Unbounded Generators of Dynamical Semigroups

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Abstract

The time evolution of a closed quantum system was described quite early in the history of quantum physics. These dynamics are reversible, and the time evolution is implemented by a continuous unitary group, which is in turn generated by a self-adjoint Hamiltonian operator. So, we have a complete mathematical characterization of all such evolutions. For open quantum systems the time evolution is given by dynamical semigroups. In the case of uniform continuity the generator of the dynamical semigroup is a bounded operator in the famous GKLS-form that has been found by V. Gorini, A. Kossakowski, G. Sudarshan and, independently, G. Lindblad. But the problem of characterizing also the merely strongly continuous dynamical semigroups or, equivalently, their unbounded generators, is open.

In the first part of this thesis we introduce a standard form for the generator of quantum dynamical semigroups that is an unbounded version of the GKLS-form. The basis of the standard form are so-called no-event semigroups, describing an evolution of a quantum system, that maps pure states to multiples of pure states, and completely positive perturbations of their generator that correspond to jumps in this evolution, like absorption by a measurement device. We will give examples of standard semigroups, which appear to be probability preserving to first order (i.e., when looking only at the generator on the finite-rank part of its domain) but not for finite times. Additionally, we construct examples of generators *not* of standard form by modifying the previous examples.

In the second part we relate the notion of standardness to W. Arveson's classification of endomorphism semigroups. He divided them into three classes, Type I, Type II and Type III. We show that a conservative dynamical semigroup is standard if and only if the minimal dilation of its adjoint is of Type I. The key feature is the set of ket-bras in the domain of the no-event generator and whether it is a core for the standard generator. With this knowledge, we suggest to extend this classification to (not necessarily conservative) semigroups that are standard or can be constructed as a series of completely positive perturbations of a no-event semigroup. By construction these are either of Type I or Type II.

Keywords: Open system dynamics, quantum dynamical semigroups, unbounded standard generators

Zusammenfassung

Die Zeitentwicklung eines geschlossenen Quantensystems wurde recht früh in der Geschichte der Quantenphysik beschrieben. Ihre Dynamik ist reversibel und die Zeitentwicklung wird implementiert durch eine stetige unitäre Gruppe, die wiederum von einem Hamilton-Operator erzeugt wird. Somit haben wir eine vollständige mathematische Charakterisierung solcher Entwicklungen. Für offene Quantensysteme wird die Zeitentwicklung durch dynamische Halbgruppen beschrieben. Ist die Halbgruppe gleichmäßig stetig, so ist ihr Erzeuger ein beschränkter Operator in der bekannten GKLS-Form, die von V. Gorini, A. Kossakowski, G. Sudarshan und, unabhängig davon, G. Lindblad entwickelt wurde. Aber die Charakterisierung von nur stark-stetigen quantendynamischen Halbgruppen ist ein noch offenes Problem.

Im ersten Teil dieser Arbeit führen wir eine Standardform für die Erzeuger quantendynamischer Halbgruppen ein, die eine unbeschränkte Version der bekannten GKLS-Form ist. Grundlage dieser Standardform sind sogenannte No-Event Halbgruppen, die eine Evolution des Quantensystem beschreiben, bei der reine Zustände auf Vielfache von reinen Zuständen abgebildet werden, und vollständig positive Störungen ihres Erzeugers, die Sprünge in dieser Entwicklung beschreiben, wie Absorption durch Messapparate. Wir geben Beispiele von Standardhalbgruppen, die in erster Ordnung (also bei Betrachtung des Erzeugers auf dem zum endlichen Rang gehörenden Definitionsbereich) wahrscheinlichkeitserhaltend sind, aber nicht bei der Betrachtung endlicher Zeiten. Zusätzlich konstruieren wir Beispiele von Erzeugern, die *nicht* von Standardform sind, indem wir die vorherigen Beispiele modifizieren.

Im zweiten Teil setzen wir die Standardform-Eigenschaft mit der von W. Arveson gefundenen Klassifikation von Endomorphismus-Halbgruppen in Verbindung. Er unterteilte sie in drei Klassen, Typ I, Typ II und Typ III. Wir zeigen, dass eine konservative dynamische Halbgruppe genau dann standard ist, wenn die minimale Dilatation ihrer Adjungierten Typ I ist. Wichtigster Bestandteil des Beweises ist die Menge an Ketbras im Definitionsbereich des No-Event-Erzeugers und ob diese einen definierenden Bereich für den Standarderzeuger bilden. Mit dieser Kenntnis schlagen wir eine Ausweitung der Klassifikation auf (nicht notwendigerweise konservative) Halbgruppen vor, die entweder standard sind oder durch eine Reihe an vollständig positiven Störungen aus einer No-Event-Halbgruppe konstruiert werden können. Durch diese Konstruktion sind sie immer vom Typ I oder vom Typ II.

Schlagworte: Dynamik offener Systeme, quantendynamische Halbgruppen, unbeschränkte Standard-Erzeuger

Contents

Introduction

Dynamical semigroups are the key structure for describing open system dynamics, yet our structural understanding of them is curiously limited. On the other hand, the time evolution of a closed quantum system was described quite early in the history of quantum physics. Erwin Schrödinger postulated his famous equation in 1925. It gives the time evolution of a closed system in a pure quantum state. Its generalization to mixed states is known as the von Neumann equation. These dynamics are reversible, and the time evolution is implemented by a continuous unitary group, which is in turn generated by a self-adjoint Hamiltonian operator. So, we have a complete mathematical characterization of all such evolutions. In fact, the spectral theorem for unbounded self-adjoint operators was one of the first elements of the mathematical structure of quantum mechanics that von Neumann developed, and he did it for just this purpose. The analogous open systems problem then would be the following:

Problem. *Consider a Hilbert space $\mathcal{H}$. Characterize all one-parameter semigroups $t \mapsto \mathcal{T}_t$, $(t \geq 0)$ such that each $\mathcal{T}_t$ is a completely positive map on the trace class $\mathfrak{T}(\mathcal{H})$, and, for any $\rho \in \mathfrak{T}(\mathcal{H})$ and any bounded operator $A \in \mathcal{B}(\mathcal{H})$, we have $\lim_{t\to 0} \operatorname{tr}\big(\mathcal{T}_t(\rho)A\big) = \operatorname{tr}(\rho A)$.*

The solution to this problem is well known in the case of bounded generators, which is equivalent to the uniform continuity condition $\lim_{t\to 0} \|\mathcal{T}_t - \mathcal{I}\| = 0$. It was found by V. Gorini, A. Kossakowski and G. Sudarshan [GKS76], and independently by G. Lindblad [Lin76] in 1976, and more information about this special case is given in Section 3.5. But the problem as written above, i.e., of characterizing also the merely strongly continuous dynamical semigroups or, equivalently, their unbounded generators, is open.

In spite of this, many applications use unbounded versions of the GKLS-form of the generator, which we will call the *standard form* in the sequel. The basic idea for the standard form goes back to E. B. Davies [Dav77], and has been somewhat further developed since [Fag99; Hol96a; Che91]. The typical attitude towards this problem is currently to use unbounded standard forms where it seems natural but to avoid the general unbounded case. Indicative of this state of affairs is that the papers by A. Holevo [Hol95; Hol96c] from 1995/96, which present an example of a non-standard generator, have practically not been cited. Likewise underrated in the physics community is the work of W. Arveson [Arv03], which also goes well beyond the standard form [Arv02a]. He gave a complete classification for semigroups of endomorphisms, which can be extended to dynamical semigroups via his dilation theory [Arv89a; Arv90a; Arv89b; Arv90b].

In this thesis, we pursue two purposes. We give a generalization of the standard form in the unbounded case and integrate this form into Arveson's classification. Chapter 1 and Chapter 2 give a rather elaborated introduction to the mathematical basics. The reason for this level of detail is twofold. Firstly, we want to present the reader with a self-contained work, giving all the information necessary for a graduated physicist - regardless of his line of research - to comprehend the main results. Secondly, this settles the vocabulary and notation, as we often experienced differences and variations, especially between physicists and mathematicians. One of the main difficulties in this thesis was bringing together the views and languages of different research communities. We hope to have found a compromise in expressing the results that is easy to read, gives a clear understanding of the different structures, but also enables the reader to recognize the concepts from the underlying literature.

In Chapter 3 we review quantum mechanics, starting from quantum states and measurements according to what are sometimes called the Hanover rules, up to quantum dynamics and the GKLS-standard form in the case of uniform continuity. We will then introduce a description of the standard form in the unbounded case (Chapter 4), emphasizing intuition and collecting and even proving the basic results around it. We also give a cautionary example showing that the standard form must not be read too naively. Further examples are given of semigroups [Dav77; Hol96c], which appear to be probability preserving to first order (i.e., when looking only at the generator on the finite-rank part of its domain) but not for finite times. This phenomenon is akin to classical processes allowing escape to infinity in finite time. We will also give examples of generators *not* of standard form by modifying the previous examples. Most but not all results of this chapter have already been published in [SHW17].

Chapter 5 collects all necessary information on endomorphism semigroups and Arveson's dilation thereof. He divided them into three classes, Type I, Type II and Type III, where the first two types are also called *spatial semigroups*. We will closely examine the role of pure states in the domain of generators as described by R. T. Powers [Pow91]. Even at first glance, the structure of spatial semigroups closely resembles the construction of the standard form in the unbounded case. We will explore these similarities and prove their relations in the final chapter (Chapter 6) before giving our conclusions and ideas for further research.

At the end of each chapter, there is a section with notes and remarks where we assemble the literature used in this chapter or give further information that is not necessarily needed in the course of this thesis.

Chapter 1

Operator algebras

In this chapter, we will briefly introduce the basics of operator algebras. This will give the relevant mathematical background needed in the following chapters and set the used notation. The obligatory mathematical starting point for Quantum mechanics is the Hilbert space, used to describe a quantum system. The introduction follows operators on Hilbert spaces, positive maps and their dilations, and the Fock space. Proofs are mainly omitted in the chapters of mathematical preliminaries. However, the sources with proofs are given for each result and the reader can find a detailed description of the corresponding literature in the last section "Notes and Remarks".

1.1 Hilbert spaces

Let us consider a vector space $\mathcal{H}$ over the complex numbers $\mathbb{C}$. A *scalar* or *inner product* on $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that the following conditions hold:

1. conjugate symmetry: $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$
2. linearity in second variable: $\langle \phi, \lambda(\psi + \eta) \rangle = \lambda \langle \phi, \psi \rangle + \lambda \langle \phi, \eta \rangle$
3. positive definiteness: $\langle \phi, \phi \rangle = 0$ if and only if $\phi = 0$

for all vectors ϕ, ψ and $\eta \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. A complex vector space with an inner product is called *inner* or *scalar product space*.

Definition 1.1.1. *Two inner product spaces $\mathcal{H}$ and $\mathcal{H}'$ are* isomorphic *if there is a bijective linear mapping $U : \mathcal{H} \to \mathcal{H}'$ such that the inner products of $\mathcal{H}$ and $\mathcal{H}'$ satisfy*

$$\langle U\phi, U\psi \rangle_{\mathcal{H}'} = \langle \phi, \psi \rangle_{\mathcal{H}} \tag{1.1}$$

for all $\phi, \psi \in \mathcal{H}$. The mapping U is an isomorphism.

Let ϕ and ψ be vectors in the inner product space $\mathcal{H}$. Then inner products satisfy the *Cauchy-Schwarz inequality*

$$\|\langle \phi, \psi \rangle\|^2 \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle. \tag{1.2}$$

ϕ, ψ are called *orthogonal* if $\langle \phi, \psi \rangle = 0$ and we write $\psi \perp \phi$.

A norm on a vector space $\mathcal{H}$ is a map $\| \cdot \| : \mathcal{H} \to \mathbb{R}$ that satisfies

1. subadditivity: $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$
2. absolute homogeneity: $\|\lambda\psi\| = |\lambda| \, \|\psi\|$
3. positive definiteness: $\|\phi\| = 0$ if and only if $\phi = 0$

for all $\phi, \psi \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. If the first two conditions are satisfied, we call the map a *seminorm*. A seminorm is non-negative, i.e. $\|\phi\| \geq 0$ for all $\phi \in \mathcal{H}$, but not necessarily positive definite.

For each inner product on $\mathcal{H}$, we can define a norm on $\mathcal{H}$ by $\|\phi\|^2 = \langle \phi, \phi \rangle$ for all $\phi \in \mathcal{H}$. A normed space can be endowed with a metric and, therefore, with a topology so that we can talk about the convergence of sequences and continuity. In a metric space, every convergent sequence must be a Cauchy sequence. A metric space is called *complete* if the converse is also true. Normed vector spaces that are complete with respect to this norm are called *Banach spaces*.

Definition 1.1.2. *An inner product space over $\mathbb{C}$ that is also a Banach space, i.e. complete with respect to the norm induced by the inner product, is called a* Hilbert space*.*

A maximal orthonormal set on $\mathcal{H}$ is a set of orthogonal vectors where each vector has unit norm, and that is not contained in another orthonormal set as a proper subset. Such a maximal orthonormal set is called an *orthonormal basis* for $\mathcal{H}$. If for any positive integer d there exists an orthonormal set of d vectors, then $\mathcal{H}$ is *infinite dimensional*. Otherwise $\mathcal{H}$ is *finite dimensional* and its dimension d is the cardinality of a maximal orthonormal set on $\mathcal{H}$.

Proposition 1.1.3 ([Naa17, Prop. 2.1.8])**.** *Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if and only if they have the same dimension.*

As a topological space, a Hilbert space is called *separable* if it contains a countable dense subset. This is the case if it has a countable orthonormal basis. In this thesis, all Hilbert spaces are assumed to be separable.

We will next consider linear maps between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. These are functions $A : \mathcal{H}_1 \to \mathcal{H}_2$ such that $A(\lambda\phi) = \lambda A(\phi)$ and $A(\phi + \psi) = A(\phi) + A(\psi)$. We will usually omit the brackets.

Proposition 1.1.4 ([Naa17, Prop. 2.1.4])**.** *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear map between two Hilbert spaces. Then the following are equivalent:*

1. *A is continuous with respect to the norm topology.*
2. *A is bounded, i.e. there exists a constant $M > 0$ such that $\|A\phi\| \leq M\|\phi\|$ for all $\phi \in \mathcal{H}_1$.*

The concept of boundedness is only relevant in the infinite-dimensional setting. If the dimension of $\mathcal{H}$ is finite, then all linear maps are bounded and, hence, continuous. However, the following statements for bounded maps are true for either finite or infinite dimensional Hilbert spaces.

Definition 1.1.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. We write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of bounded linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ or simply $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ for bounded linear maps* *on $\mathcal{H}$. An element of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ will be called a* (bounded) operator *and maps in $\mathcal{B}(\mathcal{H})$ are simply called* operators on $\mathcal{H}$.

If A is a bounded linear map $A : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces, the *adjoint* or A^* of A is the unique linear map $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\langle \phi, A^* \psi \rangle_{\mathcal{H}_1} = \langle A\phi, \psi \rangle_{\mathcal{H}_2} \tag{1.3}$$

for all $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$. In physics, the adjoint is often called *hermitian conjugate* and written $A^\dagger$. We will instead use A^* as is common in mathematics and mathematical physics.

It is possible to combine two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ into a new, composed Hilbert space. One way to do this is taking the *direct sum* $\mathcal{H}_1 \oplus \mathcal{H}_2$, that consists of all tuples (ϕ_1, ϕ_2) with $\phi_1 \in \mathcal{H}_1$ and $\phi_2 \in \mathcal{H}_2$. This is a Hilbert space if we define the following rules for addition and inner product:

$$\lambda(\phi_1, \phi_2) + \mu(\psi_1, \psi_2) = (\lambda\phi_1 + \mu\psi_1, \lambda\phi_2 + \mu\psi_2) \tag{1.4}$$
$$\langle (\phi_1, \phi_2), (\psi_1, \psi_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle \phi_1, \psi_1 \rangle_{\mathcal{H}_1} + \langle \phi_2, \psi_2 \rangle_{\mathcal{H}_2} \tag{1.5}$$

where λ and μ are scalars and $\phi_i, \psi_i \in \mathcal{H}_i$. If $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces and $L_i : \mathcal{H}_i \to \mathcal{K}_i$ linear maps, we can define a linear map $L_1 \oplus L_2 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2$ by

$$(L_1 \oplus L_2)(\phi_1, \phi_2) = (L_1\phi_1, L_2\phi_2). \tag{1.6}$$

For the direct sum of n Hilbert spaces, we will write $\bigoplus_{i=1}^n \mathcal{H}_i$. If L_1, L_2 are bounded, then so is $L_1 \oplus L_2$ and we have $(L_1 \oplus L_2)^* = L_1^* \oplus L_2^*$.

Another possibility is combining two Hilbert spaces to an inner product space. Consider the vector space V consisting of formal (finite) linear combinations of elements of the form $\phi \otimes \psi$ for $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$, so that a vector in V is of the form $\sum_{i=1}^n \phi_i \otimes \psi_i$ with $n \in \mathbb{N}$. We then impose the following identifications:

$$\lambda(\phi \otimes \psi) = (\lambda\phi) \otimes \psi = \phi \otimes (\lambda\psi) \tag{1.7}$$
$$(\phi_1 + \phi_2) \otimes \psi = \phi_1 \otimes \psi + \phi_2 \otimes \psi \tag{1.8}$$
$$\phi \otimes (\psi_1 + \psi_2) = \phi \otimes \psi_1 + \phi \otimes \psi_2 \tag{1.9}$$

for $\lambda \in \mathbb{C}$, $\phi, \phi_i \in \mathcal{H}_1$ and $\psi, \psi_i \in \mathcal{H}_2$. We obtain a vector space H whose elements will be written as $\phi \otimes \psi$ again. On this space, we can define an inner product by setting

$$\langle \phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2 \rangle_H = \langle \phi_1, \psi_1 \rangle_{\mathcal{H}_1} \langle \phi_2, \psi_2 \rangle_{\mathcal{H}_2} \tag{1.10}$$

and extending by linearity. In general, H is not a Hilbert space since it is not complete with respect to the norm induced by this inner product, but by the following construction we can complete H to obtain a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which is called the *tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Let $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ be two Cauchy sequences in $\mathcal{H}'$. We will call them equivalent if for every $\epsilon > 0$ there exists a number $N > 0$ such that $\|\xi_n - \eta_n\| < \epsilon$ for all $n > N$. That means that two Cauchy sequences belong to the same equivalence class if their difference is equivalent to the sequence $(0)_{n=1}^\infty$ of all zeros. The sum of two Cauchy sequences $(\xi_n + \eta_n)_{n=1}^{\infty}$ as well as the multiplication of a Cauchy sequence $(\lambda\xi_n)_{n=1}^{\infty}$ with a scalar $\lambda \in \mathbb{C}$ are again Cauchy sequences in $\mathcal{H}'$. Together with these two operations, the set of Cauchy sequences in $\mathcal{H}'$ forms a vector space over $\mathbb{C}$ that we denote as V. The set of Cauchy sequences equivalent to $(0)_{n=1}^{\infty}$ is also a vector space, which we denote V_0. Then the space $\mathcal{H} = V/V_0$ is a vector space over $\mathbb{C}$ and can be formed into an inner product space with the inner product

$$\langle(\xi_n)_{n=1}^{\infty}, (\eta_n)_{n=1}^{\infty}\rangle_{\mathcal{H}} = \lim_{n\to\infty}\langle\xi_n, \eta_n\rangle_{\mathcal{H}'}. \tag{1.11}$$

$\mathcal{H}'$ can be embedded into $\mathcal{H}$ via $\iota(\phi) = [(\phi_n)_{n=1}^{\infty}]$, the equivalence class of the constant sequence with values ϕ.

Theorem 1.1.6 ([Naa17, Thm. 2.1.10])**.** *Let $\mathcal{H}'$ be a (not necessarily complete) inner product space. Then there is a Hilbert space $\mathcal{H}$ and a linear embedding $\iota : \mathcal{H}' \to \mathcal{H}$ such that $\iota(\mathcal{H}')$ is dense in $\mathcal{H}$ and $\langle\phi, \psi\rangle_{\mathcal{H}'} = \langle\iota(\phi), \iota(\psi)\rangle_{\mathcal{H}}$.*

If $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces and $L_i : \mathcal{H}_i \to \mathcal{K}_i$ are bounded linear maps, we can define a map $L_1 \oplus L_2 : H \to \mathcal{K}_1 \otimes \mathcal{K}_2$ by

$$(L_1 \otimes L_2)(\phi \otimes \psi) = (L_1\phi \otimes L_2\psi). \tag{1.12}$$

We write $\mathcal{H}^{\otimes n}$ for the n-fold tensor product of $\mathcal{H}$, and for the Hilbert space completion of the tensor product of n Hilbert spaces, we will write $\bigotimes_{i=1}^{n} \mathcal{H}_i$. This notation will also be applied to vectors in these Hilbert spaces. If L_1, L_2 are bounded, then $L_1 \otimes L_2$ is bounded on H. With 1.1.4 and extension by continuity (see PN 2.1.4), we get that $L_1 \otimes L_2$ is well defined on all of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (and therefore bounded). For adjoint tensor products we have $(L_1 \otimes L_2)^* = L_1^* \otimes L_2^*$.

1.2 Operators on a Hilbert space

Let $\mathcal{A}$ be a Banach space over $\mathbb{C}$. If $\mathcal{A}$ is an algebra over $\mathbb{C}$, i.e. if it is equipped with a bilinear multiplication, and if the multiplication satisfies the inequality $\|AB\| \leq \|A\|\|B\|$ for $A, B \in \mathcal{A}$, then $\mathcal{A}$ is called a Banach algebra. We say $\mathcal{A}$ is a Banach-$*$-algebra when there is a anti-linear *involution* $* : \mathcal{A} \mapsto \mathcal{A}$ that satisfies the following properties for all $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(\lambda A)^* = \bar{\lambda}A^*$
4. $(AB)^* = B^*A^*$
5. $\|A^*\| = \|A\|$.

Definition 1.2.1. *A $*$-algebra $\mathcal{A}$ is called a C^*-*algebra *if the involution satisfies $\|A^*A\| = \|A\|^2$. If $\mathcal{A}$ has an identity $\mathbb{1}$, it is called* unital.

In this thesis, we will assume all C^*-algebras to be unital. We will call a C^*-algebra generated by a set of operators if it is the smallest C^*-algebra that contains all these operators.

Definition 1.2.2. *Let $\mathcal{A}$ be an algebra, $\mathcal{B}$ a subspace of $\mathcal{A}$ and assume $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The subspace $\mathcal{B}$ is called a* left ideal *if $AB \in \mathcal{B}$, it is called a* right ideal *if $BA \in \mathcal{B}$. If it is both a left and a right ideal, it is called a* two-sided ideal*.*

We look again at the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. For a bounded operator A on $\mathcal{H}$ we will use the following notation:

- kernel $ker(A) := \{\phi \in \mathcal{H} | A\phi = 0\}$
- range $ran(A) := \{\phi \in \mathcal{H} | \phi = A\psi \text{ for some } \psi \in \mathcal{H}\}$
- support $supp(A) := \{\phi \in \mathcal{H} | \phi \perp \psi \text{ for all } \psi \in ker(A)\}$

On $\mathcal{B}(\mathcal{H})$ we can define a norm by

$$\|A\| := \sup_{\phi \in \mathcal{H}, \phi \neq 0} \frac{\|A\phi\|}{\|\phi\|} = \sup_{\phi \in \mathcal{H}, \|\phi\|=1} \|A\phi\|. \tag{1.13}$$

Theorem 1.2.3 ([Naa17, Sect. 2.2.1])**.** *$\mathcal{B}(\mathcal{H})$ is complete with respect to* (1.13)*, and together with the composition of operators as multiplication and the adjoint as involution, $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.*

An operator $A \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $A = A^*$, and *normal* if $AA^* = A^*A$. If $A^*A = \mathbb{1}$, then it is called an *isometry*. We say A is a *contraction* if $\|A\| \leq 1$. A self-adjoint operator A is *positive* if $\langle \phi, A\phi \rangle \geq 0$ for all $\phi \in \mathcal{H}$. This is the case if and only if it can be written as $A = B^*B$ for some operator $B \in \mathcal{B}(\mathcal{H})$. For operators $A, B \in \mathcal{B}(\mathcal{H})$ we write $A \geq B$ if the operator $A - B$ is positive. Each operator $A \in \mathcal{B}(\mathcal{H})$ can be written as a linear combination of at most four positive operators. We have $A = H + iK$, with both H and K self-adjoint operators in $\mathcal{B}(\mathcal{H})$, and then write both self-adjoint operators as the difference of two positive ones, $H = H_+ - H_-$ and $K = K_+ - K_-$.

A self-adjoint operator $P \in \mathcal{B}(\mathcal{H})$ that satisfies $P^2 = P^* = P$ is called a *projection*. The image $ran(P)$ is a closed subspace of $\mathcal{H}$. For example, if V is an isometry, then VV^* is a projection.

Proposition 1.2.4 ([HZ12, Prop. 1.42])**.** *Let Q, P be projections. The following are equivalent:*

1. $P \geq Q$
2. $PQ = Q$
3. $QP = Q$
4. $QP = PQ = Q$
5. $P - Q$ *is a projection*

A bounded operator $U \in \mathcal{B}(\mathcal{H})$ is *unitary* if it satisfies $UU^* = U^*U = \mathbb{1}$. For $U \in \mathcal{B}(\mathcal{H})$ the following are equivalent:

1. U is an isomorphism
2. U is a surjective isometry

3. U is unitary.

We will regard two important subspaces of $\mathcal{B}(\mathcal{H})$ in this thesis: The class of compact operators on a Hilbert space and the class of trace class operators on a Hilbert space.

Definition 1.2.5. *An operator $A \in \mathcal{B}(\mathcal{H})$ is called* compact *if A takes bounded sets into relatively compact sets. We denote the set of compact operators by $\mathfrak{K}(\mathcal{H})$.*

The following theorem gives another possible and more intuitive way to characterize compact operators.

Theorem 1.2.6 ([RS80, Thm. 6.13])**.** *Let $\mathcal{H}$ be a separable Hilbert space, then every compact operator on $\mathcal{H}$ is the norm limit of a sequence of operators of finite rank, i.e. operators whose range is finite-dimensional.*

The set of compact operators $\mathfrak{K}(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$, and it is a C^*-algebra in itself.

In finite dimensions, an important concept is the trace of a matrix. It is the sum of its diagonal elements. It is possible to extend the definition of the trace to a subset of $\mathcal{B}(\mathcal{H})$. Let $\{\phi_i\}$ be an orthonormal basis for $\mathcal{H}$. For a positive operator $A \in \mathcal{B}(\mathcal{H})$ we write $\mathrm{tr}(A) = \sum_{j=1}^{\infty}\langle\phi_j, A\phi_j\rangle$. The right-hand side is the sum of non-negative numbers, so it might not converge. This leads to the following definition.

Definition 1.2.7. *An operator $A \in \mathcal{B}(\mathcal{H})$ is called a* trace class operator *if* $\mathrm{tr}(|A|) < \infty$, *with $|A| = (A^*A)^{\frac{1}{2}}$. We denote by $\mathfrak{T}(\mathcal{H})$ the set of trace class operators.*

For $A \in \mathfrak{T}(\mathcal{H})$ we have $\sum_{j=1}^{\infty} |\langle\phi_j, A\phi_j\rangle| < \infty$. In that case, the number

$$\mathrm{tr}(A) := \sum_{j=1}^{\infty}\langle\phi_j, A\phi_j\rangle \tag{1.14}$$

is called the *trace of the operator A* and is independent of the choice of orthonormal basis. The map

$$A \mapsto \|A\|_{\mathrm{tr}} = \mathrm{tr}(|A|) \tag{1.15}$$

is a norm on the vector space of trace class operators, and $\mathfrak{T}(\mathcal{H})$ is also an ideal in $\mathcal{B}(\mathcal{H})$; it is, however, not closed.

1.2.1 Dual spaces and topologies on $\mathcal{B}(\mathcal{H})$

A *linear functional* is a linear mapping from a complex vector space into the complex numbers $\mathbb{C}$. We are mainly interested in continuous linear functionals. The set of all continuous linear functionals from a complex vector space V into $\mathbb{C}$ is called the *dual space* of V, and we will denote it by V^*. It can be made into a vector space by itself by setting $(\omega_1 + c\omega_2)(v) = \omega_1(v) + c\omega_2(v)$ with $\omega_i \in V^*$ and for all $v \in V$ and $c \in \mathbb{C}$. With the norm given by

$$\|\omega\| := \sup_{\|v\|=1} |\omega(v)| \tag{1.16}$$

V^* becomes a normed vector space. In this situation we call V the *predual space* of V^*.

On a Hilbert space $\mathcal{H}$ each vector $\phi \in \mathcal{H}$ defines a linear functional ω_ϕ on $\mathcal{H}$ by

$$\omega_\phi(\psi) = \langle \phi, \psi \rangle. \tag{1.17}$$

As ω_ϕ is continuous, it is an element of the dual space $\mathcal{H}^*$. In fact, every linear functional from $\mathcal{H}$ to $\mathbb{C}$ can be written in this form:

Theorem 1.2.8 (Fréchet-Riesz theorem, [HZ12, Thm. 1.67])**.** *Let $\omega \in \mathcal{H}^*$ be a linear functional $\omega : \mathcal{H} \to \mathbb{C}$. Then there exists a unique vector $\phi \in \mathcal{H}$ such that $\omega = \omega_\phi$ as in* (1.17)*. Moreover we have $\|\omega_\phi\| = \|\phi\|$.*

Taking this lemma into account, P. Dirac introduced a convenient notation. He suggested writing a vector $\psi \in \mathcal{H}$ as $|\psi\rangle$ and the linear functional $\omega_\phi \in \mathcal{H}^*$ as $\langle\phi|$. The inner product on a Hilbert space is written as $\langle\phi|\psi\rangle$ and called *bracket*. He decomposed it as applying the functional $\langle\phi|$ on a vector $|\psi\rangle$, and he called $\langle\phi|$ a *bra vector* and $|\phi\rangle$ a *ket vector*. To view the bra vectors as the adjoints of ket vectors, we make a slight difference between ψ and $|\psi\rangle$. Instead of identifying $|\psi\rangle$ with the vector $\psi \in \mathcal{H}$, one can introduce $|\psi\rangle$ as the linear map from $\mathbb{C} \to \mathcal{H}$ with $|\psi\rangle(c) = c\psi$. In practice, however, ψ and $|\psi\rangle$ are used equivalently.

With this notation, we can define a linear mapping on $\mathcal{H}$ by

$$|\psi\rangle\langle\phi|\, \eta = \langle \phi, \eta \rangle\, \psi \tag{1.18}$$

with $\phi, \psi \in \mathcal{H}$. The operator $|\phi\rangle\langle\psi|$ is bounded, and if ϕ, ψ are both nonzero, then the range of $|\phi\rangle\langle\psi|$ is the one-dimensional subspace $\mathbb{C}\phi$. $|\phi\rangle\langle\psi|$ is therefore called a *rank-1 operator*. The operator $P_\phi = |\phi\rangle\langle\phi|$ with $\|\phi\| = 1$ is a the one-dimensional projection onto the subspace $\mathbb{C}\phi$.

Every normal and, therefore, every selfadjoint trace class operator on $\mathcal{H}$ can be written as a linear combination of rank-1 operators.

Theorem 1.2.9 (Spectral decomposition, [HZ12, Thm. 1.65])**.** *Let A be a normal trace class operator. Then there exists a sequence $\{\lambda_i\}$ of complex numbers and an orthonormal basis $\{\phi_i\}$ of $\mathcal{H}$ such that*

$$A = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|. \tag{1.19}$$

This is called the spectral decomposition *of A.*

As $\mathfrak{T}(\mathcal{H})$ is a normed space with trace norm, we can also look at its dual space. For each bounded operator $A \in \mathcal{B}(\mathcal{H})$ in $B \in \mathfrak{T}(\mathcal{H})$, AB is again a trace class operator. Therefore we can define a linear functional ω_A on $\mathfrak{T}(\mathcal{H})$ by

$$\omega_A(B) = \mathrm{tr}(AB). \tag{1.20}$$

The following theorem ensures that every linear functional on $\mathfrak{T}(\mathcal{H})$ can be written in this form.

Theorem 1.2.10 ([HZ12, Thm. 1.68])**.** *The mapping $A \mapsto \omega_A$ given in* (1.20) *is a linear bijection from $\mathcal{B}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H})^*$ and $\|A\| = \|\omega_A\|$ for every $A \in \mathcal{B}(\mathcal{H})$. In other words, the dual space of $\mathfrak{T}(\mathcal{H})$ can be identified with $\mathcal{B}(\mathcal{H})$.*

We say a linear functional ω on $\mathfrak{T}(\mathcal{H})$ is *positive* if $\omega(A) \geq 0$ whenever $A \geq 0$.

In this thesis we encounter three different topologies on $\mathcal{B}(\mathcal{H})$, besides the norm topology defined by (1.13).

Definition 1.2.11. *We say a sequence A_i in $\mathcal{B}(\mathcal{H})$ converges*

- *in* (operator-) norm topology *if*

$$\lim_i \|A - A_i\| = 0 \tag{1.21}$$

- *in* weak operator topology *or* weakly *if for $\phi, \psi \in \mathcal{H}$*

$$\lim_i |\langle \psi, A\phi \rangle - \langle \psi, A_i\phi \rangle| = 0 \tag{1.22}$$

- *in* strong operator topology *or* strongly *if for $\phi \in \mathcal{H}$*

$$\lim_i \|A\phi - A_i\phi\| = 0 \tag{1.23}$$

- *in* weak* topology *or* ultra weakly *if for every $T \in \mathfrak{T}(\mathcal{H})$*

$$\lim_i |\operatorname{tr} AT - \operatorname{tr} A_iT| = 0 \tag{1.24}$$

The weak* operator topology and the strong operator topology are both stronger than the weak operator topology and weaker than the operator norm topology. When speaking of continuity, in particular, but not only in the case of one-parameter semi-groups, it is of utmost importance to specify to which continuity we refer, as they lead to completely different properties.

The introduction of weak*-continuity leads to a different way to describe the *predual* of $\mathcal{B}(\mathcal{H})$. It is the space of all weak*-continuous linear functionals on $\mathcal{B}(\mathcal{H})$ denoted by $\mathcal{B}_*(\mathcal{H})$ which we identify with $\mathfrak{T}(\mathcal{H})$.

1.2.2 The Gel′fand-Naimark-Segal construction

A *state* on a (unital) C^*-algebra $\mathcal{A}$ is a positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ of norm one, such that $\omega(\mathbb{1}) = 1$. We will later (in Chapter 3) see that this is an abstraction of the notion of a state in quantum mechanics. The set of all states, also called the *state space*, is a convex set, i.e. if $0 \leq \lambda \leq 1$ and ω_1, ω_2 are states, then

$$\omega(A) = \lambda\omega_1(A) + (1 - \lambda)\omega_2(A) \tag{1.25}$$

is a state again. ω is called a *pure* state if $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ implies that $\omega = \omega_1 = \omega_2$. If ω is not pure, it is called *mixed*.

In Theorem 1.2.3 we saw that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. In fact, given a state ω on an arbitrary C^*-algebra $\mathcal{A}$, we can construct a Hilbert space $\mathcal{H}_\omega$ and a $*$-*representation* π of $\mathcal{A}$ on this Hilbert space.

A $*$-representation is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ that is *non-degenerate*, i.e. the set $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$ and therefore (if $\mathcal{A}$ is unital) $\pi(\mathbb{1}) = \mathbb{1}$. Such a representation identifies elements of an abstract C^*-algebra with bounded operators on the Hilbert space and is automatically continuous with respect to the norm topology. It is called *cyclic* if there is some vector $\Omega \in \mathcal{H}$ such that $\pi(\mathcal{A})\Omega$ is a dense subset of $\mathcal{H}$. In this case, Ω is called a *cyclic vector*.

Theorem 1.2.12 (GNS construction, [Naa17, Thm. 2.5.3]). *Let $\mathcal{A}$ be a unital C^*-algebra and let ω be a state on $\mathcal{A}$. Then there is a $*$-representation π_ω of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ with a cyclic vector Ω such that*

$$\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle \tag{1.26}$$

for $A \in \mathcal{A}$. The triple $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ is unique in the sense that if $(\pi, \mathcal{H}, \Psi)$ is another such triple, there is a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ such that $U\Omega = \Psi$ and $\pi(A) = U\pi_\omega(A)U^*$.

The GNS construction is the basis of the following important result.

Theorem 1.2.13 (Gel'fand-Naimark, PN 2.5.11). *Every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of bounded operators on some Hibert space $\mathcal{H}$.*

1.2.3 Von Neumann algebras

Let $\mathcal{M}$ be a subset of $\mathcal{B}(\mathcal{H})$. Then we write $\mathcal{M}'$ for its commutant, i.e. the set of all bounded operators on $\mathcal{H}$ commuting with every operator in $\mathcal{M}$. $\mathcal{M}'$ is a Banach algebra of operators containing $\mathbb{1}$.

Definition 1.2.14. *A* von Neumann algebra *on $\mathcal{H}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ such that*

$$\mathcal{M} = \mathcal{M}''. \tag{1.27}$$

The *centre* $c(\mathcal{M})$ of a von Neumann algebra is the set of elements in $\mathcal{M}$ that commute with all elements in $\mathcal{M}$, so it is defined by $c(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$. We call a von Neumann algebra a *factor* if it has a trivial center, i.e. $c(\mathcal{M}) = \mathbb{C}\mathbb{1}$.

An important result for von Neumann algebras is the following.

Theorem 1.2.15 (Bicommutant theorem, [BR87, Thm. 2.4.11]). *A sub-$*$-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ satisfies $\mathcal{A}'' = \mathcal{A}$ if and only if it is closed in one and therefore all of the following topologies:*

1. *weak operator topology*
2. *strong operator topology*
3. *weak* topology*

If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a $*$-algebra that contains the identity, then $\mathcal{A}$ is dense in $\mathcal{A}''$ in the strong, weak, and weak* operator topologies. This is called the *von Neumann density theorem*. Any von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ has a predual. We can therefore find states ω on $\mathcal{M}$ that are weak* continuous; or equivalently, there exists a positive trace class operator T on $\mathcal{H}$ with $\mathrm{tr}(T) = 1$, that satisfies $\omega(A) = \mathrm{tr}(TA)$. Such a state ω is called *normal*.

A von Neumann algebra $\mathcal{M}$ is said to be of *Type I* if every nonzero projection in the centre of $\mathcal{M}$ contains an abelian projection e, i.e. a projection, such that the subalgebra $e\mathcal{M}e$ is abelian. There are two other types of von Neumann algebras, but in this thesis we only look at Type I factor von Neumann algebras. For more information, look for example at [Tak79, Ch. 5], [Sak98] or [KR97].

1.2.4 Unbounded operators on Banach spaces

In Definition 1.1.5, we introduced bounded operators. We will now look at the basic properties of unbounded operators.

Definition 1.2.16. *A* (*densely defined*) unbounded operator *A on a Banach space $\mathcal{A}$ is a linear map $A : \mathrm{dom}(A) \to \mathcal{A}$, where $\mathrm{dom}(A)$ is a dense linear subspace of $\mathcal{A}$.*

The *graph* $\mathcal{G}(A)$ of an operator A is the set of pairs

$$\{(\phi, A\phi)|\phi \in \mathrm{dom}(A)\}. \tag{1.28}$$

If $\mathcal{A}$ is a Hilbert space $\mathcal{H}$ than this set is a subset of $\mathcal{H} \times \mathcal{H}$ which is a Hilbert space with inner product

$$\langle(\phi_1, \psi_1), (\phi_2, \psi_2)\rangle = \langle\phi_1, \phi_2\rangle + \langle\psi_1, \psi_2\rangle. \tag{1.29}$$

The norm defined by this inner product is

$$\|\phi\|_A = \|\phi\| + \|A\phi\| \tag{1.30}$$

and called the *graph norm*.

If A and B on $\mathcal{A}$ are operators with $\mathcal{G}(A) \supset \mathcal{G}(B)$, then A is said to be an *extension* of B. An unbounded operator A is said to be *closed* if, whenever a sequence $\phi_n \in \mathrm{dom}(A)$ converges in norm to ϕ and the sequence $A\phi_n$ converges in norm to ψ, we have $\phi \in \mathrm{dom}(A)$ and $A\phi = \psi$. It is said to be *closable* if it has a closed extension, and its *closure* is the smallest closed extension.

We can define an *adjoint* A^* of an unbounded operator; however, we need to consider the domains in the definition. The domain $\mathrm{dom}(A^*)$ of A^* is the set of all $\psi \in \mathcal{A}^*$, such that the linear functional $\phi \mapsto \langle A\phi, \psi\rangle$ is norm bounded on the domain $\mathrm{dom}(A)$. If $\psi \in \mathrm{dom}(A^*)$ then we can define $A^*\psi$ by

$$\langle A\phi, \psi\rangle = \langle\phi, A^*\psi\rangle \tag{1.31}$$

for all $\phi \in \mathrm{dom}(A)$. A^* is densely defined if and only if A is closable. We call A symmetric if

$$\langle A\phi, \psi\rangle = \langle\phi, A\psi\rangle \tag{1.32}$$

for all $\phi, \psi \in \mathrm{dom}(A)$. We say A is self-adjoint if $A = A^*$ and A is called positive $(A \geq 0)$ if $\langle A\phi, \phi\rangle \geq 0$ for all $\phi \in \mathrm{dom}(A)$.

In Chapter 2, we will take a closer look at the properties of densely defined closable operators as generators of dynamical semigroups.

1.3 Positive maps and their Dilations

We will now look at maps between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ that map operators in $\mathcal{A}$ to operators in $\mathcal{B}$. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *positive*, if for every $A \geq 0 \in \mathcal{A}$ we have that $\Phi(A) \geq 0$. We will write $\Phi \geq 0$. Since it maps self-adjoint operators to self-adjoint operators, it preserves adjoints, i.e. we have

$$\Phi(A^*) = \Phi(A)^*. \tag{1.33}$$

Every positive linear map is bounded and, therefore, continuous. A $*$-homomorphism is a homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ that preserves adjoints, so in addition to $\Phi(AB) = \Phi(A)\Phi(B)$ equation (1.33) is satisfied. We will call a $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ from $\mathcal{A}$ to itself $*$-endomorphism.

Let M_n denote the set of $n \times n$-matrices with entries in $\mathbb{C}$ and $\mathbb{I}_n$ the $n \times n$ identity matrix.

Definition 1.3.1. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear map and $k \in \mathbb{N}$. We say Φ k-positive if $\Phi \otimes \mathbb{I}_k : \mathcal{A} \otimes M_k \to \mathcal{B} \otimes M_k$ is positive. Φ is called completely positive if it is k-positive for all $k \in \mathbb{N}$.*

We will take a look at a few conditions for complete positivity. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an adjoint preserving map between two C^*-algebras $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{K})$. If $\Phi(A) = V^*AV$ for all $A \in \mathcal{A}$ with bounded operator $V : \mathcal{H} \to \mathcal{K}$, then Φ is completely positive. Φ is also completely positive if it is a $*$-homomorphism. If Φ is k-positive, then the concatenation with another k-positive linear map is again k-positive.

Theorem 1.3.2 ([Stø13, Thm. 1.2.4 and Thm. 1.2.5])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras, and let either $\mathcal{A}$ or $\mathcal{B}$ be abelian. Then every positive map $\Phi : \mathcal{A} \to \mathcal{B}$ is completely positive. In particular, every state on $\mathcal{A}$ as a map from $\mathcal{A}$ to $\mathbb{C}$ is completely positive.*

The main result on completely positive maps is the Stinespring theorem. It can be seen as an extension of the GNS construction for states to completely positive maps. We will get a hint of its importance in Quantum mechanics in Section 3.4.

Theorem 1.3.3 (Stinespring theorem, [Stø13, Thm. 1.2.7])**.** *Let $\mathcal{A}$ be a unital C^*-algebra and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Φ is completely positive if and only if there exists a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that*

$$\Phi(A) = V^*\pi(A)V \tag{1.34}$$

for all $A \in \mathcal{A}$. Additionally we have $\|V\|^2 \leq \|\Phi(1)\|$.

The pair $(V, \mathcal{K})$ is called a Stinespring representation, but this representation is not unique. It is, however, possible to find a *minimal* Stinespring representation in the sense that it is unique up to unitary equivalence. The condition for minimality is given by

$$\mathcal{K} = [\pi(\mathcal{A})V\mathcal{H}] = \overline{\operatorname{span}}\{\pi(A)V\xi | A \in \mathcal{A}, \xi \in \mathcal{H}\}, \tag{1.35}$$

so $\mathcal{K}$ is the closure of the linear span of the elements in the given set.

It is possible to define an adjoint of a positive linear map. First, let us assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional Hilbert spaces. In that case, the spaces $\mathcal{B}(\mathcal{H}_1)$ and

$\mathcal{B}(\mathcal{H}_2)$ together with the Hilbert-Schmidt inner product $\langle A, B\rangle = \mathrm{tr}(AB^*)$ are Hilbert spaces by themselves, and we can view $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ as a bounded operator. The adjoint is then given by

$$\mathrm{tr}\,(\Phi(A)B) = \mathrm{tr}\,(A\Phi^*(B)), \quad A \in \mathcal{B}(\mathcal{H}_1), B \in \mathcal{B}(\mathcal{H}_2). \tag{1.36}$$

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional, we need to consider the definition more carefully. The map Φ must be assumed to be normal, i.e. defined by a positive trace class operator. Then there is an adjoint map Φ^* mapping operators in $\mathfrak{T}(\mathcal{H}_2)$ to $\mathfrak{T}(\mathcal{H}_1)$, that is defined by (1.36).

The adjoint of a positive map is also positive, and $\Phi(\mathbb{1}) = \mathbb{1}$ if and only if $\mathrm{tr}_{\mathcal{H}_1}(\Phi^*(B)) = \mathrm{tr}_{\mathcal{H}_2}(B)$ for all operators $B \in \mathcal{B}(\mathcal{H}_2)$.

1.4 Fock space, CAR and CCR algebra

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^{\otimes n}$ the n-fold tensor product where we set $\mathcal{H}^0 = \mathbb{C}$. The Hilbert space completion of the direct sum

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \tag{1.37}$$

is called the *Fock space* over $\mathcal{H}$. A vector $\Phi \in \Gamma(\mathcal{H})$ is a sequence $\{\phi_n\}$ of vectors $\phi_n \in \mathcal{H}^{\otimes n}$.

Let $\mathcal{K}$ be another Hilbert space. Then every contraction $T : \mathcal{H} \to \mathcal{K}$ gives rise to a contraction

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} T^{\otimes n} \tag{1.38}$$

from $\Gamma(\mathcal{H})$ to $\Gamma(\mathcal{K})$.

Lemma 1.4.1 ([EL77, Lem. 6.1])**.** *Γ is a functor on the category whose objects are Hilbert spaces and whose morphisms are contractions, i.e. $\Gamma(ST) = \Gamma(S)\Gamma(T)$ and $\Gamma(\mathbb{1}_{\mathcal{H}}) = 1_{\Gamma(\mathcal{H})}$. In addition Γ is a $*$-map, so we have $\Gamma(T^*) = \Gamma(T)^*$.*

In quantum mechanics, we are usually not interested in $\Gamma(\mathcal{H})$ itself but in two of its subspaces, namely the *symmetric* or *Boson Fock space* $\Gamma^+(\mathcal{H})$ and the *antisymmetric* or *Fermion Fock space* $\Gamma^-(\mathcal{H})$.

For the construction of those two subspaces, we look at the group $\mathfrak{P}_n$ of all permutation on n elements and on its unitary action on basis elements $\phi_1, \ldots \phi_n$ of $\mathcal{H}^{\otimes n}$:

$$U_\pi(\phi_1 \otimes \ldots \otimes \phi_n) = \phi_{\pi(1)} \otimes \ldots \otimes \phi_{\pi(n)}, \tag{1.39}$$

for all $\pi \in \mathfrak{P}_n$. By linearity, this extends to a bounded operator on $\mathcal{H}^{\otimes n}$ and we can define two operators

$$P_n^+(\phi_1 \otimes \ldots \otimes \phi_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{P}_n} \phi_{\pi(1)} \otimes \ldots \otimes \phi_{\pi(n)} \tag{1.40}$$

$$P_n^-(\phi_1 \otimes \ldots \otimes \phi_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{P}_n} \epsilon(\pi)\phi_{\pi(1)} \otimes \ldots \otimes \phi_{\pi(n)} \tag{1.41}$$

in $\mathcal{H}^{\otimes n}$ where $\epsilon(\pi)$ is the signature of the permutation π. We can now define the two subspaces of $\Gamma(\mathcal{H})$.

Definition 1.4.2. *The* symmetric (or Boson) Fock space $\Gamma^+(\mathcal{H})$ *is defined by*

$$\Gamma^+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} P_n^+ \mathcal{H}^{\otimes n} \tag{1.42}$$

and the antisymmetric (or Fermion) Fock space $\Gamma^-(\mathcal{H})$ *is defined by*

$$\Gamma^-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} P_n^- \mathcal{H}^{\otimes n}. \tag{1.43}$$

We write P^+ and P^- for the projections from $\Gamma(\mathcal{H})$ to $\Gamma^+(\mathcal{H})$ and $\Gamma^-(\mathcal{H})$, respectively.

If $T : \mathcal{H} \to \mathcal{K}$ is a contraction, then T_n maps $P_n^+\mathcal{H}^{\otimes n}$ into $P_n^+\mathcal{K}^{\otimes n}$ and $P_n^-\mathcal{H}^{\otimes n}$ into $P_n^-\mathcal{K}^{\otimes n}$. $\Gamma(T)$ then induces contractions $\Gamma^+(T) : \Gamma^+(\mathcal{H}) \to \Gamma^+(\mathcal{K})$ and $\Gamma^-(T) : \Gamma^-(\mathcal{H}) \to \Gamma^-(\mathcal{K})$. Γ^+ and Γ^- inherit the properties of Lemma 1.4.1 from Γ.

The symmetric Fock space $\Gamma^+(\mathcal{H})$ is spanned by vectors of the form

$$\exp(\phi) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{(n!)}} \phi^{\otimes n} \tag{1.44}$$

with $\phi \in \mathcal{H}$ and $\exp : \mathcal{H} \to \Gamma^+(\mathcal{H})$ and we have

$$\langle \exp(\phi), \exp(\psi) \rangle = e^{\langle \phi, \psi \rangle}. \tag{1.45}$$

For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ this gives us a natural identification

$$\Gamma^+(\mathcal{H}_1 \oplus \mathcal{H}_2) = \Gamma^+(\mathcal{H}_1) \otimes \Gamma^+(\mathcal{H}_2) \tag{1.46}$$

such that for $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$

$$\exp(\phi \oplus \psi) = \exp(\phi) \otimes \exp(\psi) \tag{1.47}$$

and for contractions $T : \mathcal{H}_1 \to \mathcal{K}_1$ and $S : \mathcal{H}_1 \to \mathcal{K}_2$ with two Hilbert space $\mathcal{K}_1$ and $\mathcal{K}_2$ we get

$$\Gamma^+(T \oplus S) = \Gamma^+(T) \otimes \Gamma^+(S). \tag{1.48}$$

Two important operators on the Fock space in quantum mechanics are the creation and annihilation operators, which describe the addition or removal of a particle from a system. We focus here on their mathematical aspects, as they are important in the construction of the CCR and the CAR algebra.

Definition 1.4.3. *Let $\phi, \phi_1, \ldots, \phi_n \in \mathcal{H}$. Then the* creation operator $a^*(\phi)$ *and the* annihilation operator $a(\phi)$ *are defined by linear extension of*

$$a(\phi)(\phi_1 \otimes \ldots \otimes \phi_n) = \sqrt{n}\, \langle \phi, \phi_1 \rangle \, (\phi_2 \otimes \ldots \otimes \phi_n) \tag{1.49}$$

$$a^*(\phi)(\phi_1 \otimes \ldots \otimes \phi_n) = \sqrt{n+1}\, (\phi \otimes \phi_1 \otimes \ldots \otimes \phi_n). \tag{1.50}$$

These operators are adjoints but are not bounded and, therefore, only defined on a dense domain. The annihilation and creation operators of the symmetric and the antisymmetric Fock space are then given by

$$a_\pm(\phi) = P_\pm a(\phi) P_\pm \tag{1.51}$$
$$a^*_\pm(\phi) = P_\pm a^*(\phi) P_\pm \tag{1.52}$$

If $\Omega = (1, 0, \ldots) \in \Gamma^\pm(\mathcal{H})$ is the zero-particle state, the *vacuum*, the vectors $a^*_\pm(\phi)\Omega$ correspond to elements of the one-particle Hilbert space $\mathcal{H}$ and thus create particles of the state ϕ.

Definition 1.4.4. *The relations*

$$[a_+(\phi), a_+(\psi)] = [a^*_+(\phi), a^*_+(\psi)] = 0 \tag{1.53}$$
$$[a_+(\phi), a^*_+(\psi)] = \langle \phi, \psi \rangle \mathbb{1} \tag{1.54}$$

are called the canonical commutation relations (*CCR*). *Analogously, the relations*

$$\{a_-(\phi), a_-(\psi)\} = \{a^*_-(\phi), a^*_-(\psi)\} = 0 \tag{1.55}$$
$$\{a_-(\phi), a^*_-(\psi)\} = \langle \phi, \psi \rangle \mathbb{1} \tag{1.56}$$

are called the canonical anti-commutation relations (*CAR*), *where* $\{\cdot, \cdot\}$ *is the so-called anti-commutator defined by* $\{A, B\} = AB + BA$.

The CAR algebra The annihilation and creation operators on the antisymmetric Fock space satisfy

$$\|a_-(\phi)\| = \|\phi\| = \|a^*_-(\phi)\|, \tag{1.57}$$

and therefore have bounded extensions. If $\Omega = (1, 0, \ldots)$ is the vacuum vector and $\{\phi_i\}$ is an orthonormal basis of $\mathcal{H}$, then

$$a^*_-(\phi_{i_1}) \cdots a^*_-(\phi_{i_n}) \tag{1.58}$$

is an orthonormal basis of $\Gamma^-(\mathcal{H})$ when $\phi_{i_1}, \ldots, \phi_{i_n}$ runs over the finite subsets of $\{\phi_i\}$.

Let $a_1(\phi)$ and $a_2(\phi)$ with $\phi \in \mathcal{H}$ be two sets of operators that satisfy the canonical anti-commutation relations (1.55) and (1.56), each generating a C^*-algebra $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, then there exists a unique $*$-isomorphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$\alpha(a_1(\phi)) = a_2(\phi) \tag{1.59}$$

for all $\phi \in \mathcal{H}$. In an abstract formulation, we get the following result.

Theorem 1.4.5 ([BR81, Thm. 5.2.5]). *There exists a unique C^*-algebra $\mathcal{A} = \mathcal{A}(\mathcal{H})$, up to $*$-isomorphism, generated by the antilinear elements $a_-(\phi), a^*_-(\phi)$, satisfying* (1.55) *and* (1.56). *We call $\mathcal{A}$ the* CAR algebra.

The CCR algebra Contrary to the antisymmetric case, the annihilation and creation operators $a_+(\phi)$ and $a_+^*(\phi)$ of the symmetric Fock space do not have a bounded extension. To avoid some of the difficulties that come with this unboundedness, we introduce the operators

$$R(\phi) = \frac{a_+(\phi) + a_+^*(\phi)}{\sqrt{2}} \tag{1.60}$$

with the property that

$$R(i\phi) = \frac{a_+(\phi) - a_+^*(\phi)}{\sqrt{2}i}. \tag{1.61}$$

More specifically, we look at their self-adjoint closure, in this context also denoted by $R(\phi)$. These operators are often called *field operators*. Formally (on an appropriately chosen domain), they satisfy the commutation relation

$$[R(\phi), R(\psi)] = i\Im m\langle\phi, \psi\rangle. \tag{1.62}$$

Definition 1.4.6. *For each $\phi \in \mathcal{H}$, we can define a* Weyl operator *as the unitary operator given by*

$$W(\phi) = e^{iR(\phi)}, \tag{1.63}$$

where $R(\phi)$ is a field operator, the selfadjoint closure of (1.60)*.*
For each $\phi, \psi \in \mathcal{H}$ one can formulate the canonical commutation relations in Weyl form*:*

$$W(\phi)W(\psi) = e^{-\frac{i}{2}\Im m\langle\phi,\psi\rangle}W(\phi + \psi) = e^{-i\Im m\langle\phi,\psi\rangle}W(\psi)W(\phi). \tag{1.64}$$

The action of a Weyl operator on a vector e^ψ is given by

$$W(\phi)e^\psi = e^{-\frac{1}{2}\|\phi\|^2 - \langle\phi,\psi\rangle}\, e^{\phi+\psi}. \tag{1.65}$$

Theorem 1.4.7 ([BR81, Thm. 5.2.8])**.** *Let $\mathcal{H}$ be a Hilbert space and let $W(\Phi)$ with $\phi \in \mathcal{H}$ denote the Weyl operators satisfying*

$$W(-\phi) = W(\phi)^* \tag{1.66}$$

$$W(\phi)W(\psi) = e^{-\frac{i}{2}\Im m\langle\phi,\psi\rangle}W(\phi + \psi) \tag{1.67}$$

for all $\phi, \psi \in \mathcal{H}$. Then there exists a unique (up to $*$-isomorphism) C^*-algebra $\mathcal{A}(\mathcal{H})$ generated by the Weyl operators which we call the CCR algebra*. In addition, the Weyl operators satisfy*

1. *$W(0) = \mathbb{1}$,*
2. *$W(\phi)$ is unitary for all $\phi \in \mathcal{H}$.*

1.5 Notes and Remarks

While the main results of this chapter can be found - with varying level of detail - in every textbook on operator algebras, their phrasing differs and we tried to find the best version for later use. Section 1.1 follows [Naa17] and [HZ12]. Both are rather mathematically rigorous in their approach to Hilbert spaces as a basic starting point for quantum mechanics.

They are also the basis for Section 1.2. The topologies and the subsection on von Neumann algebras can be found in [BR87]. The subsection on unbounded operators is taken from [Dav76, Sect. 1.8]. Further sources for this section are [Tak79], [Sak98] and [Wer16].

Section 1.3 is gathered from [Stø13] and [Pau02]. For Section 1.4 we combined various sources to find the most suitable version. Basis was [EL77], however, the notation is rather oldfashioned. Much more approachable are [Naa17], [Arv03], [RS80] and in particular [BR81]. The latter was the main source for the construction of the CCR algebra.

Chapter 2

Operator semigroups

Let X be a Banach space and $\mathcal{T} : X \to X$ a bounded linear map. Just as bounded linear maps on Hilbert spaces, we will call T a *bounded linear operator* on X. A family of bounded linear operators $\mathcal{T}_t$ with $t \geq 0$ is strongly continuous if

$$\lim_{t \to 0} \mathcal{T}_t x = x \tag{2.1}$$

for all $x \in X$.

Definition 2.0.1. *A family $\mathcal{T}_t$ with $t \geq 0$ is called a* strongly continuous (one-parameter) semigroup *if*

1. $\mathcal{T}_0 = \mathbb{1}$
2. $\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}$
3. *$\mathcal{T}_t$ is strongly continuous.*

The property 2 is called the semigroup property *of $\mathcal{T}_t$.*

Such semigroups describe the dynamics of quantum systems, as we will see in Chapter 3. Here we will introduce the most important definitions, theorems and concepts needed throughout this thesis.

2.1 Strongly continuous semigroups and their generators

For every strongly continuous semigroup $\mathcal{T}_t$ one can find $w \in \mathbb{R}$ and $M \geq 1$ such that

$$\|\mathcal{T}_t\| \leq M e^{wt} \tag{2.2}$$

for all $t \geq 0$. The infimum

$$w_0 := \inf \left\{ w \in \mathbb{R} \,|\, \exists M \geq 1 \text{ with } \|\mathcal{T}_t\| \leq M e^{wt} \;\; \forall t \geq 0 \right\} \tag{2.3}$$

is called the *growth bound* of $\mathcal{T}_t$. A strongly continuous semigroup is *bounded* if $w = 0$ satisfies equation (2.2) and if we even have

$$\|\mathcal{T}_t\| \leq 1 \tag{2.4}$$

for all $t \geq 0$, then $\mathcal{T}_t$ is called *contractive*.

It is possible to introduce an adjoint of the semigroup $\mathcal{T}_t$. For this purpose, let X^* denote the dual space of the Banach space X. For the pairing of two element $x \in X$ and $\omega \in X^*$ we write $\omega(x) = [x, \omega] \in \mathbb{C}$.

Lemma 2.1.1 ([EN06, Thm. 1.1.6]). *A semigroup $\mathcal{T}_t$ on X is strongly continuous if and only if it is weakly continuous, i.e. if*

$$t \mapsto [\mathcal{T}_t x, \omega] \tag{2.5}$$

is continuous for each $x \in X$ and $\omega \in X^*$.

The adjoint semigroup of $\mathcal{T}_t$ on X^* is denoted by $\mathcal{T}_t^*$ and given by

$$[x, \mathcal{T}_t^* \omega] = [\mathcal{T}_t x, \omega]. \tag{2.6}$$

In general, for a strongly continuous semigroup $\mathcal{T}_t$, its adjoint $\mathcal{T}_t^*$ is not strongly continuous, but by Lemma 2.1.1, it is always *weak*-continuous*. In fact, we have a one-to-one correspondence between strongly continuous semigroups on X and weak*-continuous semigroups on X^*.

Infinitesimally, a semigroup $\mathcal{T}_t$ can be described by its *generator* $\mathcal{L} : \operatorname{dom} \mathcal{L} \subset X \to X$ given by

$$\mathcal{L}x := \lim_{t \searrow 0} \frac{1}{t} \left(\mathcal{T}_t x - x \right). \tag{2.7}$$

It is defined for all x in the domain

$$\operatorname{dom} \mathcal{L} := \left\{ x \in X \mid \lim_{t \searrow 0} \frac{1}{t} \left(\mathcal{T}_t x - x \right) \text{ exists} \right\}. \tag{2.8}$$

Lemma 2.1.2 ([EN06, Lem. 2.1.3]). *Let $\mathcal{L}$ be the generator of the strongly continuous semigroup $\mathcal{T}_t$ on X. Then the following hold*

1. *$\mathcal{L} : \operatorname{dom} \mathcal{L} \subset X \to X$ is a linear operator*

2. *if $x \in \operatorname{dom} \mathcal{L}$ then $\mathcal{T}_t x \in \operatorname{dom} \mathcal{L}$ and for all $t \geq 0$*

$$\frac{d}{dx} \mathcal{T}_t x = \mathcal{T}_t \mathcal{L} x = \mathcal{L} \mathcal{T}_t x \tag{2.9}$$

3. *for every $t \geq 0$ and $x \in X$*

$$\int_0^t \mathcal{T}_s x \, ds \in \operatorname{dom} \mathcal{L} \tag{2.10}$$

4. *for every $t \geq 0$ and $x \in X$*

$$\mathcal{T}_t x - x = \mathcal{L} \int_0^t \mathcal{T}_s x \, ds \quad \text{if } x \in X \tag{2.11}$$

$$= \int_0^t \mathcal{T}_s \mathcal{L} x \, ds \quad \text{if } x \in \operatorname{dom} \mathcal{L}. \tag{2.12}$$

This lemma leads to the following theorem.

Theorem 2.1.3 ([EN06, Thm. 2.1.4]). *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

As a consequence, the properties of a generator translate to corresponding properties of the strongly continuous semigroup and vice versa.

A subspace $\mathcal{D}$ of the domain $\operatorname{dom}\mathcal{L}$ of the linear operator $\mathcal{L} : \operatorname{dom}\mathcal{L} \subset X \to X$ is is called a *core* for $\mathcal{L}$ if it is dense in $\operatorname{dom}\mathcal{L}$ for the graph norm. The action of $\mathcal{L}$ on a core determines the generator completely. The following proposition will be helpful to determine whether a subspace of $\operatorname{dom}\mathcal{L}$ is a core.

Proposition 2.1.4 ([EN06, Prop. 2.1.7]). *Let $\mathcal{L}$ be the generator of a strongly continuous semigroup $\mathcal{T}_t$ on a Banach space X. A subspace $\mathcal{D} \subset \operatorname{dom}\mathcal{L}$ that is $\|\cdot\|$-dense in X and invariant under the semigroup $\mathcal{T}_t$ is always a core for $\mathcal{L}$.*

A special subgroup of strongly continuous semigroups is that of *uniformly* or *norm-continuous semigroups*, i.e. of those that satisfy

$$\lim_{t\to 0} \|\mathcal{T}_t - \mathbb{1}\| = 0. \tag{2.13}$$

Corollary 2.1.5 ([EN06, Cor. 2.1.5]). *Let $\mathcal{T}_t$ be a strongly continuous semigroup on a Banach space X with generator $\mathcal{L}$. Then the following statements are equivalent:*

1. *$\mathcal{L}$ is bounded, i.e. there is an $M > 0$ such that $\|\mathcal{L}x\| \leq M\|x\|$ for all $x \in \operatorname{dom}\mathcal{L}$,*
2. *$\operatorname{dom}\mathcal{L}$ is all of X,*
3. *$\operatorname{dom}\mathcal{L}$ is closed in X,*
4. *$\mathcal{T}_t$ is norm-continuous.*

In fact, every norm-continuous semigroup is given by

$$\mathcal{T}_t = e^{t\mathcal{L}} := \sum_0^\infty \frac{t^n\mathcal{L}^n}{n!}. \tag{2.14}$$

2.2 Resolvents and generation theorems

For every closed linear operator $\mathcal{L}$ on a Banach space X with domain $\operatorname{dom}\mathcal{L} \subset X$ we can introduce the *spectrum* of $\mathcal{L}$

$$\sigma(\mathcal{L}) := \{\lambda \in \mathbb{C} |\, \lambda\mathbb{1} - \mathcal{L} \text{ is not bijective}\} \tag{2.15}$$

and define its *resolvent set* as

$$\rho(\mathcal{L}) := \mathbb{C}\backslash\sigma(\mathcal{L}). \tag{2.16}$$

The resolvent of $\mathcal{L}$ is then given by

$$\mathcal{R}_\lambda(\mathcal{L}) := (\lambda\mathbb{1} - \mathcal{L})^{-1} \tag{2.17}$$

for all $\lambda \in \rho(\mathcal{L})$. We often omit the operator in brackets if there is no ambiguity in the given context, and we usually write $\lambda - \mathcal{L} = \lambda\mathbb{1} - \mathcal{L}$.

Theorem 2.2.1 ([EN06, Thm. 2.1.10]). *Let $\mathcal{T}_t$ be a strongly continuous semigroup on a Banach space X and let $w \in \mathbb{R}$ and $M \geq 1$ s.t.*

$$\|\mathcal{T}_t\| \leq Me^{wt} \tag{2.18}$$

for $t \geq 0$. Then for the corresponding generator $\mathcal{L}$ and its resolvent $\mathcal{R}_\lambda$ the following hold:

1. *If $\lambda \in \mathbb{C}$ and the integral $\int_0^\infty e^{-\lambda s}\mathcal{T}_s x\, ds$ exists for all $x \in X$, then $\lambda \in \rho(\mathcal{L})$ and*

$$\mathcal{R}_\lambda = \int_0^\infty e^{-\lambda s}\mathcal{T}_s\, ds. \tag{2.19}$$

2. *If $\Re e\lambda > w$, then $\lambda \in \rho(\mathcal{L})$, and $\mathcal{R}_\lambda$ is given by the above integral expression.*
3. *If $\Re e\lambda > w$, then*

$$\|\mathcal{R}_\lambda\| \leq \frac{M}{\Re e\lambda - w}. \tag{2.20}$$

Remark 2.2.2. *Equation* (2.19) *is called the* integral representation *of $\mathcal{R}_\lambda$ and the integral is understood as an improper Riemann integral*

$$\mathcal{R}_\lambda x = \lim_{t\to\infty} \int_0^t e^{-\lambda s}\mathcal{T}_s x\, ds. \tag{2.21}$$

From Theorem 2.19, it is clear that $\mathcal{R}_\lambda$ is completely positive and it satisfies the *resolvent identity*

$$\mathcal{R}_\lambda - \mathcal{R}_\mu = (\mu - \lambda)\mathcal{R}_\lambda\mathcal{R}_\mu. \tag{2.22}$$

Conversely, the dynamical semigroup can be recovered by the formula

$$\mathcal{T}_t = \lim_{n\to\infty} \left(\frac{n}{t}\mathcal{R}_{n/t}\right)^n. \tag{2.23}$$

Since $\mathcal{L}\mathcal{R}_\lambda x = \lambda\mathcal{R}_\lambda x - x$, the resolvent also gives us access to the domain of the generator, and for every $\lambda > 0$, we have

$$\operatorname{dom}\mathcal{L} = \mathcal{R}_\lambda(X). \tag{2.24}$$

After establishing the relationships between a strongly continuous semigroup, its generator and its resolvent, we can now look at the generation theorems.

Theorem 2.2.3 (Hille-Yosida generation theorem, [EN06, Thm. 2.3.5]). *Let $\mathcal{L}$ be a linear operator on a Banach space X. Then the following statements are equivalent:*

1. *$\mathcal{L}$ generates strongly continuous contraction semigroup.*
2. *$\mathcal{L}$ is closed, densely defined, and for every $\lambda > 0$ we have $\lambda \in \rho(\mathcal{L})$ and*

$$\|\lambda\mathcal{R}_\lambda\| \leq 1. \tag{2.25}$$

3. *$\mathcal{L}$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\Re e\lambda > 0$ we have $\lambda \in \rho(\mathcal{L})$ and*

$$\|\mathcal{R}_\lambda(\mathcal{L})\| \leq \frac{1}{\Re e\lambda}. \tag{2.26}$$

A linear operator $\mathcal{L}$ is called *dissipative* if it satisfies

$$\|(\lambda - \mathcal{L})x\| \geq \lambda\|x\| \tag{2.27}$$

for all $\lambda > 0$ and $x \in \operatorname{dom}\mathcal{L}$.

Theorem 2.2.4 (Lumer-Phillips generation theorem, [EN06, Thm. 2.3.15])**.** *Let $\mathcal{L}$ be a densely defined, dissipative operator on a Banach space X. Then the following are equivalent:*

1. *The closure of $\mathcal{L}$ generates a contraction semigroup.*
2. *The range $\operatorname{ran}(\lambda - \mathcal{L})$ is dense in X for some (and hence all) $\lambda > 0$.*

The next proposition will give us an other way to determine, whether a subspace is a core for the generator of a semigroup.

Proposition 2.2.5 ([Dav80, Sect. 2.1])**.** *Let $\mathcal{L}$ be a closed linear operator on a Banach space X, and let $\lambda \in \rho(\mathcal{L})$. Then a subspace $\mathcal{D} \subset \operatorname{dom}\mathcal{L}$ is a core for $\mathcal{L}$ if and only if $(\lambda - \mathcal{L})\mathcal{D}$ is dense in X.*

2.3 Perturbations of generators

In the last section, we saw under which conditions an operator $\mathcal{L}$ generates a strongly continuous semigroup. We will now consider under which conditions a generator perturbed by another operator is again the generator of a semigroup.

Theorem 2.3.1 ([Dav80, Thm. 3.1])**.** *Let $\mathcal{L}$ be the generator of a strongly continuous semigroup $\mathcal{T}_t$ on a Banach space X with*

$$\|\mathcal{T}_t\| \leq Me^{wt} \tag{2.28}$$

for all $t \geq 0$. If A is a bounded operator on X, then $\mathcal{L} + A$ is the generator of a strongly continuous semigroup S_t on X, s.t.

$$\|S_t\| \leq Me^{(w+M\|A\|)t} \tag{2.29}$$

for all $t \geq 0$.

This theorem leads to the identities

$$S_t x = \mathcal{T}_t x + \int_0^t S_{t-s} A \mathcal{T}_s x ds \tag{2.30}$$

$$= \mathcal{T}_t x + \int_0^t \mathcal{T}_{t-s} A S_s x ds \tag{2.31}$$

If the operator A on X is unbounded, Theorem 2.3.1 is not necessarily true. However, if we require A to be *relatively bounded* with respect to $\mathcal{L}$, i.e.

$$\operatorname{dom}\mathcal{L} \subset \operatorname{dom} A \tag{2.32}$$

and there are $a, b > 0$ such that

$$\|Ax\| \leq a\|\mathcal{L}x\| + b\|x\|, \tag{2.33}$$

then the following theorem shows that the perturbed generator generates a semigroup again. The constant a is called the *relative bound* of A with respect to $\mathcal{L}$.

Theorem 2.3.2 ([Dav80, Thm. 3.7 and Cor. 3.8])**.** *Let $\mathcal{L}$ be the generator of a strongly continuous contraction semigroup $\mathcal{T}_t$ on a Banach space X. If A is a perturbation of $\mathcal{L}$ with a relative bound less than 1, such that $\mathcal{L} + A$ is dissipative, then $\mathcal{L} + A$ is the generator of a strongly continuous contraction semigroup.*

As the following lemma shows, it is sufficient for A to be defined on a core of $\mathcal{L}$.

Lemma 2.3.3 ([Dav80, Lem. 3.9])**.** *Let $\mathcal{D}$ be a core for the generator $\mathcal{L}$ of a strongly continuous contraction semigroup $\mathcal{T}_t$ on a Banach space X. If A has domain $\mathcal{D}$ and satisfies equation* (2.33) *for all $x \in \mathcal{D}$, then A can be uniquely extended to* $\operatorname{dom} \mathcal{L}$ *such that 2.33 is satisfied for all $x \in \operatorname{dom} \mathcal{L}$.*

2.4 Notes and Remarks

The first to section are taken from the textbook [EN06] that we highly recommend as a starting point for the topic of one parameter semigroups. Section 2.3 follows [Dav80, Ch. 3]. Another good source for one parameter semigroups that we used for our research but not cited explicitly in this section is [Paz83].

Chapter 3

Quantum mechanics

The objective of quantum mechanics is to describe quantum experiments and predict their outputs. The results it gives us are stochastical; instead of deterministic numbers, we get probability distributions and expectation values. Every physical (quantum) experiment is essentially composed of two parts: preparation and measurement.

The preparation is characterised by a *state*, given by a positive operator ρ on a Hilbert space $\mathcal{H}$ with $\operatorname{tr}\rho = 1$. As many physical preparations may lead to the same statistical outcomes, a state is the description of the ensemble of all these similar preparation procedures. An effect is the most basic measurement apparatus that produces either 'yes' or 'no' as an outcome. It is given by a selfadjoint operator F on $\mathcal{H}$ with $0 \leq F \leq \mathbb{1}$.

The basic concept of this statistical framework was introduced by Ludwig [Lud83], and it is depicted in the following figure.

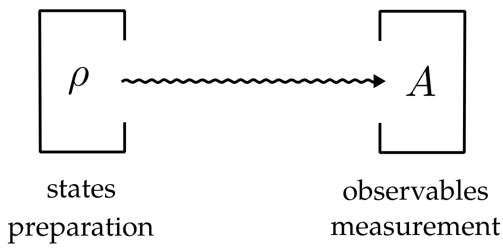

Figure 3.1: Concept of a quantum experiment.

3.1 States and effects

The positive operators of trace 1 describing a quantum state are called *density operators*. The set of all states of the quantum system is denoted by

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathfrak{T}(\mathcal{H}) | \rho \geq 0,\ \operatorname{tr}(\rho) = 1\}. \tag{3.1}$$

The dimension of the underlying Hilbert space is a property of the quantum system. If $\mathcal{H}$ is finite-dimensional, then ρ is a *desity matrix*. Sometimes, a unit vector $\phi \in \mathcal{H}$ is also called a state. In this case, we are actually referring to the rank-1 operator $|\phi\rangle\langle\phi|$ that is an element of $\mathcal{S}(\mathcal{H})$, and call ϕ a *vector state*.

$\mathcal{S}(\mathcal{H})$ is a convex set and by Theorem 1.2.9, every state has a *canonical convex decomposition* of the form

$$\rho = \sum_i \lambda_i P_i \tag{3.2}$$

where $\{\lambda_i\}$ is a finite or infinite sequence of positive numbers adding to 1 and $\{P_i\}$ is a sequence of orthogonal one-dimensional projections, i.e. $P_i P_j = \delta_{ij} P_i$. An element ρ of a convex set is *extremal* if it cannot be written as a proper convex combination of other elements, i.e. if $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ then $\rho = \rho_1 = \rho_2$. An extremal element of the convex set $\mathcal{S}(\mathcal{H})$ is called a *pure state*. Any other element is called a *mixed state*.

It is important to distinguish the notion of mixed states from that of a *superposition* of states. If $\phi, \psi \in \mathcal{H}$ are two linearly independent unit vectors we write

$$\omega = \frac{1}{\|a\phi + b\psi\|}(a\phi + b\psi) \tag{3.3}$$

with $0 \neq a, b \in \mathbb{C}$. Then the pure state $|\omega\rangle\langle\omega|$ is called a *superposition* of ϕ and ψ.

The probability of getting the outcome 'yes' for an effect F on a system in state ρ is given by $\operatorname{tr}\rho F$. This is an affine mapping from $\mathcal{S}(\mathcal{H})$ to $[0,1]$. In analogy to the set of states, one can introduce the set of all effects

$$\mathcal{F}(\mathcal{H}) = \{F \in \mathcal{B}(\mathcal{H}) | F = F^* \text{ and } 0 \leq F \leq \mathbb{1}\}. \tag{3.4}$$

This set is also convex, and its extremal elements are the projections.

3.2 Composite Systems

Let A and B be quantum systems with corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. We assume that A and B are *subsystems* of the compound system $A+B$. The Hilbert space of the composite system is given by $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Suppose we have effects F_A on $\mathcal{H}_A$ and F_B on $\mathcal{H}_B$ corresponding to measurements on A and B, respectively. Then on the compound system $A+B$, we have an effect $F_A \otimes F_B$ that describes both these separate measurements.

For separate preparations of subsystems, the preparation of the compound system is given by $\rho_A \otimes \rho_B$. We then have

$$\operatorname{tr}[(\rho_A \otimes \rho_B)(F_A \otimes F_B)] = \operatorname{tr}[\rho_A F_A] \operatorname{tr}[\rho_B F_B] \tag{3.5}$$

and the system is statistically independent if measurements and preparations are made separately. However, states of the compound system $A+B$ need not be in product form.

The state of a subsystem can be determined from the state of the compound system via the *partial trace*. The *partial trace* over the system A is the linear mapping

$$\operatorname{tr}_A : \mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathfrak{T}(\mathcal{H}_B) \tag{3.6}$$

satisfying

$$\mathrm{tr}[\mathrm{tr}_A[T]F] = \mathrm{tr}[T(\mathbb{1} \otimes F)] \tag{3.7}$$

for all $T \in \mathfrak{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $F \in \mathcal{F}(\mathcal{H}_B)$. Since

$$\mathrm{tr}[T] = \mathrm{tr}[\mathrm{tr}_A[T]] = \mathrm{tr}[\mathrm{tr}_B[T]] \tag{3.8}$$

together with $T \geq 0$ implies that $\mathrm{tr}_A[T] \geq 0$ and $\mathrm{tr}_B[T] \geq 0$, the partial trace preserves all relevant properties of the operator such that the partial trace of a state is again a state. So, if $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a state of the composite system $A + B$, then the operators $\mathrm{tr}_B[\rho]$ and $\mathrm{tr}_A[\rho]$ describe the states of the subsystems A and B, respectively and are called *reduced states*. The state ρ of the composite system is called a *joint state* and often denoted $\rho_{AB} = \rho$. If the reduced states $\rho_A = \mathrm{tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{tr}_A[\rho_{AB}]$ are pure states then ρ_{AB} is of the product form $\rho_{AB} = \rho_A \otimes \rho_B$.

Any mixed state can be seen as a reduced pure state of a composite system. For this case, one introduces an *ancillary system* in the form of an additional Hilbert space $\mathcal{H}_{anc}$. Let ρ be a state on $\mathcal{H}$, then we call a pure state $\widehat{\rho}$ on a composite system $\mathcal{H} \otimes \mathcal{H}_{anc}$ a *purification* of ρ if $\mathrm{tr}_{anc}[\widehat{\rho}] = \rho$.

3.3 Observables

In the first sections of this chapter, we used effects to describe the measurement process and thereby restricted the outcome set Ω to 'yes' and 'no'. More complex measurement results are generally reducible to whether an output lies in a specific subset X of all possible measurement outcomes Ω. This is again a 'yes' or 'no' measurement that we can associate with an effect $F(X)$. We call the set of operators $F(X)$ for all subsets X an *observable*.

More formally, let $\mathcal{P}(\Omega)$ denote the σ-algebra on Ω, such that the pair $(\Omega, \mathcal{P}(\Omega))$ is a *measurable space*. The set $X \in \mathcal{P}(\Omega)$ is called an *event*. An observable is then a *positive operator valued measure* (*POVM*), i.e. a mapping $A : \mathcal{P}(\Omega) \to \mathcal{F}(\mathcal{H})$ such that

1. $A(\emptyset) = 0$
2. $A(\Omega) = \mathbb{1}$
3. $A(\bigcup_i X_i) = \sum_i A(X_i)$ (in the weak sense) for any sequence $\{X_i\}$ of disjoint sets in $\mathcal{P}(\Omega)$.

For an observable A defined on a measurable space $(\Omega, \mathcal{F})$ we say that Ω is the *sample space* of A and $(\Omega, \mathcal{F})$ is the *outcome space* of A.

The mapping Φ_A from $\mathcal{S}(\mathcal{H})$ to the set of all probability measures on an outcome space $(\Omega, \mathcal{F})$ given by

$$\Phi_A(\rho) := \mathrm{tr}[\rho A(\cdot)] \tag{3.9}$$

is called the *statistical map* corresponding to the observable A. It is often described as an input-output device, which takes states as inputs and gives probability distributions as outputs. We will call an observable *discrete* if there is a countable set $\Omega_0 \in \mathcal{P}(\Omega)$ such that $A(\Omega_0) = \mathbb{1}$, and say it is *real* if its outcome set is either $\mathbb{R}$ or a subset of $\mathbb{R}$. In this case, the σ-algebra is the corresponding Borel σ-algebra, i.e. the smallest σ-algebra containing all open sets in $\mathbb{R}$.

3.4 Quantum channels and operations

So far, we described a physical experiment as essentially composed of two parts, preparation and measurement. The preparation device produces a quantum output; a measurement accepts quantum inputs and produces a classical output in the form of an outcome distribution. We can extend this description by introducing a third device that accepts quantum states as input and produces a quantum state as an output. Such devices are called *quantum channels*.

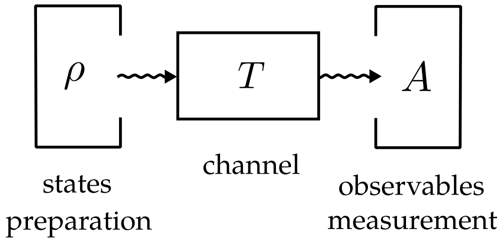

Figure 3.2: Concept of a quantum experiment with channel.

A slightly more general transformation on quantum states is called an *operation*. We will allow the possibility that an operation destroys some fraction of the system so that some probability is lost. It is, therefore, convenient to introduce the set of *subnormalized states* $\widetilde{\mathcal{S}}(\mathcal{H})$ that consists of positive trace class operators ρ with $\mathrm{tr}[\rho] \leq 1$. However, since any trace class operator $\rho \in \mathfrak{T}(\mathcal{H})$ can be written as a linear combination of positive operators in $\mathfrak{T}(\mathcal{H})$, we can extend most statements to the whole class of trace class operators. We will call a linear mapping T on $\mathfrak{T}(\mathcal{H})$ *trace nonincreasing* if $\mathrm{tr}\, T\rho \leq \mathrm{tr}\, \rho$ for all positive $\rho \in \mathfrak{T}(\mathcal{H})$. If $\mathrm{tr}\, T\rho = \mathrm{tr}\, \rho$ we say T is *trace preserving*.

Definition 3.4.1. *A mapping T on $\mathfrak{T}(\mathcal{H})$ is an* operation *if it is*

1. *linear*
2. *completely positive*
3. *trace nonincreasing*

If it is trace preserving, it is called a channel.

The definition above ensures that an operation maps states to subnormalized states and channels map states to states. We need complete positivity to ensure that in a composite system $A + B$, we can extend any operation T_A on subsystem A to a mapping $T_A \otimes \mathbb{1}_B$, that is then an operation acting on the composite system $A + B$.

States and effects are dual objects. Each linear mapping T on $\mathfrak{T}(\mathcal{H})$ induces a linear mapping T^* on the dual $\mathcal{B}(\mathcal{H})$. The connection between T and T^* is given by

$$\mathrm{tr}[T(\rho)F] = \mathrm{tr}[\rho T^*(F)] \tag{3.10}$$

for all $\rho \in \mathfrak{T}(\mathcal{H})$ and all $F \in \mathcal{F}(\mathcal{H})$. The complete positivity of T is equivalent to the complete positivity of T^* on $\mathcal{B}(\mathcal{H})$, T is trace nonincreasing if and only if $T^*(\mathbb{1}) \leq \mathbb{1}$, and T is trace-preserving if and only if T^* is unital, i.e. if $T^*(\mathbb{1}) = \mathbb{1}$. The mapping T^*

describes the same system transformation as T, but instead of transforming the states, it transforms the effects. We call T an operation in the *Schrödinger picture* and T^* the corresponding operation in the *Heisenberg picture*. To define an operation directly in the Heisenberg picture, we additionally need to require that T^* is normal to ensure that a corresponding operation in the Schrödinger picture exists. In this thesis, we use both pictures since each has advantages.

Now we consider two operations T_1 and T_2 acting consecutively on a fixed space in $\mathfrak{T}(\mathcal{H})$. This is described by a composition of the two operations $T_1 \circ T_2$ called the concatenation of T_1 and T_2. Concatenation is an associative operation but not commutative, so $T_1 \circ T_2 \neq T_2 \circ T_1$. A channel T_1 is called the *inverse* of another channel T_2 if $T_1 \circ T_2 = T_2 \circ T_1 = \mathbb{1}$ and the inverse channel for a channel T_1 exists if and only if T_1 is a unitary channel.

Quantum operations also describe the dynamics of a quantum system. A system is called *isolated* or *closed* if all its changes are reversible, i.e. if all system transformations are unitary or antiunitary. Otherwise, a system is called *open*. It is common to view an open system as part of a larger closed system, and the additional part is called an *environment*.

Proposition 3.4.2 ([HZ12, Prop. 4.13]). *Let $\mathcal{H}$ be the Hilbert space of a quantum system and let $\mathcal{H}_E$ be the Hilbert space describing the environment. If U is a unitary operator on $\mathcal{H} \otimes \mathcal{H}_E$ and ξ a fixed state of the environment, then the induced mapping*

$$T : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}),\ T(\rho) = \mathrm{tr}_E[U\rho \otimes \xi U^*] \tag{3.11}$$

is a channel.

The counterpart of this theorem is the theorem of Stinespring (Thm 1.3.3). Applied to channels, it states that every channel in the Heisenberg picture can be written as a unitary channel acting on the system and its environment

$$T^*(A) = V^*(\pi(A))V \tag{3.12}$$

with bounded operators $V : \mathcal{H} \to \mathcal{K}$ and a $*$-homomorphism $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ for a Hilbert space $\mathcal{K}$. We can set $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}_E$ and get the form

$$T^*(A) = V^*(A \otimes \mathbb{1}_E)V. \tag{3.13}$$

As T^* is unital, the operators $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_E$ satisfy $VV^* = \mathbb{1}$. In the Schrödinger picture, the counterpart property of the Stinespring theorem is more obvious.

Corollary 3.4.3 ([HZ12, Cor. 4.19]). *If $T : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ is a channel then there exists a Hilbert space $\mathcal{H}_E$, a pure state $\xi \in \mathcal{S}(\mathcal{H}_E)$ and a unitary operator U on $\mathcal{H} \otimes \mathcal{H}_E$ such that*

$$T(\rho) = \mathrm{tr}_E[U\rho \otimes \xi U^*] \tag{3.14}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$.

As a consequence of Stinespring's theorem, a linear mapping $T : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ is a channel if and only if there exists a (finite or infinite) sequence of bounded operators $K_1, K_2, \ldots$ with $\sum_i K_i^* K_i = \mathbb{1}$ such that

$$T(\rho) = \sum_i K_i \rho K_i^*. \tag{3.15}$$

This form is called the *Kraus form* or *operator-sum form* of the channel T, and the operators K_i are called *Kraus operators*. The choice of operators is not unique. Two finite sets $\{K_1, \ldots, K_n\}$ and $\{\widehat{K}_1, \ldots, \widehat{K}_m\}$ of bounded operators define the same operation via Kraus form if and only if

$$K_i = \sum_{j=1}^{m} u_{ij} \widehat{K}_j \tag{3.16}$$

with $u_{ij} \in \mathbb{C}$ and $\sum_j u_{jk} \bar{u}_{jl} = \delta_{kl}$. If $\dim \mathcal{H} < \infty$ then it is possible to choose $(\dim \mathcal{H})^2$ or fewer operators K_i. The relation between the Kraus operators K_i and the Stinespring operator $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_E$ is given by

$$V\phi = \sum_i K_i \phi \otimes e_i, \tag{3.17}$$

where e_i is a basis of $\mathcal{H}_E$, and the relation for the adjoints is

$$V^*(\phi \otimes e_i) = K_i^* \phi. \tag{3.18}$$

Therefore, the change from $\{K_1, \ldots, K_n\}$ to $\{\widehat{K}_1, \ldots, \widehat{K}_m\}$ simply corresponds to a change of basis in $\mathcal{H}_E$.

The Stinespring theorem gives rise to another theorem that, in analogy to a result in classical measure theory, is called the *Radon-Nikodym theorem*.

Theorem 3.4.4 (Radon-Nikodym theorem, [BL07, Thm. 5.8])**.** *Let $T_x^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a family of completely positive maps and let $(V, \mathcal{K})$ be a Stinespring representation of $T^* = \sum_x T_x^*$. Then there exist positive operators F_x in $\mathcal{B}(\mathcal{K})$ with $\sum_x F_x = \mathbb{1}$ and*

$$T_x^*(A) = V^*(A \otimes F_x)V. \tag{3.19}$$

If $(V, \mathcal{K})$ is the minimal Stinspring representation, then the operators F_x are uniquely determined.

3.5 Quantum dynamics

So far, our picture of an experiment does not consider the notion of time evolution. Such a description is important if the state of a system changes over time or if we want to perform any continuous measurement.

In 1926 Erwin Schrödinger developed the first quantum evolution equation. It describes the dynamics of a closed quantum system that is in a pure state $\psi_t \in \mathcal{H}$ at a time t:

$$\frac{d}{dt}|\psi_t\rangle = -iH|\psi_t\rangle. \tag{3.20}$$

The operator H is the self-adjoint Hamiltonian operator of the system, and the solution to this equation is given by

$$|\psi_t\rangle = U_t|\psi_0\rangle, \tag{3.21}$$

where U_t is a unitary operator given by $U_t = e^{-itH}$. If the system is in a mixed state ρ_t, the Schrödinger equation induces the evolution equation

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] = \mathcal{L}\rho_t, \tag{3.22}$$

sometimes called the *von Neumann* or *Liouville-von Neumann equation* and the operator $\mathcal{L}$ is accordingly called the *Liouvillian*. The solution is given by

$$\rho_t = U_t \rho_0 U_t^*. \tag{3.23}$$

Still, the time evolution is governed by a unitary operator U_t, and no information is interchanged with the environment.

As soon as we want to interact with the quantum system, for example, by performing a measurement, the system is perturbed and no longer isolated. Thus, its evolution can not be described by the above equations. However, for each time t_i the map $T^i : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ that maps an initial state ρ_0 of the open system to a state $\rho_{t_i} = T^i \rho_0$ is a quantum channel (or operation if we allow for probability loss). So, following Corollary 3.4.3, for each time t_i, we can describe the open system as embedded in an environment, and the evolution of the extended system is again unitary. The channel T^i can then be written in Kraus form (3.15) for each t_i as

$$T^i \rho_0 = \sum_j K_j^i \rho_0 K_j^{i*}. \tag{3.24}$$

Although this formally looks quite similar to equation (3.23), there is a fundamental difference. The equation (3.24) is not the solution of a differential equation but merely describes the quantum state at a particular time t_i depending on the initial state ρ_0. This problem becomes apparent if we look at the evolution of an open system in initial state ρ_0 from t_0 to t_1 described by a channel T^{01} and then from t_1 to t_2 described by T^{12}. The evolution between t_0 and t_2 can be described by a channel T^{02}. We implicitly assumed that the joint initial state of the composite system is of product form $\widehat{\rho}_0 = \rho_0 \otimes \rho_0^E$. The evolution of the composite system is given by unitary operators U_{t_1} from t_0 to t_1 and $U_{t_2 - t_1}$ between t_1 and t_2 such that $U_{t_2} = U_{t_2 - t_1} U_{t_1}$. At $t = t_1$ the open system is in a state

$$\rho_{t_1} = T^{01} \rho_0 = \mathrm{tr}_E \left[U_{t_1} \rho_0 \otimes \rho_0^E U_{t_1}^* \right] = \sum_j K_j^1 \rho_0 K_j^{1*} \tag{3.25}$$

and at $t = t_2$

$$\rho_{t_2} = T^{02} \rho_0 = \mathrm{tr}_E \left[U_{t_2} \rho_0 \otimes \rho_0^E U_{t_2}^* \right] = \sum_j K_j^{12} \rho_0 K_j^{12*}. \tag{3.26}$$

However, it is not possible to find such a simple Krausform for the channel T^{12}. We can write

$$T^{12} \rho_0 = \mathrm{tr}_E \left[U_{t_2 - t_1} \widehat{\rho}_{t_1} U_{t_2 - t_1}^* \right] \tag{3.27}$$

with $\widehat{\rho}_{t_1} = U_{t_1} \rho_0 \otimes \rho_0^E U_{t_1}^*$, but since this is in general not a product state, the evolution from t_1 to t_2 depends on the initial states of the system and the environment at $t = t_0$.

To avoid this problem, we will choose the *Markov approximation*, i.e., we can assume that to determine the evolution of an open system between $t = t_1$ and $t = t_2$ we only need to know the state ρ_{t_1} of the system at $t = t_1$. This approximation is justified if any correlation between the system and the environment is very short-lived and there are no long-time memory effects. With this assumption, the family of channels T_t obtains the semigroup property

$$T_{t_1+t_2} = T_{t_1} T_{t_2}. \tag{3.28}$$

This leaves us with the question of continuity in the time parameter. We want to be able to describe a continuous measurement, i.e. a time evolution with continuous expectation values

$$\lim_{t\to 0} \operatorname{tr}(T_t(\rho)A) = \operatorname{tr}(\rho A). \tag{3.29}$$

Therefore, we conclude that the time evolution of an open quantum (Markov) system is given by a so-called *quantum dynamical semigroup* $\mathcal{T}_t$, i.e. it is a strongly continuous (one-parameter) semigroup as characterised in Definition 2.0.1. It is the solution of the differential *master equation*

$$\rho_t = \mathcal{T}_t \rho_0. \tag{3.30}$$

It remains to determine what kind of semigroups are solutions to equation (3.30) or, equivalently, what kind of generators $\mathcal{L}$ lead to corresponding quantum dynamical semigroups that solve this equation. In 1976, two papers were independently published that gave an answer for certain subclasses of dynamical semigroups.

V. Gorini, A. Kossakowski and G. Sudarshan (GKS) considered systems with finite-dimensional Hilbert spaces [GKS76]. Let M_n denote the algebra of $n \times n$-matrices with entries in $\mathbb{C}$. They found that a linear operator $\mathcal{L} : M_n \to M_n$ generates a quantum dynamical semigroup $\mathcal{T}_t$ on M_n if it is given by

$$\mathcal{L}\rho = -i[H,\rho] + \frac{1}{2}\sum_{k,l=1}^{n^2-1} C_{kl}\left([F_k, \rho F_l^*] + [F_k\rho, F_l^*]\right), \tag{3.31}$$

where H is the self-adjoint Hamiltonian operator of the system, F_k is a basis of M_n with $\operatorname{tr} F_k = 0$ and $\operatorname{tr}(F_k F_l^*) = \delta_{kl}$, and C_{kl} is a complex positive matrix, called the *Kossakowski matrix*.

G. Lindblad (L) looked at systems with infinite-dimensional Hilbert spaces but replaced strong continuity with norm continuity, thus restricting his solutions to bounded generators [Lin76]. He states that in the Heisenberg picture, a linear operator $\mathcal{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the generator of a norm continuous quantum dynamical semigroup $\mathcal{T}_t^*$ if and only if

$$\mathcal{L}^* X = i[H,X] + \sum_i \left(L_i^* X L_i - \frac{1}{2}\{L_i^* L_i, X\}\right), \tag{3.32}$$

where H is a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ and $L_i \in \mathcal{B}(\mathcal{H})$ such that $\sum_i L_i^* L_i$ is bounded. In the Schrödinger picture, this corresponds to the form

$$\mathcal{L}\rho = -i[H,\rho] + \frac{1}{2}\sum_i \left([L_i, \rho L_i^*] + [L_i\rho, L_i^*]\right). \tag{3.33}$$

In this thesis, we will choose a slightly different notation that underlines the intuitive approach in Chapter 4. Furthermore, we will allow the system to lose probability, so we demand $\operatorname{tr} \mathcal{T}_t\rho \leq \operatorname{tr}\rho$ for all $\rho \in \mathfrak{T}(\mathcal{H})$. If equality holds for all $\rho \in \mathfrak{T}(\mathcal{H})$, we call $\mathcal{T}_t$ *conservative*.

Definition 3.5.1. *By a* bounded standard (GKLS-)form *of a generator in the Schrödinger picture, we understand*

$$\mathcal{L}\rho = K\rho + \rho K^* + \sum_i L_i \rho L_i^* \tag{3.34}$$

for all $\rho \in \widetilde{\mathcal{S}}(\mathcal{H})$*, and with bounded operators* $K, L_i \in \mathcal{B}(\mathcal{H})$ *that satisfy*

$$0 \leq K + K^* + \sum_i L_i^* L_i. \tag{3.35}$$

The set of labels i *may be infinite, in which case the sum in* (3.34) *is taken in the weak limit.*

Equivalently, by a bounded standard (GKLS-)form *of a generator in the Heisenberg picture, we understand*

$$\mathcal{L}^* X = K^* X + XK + \sum_i L_i^* X L_i \tag{3.36}$$

for all $X \in \mathcal{B}(\mathcal{H})$ *and with bounded operators* $K, L_i \in \mathcal{B}(\mathcal{H})$ *satisfying equation* (3.35).

We will regain equation (3.33) and for finite dimensions (3.31) if we require equality in (3.35) so that the generated semigroup is trace-preserving and set

$$K = -iH - \frac{1}{2}\sum_i L_i^* L_i. \tag{3.37}$$

3.6 Notes and Remarks

This whole chapter is strongly influenced by [Wer16], [Wer17] and [Osb17]. A good textbook source for the first sections is [HZ12]: Section 3.1 and 3.2 are based on Chapter 2, and Section 3.3 follows Chapter 3. Section 3.4 is also in part taken from [HZ12] (Chapter 4) with additional input from [BL07, Ch. 5], written by Michael Keyl and Reinhard F. Werner.

For Section 3.5 we used multiple sources. [RH12] is a neat summery of the development of the time evolution of open systems and [CP17] gives an interesting insight into the history of the GKLS-form. The results can be found in most textbooks on quantum dynamics, like [BR81] and [AL07]. We also used the two original papers for the GKLS-form, [GKS76] and [Lin76].

Chapter 4

Standard semigroups

In the last chapter, we solved the Markovian master equation for norm continuous quantum dynamical semigroups. Now, we turn to the problem of characterizing all solutions of the Markovian master equation if we assume strong continuity, i.e. all one-parameter semigroups $t \mapsto \mathcal{T}_t$, $(t \geq 0)$ such that each $\mathcal{T}_t$ is a completely positive map on the set of trace class operators $\mathfrak{T}(\mathcal{H})$, and, for any $\rho \in \mathfrak{T}(\mathcal{H})$ and any bounded operator $A \in \mathcal{B}(\mathcal{H})$, we have $\lim_{t\to 0} \operatorname{tr}\big(\mathcal{T}_t(\rho)A\big) = \operatorname{tr}(\rho A)$.

Firstly, we will expand the definition of a standard form (3.34) to strongly continuous semigroups, thereby emphasizing the intuition behind the equation. We start our survey by looking at the pure states in the domain of the generator $\mathcal{L}$ of $\mathcal{T}_t$. At the same time, this will give us a useful definition of the standard form and point to the possibility of non-standard generators.

4.1 Standard generators

The bounded standard (GKLS-) form of the generator established in Definition 3.5.1 can quite conspicuously be separated into a part associated with K, and another which is associated with the operators L_i which we will call *jump operators*. An intuitive way to understand this is the observation that $\exp(t\mathcal{L})$ must be a completely positive map norm close to the identity. This means [KSW07] that it must also have a Stinespring dilation close to that of the identity. Now the only Kraus operator in the decomposition of the identity is the unit operator $\mathbb{1}$, so one of the Kraus operators of $\exp(t\mathcal{L})$ can be chosen to be close to $\mathbb{1}$, say $\approx \mathbb{1} + tK$. The others will then have to scale like $\approx \sqrt{t}L_i$, which gives $\mathcal{T}_t\rho = \rho + t\mathcal{L}(\rho) + \mathbf{O}(t^2)$ with the generator in (3.34). The dominant Kraus operator $(\mathbb{1}+tK)$ belongs to a pure operation, i.e., an operation taking pure states into pure states [Dav76, Sect. 2.3]. The only difference to the unitary case is that this part now typically loses normalization, so the evolution takes pure states to multiples of pure states.

To summarize, the generator splits into one part, which by itself generates an evolution taking pure states to pure states, and a second part, which is completely positive. The work of Davies and the stochastic calculus suggest the following terminology:

Definition 4.1.1. *A* **no-event semigroup** *on a Hilbert space $\mathcal{H}$ is a dynamical semigroup $\mathcal{T}_t^0$, $t > 0$ such that every pure state $\rho = |\psi\rangle\langle\psi|$ is mapped to a multiple of a pure state. It is necessarily of the form $\mathcal{T}_t^0\rho = C_t\rho C_t^*$ with $C_t = \exp(tK)$ a strongly continuous contraction semigroup of Hilbert space operators.*

Note that this definition no longer requires K to be bounded. Moreover, it also makes sense in the discrete classical case. Let $\ell^1(X)$ be the Banach space of absolutely convergent sequences for some countable set X. Pure states δ_x are then of the form concentrated on a single point $x \in X$, corresponding to the probability distribution $\delta_x(y) = \delta_{x,y}$. It is easy to see that a no-event semigroup cannot change x, i.e., it must be of the form

$$\big(\mathcal{T}_t^0\big)(\delta_x) = e^{-t\mu_x}\,\delta_x, \tag{4.1}$$

where $\mu : X \to \mathbb{R}^+$ describes the loss rate from state x. The function μ need not be bounded. Just as in the quantum case, the whole generator will differ from the no-event part by a positive term, which describes the rates of transitions from x to other states y, resulting in the usual rate matrix.

The basic idea of constructing the generator (classical or quantum) is that the positive term in the generator will make the semigroup more nearly conservative, i.e., it will compensate some of the normalization loss in $\mathcal{T}_t^0$. However, due to the overall (sub-)normalization condition $\operatorname{tr}\mathcal{T}_t(\rho) \leq \operatorname{tr}\rho$, there cannot be more transitions than there is loss. This means the positive part must be *bounded with respect to the normalization loss* of the no-event part. Thus, all unboundedness is tamed once it is under control for the no-event part.

Definition 4.1.2. *A dynamical semigroup is called* **standard** *if it is the minimal solution arising from a completely positive perturbation of the generator of a no-event semigroup.*

We have not yet defined "the minimal solution" in this sentence, and this will be the task of Section 4.3. Standard generators look just like (3.34), with the following changes: K is the generator of an arbitrary contraction semigroup on $\mathcal{H}$, and the jump operators need to be operators

$$L_i : \operatorname{dom} K \to \mathcal{H} \quad \text{with} \sum_i \|L_i\phi\|^2 \leq -2\Re e\langle\phi, K\phi\rangle. \tag{4.2}$$

The generator is thus naturally split into $\mathcal{L} = \mathcal{L}^0 + \mathcal{P}$, i.e., no-event part and completely positive perturbation, namely

$$\mathcal{L}^0(|\phi\rangle\langle\psi|) = |K\phi\rangle\langle\psi| + |\phi\rangle\langle K\psi| \quad \text{and} \quad \mathcal{P}(|\phi\rangle\langle\psi|) = \sum_i |L_i\phi\rangle\langle L_i\psi|. \tag{4.3}$$

The natural domain for all these operators is

$$(\operatorname{dom} K)^{)(} = \operatorname{span}\{|\phi\rangle\langle\psi| \,|\, \phi, \psi \in \operatorname{dom K}\}, \tag{4.4}$$

the set of finite linear combinations of rank 1 operators $|\phi\rangle\langle\psi|$ with $\phi, \psi \in \operatorname{dom} K$. Since $\operatorname{dom} K$ is dense in $\mathcal{H}$, $(\operatorname{dom} K)^{)(}$ is dense in $\mathfrak{T}(\mathcal{H})$, and as Lemma 2.1.2 part 2. shows that $\operatorname{dom} K$ is invariant under C_t, we know that $(\operatorname{dom} K)^{)(}$ is invariant under $\mathcal{T}_t^0$. Therefore, $(\operatorname{dom} K)^{)(}$ is a core for $\mathcal{L}^0$ (see Proposition 2.1.4). The expression for

$\mathcal{P}$ on $(\mathrm{dom}\,K)^{\between}$ does explicitly not require the adjoint L_i^* to be even defined, which is important because it might not exist (see Section 4.5.1 below).

So on $(\mathrm{dom}\,K)^{\between}$ the standard generator $\mathcal{L}$ takes on the form

$$\mathcal{L}|\phi\rangle\langle\psi| = |K\phi\rangle\langle\psi| + |\phi\rangle\langle K\psi| + \sum_i |L_i\phi\rangle\langle L_i\psi| \tag{4.5}$$

The effect of the minimal solution construction is then to extend the domain of $\mathcal{L}$ beyond $(\mathrm{dom}\,K)^{\between}$, so that in the end, we may well get some $\rho \in \mathrm{dom}\,\mathcal{L}$, for which the individual terms $\mathcal{L}^0\rho$ and $\mathcal{P}\rho$ are no longer well defined.

We need to take into consideration that the choice of the operators K and L_i in equation (4.3) is not unique since the Kraus decomposition of a completely positive map is not, as it depends on the choice of a basis in the dilation space. Thus, we may transform the jump operators linearly among each other by a unitary matrix without changing the generator. In addition, there is a change of Kraus operators of $\mathcal{T}_t$ for small t, which mixes the $\sqrt{t}L_i$ and $\mathbb{1} + tK$. This is well-known in the bounded case and is sometimes called a change of gauge. We will verify here that it survives mutatis mutandis in the unbounded case.

Lemma 4.1.3. *Let K and L determine a standard generator as in (4.2), and let $\lambda_i \in \mathbb{C}$ with $\sum_i |\lambda_i|^2 < \infty$, and $\beta \in \mathbb{R}$. Then for $\phi \in \mathrm{dom}\,K$ set*

$$L_i'\phi = L_i\phi + \lambda_i\phi \tag{4.6}$$

$$K'\phi = K\phi + \sum_i \overline{\lambda_i}\, L_i\phi + \frac{1}{2}\Big(i\beta + \sum_i |\lambda_i|^2\Big)\phi \tag{4.7}$$

Then the sum in the second term in (4.7) converges in norm. Moreover, K' is a contraction generator with $\mathrm{dom}\,K' = \mathrm{dom}\,K$. The standard generators for (K, L) and (K', L') coincide on $(\mathrm{dom}\,K)^{\between}$ so that they determine the same minimal solution.

Proof. First we show that $\|\sum_i \overline{\lambda}_i L_i\phi\|$ is K-bounded. Using the Cauchy-Schwarz inequality we have, for arbitrary $\psi \in \mathcal{H}$, $\phi \in \mathrm{dom}\,K$, and $\epsilon > 0$

$$\begin{aligned}
\Big|\sum_i \overline{\lambda}_i\langle\psi, L_i\phi\rangle\Big|^2 &\leq \sum_i |\overline{\lambda}_i|^2 \sum_i |\langle\psi, L_i\phi\rangle|^2 \leq A\,\|\psi\|^2 \sum_i \|L_i\phi\|^2 \\
&\leq A\,\|\psi\|^2\, |2\Re e\langle\phi, K\phi\rangle| \leq \|\psi\|^2\, 4\Big(\frac{A}{2\epsilon}\|\phi\|\Big)(\epsilon\|K\phi\|) \\
&\leq \|\psi\|^2\big(\epsilon\|K\phi\| + \frac{A}{2\epsilon}\|\phi\|\big)^2
\end{aligned}$$

where we have introduced the abbreviation $A = \sum_i |\overline{\lambda}_i|^2$, used (4.2) at the second line, and the estimate $4xy \leq (x+y)^2$ at the last. Taking the square root and using that ψ is arbitrary, we get $\|\sum_i \overline{\lambda}_i L_i\phi\| \leq \epsilon\|K\phi\| + (A/(2\epsilon))\|\phi\|$, and, including the last term in (4.7), $\|(K' - K)\phi\| \leq \epsilon\|K\phi\| + C\|\phi\|$, for some constant C. That is, the perturbation is infinitesimally K-bounded. According to [Kat95, Theorem IV.1.1], $\epsilon < 1$ is enough to conclude that K' generates a semigroup with the same domain as K. It remains to show that K' is the generator of a contraction semigroup, i.e. that it is dissipative, which for a Hilbert space operator just means $2\Re e\langle\phi, K'\phi\rangle \leq 0$. For this, we get

$$\begin{aligned} 2\Re e\langle\phi, K'\phi\rangle &= 2\Re e\langle\phi, K\phi\rangle - 2\Re e\sum_i\langle\lambda_i\phi, L_i\phi\rangle - \sum_i\langle\lambda_i\phi, \lambda_i\phi\rangle \\ &= 2\Re e\langle\phi, K\phi\rangle + \sum_i\|L_i\phi\|^2 - \sum_i\|L_i\phi + \alpha_i\phi\|^2. \end{aligned}$$

Then the first two terms together are ≤ 0 because of (4.2), and the third is obviously ≤ 0. The equality of the generator then follows by the same elementary algebra as in the bounded case. □

This *gauging* is all the freedom we have on $(\operatorname{dom} K)^{\mathbb{X}}$ in writing the generator.

4.2 Exit spaces and reinsertions

This section will give a dynamical interpretation of the standard form, which forms the background for the term "no-event" semigroup. This interpretation is consistent also with the unbounded standard form. It provides the basis for the more technical statement that, for a standard generator, all the unboundedness is already determined by the no-event part, relative to which the positive perturbation $\mathcal{P}$ is bounded.

Let us consider a simplified description of a measurement, a quantum system in an environment that contains one or more measurement apparatuses like counters and other absorbing objects like walls. The idea behind the term "no-event semigroup" is that $\mathcal{T}_t^0$ describes the evolution for as long the system has not yet been captured, i.e., up until a detection or "arrival" event [Wer87; Hol95]. We will interpret $0 \leq \operatorname{tr}\mathcal{T}_t^0\rho \leq 1$ as the probability that the system survives at least until time t. A standard way to describe a detection process is to modify a Hamiltonian by absorbing terms $-iH_{abs}$ with $H_{abs} \geq 0$. By choosing H_{abs} to be spatially localized in a region, we get a model of a detector in that region. $K = -iH_{abs}$ is then the generator of a strongly continuous contraction semigroup C_t on the Hilbert space, and the time evolution is of the form described in Definition 4.1.1.

The probability for detection in the time interval $[t, s]$ starting from an initially normalized state ρ is, by definition, $\operatorname{tr}\mathcal{T}_t^0\rho - \operatorname{tr}\mathcal{T}_s^0\rho$, and it allows for the possibility that the particle never arrives. This defines a POVM $\widehat{G}$ on the positive time axis $\mathbb{R}_+$ for the arrival time distribution by

$$\operatorname{tr}\mathcal{T}_t^0\rho - \operatorname{tr}\mathcal{T}_s^0\rho = \operatorname{tr}\rho\,\widehat{G}(t, s). \tag{4.8}$$

$\widehat{G}$ is given by $\widehat{G}(0, t) = 1 - C_t^*C_t$ and it is a covariant observable with respect to C_t, since for any $s > t > 0$ and $r > 0$ we have $C_r^*\widehat{G}(t, s)C_r = \widehat{G}(t + r, s + r)$.

We would also like to describe certain events at the arrival time; for example, when there are several detectors, we need to know which of them fired. Thus, we need to find the observables jointly measurable with the arrival time observable. This is naturally captured by the notion of the *exit space* of a contraction semigroup [Wer87].

For a semigroup $C_t = e^{tK}$ we consider the normalization loss as a quadratic form on $\operatorname{dom} K$,

$$\phi \mapsto -\frac{d}{dt}\|C_t\phi\|^2|_{t=0}. \tag{4.9}$$

It is the probability density for arrival at time $t = 0$ for a system prepared in a pure state ϕ. We then define an exit space for K as a pair $(\mathcal{E}, j)$ of a Hilbert space $\mathcal{E}$ and a linear map $j : \operatorname{dom} K \to \mathcal{E}$ such that, for $\psi, \phi \in \operatorname{dom} K$,

$$\langle j\psi, j\phi\rangle = -\frac{d}{dt}\Big\langle e^{tK}\psi \;\Big|\; e^{tK}\phi\Big\rangle = -\big(\langle K\psi, \phi\rangle + \langle \psi, K\phi\rangle\big). \tag{4.10}$$

There is always a unique minimal exit space: The separated completion of $\operatorname{dom} K$ with respect to the above scalar product. In this case, j is the canonical embedding. However, for reasons that will be apparent later, we also allow non-minimal exit spaces, possibly even with an inequality $\leq$ instead of equality in (4.10).

Now, if $F \in \mathcal{B}(\mathcal{E})$ is an effect operator describing some yes-no-question asked at exit time, we set the probability density for obtaining that result at time t, on an initial preparation $|\phi\rangle\langle\phi|$ with $\phi \in \operatorname{dom} K$, to be $\langle je^{tK}\phi|F|je^{tK}\phi\rangle$. More formally, we consider a map $J : \mathcal{H} \to L^2(\mathbb{R}_+, dt; \mathcal{E})$. The range of J is the space of $\mathcal{E}$-valued functions on $\mathbb{R}_+$, which is canonically isomorphic to $L^2(\mathbb{R}_+, dt) \otimes \mathcal{E}$, but the function notation is more helpful for our purpose. We set

$$(J\phi)(t) = j\big(C_t\phi\big) \in \mathcal{E} \tag{4.11}$$

for $\phi \in \operatorname{dom} K$. Then J extends to $\mathcal{H}$ by continuity, because

$$\begin{aligned} \|J\phi\|^2 &= \int_0^\infty dt\; \|jC_t\phi\|^2_{\mathcal{E}} = -\int_0^\infty dt\; \frac{d}{dt}\|C_t\phi\|^2 \\ &= \|\phi\|^2 - \lim_{t\to\infty} \|C_t\phi\|^2 \leq \|\phi\|^2. \end{aligned} \tag{4.12}$$

The natural time observable on the space $L^2(\mathbb{R}_+, dt)$ is given by multiplication with the characteristic function $\chi_{[t,s]}$. Therefore, we can write the arrival time observable $\widehat{G}$ as

$$\widehat{G}(t, s) = J^*(\chi_{[t,s]} \otimes \mathbb{1})J. \tag{4.13}$$

The joint probability for an F-detection in the time interval $[t, s]$ on the initial state $\rho = |\phi\rangle\langle\phi|$ is then

$$\operatorname{tr}\big(\rho\, J^*(\chi_{[t,s]} \otimes F)J\big) = \int_t^s d\tau\; \langle je^{\tau K}\phi|F|je^{\tau K}\phi\rangle. \tag{4.14}$$

Here the right-hand side uses the density mentioned for $\phi \in \operatorname{dom} K$, but the left-hand side also makes sense for arbitrary ρ by continuous extension.

We can turn the arrival time detection into a dynamical, repeatable process on $\mathcal{H}$ by introducing a *reinsertion map*, which transforms the "state upon exit" into a new state of the system. This is done by a completely positive, trace non-increasing map $\mathcal{S} : \mathfrak{T}(\mathcal{E}) \to \mathfrak{T}(\mathcal{H})$. Then the effect F in (4.14) may arise from a measurement on the original system, including an arrival time measurement of just the same kind.

Since $\mathcal{S}$ is completely positive, we can use the Stinespring dilation and introduce a contraction $V : \mathcal{E} \to \mathfrak{N} \otimes \mathcal{H}$, so that

$$\mathcal{S}(\sigma) = \mathrm{tr}_{\mathfrak{N}} V\sigma V^*. \tag{4.15}$$

Observables on $\mathfrak{N}$ then describe the information that can be extracted at the moment of a jump, so we call $\mathfrak{N}$ the *transit space*. Composing V with j we get a map

$$\tilde{\jmath} = Vj : \mathrm{dom}\, K \to \mathfrak{N} \otimes \mathcal{H}, \tag{4.16}$$

which, apart from the special form of the image space, satisfies exactly the requirements (4.10) for an exit space (possibly with an inequality, if $\mathcal{S}$ can reduce the trace). In this sense, a process of exit and reinsertion is completely specified by an exit space of the special form $(\mathfrak{N} \otimes \mathcal{H}, \tilde{\jmath})$.

From now on, we will take J to be defined by $\tilde{\jmath}$. We can iterate this operator to a sequence of maps

$$J^{(n)} : \mathcal{H} \to \left(L^2(\mathbb{R}_+, dt; \mathfrak{N})\right)^{\otimes n} \otimes \mathcal{H}, \tag{4.17}$$

with $J^{(0)} = \mathbb{1}_{\mathcal{H}}$, $J^{(1)} = J$, and $J^{(n+1)} = (\mathbb{1}^{\otimes n} \otimes J)J^{(n)}$. This has the same interpretation as J, only that we are now looking at n consecutive events. The n time arguments of wave functions in this space, the elements of the factors $L^2(\mathbb{R}_+, dt; \mathfrak{N})$, are the time increments between successive events. In order to get a dynamical semigroup out of this iteration, we need to fix a time interval $[0, \tau]$ and look only at events happening during this interval. We also need to evolve the system up to time τ after the last event with a further application of the no-event semigroup. Thus we set $J_\tau^{(n)}$ to be a map between the same spaces as $J^{(n)}$, but modified as

$$(J_\tau^{(n)}\phi)(t_1, \ldots, t_n) = (\mathbb{1}^{\otimes n} \otimes C_{\tau - \sum_i t_i})(J^{(n)}\phi)(t_1, \ldots, t_n), \tag{4.18}$$

whenever $\sum_i t_i \leq \tau$, and zero otherwise. So $J_\tau^{(n)}$ is a dilation of the evolution conditional on exactly n events happening in that interval. The conditional evolution up to the end of this interval is

$$\mathcal{T}_\tau^{(n)}\rho = \mathrm{tr}_{\mathrm{events}}\, J_\tau^{(n)}\rho J_\tau^{(n)\,*}, \tag{4.19}$$

where the trace is the partial trace over the tensor factor $\left(L^2(\mathbb{R}_+, dt; \mathcal{E})\right)^{\otimes n}$. Then

$$\mathcal{T}_\tau\rho = \sum_{n=0}^{\infty} \mathcal{T}_\tau^{(n)} \tag{4.20}$$

is a dynamical semigroup. In fact, it is the same minimal semigroup as constructed in the next section. We will not go through the proof of this assertion, which is best done via the Laplace transforms of the $\mathcal{T}_\tau^{(n)}$, which turn out to be exactly the terms in the sequence (4.25) below.

Experts in stochastic calculus will easily recognize the dilation construction here. In fact, when we write the time arguments in the space $\left(L^2(\mathbb{R}_+, dt; \mathcal{E})\right)^{\otimes n}$ not as increments but as the absolute event times $\tau_i = \sum_{k=1}^{i} t_i$, we get wave functions defined on ordered time arguments, which have unique symmetric and antisymmetric extensions to arbitrary n tuples of times, yielding the Fermionic and the Bosonic stochastic integrals. However, our focus here was just the dynamical semigroup, specifically to trace the implications of unboundedness through the construction. Indeed, the key point is (4.12): Once J has been extended from $\operatorname{dom} K$ to a bounded operator on all of $\mathcal{H}$, the entire further construction is in terms of bounded operators, and no more domain questions need to be addressed.

The exit & reinsertion picture suggests other standard ways to look at the generator, which are brought together with the form (3.34) and (4.2) in the following proposition. It also lists (in (d)) the form we prefer for the next section. For the action of the exit space injection j on mixed states we introduce the linear operator $j^{\rangle\langle} : (\operatorname{dom} K)^{\rangle\langle} \to \mathfrak{T}(\mathcal{E})$ given by

$$j^{\rangle\langle}(|\phi\rangle\langle\psi|) = |j\phi\rangle\langle j\psi|. \tag{4.21}$$

Then we have:

Proposition 4.2.1. *Let $t \mapsto \exp(tK)$ be a contraction semigroup on $\mathcal{H}$ with generator K and minimal exit space $(\mathcal{E}, j)$. Then standard generators with no-event semigroup $\mathcal{T}_t^0\rho = e^{tK}\rho e^{tK^*}$ are equivalently characterized by any of the following sets:*

- (a) *Completely positive "reinsertion" maps $\mathcal{S} : \mathfrak{T}(\mathcal{E}) \to \mathfrak{T}(\mathcal{H})$ with $\operatorname{tr}\mathcal{S}(\sigma) \leq \operatorname{tr}\sigma$.*
- (b) *Non-minimal exit spaces of product form, i.e., maps $\tilde{\jmath} : \operatorname{dom} K \to \mathfrak{N} \otimes \mathcal{H}$ such that $\|\tilde{\jmath}\phi\|^2 \leq 2\Re e\langle\phi, K\phi\rangle$.*
- (c) *Maps $\mathcal{P} : (\operatorname{dom} K)^{\rangle\langle} \to \mathfrak{T}(\mathcal{H})$, which can be written in the form (4.3) with jump operators L_i satisfying (4.2).*
- (d) *Completely positive maps $\mathcal{P} : \operatorname{dom}\mathcal{L}^0 \to \mathfrak{T}(\mathcal{H})$, with $\operatorname{tr}\mathcal{P}\rho \leq -\operatorname{tr}\mathcal{L}^0\rho$ for all positive $\rho \in \operatorname{dom}\mathcal{L}^0$.*

The correspondence is given by restriction from (d) to (c) and by unique $\mathcal{L}^0$-graph-norm continuous extension in the other direction. Between (a),(b),(c) it is given on $(\operatorname{dom} K)^{\rangle\langle}$ by $\mathcal{P} = \mathcal{S}j^{\rangle\langle} = (\mathcal{I} \otimes \operatorname{tr}_{\mathfrak{N}})\tilde{\jmath}^{\rangle\langle}$. Possible choices of jump operators correspond precisely to choices of Kraus operators for $\mathcal{S}$ or a basis $e_i \in \mathfrak{N}$, with $\mathcal{S}(\sigma) = \sum_i M_i\sigma M_i^*$, via $L_i = M_i j$ and $\tilde{\jmath}\phi = \sum_i e_i \otimes L_i\phi$.

Proof. The equivalences are largely trivial to verify on $(\operatorname{dom} K)^{\rangle\langle}$ or have already been described in the text above. The only statement not of this kind is the continuous extension $(c)\to(d)$. $(\operatorname{dom} K)^{\rangle\langle} \subset \operatorname{dom}\mathcal{L}^0$ is a core, so continuity will guarantee an extension to $\operatorname{dom}\mathcal{L}^0$. Since $\mathcal{S}$ is clearly trace norm continuous, the identity $\mathcal{P} = \mathcal{S}j^{\rangle\langle}$ shows that we only need to prove the continuity of $j^{\rangle\langle}$, i.e., the statement that $j^{\rangle\langle}\rho_n \to 0$, whenever $\rho_n \to 0$ and $\mathcal{L}^0\rho_n \to 0$ (each limit in trace norm). We will do this by establishing the estimate $\|j^{\rangle\langle}\rho\| \leq \|\mathcal{L}^0\rho\|$. By definition of $(\operatorname{dom} K)^{\rangle\langle}$, we can write $\rho = \sum_\ell^N r_\ell|\phi_\ell\rangle\langle\psi_\ell|$ with $r_\ell \in \mathbb{C}$ and $\phi_\ell, \psi_\ell \in \operatorname{dom} K$. Now on the finite-dimensional span of the ϕ_ℓ, ψ_ℓ we can perform a singular value decomposition and get a more canonical form of ρ, where $r_\ell > 0$, and each of the families $\{\phi_\ell\}, \{\psi_\ell\}$ is orthonormal.

Then we have

$$
\begin{aligned}
\|j^{\chi}\rho\| &= \Big\|\sum_\ell r_\ell |j\phi_\ell\rangle\langle j\psi_\ell|\Big\| \le \sum_\ell r_\ell \|j\phi_\ell\|\|j\psi_\ell\| \\
&\le \sum_\ell \frac{r_\ell}{2}\big(\|j\phi_\ell\|^2 + \|j\psi_\ell\|^2\big) = -\Re e \sum_\ell r_\ell\big(\langle\phi_\ell, K\phi_\ell\rangle + \langle K\psi_\ell, \psi_\ell\rangle\big) \\
&= -\Re e \sum_{\ell,m} r_\ell\big(\langle\phi_m, K\phi_\ell\rangle\langle\psi_\ell,\psi_m\rangle + \langle\phi_m,\phi_\ell\rangle\langle K\psi_\ell,\psi_m\rangle\big) \\
&= -\Re e\,\mathrm{tr}\, W\mathcal{L}^0\rho, \qquad (4.22)
\end{aligned}
$$

where $W = \sum_m |\psi_m\rangle\langle\phi_m|$. This is a partial isometry, so $\|W\| = 1$, and hence $\|j^{\chi}\rho\| \le \|\mathcal{L}^0\rho\|$. □

4.3 The minimal solution

Adding a further term ("a perturbation") to a well-known "simple" generator is, of course, commonplace throughout quantum mechanics and more general evolution equations. Very often, one considers perturbations relatively bounded with respect to the given generator. In this case [RS80], the domain of the perturbed generator remains the same. The perturbations considered here will usually *not* be of this kind. There are two equivalent versions of the construction. One is based on the resolvent series [Dav77], and one on the iteration of integral equations [Hol95]. Since the resolvent version can be stated slightly more compactly, and we will need to consider resolvents anyhow, we will choose this version. Now consider a generator $\mathcal{L}^0$, typically (but not necessarily) of a no-event semigroup, from which we would like to construct a new generator $\mathcal{L} = \mathcal{L}^0 + \mathcal{P}$ with $\mathcal{P}$ completely positive. For the construction of standard generators, the forms of $\mathcal{L}^0$ and $\mathcal{P}$ are given in (4.3). The domain of $\mathcal{L}$ should be at least $\mathrm{dom}\,\mathcal{L}^0$, and we want the normalization of the new semigroup to be non-increasing. This fixes the normalization condition (4.2). Moreover, for $\rho \ge 0$,

$$0 \ge \mathrm{tr}(\mathcal{L}^0 + \mathcal{P})\mathcal{R}^0_\lambda\rho = \mathrm{tr}\big(\lambda\mathcal{R}^0_\lambda\rho - \rho + \mathcal{P}\mathcal{R}^0_\lambda\rho\big) \ge \mathrm{tr}\,\mathcal{P}\mathcal{R}^0_\lambda\rho - \mathrm{tr}\,\rho. \qquad (4.23)$$

Hence, $\mathcal{P}\mathcal{R}^0_\lambda$ is everywhere defined, completely positive, and trace non-increasing. Therefore, $\|\mathcal{P}\mathcal{R}^0_\lambda\| \le 1$. Formally, we get the resolvent $\mathcal{R}_\lambda$ of the perturbed semigroup from

$$\mathcal{R}_\lambda - \mathcal{R}^0_\lambda = \mathcal{R}_\lambda\Big((\lambda - \mathcal{L}^0) - (\lambda - \mathcal{L})\Big)\mathcal{R}^0_\lambda = \mathcal{R}_\lambda\mathcal{P}\mathcal{R}^0_\lambda. \qquad (4.24)$$

Still proceeding formally, we can use this to determine $\mathcal{R}_\lambda$ by iteration, or equivalently to solve the Neumann series for $(\mathcal{I} - \mathcal{P}\mathcal{R}^0_\lambda)^{-1}$ to find:

$$\mathcal{R}_\lambda = \sum_{n=0}^{\infty} \mathcal{R}^0_\lambda\big(\mathcal{P}\mathcal{R}^0_\lambda\big)^n. \qquad (4.25)$$

The basic algebra here is quite standard and is also used for the relatively bounded perturbation theory of generators. In that case, $\|\mathcal{P}\mathcal{R}^0_\lambda\| < 1$, so the series obviously converges in norm. Moreover, one can then write the factor $\mathcal{R}^0_\lambda$ outside the sum so that $\mathrm{dom}\,\mathcal{L} = \mathcal{R}_\lambda(\mathfrak{T}(\mathcal{H})) \subset \mathcal{R}^0_\lambda(\mathfrak{T}(\mathcal{H})) = \mathrm{dom}\,\mathcal{L}^0$, and the domain will not increase. This will be different now. We state the basic construction result without assuming

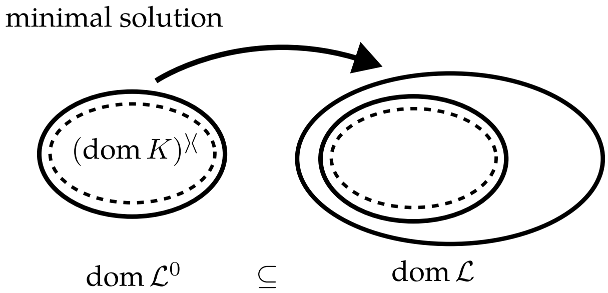

Figure 4.1: Generators and their domains in the construction of a standard generator $\mathcal{L}$.

that $\mathcal{L}^0$ is a no-event semigroup. This generalization will be needed in Section 4.6. For use in that section, we also provide Lemma 4.3.2, showing that sometimes the domain does not increase.

Proposition 4.3.1. *Let $\mathcal{L}^0$ be the generator of a dynamical semigroup, and let $\mathcal{P} : \mathrm{dom}\,\mathcal{L}^0 \to \mathfrak{T}(\mathcal{H})$ be a completely positive map such that, for $0 \le \rho \in \mathrm{dom}\,\mathcal{L}^0$,*

$$\mathrm{tr}\,\mathcal{P}(\rho) \le -\,\mathrm{tr}\,\mathcal{L}^0(\rho). \tag{4.26}$$

Then $\mathcal{P}\mathcal{R}^0_\lambda$ is a completely positive operator on $\mathfrak{T}(\mathcal{H})$, and the series (4.25) converges strongly to the resolvent $\mathcal{R}_\lambda$ of a dynamical semigroup. $\mathcal{X} = \mathcal{R}_\lambda$ is the smallest completely positive solution of the equation $\mathcal{X} = \mathcal{R}^0_\lambda + \mathcal{X}\mathcal{P}\mathcal{R}^0_\lambda$ in completely positive ordering, and is hence called the minimal resolvent solution *associated with the perturbation $\mathcal{P}$.*

Proof. We only sketch the key idea, which makes clear why the series indeed converges, even without assuming $\|\mathcal{P}\mathcal{R}^0_\lambda\| < 1$. The the partial sum truncated at n is just the n^{th} iterate $\mathcal{R}^{(n)}_\lambda$ defined by $\mathcal{R}^{(0)}_\lambda = \mathcal{R}^0_\lambda$ and

$$\mathcal{R}^{(n+1)}_\lambda = \mathcal{R}^0_\lambda + \mathcal{R}^{(n)}_\lambda \mathcal{P}\mathcal{R}^0_\lambda. \tag{4.27}$$

We will prove by induction that for positive ρ, we have $\mathrm{tr}\,\lambda\mathcal{R}^{(n)}_\lambda\rho \le \mathrm{tr}\,\rho$. Indeed, this is true for $n = 0$, like for the resolvent of any dynamical semigroup and, by the induction hypothesis,

$$\begin{aligned}\mathrm{tr}\,\lambda\mathcal{R}^{(n+1)}_\lambda\rho &\le \mathrm{tr}\,\lambda\mathcal{R}^0_\lambda\rho + \mathrm{tr}\,\mathcal{P}\mathcal{R}^0_\lambda\rho \\ &\le \mathrm{tr}\,\lambda\mathcal{R}^0_\lambda\rho - \mathrm{tr}\,\mathcal{L}\mathcal{R}^0_\lambda\rho = \mathrm{tr}\,\lambda\mathcal{R}^0_\lambda\rho - \mathrm{tr}\big(\lambda\mathcal{R}^0_\lambda\rho - \rho\big) = \mathrm{tr}\,\rho.\end{aligned}$$

Hence, the sequence $\lambda\mathcal{R}^{(n)}_\lambda\rho$ is increasing and uniformly bounded in trace norm, and therefore convergent in norm. By linearity, this extends to the trace class, and applying it to a matrix of trace class operators, we conclude that the limit $\mathcal{R}_\lambda$ is a completely positive operator. If $\mathcal{S}$ is any completely positive solution of the equation in the proposition, we have that $(\mathcal{S} - \mathcal{R}^{(0)}_\lambda) = (\mathcal{S} - \mathcal{R}^0_\lambda)$ is completely positive, and because

$$(\mathcal{S} - \mathcal{R}^{(n+1)}_\lambda) = (\mathcal{S} - \mathcal{R}^{(n)}_\lambda)\mathcal{P}\mathcal{R}_\lambda \tag{4.28}$$

this persists through iteration, and the result follows by taking the limit. □

The semigroup that corresponds to this minimal solution is larger than the no-event semigroup in completely positive ordering, i.e.,

$$\mathcal{T}_t - \mathcal{T}_t^0 \geq 0, \tag{4.29}$$

as was shown by Holevo in [Hol95, Prop. 3].

Lemma 4.3.2. *If, in the setting Proposition 4.3.1, the perturbation $\mathcal{P}$ has finite rank, we have* $\operatorname{dom}\mathcal{L} = \operatorname{dom}\mathcal{L}^0$.

Proof. We will show that, for some n, $\|(\mathcal{P}\mathcal{R}_\lambda^0)^n\| < 1$. Then the resolvent series (4.25) converges in norm, even without the factor $\mathcal{R}_\lambda^0$ in each term, so as argued after that equation, the domain will not increase. By definition, a finite rank operator and its adjoint can be written as

$$\mathcal{P}\mathcal{R}_\lambda^0\rho = \sum_i \sigma_i \operatorname{tr}(S_i\rho) \quad \text{and } (\mathcal{P}\mathcal{R}_\lambda^0)^* X = \sum_i S_i \operatorname{tr}(\sigma_i X), \tag{4.30}$$

where the sum is finite and the $\sigma_i \in \mathfrak{T}(\mathcal{H})$ and the $S_i \in \mathcal{B}(\mathcal{H})$ are chosen linearly independent. The action on the linear span of the σ_i is given by the finite-dimensional matrix $P_{ij} = \operatorname{tr} S_i\sigma_j$ in the sense that $\mathcal{P}\mathcal{R}_\lambda^0 \sum_j x_j\sigma_j = \sum_i(\sum_j P_{ij}x_j)\sigma_i$. Because $\|\mathcal{P}\mathcal{R}_\lambda^0\| \leq 1$, all the eigenvalues of the matrix P must be in the unit circle. If there are no eigenvalues of modulus one, the powers of P and hence of $\mathcal{P}\mathcal{R}_\lambda^0$ contract exponentially to zero, and we are done. Now, suppose P has an eigenvalue of modulus one. Then so does its transpose, and we hence have an operator X with $(\mathcal{P}\mathcal{R}_\lambda^0)^* X = \omega X$ with $|\omega| = 1$. Then 2-positivity implies

$$(\mathcal{P}\mathcal{R}_\lambda^0)^*(X^*X) \geq (\mathcal{P}\mathcal{R}_\lambda^0)^*(X)^*(\mathcal{P}\mathcal{R}_\lambda^0)^*(X) = X^*X. \tag{4.31}$$

Hence, iterating $(\mathcal{P}\mathcal{R}_\lambda^0)^*$ on X^*X gives an increasing sequence, which is, however, bounded by $\|X^*X\|\mathbb{1}$, because $\|(\mathcal{P}\mathcal{R}_\lambda^0)^*\| \leq 1$. Hence, this sequence must have a weak limit, and because $(\mathcal{P}\mathcal{R}_\lambda^0)^*$ is normal, this limit is a fixed point. Therefore, P and its transpose, and consequently $(\mathcal{P}\mathcal{R}_\lambda^0)$ must have a nonzero fixed point σ. But then the resolvent series for $\mathcal{R}_\lambda\sigma$ has all equal terms and hence diverges, contradicting the trace estimate in the proof of Proposition 4.3.1. □

4.4 Strictly and strongly standard generators

In Proposition 4.3.1, we explicitly allowed non-equality in equation (4.26). In most literature, the minimal solution construction demands $\operatorname{tr}\mathcal{L}\rho = 0$ for all $0 \leq \rho \in \operatorname{dom}\mathcal{L}^0$ (see for example [Dav77]). For the following examples and results, we will often require this additional condition for standard semigroups so that we will introduce the following definition.

Definition 4.4.1. *A dynamical semigroup is called* strictly standard *if it is the minimal solution arising from a completely positive perturbation $\mathcal{P}$ of the generator $\mathcal{L}^0$ of a no-event semigroup and if it additionally satisfies*

$$\operatorname{tr}\mathcal{P}(\rho) = -\operatorname{tr}\mathcal{L}^0(\rho) \tag{4.32}$$

for all $0 \leq \rho \in \operatorname{dom}\mathcal{L}^0$.

A question that naturally arises is whether $(\mathrm{dom}\, K)^{\between}$ is a core for the generator $\mathcal{L}$ of the minimal solution. The answer is quite simple if we look at strictly standard semigroups.

Proposition 4.4.2. *A strictly standard semigroup $\mathcal{T}_t$ is conservative if and only if $(\mathrm{dom}\, K)^{\between}$ is a core for its generator $\mathcal{L}$.*

Proof. First, let $\mathcal{T}_t$ be conservative. From the integral (2.19), we see that this is equivalent to

$$\mathrm{tr}\, \mathcal{T}_t \rho = \mathrm{tr}\, \lambda \mathcal{R}_\lambda \rho = \mathrm{tr}\, \rho \tag{4.33}$$

for all $\rho \in \mathfrak{T}(\mathcal{H})$ and

$$\mathrm{tr}(\mathcal{L}^0 + \mathcal{P})\mathcal{R}^0_\lambda \rho = \mathrm{tr}\big(\lambda \mathcal{R}^0_\lambda \rho - \rho + \mathcal{P}\mathcal{R}^0_\lambda \rho\big) = 0. \tag{4.34}$$

By induction, this equation can be generalized for all $\rho \in \mathfrak{T}(\mathcal{H})$ to

$$\mathrm{tr}\left(\lambda \sum_{k=0}^{n} \mathcal{R}^0_\lambda (\mathcal{P}\mathcal{R}^0_\lambda)^k \rho + (\mathcal{P}\mathcal{R}^0_\lambda)^{n+1}\rho - \rho\right) = 0. \tag{4.35}$$

For $k = 0$, this is true by equation (4.34). If it is satisfied for n, then we can choose $\rho = (\mathcal{P}\mathcal{R}^0_\lambda)\mu$ and get

$$\mathrm{tr}\left(\lambda \sum_{k=1}^{n+1} \mathcal{R}^0_\lambda (\mathcal{P}\mathcal{R}^0_\lambda)^k \mu + (\mathcal{P}\mathcal{R}^0_\lambda)^{n+2}\mu - (\mathcal{P}\mathcal{R}^0_\lambda)\mu\right) = 0, \tag{4.36}$$

thus proving equation (4.35) for $n + 1$.

By Proposition 2.2.5, $(\mathrm{dom}\, K)^{\between}$ is a core for $\mathcal{L}$, if the orthogonal complement of $(\lambda - \mathcal{L})((\mathrm{dom}\, K)^{\between})$ in $\mathcal{B}(\mathcal{H})$ is trivial, i.e. there is no nonzero $X \in \mathcal{B}(\mathcal{H})$ that satisfies

$$\mathrm{tr}(\lambda - \mathcal{L})\rho X = 0 \tag{4.37}$$

for all $\rho \in (\mathrm{dom}\, K)^{\between}$. So, let us assume $(\mathrm{dom}\, K)^{\between}$ is not a core, and there exists a nonzero $X \in \mathcal{B}(\mathcal{H})$ in the orthogonal complement. Since $(\mathrm{dom}\, K)^{\between}$ is a core for $\mathcal{L}^0$, we can set $\rho = \mathcal{R}^0_\lambda \mu$ and get

$$\mathrm{tr}(\mathcal{L}^0 + \mathcal{P})\mathcal{R}^0_\lambda \mu X = \mathrm{tr}\big(\lambda \mathcal{R}^0_\lambda \mu X - \mu X + \mathcal{P}\mathcal{R}^0_\lambda \mu X\big) = \mathrm{tr}\, \lambda \mathcal{R}^0_\lambda \mu X. \tag{4.38}$$

Thus, equation (4.37) simplifies to $\mathrm{tr}\, \mu X = \mathrm{tr}(\mathcal{P}\mathcal{R}^0_\lambda)\mu X$ and in the Heisenberg picture this reads

$$(\mathcal{P}\mathcal{R}^0_\lambda)^* X = X. \tag{4.39}$$

If X is a fixpoint of $(\mathcal{P}\mathcal{R}^0_\lambda)^*$, this is also the case for X^*. We apply $(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}$ to the inequalities $-2\|X\|\mathbb{1} \le X + X^* \le 2\|X\|\mathbb{1}$ and $-2\|X\|\mathbb{1} \le i(X - X^*) \le 2\|X\|\mathbb{1}$ and have

$$-2\|X\|(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1}) \le X + X^* \le 2\|X\|(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1}) \tag{4.40}$$

$$-2\|X\|(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1}) \le i(X - X^*) \le 2\|X\|(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1}). \tag{4.41}$$

The sequence of postive operators $(\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1})$ is positive and decreasing, since equation (4.35) gives

$$\operatorname{tr}\rho\left((\mathcal{P}\mathcal{R}^0_\lambda)^{*n}(\mathbb{1}) - (\mathcal{P}\mathcal{R}^0_\lambda)^{*n+1}(\mathbb{1})\right) = \operatorname{tr}\left(\lambda\mathcal{R}^0_\lambda(\mathcal{P}\mathcal{R}^0_\lambda)^n\rho\right). \tag{4.42}$$

Thus, it converges strongly to a positive operator Y. Taking the limit $n \to \infty$ in equation (4.35), we get

$$\operatorname{tr}(\lambda\mathcal{R}_\lambda\rho + \rho Y) = \operatorname{tr}\rho. \tag{4.43}$$

Comparing with (4.33), we conclude $Y = 0$, contradictory to equations (4.40). We conclude that $(\operatorname{dom}K)^{\between}$ is a core for $\mathcal{L}$.

On the other hand, $\mathcal{T}_t$ is strictly standard, so $\operatorname{tr}\mathcal{L}\rho = 0$ for $\rho \in (\operatorname{dom}K)^{\between}$, and if this is a core for gen, then $\operatorname{tr}\mathcal{L}\rho = 0$ for all $\rho \in \operatorname{dom}\mathcal{L}$. This implies

$$\frac{d}{dt}\operatorname{tr}\mathcal{T}_t\rho = \operatorname{tr}\mathcal{L}\mathcal{T}_t\rho = 0 \tag{4.44}$$

for all $t \geq 0$ and $\operatorname{tr}\mathcal{T}_t\rho = \operatorname{tr}\rho$. $\operatorname{dom}\mathcal{L}$ in dense in $\mathfrak{T}(\mathcal{H})$, so this holds for all $\rho \in \mathfrak{T}(\mathcal{H})$ and $\mathcal{T}_t$ is conservative. □

We emphasize that Proposition 4.4.2 also states that there are strictly standard and, therefore, standard semigroups in general, for which $(\operatorname{dom}K)^{\between}$ is not a core for $\mathcal{L}$ (for example, those that are strictly standard but lose normalization), and in Section 4.6 we will see that this is a key feature in our construction of non-standard generators.

Following [Hol19], we introduce another subclass of standard semigroups.

Definition 4.4.3. *A dynamical semigroup is called* strongly standard *if it is the minimal solution arising from a completely positive perturbation $\mathcal{P}$ of the generator $\mathcal{L}^0$ of a no-event semigroup and if additionally, each L_i is closable with* $\operatorname{dom}K^* \subset \operatorname{dom}L_i^*$ *and* $\sum_i \|L_i^* f\|^2 < \infty$ *for* $f \in \operatorname{dom}K^*$.

This condition on standard semigroups, first introduced in [Hol96b], is related to whether the semigroup on the trace class is the dual of a semigroup on the compact operators. We show in Section 4.5.1 that it may be violated. On the other hand, it is quite easy to verify in the two main examples of this chapter.

Proposition 4.4.4. *Let $\mathcal{T}_t$ be a strongly standard semigroup with $\mathcal{L}$ and $\mathcal{L}^0$ as in* (4.3) *and* (4.5)*. Then $\mathcal{T}_t$ is the dual of a semigroup on the compact operators.*

Proof. The formal generator on $(\operatorname{dom}K^*)^{\between}$ is given by

$$\mathcal{L}_*(|f\rangle\langle g|) = |K^*f\rangle\langle g| + |f\rangle\langle K^*g| + \sum_i |L_i^*f\rangle\langle L_i^*g|, \tag{4.45}$$

and it maps $|f\rangle\langle g| \in (\operatorname{dom}K^*)^{\between}$ to a compact operator since the last term converges in norm

$$\|\sum_j |L_j^*f\rangle\langle L_j^*g|\| \leq \sum_j \|L_j^*f\|\cdot\|L_j^*g\| \leq \left(\sum_j \|L_j^*f\|^2\right)^{\frac{1}{2}} \cdot \left(\sum_j \|L_j^*g\|^2\right)^{\frac{1}{2}}. \tag{4.46}$$

So we have $\operatorname{tr} \rho\mathcal{L}_*(X) = \operatorname{tr} \mathcal{L}(\rho)X$ for $\rho \in (\operatorname{dom} K)^{\between}$ and $X \in (\operatorname{dom} K^*)^{\between}$. Now, let X_n be a sequence in $(\operatorname{dom} K^*)^{\between}$ that converges in norm to X such that $L_*(X_n) \to Y$ in norm. Then Y is a compact operator. Henceforth, we will denote this closure of the formal generator with $\mathcal{L}_*$.

It remains to show that $\mathcal{L}_*$ generates a semigroup $\mathcal{T}_t^*$ that maps compact operators to compact operators. The graph of the resolvent $\mathcal{R}_{*\lambda}$ of $\mathcal{L}_*$ is given by

$$\mathcal{G} = \{(X, \mathcal{R}_{*\lambda}(X))|X \in \mathfrak{K}(\mathcal{H})\} = \{(\lambda Y - \mathcal{L}_*Y, Y)|Y \in \operatorname{dom} \mathcal{L}_*\}. \tag{4.47}$$

We can introduce its "dual graph", given by elements $\{\rho_1, \rho_2\} \in \mathcal{T}(\mathcal{H})^2$, such that $\operatorname{tr} \rho_2 X = \operatorname{tr} \rho_1 \mathcal{R}_{*\lambda(X)}$ and we get

$$\begin{aligned} \mathcal{G}^\perp &= \{(\rho_1, \rho_2) \in \mathcal{T}(\mathcal{H})^2 | \operatorname{tr} \rho_1 X_2 = \operatorname{tr} \rho_2 X_1 \text{ for } (X_1, X_2) \in \mathcal{G}\} & (4.48) \\ &= \{(\rho_1, \rho_2) \in \mathcal{T}(\mathcal{H})^2 | \rho_1 = (\lambda - \mathcal{L})\rho_2\}. & (4.49) \end{aligned}$$

This is the graph of the resolvent of $\mathcal{T}_t$ on $\mathfrak{T}(\mathcal{H})$. Analogously, we can compute $\mathcal{G}^{\perp\perp} \subset \mathcal{B}(\mathcal{H})^2$. It is easy to see that $\mathcal{G}^\perp \subset \mathcal{G}^{\perp\perp}$, so the semigroup $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H})$ maps compact operators to compact operators. Thus, $\mathcal{T}$ is the dual of a semigroup on the compact operators, namely of the restriction of $\mathcal{T}_t^*$ to $\mathfrak{K}(\mathcal{H})$. □

With Definition 4.4.3, we can reach another key result for constructing non-standard generators.

Proposition 4.4.5. *Let $\mathcal{T}_t$ be a strongly standard dynamical semigroup. Then $|\phi\rangle\langle\psi| \in \operatorname{dom} \mathcal{L}$ for some $\phi, \psi \in \mathcal{H}$ implies that $\phi, \psi \in \operatorname{dom} K$.*

Proof. Following Theorem A.2 in [Hol96b] (see also Section 4.7), the semigroup satisfies the so called "forward master equation" with the generator

$$\langle f, (\mathcal{L}\omega)g\rangle = \langle K^* f, \omega g\rangle + \langle f, \omega K^* g\rangle + \sum_i \langle L_i^* f, \omega L_i^* g\rangle \tag{4.50}$$

for $\omega \in \operatorname{dom} \mathcal{L}$. Now let $\omega = |\phi\rangle\langle\psi|$ with ϕ, ψ not necessarily in $\operatorname{dom} K$, and pick a vector $g \in \operatorname{dom} K^*$ such that $\langle\psi, g\rangle = 1$. This is possible because $\operatorname{dom} K^*$ is dense. Applying Lemma 4.1.3 with $\lambda_i = -\overline{\langle\psi, L_i^* g\rangle}$ leads to an equivalent form of the generator, for which, however, $\langle\psi, L_i^* g\rangle = 0$. Therefore, (4.50) simplifies to

$$\langle f, (\mathcal{L}\omega)g\rangle = \langle K^* f, \phi\rangle\,\langle\psi, g\rangle + \langle f, \phi\rangle\langle\psi, K^* g\rangle. \tag{4.51}$$

Solving for the first term on the right, using $\langle\psi, g\rangle = 1$, we find

$$\langle K^* f, \phi\rangle = \Big\langle f \,\Big|\, \mathcal{L}(\omega)g - \phi\langle\psi, K^* g\rangle \Big\rangle. \tag{4.52}$$

Therefore $\phi \in \operatorname{dom} K^{**} = \operatorname{dom} K$, and $K\phi = \mathcal{L}(\omega)g - \phi\langle\psi, K^* g\rangle$. By the same argument applied to the hermitian conjugates, we get $\psi \in \operatorname{dom} K$. □

The conditions of strict and strong standardness are not even necessarily satisfied if the semigroup is norm continuous. In this case, however, a semigroup is strictly standard simply if it is conservative, as no domain issues arise.

4.5 Examples of standard generators

4.5.1 Non-closable jump operators

A fundamental example of a contraction semigroup with an unbounded generator is the half-sided shift on $\mathcal{H} = L^2(\mathbb{R}^+, dx)$, given by

$$\big(S_t\psi\big)(x) = \psi(x+t). \tag{4.53}$$

Its generator K is differentiation, so $\operatorname{dom} K$ consists of functions that have an L^2-derivative. In particular, they are continuous; hence, for $\psi \in \operatorname{dom} K$, the boundary value $\psi(0)$ is well-defined. This directly determines the exit space $\mathcal{E} = \mathbb{C}$ with $j\psi = \psi(0)$. Indeed,

$$-\frac{d}{dt}\,\langle S_t\psi, S_t\phi\rangle|_{t=0} = -\frac{d}{dt}\int_t^\infty dx\;\overline{\psi(x)}\phi(x) = \overline{\psi(0)}\phi(0) = \langle j\psi, j\phi\rangle. \tag{4.54}$$

Hence, the standard generators with no-event semigroup implemented by S are parameterized by the cp map taking a one-dimensional system on exit $\mathcal{E}$ to the system Hilbert space, i.e., by a state $\Omega \in \mathfrak{T}(\mathcal{H})$. The intuitive picture is that whenever the system hits the boundary, it is reset to the "rebound" state Ω. The number of jump operators needed here depends on the mixedness of the rebound state Ω. When $\Omega = \sum_i |\phi_i\rangle\langle\phi_i|$ is the spectral resolution (ϕ_i orthogonal but not normalized), we can set $L_i : \mathcal{E} \to \mathcal{H}$ to be $L_i z = z\phi_i$. As operators on Hilbert space, these jump operators are very ill-behaved. Formally, they would come out as $L_i = |\phi_i\rangle\langle\delta|$, where δ is the Dirac-δ at the origin. This L_i is not a closable operator, intuitively, because the value of a general L^2-function at a point is an ill-defined notion. More formally, we can find a sequence $\psi_n \in \operatorname{dom} K = \operatorname{dom} L_i$ such that $\|\psi_n\| \to 0$, but $\psi_n(0) = 42$. Then $L_i\psi_n = 42\phi_i \neq 0$, independently of n. Hence, the closure of L_i would have to map 0 into $42\phi_i$, which is impossible for a linear operator. Since the usual definition of adjoint works well only for closable operators, the jump operators in the standard form (4.2), and even more so their adjoints, have to be interpreted with care. One can build a special notion of adjoint for this purpose [AB15], but it is better to take the view of Proposition 4.2.1 and take $L_i = M_i j$, i.e., as completely determined by the *bounded* operators M_i. In this way, all the difficulties with singular L_i are controlled by the normalization loss of the no-event semigroup. This is analogous to a well-known example of a generator perturbation for which the added term by itself makes little sense, namely point potentials (δ-function potentials) for Schrödinger operators. Again, multiplication by a δ-function, which is formally the potential "added" to the Laplacian, is a crazy operator by itself. However, as a perturbation of the Laplacian, it makes sense and leads to a well-defined self-adjoint operator, which has an alternative description as the Laplacian with a modified boundary condition at the origin. The whole construction is quite stable, and we can also obtain the perturbed operator as the strong resolvent limit of Schrödinger operators with suitably scaled potentials with small support around the origin. The example of this section is also discussed in [Hol18], where it is shown that Arveson's "domain algebra" [Arv02b] can be trivial.

4.5.2 CCR-flow

A basic example of a quantum dynamical semigroup, whose adjoint is a semigroup of endomorphisms (see Chapter 5), is the so-called *CCR flow*. It was first constructed by Powers and Robinson in [PR89] and was used by William Arveson as a main example in his classification of endomorphism semigroup (see for example [Arv03]). We start with the one-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, dt; \mathcal{K})$, the $\mathcal{K}$-valued square integrable functions on $\mathbb{R}_+$, where $\mathcal{K}$ is some Hilbert space of dimension d. On $\mathcal{K}$ we consider the group S_t of co-isometries as in Example 4.5.1. For every t the map S_t gives us an isometric identification of $L^2([t,\infty), dt; \mathcal{K})$ with $\mathcal{H}$, and $P_t = \mathbb{1} - S_t^* S_t$ is the projection onto the functions supported on the interval $[0,t]$. The system Hilbert space is then the symmetric Fock space $\Gamma^+(\mathcal{H})$ introduced in Definition 1.4.2.

If we fix some $t > 0$, every vector ϕ can be split as $\phi = P_t\phi + (\mathbb{1} - P_t)\phi \equiv \phi' + \phi''$. Then

$$\left\langle e^{\psi} \,\middle|\, e^{\phi} \right\rangle = e^{\langle \phi', \psi' \rangle + \langle \phi'', \psi'' \rangle} = \left\langle e^{\psi'} \otimes e^{\psi''} \,\middle|\, e^{\phi'} \otimes e^{\phi''} \right\rangle. \tag{4.55}$$

This defines an isomorphism $e^{\mathcal{H}} \cong e^{\mathcal{H}'} \otimes e^{\mathcal{H}''}$ under which $e^{\psi} \cong e^{\psi'} \otimes e^{\psi''}$. Now using S_t to identify $\mathcal{H}''$ with $\mathcal{H}$, we can write a channel in which the system part in $e^{\mathcal{H}'}$ is traced out, and $e^{\mathcal{H}''}$ is identified with $e^{\mathcal{H}}$:

$$\sigma_t\Big(|e^{\psi}\rangle\langle e^{\phi}|\Big) = e^{\langle \phi, P_t \psi \rangle} |e^{S_t\psi}\rangle\langle e^{S_t\phi}| \tag{4.56}$$

The part $|e^{S_t\phi}\rangle\langle e^{S_t\psi}|$ in 4.56 is the no-event semigroup σ_t^0. Since the exit space of the Shift semigroup S_t is given by $\mathcal{E}_1 = \mathbb{C}$ with $(j_1\psi)(x) = \psi(0)$ we can deduce the exit-space of the CCR flow as $\mathcal{E} = \mathcal{E}_1 \otimes e^{\mathcal{H}}$ with $je^{\phi} = (j_1\phi) \otimes e^{\phi}$, because

$$-\frac{d}{dt}\langle e^{S_t\phi}, e^{S_t\psi}\rangle|_{t=0} = -\frac{d}{dt} e^{\langle S_t\phi, S_t\psi\rangle}|_{t=0} = \langle j_1\phi, j_1\psi\rangle e^{\langle \phi,\psi\rangle}. \tag{4.57}$$

So this is a semigroup of standard form. Its generator is given by

$$\mathcal{L}|e^{\psi}\rangle\langle e^{\phi}| = |De^{\psi}\rangle\langle e^{\phi}| + |e^{\psi}\rangle\langle De^{\phi}| + \overline{\psi(0)}\phi(0)|e^{\psi}\rangle\langle e^{\phi}| \tag{4.58}$$

where D denotes differentiation.

Clearly, this defines a conservative dynamical semigroup, whose adjoint is a semigroup of endomorphisms. The adjoint is best characterized in terms of its action on the Weyl operators, and one easily verifies that

$$\sigma_t^*(W(\psi)) = W(S_t^*\psi). \tag{4.59}$$

Since the linear hull of the Weyl operators is weakly dense in $\mathcal{B}(\mathcal{H})$, this also characterizes the semigroup completely. Moreover, because S_t^* is an isometry, the Weyl relations are preserved. Thus, we arrive at Arveson's definition of the CCR flow.

Definition 4.5.1. *The CCR flow of rank d is the semigroup on $\mathcal{B}(\Gamma^+(\mathcal{H}))$ where $\mathcal{H} = L^2(\mathbb{R}_+, dt; \mathcal{K})$ associated with the shift S_t^* defined by its action on the Weyl operators given in* (4.59)*, where d is the dimension of $\mathcal{K}$.*

The CCR flow, however, does not map compact operators to compact operators; therefore, σ_t is not strongly standard. To see this, we take a look at the action of σ_t^* on a ketbra

$$\langle e^{\xi}|\sigma_t^*(|e^{\phi}\rangle\langle e^{\psi}|)e^{\eta}\rangle = e^{\langle \xi, P_t\eta\rangle}\langle e^{\xi}|e^{S_t^*\phi}\rangle\langle e^{S_t^*\psi}|e^{\eta}\rangle \tag{4.60}$$

Thus, it can be decomposed into the tensor product of the identity operator for $t \le 1$ and shifted part for $t > 0$

$$\sigma_t^*(|e^{\phi}\rangle\langle e^{\psi}|) = \mathbb{1}_{\le t} \otimes \Gamma(S_t^*)|e^{\phi}\rangle\langle e^{\psi}|\Gamma(S_t^*)^*. \tag{4.61}$$

and so we have $\sigma_t^*(\mathfrak{K}(\mathcal{H})) \cap \mathfrak{K}(\mathcal{H}) = \{0\}$.

4.5.3 Quantum birth process

The process

A standard example of the classical theory is the so-called pure birth process. The state of the system at any time is given by an integer n, from where it can jump to $n+1$ with rate $\mu_n > 0$. The generator thus acts on $\rho \in \ell^1(\mathbb{N})$ as

$$(\mathcal{L}\rho)(n) = \begin{cases} \mu_{n-1}\rho(n-1) - \mu_n\rho(n) & \text{for } n > 0 \\ \qquad\qquad\; - \mu_0\rho(0) & \text{for } n = 0. \end{cases} \tag{4.62}$$

The case distinction can be avoided by the convention $\rho(-1) = 0$. By telescoping sum, one verifies $\sum_n (\mathcal{L}\rho)(n) = 0$, so the process appears to be conservative. On the other hand, noting that the expected time for the transition from n to $n+1$ is μ_n^{-1}, it seems possible that the process reaches infinity in finite time when μ_n increases sufficiently rapidly, i.e.,

$$\sum_n \frac{1}{\mu_n} = \tau < \infty. \tag{4.63}$$

Indeed, this is part of the well-established lore on this process (see [Fel57, Section XVII.4] and below). Our interest here is in a closely related quantum process, which is a standard semigroup on $\mathcal{H} = \ell^2(\mathbb{N})$ with K and a single jump operator L given by

$$\begin{aligned} K|n\rangle &= -\frac{1}{2}\mu_n|n\rangle, \quad \operatorname{dom} K = \Big\{\psi \in \ell^2(\mathbb{N}) : \sum_{n=0}^{\infty} \mu_n^2 |\langle\psi|n\rangle|^2 < \infty\Big\}, \\ L|n\rangle &= \sqrt{\mu_n}\,|n+1\rangle, \quad \operatorname{dom} L \subset \operatorname{dom} K, \end{aligned}$$

where $\{|n\rangle\}$ is the canonical basis of the Hilbert space. As usual, we denote by $\mathcal{L}^0\rho = K\rho + \rho K^*$ the no-event generator, which corresponds to the first term in the expression for the standard generator

$$\langle n|\mathcal{L}\rho|m\rangle = -\frac{1}{2}(\mu_n + \mu_m)\langle n|\rho|m\rangle + \sqrt{\mu_{n-1}\mu_{m-1}}\,\langle n-1|\rho|m-1\rangle. \tag{4.64}$$

This is the quantum analogue of (4.62), a simplified and generalized version of a process first studied in [Dav77, Example 3.3]. It reduces precisely to the classical case for purely diagonal density operators. We, therefore, call the process generated by K and L the *quantum birth process*. Like its classical counterpart, it is formally conservative, but it may fail to be conservative due to the possibility of escape to infinity. It will then be interesting to look at the details of the escape: Is there any quantum information "coherently" pushed to infinity? For this simple example, the resolvent series (4.25) can be summed explicitly. We get, for any $\rho \in \mathfrak{T}(\mathcal{H})$,

$$\langle n|\mathcal{R}_\lambda\rho|m\rangle = \frac{1}{\lambda + \frac{1}{2}(\mu_n + \mu_m)} \sum_{k=0}^{\min(n,m)} p_{nm}^k \,\langle n-k|\rho|m-k\rangle \tag{4.65}$$

$$p_{nm}^k = \prod_{j=1}^{k} \frac{\sqrt{\mu_{n-j}\mu_{m-j}}}{\lambda + \frac{1}{2}(\mu_{n-j} + \mu_{m-j})}. \tag{4.66}$$

Thus, the domain of the generator of the minimal solution is $\operatorname{dom}\mathcal{L} = \{\mathcal{R}_\lambda\rho' | \rho' \in \mathfrak{T}(\mathcal{H})\}$, and $\mathcal{L}\rho = \mathcal{L}\mathcal{R}_\lambda\rho' = \lambda\mathcal{R}_\lambda\rho' - \rho'$. In general, it is not easy to determine $\operatorname{dom}\mathcal{L}$

from the expression (4.5), here (4.64), which merely expresses the generator on the domain $(\mathrm{dom}\, K)^{)(}$. On the other hand, the matrix elements on the right-hand side of (4.64) make sense for any bounded operator ρ. It turns out that this reading of (4.64) correctly expresses the extension by minimal solution:

Lemma 4.5.2. *For $\rho \in \mathrm{dom}\,\mathcal{L}$, and all $n, m \in \mathbb{N}$, Eq.* (4.64) *holds. Conversely, if, for some trace class operator ρ, the right-hand side of Eq.* (4.64) *gives the matrix elements of a trace class operator, then $\rho \in \mathrm{dom}\,\mathcal{L}$.*

Proof. Both (4.64) and (4.65) involve finite sums only for fixed n, m. Therefore, we can consider them to define extensions $\mathcal{L}^\sharp$ and $\mathcal{R}_\lambda^\sharp$ of $\mathcal{L}$ and $\mathcal{R}_\lambda$ to arbitrary matrices ρ. It is straightforward to verify that $\mathcal{L}^\sharp\mathcal{R}_\lambda^\sharp = \lambda\mathcal{R}_\lambda^\sharp - \mathcal{I}^\sharp = \mathcal{R}_\lambda^\sharp\mathcal{L}^\sharp$. Take the first equation and apply it to some $\rho' \in \mathfrak{T}(\mathcal{H})$. This shows that $\mathcal{L}^\sharp\mathcal{R}_\lambda\rho' = \mathcal{L}^\sharp\mathcal{R}_\lambda^\sharp\rho' = \lambda\mathcal{R}_\lambda^\sharp\rho' - \rho' = \lambda\mathcal{R}_\lambda\rho' - \rho' = \mathcal{L}\mathcal{R}_\lambda\rho'$, i.e., $\mathcal{L}^\sharp$ and $\mathcal{L}$ coincide on $\mathrm{dom}\,\mathcal{L}$. Now suppose that ρ and $\mathcal{L}^\sharp\rho$ are both trace class. Then by the second equation $\rho = \mathcal{R}_\lambda^\sharp(\lambda\rho - \mathcal{L}^\sharp\rho) \in \mathcal{R}_\lambda^\sharp\mathfrak{T}(\mathcal{H}) = \mathcal{R}_\lambda\mathfrak{T}(\mathcal{H}) = \mathrm{dom}\,\mathcal{L}$. □

Conservativity

From the integral (2.19), one sees that $\mathcal{T}_t$ is conservative if and only if $\mathrm{tr}\,\lambda\mathcal{R}_\lambda\rho = \mathrm{tr}\,\rho$ for all ρ. The trace of (4.65) depends only on the sums with $n = m$; hence, the conservativity is exactly the same as for the classical problem. The resolvent actually contains more information. Let $m(t) = -d/(dt)\,\mathrm{tr}\,\mathcal{T}_t\rho$ be the "arrival probability density" at infinity. Then its Laplace transform is

$$\widehat{m}(\lambda) = \int_0^\infty dt\, e^{-\lambda t}\, m(t) = 1 - \mathrm{tr}\,\lambda\mathcal{R}_\lambda\rho. \tag{4.67}$$

Starting from $\rho = |n\rangle\langle n|$, and introducing the abbreviation $c_i = \mu_{n+j}/(\lambda + \mu_{n+j})$ we get from (4.65)

$$\begin{aligned}
\widehat{m}(\lambda) &= 1 - \mathrm{tr}\,\lambda\mathcal{R}_\lambda|n\rangle\langle n| = 1 - \sum_{k=0}^{\infty}\frac{\lambda}{\lambda+\mu_{n+k}}\prod_{j=0}^{k-1}\frac{\mu_{n+j}}{\lambda+\mu_{n+j}} \\
&= 1 - \sum_{k=0}^{\infty}(1-c_k)\prod_{j=0}^{k-1}c_i = \lim_{N\to\infty}\prod_{j=0}^{N}c_i \\
&= \prod_{j=n}^{\infty}\frac{1}{1+\lambda\mu_i^{-1}}.
\end{aligned} \tag{4.68}$$

This has a straightforward probabilistic interpretation: The probability density of a sum of independent random variables is the convolution of the individual densities corresponding to the product of the Laplace transforms. Hence, the "arrival time at infinity" is the sum of infinitely many independent contributions, each exponentially distributed with density $\mu_i e^{-\mu_i t}$. When $\tau = \sum_i \mu_i^{-1} = \infty$, this sum is actually infinite with probability 1, and $\widehat{m}(\lambda) = 0$.

Domain increase

Next, we consider whether the inclusion $\operatorname{dom}\mathcal{L} \supset \operatorname{dom}\mathcal{L}^0$ is strict. For this, it is helpful to note that for any $\rho \in \operatorname{dom}\mathcal{L}$ and $q \in \mathbb{Z}$ the limit

$$\Phi_q(\rho) = \lim_{n\to\infty} \frac{1}{2}(\mu_n + \mu_{n+q})\langle n|\rho|n+q\rangle \tag{4.69}$$

exists. Indeed, setting $\rho = \mathcal{R}_\lambda \rho'$ this is clear from (4.65), using $p^k_{nm} \leq 1$ and $\rho' \in \mathfrak{T}(\mathcal{H})$. Moreover, if $\rho \in \operatorname{dom}\mathcal{L}^0$ the matrix elements in the above limit belong to the trace class operator $\mathcal{L}^0\rho$, and therefore are summable and have to go to zero, so $\Phi_q(\operatorname{dom}\mathcal{L}^0) = \{0\}$. We note that Φ_0 plays a special role since, for $\rho \in \operatorname{dom}\mathcal{L}$,

$$\Phi_0(\rho) = \lim_{n\to\infty} \sum_{m=0}^{n} \Big(\mu_m\langle m|\rho|m\rangle - \mu_{m-1}\langle m-1|\rho|m-1\rangle\Big) = -\operatorname{tr}\mathcal{L}\rho \tag{4.70}$$

is exactly the infinitesimal normalization loss. When the semigroup is not conservative (the only case we consider now), we can directly find an element on which this does not vanish:

$$\sigma = \sum_n \frac{1}{\mu_n}|n\rangle\langle n|, \qquad \text{with } \Phi_0(\sigma) = 1. \tag{4.71}$$

For the other values of q, the existence of such elements depends, in fact, on how fast the μ_n grow.

Proposition 4.5.3. *Let the rates μ_n* grow moderately *in the sense that, for all q, n,*

$$\left|1 - \frac{\mu_{n+q}}{\mu_n}\right| \leq \frac{c}{n} \tag{4.72}$$

for some constant c independent of n. Then, for any $q \in \mathbb{Z}$, let

$$\sigma^q = \sum_n \frac{2}{\mu_n + \mu_{n+q}}|n\rangle\langle n+q|. \tag{4.73}$$

Then $\sigma^q \in \operatorname{dom}\mathcal{L}$, and $\Phi_q(\sigma^{q'}) = \delta_{qq'}$.

Moderate growth covers rational functions, stretched exponentials like $\mu_n \sim \exp(an^\alpha)$ with $\alpha < 1$, but exponentials $\mu_n = e^{an}$ clearly do not satisfy this condition.

Proof. The matrix (4.73) is clearly positive definite, and $\operatorname{tr}\sigma^q = \tau < \infty$. The critical question is whether $\mathcal{L}\sigma^q$, as defined by (4.64), is trace class. Like σ^q itself, $\mathcal{L}\sigma^q$ is of the form $\sum_n a_n|n\rangle\langle n+q|$, and such an operator is trace class if and only if $\sum_n |a_n| < \infty$ (Think of this as a diagonal operator multiplied with a shift from one side). Thus, we have to show that the sum

$$\sum_n |\langle n|\mathcal{L}\sigma^q|n+q\rangle| = \sum_n \left|-1 + \frac{2\sqrt{\mu_{n-1}\mu_{n+q-1}}}{\mu_{n-1} + \mu_{n+q-1}}\right| \tag{4.74}$$

is finite. Introducing the function

$$g(a,b) = 1 - \frac{2\sqrt{ab}}{a+b} = \frac{(\sqrt{a}-\sqrt{b})^2}{a+b} \leq \left(1 - \frac{b}{a}\right)^2 \tag{4.75}$$

we find that for moderately growing μ_n the terms in the sum (4.74) are bounded by $(c/n)^2$, so the sum converges. □

Example: Exponentially growing μ_n
Let us put $\mu_n = a^n$ for some $a > 1$. Then, for $q \neq 0$, the sum (4.74) has all equal terms and hence diverges. While the limit (4.69) still exists for $\rho = \sigma^q$ and is equal to 1, this does not help to establish domain increase because $\sigma^q \notin \operatorname{dom} \mathcal{L}$. Nor is there any other choice of ρ of which we can prove in this way that $\rho \in \operatorname{dom} \mathcal{L} \setminus \operatorname{dom} \mathcal{L}^0$: For *any* $\rho \in \operatorname{dom} \mathcal{L}$ we get $\Phi_q(\rho) = 0$.

Proof. Consider the resolvent sum (4.65). Each factor in $p^k_{n,m}$ with $m = n + q$ is

$$\frac{\sqrt{\mu_{n-j}\mu_{m-j}}}{\lambda + \frac{1}{2}(\mu_{n-j} + \mu_{m-j})} \leq \frac{2\sqrt{\mu_{n-j}\mu_{m-j}}}{\mu_{n-j} + \mu_{m-j}} \leq \frac{2a^{q/2}}{1 + a^q} =: \gamma. \tag{4.76}$$

Hence $p^k_{n,n+q} \leq \gamma^k$, which is summable with respect to k. Assuming $q \geq 1$ without loss,

$$\begin{aligned} |\Phi_q(\mathcal{R}_\lambda \rho)| &= \lim_n \frac{\frac{1}{2}(\mu_n + \mu_{n+q})}{\lambda + \frac{1}{2}(\mu_n + \mu_{n+q})} \left| \sum_k p^k_{n,n+q} \langle n|\rho|n+q\rangle \right| \\ &\leq \lim_n \sum_{k=0}^{n} \gamma^k r_{n-k}, \end{aligned} \tag{4.77}$$

where we abbreviated $r_n = |\langle n|\rho|n+q\rangle|$. This is a summable sequence because ρ is trace class. The sum is consequently summable as the convolution of two such sequences and therefore goes to zero as $n \to \infty$. □

No new pure states

We have seen that $\operatorname{dom} \mathcal{L}$ is properly larger than $\operatorname{dom} \mathcal{L}^0$. But are there also additional pure states in this larger domain? We could use Prop. 4.4.5 to answer this in the negative. Instead, we give a simple alternative argument based on the range of resolvents.

Proposition 4.5.4. *Let $\mathcal{L}$ and $\mathcal{L}^0$ be as above and $|\phi\rangle\langle\psi| \in \operatorname{dom} \mathcal{L}$. Then $|\phi\rangle\langle\psi| \in \operatorname{dom} \mathcal{L}^0$, i.e., $\phi, \psi \in \operatorname{dom} K$.*

Proof. Since $|\phi\rangle\langle\psi| \in \operatorname{dom} \mathcal{L}$ we may write $|\phi\rangle\langle\psi| = \mathcal{R}_\lambda \rho$ for some $\rho \in \mathfrak{T}(\mathcal{H})$. Let m be the smallest index for which $\rho|m\rangle \neq 0$. Then, in the formula (4.65) for the resolvent, only the term $k = 0$ gives a nonzero contribution. Noting that $p^0_{nm} = 1$ we get

$$\begin{aligned} \langle n|\mathcal{R}_\lambda \rho|m\rangle &= \frac{1}{\lambda + \frac{1}{2}(\mu_n + \mu_m)} \langle n|\rho|m\rangle \quad \text{for all } n \\ \mathcal{R}_\lambda \rho|m\rangle &= (\lambda + \mu_m/2 - K)^{-1} \rho|m\rangle. \\ \phi\,\langle\psi, m\rangle &\in (\lambda + \mu_m/2 - K)^{-1}\mathcal{H} = \operatorname{dom} K. \end{aligned}$$

Now $\langle\psi|m\rangle$ cannot vanish because $\rho|m\rangle \neq 0$. Hence $\phi \in \operatorname{dom} K$, and the same argument applied to ρ^* gives also $\psi \in \operatorname{dom} K$. □

4.5.4 Diffusion on diagonals

This is the basis for the example of a non-standard generator given in [Hol96c]. The basic idea is very similar to the quantum birth process, and the main conclusion is the same. However, the presentation in [Hol96c] was rather sketchy and incomplete and did not mention an argument along the lines of Proposition 4.4.5. These clarifications were the focus of our collaboration and have been independently summarized in [Hol18]. The system Hilbert space in this case is $\mathcal{H} = L^2(\mathbb{R}_+, dx)$. In order to stress the analogies, we use the same notations as above for the generators. They are

$$K = \frac{d^2}{dx^2} \qquad \operatorname{dom} K = \{\psi \in \mathcal{H} \mid \psi(0) = 0,\ \psi'' \in \mathcal{H}\} \tag{4.78}$$

$$L = \sqrt{2}\,\frac{d}{dx} \qquad \operatorname{dom} L = \operatorname{dom} K. \tag{4.79}$$

K generates a diffusion with absorption at the boundary point 0. Similarly, when seen as acting on integral kernels $\rho(x,y)$, $\mathcal{L}$ generates a diffusion with a degenerate diffusion operator $(\frac{d}{dx} + \frac{d}{dy})^2$, corresponding to diffusion along the diagonals $x - y = const$ with absorption at the boundary of the positive quadrant. Both semigroups can be solved explicitly by the reflection trick: The semigroup without the absorbing boundary condition is translation invariant and acts by convolution with a Gaussian kernel. The solution with absorption is then obtained by first extending the initial function to an antisymmetric one on the whole line, applying the Gaussian kernel, and restricting to the half line afterwards. In this way, we get the time evolution (see [Hol18], correcting [Hol96c]), written in terms of its action on integral kernels $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ representing trace class operators:

$$\begin{aligned}(\mathcal{T}_t\omega)(x,y) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty d\xi \sum_{n=0,1} (-1)^n \exp\left\{-\frac{1}{4t}\left|\min(x,y) - (-1)^n\xi\right|^2\right\} \\ &\qquad \times \omega(\xi + [x-y]_+, \xi + [y-x]_+)\end{aligned} \tag{4.80}$$

Here we wrote $x_+ = \max\{0, x\}$ for the positive part of a number. By integration (2.19), we get the resolvent

$$(\mathcal{R}_\lambda\omega)(x,y) = \int_0^\infty d\xi\, f_\lambda^{x,y}(\xi)\, \omega\big(\xi + [x-y]_+, \xi + [y-x]_+\big) \tag{4.81}$$

$$f_\lambda^{x,y}(\xi) = \frac{1}{2\sqrt{\lambda}} \sum_{n=0,1} (-1)^n \exp\left\{-\sqrt{\lambda}\left|\min(x,y) - (-1)^n\xi\right|\right\}. \tag{4.82}$$

Since $f_\lambda^{0,y} = f_\lambda^{x,0} = 0$ we must have $\mathcal{R}_\lambda\omega(0,y) = \mathcal{R}_\lambda\omega(x,0) = 0$ for all ω. Hence, for all $\omega \in \operatorname{dom}\mathcal{L}$, $\omega(0,y) = \omega(x,0) = 0$. Similarly, one sees that the kernel $\omega(x,y)$ has to be continuous for $\omega \in \operatorname{dom}\mathcal{L}$. To find the normalization loss, we can integrate (4.80) to get

$$\operatorname{tr}\mathcal{T}_t\omega = \operatorname{tr}\omega - \int_0^\infty d\xi\, \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)\omega(\xi,\xi), \tag{4.83}$$

where erfc denotes the complementary error function. We substitute $\xi \mapsto 2\sqrt{t}\eta$ and take from [Hol18] the information that, for $\omega \in \operatorname{dom}\mathcal{L}$, we have $\omega(x,x) = \Lambda x + \mathbf{o}(x)$ as $x \to 0$. Then by dominated convergence, and using $\int_0^\infty dx\, x\, \operatorname{erfc}(x) = 1/4$, we find

$$\operatorname{tr}\mathcal{T}_t\omega = \operatorname{tr}\omega - 2\sqrt{t}\int_0^\infty d\eta\, \operatorname{erfc}(\eta)\omega(2\sqrt{t}\,\eta, 2\sqrt{t}\,\eta) = \operatorname{tr}\omega - t\Lambda\ + \mathbf{o}(t) \tag{4.84}$$

The diagonal derivative $\Lambda = -d/(dx)\omega(x,x)|_{x=0}$ plays the same role as $\Phi_0(\rho)$ in the previous section (compare (4.70)). The crucial observation is, once again, that $|\phi\rangle\langle\psi| \in \operatorname{dom}\mathcal{L}$ implies $|\phi\rangle\langle\psi| \in \operatorname{dom}\mathcal{L}^0$. Two techniques are available for showing this: In analogy to Prop. 4.5.4, one can directly show that $(\mathcal{R}_\lambda\omega)\chi \in \operatorname{dom}K$ for suitable χ. But in this case, it is preferable to invoke Prop. 4.4.5.

4.6 Examples of non-standard generators

We focus here on the examples that come immediately from the two examples studied in the previous section: The quantum birth and the diagonal diffusion semigroups. In both cases, we considered a standard generator $\mathcal{L}$, arising from positive perturbation of a no-event generator $\mathcal{L}^0$. Since both semigroups are strictly standard but not conservative, we go one step further and add to $\mathcal{L}$ another positive term, leading to the generator $\operatorname{dom}\widehat{\mathcal{L}}$ of a conservative semigroup. This perturbation again follows the minimal solution pattern (Section 4.3) with a rank one perturbation for simplicity. That is, we set

$$\widehat{\mathcal{L}}\rho = \mathcal{L}\rho - \operatorname{tr}(\mathcal{L}\rho)\widehat{\rho}, \qquad \operatorname{dom}\widehat{\mathcal{L}} = \operatorname{dom}\mathcal{L}. \tag{4.85}$$

The added term is completely positive on $\operatorname{dom}\mathcal{L}$ because normalization loss is negative. The equality of domains follows from Lemma 4.3.2. In dynamical terms, the process will reset to $\widehat{\rho}$ whenever there is an "arrival event", which under $\mathcal{L}$ would mean a loss of normalization: in the quantum birth case, this will be an arrival at infinity, and in the diagonal diffusion case an arrival at the origin.

We have here two construction steps in which a completely positive term is added to the generator. Why can they not be fused into a single step, adding both terms simultaneously? Indeed, if this were possible, $\widehat{\mathcal{L}}$ would be, by definition, a standard generator. The key observation is that $\mathcal{L}$ is strictly standard and therefore infinitesimally trace-preserving on $\operatorname{dom}\mathcal{L}^0$, so $\mathcal{P} = \mathcal{L} - \mathcal{L}^0$ is already as large as it can be. However, since the semigroup $\exp(t\mathcal{L})$ is *not* conservative, $\operatorname{dom}\mathcal{L}$ must be properly larger than $\operatorname{dom}\mathcal{L}^0$ because a generator that is infinitesimally conservative on its full domain would generate a conservative semigroup. The term added when passing from $\mathcal{L}$ to $\widehat{\mathcal{L}}$ vanishes on $\operatorname{dom}\mathcal{L}^0$, so it is only associated with the "new" part of the domain.

The various generators and domains are graphically summarized in Figure 4.2. The same relations may hold in the discrete classical case, namely whenever $\mathcal{L}$ is a standard generator that appears conservative on the pure states in its domain but actually allows some escape to infinity and hence generates a non-conservative semigroup. Indeed, any standard generator is completely determined by its action on the pure states (even though its full domain might not be spanned just by these). If $\widehat{\mathcal{L}}$ were standard, since it coincides with $\mathcal{L}$ on the pure states, we would have $\widehat{\mathcal{L}} = \mathcal{L}$. On the other hand, these generators are clearly different since one generates a conservative semigroup and the other does not.

In the quantum case, this argument is too simple since not all pure states are in the domain, but only those $|\psi\rangle\langle\psi|$ with $\psi \in \operatorname{dom}K$. So the possibility we have to discuss

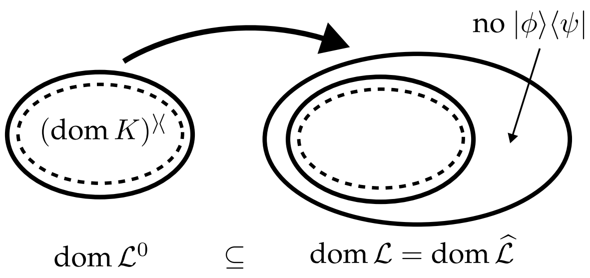

Figure 4.2: Generators and their domains in the construction of a non-standard generator $\widehat{\mathcal{L}}$.

is that there might be another contraction generator $\widetilde{K}$ and associated no-event semigroup generator $\widetilde{\mathcal{L}}^0$, from which $\widehat{\mathcal{L}}$ arises in a one-step minimal solution construction. It is here that we can use the fact from Proposition 4.4.5 that for strongly standard semigroups, we actually have $\phi, \psi \in \mathrm{dom}\, K$ for all $|\phi\rangle\langle\psi| \in \mathrm{dom}\, \widehat{\mathcal{L}}$. So even if we had started from some other $\widetilde{K}$, we could still reconstruct $\mathrm{dom}\, K$ from $\mathrm{dom}\, \widehat{\mathcal{L}} = \mathrm{dom}\, \mathcal{L}$. Since $\widehat{\mathcal{L}}$ and $\mathcal{L}$ coincide on $(\mathrm{dom}\, K)^{)(}$, they would arise as minimal solutions from the same equation on $(\mathrm{dom}\, K)^{)(}$. Therefore, in both examples, $\widehat{\mathcal{L}}$ is non-standard.

In fact, this construction gives us a whole class of non-standard generators, namely those arising as the minimal solution from a finite-rank perturbation of non-conservative but strictly and strongly standard semigroups.

4.7 Notes and Remarks

A large part of this chapter has already been published in [SHW17]. Starting point of our considerations was the example of a non-standard generator given in [Hol96c]. We felt the need to clarify what exactly we mean by an unbounded version of a GKLS-form, and to get a structural understanding of the example's construction. The quantum birth process (3.4) gave us an insight into the behaviour of domains under the minimal solution construction; thus, we were able to get a whole class of non-standard generators.

Additional content is the notion of strictly and strongly standard generators and the corresponding results. Proposition 4.4.2 is a combination of [Dav77, Thm. 3.2], and [Fag99, Thm. 3.32]. Proposition 4.4.5 can already be found in [SHW17], but with slightly different phrasing. The idea of strongly standard semigroups can be found in [Hol96b], where A. Holevo introduced the *forward and backward Markovian Master Equation* (*MME*), and can be found more explicitly in [Hol18]. The backward MME for a strongly standard semigroup $\mathcal{T}_t$ is given by

$$\frac{d}{dt}\langle\phi|\mathcal{T}_t^*[X]|\psi\rangle = \sum_j \langle L_j\phi|\mathcal{T}_t^*[X]|L_j\psi\rangle - \langle K\phi|\mathcal{T}_t^*[X]|\psi\rangle - \langle\phi|\mathcal{T}_t^*[X]|K\psi\rangle \tag{4.86}$$

with $\phi, \psi \in \mathcal{D}$ and $X \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{T}_t^*$ is also the solution to the Master equation

$$\frac{d}{dt}\langle\phi|\mathcal{T}_t^*[X]|\psi\rangle = \mathrm{tr}\left(\mathcal{L}[|\psi\rangle\langle\phi|\mathcal{T}_t^*[X]]\right) \tag{4.87}$$

where $\mathcal{L}$ is the Lindblad generator

$$\mathcal{L}[|\psi\rangle\langle\phi|] = \sum_j |L_j\psi\rangle\langle L_j\phi| - |K\psi\rangle\langle\phi| - |\psi\rangle\langle K\phi|. \tag{4.88}$$

Then a dynamical semigroup is standard if $\mathcal{T}_t^*$ can be obtained from the minimal solution of the backward MME (4.86) or (4.87). The Forward MME on the other hand is formulated with a generator defined in the Heisenberg picture, so we have

$$\frac{d}{dt}\langle\phi|\mathcal{T}_t[\omega]|\psi\rangle = \sum_i \langle L_i^*\phi|\mathcal{T}_t[\omega]|L_i^*\psi\rangle - \langle K^*\phi|\mathcal{T}_t[\omega]|\psi\rangle - \langle\phi|\mathcal{T}_t[\omega]|K^*\psi\rangle \tag{4.89}$$

with $\phi, \psi \in \mathcal{D}^*$ and $\omega \in \mathfrak{T}(\mathcal{H})$ and

$$\mathcal{L}^*[|\psi\rangle\langle\phi|] = \sum_i |L_i^*\psi\rangle\langle L_i^*\phi| - |K^*\psi\rangle\langle\phi| - |\psi\rangle\langle K^*\phi|. \tag{4.90}$$

Although this looks quite similar to (4.87) and (4.88), we need a slightly different definition for the jump operators L_i. In (4.88) and our definition of standardness the L_i were defined on $\operatorname{dom} K$, for (4.90) one assumes that all L_i^* are defined on $\operatorname{dom} K^*$. This is the case if $\sum_i \|L_i^* f\|^2 < \infty$ for $f \in \operatorname{dom} K^*$. In this sense, we can rephrase Definition 4.4.3, and call standard dynamical semigroup strongly standard if $\mathcal{T}_t$ can be obtained from the minimal solution of the forward MME (4.89) and the representation (4.90) holds.

We also looked at another example, the CCR-flow (4.5.2). It is possible to replace the shift semigroup here with a more general coisometry. However, in this thesis we chose the simpler version, as it is the main example used by W. Arveson [Arv03] for semigroups of endomorphism and serves as a link between this chapter and Chapter 5.

Chapter 5

Endomorphism semigroups

A quantum operation T^* (in the Heisenberg picture) with the property that $T^*(AB) = T^*(A)T^*(B)$ is called an endomorphism. A particular case is that of unitary channels, but the loss of a complete subsystem is also allowed. Semigroups of endomorphisms arise naturally in the dilation theory of dynamical semigroups. Deviating from the literature on endomorphism semigroups, we will continue to label channels and semigroups in the Heisenberg picture with a $*$. We will, however, adopt the custom of denoting endomorphism semigroups with letters from the beginning of the Greek alphabet, thus distinguishing them from dynamical semigroups in general.

5.1 Endomorphism semigroups and product systems

The study and classification of endomorphism semigroups on $\mathcal{B}(\mathcal{H})$ was initiated by Powers [Pow88; Pow87] and was later reduced by Arveson in his series of papers [Arv89a; Arv90a; Arv89b; Arv90b] to the classification of corresponding product systems.

Definition 5.1.1. *A one-parameter semigroup α_t^* is called an E_0-semigroup on $\mathcal{B}(\mathcal{H})$ if the following conditions are satisfied:*

1. *α_t^* is the adjoint of a strongly continuous one-parameter group α_t on $\mathfrak{T}(\mathcal{H})$ as defined in Definition 2.0.1,*
2. *α_t^* is a $*$-endomorphism of $\mathcal{B}(\mathcal{H})$ for each $t \geq 0$,*
3. *$\alpha_t^*(\mathbb{1}) = \mathbb{1}$ for each $t \geq 0$.*

If a semigroup satisfies the first two conditions but is not necessarily unital, it is called an e_0-semigroup.

If α_t^* is an e_0-semigroup, then it is of one of the following two forms.

1. There is a strongly continuous semigroup S_t of isometries in $\mathcal{B}(\mathcal{H})$ such that

$$\alpha_t^*(A) = S_t A S_t^*, \quad t \geq 0, \quad A \in \mathcal{B}(\mathcal{H}). \tag{5.1}$$

If S_t is a unitary operator, then α_t^* is a semigroup of automorphisms; otherwise, $\alpha_t^*(\mathbb{I})$ is a proper projection.

2. For every $t > 0$, there is an infinite sequence of isometries $V_1(t), V_2(t), \dots$ having mutually orthogonal ranges, that satisfy

$$\alpha_t^*(A) = \sum_n V_n(t) A V_n^*(t), \quad A \in \mathcal{B}(\mathcal{H}). \tag{5.2}$$

Moreover, $\alpha_t^*(A)V_n(t) = V_n(t)A$, for every $A \in \mathcal{B}(\mathcal{H})$ and every $n = 1, 2, \dots$.

A strongly continuous family of unitary operators W_t on $\mathcal{H}$ is called a cocycle for α_t^* if it satisfies the cocycle equation

$$W_{s+t} = W_s \alpha_s^*(W_t), \quad s, t \geq 0. \tag{5.3}$$

Such cocycles for α_t^* lead to other endomorphism semigroups of the form

$$\alpha_t^{*W}(A) = W_t \alpha_t^*(A) W_t^* \tag{5.4}$$

that are closely related to α_t^*. These semigroups are called *cocycle perturbations* of α_t^*.

Definition 5.1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let α_t^* and β_t^* be E_0-semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. Then α_t^* and β_t^* are called* cocycle conjugate *if β_t^* is conjugate to a cocycle perturbation of α_t^*, i.e. there is a $*$-isomorphism $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and a cocycle W_t of α_t^* such that for all $t \geq 0$*

$$\Phi \circ \alpha_t^{*W} = \beta_t^* \circ \phi. \tag{5.5}$$

Endomorphism semigroups that are cocycle conjugate share many properties. In fact, Arveson's classification of endomorhism semigroups is up to cocycle conjugacy. Let α_t^* be an e_0-semigroup. For each $t > 0$ we can consider the intertwining space of α_t^*

$$\mathcal{E}_{\alpha^*}(t) = \{R \in \mathcal{B}(\mathcal{H}) | \alpha_t^*(A)R = RA\} \tag{5.6}$$

for $A \in \mathcal{B}(\mathcal{H})$. This leads to a family of vector spaces over the interval $(0, \infty)$

$$\mathcal{E}_{\alpha^*} = \{(t, R) | t > 0, \; R \in \mathcal{E}_{\alpha^*}(t)\} \tag{5.7}$$

with a projection $p : \mathcal{E}_{\alpha^*} \to (0, \infty)$ given by $p(t, R) = t$ and fibers $p^{-1}(t) = \mathcal{E}_{\alpha^*}(t)$. Due to the intertwining property, the term R^*S with $R, S \in \mathcal{E}_{\alpha^*}(t)$ commutes with each $A \in \mathcal{B}(\mathcal{H})$, so it is a scalar multiple of the identity, and its value defines an inner product on $\mathcal{E}_\alpha$ via

$$R^*S = \langle R, S \rangle \mathbb{I}. \tag{5.8}$$

As $\|R\|^2 = \|R^*R\| = \|\langle R, R\rangle \mathbb{I}\| = \langle R, R\rangle$, the norm defined by this inner product coincides with the operator norm on $\mathcal{E}_{\alpha^*}(t)$ and the vector space thus becomes a Hilbert space.

If α_t^* is of the form (5.1), then all fiber spaces $\mathcal{E}_{\alpha^*}(t)$ are simply one-dimensional. If α_t^* is of the form (5.2), then for a fixed $t > 0$ we find that the isometries $\{V_1(t), V_2(t), \dots\}$ are an orthonormal basis for the Hilbert space $\mathcal{E}_{\alpha^*}(t)$.

It is possible to establish a semigroup structure on this family of Hilbert spaces associated with an e_0-semigroup $\mathcal{E}_{\alpha^*}$. For every $s, t > 0$ we have:

1. $\mathcal{E}_{\alpha^*}(s)\mathcal{E}_{\alpha^*}(t) \subseteq \mathcal{E}_{\alpha^*}(s+t)$
2. For $S, S' \in \mathcal{E}_{\alpha^*}(s)$ and $R, R' \in \mathcal{E}_{\alpha^*}(t)$ we have
$$\langle SR, S'R'\rangle_{s+t} = \langle S, S'\rangle_s \langle R, R'\rangle_t. \tag{5.9}$$
3. The set of products $\{SR | S \in \mathcal{E}_{\alpha^*}(s), T \in \mathcal{E}_{\alpha^*}(t)\}$ has $\mathcal{E}_{\alpha^*}(s+t)$ as its closed linear span.

Moreover, for every $t > 0$, $\alpha_t^*(\mathbb{1})$ is the projection onto the subspace $[\mathcal{E}_{\alpha^*}(t)\mathcal{H}]$ of $\mathcal{H}$ spanned by the ranges of the operators in $\mathcal{E}_{\alpha^*}(t)$.

As the σ-algebra of subsets of $\mathcal{B}(\mathcal{H})$ generated by the weak operator topology on $\mathcal{B}(\mathcal{H})$ makes $\mathcal{B}(\mathcal{H})$ into a standard Borel space, it gives us a context for the structure of $\mathcal{E}_{\alpha^*}$.

Definition 5.1.3. *A* concrete product system *is a Borel subset $\mathcal{E}$ of the cartesian product of Borel spaces $(0,\infty) \times \mathcal{B}(\mathcal{H})$ with the following properties. Let $p : \mathcal{E} \to (0,\infty)$ be the natural projection $p(t, R) = t$. We require that p should be surjective, and in addition:*

1. *For every $t > 0$, the set of operators $\mathcal{E}(t) = p^{-1}(t)$ is a norm-closed linear subspace of $\mathcal{B}(\mathcal{H})$ with the property that B^*A is a scalar for every $A, B \in \mathcal{E}(t)$.*
2. *For every $s, t > 0$, $\mathcal{E}(s+t)$ is the norm-closed linear span of the set of products $\mathcal{E}(s)\mathcal{E}(t)$.*
3. *There is a sequence of Borel-measurable operator functions $V_n : (0,\infty) \to \mathcal{B}(\mathcal{H})$, $n = 1, 2, \ldots$, such that $\{V_1(t), V_2(t), \ldots\}$ is an orthonormal basis for $\mathcal{E}(t)$ for every $t > 0$.*

Two concrete product systems $\mathcal{E} \subseteq (0,\infty) \times \mathcal{B}(\mathcal{H})$ and $\mathcal{F} \subseteq (0,\infty) \times \mathcal{B}(\mathcal{K})$ are said to be *isomorphic* if there is an isomorphism of Borel structures $\theta : \mathcal{E} \to \mathcal{F}$ that satisfies $\theta(xy) = \theta(x)\theta(y)$ for $x, y \in \mathcal{E}$, and restricts to a unitary operator from $\mathcal{E}(t)$ to $\mathcal{F}(t)$ for every $t > 0$.

We summarize these thoughts in the following theorem.

Theorem 5.1.4 ([Arv03, Thm. 2.4.7])**.** *Let α_t^* be an e_0-semigroup acting on $\mathcal{B}(\mathcal{H})$. Then the structure $\mathcal{E}_{\alpha^*}$ is a concrete product system whose fibers are either all one-dimensional or all infinite-dimensional.*

In this way, every e_0-semigroup gives rise to a unique concrete product system. On the other hand, every concrete product system uniquely determines an e_0-semigroup associated with it.

Theorem 5.1.5 ([Arv03, Prop. 2.4.9])**.** *Let $\mathcal{E} \subseteq (0,\infty) \times \mathcal{B}(\mathcal{H})$ be a concrete product system. Then there is a unique e_0-semigroup α_t^* acting on $\mathcal{B}(\mathcal{H})$ whose endomorphisms satisfy the following two conditions for every $t > 0$:*

1. *$\alpha_t^*(A)R = RA$, for every $R \in \mathcal{E}(t)$ and $A \in \mathcal{B}(\mathcal{H})$.*
2. *$\alpha_t^*(\mathbb{1})$ is the projection on $[\mathcal{E}(t)\mathcal{H}]$.*

Moreover, $\mathcal{E} = \mathcal{E}_{\alpha^*}$ is the concrete product system associated with α_t^*

The following theorem finally shows the importance of product systems in classifying E_0-semigroups.

Theorem 5.1.6 ([Arv03, Thm. 2.4.10]). *Two E_0-semigroups α_t^* and β_t^* acting, respectively, on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ are cocycle conjugate if and only if their product systems are isomorphic.*

Product systems are thus a complete invariant for cocycle conjugacy. This theorem is, however, explicitly formulated for E_0-semigroups, so α_t^* and β_t^* are assumed to be unital.

5.2 The classification of endomorphism semigroups

A key part in the classification of E_0-semigroups or, equivalently their product systems are units.

Definition 5.2.1. *A* unit *of an E_0-semigroup α_t^* is a semigroup of bounded operators U_t on $\mathcal{H}$ that satisfies*

1. *U_t is strongly continuous*
2. *$U_0 = \mathbb{1}$*
3. *$\alpha_t^*(A)U_t = U_t A, \quad t \geq 0, A \in \mathcal{B}(\mathcal{H})$.*

Thus, a unit is a section of the product system $p : \mathcal{E}_{\alpha^*} \to (0, \infty)$ that is strongly continuous and converges to $\mathbb{1}$ for $t \to 0$. We will write $\mathcal{U}_{\alpha^*}$ for the set of all units of α_t^*. It is possible to construct a Hilbert space associated with $\mathcal{U}_{\alpha^*}$. There is a unique function $c : \mathcal{U}_{\alpha^*} \times \mathcal{U}_{\alpha^*} \to \mathbb{C}$ such that for two units U_t and S_t of α^*

$$S_t^* U_t = e^{c(U,S)t}\mathbb{1}. \tag{5.10}$$

c is called the *covariance function* of α_t^*.

Occasionally, units are assumed to be semigroups of isometries ([PP90; Pow91]). However, equation (5.10) implies that $U_t^* U_t$ is a multiple of the identity, so the difference lies only in normalization and can be neglected for most purposes.

The amount of units associated with an E_0-semigroup is a characteristic of the semigroup itself. One can observe that some semigroups do not have any units while others are even completely determined by them.

Definition 5.2.2. *An E_0-semigroup α_t^* on a von Neumann algebra $\mathcal{B}(\mathcal{H})$ is called* spatial *if it has at least one unit U_t, i.e. if $\mathcal{U}_{\alpha^*} \neq \emptyset$.*

If α_t^* is spatial, we can define a family of subspaces of $\mathcal{H}$

$$\mathcal{H}_t = \overline{\operatorname{span}}\{U_{t_1}^1 U_{t_2}^2 \cdots U_{t_n}^n \psi | \psi \in \mathcal{H}, U^i \in \mathcal{U}_{\alpha^*}, t_i > 0, \sum_{i=1}^n t_i = t\}. \tag{5.11}$$

Definition 5.2.3. *If $\mathcal{H}_t = \mathcal{H}$ for all $t > 0$, then α_t^* is called* completely spatial.

This is equivalent to requiring that for every $t > 0$, the Hilbert space $\mathcal{E}_{\alpha^*}(t)$ is the closed linear span of the set of products $U_{t_1}^1 \cdots U_{t_n}^n$, where $U_{t_i}^i \in \mathcal{U}_{\alpha^*}$ and $\sum_{i=1}^n t_i = t$ with $t_i > 0$. In fact, it suffices to find a $t_0 > 0$ that satisfies one and, therefore, both of these conditions.

The properties of being spatial and of being completely spatial are cocycle conjugacy invariants since there is a bijection between the sets of units of cocycle conjugate E_0-semigroups ([Pow99, Def. 2.3]). We can, therefore, formulate a classification of E_0-semigroups up to cocycle conjugacy.

Definition 5.2.4. *An E_0-semigroup is said to be of* Type I *if it is completely spatial. It is of* Type II *if it is spatial but not completely spatial, and it is* Type III *if its set of units is empty.*

In this thesis, we are concerned with spatial E_0-semigroups on $\mathcal{B}(\mathcal{H})$ and their connection to standard semigroups on $\mathfrak{T}(\mathcal{H})$ as described in Chapter 4. The construction of Type III E_0-semigroups, on the other hand, is rather involved, and we refer to [Pow87; Arv03; Sch18] for more information.

For every E_0-semigroup α_t^* on $\mathcal{B}(\mathcal{H})$ there is naturally defined Type I semigroup $\widehat{\alpha}_t^*$ subordinate to it. If α_t^* is of Type III, then $\widehat{\alpha}_t^*$ is trivial and can be seen as simply acting on the von Neumann algebra $\mathcal{M} = \{0\}$, so from here on, we assume α_t^* to be spatial and $\mathcal{U}_{\alpha^*} \neq \emptyset$. On the other hand, if α_t^* is Type I, $\widehat{\alpha}_t^*$ coincides with α_t^*. $\widehat{\alpha}_t^*$ acts on a subalgebra of $\mathcal{B}(\mathcal{H})$ if α_t^* is Type II.

To construct $\widehat{\alpha}_t^*$ we look at the subspace of the product system $\mathcal{E}_{\alpha_t^*}$ spanned by its units

$$\mathcal{D}(t) = \overline{\operatorname{span}}\{U_{t_1}^1 U_{t_2}^2 \cdots U_{t_n}^n | U^i \in \mathcal{U}_{\alpha^*}, t_i > 0, \sum_{i=1}^n t_i = t, n \geq 1\}. \tag{5.12}$$

This subsystem determines a family of $*$-endomorphisms β_t^* via

$$\beta_t^*(A) = \sum_k V_k(t) A V_k(t)^* \tag{5.13}$$

where $\{V_1(t), V_2(t), \dots\}$ is an orthonormal basis for $\mathcal{D}(t)$. By definition we have $\beta_{s+t}^* = \beta_s^* \beta_t^*$. We can set $\beta_0^*(A) = A$, and β_t^* is an e_0-semigroup. However, β_t^* is not necessarily unit preserving, but we have $\beta_t^* \leq \alpha_t^*$ for $t \geq 0$. The following conditions are equivalent for every $t > 0$ (see [Arv89a, Ch. 7] and [Arv03, Sect. 8.8]):

1. $\mathcal{H}$ is spanned by $\{R\xi | R \in \mathcal{D}(t), \xi \in \mathcal{H}\}$
2. $\beta_t^*(\mathbb{1}) = \mathbb{1}$
3. $\mathcal{E}_{\alpha^*}(t) = \mathcal{D}(t)$
4. $\beta_t^* = \alpha_t^*$

If one of these conditions is satisfied for some t then it satisfied for all t and α_t^* is completely spatial (i.e. of Type I).

Now we look at the projections $\beta_t^*(\mathbb{1})$. They form a sequence decreasing in t and

$$\beta_t^*(\mathbb{1})\mathcal{H} = \overline{\operatorname{span}}\{R\xi | R \in \mathcal{D}(t), \xi \in \mathcal{H}\} \tag{5.14}$$

for $t > 0$. Let q denote the limit for

$$q = \lim_{t \to \infty} \beta_t^*(\mathbb{1}). \tag{5.15}$$

Then for all $s > 0$ we have $\beta_s^*(q) = q$. Thus we can restrict β_t^* to the $q\mathcal{B}(\mathcal{H})q \cong \mathcal{B}(q\mathcal{H})$ and get a semigroup of endomorphism on $\mathcal{B}(q\mathcal{H})$ with $\beta_t^*(\mathbb{I}_{q\mathcal{H}}) = \mathbb{I}_{q\mathcal{H}}$ that is completely spatial. Additionally, there is a bijection $\theta : \mathcal{U}_{\alpha^*} \to \mathcal{U}_{\beta^*}$, defined by

$$\theta(U_t) = U_t|_{q\mathcal{H}} \tag{5.16}$$

for all $U_t \in \mathcal{U}_{\alpha^*}$. By construction, we know that $\alpha_t^*(q) \geq q$, and one can find

$$q\alpha_t^*(A)q = \beta_t(A), \tag{5.17}$$

for all $X \in q\mathcal{B}(\mathcal{H})q$ and $t \geq 0$ [Arv03, Sect. 8.8].

Definition 5.2.5. *The semigroup $\widehat{\alpha}_t^*$ defined by restricting β_t^* to $\mathcal{B}(q\mathcal{H})$ is called the* Type I part *of α_t^*.*

Note that if α_t^* is a Type II semigroup, we might get $q = 0$, and the Type I part of α_t^* is trivial even though $\mathcal{U}_\alpha \neq \emptyset$.

5.3 Generators of spatial semigroups and their domain

Let α_t^* be a spatial E_0-semigroup of $\mathcal{B}(\mathcal{H})$ and let U_t be a unit of α_t^*. In this section, we assume all units to be semigroups of isometries (by normalizing U_t if necessary, see Section 5.1). Let $-d$ be the generator of U_t so

$$d\phi = \lim_{t\to 0} \frac{1}{t}(\phi - U_t\phi) \tag{5.18}$$

and $\operatorname{dom} d$ is the set of all ϕ so that the limit exists in the strong operator topology. At this point, we will follow convention in literature and explicitly use the minus sign for the generator of the unit, i.e. we have $U_t = e^{-td}$. Let δ be the generator of α_t^* so

$$\delta(A) = \lim_{t\to 0} \frac{1}{t}(\alpha_t^*(A) - A) \tag{5.19}$$

and the domain is given by all A so that the limit exists in weak* convergence. Then δ is a $*$-derivation of $\operatorname{dom}(\delta)$ into $\mathcal{B}(\mathcal{H})$. Since $U_tA = \alpha_t^*(A)U_t$ we have by differentiation that if $A \in \operatorname{dom}\delta$, then $A \operatorname{dom} d \subset \operatorname{dom} d$ and

$$dA\phi = Ad\phi - \delta(A)\phi \tag{5.20}$$

for all $\phi \in \operatorname{dom} d$. Let d^* be the hermitian adjoint of d. Since the U_t are isometries, d is skew-hermitian and therefore $d^* \supset -d$ (i.e. $\operatorname{dom} d^* \supset \operatorname{dom} d$ and $d^*\phi = -d\phi$ for all ϕ in $\operatorname{dom} d$). Replacing $A \in \operatorname{dom}\delta$ in (5.20) by $A^* \in \operatorname{dom}\delta$ and taking adjoints one finds that $A \operatorname{dom} d^* \subset \operatorname{dom} d^*$ and

$$d^*A\phi = Ad^*\phi + \delta(A)\phi \tag{5.21}$$

for all $\phi \in \operatorname{dom} d^*$. Each vector $\phi \in \operatorname{dom} d^*$ can be uniquely decomposed as $\phi = \phi_0 + \phi_+$ with $\phi_0 \in \operatorname{dom} d$ and $d^*\phi_+ = \phi_+$. This is basically the von Neumann decomposition (see [v N32]). On $\operatorname{dom} d^*$ one can define a non-negative inner product $[\cdot,\cdot]$ via

$$[\phi, \psi] = \frac{1}{2}\langle d^*\phi, \psi\rangle + \frac{1}{2}\langle \phi, d^*\psi\rangle \tag{5.22}$$

for $\phi, \psi \in \operatorname{dom} d^*$. For $\phi = \phi_0 + \phi_+$ and $\psi = \psi_0 + \psi_+$ in $\operatorname{dom} d^*$ we get

$$[\phi, \psi] = \langle \phi_+, \psi_+ \rangle, \tag{5.23}$$

so $\phi \in \operatorname{dom} d$ if and only if $[\phi, \phi] = 0$. Thus, one can construct the space

$$\mathcal{K} = \operatorname{dom} d^* \big/ \operatorname{dom} d, \tag{5.24}$$

that is a Hilbert space with inner product $\langle [\phi], [\psi] \rangle_{\mathcal{K}} = [\phi, \psi]$, where $[\phi]$ denotes the equivalence class of $\phi \in \operatorname{dom} d^*$ in $\mathcal{K}$.

Let S_t denote another unit of α_t^* with generator $-D$. By rescaling if necessary, we can simplify equation (5.10) to

$$S_t^* U_t = e^{-\mu t} \mathbb{1} \tag{5.25}$$

with $0 \leq \mu < \infty$.

Lemma 5.3.1 ([PP90, Lem. 2.1 and Lem. 2.2])**.** *Let U_t and S_t be units of α_t^* that satisfy* (5.25) *with generators $-d$ and $-D$, respectively. Then* $\operatorname{dom} D \subset \operatorname{dom} d^*$ *and for* $\phi \in \operatorname{dom} D$ *we have*

$$D\phi = -d^*\phi + \mu\phi. \tag{5.26}$$

Additionally, $\operatorname{dom} D \cap \operatorname{dom} d = \{0\}$.

The first part is straightforward computation. For the statement about the domains, we write $\|f\|^2 = \langle S_t f, S_t f \rangle$, since we assumed S_t to be an isometry, and take the derivative at $t = 0$. Thus, $-\mu \|f\|^2 = 2\,\Re e \langle f, df \rangle$, and as d is skew-hermitian, $f = 0$ follows.

Let $V_0 : \operatorname{dom} D \to \mathcal{K}$ be the one-to-one linear mapping given by $V_0 \phi = [\phi]$. By differentiating $\langle \phi, \psi \rangle = \langle S_t \phi, S_t \psi \rangle$ we derive

$$0 = \langle D\phi, \psi \rangle + \langle \phi, D\psi \rangle = 2\mu \langle \phi, \psi \rangle - 2\langle [\phi], [\psi] \rangle_{\mathcal{K}}, \tag{5.27}$$

so $\frac{1}{\sqrt{\mu}} V_0$ is an isometry on $\operatorname{dom} D$ and we denote by V its isometric extension to $\mathcal{H}$. The domain of D then consist of precisely those $\phi \in \operatorname{dom} d^*$ that satisfy

$$\sqrt{\mu} V \phi = [\phi] \tag{5.28}$$

for all $\psi \in \operatorname{dom} d^*$.

Lemma 5.3.2 ([PP90, Lem. 2.4])**.** *Let U_t and S_t be units of α_t^* that satisfy* (5.25) *with generators $-d$ and $-D$, respectively. Then the domains of d^* and D^* coincide and for $\phi \in$* $\operatorname{dom} d^*$ *we have*

$$D^* \phi = d^* \phi + \mu \phi - \sqrt{\mu} V^*(\phi_+). \tag{5.29}$$

The isometry $V : \mathcal{H} \to \mathcal{K}$ can be written as $V = \sum_{i=1}^n s_i V_i$ with complex numbers s_i that satisfy $\sum_{i=1}^n |s_i|^2 = 1$, where the V_i are a family of isometries from $\mathcal{H} \to \mathcal{K}$ with orthogonal ranges, i.e. $V_i V_j = 0$ for $i \neq j$. The number n of isometries is a positive integer or $n = \infty$. If V_i' is another family of isometries with orthogonal ranges that satisfy $V = \sum_{i=1}^m s_i' V_i'$, then $m = n$. This number n is called the *Arveson index* of α_t^*, and it is a numerical invariant for cocycle conjugacy.

The map *-representation Φ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$ given by

$$\Phi(A) = \sum_{i=1}^{n} V_i A V_i^* \tag{5.30}$$

is called the *normal boundary representation* associated with α_t^* and U_t. It does not depend on the specific choice of the V_i.

If δ is the generator of the E_0-semigroup α_t^* on $\mathcal{B}(\mathcal{H})$, then we denote by δ_* the generator of the semigroup α_t on $\mathcal{B}(\mathcal{H})_* = \mathfrak{T}(\mathcal{H})$. The property of an E_0-semigroup of being spatial or being completely spatial is closely connected to the number of pure states in the domain of δ_*.

Theorem 5.3.3 ([Pow91, Thm. 3.1])**.** *An E_0-semigroup α_t^* on $\mathcal{B}(\mathcal{H})$ is spatial if and only if the domain of δ_* contains a pure state of $\mathcal{B}(\mathcal{H})$.*

Let Δ be the linear map on $\mathfrak{T}(\mathcal{H})$ given by

$$\operatorname{tr} \Delta(|\phi\rangle\langle\psi|)A = -\langle d^*\phi, A\psi\rangle - \langle\phi, Ad^*\psi\rangle + \langle[\phi], \Phi(A)[\psi]\rangle \tag{5.31}$$

for $\phi, \psi \in \operatorname{dom} d^*$. It is dissipative, closable, and its closure is the generator of a dynamical semigroup if the range of $\lambda - \Delta$ is dense in $\mathfrak{T}(\mathcal{H})$. In that case, the set

$$(\operatorname{dom} d^*)^{)(} = \operatorname{span}\{|\phi\rangle\langle\psi| | \phi, \psi \in \operatorname{dom} \mathrm{d}^*\} \tag{5.32}$$

is a core for Δ.

Theorem 5.3.4 ([Pow91, Thm. 4.7])**.** *Let α_t^* be a spatial E_0-semigroup on $\mathcal{B}(\mathcal{H})$ with generator δ, let U_t be a unit of α_t^* with generator $-d$ and let Φ denote the normal boundary representation concerning α_t^* and U_t. Then the following assertions are equivalent.*

1. *α_t^* is completely spatial.*
2. *Φ is unital and the action of α_t^* on its predual is given by the generator $\delta_* = \Delta$ as defined in* (5.31)*.*
3. *The linear span of pure states in $\operatorname{dom} \delta_*$ is a core for δ_*.*

5.4 E_0-semigroups with standard preadjoint

Endomorphism semigroups are, of course, the adjoints of particular quantum dynamical semigroups in the Schrödinger picture. The question arises, what characteristics E_0-semigroups have if we assume these particular semigroups to be standard. The results of the last section bear a close resemblance with the ideas in Chapter 4.

Remark 5.4.1. *The inner product given by* (5.22) *directly defines a minimal exit space $(\mathcal{E}, j)$ for the preadjoint semigroup, as in* (4.10)

$$\langle j\phi, j\psi\rangle = -2[\phi, \psi] = -\big(\langle d^*\phi, \psi\rangle + \langle\phi, d^*\psi\rangle\big). \tag{5.33}$$

$\mathcal{E}$ is given as the completion of $\operatorname{dom} d^*$ with respect to this inner product; thus, we have

$$\mathcal{E} = \mathcal{K} \tag{5.34}$$

where $\mathcal{K}$ is given by (5.24)*. The normal boundary representation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{K}$ (see* (5.30)*) corresponds in the Schrödinger picture to completely positive reinsertion maps $\mathcal{S} : \mathfrak{T}(\mathcal{K}) \to \mathfrak{T}(\mathcal{H})$, as defined in Proposition 4.2.1.*

In this sense, the construction of standard semigroups can be viewed as a generalization of the underlying ideas concerning spatial semigroups to semigroups of completely positive maps.

Let us take a closer look at two special cases of standard semigroups, whose adjoints are E_0-semigroups.

Proposition 5.4.2. *Let $\mathcal{T}_t$ be a dynamical semigroup with a bounded generator, and assume that $\mathcal{T}_t^*$ is an E_0-semigroup. Then $\mathcal{T}_t$ is unitarily implemented.*

Proof. We can use the form (4.5) of the generator and its adjoint $\mathcal{L}^*$. The infinitesimal version of the endomorphism property is the derivation property

$$\begin{aligned} 0 &= \mathcal{L}^*(AB) - \mathcal{L}^*(A)B - A\mathcal{L}^*(B) \\ &= -A(K+K^*)B + \sum_j \big(L_j^* ABL_j - L_j^* AL_j B - AL_j^* BL_j\big) \\ &= \sum_j [A^*, L_j]^*[B, L_j] - A\big(K + K^* + \sum_j L_j^* L_j\big)B \end{aligned} \tag{5.35}$$

Now, the parenthesis is $\mathcal{L}^*(\mathbb{1}) = 0$, and for $A = B^*$ the sum has positive terms, which consequently all have to vanish. Thus the jump operators must be multiples of the identity, and can be included in K. This yields an evolution generated by the Hamiltonian iK. □

Proposition 5.4.3. *Let $\mathcal{T}_t$ be a strongly standard semigroup such that $\mathcal{T}_t^*$ is an E_0-semigroup. Then $\mathcal{T}_t$ is unitarily implemented.*

Proof. An E_0-semigroups that maps compact operators to compact operators is necessarily of the form given in 5.1 with isometries S_t. Additionally, we have

$$\alpha_t^*(\mathbb{1}) = S_t S_t^* = \mathbb{1}, \tag{5.36}$$

so the implication follows. □

These propositions show that in some cases the differences between E_0-semigroups and CP_0-semigroups in general are surprisingly large and we need to be careful when transferring ideas from endomorphism semigroups to quantum dynamical maps.

One example of an E_0-semigroup with standard preadjoint, the CCR-flow, was already given in Section 4.5.2. Its product system is given by its fibers

$$\mathcal{E}_{\sigma^*}(t) = \Big\{ R \in \mathcal{B}(e^{L^2((0,\infty),\mathcal{K})}) | \sigma_t^*(A)R = RA \Big\} \tag{5.37}$$

for $t > 0$ and all $A \in \mathcal{B}(\Gamma^+(\mathcal{H}))$. The elements of $\mathcal{E}_{\sigma^*}(t)$ are of the form

$$R_\xi |e^\phi\rangle = |\xi\rangle \otimes |e^{S_t^* \phi}\rangle, \tag{5.38}$$

where $\xi \in \Gamma^+(L^2((0,t), dt; \mathcal{K}))$ and $\phi \in \mathcal{H}$. For an element $\psi \in \mathcal{H}$ we write $\psi_{(s,t)}$ for the function in $L^2(\mathbb{R}_+, dt; \mathcal{K})$ that is identically ψ in (s,t) and zero everywhere else. Then the units of the CCR flow are given by

$$U_t^{(a,\psi)}|e^{\phi}\rangle = e^{at}|e^{\psi_{(0,t)}+S_t^*\phi}\rangle. \tag{5.39}$$

with $a \in \mathbb{C}$, and for $t > 0$ the ranges of all finite products

$$U_{t_1}^{(a_1,\psi_1)} U_{t_2}^{(a_2,\psi_2)} \cdots U_{t_n}^{(a_n,\psi_n)} \tag{5.40}$$

span the Hilbert space $\Gamma^+(\mathcal{H})$. Therefore, the CCR flow is completely spatial. In fact, every completely spatial E_0-semigroup is cocycle conjugate to a CCR flow of a certain rank.

5.5 Dilations of cp_0-semigroups

A *cp_0-semigroup* $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H}_0)$ is the adjoint of a quantum dynamical semigroup $\mathcal{T}$ on $\mathfrak{T}(\mathcal{H}_0)$. We say $\mathcal{T}_t^*$ is a *CP_0-semigroup* if $\mathcal{T}_t^*$ is unital. In this section we describe how to dilate cp_0-semigroups to corresponding e_0-semigroups, thus integrating them in the classification given in Section 5.2.

Let α_t^* be an e_0-semigroup acting on $\mathcal{B}(\mathcal{H})$. A *corner* of $\mathcal{B}(\mathcal{H})$ is a von Neumann subalgebra $p\mathcal{B}(\mathcal{H})p = \mathcal{B}(\mathcal{H}_0)$, where p is a projection in $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}_0 = p\mathcal{H}$. We will assume p to be a *coinvariant* projection under α_t^*, i.e.

$$\alpha_t^*(1-p) \leq 1-p \tag{5.41}$$

for $t \geq 0$. If α_t^* is an E_0-semigroup, p is covariant if and only if p is *increasing*, i.e. if

$$\alpha_t^*(p) \geq p \tag{5.42}$$

for $t \geq 0$.

Proposition 5.5.1 ([Arv03, Prop. 8.1.2]). *Let p be a coinvariant projection for an e_0-semigroup α_t^* on $\mathcal{B}(\mathcal{H})$, and let $\mathcal{T}_t^*$ on $p\mathcal{B}(\mathcal{H})p = \mathcal{B}(\mathcal{H}_0)$ be defined by*

$$\mathcal{T}_t^*(A) = p\alpha_t^*(A)p \tag{5.43}$$

for $A \in \mathcal{B}(\mathcal{H}_0)$ and $t \geq 0$. Then $\mathcal{T}_t^*$ is a cp_0-semigroup. If α_t^* is unital, then $\mathcal{T}_t^*$ is unital in the sense that $\mathcal{T}_t^*(p) = p$.

In this setting, one calls $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$ the *dilation* of $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$, and $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ is called the *compression* of $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$.

Given $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ with $\|\mathcal{T}_t^*\| \leq 1$, there is a family $\mathcal{D}(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ of all dilations of $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ and there exists a natural hierarchy. Given two dilations $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$, $(\mathcal{B}(\tilde{\mathcal{H}}), \tilde{\alpha}_t^*, \tilde{p})$, we write

$$(\mathcal{B}(\mathcal{H}), \alpha_t^*, p) \geq (\mathcal{B}(\tilde{\mathcal{H}}), \tilde{\alpha}_t^*, \tilde{p}) \tag{5.44}$$

if there is $*$-homomorphism $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\tilde{H})$ with $\theta(A) = A$ for $A \in \mathcal{B}(\mathcal{H}_0)$ and

$$\theta \circ \alpha_t^* = \tilde{\alpha}_t^* \circ \theta. \tag{5.45}$$

The two dilations are called *equivalent* if θ is an isomorphism, i.e., if each dilation dominates the other. If a dilation $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$ is dominated by every other dilation in $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$, it can be characterized, up to isomorphism, in the following way.

Definition 5.5.2. *A dilation* $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$ *of* $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ *is called* minimal *if* $\mathcal{B}(\mathcal{H})$ *is a von Neumann algebra generated by* $\bigcup_{t\geq 0} \alpha_t^*(\mathcal{B}(\mathcal{H}_0))$.

We will also use the terminology α_t^* is minimal over $\mathcal{T}_t^*$ if $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$ is a minimal dilation of $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$.

Let α_t^* be an E_0-semigroup acting on $\mathcal{B}(\mathcal{H})$. A *multiplicative projection* for α_t^* is an increasing projection $q \in \mathcal{B}(\mathcal{H})$ with the property that the CP_0-semigroup β_t^* defined on $q\mathcal{B}(\mathcal{H})q$ by

$$\beta_t^*(A) = q\alpha_t^*(A)q \tag{5.46}$$

for $A \in q\mathcal{B}(\mathcal{H})q$ is an E_0-semigroup, i.e. for $A, B \in q\mathcal{B}(\mathcal{H})q$

$$\beta_t^*(AB) = \beta_t^*(A)\beta_t^*(B). \tag{5.47}$$

So, given a dilation $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$ of $(\mathcal{B}(\mathcal{H}_0), \mathcal{T}_t^*)$ and a multiplicative projection $q \geq p$, the semigroup $\beta_t^*(A) = q\alpha_t^*(A)q$ gives rise to another dilation $(q\mathcal{B}(\mathcal{H})q, \beta_t^*, p)$. An example of this is the construction of the Type I part of an E_0-semigroup, given in Section 5.2. If q is an increasing projection in $\mathcal{B}(\mathcal{H})$, then it is multiplicative if and only if it commutes with $\alpha_t^*(q\mathcal{B}(\mathcal{H})q)$ for every $t \geq 0$.

Lemma 5.5.3 ([Arv03, Cor. 8.9.6])**.** *An E_0-semigroup α_t^* is minimal over a CP_0-semigroup $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H}_0) = p\mathcal{B}(\mathcal{H})p$ if and only if the only multiplicative projection $q \in \mathcal{B}(\mathcal{H})$ satisfying $q \geq p$ is the trivial projection $q = \mathbb{1}$.*

For the remainder of this section, we will assume $\mathcal{T}_t^*$ to be a CP_0-semigroup, so its minimal dilation is given by $(\mathcal{B}(\mathcal{H}), \alpha_t^*, p)$, where α_t^* is an E_0-semigroup and p is a nonzero increasing projection. The Hilbert space $\mathcal{H}$ is spanned by the set of vectors

$$\{\alpha_{t_1}^*(A_1)\alpha_{t_2}^*(A_2)\cdots\alpha_{t_n}^*(A_n)\xi | A_i \in p\mathcal{B}(\mathcal{H})p, t_i \geq 0, \xi \in p\mathcal{H}\}. \tag{5.48}$$

Definition 5.5.4. *A* unit *for $\mathcal{T}_t^*$ is a strongly continuous semigroup of bounded operators U_t on $\mathcal{H}_0$ with the following property: there is a real constant k such that for every $t \geq 0$ the operator mapping $\Lambda_t : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_0)$ given by*

$$\Lambda_t(A) = e^{kt}\mathcal{T}_t^*(A) - U_t A U_t^* \tag{5.49}$$

is completely positive.

We write $\mathcal{U}_{\mathcal{T}^*}$ for the set of units of $\mathcal{T}_t^*$. Let α_t^* on $\mathcal{B}(\mathcal{H})$ be an E_0-semigroup, then every unit of α_t^* in the sense of Definition 5.2.1 is also a unit in the more general sense of Definition 5.5.4.

Proposition 5.5.5 ([Arv03, Prop. 8.10.2])**.** *Let α_t^* on $\mathcal{B}(\mathcal{H})$ be a dilation over $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H}_0)$, (not necessarily minimal), and let $\mathcal{E}_{\alpha^*}$ be its product system. For every $t \geq 0$, the subspace $\mathcal{H}_0 = p\mathcal{H}$ of $\mathcal{H}$ is invariant under the set of operators in $(\mathcal{E}_{\alpha^*})^*$. Moreover, for every unit S_t of α^*, the semigroup of operators U_t in $\mathcal{B}(\mathcal{H}_0)$ defined by*

$$U_t^* = S_t^*|_{\mathcal{H}_0} \tag{5.50}$$

for $t \geq 0$ is a unit of $\mathcal{T}_t^*$.

This means, there is a natural map $\theta : \mathcal{U}_{\alpha^*} \to \mathcal{U}_{\mathcal{T}^*}$ given by $\theta(S_t)^* = S(t)^*|_{\mathcal{H}_0}$. If α_t^* is minimal over $\mathcal{T}_t^*$, then θ is a bijection.

Lemma 5.5.6 ([Arv03, Lem. 8.10.5] and [Mar03, Lem. 1.2]). *Let U_t be a unit of $\mathcal{T}_t^*$ and for every $t \geq 0$ let q_t be the projection onto*

$$\overline{span}\{A\xi | A \in \alpha_t^*(\mathcal{B}(\mathcal{H})), \xi \in \mathcal{H}_0 = p\mathcal{H}\} \tag{5.51}$$

Then for every $t > 0$ there is a unique operator $V_t \in \mathcal{E}_{\alpha^*}(t)$ satisfying $q_t V_t = V_t$ and $V_t^*|_{\mathcal{H}_0} = U_t^*$. Additionally, there is a real constant k such that $\|V_t\| = e^{kt}$.

This family V_t is a section of the product system of α, but it does not satisfy the semigroup property. It is, however, possible to construct a unit of α_t^* from these operators. For a fixed $t > 0$ and a finite partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$ consider the operator

$$V_{\mathcal{P},t} = V_{t_1 - t_0} V_{t_2 - t_1} \cdots V_{t_n - t_{n-1}}. \tag{5.52}$$

$V_{\mathcal{P},t}$ belongs to $\mathcal{E}_{\alpha^*}(t)$ and since $\|V_{\mathcal{P},t}\| = e^{kt}$, the map $\mathcal{P} \mapsto V_{\mathcal{P},t}$ defines a bounded net of operators belonging to the weak*-closed operator space $\mathcal{E}_{\alpha^*}(t)$. This net converges in weak operator topology and for its limit

$$S_t = \lim_{\mathcal{P}} V_{\mathcal{P},t} \tag{5.53}$$

we have $S_{s+t} = S_s S_t$. S_t is strongly continuous in t and, thus, a unit of α_t^* that satisfies equation (5.50).

5.6 The product system of a CP_0-semigroup

Let $\mathcal{E}$ be a linear subset of $\mathcal{B}(\mathcal{H}_0)$, not necessarily norm close or stable under the adjoint operation. Let $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ denote an inner product on $\mathcal{E}$. Then a family X_i of operators in $\mathcal{E}$ is called an orthonormal basis of $\mathcal{E}$, if it satisfies $\langle X_i, X_j \rangle = \delta_{ij}$ and there is no nonzero element $A \in \mathcal{E}$ such that $\langle X_i, A \rangle = 0$ for all i. One can define an inner product on the tensor product $\mathcal{H}_0 \otimes \mathcal{E}$ via

$$\langle \phi \otimes X, \psi \otimes Y \rangle = \langle \phi, \psi \rangle \langle X, Y \rangle_{\mathcal{E}}, \tag{5.54}$$

so that its completion becomes a Hilbert space. Let $M : \mathcal{H}_0 \otimes \mathcal{E} \to \mathcal{H}_0$ denote the linear multiplication map

$$M(\phi \otimes X) = X\phi. \tag{5.55}$$

Definition 5.6.1. *The operator space $\mathcal{E} \subset \mathcal{B}(\mathcal{H}_0)$ together with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is called a* metric operator space *if the inner product satisfies one and therefore all of the following conditions.*

1. *The multiplication $M : \mathcal{H}_0 \otimes \mathcal{E} \to \mathcal{H}_0$ is a bounded linear operator.*
2. *For every orthonormal basis X_i of $\mathcal{E}$ and $\phi \in \mathcal{H}_0$, we have*

$$\sum_i \|X_i^* \phi\|^2 < \infty. \tag{5.56}$$

3. *There is an orthonormal basis X_i of $\mathcal{E}$, such that for all $\phi \in \mathcal{H}_0$*

$$\sum_i \|X_i^* \phi\|^2 < \infty. \tag{5.57}$$

In this case, the adjoint of M is given by $M^* : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{E}$ with

$$M^*(\phi) = \sum_i X_i^* \phi \otimes X_i, \tag{5.58}$$

where X_i is an orthonormal basis of $\mathcal{E}$.

Every metric operator space gives rise to a completely positive map on $\mathcal{B}(\mathcal{H}_0)$ via

$$T_{\mathcal{E}}^*(A) = \sum_i X_i A X_i^*. \tag{5.59}$$

On the other hand, every completely positive map on $\mathcal{B}(\mathcal{H}_0)$ determines a metric operator space

$$\mathcal{E}_{T^*} = \{X \in \mathcal{B}(\mathcal{H}_0) | \exists k > 0 \text{ s.t. } XAX^* \leq kT^*\} \tag{5.60}$$

The connection is given by the minimal Stinespring dilation of T^*,

$$T^*(A) = V^*(A \otimes \mathbb{1})V, \tag{5.61}$$

where $V : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_E$ can be written in terms of Kraus operators $K_i \in \mathcal{B}(\mathcal{H}_0)$

$$V\phi = \sum_i K_i \phi \otimes e_i, \tag{5.62}$$

(as described in Section 3.4). One can identify $\mathcal{H}_E$ with the Hilbert space completion of $\mathcal{E}_{T^*}$, denoted by $\bar{\mathcal{E}}_{T^*}$, i.e. there is a unitary operator $W : \mathcal{H}_E \to \bar{\mathcal{E}}_{T^*}$ with

$$W\xi = \big((\mathbb{1} \otimes \langle \xi |)V\big)^*. \tag{5.63}$$

W maps the basis elements e_i to the adjoints of the corresponding Kraus operators that are a basis for $\bar{\mathcal{E}}_{T^*}$. With this identification, V coincides with the adjoint of multiplication M^*, and we have

$$T^*(A) = M(A \otimes \mathbb{1})M^*. \tag{5.64}$$

Therefore, the association between completely positive maps and metric operator spaces is a bijection.

We now take a look at the concatenation $T^* = T_1^* T_2^*$ of two completely positive maps T_1^* and T_2^* in terms of their metric operators spaces $\mathcal{E}_i = \bar{\mathcal{E}}_{T_i^*}$ and $\mathcal{E}_{T^*}$. With the Stinespring isometries $V_i : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{E}_i$ one can write T^* as

$$T^*(A) = V_1^*(V_2^*(A \otimes \mathbb{1}_2)V_2 \otimes \mathbb{1}_1)V_1 \tag{5.65}$$

where $\mathbb{1}_i$ denotes the identity operator in $\mathcal{B}(\mathcal{E}_i)$. This defines an operator $V : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{E}_2 \otimes \mathcal{E}_1$ with

$$V\phi = \sum_{i,j} S_i^* X_j^* \phi \otimes S_i \otimes X_j \tag{5.66}$$

where X_i and S_i are a basis of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. The Stinespring dilation determined by V is not necessarily minimal. However, the set $\mathcal{E}_{T^*}$ contains the set of all products $\{SX|X \in \mathcal{E}_1, S \in \mathcal{E}_2\}$, and there is a unique bounded linear map $M : \mathcal{E}_2 \otimes \mathcal{E}_1 \to \mathcal{E}_{T^*}$ given by

$$M(S \otimes X) = SX \tag{5.67}$$

for $X \in \mathcal{E}_1$ and $S \in \mathcal{E}_2$, such that $V : \mathcal{H}_0 \to \mathcal{E}_{T^*}$ is the minimal Stinespring dilation for T^* with

$$\mathcal{H}_0 \otimes \mathcal{E}_{T^*} = \overline{\text{span}}\{(A \otimes \mathbb{1})V\phi|A \in \mathcal{B}(\mathcal{H}_0), \phi \in \mathcal{H}_0\}, \tag{5.68}$$

The adjoint of M is a Hilbert space isometry $M^* : \mathcal{E}_{T^*} \to \mathcal{E}_2 \otimes \mathcal{E}_1$ called *comultiplication*, so the range of M is all of $\mathcal{E}_{T^*}$, and $\mathcal{E}_{T^*}$ is the span of products $\{SX|X \in \mathcal{E}_1, S \in \mathcal{E}_2\}$.

In the case in which T^* is a normal endomorphism, the metric operator space reduces to the intertwining space

$$\mathcal{E}_{T^*} = \{R \in \mathcal{B}(\mathcal{H}_0)|T^*(A)R = RA\}. \tag{5.69}$$

With these preparations, it is possible to construct a product system for a CP_0-semigroup $\mathcal{T}_t^*$. Given a partition $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of the interval $[0, t]$, we can define a Hilbert space by

$$\mathcal{E}_{T^*}^{\mathcal{P}} = \mathcal{E}_{T^*}(t_1 - t_0) \otimes \ldots \otimes \mathcal{E}_{T^*}(t_n - t_{n-1}) \tag{5.70}$$

where $\mathcal{E}_{T^*}(s)$ denotes the metric operator space corresponding to the completely positive map T_s^*. Then for two partitions $\mathcal{P} \subset \mathcal{Q}$ of $[0, t]$ there is an isometric comultiplication map

$$M_{\mathcal{QP}}^* : \mathcal{E}_{T^*}^{\mathcal{P}} \to \mathcal{E}_{T^*}^{\mathcal{Q}} \tag{5.71}$$

such that for $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{R}$ we have

$$M_{\mathcal{RP}}^* = M_{\mathcal{RQ}}^* M_{\mathcal{QP}}^*. \tag{5.72}$$

This allows us to take the inductive limit of the Hilbert spaces $\mathcal{E}_{T^*}^{\mathcal{P}}$ to obtain the Hilbert space

$$\mathcal{E}_{T^*}(t) = \lim \mathcal{E}_{T^*}^{\mathcal{P}}. \tag{5.73}$$

In his paper [Mar03], D. Markiewicz showed that if α_t^* is a minimal E_0-semigroup over $\mathcal{T}_t^*$, then there exists an admissible Hilbert bundle structure on $\mathcal{E}_{T^*}(t)$ with respect to which there is an isomorphism of measurable Hilbert bundles between $\mathcal{E}_{T^*}(t)$ and the product system $\mathcal{E}_{\alpha^*}$.

5.7 Notes and Remarks

The main reference for E_0-semigroups is certainly [Arv03]. It is also a compilation of his results in various papers, most importantly [Arv89a; Arv90a; Arv89b; Arv90b], and offers a high level of detail. Section 5.1 is basically a selection of [Arv03, Ch. 2], as is Section 5.2. Most of the definitions and concepts can also be found (in a rather compressed version) in [Pow99]. Note, that W. Arveson uses a slightly different terminology in his early papers. Most importantly, when writing about *spatial* endomorphism semigroups in [Arv89a], he actually refers to endomorphism semigroups that are *completely spatial*.

The main objective of R. T. Powers, W. Arveson and their coauthors was defining a numerical invariant for the classification of E_0-semigroups. The idea for an *index* was introduced by R. T. Powers [Pow88], still containing some ambiguity, and he and D. W. Robinson proposed a different approach in [PR89]. The final definition taking into account cocycle conjugacy was offered by W. Arveson in [Arv89a]: The covariance function defined in (5.10) defines a positive semidefinite inner product on $C_0(\mathcal{U}_{\alpha^*})$, the space of complex functions $f : \mathcal{U}_{\alpha^*} \to \mathbb{C}$ that vanish off some finite subset of $\mathcal{U}_{\alpha^*}$ and satisfy $\sum_{U_t \in \mathcal{U}_{\alpha^*}} f(U_t) = 0$. Thus $C_0(\mathcal{U}_{\alpha^*})$ can be completed to a Hilbert space $\mathcal{H}(\mathcal{U}_{\alpha^*})$. The Arveson index is then the dimension of this Hilbert space if $\mathcal{U}_{\alpha^*} \neq \emptyset$.

Section 5.3 is composed of the work of R. T. Powers and G. Price [PP90; Pow91]. The proofs for Theorems 5.3.3 and 5.3.4 are elaborate, and we highly encourage the interested reader to work through Chapter 3 and 4 of [Pow91] to ge a better understanding of the structure of pure states in the domain of the preadjoint generator of endomorphism semigroups.

The Chapters 8 and 9 of [Arv03] (stated there in more general terms) are the main references for Section 5.5 and 5.6, respectively. As a side note, the construction of a product system for CP_0-semigroups bears a close resemblance to the ideas in [Neu16] in the construction of a GKLS-generator decribing a measurement process. In this context, we would also like to recommend the works of B. V. R. Bhat. The dilation of CP_0-semigroups on $\mathcal{B}(\mathcal{H})$ to semigroups of endomorphism was already developed in [Bha96; Bha99]. More details on dilations and product systems gives [BS00], and a study of product system in the language of Hilbert modules can be found in [Bha05].

In the context of this chapter, Section 5.4 is a bit out of line, since it does not build on the sources mentioned, but rather connects Section 5.3 with our results of Chapter 4. We thought it would be helpful to have such a link and answer the obvious questions arising from the similarities at this point.

Chapter 6

The type of standard semigroups

This chapter will examine how the standard semigroups defined in Chapter 4 fit into the characterization given in Section 5.1. Standard semigroups are defined in the Schrödinger picture and the classification of endomorphism semigroups is in the Heisenberg picture. As we see those pictures as equivalent, we feel the need to simplify the terminology.

Definition 6.0.1. *We call a quantum dynamical semigroup in the Heisenberg picture Type I, II or III if its minimal dilation is Type I, II or III, respectively. In the same way, we say a quantum dynamical semigroup in the Schrödinger picture is of Type I, II or III if its adjoint in the Heisenberg picture is.*

On the other hand, we will speak of standard semigroups in the Heisenberg picture if the preadjoint is a standard semigroup as defined in Chapter 4. Note, that we must require standard semigroups to be conservative, since the classification of E_0-semigroups specifically demands unitality.

6.1 The type of a conservative standard semigroup

In his paper [Arv99], W. Arveson determined the type of norm continuous CP_0-semigroups. This section will follow the general idea of his proof and apply it to conservative standard semigroups, whose generators have the set $(\mathrm{dom}\, K)^{\chi}$ as a core (see Section 4.4). We begin with a simple observation.

Lemma 6.1.1. *Let $\mathcal{T}_t^*$ be a CP_0-semigroup on $\mathcal{B}(\mathcal{H}_0) = \mathcal{B}(p\mathcal{H})$ and let its minimal dilation α_t^* on $\mathcal{B}(\mathcal{H})$ be spatial. Then the domains of the generators of S_t^* coincide for all units $S_t \in \mathcal{U}_{\mathcal{T}^*}$.*

Proof. This is a direct consequence of Lemma 5.3.2. Every unit S_t of $\mathcal{T}_t^*$ is determined by a unit U_t of α_t^* via

$$S_t^* = U_t^*|_{\mathcal{H}_0}. \tag{6.1}$$

Since the domains of all U_t^* with $U_t \in \mathcal{U}_{\alpha^*}$ coincide and $U_t^*(\mathcal{H}_0) \subset \mathcal{H}_0$ (see Proposition 5.5.5), the domains of S_t^* with $S_t \in \mathcal{U}_{\mathcal{T}^*}$ are simply given by applying p and the statement follows. □

Let $\mathcal{T}_t$ be a standard semigroup on $\mathfrak{T}(\mathcal{H}_0)$ with generator

$$\mathcal{L}\rho = K\rho + \rho K^* + \sum_i L_i \rho L_i^* \tag{6.2}$$

for $\rho \in (\operatorname{dom} K)^{)(}$ that has $(\operatorname{dom} K)^{)(}$ as a core. As the semigroup is larger than the no-event semigroup $\mathcal{T}_t^0$ in completely positive ordering (see Chapter 4, equation (4.29)), the same is true in the Heisenberg picture, and

$$\Lambda_t(A) = e^{kt}\mathcal{T}_t^*(A) - C_t^* A C_t \tag{6.3}$$

is completely positive for contraction semigroups $C_t = e^{tK}$ on $\mathcal{H}_0$, even for $k = 0$. Therefore, C_t^* is a unit for $\mathcal{T}_t^*$. The same holds for the contraction semigroups we get by gauging (see Lemma 4.1.3), so the (multiples of)

$$U_t = e^{tK'^*}, \tag{6.4}$$

where K' is of the form

$$K'\phi = K\phi + \sum_i \overline{\lambda_i}\, L_i\phi + \frac{1}{2}\Big(i\beta + \sum_i |\lambda_i|^2\Big)\phi, \tag{6.5}$$

certainly are units of $\mathcal{T}_t^*$. In fact, every unit of $\mathcal{T}_t^*$ is of this form.

Lemma 6.1.2. *Let $\mathcal{T}_t^*$ be a CP_0-semigroup and let S_t be a unit of $\mathcal{T}_t^*$ with generator D. Assume that $(\operatorname{dom} D^*)^{)(}$ is a core for the generator of $\mathcal{T}_t$. Then $\mathcal{T}_t$ is the minimal solution over the no-event semigroup $\mathcal{T}_t^0\rho = S_t^*\rho S_t$.*

Proof. By replacing S_t with $e^{-t\frac{k}{2}}S_t$ if necessary, we get that

$$\mathcal{T}_t^*(A) - S_t A S_t^* \tag{6.6}$$

is completely positive. In the Schrödinger picture, $(\operatorname{dom} D^*)^{)(}$ is a core for both terms, and the map

$$\mathcal{T}_t(\rho) - S_t^*\rho S_t = (\mathcal{T}_t(\rho) - \rho) - (S_t^*\rho S_t - \rho) \tag{6.7}$$

is completely positive for all $\rho \in (\operatorname{dom} D^*)^{)(}$. Thus, we can divide by t and take the limit $t \to \infty$ to see that on $(\operatorname{dom} D^*)^{)(}$ the generator $\mathcal{L}$ is given by

$$\mathcal{L}\rho = D^*\rho + \rho D + \mathcal{P}(\rho), \tag{6.8}$$

where $\mathcal{P}(\rho)$ is a completely positive perturbation of the no-event semigroup with generator $D^*\rho + \rho D$. In this case, D^* is of the form (6.5). $\square$

Lemma 6.1.3. *Let $\mathcal{T}_t^*$ and $\mathcal{Q}_t^*$ be two adjoints of conservative standard semigroups with the same set of units, obtained by gauging the adjoints $C_t = e^{tK}$. Then $\mathcal{T}_t = \mathcal{Q}_t$.*

Proof. Assume we have $\mathcal{T}_t \neq \mathcal{Q}_t$. Since $(\operatorname{dom} K)^{)(}$ is a core for both generators (see Proposition 4.4.2), their standard generators need to differ on this set. However, since they have the same sets of units, there are operators K and K' that generate contraction semigroups on $\mathcal{H}_0$ and satisfy

$$K'\phi = K\phi + \sum_i \overline{\lambda_i}\, L_i\phi + \frac{1}{2}\Big(i\beta + \sum_i |\lambda_i|^2\Big)\phi \tag{6.9}$$

and

$$K'\phi = K\phi + \sum_j \overline{\mu_j}\, \tilde{L}_j\phi + \frac{1}{2}\Big(i\gamma + \sum_j |\mu_j|^2\Big)\phi \tag{6.10}$$

with jump operators L_i and $\tilde{L}_i$ that determine completely positive perturbations

$$\mathcal{P}(|\phi\rangle\langle\psi|) = \sum_i |L_i\phi\rangle\langle L_i\psi| \quad \text{and} \quad \tilde{\mathcal{P}}(|\phi\rangle\langle\psi|) = \sum_i |\tilde{L}_i\phi\rangle\langle\tilde{L}_i\psi|. \tag{6.11}$$

We can assume, that L_i and $\tilde{L}_i$ are not multiples of the identity, so they satisfy

$$\sum_i \overline{\lambda_i}\, L_i\phi = \sum_j \overline{\mu_j}\, \tilde{L}_j\phi \tag{6.12}$$

Thus, the two sets of jump operators are linear combinations of each other, and we have

$$\mathcal{P}(|\phi\rangle\langle\psi|) = c \cdot \tilde{\mathcal{P}}(|\phi\rangle\langle\psi|). \tag{6.13}$$

Since these are completely positive and we assumed both semigroups to be conservative, the generators of both semigroups coincide on a core, and the statement follows. □

Lemma 6.1.4. *Let α_t^* on $\mathcal{B}(\mathcal{H})$ be a minimal dilation over $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H}_0) = \mathcal{B}(p\mathcal{H})$. Let $U_t \in \mathcal{U}_{\alpha^*}$ be a unit of α with generator $-d$ and let $S_t \in \mathcal{U}_{\mathcal{T}^*}$ be given by $S_t^* = U_t^*|_{\mathcal{H}_0}$ with generator K. If $(\operatorname{dom} d^*)^{\between}$ is a core for the generator δ_* of α_t on $\mathfrak{T}(\mathcal{H})$, then $(\operatorname{dom} K)^{\between}$ is a core for the generator $\mathcal{L}$ of $\mathcal{T}_t$ on $\mathfrak{T}(\mathcal{H}_0)$.*

Proof. With Proposition 5.5.5 we know that the contraction semigroup U_t^* leaves the subspace $p\mathcal{H}$ invariant, so $U_t^*(\mathcal{H}_0) \subset \mathcal{H}_0$. Thus, we have

$$\lim_{t\to 0} \frac{1}{t}\|S_t^*\phi - \phi\| = \lim_{t\to 0} \frac{1}{t}\|U_t^*\phi - \phi\| \tag{6.14}$$

for all $\phi \in \mathcal{H}_0$. Therefore, $\operatorname{dom} K = p \operatorname{dom} d^*$. As $(\operatorname{dom} d^*)^{\between}$ is a core for δ_*, it is dense in $\mathcal{B}(\mathcal{H})$ and invariant under the action of α_t. Thus,

$$p(\operatorname{dom} d^*)^{\between} p = \text{span}\big\{|\mu\rangle\langle\nu| \;|\mu, \nu \in \operatorname{dom} K\big\} \tag{6.15}$$

is dense in $p\mathcal{B}(\mathcal{H})p = \mathcal{B}(p\mathcal{H})$. It remains to show that $(\operatorname{dom} K)^{\between}$ is invariant under $\mathcal{T}_t$. For $\phi, \psi \in \operatorname{dom} d^*$, the action of $\mathcal{T}_t$ is given by

$$\mathcal{T}_t(|p\phi\rangle\langle p\psi|) = p\alpha_t(|p\phi\rangle\langle p\psi|)p. \tag{6.16}$$

As $(\operatorname{dom} d^*)^{\between}$ is invariant under the action of α_t, $\alpha_t(|p\phi\rangle\langle p\psi|)$ is again an element in $(\operatorname{dom} d^*)^{\between}$, so

$$\mathcal{T}_t\left((\operatorname{dom} K)^{\between}\right) \subset (\operatorname{dom} K)^{\between} \tag{6.17}$$

and $(\operatorname{dom} K)^{\between}$ is a core for $\mathcal{L}$. □

Theorem 6.1.5. *A conservative semigroup $\mathcal{T}_t$ is standard if and only if it is of Type I.*

Proof. Let $\mathcal{T}_t$ be a conservative standard semigroup. Then Proposition 4.4.2 says that $(\operatorname{dom} K)^{\between}$ is a core for the generator of $\mathcal{T}_t$. Let α_t^* on $\mathcal{B}(\mathcal{H})$ be minimal over $\mathcal{T}_t^*$ such that $\mathcal{T}_t^*(A) = p\alpha_t^*(A)p$. Assume α_t^* is not of Type I. Then we can construct a semigroup β_t^* as in Section 5.1 from the units of α_t^*, and $\beta_t^*(\mathbb{1}_{\mathcal{H}})$ converges to a projection q as $t \to \infty$. The restriction of β_t^* to $q\mathcal{B}(\mathcal{H})q$ is the Type I part of α_t^*, which we can also obtain by compressing α_t^*. Since p is coinvariant under β_t^*, the compression

$$\mathcal{Q}_t^*(A) = p\beta_t^*(A)p \tag{6.18}$$

of β_t^* to $\mathcal{B}(\mathcal{H}_0)$ is a cp_0-semigroup, such that

$$\mathcal{T}_t^* - \mathcal{Q}_t^* \tag{6.19}$$

is completely positive. Obviously, every unit of $\mathcal{Q}_t^*$ is also a unit of $\mathcal{T}_t^*$. So, let $S_t \in \mathcal{U}_{T^*}$ be a unit of $\mathcal{T}_t^*$. Then there exists a unit $U_t \in \mathcal{U}_{\alpha^*}$ such that

$$S_t^* = U_t^*|_{\mathcal{H}_0} \tag{6.20}$$

By definition, U_t is also a unit of β, so

$$e^{tk}\beta_t^*(A) - U_tAU_t^* \tag{6.21}$$

is completely positive for a $k \geq 0$. Multiplying from both sides with p, we see that S_t is also a unit of $\mathcal{Q}_t^*$. So $\mathcal{U}_{\mathcal{T}}^* = \mathcal{U}_{\mathcal{Q}}^*$ and with Lemma 6.1.3 it follows that $\mathcal{T}_t^* = \mathcal{Q}_t^*$. We can now see that

$$p\beta_t^*(p)p = \mathcal{Q}_t^*(p) = \mathcal{T}_t^*(p) = p, \tag{6.22}$$

so $\beta_t^*(p) \geq p$ and taking the limit $t \to \infty$ we get $q \geq p$. Since α_t^* is minimal over $\mathcal{T}_t^*$, Lemma 5.5.3 gives $q = \mathbb{1}_{\mathcal{H}}$ and α_t^* is of Type I.
Now assume, that $\mathcal{T}_t^*$ is a CP_0-semigroup on $\mathcal{B}(\mathcal{H}_0)$ of Type I. Let α_t^* on $\mathcal{B}(\mathcal{H})$ be minimal over $\mathcal{T}_t^*$ on $\mathcal{B}(\mathcal{H}_0) = \mathcal{B}(p\mathcal{H})$ and let δ_* denote the generator of α_t. Let $S_t \in \mathcal{U}_{\mathcal{T}^*}$ be a unit of $\mathcal{T}_t^*$ and let K be the generator of S_t^*. Since α_t^* is of Type I, the set $(\operatorname{dom} \delta_*)^{\between}$ is a core (see Theorem 5.3.4), and with Lemma 6.1.4 we know that $(\operatorname{dom} K)^{\between}$ is a core for $\mathcal{T}_t$. By Lemma 6.1.2, $\mathcal{T}_t$ is standard. □

6.2 A type classification for standard semigroups

It remains the question for a classification of non-conservative standard semigroups and of the examples given in 4.6. The common factor of all the semigroups in Chapter 4 is that they are constructed as a series of one or more perturbations with corresponding minimal solutions of a no-event semigroup. If such semigroups are conservative, then by construction, they are either of Type I or of Type II. They are of Type I if its generator has $(\operatorname{dom} K)^{\between}$ as a core. The proof of this is quite simple. If a semigroup is constructed as one or more perturbations of a no-event semigroup and has $(\operatorname{dom} K)^{\between}$ as a core, then it can be constructed with just one perturbation, and it is standard. Therefore, if it is conservative, it is of Type I. Even if $(\operatorname{dom} K)^{\between}$ is not a core, it is contained in the domain of its generator, as are all pure states in $(\operatorname{dom} K)^{\between}$. Theorem 5.3.3 ensures that $\mathcal{T}_t$ is spatial, and therefore of Type II. With this reasoning, it is easy to see that all conservative non-standard examples in Chapter 4 are of Type II.

Arveson's classification of endomorphism semigroups, and therefore of completely positive on parameter semigroups, is restricted to *unital* semigroups. The main reason is that he based his classification on the classification of isomorphic product systems, and Theorem 5.1.6 is explicitly only valid for conservative semigroups.

The key feature in our proof of Theorem 6.1.5 is whether the ketbra content in the domain of the generator is a core. So, with this in mind, we suggest the following classification:

Definition 6.2.1. *We call a standard semigroup of* Type I *if the ketbra domain is a core for its generator. Otherwise it is of* Type II.

In other words, a standard semigroup is of Type I if the generator of its minimal solution is the closure of the perturbed no-event generator. This definition keeps the intuition that Type I semigroups are completely determined by their set of units. For Type II standard semigroups, however, we see a perturbation part with support outside the ketbra domain. The broadening of this classification to all semigroups that are constructed as a series of positive perturbations of a no-event semigroup is then straightforward, as by construction, they also are always of Type I or Type II.

Conclusions and outlook

We have explicitly defined a notion of unbounded GKLS-generators, which we feel summarizes an agreement in the literature [Dav77; Dav79; Fag99; Hol95; Hol96b]. While in the conclusion of [SHW17] we wrote that it was unclear how our notion of standardness and Arveson's Type I are exactly related, this question has been answered in Chapter 6 of this thesis. The main result is certainly Theorem 6.1.5, stating that conservative semigroups are standard if and only if they are of Type I. The class of non-standard semigroups given in Section 4.6 is of Type II. The key feature in our answer was the mutual domain of all generators of preadjoints of units and whether the corresponding ketbra domain was a core for the generator of the standard semigroup.

Therefore, we suggested to extend this classification to non-conservative standard semigroups and even to all semigroups that are constructed as a series of positive perturbations of a no-event semigroup. Within this classification, not only the non-standard semigroups in Section 4.6 are of Type II. Those examples were constructed based on standard semigroups, for which the minimal solution led to a domain increase, such that the ketbra domain was no longer a core. These standard semigroups are then also of Type II in the broader meaning.

In many ways, this gives us a much better understanding of non-standard semigroups, especially of those that are spatial. The question remains, how does Type I lie within Type II. Can all spatial semigroups be constructed as a series of perturbations of no-event semigroups? Or can we always write a Type II semigroup as a direct sum of a Type I and a Type III semigroup? The answer may lie in the lines of a Type I part of such semigroups, analogue to the Type I part of an endomorphism semigroup defined in Section 5.2. A big issue here lies with the differences between the domains involved. It could be helpful to understand, how to construct a unit of an endomorphism semigroup from a pure state in the domain of its preadjoint generator. Remark 5.4.1 already hints to a relation between spatialness and standardness.

Another starting point for further research is the set of ketbras in the domains. Proposition 4.4.5 is explicitly formulated for strongly standard semigroups. It is not known, whether strong standardness is necessary here. Is it possible to prove a generalization to all standard semigroups? Or is there a counter example, where the minimal solution leads to additional ketbras? To phrase it in a diferent way: Can we reconstruct $\operatorname{dom} K$ from the standard generator $\mathcal{L}$?

Finally there is the question about the interpretation of Type III semigroups, that is of semigroups with no units at all [Arv03; Pow87]. As we have seen in Section 5.3, these have no pure states in the domain of their generator. As standard semigroup are always either Type I or Type II, there is no part of the semigroup, that maps pure states to multiples of pure states. The implications for physical interpretation here remain vague, and this certainly leaves room for further research.

List of Figures

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List of Publications

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