IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received July 28, 2021, accepted August 6, 2021, date of publication August 11, 2021, date of current version August 18, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3104013

A Contribution to the Theory of Rotating Electrical Machines

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ABSTRACT This contribution introduces a vector interpretation to unify the description of torque development in rotating electrical machines. The motivation behind this attempt is the necessity of applying different physical interpretation and calculation methods, such as Lorentz force, Maxwell stress tensor, co-energy methods etc., for predicting and estimating different torque components, e.g. synchronous torque, oscillating torque, cogging torque, reluctance torque etc., in different rotating electrical machines. The unified vector model describes and estimates the different torque components in rotating electrical machines with an apprehensible concise formulation. Beyond that, the other most frequently used tools and terms in the field of electrical machines, such as Park and Clark transformations, phasor diagram etc., can be derived directly from this model, which has also a simplifying didactic contribution to a conventional course of electrical machines.

INDEX TERMS Rotating electrical machines, Lorentz force, Maxwell stress tensor, reluctance force, cogging torque.

I. INTRODUCTION

For a complete and convenient description of torque formation in electrical machines, several interpretations are used in different cases and thus terms like reluctance torque, synchronous torque, cogging torque, hysteresis torque, which are estimated with different methods such as rotating field theory, Lorentz force [1], co-energy [2], Maxwell Stress Tensor [3], MMF method [4] etc., are useful. In addition, there are other mathematical tools for a simple analysis and description of the behavior of the electrical machines, such as phasor diagram, Park and Clark transformation, Goerges diagram, etc. The fact that the complete description of the functioning principle of electrical machines requires different tools and interpretations raises the question whether there is a unified description, from which different torque interpretations and mathematical tools can be derived. Any attempt to create a unified model to describe electrical machines will only be successful, if a compressed mathematical model can be used to describe all the properties and behaviors of electrical machines in a unified manner.

A conventional course of electrical machines usually begins with the fundamentals of magnetic circuits and traditionally continues with describing the functionality of permanent magnet DC machines with Lorentz force formulation at least in case the armature conductors are placed around a slotless rotor. This method describes correctly the resulting torque. However, a paradox arises as the conductors rest in the slot. To solve this, an alternative method for describing the force in DC machines is usually used based on the formulation of the mechanical output power, as a product results of armature current and the induced voltage ($U_{ind}I_a = T\omega$). This will result in the same Lorentz force formulation in the case of a slotless armature [8]. However, it is not possible to calculate the starting torque, since the output power should be divided by zero speed. The mechanical power method can also be used for other electrical machines [5]–[7]. However, it is not possible to estimate the cogging torque using this method. For synchronous and asynchronous machines and especially for calculating the oscillating torques, and also winding factors and harmonics of induced voltage, the rotating field theory is used [9]. This method assumes a fine distribution of a so-called electrical loading around the internal circumference of a stator with an infinite permeability. It also assumes that this electrical loading is proportional to the tangential component of the air gap flux density. Knowing this, the rotating field theory is used for calculating the constant and oscillating components of the torque, but cannot be directly used for estimating the reluctance torque. For calculating the cogging torque,

The associate editor coordinating the review of this manuscript and approving it for publication was Paolo Giangrande.

the energy or rather co-energy method is a favorable method, in order to avoid the labor-intensive process of calculating the tangential component of the air gap flux density [10], [11]. Ignoring the oscillating torque components, alone for describing the different torque components of a V-type permanent magnet synchronous machine, we need different torque interpretations.

The synchronous torque ($\frac{3}{2}pi_q\varphi_{PM}$ term) is described as intention to align the rotor flux with the stator flux, the reluctance part of the torque ($\frac{3}{2}p(L_d - L_q)i_d i_q$ term) is described with the principle of the reluctance force, the intrinsic intention to minimize the air gap and the cogging torque as the change of energy due to the rotation [12]. The question is, weather there exists a unified description of electromagnetic torque generation in rotating electrical machines, which can physically and also mathematically describe all different torques, regardless of their originations. We answer this question by first introducing, a mathematical object, hereinafter called Tensor. In the second part, the vector model is described and estimated for different boundary cases. For the sake of simplicity, all estimations are done for two-pole electrical machines. Naturally, the theory described in the next sections is applicable to multi-phase multi-pole rotating electrical machines.

II. ROTATING TENSOR

For a physical parameter with a sinusoidal spatial distribution on its domain θ, the Tensor $T_{i,j}^{k,l}$ is defined by its four attributes i, j, k and l:

$$T_{i,j}^{k,l} = A_m cos\left(k2\pi f_1 t + \beta\right) \cdot e^{j^*(ij)} \cdot e^{j^*(2\pi ftl)} \cdot \begin{bmatrix} 1 \\ e^{j^*(\frac{2\pi}{i})} \\ \vdots \\ e^{j^*(i-1)\frac{2\pi}{i}} \end{bmatrix} \tag{1}$$

and describes time-varying rotating vectors. By definition is the $T_{0,0}^{0,0}$ the amplitude of the Tensor and f_1 is the amplitude frequency. j^* is the complex number. The attributes i, j, k and l represent the order, position, frequency in time and spatial domain. f is the fundamental frequency of the rotating Tensor in spatial domain (θ) of Tensor $T_{i,j}^{k,l}$. Figure 1 shows several Tensors in different cases. It is to be noted that this formulation should result in i vectors beginning at a phase angle of j. The decisive aspect is the validity range of the vectors. Assuming a fifth order Tensor $T_{5,0}^{0,0}$. According to the equation (1) this formulation results into five vectors in positions [0, 72°, 144°, 216°, 288°] indicated in Figure 2-a. The first vector has a validity range of [−36° 36°], which means that this vector can only interact with vectors locating in this range. For clarification assume the summation of the three Tensors $\mathrm{T}_{\mathrm{u}1,0}^{1,0}$, $\mathrm{T}_{\mathrm{v}1,-2\pi/3}^{1,0}$ and $\mathrm{T}_{\mathrm{w}1,-4\pi/3}^{1,0}$. Each Tensor represents five vectors indicated in Figure 2-b. The first vector of $\mathrm{T}_{\mathrm{u}1,0}^{1,0}$ and the fourth vector of $\mathrm{T}_{\mathrm{v}1,-2\pi/3}^{1,0}$ and the thrid vector of $\mathrm{T}_{\mathrm{w}1,-4\pi/3}^{1,0}$ can be added together to build the first vector

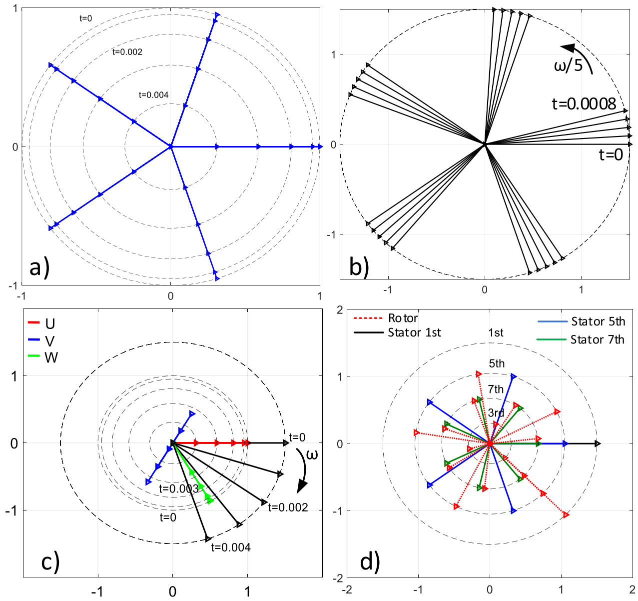

FIGURE 1. Rotating Tensor a) $T_{5,0}^{1,0}$: 5th-order time varying vectors, b) $T_{5,0}^{0,-1}$: 5 × 5th-order rotating fields (counter clockwise), c) three-phase time-varying field and fundamental rotating field, d) rotor fields with 1st, 3rd, 5th and 7th-order spatial harmonics interacts with stator fields (*MMF*) with 1st, 5th and 7th order spatial harmonics to generate a constant torque and a pulsating torque with a 6f component (f = 50 *Hz*).

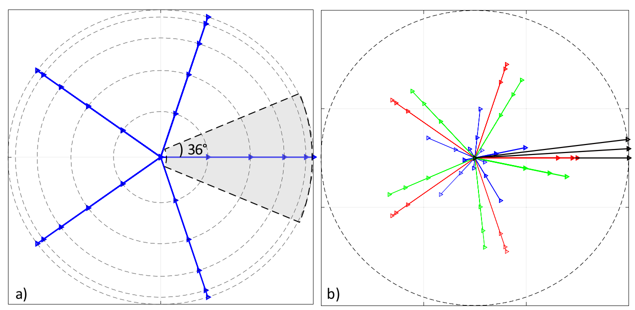

FIGURE 2. Validity region of the vectors: a) fifth order Tensor and the validity range of the first vector at zero, [−36° 36°], b) the summation of three Tensors which results into a rotating vector; $\mathrm{T}_{\mathrm{u}5,0}^{1,0}$ in red at zero, $\mathrm{T}_{\mathrm{v}5,-2\pi/3}^{1,0}$ in blue at 24° and $\mathrm{T}_{\mathrm{w}5,-4\pi/3}^{1,0}$ in green at −24°.

of the summation result since the fourth vector of $\mathrm{T}_{\mathrm{v}5,-2\pi/3}^{1,0}$ and the third vector of $\mathrm{T}_{\mathrm{w}5,-4\pi/3}^{1,0}$ are at angles 24° and −24° respectively an hence in the validity range of the first vector of $\mathrm{T}_{u5,0}^{1,0}$, [−36° 36°]. Since the vectors are built with exponential functions with a natural validity of [0° 360°], the angles of the vectors have to be multiplied with factor 5 for the correct summation result. The angle of the resulted vector has to be then divided by 5.

For a comprehensive description of torque generation in electrical machines, three mathematical operations, namely summation, cross multiplication and projection, are sufficient. If the Tensor is reduced to a first-order vector, e.g. $T_{1,\delta}^{0,0}$, the summation is exactly the vector summation. For a three-phase symmetrical machine, the summation of three time-variant Tensors for the first, fifth, seventh etc. (but not third, sixth, ninth etc.) will result in:

$$U_{i,0}^{k,0}+V_{i,\frac{-2\pi}{3}}^{k,0}+W_{i,\frac{-4\pi}{3}}^{k,0}=\frac{3}{2}T_{i,0}^{0,\sqrt{\frac{2}{3}}k\sin(\frac{\pi i}{3})} \tag{2}$$

For a two-pole three-phase machine with a three-phase *MMF* of $\Theta_{u_{1,0}}^{1,0}$, $\Theta_{w_{1,-2\pi/3}}^{1,0}$ and $\Theta_{v_{1,-4\pi/3}}^{1,0}$, the rotating fields can be estimated directly from equation 2 [13].

For the first harmonic ($i=1, k=1$), the summation gives the fundamental harmonic field rotating with fundamental frequency ($\omega=2\pi f_1 t$), for the third harmonic ($i=3, k=1$), the summation does not result in a rotating vector and for the seventh harmonics ($i=7, k=1$), it results into seven rotating vectors rotating with $\omega/7$. For a machine with only the fifth spatial harmonic ($i=5$) and f_1 current harmonics ($k=1$), this results in five vectors rotating counterclockwise with $\omega/5$ (Figure 1-b). For the fifth spatial harmonics ($i=5$) and $5f_1$ current frequency ($k=5$), e.g. coming from power electronic, it results in five vectors rotating counterclockwise with ω.

The multiplication of the two Tensors $A_{i_1,j_1}^{k_1,l_1}$ and $B_{i_2,j_2}^{k_2,l_2}$ is only allowed for tensors with the same order i. The cross multiplication results in:

$$C_{0,0}^{0,0}=A_{0,0}^{k_1,0}B_{0,0}^{k_2,0}\sin i\cdot((l_1-l_2)2\pi ft+j_1-j_2) \tag{3}$$

where $A_{0,0}^{k_1,0}$ is the instantaneous value of the Tensor $A_{i_1,j_1}^{k_1,l_1}$. For reduced Tensors $T_{1,0}^{0,0}$ (vectors), the cross multiplication is geometrically the area of a parallelogram, the sides of which are the two Tensors. The cross multiplication of a rotating time-invariant Tensor ($k=0$) and a stationary time-variant Tensor ($l=0$) with the same initial positon ($j_1=j_2=j$) and ($l_1=k_2$) results in:

$$A_{1,j}^{0,l_1}\times B_{1,j}^{k_2,0}=\frac{1}{2}A_{0,0}^{0,0}B_{0,0}^{0,0}\sin(2\theta) \tag{4}$$

This (1/2) factor is the same factor present when formulating the induced voltage of a reluctance machine and will be mentioned later. Since each Tensor describes a sinusoidal distribution in spatial domain, they could also be considered as fields. Hence, equation (4) is the summation of a rotating field with a time-variable field, which will be used to estimate the reluctance torque in the next sections. Although summation and multiplication of Tensors are sufficient for estimating the electromagnetic torque by the vector theory, another useful operator is the projection of Tensor $A_{i_1,j_1}^{k_1,l_1}$ on Tensor $B_{i_2,j_2}^{k_2,l_2}$ and can generally be formularized as:

$$A_{i,j_1}^{k_1,l_1}\perp B_{i,j_2}^{k_2,l_2}=C_{i,j2+2\pi ftl_2}^{k_1,0}cos((l_1-l_2)2\pi ft+j_1-j_2) \tag{5}$$

Equation (5) can be used to estimate the projection of a vector on a vector ($A_{1,\delta}^{0,0}\perp B_{1,0}^{0,0}$), a rotating vector on a vector ($A_{1,\delta}^{0,1}\perp B_{1,0}^{0,0}$), Figure 3-b, a rotating vector on a rotating vector ($A_{1,\delta}^{0,1}\perp B_{1,0}^{0,1}$), Figure 3-d a time-variant vector on a stationary vector ($A_{1,\delta}^{1,0}\perp B_{1,0}^{0,0}$), Figure 3-a and a time-varying vector on a rotating vector, Figure 3-c, e.g.:

$$A_{1,\delta}^{1,0}\perp B_{1,0}^{0,-1}=A_{1,\omega t}^{0,0}cos(\delta+\omega t) \tag{6}$$

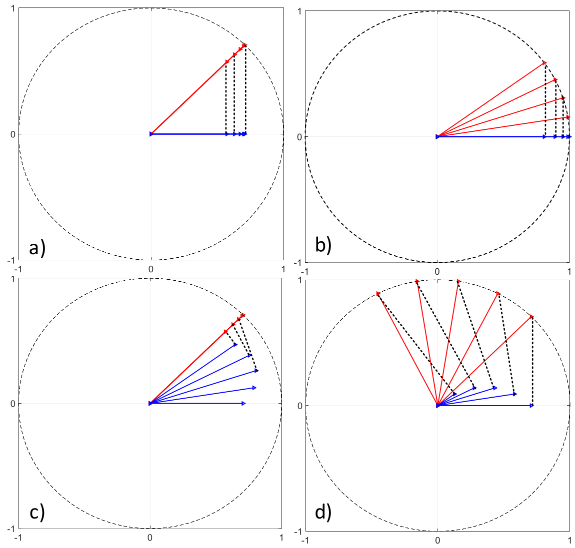

FIGURE 3. Projection of one vector (red) on the other vector: a) a time-varying vector on a vector, b) a rotating vector on a vector, c) a time-varying vector on a rotating vector, d) a rotating vector with a rotational speed of 2ω on another rotating vector with a rotational speed of ω.

The projection of a time-variant vector on a stationary vector ($A_{1,\delta}^{1,0}\perp B_{1,0}^{0,0}$) is also known as Clark transformation and the projection of a time-varying vector on a rotating vector is also known as Park transformation [14]. Applying the projection rule of equation (5) of a three-phase system on $d_{1,0}^{0,1}$ and $q_{1,-\frac{\pi}{2}}^{0,1}$ leads to:

$$\begin{bmatrix} I_{u_{1,0}}^{1,0} \\ I_{v_{1,-\frac{2\pi}{3}}}^{1,0} \\ I_{w_{1,-\frac{4\pi}{3}}}^{1,0} \end{bmatrix}\perp d_{1,0}^{0,1}=\begin{bmatrix} I_{u_{1,\omega t}}^{1,0}\cos(\theta-0) \\ I_{v_{1,\omega t}}^{1,0}\cos(\theta-\frac{2\pi}{3}) \\ I_{w_{1,\omega t}}^{1,0}\cos(\theta-\frac{4\pi}{3}) \end{bmatrix} \tag{7}$$

and

$$\begin{bmatrix} I_{u_{1,0}}^{1,0} \\ I_{v_{1,-\frac{2\pi}{3}}}^{1,0} \\ I_{w_{1,-\frac{4\pi}{3}}}^{1,0} \end{bmatrix}\perp q_{1,\frac{\pi}{2}}^{0,1}=\begin{bmatrix} I_{u_{1,\omega t-\frac{\pi}{2}}}^{1,0}\cos(\theta-0+\frac{\pi}{2}) \\ I_{v_{1,\omega t-\frac{\pi}{2}}}^{1,0}\cos(\theta-\frac{2\pi}{3}+\frac{\pi}{2}) \\ I_{w_{1,\omega t-\frac{\pi}{2}}}^{1,0}\cos(\theta-\frac{4\pi}{3}+\frac{\pi}{2}) \end{bmatrix} \tag{8}$$

The factor $\theta=\omega t$ in $T_{1,\omega t}^{1,0}$ is the position of the d and q axis, respectively, and the projection result of equation (7) and (8) is exactly the power invariant Park transformation:

$$\begin{bmatrix} I_d \\ I_q \end{bmatrix}=\begin{bmatrix} cos(\theta) & cos(\theta-\frac{2\pi}{3}) & cos(\theta-\frac{4\pi}{3}) \\ -sin(\theta) & -sin(\theta-\frac{2\pi}{3}) & -sin(\theta-\frac{4\pi}{3}) \end{bmatrix} \times\begin{bmatrix} I_u \\ I_v \\ I_w \end{bmatrix}$$

It is obvious that also the higher order of the dq transformation can be estimated using the projection rule.

The contributions of each phase on the d and q axes are illustrated in Figure 4.

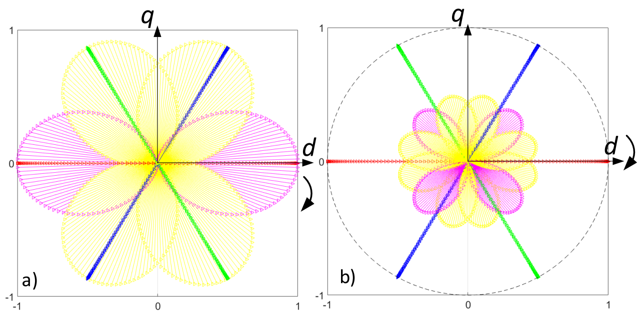

FIGURE 4. Estimation and illustration of Park transformation: a) magenta: projection of the phase U (red) on the rotating d-axis, yellow: projection of the phases V and W (blue and green) on rotating d-axis b) projection of the phases U, V, W on the rotating q-axis.

III. VECTOR THEORY OF ELECTRICAL MACHINES

The electromagnetic torque can be formulated as the multiplication result of two Tensors:

$$M = A_{i,j_1}^{k_1,l_1} \times B_{i,j_2}^{k_2,l_2} \tag{9}$$

where $A_{i,j_1}^{k_1,l_1}$ is the cause and $B_{i,j_1}^{k_1,l_1}$ is the effect. For rotating electrical machines, A is the magnetomotive force (current) and B is the flux (linkage). The torque arises as a response to the deviation of the resulting flux from the flux which would have occurred as a result of the applied magnetomotive force.

$A_{i,j_1}^{k_1,l_1}$ is the i^{th} MMF harmonic of a rectangular coil spanned along a cylinder, e.g. stator of a rotating electrical machine. The well-known winding coefficients can either be considered in $A_{i,j_1}^{k_1,l_1}$, in this case $B_{i,j_2}^{k_2,l_2}$ is the flux, or the winding coefficients are considered in $B_{i,j_2}^{k_2,l_2}$ (flux linkage), and $A_{i,j_1}^{k_1,l_1}$ is the current. It should be noted that the flux Tensor has to be calculated directly from the air gap flux density distribution (including also the spatial harmonics) multiplied by the pole pitch area. The cross multiplication creates an analogy to the classical mechanics in which the altitude over the ground is similar to the angle between MMF and flux Tensors.

In the next sections, a comparison of results estimated with the vector model is compared to other known formulations for some boundary cases.

For an ideal permanent magnet synchronous machine with only the first spatial harmonic, the torque can be estimated from rotating field theory. This method assumes a fine distribution of a so-called electrical loading A around the internal circumference of a stator with an infinite permeability and assuming that the tangential component of the air gap field intensity originates from this electrical loading:

$$\int_{x_1}^{x_2} A(x,t)dx = \Theta(x,t) = \oint \vec{H}\cdot d\vec{s} = \int_{x_1}^{x_2} \left(H_{t,\delta} - H_{t,Fe}\right) dx \tag{10}$$

where $H_{t,\delta}$ and $H_{t,Fe}$ are the tangential components of the field intensity in the air gap [9]. With sinusoidal distributions, according to the rotating field theory, the torque can be given by:

$$\begin{aligned} T = \frac{1}{2}R^2 l\hat{A}_{\nu'}\hat{B}_{\mu'}\int_0^{2\pi} & [cos((\nu'+\mu')\gamma' - 2\pi\left(f_{\nu'}+f_{\mu'}\right)t \\ & - (\varphi_{\nu'}+\varphi_{\mu'})) + cos((\nu'-\mu')\gamma' - 2\pi\left(f_{\nu'}-f_{\mu'}\right)t \\ & - (\varphi_{\nu'}-\varphi_{\mu'}))]d\gamma \end{aligned} \tag{11}$$

where ν', μ' are the stator and rotor field harmonics, and γ' is the mechanical angle. For the first harmonic ($\nu' = \mu' = 1$), only the second part of the integral has the non-zero value and results in:

$$T = R^2 l\hat{A}_1\hat{B}_1 2\pi\frac{1}{2}cos(-\varphi_{\nu'} - \varphi_{\mu'}) \tag{12}$$

where R and l are the radius and length of the machine. For $\varphi_{\nu'} - \varphi_{\mu'} = \delta$ and considering the relations for MMF and flux, we obtain:

$$RA_1 = \Theta_s \quad \text{and } \psi = \pi RlB_1 \tag{13}$$

Due to the fact that the original formulation of the rotating field theory supposes a sinusoidal distribution of electrical loading and the here presented vector theory a cosine distribution ($\cos\left(\delta - \frac{\pi}{2}\right) = \sin(\delta)$), the equation (9) will result in:

$$T = \Theta_s\psi\,\sin(\delta) \tag{14}$$

where $\Theta_s = \frac{3}{2}I_s$ and $\psi = \psi_{PM}$. In the next step, the same torque will estimated using the vector model. Assuming a three-phase distributed current with $I_{u0,0}^{0,0} = I_m$, the sum of all phases results in a rotating field according to equation (2):

$$I_{u1,0}^{1,0} + I_{v1,0}^{1,0} + I_{w1,0}^{1,0} = \frac{3}{2}I_{s1,0}^{0,1} = \Theta_{s1_{1,0}}^{0,1} \tag{15}$$

Assuming a two-pole permanent magnet rotor with sinusoidal flux distribution, according to the vector theory, the torque is given by:

$$\begin{aligned} T &= \Theta_{s1_{1,0}}^{0,1} \times \psi_{1,\delta}^{0,1} = \Theta_{s1_{0,0}}^{0,0}\cdot\psi_{1_{0,0}}^{0,0}\cdot sin(\delta) \\ &= \frac{3}{2}I_s\psi_1\sin(\delta) = \frac{3}{2}I_q\psi_{PM} \end{aligned} \tag{16}$$

which is the same result as predicted by the rotating field theory using equation (11). For the 5^{th} spatial harmonic ($\nu' = \mu' = 5$), according to the rotating field theory ($f_{\nu'} = f_1, f_{\mu'} = 5f_1$), the oscillating torque with six times the fundamental frequency is given by:

$$T = R^2 lA_5B_5 2\pi\frac{1}{2}cos(-6\omega t + \delta - \frac{\pi}{2}) \tag{17}$$

The vector interpretation also predicts the same torque. As it is mentioned in the previous section the rotational speed of the fifth order Tensor of the stator MMF is $\omega/5$ and counterclockwise. Since the rotor field and its harmonics are mechanically connected to the rotor the fifth order field of the rotor also rotates with ω in the direction of the fundamental field so clockwise. That means for each $\Delta\theta$ that the stator

MMF rotates in counterclockwise direction the rotor rotates $5\Delta\theta$ in clockwise direction. Hence the angle between the Two Tensors changes with $6\Delta\theta$ respectively $6\omega t$:

$$T = \Theta_{s_{5,\delta/5}}^{0,-1/5} \times \psi_{5,0}^{0,1} = \Theta_{s1_{0,0}}^{0,0} \cdot \psi_{1_{0,0}}^{0,0} \cdot sin(\delta - 6\omega t) \quad (18)$$

IV. RELUCTANCE TORQUE

The torque of a salient-pole machine is normally estimated from the power equation after some mathematical derivation. The beauty of the vector theory is the fast estimation of the same torque completely geometrically. As mentioned before, the multiplication of Tensors can also be represented as the area of a parallelogram, the sides of which are the two Tensors. According to Figure 5, the area between the Tensors $\Theta_{s1_{1,0}}^{0,1}$ and $\psi_{1_{1,0}}^{0,1}$ can be estimated as the geometrical difference of the areas under the large rectangle and the areas 1 to 6:

$$T = (\varphi_{Rd} + \Theta_{sd})(\varphi_{Rq} + \Theta_{sq}) - \frac{1}{2}\varphi_{Rd}\varphi_{Rq} - \frac{1}{2}\Theta_{sd}\Theta_{sq} - \Theta_{sd}\varphi_{Rq} - \frac{1}{2}\Theta_{sd}\Theta_{sq} - (\Theta_{sd}\varphi_{Rq} + \frac{1}{2}\varphi_{Rd}\varphi_{Rq}) \quad (19)$$

Substituting Θ_s with $\frac{3}{2}I_s$ will result in

$$\Theta_{sq}\varphi_{Rd} - \Theta_{sd}\varphi_{Rq} = \frac{3}{2}(L_d - L_q)i_d i_q \quad (20)$$

and thereby the synchronous torque (equation 16) and the reluctance torque (equation 20) can be derived with the same theory.

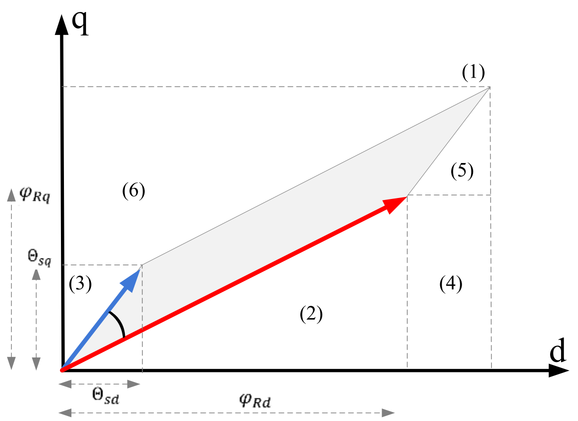

FIGURE 5. **Reluctance torque. The Tensor multiplication of stator *MMF* (blue) and rotor flux (red) is equal to the gray area bordered with *MMF* and flux Tensors.**

Also, the Lorentz force can be estimated using the vector theory. Assume a rectangular armature coil with a width and length of $2r$ and l with an area of $2rl$, imposed to a constant flux of a DC machine's stator poles with constant flux density $\vec{B}$. Suppose the rectangular coil rests perpendicular to the stator flux so that the surface unit vector of the coil is perpendicular to the stator flux density vector. The Lorentz force acts on the two conductors of the armature coil, which are carrying currents in opposite direction. The starting torque can be easily estimated using the Lorentz force (for each conductor):

$$\vec{F} = I \cdot \vec{l} \times \vec{B} \quad (21)$$

The Torque is then given by:

$$M = \vec{r} \times \vec{F} = 2rlBI \quad (22)$$

where the term $2rlB$ is the flux and hence, the torque can be summarized as:

$$T = \psi I \quad (23)$$

The same can be estimated also with the vector theory according to equation (9). Since the vector theory is based on cosine distributions, and the *MMF* and flux of the sample DC machine of this example have rectangular shapes, the torque can estimated as the sum of all harmonics:

$$T = \sum\nolimits_p \Theta_{p,0}^{0,0} \times \psi_{p,-\frac{\pi}{2}}^{0,0} = \frac{1}{2}(\frac{4}{\pi}I_1\psi_1 - \frac{4}{3\pi}I_3\psi_3 + \frac{4}{5\pi}I_5\psi_5 - \frac{4}{7\pi}I_7\psi_7 + \cdots) = \sum\nolimits_{n=p}^{\infty} \frac{2}{n\pi}\psi_n I_n sin\left(\frac{n\pi}{2}\right) \quad (24)$$

The factor $\frac{4}{n\pi}$ comes from estimating the *MMF* harmonics of a rectangular coil. The resulting series of equation (24) can also result from the following product (based on trigonometric relations) which reconstructs the original current and flux:

$$\left\{\sum\nolimits_{v=1,3,\ldots}^{\infty} I_v sin(v\theta)\right\} \cdot \left\{\sum\nolimits_{n=1,3,\ldots}^{\infty} \psi_n \sin\left(n\theta + \frac{n\pi}{2}\right)\right\} = I\psi \quad (25)$$

V. ENERGY

The co-energy method is also a well-known method for torque calculation in electrical machines which can be obtained by considering the change in co-energy of the system produced by a small change in rotor position when the currents are held constant. Suppose a basic two-pole non-salient machine with infinite permeability. Thus, the whole energy is stored in the air gap with a length of g. The peak value of the resultant *MMF* in the air gap can be estimated from the Tensor summation result of stator $\Theta_{S_{1,0}}^{0,1}$ and rotor $\Theta_{R_{1,0}}^{0,1}$ according to the angle between them γ:

$$(\Theta_{T_{1,0}}^{0,1})^2 = \left(\Theta_{S_{1,0}}^{0,1}\right)^2 + \left(\Theta_{R_{1,0}}^{0,1}\right)^2 + 2\left(\Theta_{S_{1,0}}^{0,1}\right)\left(\Theta_{R_{1,0}}^{0,1}\right) cos\,(\gamma) \quad (26)$$

The RMS value of the resultant field intensity $\underline{H}$ in the air gap is:

$$\underline{H} = \frac{H_m}{\sqrt{2}} \quad (27)$$

The average energy density (energy per volume) can be given by:

$$w' = \frac{\mu_0}{2}\underline{H}^2 = \frac{\mu_0}{4}\frac{(\Theta_{T_{1,0}}^{0,1})^2}{g^2} \quad (28)$$

For a machine with a diameter D and length L, the stored energy is given by:

$$W = \left[\left(\Theta^{0,1}_{S_{1,0}}\right)^2 + \left(\Theta^{0,1}_{R_{1,0}}\right)^2 + 2\left(\Theta^{0,1}_{S_{1,0}}\right)\left(\Theta^{0,1}_{R_{1,0}}\right) cos\,(\gamma)\right] \times \frac{\mu_0}{4g^2}[\pi DLg] \tag{29}$$

The torque can be estimated as follows:

$$T = -\frac{\partial W}{\partial g} = \frac{\mu_0}{2g}\Theta^{0,0}_{S_{0,0}}\Theta^{0,0}_{R_{0,0}} sin(\gamma)(\pi DL) \tag{30}$$

The rotor flux can be estimated from $\Theta^{0,0}_{R_{0,0}}$:

$$\psi_R = \frac{\mu_0}{g}\Theta^{0,1}_{R_{0,0}}\frac{\pi DL}{2} \tag{31}$$

And finally substituting $\Theta^{0,1}_{S_{1,0}} = \frac{3}{2}I_m$ in equation (31), will result in equation (16).

VI. OTHER TORQUES IN ELECTRICAL MACHINES

Suppose a simplified two-pole salient-pole machine of Figure 6 which has a zero field current (only reluctance torque) and sinusoidal air gap reluctance with a locked rotor (locked rotor position).

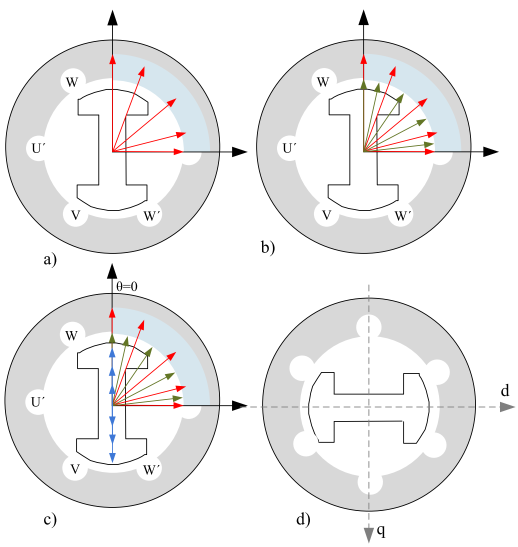

FIGURE 6. Reluctance Torque, a) rotating time-varying field, b) constant rotating MMF in green and time-varying rotating field in red, c) time-varying rotating field as a sum of a rotating field (green) and a time-varying field (blue), dq-axes.

Applying a three-phase voltage to the motor, the reluctance torque according to the well-known reluctance torque formulation is given by:

$$T = \frac{3}{\omega}\frac{U^2}{2}\left(\frac{1}{X_d} - \frac{1}{X_q}\right) sin(2\theta) \tag{32}$$

The conventional derivation of equation (32) originates from the formulation of the output power. For estimating this torque using the vector theory, we need two Tensors according to equation (9). Applying the rotating *MMF* starting at zero position, it can be observed that a time-variable rotating field (red arrow in Figure 6) rotates at fundamental frequency. At 90 degree, the flux value is at its minimum. The rotating time-variant flux could be reconstructed as a summation of a rotating field with constant amplitude (green), $\psi^{0,1}_{1,0}$, and a time-variable stationary field (blue), $\psi^{1,0}_{1,0}$. According to the vector theory, the torque is a result of the Tensor multiplication of two Tensors, in this case a rotating field and a time-variant field, which according to equation (4), is given by:

$$A^{0,1}_{1,j} \times B^{1,0}_{1,j} = \frac{1}{2}C^{0,0}_{0,0}\sin(2\theta) \tag{33}$$

revealing the term $\frac{1}{2}\sin(2\theta)$ in equation (32), that is also present in all reluctance torques. Also, the correct value of the torque $C^{0,0}_{0,0}$ can be estimated using the vector theory. The rotating *MMF*, $\Theta^{0,1}_{1,0}$, induces two time-varying vectors in d and q axes, $\psi^{1,0}_{d_{1,0}}$ and $\psi^{1,0}_{q_{1,0}}$. According to the multiplication rules, equation (3), the torque can be given by:

$$T = {\Theta_{1,0}}^{0,1} \times (\psi^{1,0}_{d_{1,0}} + \psi^{1,0}_{q_{1,\frac{\pi}{2}}}) = \Theta^{0,0}_{0,0}\psi^{0,0}_{d_{0,0}} sin\,(\omega t) - \Theta^{0,0}_{0,0}\psi^{0,0}_{q_{0,0}} cos(\omega t) \tag{34}$$

Since the time variation is included in the (ωt) term, the peak values can be substituted in $\psi^{0,0}_{0,0}$, ψ_d and ψ_q, and $\Theta^{0,0}_{0,0}$, $\frac{3}{2}I_q$ and $\frac{3}{2}I_d$. Hence, the equation (34) results in:

$$T = \frac{3}{2}(\psi_d I_q - \psi_q I_d) = \frac{3}{2}\left(L_d - L_q\right)I_d I_q = 3\frac{I_d}{\sqrt{2}}\frac{I_q}{\sqrt{2}}\left(X_d - X_q\right)\frac{1}{\omega} \tag{35}$$

Using the trigonometric function relations:

$$\frac{1}{2}sin\,(2\gamma) = sin\,(\gamma)\,cos(\gamma) \tag{36}$$

and substituting U_d and U_q:

$$U_q = X_d i_d,\quad U_d = -X_q i_q,\ U_q = Ucos(\gamma),\ U_d = Usin(\gamma) \tag{37}$$

will result into:

$$T = 3\left(\frac{U}{X_d}\right)\left(\frac{U}{X_q}\right)\left(X_d - X_q\right)\frac{1}{\omega} = \frac{3}{2\omega}\left(\frac{U^2}{X_q X_d}\right)\left(X_d - X_q\right)\sin(2\gamma) \tag{38}$$

While $\Theta^{0,1}_{1,0}$ can generate a double-frequency torque ($T^{0,0}_{0,0}\sin(2\theta)$), a $\Theta^{0,1}_{2,0}$ could generate a torque four times the fundamental frequency ($T^{0,0}_{0,0}\sin(4\theta)$). This fact will be used in the next step to estimate the cogging torque.

Figure 7 shows a 12-slot, 2-pole motor with radially magnetized permanent magnets in no-load operation. The field of application of the method presented in the previous section (Figure 6) is reduced to two times the slot pitch (here 60°). Hence, comparing to Figure 6, the two slots are equivalent to the q-axis of the machine in Figure 6 and the two teeth are comparable to the salient poles (d-axis). Reduced on a domain of 60°, the 5^{th} and 7^{th} harmonics of the permanent magnets generate the fundamental component of the *MMF* and the 11^{th} and 13^{th} harmonics generate the second-order *MMF* (compare to $\Theta_{1,0}^{0,1}$ and $\Theta_{2,0}^{0,1}$ of the previous example of Figure 6). In other words, for the machine of Figure 7, according to the vector theory, $\Theta_{5,0}^{0,0}$ and $\Theta_{7,0}^{0,0}$ will generate a $6f$ torque pulsation and the $\Theta_{11,0}^{0,0}$ and $\Theta_{13,0}^{0,0}$ produce a $12f$ torque pulsation.

The mentioned *MMFs* can be estimated as a function of the coercivity force H_{PM} and the height of the permanent magnet l_{PM}. The corresponding cross section for calculating the flux along the slot axis ψ_N and along the tooth axis ψ_Z should accordingly be estimated with a cross-sectional area spanned over two teeth. Although $\Theta_{5,0}^{0,0}$, $\Theta_{7,0}^{0,0}$, $\Theta_{11,0}^{0,0}$ and $\Theta_{13,0}^{0,0}$ are the dominant *MMF*s for generating the cogging torque of Figure 7, other orders of *MMFs* also have their contributions to the cogging torque. Nevertheless, estimating the cogging torque by vector theory:

$$T_{cog} \propto \sum\nolimits_{n=5,7,11,13} \frac{1}{n} \cdot H_{PM} \cdot l_{PM} \cdot (\psi_Z^n - \psi_N^n) \quad (39)$$

will lead to the results indicated in Figure 8.

The examples and special cases considered in this paper are solely handled to compare the compatibility of the vector theory to other conventional methods. The method can also be verified for other cases. Also, the phasor diagrams can be modified using the vector theory. To include the effect of higher harmonics of the induced voltage into the phasor diagram, the principle of the first harmonic linkage is applied. In this case, the effect of higher harmonics is simply added to the stray reactances in the phasor diagram. Using the vector model, this could be handled separately.

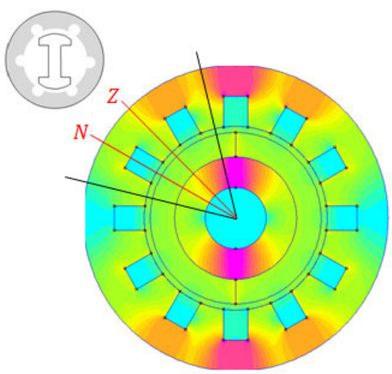

FIGURE 7. **Cogging torque; the domain of Figure 6-d is reduced to a domain of two tooth pitches. Hence, the N- and Z-axes are analog to q- and d-axis of figure 6-d.**

By application of the projection, summation and multiplication rules of the introduced vector theory, it is possible to explain different effects in electrical machines with a unified theory. The rotation tensor of equation (1) or Figure 1-a is a symmetrical Tensors. Hence, all i components of the Tensor

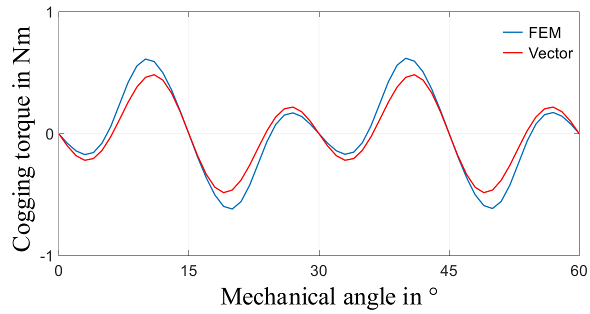

FIGURE 8. **Cogging torque; Comparison between FEM results obtained from the Maxwell stress tensor method with vector theory calculated with 5^{th}, 7^{th}, 11^{th} and 13^{th} harmonics of MMF and flux Tensors.**

$T_{i,j}^{k,l}$ have the same character. It is also possible to introduce asymmetrical tensors in which the last term of equation (1), the matrix, is not symmetrical anymore. This result in a Tensor with i unsimilar components. Since the amplitude values (f_1, β, k) can be defined independent from spatial attributes (f, j, l), it is also possible to perform unsymmetrical and transient calculations. In this case, the winding parameters (current, voltage, frequency) can be influenced independently from the coil parameters (winding factor, position, etc.). In general case for a 2p-pole machine the i component of the Tensor should be substituted with pi.

The examples explained in this contribution are solely supposed to introduce the idea and show its potential of simplifying and unifying the calculation of electrical machines. The idea will be further developed for calculating the other aspects in electrical machines, such as winding factors, radial forces, eccentricity etc.

VII. CONCLUSION

A conventional course in an electrical machine applies different methods and tools to describe its functionality. These methods are best optimized for a precise and simple description of different effects in electrical machines. Besides using these interpretation and estimation tools, it would be didactically helpful to uniformly explain different phenomena of electrical machines best with a compact formulation in order to create (at least didactically) a coherency between several tools, methods and interpretations.

This contribution is answering the fundamental question: Is there a compact general description of rotating electrical machines which can unify different models, tools and interpretations, such as rotating field theory, reluctance torque, Lorentz force, Park transformation, phasor diagram, etc.?

To answer this question, a general model of rotating electrical machines is introduced. The mathematical tool for describing the theory is based on a four-dimensional rotating time-variant vector. The Tensor represents a powerful tool, from which also other essential tools in the field of rotating electrical machines, such as Clark and Park transformation, phasor diagrams, Goerges diagram and winding factors can be derived. This has the advantage of bringing all these tools together relating them to one another. Based on the rotating Tensor the vector model is introduced.

The vector theory has a concise formulation that is able to explain and estimate different torque generation principles normally derived from Lorentz force, Maxwell Stress Tensor, reluctance force, cogging torque, etc. using a unified theory. The comparison to other conventional methods is described for some simple boundary cases in this contribution to clarify the idea behind this development. However, this method can be used for calculating more complex phenomena in electrical machines which will be pursued in future works.

The tool and the torque formulation has to and will be further developed for the prediction of other phenomena in rotating electrical machines, such as radial forces and acoustic noises, winding factors, eccentricity, etc.

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