum system to itself act by unitary transformation, where the unitary operator belongs to the metaplectic representation of the affine symplectic group (see Sect. 5.1.3).

The following proposition characterizes a further class of noiseless channels, namely those with a right inverse in the Schrödinger picture.

Proposition 129. *Let $\mathcal{T}_1 : C^*(\Xi_1, \sigma_1)^* \rightarrow C^*(\Xi_2, \sigma_2)^*$ and $\mathcal{T}_2 : C^*(\Xi_2, \sigma_2)^* \rightarrow C^*(\Xi_1, \sigma_1)^*$ be quasifree channels such that $\mathcal{T}_1\mathcal{T}_2 = \mathbb{1}$. Then*

- (1) \mathcal{T}_1 is noiseless, and $S_1 : \Xi_2 \rightarrow \Xi_1$ is injective.
- (2) \mathcal{T}_2 is an expansion, i.e., there is a system (Ξ_e, σ_e) such that there is an isomorphism

$$(\Xi_1, \sigma_1) \cong (\Xi_2 \oplus \Xi_e, \sigma_2 \oplus \sigma_e),$$

and $\mathcal{T}_2 = \mathbb{1}_2 \otimes \mathcal{P}_e$, where \mathcal{P}_e is a preparation of a (Ξ_e, σ_e) -system.

- (3) Under the isomorphism from (2), $S_1 : \Xi_2 \rightarrow \Xi_2 \oplus \Xi_e$ is the embedding into the first summand.

Moreover, for $i = 1, 2$ if a channel \mathcal{T}_i satisfies the condition (i), then there is a channel \mathcal{T}_{3-i} so that \mathcal{T}_1 and \mathcal{T}_2 satisfy all the above conditions.

Proof. The composition relation in Eq. (5.213) gives that $\mathbb{1} = S_2S_1$ and similarly $f_1(\xi)f_2(S_1\xi) = 1$. Since $|f_i(\xi)| \leq 1$ by positive definiteness, we must have $|f_1(\xi)| = |f_2(S_1\xi)| = 1$ for all ξ . In particular, \mathcal{T}_1 must be noiseless, and since S_1 has a left inverse, it is injective. This shows (1).

Now consider (2). f_2 is twisted positive definite for $\Delta\sigma = \sigma_1 - S_2^\top\sigma_2S_2$. Moreover, on the subspace $S_1\Xi_1$ this function has the maximal modulus, so by Lem. 128 the range of S_1 is in the null space of $\Delta\sigma$. This is equivalent to the matrix equation $(\Delta\sigma)S_1 = 0$. Using $S_2S_1 = \mathbb{1}$ gives

$$\sigma_1S_1 = S_2^\top\sigma_2. \quad (5.223)$$

Since S_1S_2 is an idempotent operator, every $\xi \in \Xi_1$ is naturally split as $\xi = S_1S_2\xi + (\mathbb{1} - S_1S_2)\xi$, where the first summand is obviously in the range $S_1\Xi_2$ and the second satisfies $S_2(\mathbb{1} - S_1S_2)\xi = (S_2 - S_2)\xi = 0$. Therefore, by transposing Eq. (5.223), these parts are σ_1 -orthogonal:

$$S_1\xi \cdot \sigma_1(\mathbb{1} - S_1S_2)\eta = \xi \cdot \sigma_2S_2(\mathbb{1} - S_1S_2)\eta = 0. \quad (5.224)$$

Moreover, $S_1 : \Xi_2 \rightarrow \Xi_1$ is an isomorphism onto its range, changing σ_2 to the restriction of σ_1 . This proves the decomposition with $\Xi_e = (\mathbb{1} - S_1S_2)\Xi_1$ and σ_e the restriction of σ_1 to this subspace. The action of S_2 is very simple in these terms: It acts separately on the two summands, which makes the corresponding channel a tensor product. On the first summand, $S_1\Xi_2$, it just inverts the isomorphism S_1 .

Hence, after identifying the 2 subsystem of Ξ_1 with Ξ_2 , it acts like the identity channel on this part. The second summand Ξ_e is annihilated by S_2 , which is the hallmark of a preparation (see above). The state prepared lives on (Ξ_e, σ_e) and has characteristic function $\chi_e(\xi_e) = f_2(\xi_e)$. \square

5.4.6 Noise factorization and dilations

The Stinespring dilation we have introduced in Sect. 2.4.2 is one of the most powerful tools in quantum information theory, so it is naturally interesting in the hybrid scenario. In the standard setting, it is a structure theorem for completely positive maps $\mathcal{T}^* : \mathcal{A}_{\text{out}} \rightarrow \mathcal{B}(\mathcal{H}_{\text{in}})$, where we have added the star on \mathcal{T} and the labels “in” and “out” to be consistent with the above notation. It provides an additional Hilbert space \mathcal{K} , an isometry $V : \mathcal{H}_{\text{in}} \rightarrow \mathcal{K}$, and a representation $\pi : \mathcal{A}_{\text{out}} \rightarrow \mathcal{B}(\mathcal{K})$ such that

$$\mathcal{T}^*(A) = V^* \pi(A) V. \quad (5.225)$$

For a quasifree channel, \mathcal{T}^* will map some subalgebra $\mathcal{A}_{\text{out}} \subset C^*(\Xi_{\text{out}}, \sigma_{\text{out}})^{**}$ into a representation of a subalgebra $\mathcal{A}_{\text{in}} \subset C^*(\Xi_{\text{in}}, \sigma_{\text{in}})^{**}$, so that there are many choices to be made, and consequently many variations on the dilation theme. All these variations have the structure of factorizations through an intermediate system, $\mathcal{B}(\mathcal{K})$. The first step (in the direction from input to output) is the embedding of the input states in \mathcal{H}_{in} into this *larger system*, an expansion. The second step, done here by the representation π , is a noiseless operation in the sense of the previous paragraph. These features can be phrased entirely in the category of quasifree maps. What is more, the factorization can be done for arbitrary quasifree channels. This is the content of Thm. 130.

Note that the channels are written here in the Schrödinger picture, so in the factorization $\mathcal{T} = \mathcal{T}_N \mathcal{T}_E$, the expansion \mathcal{T}_E is applied to the physical system first, and \mathcal{T}_N acts on the expanded system. If we write the expansion channel as tensoring with a fixed state ω_E , the factorization is written as

$$\mathcal{T}\omega = \mathcal{T}_N(\omega \otimes \omega_E). \quad (5.226)$$

Theorem 130. *Every quasifree channel can be decomposed into $\mathcal{T} = \mathcal{T}_N \mathcal{T}_E$, where \mathcal{T}_E is an expansion, and \mathcal{T}_N is a noiseless channel. The phase space of the extension system is $\Xi_\Delta = \Xi_{\text{out}}$ as a vector space but with antisymmetric form $\Delta\sigma = \sigma_{\text{out}} - S^\top \sigma_{\text{in}} S$. The salient linear maps and noise functions are*

$$\begin{aligned} S_N : \Xi_{\text{out}} &\rightarrow \Xi_{\text{in}} \oplus \Xi_\Delta & S_N \xi &= S \xi \oplus \xi & f_N(\xi) &= 1 \\ S_E : \Xi_{\text{in}} \oplus \Xi_\Delta &\rightarrow \Xi_{\text{in}} & S_E(\xi_1 \oplus \xi_2) &= \xi_1 & f_E(\xi_1 \oplus \xi_2) &= f(\xi_2). \end{aligned} \quad (5.227)$$

Proof. Let us first verify that the given data for \mathcal{T}_N and \mathcal{T}_E satisfy the positivity condition for channels. The respective difference forms are

$$\begin{aligned}\xi \cdot (\Delta\sigma)_N \eta &= \xi \cdot \sigma_{\text{out}} \eta - (S_N \xi) \cdot (\sigma_{\text{in}} \oplus \sigma_\Delta) S_N \eta \\ &= \xi \cdot \sigma_{\text{out}} \eta - (S \xi) \cdot \sigma_{\text{in}} S \eta - \xi \cdot \sigma_\Delta \eta = 0\end{aligned}\quad (5.228)$$

$$\begin{aligned}(\xi_1 \oplus \xi_2) \cdot (\Delta\sigma)_E (\eta_1 \oplus \eta_2) &= (\xi_1 \oplus \xi_2) \cdot (\sigma_{\text{in}} \oplus \sigma_\Delta) (\eta_1 \oplus \eta_2) \\ &\quad - S_E (\xi_1 \oplus \xi_2) \cdot \sigma_{\text{in}} S_E (\eta_1 \oplus \eta_2) \\ &= \xi_1 \cdot \sigma_{\text{in}} \eta_1 + \xi_2 \cdot \sigma_\Delta \eta_2 - \xi_1 \cdot \sigma_{\text{in}} \eta_1 \\ &= \xi_2 \cdot \sigma_\Delta \eta_2.\end{aligned}\quad (5.229)$$

Hence $(\Delta\sigma)_N = 0$, as required of a noiseless channel, which makes $f = 1$ a legitimate choice. Moreover, $(\Delta\sigma)_E = (0 \oplus \sigma_\Delta)$, which is exactly the noise function for which f_E has to be twisted positive definite. Hence \mathcal{T}_N and \mathcal{T}_E are well defined.

It remains to verify the concatenation relation. Of course, the product of two channels in our class is again in the class, and there is a simple general formula for the data (S', f') of the product. By Eq. (5.213), this gives

$$S' = S_E S_N = S \quad \text{and} \quad f'(\xi) = f_E(S_N \xi) f_N(\xi) = f(\xi).\quad (5.230)$$

Hence, we have $\mathcal{T}_N \mathcal{T}_E = \mathcal{T}$, as claimed. \square

When only one fixed channel \mathcal{T} is under consideration, the above representation may be very wasteful. For example, when \mathcal{T} is itself noiseless, one can clearly choose \mathcal{T}_E to be the identity, and there is no need to adjoin an additional system Ξ_Δ , i.e., Eq. (5.227) is not a *minimal* factorization.

In order to get to a move in that direction, for a general quasifree channel, consider the noise function $f : \Xi_{\text{out}} \rightarrow \mathbb{C}$. Let $N \subset \Xi_\Delta$ denote the largest subspace on which $|f(\xi)| = 1$. Then by Lem. 128, f is a character on N , and hence of the form $f_1(\xi) = \exp(i\lambda \cdot \xi)$ for some $\lambda \in \Xi_{\text{out}}$. Here λ is not uniquely determined because only the scalar products $\lambda \cdot \xi$ with $\xi \in N$ appear, but any choice allows us to proceed. The remainder $f'(\xi) = f(\xi)/f_1(\xi)$ is then a legitimate noise function with $f'(\eta + \xi) = f'(\eta)$ for $\xi \in N$. We may therefore consider f' as a function f_{mid} on the quotient $\Xi_{\text{mid}} = \Xi_{\text{out}}/N$. Denoting the quotient map by $S_{\text{mid}} : \Xi_{\text{out}} \rightarrow \Xi_{\text{mid}}$, this amounts to $f(\xi) = f_1(\xi) f_{\text{mid}}(S_{\text{mid}} \xi)$. By Lem. 128, N is also contained in the null space of $\Delta\sigma$, so this form also passes to the quotient as σ_{mid} . This gives an alternative noise factorization $\mathcal{T} = \mathcal{T}_N \mathcal{T}_E$, closely related to Eq. (5.227), but with the intermediate system $(\Xi_\Delta, \Delta\sigma)$ replaced by $(\Xi_{\text{mid}}, \sigma_{\text{mid}})$ and

$$\begin{aligned}S_N : \Xi_{\text{out}} &\rightarrow \Xi_{\text{in}} \oplus \Xi_{\text{mid}} & S_N \xi &= S \xi \oplus S_{\text{mid}} \xi & f_1(\xi) &= \exp(i\lambda \cdot \xi) \\ S_E : \Xi_{\text{in}} \oplus \Xi_{\text{mid}} &\rightarrow \Xi_{\text{in}} & S_E(\xi_1 \oplus \xi_2) &= \xi_1 & f_E(\xi_1 \oplus \xi_2) &= f_{\text{mid}}(\xi_2).\end{aligned}\quad (5.231)$$

The map S_N in this construction is connected to the previous one by another noiseless channel based on the quotient map S_{mid} . That is, the modification just described moves in the direction of including as much of the channel into the noiseless part as possible.

This follows a categorical approach described in [156] as the *Paschke dilation*. It generalizes the Stinespring construction to the category of W^* -algebras with normal,

completely positive maps when the range in the Heisenberg picture (corresponding to the input in physical terms) is no longer of the form $\mathcal{B}(\mathcal{H})$. A Paschke dilation or a factorization $\mathcal{T} = \mathcal{T}_N \mathcal{T}_E$ can be turned into a Stinespring dilation in the usual sense by taking the input algebra as faithfully represented on a Hilbert space \mathcal{H} , so in the Heisenberg picture, the channel maps into $\mathcal{B}(\mathcal{H})$, and hence we realize the standard setting for the Stinespring construction, also containing a minimal one [3].

5.5 Basic Physical Operations

In the unified picture, every operation, including preparations and measurements, is given by a quasifree channel. This section aims to advertise this unification by showing how basic quantum operations fit into the framework. We will assume only the basic definitions (Sect. 5.4.1) and the parametrization of channels by a linear map $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$ and the noise function f , respectively, the noise state τ .

Typically, S specifies the kind of operation one considers, the number of classical/quantum inputs/outputs, and how they are related. It will typically be fixed at the beginning of each subsection below. This fixes a hybrid system $(\Xi_{\text{out}}, \Delta\sigma)$, and hence the possible noise states τ , resp. noise functions f . While the knowledge of the definitions suffices to verify how the respective examples fit in the general framework, we do sometimes draw on the general results above or illustrate them in the particular case.

5.5.1 States

States are the mathematical description of a system preparation. The input system is therefore a trivial one, $\Xi_{\text{in}} = \{0\}$. Hence $S = 0$, and $\Delta\sigma = \sigma_{\text{out}}$. The positivity condition for f thus demands that f is a characteristic function of a standard state for the out-system. There is no other condition, i.e., *all states* are quasifree channels in this sense. We caution the reader that this is in contrast to another well-established use of the term, by which only Gaussian states are called *quasifree* [115, 108].

In the theory of channel capacity, e.g., for the Holevo bound [157], [23, 12.3], one needs **state ensembles**, usually written as a collection of states with probability weights. When hybrids are considered as systems in their own right, this is just the same as a state on a hybrid. This view of state ensembles naturally extends also to continuous ensembles, in which a non-discrete measure replaces the convex weights.

5.5.2 Disturbance

The word disturbance always refers to a situation deviating from an ideal, which is, in our case, a deviation from the identity channel. That is, we look at how much the output states differ from the inputs. This requires input and output to be of the same type, i.e.,

$$(\Xi_{\text{in}}, \sigma_{\text{in}}) = (\Xi_{\text{out}}, \sigma_{\text{out}}) = (\Xi, \sigma). \quad (5.232)$$

Moreover, since the ideal channel, or “no disturbance”, should be a special case, we choose $S = \mathbb{1}$. Then $\Delta\sigma = 0$, so the condition on the noise function f is the classical Bochner condition. Hence, f is the Fourier transform of a probability measure τ , appropriately called the noise measure. The channel acts as

$$\mathcal{T}\omega = \int \tau(d\xi) \alpha_\xi(\omega) = \omega * \tau, \quad (5.233)$$

where the convolution is taken in the sense of Def. 114.

The size of the noise can be ascertained in different ways. For example, using Cor. 120 we get the norm bound

$$\|\mathbb{1} - \mathcal{T}\|_{\text{cb}} = \sup_{\rho} \|\rho - \mathcal{T}\rho\|_1 = \|\delta_0 - \tau\|_1, \quad (5.234)$$

where δ_0 is the point measure at 0, and the last norm is the norm for classical states, i.e., the total variation. If we decompose $\tau = (1 - \lambda)\delta_0 + \lambda\tau'$ for some probability measure with $\tau'(\{0\}) = 0$ we get $\|\delta_0 - \tau\|_1 = 2\lambda$. This norm measure of noise is only small if we have a large convex component of \mathcal{T} , which is equal to $\mathbb{1}$. In particular, a channel that introduces a small shift ($\tau = \delta_\xi$, $\xi \approx 0$) is always at a maximal distance. Better measures of the noise for many purposes are variances or, more generally, transport distances [158].

In many cases, it is not necessary to condense the size of the noise into a single number, and the most accurate description is, of course, the noise measure itself.

5.5.3 Observables

An observable is a channel with classical output, i.e., $\sigma_{\text{out}} = 0$, and Ξ_{out} is the space of measurement outputs. In the quasifree setting, the observable automatically gets a covariance property with respect to shifts of the outputs.

The theory laid out in Sect. 5.3 shows that the two ways of looking at an observable, namely as a positive operator-valued measure (POVM) on the one hand and as an operator on continuous functions on the other, are equivalent Heisenberg pictures of such a channel. For the POVM view, we have to identify, for every measurable set $M \subset \Xi_{\text{out}}$, an effect operator $F(M)$ on the input system. Thus, we need a Heisenberg picture map \mathcal{T}^* which is well defined on the indicator function $1_M \in \mathcal{U}(\Xi_{\text{out}}, 0)$. The appropriate Heisenberg picture is thus

$$\mathcal{T}^* : \mathcal{U}(\Xi_{\text{out}}, 0) \rightarrow \mathcal{U}(\Xi_{\text{in}}, \sigma_{\text{in}}), \quad (5.235)$$

and the positive operator-valued measure describing the observable is given by $F(M) = \mathcal{T}^*(1_M)$. Equivalently, we can consider the observable as a map on bounded continuous functions ϕ on Ξ_{out} , such that $\mathcal{T}^*\phi = \int F(d\xi)\phi(\xi)$.

The further characterization of the class of covariant observables so described depends on the range of $S : \Xi_{\text{out}} \rightarrow \Xi_{\text{in}}$, and especially on the restriction of σ_{in} to the range $S\Xi_{\text{out}}$. This is basically the question of whether the quantities measured are subject to a quantum uncertainty constraint or not. We will consider the two extreme cases, a position observable and a phase space or position and momentum observable, separately below. In either case, we take S to be injective because otherwise, we would have directions in the output space that have distributions not depending on the quantum input.

By virtue of our definition of S-covariant channels, i.e., Eq. (5.187), quasifree observables fit into the framework of observables covariant with respect to a projective unitary representation of a group G . In this traditional subject, [19, 18], the basic construction of all covariant observables uses a covariant version [159] of the Stinespring dilation (called Naimark's dilation for the case of classical output) to reduce the construction to the noiseless, i.e., projection-valued case, which is then solved by Mackey's theory of induced representations (see [160], and [161] for a worked example). What has apparently not been considered in detail was the nature of the noise. In our framework, there is a clear distinction of the position vs. the phase space case, requiring classical vs. quantum noise. We, therefore, treat these cases separately below.

A traditional subject in the general theory is the existence of a direct formula for the output probability density at a point in the outcome set. If such a formula exists, it will be given by the expectation value of a positive, possibly unbounded operator, which is called the operator-valued Radon-Nikodym density of the observable (see e.g., [18, Sect. IV.2.], [162, Sect. I.5.G], and [19, Thm. 4.5.2] for the compact group case). That is, we are looking for a family of positive, possibly unbounded operators $\dot{F}(x)$ such that the observable is expressed as

$$\mathcal{T}^*(g) = \int dx \dot{F}(x) g(x). \quad (5.236)$$

Recall that, following the hybrid version of Eq. (5.7), our convention for the action of translations is $(\alpha_x g)(y) = g(x + y)$. So the covariance condition in Eq. (5.187) translates to $\alpha_\xi(\dot{F}(x)) = \dot{F}(x - S^\top \xi)$. Now since S is injective, S^\top is onto, so this equation determines the function $\dot{F}(x)$ from one of the values, say $\dot{F}(0) =: \dot{F}$:

$$\dot{F}(x) = \alpha_{-\xi}(\dot{F}), \quad \text{for any } \xi \in \Xi_{\text{in}} \text{ such that } S^\top \xi = x. \quad (5.237)$$

Since S^\top might have a kernel, this also implies the invariance of \dot{F} under α_ξ with $S^\top \xi = 0$.

Of course, there is also an expression for \dot{F} in terms of the noise function f since both quantities determine the observable. For that, we put $g(x) = \exp(ik \cdot x)$, i.e., $g = W_{\text{out}}(k)$, in the above equation, and solve for \dot{F} by inverse Fourier transform. With $n = \dim \Xi_{\text{out}}$ we get:

$$\dot{F} = \frac{1}{(2\pi)^n} \int dk f(k) W_{\text{in}}(Sk). \quad (5.238)$$

In general, e.g., for the canonical position observable, neither \dot{F} nor this integral makes sense. However, with sufficient noise, seen by the decay of f at infinity, both do.

Position observables

The canonical position observable of a purely quantum system belongs to the self-adjoint operators Q_j from Sect. 5.2.1. The characteristic function of the output probability distribution is hence the expectation of $\exp(ik \cdot Q) = W(k, 0)$. So this is quasifree with $\Xi_{\text{out}} = \{k\} = \mathbb{R}^n$ and

$$Sk = (k, 0), \quad (5.239)$$

when the variables are arranged as in Sect. 5.2.1. Of course, one could also include some classical hybrid variables. Since the noise function vanishes, the observable is projection-valued, which can be said in two equivalent ways, namely that $F(M)$ is always a projection or that \mathcal{T}^* is a homomorphism (also compare Prop. 127). For any input density operator ρ , we write $\mathcal{T}\rho = \rho^Q$ and call it the position distribution of ρ . Similarly, we define ρ^P as the momentum distribution.

The beauty of the quasifree formalism is that it automatically includes noisy versions. These are characterized by choosing the same S but allowing f to be more

general. This defines the class of generalized position observables, which share the covariance condition with the canonical one. The structure theory is then immediate: Since $\Delta\sigma = 0$ the noise is necessarily classical, so the most general position observable has the output distribution $\nu * \rho^Q$, where ν is some fixed noise measure on position space which is independent of ρ , and ρ^Q is the output distribution of the standard position observable. Thus, we can always think of such a measurement as executing the standard one and then adding, from a statistically independent source, noise with distribution ν .

When the noise distribution has a Radon-Nikodym density $\dot{\nu}$ with respect to the Lebesgue measure, we have $\dot{F} = \dot{\nu}(Q)$ in the functional calculus of the commuting selfadjoint operators Q_k . In contrast, for the canonical observable itself, the expectation of \dot{F} in the state vector Ψ should be $|\Psi(x)|^2$, which might be given a meaning as a sesquilinear form on a Schwartz space. However, there is no closable operator \dot{F} corresponding to this. This is also seen in the difficulty of making sense of Eq. (5.238).

Phase space observables

Here we demand a joint measurement of all positions and momenta. So we have $\Xi_{\text{out}} = \Xi_{\text{in}}$ and $S = \mathbb{1}$, but the symplectic forms are different, namely the standard quantum one on Ξ_{in} and 0 on Ξ_{out} . Hence, $\Delta\sigma = -\sigma_{\text{in}}$ and the admissible noise functions are exactly the characteristic functions of quantum states τ . Hence, the relation in Eq. (5.190) is exactly that for a **convolution** of quantum states in the sense of [121] and Def. 114. When τ is the quantum state defining the observable, and ρ is the input state, the output distribution is thus $\tau * \rho$. Comparing the expression in Eq. (5.184) with Eq. (5.236) we find the Radon-Nikodym density of the POVM to be

$$\dot{F} = \beta_-(\tau). \quad (5.240)$$

This is a density operator in two different meanings of the word: A Radon-Nikodym density, and also a positive operator with trace 1, provided the correct normalization of phase space Lebesgue measure (see [121]) is used. This characterization of covariant phase space observables is well known [19, 121, 18]. The Gaussian special case is known in quantum optics as the Husimi distribution or Q-function of ρ . But as the quasifree formalism clearly indicates, any τ , pure or mixed, will work analogously.

Of course, such a joint position/momentum measurement necessarily includes errors, which is the subject of **measurement uncertainty** relations [62]. By this, we mean any relation expressing that one can either get a fairly good position measurement with large errors for momenta or conversely. For uncertainty relations, the covariance condition is an unwanted restriction, but the proof of the general case [62] works via showing that among the optimal solutions, there is always a covariant one. This makes the tradeoffs extremely easy to describe. Indeed, the position marginal of the output distribution is $(\tau * \rho)^Q = \tau^Q * \rho^Q$, a relation which is shown by setting one set of variables equal to zero in the product of characteristic functions of τ and ρ . In other words, the position marginal of phase space observable is a noisy position observable. That statement is obvious from the covariance conditions, but here we also learn that the noise measure is itself the position distribution τ^Q of

a quantum state τ . The same holds for momentum, and, crucially, it is *the same* quantum state τ that enters. In other words, the tradeoff between the noises in the marginals of a phase space observable is the same as the tradeoff between the concentration of the position distribution τ^Q and the momentum distribution τ^P of a quantum state. This tradeoff is known as **preparation uncertainty**. The equality of measurement uncertainty and preparation uncertainty is false for most other observable pairs but persists [163] for more general observable pairs, which are related by the Fourier transformation of some locally compact abelian group. This includes angle and number or qubit strings looked at in different Pauli bases.

5.5.4 Dynamics

In the case of time evolutions, the input and output systems are the same. Let us start with the **reversible** case, for which the time parameter t in \mathcal{T}_t is allowed to be positive or negative, i.e., the \mathcal{T}_t form a one-parameter group rather than just a semigroup. Then $\Delta\sigma$ has to vanish, and each \mathcal{T}_t must be a noiseless operation (see Sect. 5.4.5), and by Prop. 127 it follows that \mathcal{T}_t^* must be a homomorphism. Actually, this conclusion is valid even without the quasifree form: Using the existence of an inverse for equality in the generalized Schwarz inequality for 2-positive maps ($\mathcal{T}(x^*x) \geq \mathcal{T}(x)^*\mathcal{T}(x)$ [50]) implies the homomorphism property. Hence, for a reversible evolution, the center of the algebra, i.e., the classical part, must be invariant as a set, and there is a well-defined restriction of \mathcal{T}_t to the classical subsystem. That is, by observing the classical subsystem, we can never find out anything about the initial state of the quantum subsystem. This is another showcase of the **no-interaction theorem** discussed in Sect. 3.3.3, which blocks any understanding of the quantum measurement process by reversible, e.g., Hamiltonian couplings.

A traditional subject in classical probability are processes with **independent increments**. Since the increments are supposed to have the same distribution for any current state, this implies translation invariance, and since successive increments are assumed independent, we get a convolution semigroup ($S_t \equiv \mathbb{1}$). The classic result is the Lévy-Khintchine Theorem (see, e.g., [164]), characterizing the generators as a combination of a Gaussian part and a jump part. If we likewise stick to the choice of trivial S_t , this result applies verbatim to arbitrary hybrids. Even without the quasifreeness assumption, it is treated in [72].

For the general case of an arbitrary semigroup S_t , the precise and general characterization of generators is lacking so far. It is easy to see that the Lévy-Khintchine formula is still valid, but there are uncertainty-type constraints needed to ensure complete positivity. These are readily solved in the purely Gaussian case: The logarithmic derivative of the noise function at $t = 0$ has to be an admissible quantum covariance matrix for the *symplectic form* computed as the derivative of $\Delta\sigma$. It turns out that this is already all [86]: In the general case, the Lévy-Khintchine formula decomposes the generator into a Gaussian part and a jump part. The noise required for complete positivity depends only on the Gaussian part. The jump part, which belongs to a classical Lévy process, adds no further requirements, nor can it be used to ease the noise requirements for the Gaussian part. This is the situation for finite-dimensional phase spaces, but the quasifree analogs for infinite dimension offer interesting challenges, including generators not of Lindblad form ([30, 165]).

Many applications use the quasifree structure. Especially when time-dependent generators are involved, as in the case of **feedback and control**, it is vastly easier to put the process together in phase space than to multiply cp maps on the infinite-dimensional observable algebra. **Continual observation** is likewise a hybrid scenario in which the classical part can be observed completely and at all times without incurring disturbance costs. Doing justice to this field would require a book of its own, and we do not even try to review the literature. The hybrid aspects are typically neglected, as are the demands of building observable algebras.

5.5.5 Classical limit

The classical limit, $\hbar \rightarrow 0$, characterizes the behavior of states and observables that do not change appreciably over phase space regions whose size is measured by \hbar . We have suppressed this parameter, which implicitly means that we used units for quantum position and quantum momentum, which make $\hbar = 1$. For the discussion of the classical limit, it is better to make this parameter explicit as a factor to the commutation form Eq. (5.71), just as physics textbooks have it. The identity map S between universes with different \hbar is then not symplectic, but one can build a (necessarily noisy) quasifree channel between such universes, allowing the comparison of observables. Equivalently, one can scale all phase space variables by $\sqrt{\hbar}$. The connection maps are then used to formulate a notion of *convergent sequences* by a Cauchy-like condition. This approach to the classical limit [166] is as close to a limit of the entire theory (not just isolated aspects such as WKB wave functions or partition functions) as one can get. The limit is a classical canonical system, with quantum Hamiltonian dynamics going to its classical counterpart. For our context, it should be noted that it can be taken for parts of the system (like the heavy particles in a Born–Oppenheimer approximation) and, due to the complete positivity of the connection maps, composes well with further degrees of freedom, i.e., can be applied to hybrids.

5.5.6 Cloning

Cloning, also known as copying or broadcasting, is a process that generates copies of a quantum system [116]. Of course, the well-known No-cloning Theorem says that this cannot be done without error. Quasifree maps are ideally suited as a simple testbed for this basic operation and the unavoidable errors. Let us consider a fixed system type (Ξ, σ) , which also serves as the input. At the output, we have N such systems in parallel, so $\Xi_{\text{out}} = \bigoplus_j^N \Xi_j$ where Ξ_j is just an isomorphic copy of the underlying $\Xi = \Xi_{\text{in}}$. The marginals of interest forget all but one output and are thus described by a disturbance channel with $S = \mathbb{1}$ (see above). This fixes S on each of the subspaces in Ξ_{out} , and hence by linearity, the overall map S :

$$S\left(\bigoplus_j \xi_j\right) = \sum_j \xi_j. \quad (5.241)$$

In other words, this map is exactly what one would write down for an ideal copier if one had never heard of the No-cloning Theorem. The quasifree formalism then generates all possible error tradeoffs consistent with this overall behavior.

The optimal solution of this problem depends on how the quality of the clones is assessed, and in particular, whether one uses the average fidelity of the clones or the closeness of the overall output to a product state, i.e., whether one also demands the output systems to be nearly uncorrelated. The optimization problem should be stated without assuming quasifreeness, but one can *prove* that the optimal cloners will be quasifree with the above S . It turns out that for the criterion of overall product state fidelity, the optimal cloner is Gaussian, whereas for the average single state fidelity criterion, it is not, although the best Gaussian cloner performs only a few percent below optimum [167]. One can also look at asymmetric scenarios, in which the various copies satisfy different quality requirements, i.e., the output state is not permutation symmetric.

5.5.7 Instruments

An instrument, according to a now-standard terminology by Davies and Lewis [168, 19], is a channel with both a classical and a quantum output, i.e., a hybrid output. This is the setting in which one can discuss the tradeoff between information gain on the classical part of the output and disturbance on the quantum output (see Fig. 5.3).

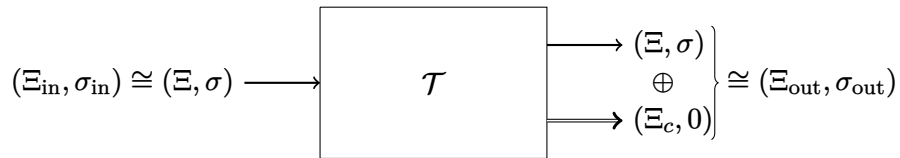


Figure 5.3: A covariant instrument: A quantum system with the phase space (Ξ, σ) is measured by the instrument \mathcal{T} . The output is a hybrid system with a quantum part on the same space (Ξ, σ) joined by a classical system, the measurement result, with some classical system $(\Xi_c, 0)$.

Concretely, let $\Xi_{\text{out}} = \Xi_{\text{in}} \oplus \Xi_c$, where Ξ_c is the classical output. As in the case of a cloner, linearity of S implies that we just have to fix our demands for the marginals, i.e., the actions on the summands Ξ_{in} and Ξ_c , to get the overall map S . On the first summands, we take the identity, in keeping with our intention to discuss the disturbance inflicted by the instrument. The case of *no disturbance* should be included, so we should take $S = \mathbb{1}$ on the summand Ξ_{in} . For the second summand, Ξ_c , we just have to say which variable or combination of variables we wish to measure, i.e., S is chosen exactly as the corresponding map S in Eq. (5.239) from the above description of observables. To distinguish it from the overall S , we denote this by S_c . Putting these parts together, we get

$$S(\xi \oplus \eta) = \xi + S_c \eta \quad (5.242)$$

or, equivalently, $S^\top \xi = \xi \oplus S_c^\top \xi$. The noise functions consistent with this choice then parametrize the class of covariant phase space instruments. Their analysis is a nice illustration of our theory. The main interest is again in the marginals, which reflect the tradeoffs between disturbance and information gain. We treat them in analogy to the corresponding observables.

Just as for observables, the theory of quasifree instruments fits into the theory of covariant instruments for more general groups [169, 170, 171, 172]. We begin by outlining a heuristic argument suggesting a form for general covariant instruments. We will verify later how this form comes out of our approach. As in the case of observables, we assume an operator density for the outputs as a function of the measured parameter: Its interpretation is the quantum channel conditioned on the classical output x . This captures a typical use of instruments, where the quantum state is updated based on the classical result. We are thus looking for a family of cp maps \mathcal{T}_x such that the following analog of Eq. (5.236) holds:

$$\mathcal{T}^*(A \otimes g) = \int dx \mathcal{T}_x^*(A)g(x). \quad (5.243)$$

Putting $A = \mathbb{1}$, it is clear that \mathcal{T}_x^* is not a channel, as it is not normalized to the identity. Instead, $\mathcal{T}_x^*(\mathbb{1}) = \dot{F}(x)$ is the Radon-Nikodym density of the classical marginal observable. Thus, if the classical marginal has no density, then \mathcal{T}_x^* cannot be defined either. On the other hand, if $\dot{F}(x)$ exists, we can look for a bona fide channel $\tilde{\mathcal{T}}_x^*$ such that, with the abbreviation $D(x) = \dot{F}(x)^{1/2}$, we have $D(x)\tilde{\mathcal{T}}_x^*(A)D(x) = \mathcal{T}_x(A)$. With the Kraus decomposition $\tilde{\mathcal{T}}_x^*(A) = \sum_j K_j(x)^*AK_j(x)$ we get

$$\mathcal{T}_x^*(A) = \sum_j (K_j(x)D(x))^*AK_j(x)D(x). \quad (5.244)$$

It is clear from this formula that $K_j(x)$ can be thought of as a map from the closed range of D to \mathcal{H} and should be normalized as $\sum_j K_j(x)^*K_j(x) = \text{supp}(D(x))$, where the right-hand side denotes the support projection of $D(x)$.

A feature shared with the observable case and the general group case is that $\mathcal{T}_x(A)^*$ needs only be known at one point because this can be transferred to all x by covariance. Indeed, the covariance of the instrument is equivalent to

$$\mathcal{T}_{x+S_c^\top \xi}^* = \alpha_{-\xi} \mathcal{T}_x^* \alpha_\xi. \quad (5.245)$$

Thus $\alpha_{-\xi} K_j(x) = K_j(x+S_c^\top \xi)$, extending the covariance condition in Eq. (5.237) for the observable F , written for D as $\alpha_{-\xi} D(x) = D(x+S_c^\top \xi)$. Since S_c^\top is surjective, we only need all values at the origin and abbreviate $D(0) =: D$ and $K_j(0) = K_j$. This gives the form

$$\mathcal{T}_x^*(A) = \sum_j \alpha_{-\xi} (K_j D)^* A \alpha_{-\xi} (K_j D), \quad \text{where } S_c^\top \xi = x. \quad (5.246)$$

In this general form, the Kraus operators are only constrained by the normalization $\sum_j K_j^* K_j = \text{supp}(D)$ and the invariance condition arising from the possibility that $S_c^\top \xi = 0$ might have non-zero solutions ξ . In that case, we must demand that the K_j and the $\alpha_\xi(K_j)$ describe the same channel. In particular, for extremal instruments, when there is only one Kraus operator, it has to be invariant up to a phase.

Position instruments

We will illustrate our formalism by executing the task of finding all position instruments twice: Once directly via the characteristic functions and Prop. 118, and once in the way inspired by general covariance theory, i.e., via Eq. (5.246). For simplicity, we look only at the pure case, i.e., we are happy to find the simplest solutions from which all others arise by mixture.

Beginning with our approach, we use the notational conventions for phase space and dual vectors outlined earlier. The map S_c comes from the position observable in Eq. (5.239), i.e., $S(\hat{p}, \hat{q}, k) = (\hat{p} + k, \hat{q})$. All these quantities can be vectors $\hat{p}, \hat{q}, k \in \mathbb{R}^n$ and with the above choice of S we have

$$\begin{aligned} (\hat{p}, \hat{q}, k) \cdot \Delta\sigma(\hat{p}', \hat{q}', k') &= \hat{p} \cdot \hat{q}' - \hat{q} \cdot \hat{p}' - (\hat{p} + k) \cdot \hat{q}' + \hat{q} \cdot (\hat{p}' + k') \\ &= \hat{q} \cdot k' - k \cdot \hat{q}'. \end{aligned} \quad (5.247)$$

Now Eq. (5.247) is the commutation form of a hybrid phase space with quantum coordinates (\hat{q}, k) and a classical direction \hat{p} . A pure state on this hybrid fixes the classical part (cf. Lem. 87) to a point a , say, and is given on the quantum part by a vector ψ on the Hilbert space of n degrees of freedom, defining the noise state τ . This gives the noise function of our position instrument as

$$f(\hat{p}, \hat{q}, k) = e^{ia \cdot \hat{p}} \chi_\tau(k, \hat{q}) = e^{ia \cdot \hat{p}} \langle \psi, W(-k, \hat{q}) \psi \rangle. \quad (5.248)$$

Here we chose the sign of k by a convention for ψ , for literal agreement with the second approach, as we will prove later on. Together with S , Eq. (5.248) is a complete description of the position instrument in our quasifree hybrid setting.

For the approach via Eq. (5.246), with a single Kraus operator K , we have to satisfy the normalization condition $K^*K = \text{supp}(D)$ and the invariance condition. Here we have $S_c^\top(\xi) \cdot k = \xi \cdot S_c(k) = \xi \cdot (k, 0) = \hat{p} \cdot k$, so our constraint for ξ translates to vectors of the form $\xi = (x, \hat{q})$, where \hat{q} is arbitrary and our invariance means

$$\alpha_\xi(K) = u(q)K \quad \text{for } \xi = (0, \hat{q}) \in \ker S_c^\top. \quad (5.249)$$

Inserting a sum for ξ , it is clear that $u(\hat{q}) = \exp(-ia \cdot \hat{q})$ is a character. The eigenvalue equation for K is satisfied by the Weyl operator $W(0, -a)$, but in contrast to Lem. 113 the Weyl operators $W(\ker S_c^\top)$ do not act irreducibly, and so K is only determined up to an operator invariant under all $\alpha_\xi(0, \hat{q})$. Such operators commute with all $W(\sigma(0, \hat{q})) = W(\hat{q}, 0)$, i.e., are multiplication operators in the position representation. Thus we have

$$K = W(0, -a)\tilde{\psi}(Q). \quad (5.250)$$

Similarly, $D = \psi_D(Q)$ is a positive multiplication operator, whose square is the noise density $\nu(Q)$ discussed above for the position observable, so $\psi_D \in L^2(\mathbb{R}^n)$. The normalization condition $K^*K = \text{supp}(D)$ means that $|\tilde{\psi}(x)| = 1$ for $x \in \text{supp } \psi_D$. Setting $\psi(x) = \tilde{\psi}(x)\psi_D(x)$, we get $KD = W(0, -a)\psi(Q)$, i.e.,

$$(KD\phi)(x) = \psi(x - a)\phi(x - a). \quad (5.251)$$

Now, we compute the characteristic function of the overall channel to compare it with Eq. (5.248) from our approach. For the start, we calculate the translates of the Kraus operator KD using the action of a Weyl operator described in Eq. (5.11) and our Kraus operator in Eq. (5.251):

$$\begin{aligned}
(\alpha_{-\xi}(KD)\phi)(y) &= \left(W(-\sigma\xi)^* KD W(-\sigma\xi)\phi \right)(y) \\
&= \left(W(\hat{q}, -x) KD W(-\hat{q}, x)\phi \right)(y) \\
&= e^{i\hat{q}y - \frac{i}{2}\hat{q}x} \left(KD W(-\hat{q}, x)\phi \right)(y-x) \\
&= e^{i\hat{q}y - \frac{i}{2}\hat{q}x} \Psi(y-x-a) \left(W(-\hat{q}, x)\phi \right)(y-x-a) \\
&= e^{i\hat{q}y - \frac{i}{2}\hat{q}x} \Psi(y-x-a) e^{-\frac{i}{2}\hat{q}x - i\hat{q}(y-x-a)} \phi(y-x-a+x) \\
&= e^{i\hat{q}a} \Psi(y-x-a) \phi(y-a). \tag{5.252}
\end{aligned}$$

Then, the characteristic function after the action of our instrument can be easily read off by evaluating $\mathcal{T}^*(W(\hat{p}, \hat{q}, k))$. So we take two vectors ϕ, ϕ' and get the following:

$$\begin{aligned}
\langle \phi, \mathcal{T}^*(W(\hat{p}, \hat{q}, k))\phi' \rangle &= \int dx e^{ikx} \langle \alpha_{-\xi}(KD)\phi, W(\hat{p}, \hat{q})\alpha_{-\xi}(KD)\phi' \rangle \\
&= \int dx \int dy \overline{(\alpha_{-\xi}(KD)\phi)(y)} e^{\frac{i}{2}\hat{p}\hat{q} + i\hat{p}y} (\alpha_{-\xi}(KD)\phi')(y + \hat{q}) \\
&= \int dx \int dy e^{i(kx + \frac{1}{2}\hat{p}\hat{q} + \hat{p}(y+a))} \\
&\quad \overline{\psi(y-x)} \psi(y-x+\hat{q}) \overline{\phi(y)} \phi'(y+\hat{q}) \\
&= \int dx \int dy e^{i(k(y-x) + \frac{1}{2}\hat{p}\hat{q} + \hat{p}(y+a))} \overline{\psi(x)} \psi(x+\hat{q}) \overline{\phi(y)} \phi'(y+\hat{q}) \\
&= e^{i\hat{p}a} \int dx \int dy e^{\frac{i}{2}\hat{p}\hat{q} + \frac{i}{2}k\hat{q} - \frac{i}{2}(\hat{p}+k)\hat{q}} \overline{\psi(x)} \phi(y) \\
&\quad (W(-k, \hat{q})\psi(x)) (W(\hat{p}+k, \hat{q})\phi'(y)) \\
&= e^{i\hat{p}a} \langle \psi, W(-k, \hat{q})\psi \rangle \langle \phi, W(S(\hat{p}, \hat{q}, k))\phi' \rangle. \tag{5.253}
\end{aligned}$$

Thus we have $f(\hat{p}, \hat{q}, k) = e^{ia\hat{p}} \langle \psi, W(-k, \hat{q})\psi \rangle$, i.e., the same characteristic function as in Eq. (5.248), so the two approaches yield the same result, only with less analytical pain in our quasifree theory.

Finally, we are interested in the tradeoffs for the marginals, namely the quantum output, which is necessarily of the type discussed prior under the keyword *disturbance* in Sect. 5.5.2, and the measurement output, which is of the type discussed under position observables in Sect. 5.5.3. From the equation

$$\chi_{\text{out}}(\hat{p}, \hat{q}, k) = f(\hat{p}, \hat{q}, k) \chi_{\text{in}}(\hat{p}+k, \hat{q}) \tag{5.254}$$

we can directly read those off by setting the according variables to zero:

$$\begin{array}{ll}
\text{classical marginal: } \hat{p} = \hat{q} = 0, & \text{noise measure} = \tau^Q, \\
\text{quantum marginal: } k = 0, & \text{noise measure} = \delta_a \times \tau^P.
\end{array} \tag{5.255}$$

This is a very concise formulation of a well-known intuition: τ^Q is the distribution of the noise added to the measurement outcomes, i.e., the *error* of the measurement. τ^P , on the other hand, is the disturbance of the momentum variable. So, these are reciprocal in exactly the way known for quantum states. We remark that noise could also occur in the quantum position direction, here given by a deterministic shift a . Non-pure instruments will have the distribution for that as well, and τ in the above description generally depends on a , allowing all the complex correlations in a hybrid noise state.

Phase space instruments

In this case, $S(\xi \oplus \eta) = \xi + \eta$, and $\Delta\sigma$ is non-degenerate, so the noise state is a quantum state of twice the number of degrees of freedom. In the pure case, it is given by a vector $\psi \in L^2(\mathbb{R}^n \times \mathbb{R}^n, dx_1, dx_2)$. Such a vector can be identified with a Hilbert-Schmidt operator over the system Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, dx)$, and we will see that this is precisely the required form of the local Kraus operator KD . This general form for phase space instruments was also obtained independently in [172]. In the following proposition, which is a straightforward application of our formalism, we also describe the resulting tradeoff between disturbance (noise in the quantum marginal) and precision (noise in the classical marginal). They are precisely related by Fourier transformation almost exactly as in the case of joint measurements of position and momentum. Only the Fourier transform is not between position and momentum but between the operator side and the function side of quantum harmonic analysis.

Proposition 131. (1) *Every extremal quasifree phase space instrument is characterized by a Hilbert-Schmidt operator $\hat{\Psi}$ with $\text{tr}(\hat{\Psi}^* \hat{\Psi}) = 1$ such that*

$$\mathcal{T}^*(A \otimes g) = \int d\xi \alpha_{-\xi}(\hat{\Psi})^* A \alpha_{-\xi}(\hat{\Psi}) g(\xi). \quad (5.256)$$

(2) *Conversely, any such operator $\hat{\Psi}$ determines an instrument and is determined by it up to a phase.*

(3) *The classical marginal is a covariant phase space observable with density*

$$\dot{F} = \hat{\Psi}^* \hat{\Psi}. \quad (5.257)$$

(4) *The quantum marginal is addition of translation noise: $\rho \mapsto \int d\xi m(\xi) \alpha_\xi(\rho)$ with $m \in L^1(\Xi)$*

$$m(\xi) = |(\mathcal{F}\hat{\Psi})(-\sigma\xi)|^2. \quad (5.258)$$

Note that since \mathcal{F} is unitary from the Hilbert-Schmidt class onto $L^2(\Xi)$, not only all operator densities \dot{F} but also all L^1 -densities m can occur. The prototype of this tradeoff is the case of a single degree of freedom with additional covariance under harmonic oscillator rotations. In particular, we can look at the Gaussians $\hat{\Psi} = c \exp(-\beta H)$ with $H = (P^2 + Q^2)/2$. Then the Fourier transform is also Gaussian and proportional to $\exp(-\coth(\beta/2))\xi^2/4$, where $\xi^2 = (p^2 + q^2)/2$. Now, for $\beta \rightarrow 0$, $\hat{\Psi}$ is a small multiple of the identity, so it can approximately be interchanged with A in

Eq. (5.256). This even works in trace norm for the action on a trace class operator for the dual channel. This means that the disturbance goes to zero, and this is borne out by the computation of m , which for small β is Gaussian with variance $\propto 1/\beta$. On the other hand, the phase space density of the classical marginal becomes very broad, and the measurement outputs reveal very little about the state. In the other direction, $\beta \rightarrow \infty$, $\hat{\Psi}$ becomes a coherent state projection, and the output distribution becomes the Husimi function. The quantum noise m is still Gaussian, with a variance on the order of standard quantum uncertainties.

Proof of Prop. 131. The difference symplectic form is now

$$(\xi, \eta) \cdot \Delta\sigma(\xi', \eta') = \xi \cdot \sigma\xi' - (\xi + \eta) \cdot \sigma(\xi' + \eta'). \quad (5.259)$$

Rather than expanding this, we just choose a twisted definite function, evaluated for the independent variables ξ and $\xi + \eta$. That is, for the extremal case, we choose a pure state on a doubled system, given by a vector $\Psi \in \mathcal{H} \otimes \mathcal{H}$ such that

$$f(\xi \oplus \eta) = \langle \Psi | W(\xi) \otimes \overline{W(\xi + \eta)} | \Psi \rangle. \quad (5.260)$$

Here, the bar indicates complex conjugation $\overline{W(\xi)} = \theta^* W(\xi) \theta$ with respect to an arbitrary antilinear involution θ , which has the effect of reversing the symplectic form and hence takes care of the minus sign in Eq. (5.259). This completes the parametrization of the family of instruments. What is left is rewriting this in the stated form and computing the marginals.

To this end, we introduce the isomorphism $\Psi \mapsto \hat{\Psi}$ from $\mathcal{H} \otimes \mathcal{H}$ to Hilbert-Schmidt operators on \mathcal{H} given by $\psi_1 \otimes \psi_2 \mapsto |\psi_1\rangle\langle\theta\psi_2|$. Note that the involution θ is needed here so that both sides of the identification are linear in ψ_2 . We next express the action of the Weyl operators in Eq. (5.260) in terms of the Hilbert-Schmidt operators. For $\Psi = \psi_1 \otimes \psi_2$, we get

$$\begin{aligned} W(\xi) \otimes \overline{W(\xi + \eta)} \Psi &= (W(\xi)\psi_1) \otimes (\theta^* W(\xi + \eta)\theta\psi_2) \\ &\mapsto |W(\xi)\psi_1\rangle\langle W(\xi + \eta)\theta\psi_2| = W(\xi) \hat{\Psi} W(\xi + \eta)^*. \end{aligned} \quad (5.261)$$

Inserting this into Eq. (5.260) gives the equivalent expression

$$f(\xi \oplus \eta) = \text{tr}(\hat{\Psi}^* W(\xi) \hat{\Psi} W(\xi + \eta)^*). \quad (5.262)$$

Denoting the Weyl elements on the classical output by W_0 , and using the identity $\int d\zeta \alpha_\zeta(A) = \text{tr}(A)\mathbb{1}$, we find

$$\begin{aligned} \mathcal{T}^*(W(\xi) \otimes W_0(\eta)) &= \text{tr}(\hat{\Psi}^* W(\xi) \hat{\Psi} W(\xi + \eta)^*) W(\xi + \eta) \\ &= \int d\zeta \alpha_\zeta(\hat{\Psi}^* W(\xi) \hat{\Psi} W(\xi + \eta)^*) W(\xi + \eta) \\ &= \int d\zeta \alpha_\zeta(\hat{\Psi})^* e^{i\zeta \cdot \xi} W(\xi) \alpha_\zeta(\hat{\Psi}) e^{-i\zeta \cdot (\xi + \eta)} W(\xi + \eta)^* W(\xi + \eta) \\ &= \int d\zeta \alpha_\zeta(\hat{\Psi})^* W(\xi) \alpha_\zeta(\hat{\Psi}) e^{-i\zeta \cdot \eta} \\ &= \int d\zeta \alpha_{-\zeta}(\hat{\Psi})^* W(\xi) \alpha_{-\zeta}(\hat{\Psi}) W_0(\eta)(\zeta). \end{aligned} \quad (5.263)$$

This coincides with Eq. (5.243) and Eq. (5.246) with $g = W_0(\eta)$, $A = W(\xi)$ and $KD = \hat{\Psi}$.

The form of the classical marginal is obvious from Eq. (5.256) by putting $A = \mathbb{1}$ (resp. $\xi = 0$ in Eq. (5.263)). For the quantum marginal, putting $g = 1$ leads to a form from which it is not even clear that it is just convolution with noise. For that, it is better to go back to the characteristic functions. Indeed, the function m in Eq. (5.258) is just the inverse Fourier transform of $f(\xi \oplus 0)$, i.e.,

$$\begin{aligned}
m(\eta) &= (2\pi)^{-2n} \int d\xi e^{i\eta \cdot \xi} f(\xi \oplus 0) = (2\pi)^{-2n} \int d\xi e^{i\eta \cdot \xi} \operatorname{tr}(\hat{\Psi}^* W(\xi) \hat{\Psi} W(\xi)^*) \\
&= (2\pi)^{-2n} \int d\xi e^{i\eta \cdot \xi} \operatorname{tr}(\hat{\Psi}^* \alpha_{\sigma\xi}(\hat{\Psi})) \\
&= (2\pi)^{-2n} \int d\xi \operatorname{tr}(\hat{\Psi}^* \alpha_{\sigma\xi}(\hat{\Psi} W(-\sigma\eta)) W(\sigma\eta)) \\
&= \operatorname{tr}(\hat{\Psi} W(-\sigma\eta)) \operatorname{tr}(\hat{\Psi}^* W(\sigma\eta)) = |(\mathcal{F}\hat{\Psi})(-\sigma\eta)|^2.
\end{aligned} \tag{5.264}$$

In the second line, we used the eigenvalue equation in Eq. (5.177) to absorb the exponential factor and Eq. (5.120) in the last line to evaluate the integral. \square

Chapter 6

Conclusion and Outlook

Summary

Let us start this part by answering a fundamental question: How difficult is the mathematical description of a quantum-classical hybrid?

After our discussion in Chap. 3, the answer is easy: It is difficult. This answer was to be expected. Quantum and classical systems behave differently, so a priori, there is no reason a combination should be straightforward.

The common ansatz of just choosing a commutative algebra for our classical system and doing a tensor product with the quantum system shows its limitations quite quickly. Unless we choose the special cases of discrete and finite-dimensional systems, this construction either has reasonable states or observables, but not at the same time. Accordingly, a description of channels with both Schrödinger and Heisenberg picture proves to be difficult in this setting.

In a broader sense, this explains why there are plenty of works about quantum-classical hybrids (Sect. 3.4) but no commonly accepted standard framework: It is possible to avoid many of the difficulties when working on hybrid systems if focused on a specific application or only on one part of a system, say hybrid states. The problems arise if we want a hybrid setting that works for all parts of an experiment, i.e., states, dynamics, and observables at the same time.

Before we presented a framework that resolves the issues from the straightforward tensor ansatz, we reassured ourselves that a hybrid treatment has the potential for real positive gains, i.e., we started answering one of our main questions: What benefits does a unified hybrid treatment offer? For this, we took a simple hybrid algebra and generalized a classical result regarding the generators of diffusions to the hybrid setting (Chap. 4). Precisely as in the classical theory, this leads to generators, which are second-order differential operators in the classical variables, having a Lindblad-type quantum part and coupling terms that describe the information flow between the systems. Here, the unified hybrid treatment shows its strength. In Thm. 76, a simple positive condition to the coefficients in the generator bounds the interaction, and thereby the information flow, between the systems. This part is crucial for describing measurements this way. The theory sets mathematical bounds on the perturbation of the quantum system associated with any information gained by the classical subsystem.

Although it is only defined on a simple hybrid algebra and there are already other works regarding this topic, it perfectly indicates possible results from a genuine hybrid approach.

Proceeding to more complex scenarios, we have described a framework for canonical hybrid systems that includes both states and observables. In this framework, quasifree channels can be discussed with remarkable ease and full generality. The key idea is straightforward and not new: Instead of requiring a symplectic form σ in the canonical commutation relations, one allows σ to be degenerate and house the classical part of the system in its null space. This way, the algebra and notation stay formally the same. The more complex part begins when we start to describe its representation and the rest of the framework.

Defining a standard representation on a von Neumann algebra and its normal states by a hybrid version of the Bochner theorem works out. The downside is that this von Neumann algebra mostly inherits the problems described in Sect. 3.3, i.e., the additional requirements of choosing a measure and taking care of the associated consequences for the state space and channels. We resolve these by introducing a measure- or μ -free approach on top of it: The μ -free approach, whose distinction from a μ -dependent one is sketched at the beginning of Sect. 5.3, enables us to include pure states from the outset, and taken together with Sect. 5.4, extremal channels. It is based on the C^* -algebra $C^*(\Xi, \sigma)$, which has our desired state space in its dual $C^*(\Xi, \sigma)^*$. While this algebra itself is too small for an observables algebra, its bidual $C^*(\Xi, \sigma)^{**}$ is way too large, so our last step in describing the basic building blocks is finding a suitable intermediate algebra $\mathcal{M} \subset C^*(\Xi, \sigma)^{**}$ that can fulfill this role. Here, we had to go to considerable functional analytic lengths, but the result is simple and easy to apply: With the multiplier algebra or the universally measurable functions, we get hybrid observable algebras that can be used systematically with an automatic Heisenberg picture description for the entire class of hybrid quasifree channels. The calculus for quasifree channels perfectly fits within the well-established framework for purely quantum ones (Sect. 5.4), including a state-channel correspondence (Sect. 5.4.2) and factorization theorem (Sect. 5.4.6). Furthermore, while the restriction to Gaussian systems is a widely chosen simplification in physics, and we have seen that this subclass does fit well into our framework (Sect. 5.2.4), working on the larger quasifree class is often easier and beneficial.

Regarding the possible applications of hybrid systems, we have described a toolbox of basic physical operations on the more practical level in Sect. 5.5. Applying these in all kinds of different scenarios is now the obvious next step, which perfectly leads over to the next section.

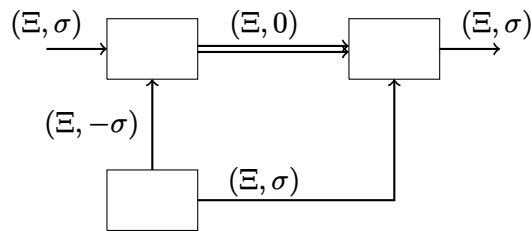
Outlook

One direction for future work is straightforward: Applying and extending the hybrid toolbox. This can indeed deliver several positive aspects: The reformulation of already-known results generally works as a benchmark for our hybrid formulation. Furthermore, a plain reformulation of standing problems can be a good problem-solving strategy, having the possibility to reveal new aspects.

In general, a unified hybrid description automatically contains all the interfacing operations between a quantum system and the classical laboratory, such as prepara-

tion, destructive and repeated measurements, feedback control, quantum information protocols, and dissipative time evolutions describing continuous measurements. While all these aspects are well and broadly studied, these works often pass over or do not fully utilize the underlying hybrid nature.

Quantum information protocols, which, like our initial example of teleportation in Sect. 1.1, are typically true hybrid systems. While they are commonly formulated in a finite and discrete setting, i.e., use bits and qubits, they often have more general formulations, including formulations for continuous variable systems and also the quasifree case [3, 173, 174, 175].



Teleportation

Figure 6.1: The protocol for teleportation. A double arrow indicates classical information. All operations in the top row are noiseless, and the noise arises from the entangled resource state. This can be chosen to be zero in the finite cases.

So, an important step towards all these different scenarios is clearly to go beyond the quasifree case. While, in several ways, our broader quasifree approach to hybrids is simpler than the more specialized versions like the Gaussian case, it is highly likely that the next step in generalizing towards general hybrids will be harder. We heavily used the quasifree nature, so what structure remains if we drop this assumption? What happens if we exchange the canonical commutation relations with the canonical anticommutation relations, i.e., switch from the CCR- to the CAR-algebra? This scenario is clearly more complicated because the classical variables differ even more from the quantum ones. Whether it be more applied or theoretical areas, there are plenty of possibilities for future work.

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