Fredholm and Index Theory for Symmetrizable Hyperbolic Systems with Nonlocal Boundary Conditions

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Abstract

In this thesis we investigate the Fredholm conditions and the index theory of symmetrizable hyperbolic systems $\partial_t u = Lu$ with nonlocal boundary conditions stated on a finite time subset $M_{[0,1]} = \Sigma \times [0,1]$ of a globally hyperbolic manifold $M = \Sigma \times \mathbb{R}$ with boundary components $\Sigma_0 = \Sigma \times \{0\}$ and $\Sigma_1 = \Sigma \times \{1\}$. For two pseudodifferential projections $P_{+,0}, P_{-,1}$ and two zero order pseudodifferential matrices $\mathcal{A}_0, \mathcal{A}_1$ we consider $\partial_t u = Lu$ together with the conditions $P_{+,0}\mathcal{A}_0 u(0) = g_0$ and $P_{-,1}\mathcal{A}_1 u(1) = g_1$ for functions $g_0 \in \mathrm{Im}(P_{+,0})$ and $g_1 \in \mathrm{Im}(P_{-,1})$. We derive the general Fredholm conditions for the problem and show that for the cases where the conjugation of $P_{-,0} =: 1 - P_{+,0}$ by the solution operator Φ_1 to $\partial_t u = Lu$ is equal to $P_{-,1}$ (up to a compact error), the Fredholm conditions can be reduced to the ellipticity of a matrix of G-operators, as long as some assumptions about the group G and the operator L are made. We also apply the results from the abstract Fredholm theory we achieved to the case of the wave equation $\partial_t^2 u = -\Delta u$ with a time dependent Laplacian Δ subject to the boundary conditions $A_0 u(0) + B_0(\partial_t u)(0) = g_0 \in L^2(\Sigma_0)$, and $A_1 u(1) + B_1(\partial_t u)(1) = g_1 \in L^2(\Sigma_1)$ (with zero order operators $B_{0/1}$ and first order operators $A_{0/1}$). The Fredholm conditions for this application of the abstract theory are expressed explicitly as conditions on the operators $A_{0/1}$ and $B_{0/1}$ and some special cases are considered, where the index formulas of the problem are given by the Fedosov index formula or some simple trace formula.

Keywords: Fredholm Theory, Index Theory, Fourier Integral Operators, Hyperbolic Systems

Zusammenfassung

In dieser Arbeit untersuchen wir die Fredholm-Bedingungen und die Indextheorie von symmetrisierbaren hyperbolischen Systemen $\partial_t u = Lu$ mit nichtlokalen Randbedingungen. Das Problem wird auf der Teilmenge $M_{[0,1]} = \Sigma \times [0,1]$ einer global hyperbolischen Raumzeit $M = \Sigma \times \mathbb{R}$ mit Randkomponenten $\Sigma_0 = \Sigma \times \{0\}$ und $\Sigma_1 = \Sigma \times \{1\}$ betrachtet. Für zwei pseudodifferentielle Projektionen $P_{+,0}, P_{-,1}$ und zwei pseudodifferentielle Matrizen $\mathcal{A}_0, \mathcal{A}_1$ der Ordnung 0 betrachten wir die Gleichung $\partial_t u = Lu$ zusammen mit den Bedingungen $P_{+,0}\mathcal{A}_0 u(0) = g_0$ und $P_{-,1}\mathcal{A}_1 u(1) = g_1$ für Funktionen $g_0 \in \mathrm{Im}(P_{+,0})$ und $g_1 \in \mathrm{Im}(P_{-,1})$. Wir leiten die allgemeinen Fredholmbedingungen für das Problem her und zeigen, dass für den Fall, bei dem die Konjugation von $P_{-,0} =: 1 - P_{+,0}$ mit dem Lösungsoperator Φ_1 zu $\partial_t u = Lu$ bis auf eine kompakte Störung mit $P_{-,1}$ übereinstimmt, die Fredholmbedingungen auf die Elliptizität einer Matrix von G-Operatoren reduziert werden kann, zumindest solange die Gruppe G und der Operator L gewissen Forderungen genügt. Die Ergebnisse der abstrakten Fredholmtheorie wenden wir auf die Wellengleichung $\partial_t^2 u = -\Delta u$ mit einem zeitabhägingen Laplace-Operator Δ an, wobei die Wellengleichung zusammen mit den Randbedingungen $A_0 u(0) + B_0(\partial_t u)(0) = g_0 \in L^2(\Sigma_0)$, und $A_1 u(1) + B_1(\partial_t u)(1) = g_1 \in L^2(\Sigma_1)$ betrachtet wird. Hierbei sind $B_{0/1}$ Operatoren der Ordnung 0 und $A_{0/1}$ Operatoren erster Ordnung. Die Fredholmbedingungen für diese Anwendung werden explizit als Bedingungen an die Operatoren $A_{0/1}$ und $B_{0/1}$ formuliert, und es werden Spezialfälle betrachtet, bei denen der Index des Problems mithilfe der Indexformel von Fedosov oder über einfache Spurformeln berechnet werden kann.

Schlagworte: Fredholm Theorie, Indextheorie, Fourier Integral Operatoren, Hyperbolische Systeme

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Contents

0 Introduction

One of the most significant results in modern analysis is the *Atiyah-Singer Index Theorem* formulated and proved 1963 by Michael Atiyah and Isadore Singer. It states that any elliptic pseudodifferential operator P on a closed manifold M is a Fredholm operator and that its Fredholm index can be computed in terms of the principal symbol of P and topological invariants of the manifold M. Fredholm operators form an important subclass of the class of all linear operators acting between sections of two vector bundles E and F over M: As function spaces like the Sobolev space $H^s(M)$ are infinite dimensional, a generic operator $P : H^s(M) \to H^r(M)$ might have infinite dimensional kernel or cokernel. Now Fredholm operators are those operators P with $\dim(\mathrm{Ker}(P)) < \infty$ and $\dim(\mathrm{coker}(P)) < \infty$, and therefore Fredholm operators can be seen as operators of almost (= up to finite dimensional errors) invertible type. For example, the solution space of the equation $\Delta u = f$ for a given function f is infinite dimensional on $\mathbb{R}^n$, while on a closed connected manifold M every solution of the equation $\Delta u = f$, with $f \in L^2(M)$ may be written as $u = c + u_f$ with a constant c and a special solution of the inhomogenous equation, making the solution space finite dimensional. While the original *Atiyah-Singer Index Theorem* was stated only for compact manifolds without boundary, in 1975 there followed another version, the *Atiyah-Patodi-Singer(APS) Index Theorem*, stating the conditions, under which elliptic operators on compact manifolds with boundary are of Fredholm type. *Atiyah, Patodi* and *Singer* (see [12]) were considering operators, which take the form

$$\mathcal{D} = \sigma(\partial_t + A(t)) \tag{0.1}$$

in a neigborhood of the boundary of M, where $A(t)$ is an elliptic (i.e. with invertible principal symbol) first order operator and where σ is some bundle isomorphism. More concretely, they considered compact Riemannian manifolds M, g with boundary $\Sigma = \partial M$, where the metric g is cylindrical near the boundary. The *Atiyah-Patodi-Singer Index Theorem* says that any operator like (0.1) with an elliptic $A(t)$ will be of Fredholm type, as long as the *APS*-boundary condition

$$P_+ u(0) = 0 \tag{0.2}$$

is imposed on the boundary. The P_+ in (0.2) is the projection onto the space spanned by the eigenfunctions associated to positive eigenvalues of $A_0 = A(0)$. Furthermore, *Atiyah, Patodi* and *Singer* gave a formula for the index of $\mathcal{D}$ under the conditions (0.2) which is given by

$$\mathrm{ind}(\mathcal{D}_{L^2}) = \int_M \alpha_0(x)dx - \frac{h + \eta(0)}{2} \tag{0.3}$$

where α_0 is an expression arising from the asymptotic expansion of $\mathrm{Tr}(\exp(-t\mathcal{D}^*\mathcal{D})) - \mathrm{Tr}(\exp(-t\mathcal{D}\mathcal{D}^*))$, h is the dimension of the kernel of $A(0)$ and the η-invariant $\eta(0)$ is some expression measuring the formal difference between the number of positive and negative eigenvalues of $A(0)$. Another example for a compact manifold with boundary would be the subset of a globally hyperbolic manifold $M = \Sigma \times \mathbb{R}$ with compact Cauchy hypersurfaces Σ_t that arises when one considers two special times t_1 and t_2, i.e. the manifold $N = \Sigma \times [t_1, t_2]$. The problem with such a setting is that the natural type of operators on such a globally hyperbolic manifold are of hyperbolic nature i.e. not elliptic operators. *Bär* and *Strohmaier* succesfully formulated a hyperbolic version of the results of the *APS Index Theorem* in [10]: As a model for a first order hyperbolic operator on a globally hyperbolic spacetime M they considered the Dirac operator $\mathcal{D}$ defined by a Lorentzian metric g and the Fredholm conditions were similar to the Fredholm conditions from the 1975 *APS* papers: They also used eigenspace projections to formulate the boundary conditions, but while the *APS* condition (0.2) made only use of the positive eigenvalue projection, *Bär* and *Strohmaier* divided the conditions into two parts, making use of both positive and negative eigenvalue projections: Denoting by $P_{\pm,t}$ the projections onto the positive/negative eigenvalue eigenspaces of the operator $A(t)$ they imposed

$$(i)\ P_+(t_1)u(t_1) = 0, \quad (ii)\ P_-(t_2)u(t_2) = 0. \tag{0.4}$$

The result *Bär* and *Strohmaier* obtained in this setting was that the hyperbolic Dirac operator $\mathcal{D}$ was Fredholm under (0.4) with an index formula similar to (0.3), involving h and η terms at both times t_1 and t_2. The results of their paper *An Index Theorem for Lorentzian Manifolds with Compact Spacelike Cauchy Boundary* provide an index theorem for a hyperbolic (and not elliptic) operator with the nonlocal boundary conditions (0.4). The interesting thing about boundary value problems with boundary data like (0.4) is that their Fredholm theory involves the use of G-operators rather than only pseudodifferential operators. G-operators form a larger class of operators, and the operators of pseudodifferential type may be considered as a subalgebra of the algebra of G-operators. They appear in problems with boundary conditions like (0.4), because the boundary data are given at two different components of the boundary and because the solution operator Φ_1 (evaluated at time 1) of the equation $\partial_t u = Lu$ matching these two components of the boundary may be seen as the quantization of some group G associated to the equation $\partial_t u = Lu$. *A.B. Antonevich* already worked with nonlocal boundary value problems in 1985, when he stated the problem

$$(+)\ \partial_t^2 u = \partial_x^2 u, \quad (++)\ u(0) = g_0, \quad (+++)\ a(x)\partial_x u(\tau) + b(x)(\partial_t u)(\tau) = g_1 \tag{0.5}$$

as a boundary value problem on the manifold $\mathbb{S}^1 \times [0, \tau]$ in his work *A two point problem for the equation of a vibrating string and functional equations connected with it* ([13]) (with given functions a, b, g_0 and g_1 on $\mathbb{S}^1$). However, the problem from (0.5) is still quite special since it is formulated only on the one dimensional closed manifold $\mathbb{S}^1$ and since the operator $R = a(x)\partial_x + b(x)\partial_t$ from $((0.5), (iii))$ is only a differential operator. But recently *Anton Savin* and *Andrei Boltachev* made a generalization of problems like (0.5) in ([11]). They worked with problems like (0.5) but the wave equation $((0.5), (+))$ was stated on a general closed manifold and they considered more general operators $A(x)$ and $B(x)$ of pseudodifferential type. *Savin* and *Boltachev* found out that the Fredholm property of problems like (0.5) on a more general closed manifold may be reduced to the Fredholm property of operators of the form

$$C\Phi(\tau) + D\Phi(-\tau) \tag{0.6}$$

where C and D are of pseudodifferential type (depending on A, B and the Laplacian Δ), $\Phi(t)$ is a time dependent Fourier integral operator associated to the equation $\partial_t^2 u = -\Delta u$, which can also be seen as the quantization of a canonical transformation on T^*M. Compositions of pseudodifferential operators and Fourier integral operators like in (0.6) are called G-operators if the Fourier integral operators involved can be associated to some group G, for example a group of canonical transformations. While in the case of a pseudodifferential operator F the Fredholm property of F is equivalent to ellipticity, i.e. the invertibility of the principal symbol $\sigma_p(F)$, which is a simple algebraic condition, Fredholm conditions for operators like (0.6) are usually more complicated, especially if one considers even more general operators like

$$F = \sum_{g\in G} A_g \Phi_g \tag{0.7}$$

for pseudodifferential operators A_g, and Fourier integral operators Φ_g associated to some group G (note that in (0.6) there is practically only one Φ_g, since the $\Phi(-\tau)$ is the inverse of $\Phi(\tau)$, see ([11], *Proposition 3.1*). If one wants the Fredholm property of operators like in (0.7) to be *equivalent* to the invertibility of a certain symbol (i.e. ellipticity) one usually needs to impose some conditions on the group G, like, for example, amenability. In this work, we want to give Fredholm conditions for hyperbolic equations on a globally hyperbolic manifold M with nonlocal boundary conditions, which are formulated in a very general way containing the works of *Savin-Boltachev* and *Bär-Strohmaier* as special examples. We will consider problems of the form

$$(*)\ \partial_t u = Lu, \quad (**)\ P_{+,0}\mathcal{A}_0 u(0) = g_0 \in \mathrm{Im}(P_{+,0}), \quad (***)\ P_{-,1}\mathcal{A}_1 u(1) = g_1 \in \mathrm{Im}(P_{-,1}) \tag{0.8}$$

where $\mathcal{A}_{0/1}$ are matrices of zero order pseudodifferential operators, $P_{+,0}$ and $P_{-,1}$ are certain projection operators and the equation $\partial_t u = Lu$ is a vector valued system of equations with a first order pseudodifferential matrix L defining a symmetrizable hyperbolic system (see [7], p.76-78). We seek solutions $u \in L^2(M)$ of (0.8) and want to find those conditions depending on $P_{+,0}, P_{-,1}, \mathcal{A}_0$ and $\mathcal{A}_1$, such that for given g_0 and g_1 the solution space of (0.8) is finite dimensional and that only finitely many linear conditions on g_0 and g_1 are needed to guarantee solvability, i.e. we are searching for the Fredholm conditions of (0.8). Note that the projection operators we are considering in this work do not have to be eigenspace projections but can be more general and note that second order problems like (0.5) can be brought into the form (0.8) with the help of an order reduction.

Let us now briefly summarize the structure of this thesis: In the first four chapters we explain the theory, which is necessary to understand all the objects from (0.8) and to develop a suitable Fredholm theory. In the first chapter we recall some important facts and theorems about the algebra of pseudodifferential operators. Since we are using the solution operator Φ_t of $((0.8), (*))$ for the development of the Fredholm theory of (0.8) and since this solution operator may be expressed via Fourier integral operators, the second chapter deals with Fourier integral operators and symplectic geometry, which is the geometrical background to understand Fourier integral operators. We also explain the algebra of G-operators and the use of the so called *trajectory symbol*, which is a symbol one can define for any G-operator D, similar to the pseudodifferential symbol σ of a pseudodifferential operator A. After we have defined all the algebras of operators needed for investigating (0.8), we will explain Fredholm theory in the *Chapter 3*. This third chapter is divided into two parts: In the first part, we explain the general Fredholm theory for arbitrary operators between Banach spaces, recall some well known index theorems like the *Atiyah-Singer Index Theorem* or the *Fedosov Index Formula* and we explain the Fredholm theory of G-operators. In the second part of *Chapter 3* we devote ourselve to the study of special systems of operators: We are considering the Fredholm theory of triangular matrices of operators and in particular the Fredholm theory of 2×2 matrices, where the first entry is of Fredholm type. Operators of this type will become very relevant in the last section of this work. In the fourth chapter we discuss some theory of systems of hyperbolic type: We define the notion of a globally hyperbolic manifold and present some types of hyperbolic systems, where the symmetrizable systems are presented as the most general ones with an invertible solution operator. In particular we consider systems which are diagonalizable with constant multiplicites up to lower order errors and we show that any strictly hyperbolic system belongs to this class. More concretely, any strictly hyperbolic system $\partial_t u = Lu$ may be transformed into the almost (= up to errors of order zero) diagonal system

$$\partial_t v = Dv + Sv, \;\; S \in \mathrm{OPS}^0 \tag{0.9}$$

where D is a diagonal matrix of first order operators with multiplicities of one. For systems of the form (0.9) with a diagonal first order operator D with diagonal entries id_j, such that the d_j have real principal symbol, there is a very convenient way to express the corresponding solution operator Φ_t with the help of G-operators. We will also explain this solution method, which was developed by *Kumano-go* in ([1], Chapter 10, p.313-323) in *Chapter 4*. At the end of *Chapter 4* we will have explained all things which are necessary for some calculus to work with (0.8). However, because projections play a central role in (0.8), we decided to create a chapter which deals exclusively with projection operators. We state and prove some important theorems about general projections, which will be used later in order to formulate necessary (but not sufficient) conditions for the Fredholmness of (0.8). Furthermore, we explain the notion of ellipticity of operators acting between the images of two projections P_1 and P_2. Then in the sixth chapter we collect some useful facts about compact or norm small perturbations of operators, which are helpful for some of the theorems stated in the main chapters of this work. The two main chapters, where the new results concerning the Fredholm theory of (0.8) are presented, are the chapters *Chapter 7* and *Chapter 8*. At the beginning of *Chapter 7* we reduce the problem (0.8) to the operator

$$\mathcal{D} = \begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ \Phi_1^{-1} P_{-,1}\mathcal{A}_1\Phi_1 \end{pmatrix} : L^2(\Sigma_0) \to \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1) \tag{0.10}$$

(with the Cauchy hypersurface $\Sigma_0 = \Sigma \times \{0\}$) which has to be regarded as an operator acting between projection spaces. Using some theorems from the section about projections, we find that the conditions

$$\operatorname{codim}(\operatorname{Ker}(P_{+,0}) + \operatorname{Ker}(\Phi_1^{-1} P_{-,1} \Phi_1)) < \infty, \quad \operatorname{codim}(\operatorname{Im}(\mathcal{A}_0^\dagger) + \operatorname{Im}(\Phi_1^\dagger \mathcal{A}_1^\dagger \Phi_1^{-\dagger})) < \infty \quad (0.11)$$

are necessary for the Fredholm property of (0.8) and we formulate the general conditions equivalent to the Fredholm property by using the notion of Fredholm operators acting between projection spaces. Because ellipticity of operators acting between (possibly different) projection spaces is more complicated than ellipticity of operators $A : \mathcal{H} \to \mathcal{H}$ for some Hilbert space $\mathcal{H}$, we make some assumptions about the projections $P_{+,0}$ and $P_{-,1}$ which are compatible with the first condition in (0.11) and reduce the operator $\mathcal{D}$ to a simpler operator $\mathcal{D}_{L^2} : L^2(\Sigma_0) \to L^2(\Sigma_0)$. Denoting by Φ_1 the time evolution operator of $((0.8), (*))$, we prove the theorem

Theorem 7.3: *Let $P_{+,0} = 1 - P_{-,0}$, $P_{+,1} = 1 - P_{-,1}$ and $P_{-,1} = \Phi_1 P_{-,0} \Phi_1^{-1} + K$ with a compact operator K. The operator $\mathcal{D}$ in* (0.10) *is Fredholm iff*

$$\mathcal{D}_{L^2} := P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 : L^2(\Sigma_0) \to L^2(\Sigma_0) \quad (0.12)$$

is Fredholm.

We show that under certain assumptions on L relying on assumptions made by *Kumanogo* in ([1]) the operator $\mathcal{D}_{L^2}$ can be expressed with G-operators like (0.7) where G is a group generated by a finite number of canonical transformations on $T^*\Sigma_0$. More precisely, if $\Phi_{j,t}$ are the G-operators solving the equations $\dot{\Phi}_{j,t} = id_j \Phi_{j,t}$ where $id_j \in \mathrm{OPS}^1$ are the diagonal entries of the D in (0.9), the explicit dependence of $\mathcal{D}_{L^2}$ on the operators $\Phi_{j,t}$ can be determined up to a compact error. We state that the Fredholm property of $\mathcal{D}_{L^2}$ is equivalent to ellipticity as long as the group G acts topologically freely and is amenable, whereby ellipticity means invertibilitiy of the trajectorial symbol, see ([8], p.147). Furthermore we show that the Fredholm property is invariant under lower order perturbations of the L in $((0.8), (*))$, which means that it only depends on the principal part of the equation. We also consider special cases of the operator in (0.12), where parametrices can be explicitely written down and where the index of the problem may be expressed through the indices of $\mathcal{A}_0$ and $\mathcal{A}_1$ (under the assumption of the Fredholm property of $\mathcal{A}_{0/1}$ and some additional assumptions). The last chapter is devoted to the study of second order problems with boundary conditions over the entire boundary, i.e problems in a similar manner like (0.5). More concretely, the problems of interest treated in *Chapter 8* are problems of the form

$$(i)\ \partial_t^2 u = Pu, \quad (ii)\ A_0 u(0) + B_0(\partial_t u)(0) = g_0 \in L^2(\Sigma_0), \quad (iii)\ A_1 u(1) + B_1(\partial_t u)(1) = g_1 \in L^2(\Sigma_0) \quad (0.13)$$

with first order pseudodifferential operators A_0, A_1, zero order operators B_0, B_1 and an elliptic differential operator P. We are seeking for solutions $u \in H^1(M)$ of (0.13) and are again asking for the combinations of $A_{0/1}$ and $B_{0/1}$ such that (0.13) is Fredholm. At the beginning of the last section we show that the Fredholm property indeed only depends on the principal part of P. Since the principal part of a second order elliptic differential operator P can be choosen to coincide with the principal part of Δ by choosing the metric tensor g_t on Σ_t in a suitable way, $((0.13),(i))$ can be replaced by the wave equation $\partial_t^2 u = -\Delta u$. After an order reduction of (0.13) and a diagonalization (see(0.9)), (0.13) can be associated to an operator $\mathcal{D}_{L^2}$ encoding the Fredholm property of a problem like (0.8). The operator $\mathcal{D}_{L^2}$ corresponding to (0.13) is given by

$$\mathcal{D}_{L^2} = \begin{pmatrix} A_0\Lambda^{-1} + iB_0 & A_0\Lambda^{-1} - iB_0 \\ (A_1\Lambda^{-1} + iB_1)^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} & (A_1\Lambda^{-1} - iB_1)^{\alpha_2} \end{pmatrix} : L^2(\Sigma_0) \times L^2(\Sigma_0) \to L^2(\Sigma_0) \times L^2(\Sigma_0). \tag{0.14}$$

Note that by α_2 and α_1 we denote the time $s = 1$ Hamiltonian flow along $\pm g_t(\xi, \xi)$, $A^{\alpha_{1/2}}$ is the conjugation of an operator A by the Fourier integral operator corresponding to the flows $\alpha_{1/2}$ and Λ is the order reducing operator $\Lambda = \sqrt{\Delta - 1}$. One of the important theorems we achieved in the last chapter is

Theorem 8.2: *If the entry $A_0\Lambda^{-1} + iB_0$ is a Fredholm operator, the operator $\mathcal{D}_{L^2}$ is Fredholm iff the operator*

$$F = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}((A_0\Lambda^{-1} + iB_0)^p(A_0\Lambda^{-1} - iB_0))^{\alpha_1^{-1}\circ\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} \tag{0.15}$$

is Fredholm. In this case, the index of $\mathcal{D}_{L^2}$ is given by

$$\operatorname{ind}(\mathcal{D}_{L^2}) = \operatorname{ind}(A_0\Lambda^{-1} + iB_0) + \operatorname{ind}(F). \tag{0.16}$$

After stating this quite general Fredholm condition, we investigate cases arising from choosing $A_{0/1}$ and $B_{0/1}$ in a special way. We show that for some choices for the boundary conditions in (0.13) the problem is never Fredholm, but always semifredholm with a existent left parametrix, which means that at least the solution space of (0.13) is finite dimensional in these cases. Moreover, we show that the Fredholm conditions achieved by *Savin-Boltachev* are a special case of the Fredholm conditions from *Theorem 8.2*. The techniques in this first part of the last section are based on the use of order reduction. In the last part of this work we show that the problem (0.13) can alternatively be treated without performing an order reduction. It turns out that the Fredholm conditions one obtains without using the method of order reduction have a more complicated form than those from *Theorem 8.2* which means that the methods developed in *Chapter 7* are indeed beneficial. At last we show that there are cases where the index of the operator in (0.16) can be calculated by using the *Fedosov Index Formula* or some simple trace formula.

1 Pseudodifferential Operators

In this section we want to introduce one of the most important calculi in modern analysis, namely the calculus of pseudodifferential operators. Remember that although differentiation and integration are introduced as inverse operations in beginner´s courses on analysis, it somehow feels as they do not belong to the same class of operators: For example, the inverse of any invertible $n \times n$ matrix A is again a matrix and belongs to the same class of objects like A. However, in introductory courses differentiation is usually not presented as an integral type operator, which inverts another integral. The basic idea of pseudodifferential operators is to introduduce a generalization of the order of a differential operator, which is usually a natural number, such that the order can have any real values. With this generalization both differential and integral operators may be written in the form of an integral, when applied to a suitable class of functions. We will state some of the well known facts about pseudodifferential operators without a proof, more details can be read for example in ([1], *Chapter 1 - Chapter 3*).

Let us first recall the definition of the Schwartz space. Afterwards we want to use properties of functions belonging to the Schwartz space to develop an intuition how to define a generalization of differential operators.

1.1 Basic Definitions

Definition 1.1: *Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of functions f on $\mathbb{R}^n$, such that*

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty \tag{1.1}$$

for all indices $\alpha, \beta \in \mathbb{N}_0$.

Lemma 1.1: *For the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ the following facts are true:*
• *The compactly supported smooth functions are contained in the Schwartz space, $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.*
• $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.
• *For the Fourier transform $\mathcal{F}$ of a function f defined by*

$$\mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi x} f(x) dx \tag{1.2}$$

we have

$$(i)\ \mathcal{F}(\partial_x^\alpha f)(\xi) = (-1)^{|\alpha|} \xi^\alpha \mathcal{F}(f)(\xi), \tag{1.3}$$

$$(ii)\ \partial_\xi^\alpha \mathcal{F}(f)(\xi) = \mathcal{F}(x^\alpha f)(\xi). \tag{1.4}$$

We want to use (1.3)-(1.4) to see how one may generalize differential operators. Recall that the Fourier transform is invertible with

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{i\xi x} g(\xi) d\xi. \tag{1.5}$$

Then, for any differential operator

$$P = \sum_{\alpha \leq m} a_\alpha(x) \partial_x^\alpha \tag{1.6}$$

one may write

$$(Pf)(x) = (\sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha f)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \mathcal{F}^{-1}(i^{|\alpha|} \xi^\alpha \mathcal{F}(f))(x) \tag{1.7}$$

$$= \int_{\mathbb{R}^n} e^{i(x-y)\xi} \left(\sum_{|\alpha| \leq m} i^{|\alpha|} \frac{1}{(2\pi)^n} a_\alpha(x) \xi^\alpha \right) f(y) dy d\xi, \tag{1.8}$$

as long as f is a Schwartz function. We realize, that the action of the differential operator may be rewritten as

$$(Pf)(x) = \int_{\mathbb{R}^n} e^{i(x-y)\xi} p(x,\xi) f(y) dy d\xi \tag{1.9}$$

with $p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha i^{|\alpha|} \xi^\alpha / (2\pi)^n$. This function p is called the *symbol* of the differential operator P.

Now, for every differential operator P of order m, the corresponding symbol $p(x,\xi)$ will be a function of order m in ξ. The crucial idea is to define operators like (1.9), but with a symbol p of an arbitrary order. The standard class of pseudodifferential operators is given by the following definition:

Definition 1.2: *Consider a function p of the two variables x and ξ satisfying*

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m - \rho\alpha + \delta\beta}, \tag{1.10}$$

for some constants $C_{\alpha,\beta}$ depending on the multi-indices α and β. For such a function we write $p \in S^m_{\rho,\delta}$. We can define an operator P on the Schwartz space associated to p via

$$(Pf)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} p(x,\xi) u(y) dy d\xi. \tag{1.11}$$

p is called the symbol of the operator P, m the order of P and we may write $p = \sigma(P)$, or equivalently, $P = \mathrm{op}(p)$. Moreover, if the order of P is m, we write $P \in \mathrm{OPS}^m$.

Remarks 1.1:

• Quite obviously, if $p(x,\xi) \in S^m_{\rho,\delta}$ then $\partial_x^\beta \partial_\xi^\alpha p(x,\xi) \in S^{m-\rho\alpha+\delta\beta}_{\rho,\delta}$. Furthermore, for $p \in S^m_{\rho,\delta}$

and $q \in S^n_{\tilde{\rho},\tilde{\delta}}$, $pq \in S^{m+n}_{\min\{\rho,\tilde{\rho}\},\max\delta,\tilde{\delta}}$.
• The definition given by (1.10) depends on the parameters ρ and δ. In this work we will focus on the special case of operators with $\rho = 1$ and $\delta = 0$, thus operators P with a symbol $\sigma(P) \in S^m_{1,0}$. For this subclass of operators we say that an operator P is of classical type, if there exist functions $p_{m-j}(x,\xi)$ which are homogenous of degree $m-j$ in ξ for $|\xi| > C$ with some constant $C > 0$, such that

$$(\sigma(P)(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}_{1,0} \tag{1.12}$$

(note that (1.12) means that asymptotically we could write $\sigma(P)(x,\xi) = \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$).
• It is often the case that in the literature pseudodifferential operators are defined via

$$(Pf)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} p(x,y,\xi) f(y) dy d\xi. \tag{1.13}$$

In this alternative definition the symbol $p = \sigma(P)$ has two position variables x and y. Although it might not seem intuitive on the first glance, such a definition does not produce a larger class of operators, which can be read in ([1], *Chapter 2*). Every pseudodifferential operator defined like in (1.11) has also different representations as an operator with symbols with two position variables and for every double symbol $p = p(x,y,\xi)$ one can find $\tilde{p} = \tilde{p}(x,\xi)$ such that the operator defined via (1.13) is equal to the operator with symbol $\tilde{p}$ defined via (1.11).
• As already mentioned in the foregoing remark, there are different representations for a given pseudodifferential operator P. One useful representation is the *Weyl* representation: Given any pseudodifferential operator P represented like in (1.13) or (1.11), there exists a *Weyl symbol* p_W such that

$$(Pf)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} p_W\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi. \tag{1.14}$$

1.2 Useful Properties and Theorems

Let us come back to the reason why we defined pseudodifferential operators: As already explained, differential operators may be written as integral operators when using pseudos, therefore one has to deal with only one type of operators. For any class of operators it is always practical to know, whether they form an algebra, i.e. if one can define addition and multiplication. Fortunality this is the case for pseudos.

Theorem 1.1: *Let $P \in \mathrm{OPS}^m$ and $Q \in \mathrm{OPS}^n$ be classical operators (which we always assume from now on), for some $m,n \in \mathbb{R}$. Then $(P+Q) \in \mathrm{OPS}^{\max\{m,n\}}$ and $PQ \in \mathrm{OPS}^{m+n}$. Moreover,*

$$\sigma(P+Q) = \sigma(P) + \sigma(Q), \quad \sigma(PQ) = \sum_{\alpha} \frac{(-i)^{|\alpha|}}{|\alpha|!} \partial_x^{\alpha} \sigma(P) \partial_{\xi}^{\alpha} \sigma(Q). \tag{1.15}$$

As we work with classical symbols, we have the possibility to define a leading or principal part for any operator P. It will turn out later that this is the essentially important part concerning Fredholm and index theory.

Definition 1.3: *If P is a pseudodifferential operator with asymptotic expansion*

$$\sigma(P) = \sum_{j=0}^{\infty} p_{m-j}(x, \xi) \tag{1.16}$$

for its symbol $\sigma(P)$, the operator P_m defined by $P_m = \mathrm{op}(p_m)$ is the princpial part of P. p_m itself is called the principal symbol of P. We may also write $\sigma_p(P)$ for the principal symbol.

Remarks 1.2:
• The principal symbol σ_p is multiplicative, $\sigma_p(PQ) = \sigma_p(P)\sigma_p(Q)$ (to see this, just consider the $\alpha = 0$ term in the expansion (1.15)). It is only additive as long as the operators P and Q have the same order.
• Let P be a pseudodifferential operator of order zero and g an analytic function, such that the operator $g(P)$ is well defined. Then the principal symbol fulfills $\sigma_p(f(P)) = f(\sigma_p(P))$. To see this, we think of $g(P)$ as a power series in P. Any power of P in this series has order zero, therefore $g(P)$ is a series of order zero operators. In this case the additivity and multiplicativity may be used on the power series to arrive at $\sigma_p(f(P)) = f(\sigma_p(P))$.
• The multiplicativity of the principal symbol still holds if one considers matrices with pseudodifferential operators as entries, i.e. operators $P : \mathcal{S}(\mathbb{R}^n)^j \to \mathcal{S}(\mathbb{R}^n)^k$.

We want to remind the reader about the Schwartz kernel theorem of an operator. This reminder shall have the purpose to help us calculating the adjoint P^* of some pseudo P.

Theorem 1.2: *Any continous operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a representation*

$$(Af)(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy \tag{1.17}$$

with some distributional kernel $k \in D'(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 1.1: *Any pseudodifferential operator $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.*

Theorem 1.2 and *Proposition 1.1* tell us that pseudos have a representation with an integral kernel k. Looking at the equivalent definitions (1.11) or (1.13) one may guess that k is given either by

$$k(x, y) = \int e^{i(x-y)\xi} p(x, \xi) d\xi \quad \text{or} \quad k(x, y) = \int e^{i(x-y)\xi} p(x, y, \xi) d\xi. \tag{1.18}$$

Both integrals do not exists in the classical sense as a function, as long as the order of P

is greater or equal to $-n$. However, it turns out that in the distributional sense, defined by the action

$$\langle k, u\rangle := \int e^{i(x-y)\xi} p(x, y, \xi) u(x, y) dy dx d\xi \tag{1.19}$$

on an arbitrary function $u \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ the kernel k is well defined and it is indeed the Schwartz kernel of P. If we want to compute the adjoint of a pseudo P, we can make use of the Schwartz kernel to get the following result:

Proposition 1.2: *Let P be a pseudodifferential operator of order m whith a double symbol representation*

$$(Pf)(x) = \int_{\mathbb{R}^n} e^{i(x-y)\xi} p(x, y, \xi) f(y) dy d\xi. \tag{1.20}$$

Then the adjoint operator $P^\dagger$ is also pseudodifferential of order m, with symbol $\sigma(P^\dagger)(x, y, \xi) = \overline{p(y, x, \xi)}$. Moreover, if P is given by its Weyl representation (1.14)*, P is self adjoint as long as the Weyl symbol p_W is real.*

Proof: Let us first show that for any bounded linear operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ with a Schwartz kernel $k = k(x, y)$ the adjoint operator $A^\dagger$ has Schwartz kernel $\tilde{k}(x, y) = \overline{k(y, x)}$. In order to see this, let us first write down $\langle Au, v\rangle$ explicitly for $u, v \in \mathcal{S}(\mathbb{R}^n)$:

$$\langle \int_{\mathbb{R}^n} k(x, y) u(y) dy, v\rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(x, y) u(y) dy \overline{v(x)} dx \tag{1.21}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(x, y) u(y) \overline{v(x)} dy dx. \tag{1.22}$$

Lets see what happens, if we try $\overline{k(y, x)}$ as the adjoint kernel:

$$\langle u, \int_{\mathbb{R}^n} \overline{k(y, x)} v(y) dy\rangle = \int_{\mathbb{R}^n} u(x) \left(\overline{\int_{\mathbb{R}^n} \overline{k(y, x)} v(y) dy}\right) dx \tag{1.23}$$

$$= \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} k(y, x) \overline{v(y)} dy dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) k(y, x) \overline{v(y)} dy dx. \tag{1.24}$$

To see that (1.22) and (1.24) are equal, just exchange y and x and then switch integration. This means that the kernel of $A^\dagger$ has to be $\overline{k(y, x)}$. As already mentioned in (1.18), for a pseudodifferential operator P the integral kernel is

$$k(x, y) = \int_{\mathbb{R}^n} e^{i(x-y)\xi} p(x, y, \xi) d\xi. \tag{1.25}$$

Complex conjugation and switching x and y both produce a minus sign in the exponent of the exponential factor, which keeps the term $\exp(i(x-y)\xi)$ unchanged. Therefore we get another pseudodifferential operator $P^\dagger$ with kernel

$$\tilde{k}(x, y) = \int e^{i(x-y)\xi} \overline{p(y, x, \xi)} d\xi. \tag{1.26}$$

But this corresponds to the fact that the symbol of $P^\dagger$ is given by $\sigma(P^\dagger) = \overline{p(y, x, \xi)}$. If the symbol of P has the special form of the Weyl representation (1.14), it is invariant under exchange of the two coordinates x and y. If $p = p_W$ is furthermore a real symbol it is also invariant under complex conjugation, meaning that the adjoint operator in this case coincides with the original P. *q.e.d.*

Up to this point we only considered pseudodifferetial operators acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This was due to the properties of the Fourier transform on the Schwartz space, which allowed us to write differential operators in a pseudodifferential form. Because the compactly supported functions are contained in the Schwartz space, we can also define pseudodifferential operators acting on smooth functions on compact manifolds. This is done via the representation in charts.

Definition 1.4: *For a compact manifold M of dimension m we say that $P : C^\infty(M) \to C^\infty(M)$ is a pseudodifferential operator, if its representation in local coordinates is a pseudodifferential operator on $\mathbb{R}^m$. One may use a partition of unity to define the action of P globally.*

Remark 1.3: Of course one could also consider operators P operating between sections of a rank j bundle E and a rank k bundle F over M. In this case P is considered to be pseudodifferential, if the local representation of every component of P considered as a matrix operator is pseudodifferential. Note that for a pseudodifferential operator P on a compact manifold M, the symbol $\sigma(P)$ may be seen as a function on the cotangent bundle T^*M.

Until now the function space where we let the pseudos act on was always some subset of smooth functions. Other interesting spaces are of course any of the Sobolev spaces $H^s(M)$. As we know, classical differential operators of order m decrease the regularity of function in a Sobolev space by the value of its order. For pseudos we have the same result, but it holds for every real order m.

Lemma 1.2: *An arbitrary pseudodifferential operator P of any real order m may be regarded as an operator $P : H^s(M) \to H^{s-m}(M)$. For any $s \in \mathbb{R}$ P is a continous operator.*

Lemma 1.2 is very helpful if one wants to see that operators of negative order are compact. Let us recall the embedding theorem of Rellich:

Theorem 1.3: (*Rellich*) *For every $r > s$, $H^r(M)$ is compactly embedded in $H^s(M)$ on a compact manifold M.*

Proposition 1.3: *On a compact manifold M, every pseudodifferential operator P of negative order m acting on some Sobolev space $H^s(M)$ is a compact operator.*

Proof: From *Lemma 1.2* we know, that $P : H^s(M) \to H^{s-m}(M)$. As long as $m < 0$,

$s-m>s$. But then it follows from the Rellich embedding theorem that the image of P is a subset of a compact set in $H^s(M)$. This means that the image is relatively compact and therefore P is a compact operator. *q.e.d.*

Remark 1.4: Note that if one considers pseudodifferential operators on $\mathbb{R}^n$ defined by symbols of the class (1.10), negative order operators are not neccesarily compact. Even operators of order $m=-\infty$, which means that the operator has every real order $m \in \mathbb{R}$, might be noncompact on $\mathbb{R}^n$, as we will see later.

1.3 Shubin Type Operators

The property of negative order operators to be always compact makes the construction of so called *parametrices* for elliptic operators much easier, as we will see later in the chapter about *Fredholm operators*. A parametrix may be seen as an almost inverse for some pseudodifferential operator and it exists for the class of elliptic operators, which we will define in a moment. If only the operators of order $-\infty$ are compact, such parametrices are harder to construct. But there exist other classes of pseudos, defined by symbols, which differ from the definition in (1.10) and define operators on $\mathbb{R}^n$ having similar properties like the usual pseudos on compact manifolds. Before we define the notion of ellipticity and make some concluding remarks and examples on the topics of this section, let us introduce the class of Shubin-type operators:

Definition 1.5: *For functions $f \in \mathcal{S}(\mathbb{R}^n)$, we may alternatively define pseudodifferential operators via* (1.11) *but with symbols p satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|}. \tag{1.27}$$

Operators of this type are called Shubin-type operators. For a symbol like (1.27) *we will write $p \in S_{Sh}^m$.*

Shubin-type operators behave very well, as long as their order fulfills certain conditions. First, we can also define an asymptotic expansion like (1.12) for Shubin operators, with the difference, that homogeneity is demanded for the whole variable $z := (x,\xi)$. Moreover, the formula for the composition (1.15) of operators and *Proposition 1.2* are still true. But there are some facts, which are really more useful than in the usual pseudo case. Let us list some of the good features of Shubin operators.

Proposition 1.4: (see [2], Chapter IV) *For the algebra $\mathcal{A}_{Sh}$ of Shubin type operators the following facts hold:*

(*a*) *Shubin operators $P \in \mathrm{OPS}_{Sh}^0$ of order zero may be extended to $L^2(\mathbb{R}^n)$. On $L^2(\mathbb{R}^n)$ operators of negative order are always compact.*
(*b*) *If the order m of a Shubin operator P is less than $-2n$, it is of trace class.*

(c) *Any Shubin operator P with order $m < -n$ is a Hilbert-Schmidt operator.*

Remark 1.5: Just as we did for the case of usual pseudos we want to restrict only to operators with symbols of the form $S^m_{Sh} = S^m_{Sh,1,0}$. In the case of standard pseudos we see from *Remarks 1* for $\rho = 1$ and $\delta = 0$ that derivatives with respect to the position variable x do not change the order. In contrast to that, for operators of Shubin type both derivatives ∂_x and ∂_ξ decrease the order of the symbol. This means that in the composition formula (1.15) for two pseudos P and Q two consecutive terms in the sum differ by two orders if we are working with Shubin operators, while the order difference is one for usual pseudos. Using this fact we can give a corollary which shows the advantages of Shubin type operators concerning the composition and commutators of two operators:

Corollary 1.1: *Given two scalar pseudodifferential operators P and Q of order m and n, the following statement for their commutators is true:*

$[P, Q], [Q, P] \in \mathrm{OPS}^{m+n-1}$ *as long as P and Q are standard pseudos and* $[P, Q], [Q, P] \in \mathrm{OPS}^{m+n-2}_{Sh}$ *if the operators are of Shubin type.*

Proof: Writing $[P, Q] = PQ - QP$, we have

$$\sigma([P, Q]) = \sigma(PQ) - \sigma(QP) = \sigma(P)\sigma(Q) + \sum_{\alpha \geq 1} \frac{i^{|\alpha|}}{\alpha!} \partial^\alpha_x \sigma(P) \partial^\alpha_\xi \sigma(Q) \tag{1.28}$$

$$-\sigma(Q)\sigma(P) - \sum_{\alpha \geq 1} \frac{i^{|\alpha|}}{\alpha!} \partial^\alpha_x \sigma(Q) \partial^\alpha_\xi \sigma(P) \tag{1.29}$$

$$= \sum_{\alpha \geq 1} \frac{i^{|\alpha|}}{\alpha!} (\partial^\alpha_x \sigma(P) \partial^\alpha_{xi} \sigma(Q) - \partial^\alpha_x \sigma(Q) \partial^\alpha_{xi} \sigma(P)). \tag{1.30}$$

All the remaining terms in (1.30) are of one order lower in the usual pseudo case and two orders lower in the Shubin case. *q.e.d.*

1.4 Ellipticity and Examples

At the end of this chapter, we want to give the definition of elliptic operators. This class of operators will get more relevant in the chapter *Fredholm operators*.

Definition 1.6: *A pseudodifferential operator P is called elliptic, if there exists a $M \in \mathbb{R}$, such that the principal symbol $\sigma_p(P)$ is invertible for all (x, ξ) with $|\xi| \geq M$.*

Remark 1.6: In the case of Shubin operators, ellipticity is defined in the sense of *Definition 1.6*, but with the requirement that the principal symbol has to be invertible for $|(x, \xi)| \geq M$.

Let us close the discussion about pseudodifferential operators with a few examples.

Example 1.1: Let (M, g) be a compact Riemannian manifold and $P = -\text{div} \circ \text{grad}_g := \Delta$ be the Laplace-Beltrami operator. Then P is a second order differential operator with principal part $\Delta_p = -g^{kj}\partial_{x_k}\partial_{x_j}$. The principal symbol can be calculated by simply exchanging ∂_{x_k} by $-i\xi_k$, $\sigma_p(\Delta) = g^{kj}\xi_k\xi_j = |\xi|^2$. Therefore the Laplace-Beltrami operator is an elliptic operator.

Example 1.2: Consider $\Lambda = \sqrt{\Delta + 1} \in \text{OPS}^1$. The principal symbol of this operator is $\sigma_p(\Lambda) = \sqrt{\sigma_p(\Delta + 1)} = \sqrt{\sigma_p(\Delta)} = |\xi|$. So, this operator is elliptic just as the Laplacian Δ. If we treat the Laplacian Δ as an unbounded operator $\Delta : \mathcal{D}(\Delta) = H^2(M) \subset L^2(M) \to L^2(M)$ it is essentially self adjoint and therefore has real spectrum. Moreover, it is known, that Δ is a positive operator. This means that $\text{spec}(\Delta) \subset [0, \infty)$. The operator $\Delta + 1$ then has a spectrum $\text{spec}(\Delta + 1) \subset [1, \infty)$ meaning that zero is not an eigenvalue. This and the self adjointness result in the fact that Λ is an invertible operator.

Example 1.3: A classical example of an elliptic operator of Shubin type is the Schrödinger operator $P = |x|^2 - \Delta$. The principal symbol is given by $\sigma_p(P) = |x|^2 + |\xi|^2$.

Example 1.4: As a standard example of a non elliptic operator, there is the wave operator $\Box = \partial_t^2 - \Delta$. The equation $\sigma_p(\Box) = 0$ has the solutions $\xi_t = \pm|\xi_x|$ and this does not fulfill the requirements of *Definition 1.6*.

2 Symplectic Geometry, Fourier Integral Operators and G-Operators

2.1 Symplectic Geometry

After introducing pseudodifferential operators, which are already quite general, we still need even larger and more general classes of operators in order to describe the solution theory of hyperbolic systems. These extensions of the algebra of pseudodifferential operators are Fourier Integral Operators, or even more general, G-operators. In order to understand these classes it is useful to introduce symplectic geometry, which is crucially connected to the definition of a Fourier integral operator. It will turn out that especially concerning the solution theory of hyperbolic equations, there is a strong connection to symplectic geometry. We will recall some basic definitions and properties concerning symplectic manifolds in the first part of this section and we will explain Fourier integral operators and G-operators in the second part. For more details on symplectic geometry, see for example ([4], *Chapter 3*) or ([15]).

2.1.1 Symplectic Manifolds and Symplectomorphisms

Definition 2.1: *Given any even dimensional manifold M of dimension dim(M)= $2n$, we call M symplectic, if there exists a closed 2-form w on M which defines pointwise a non-degenerate bilinear form on T_pM. w itself is called a symplectic form.*

Examples:

• For $M = \mathbb{R}^{2n}$, one can choose the constant matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.1}$$

as a symplectic form. It is closed, because its components are constant. The matrix J is skew symmetric, and therefore defines a two form, which acts skew symmetriallyc on two vector fields. As the determinant of J is 1, it is invertible and therefore non-degenerate.
• If M is a manifold with arbitrary dimension n, the cotangential bundle T^*M has always the even dimension $2n$. The Theorem of Darboux states that in some local coordinates (which are called the canonical coordinates), any symplectic form on T^*M will have the representation

$$w = \sum_i d\xi_i \wedge dx_i. \tag{2.2}$$

It is clear, that this two form is closed. Note, that it is also exact in local coordinates, $w = d\alpha$, with $\alpha = \sum_i \xi_i dx_i$. α is often called the canonical one form of the cotangent bundle T^*M.

As with any usual manifold, one can study diffeomorphisms on a symplectic manifold M. In the case where (M, g) is a Riemannian manifold, an interesting subclass of the diffeomorphism group is the group of isometries. These are the diffeomorphisms, which preserve lengths and angles with respect to the Riemannian metric g, meaning

$$\langle v, w\rangle_{g_p} = \langle DF(v), DF(w)\rangle_{g_{F(p)}} \tag{2.3}$$

for any diffeomorphism $F : M \to M$. One may see the symplectic form w on a symplectic manifold (M, w) analogous to the Riemannian metric, in the sense, that it defines how angles between vectors on the manifold are measured. The corresponding class of diffeomorphisms, which are compatible with w, would be all diffeomorphisms $F : M \to M$, such that

$$w_p(v, w) = w_{F(p)}(DF(v), DF(w)) \tag{2.4}$$

Such diffeomorphisms are called symplectomorphisms. Given any vector field X on M, one can induce a diffeomorphism by considering the time $t = 1$ flow $\Phi_{t=1}$ along that vector field. In that sense, one may ask how the property of being a symplectomorphism generated by a flow along a vector field X reflects in the vector field X itself. One of the crucial tools of symplectic geometry is the notion of a Lie-derivative: Given any tensor T and a vector field X on M one defines the Lie-derivative of T along X to be the tensor

$$\mathcal{L}_X T := \frac{d}{dt}_{t=0}(\Phi_t^* T) \tag{2.5}$$

where Φ_t is the time t flow along X. Now, being a symplectomorphism means for a map $F : M \to M$ that the symplectic form is conserved along X, if X is the vector field, which generates F. In other words, we want the Lie-derivative of w along X to be zero. For any two form σ, there is a nice identity, which tells us, how to compute its Lie-derivative:

Lemma 2.1: ***(Magic formula of Cartan)*** *Let σ be any 2-form on the manifold M and X a vector field. Then the Lie-derivative $\mathcal{L}_X\sigma$ is the 2-form given by*

$$\mathcal{L}_X\sigma(\cdot, \cdot) = (d\sigma)(X, \cdot, \cdot) + d(\sigma(X, \cdot)). \tag{2.6}$$

Formula (2.6) gives a way to characterize vector fields, which generate symplectomorphisms: Assume that a given vector field X has a flow Φ_t, which is a symplectomorphism. Then by (2.6) we have

$$(dw)(X, \cdot, \cdot) + d(w(X, \cdot)) = 0. \tag{2.7}$$

Using *Definition 2.1*, we know that $dw = 0$ and therefore $d(w(X, \cdot)) = 0$. But this means, that inserting the vector field X into the symplectic form w yields a closed one form.

Therefore we have shown

Proposition 2.1: *Any diffeomorphism $F : M \to M$ generated by the flow of a vector field X is a symplectomorphism, iff the one-form $w(X, \cdot)$ is closed.*

Because every exact one-form $\eta = d\alpha$ is automatically closed, one could investigate those vector fields X, such that $w(X, \cdot)$ is exact as a special case of *Proposition 2.1.* Since the exact one forms are precisely the gradients of scalar functions, this requieres $w(X, \cdot) = df$. The differential df of a function is commonly used in order to express the directional derivative with respect to some vector field Y. We can use this fact to connect the special case of exact one-forms in *Proposition 2.1* to the directional derivative of a function with the help of the following definition:

Definition 2.2: *Let $f : M \to \mathbb{C}$ be a function on the manifold M. A vector field H_f is called a Hamiltonian field to f, if one can express the directional derivative of f in direction Y as*

$$df(Y) = w(H_f, Y). \tag{2.8}$$

We derive the special case in *Proposition 2.1* directly from *Definition 2.2* and formulate it in another proposition:

Proposition 2.2: *If H_f is a Hamiltonian vector field to the function $f : M \to \mathbb{C}$, the flow Φ_t along H_f defines a symplectomorphism for any t. The corresponding closed one form $w(H_f, \cdot)$ is an exact form. We call the symplectomorphism Φ_t a Hamiltonian flow and may also write α_t for all flows which are of this type.*

2.1.2 Properties of Hamiltonian Vector Fields and Poisson Brackets

Remark 2.1: *Note that, given any function $f : M \to \mathbb{C}$, there is a unique vector field H_f satisfying* (2.8). *In canonical coordinates (x, ξ) of a cotangential bundle $M = T^*N$, where the symplectic form can be expressed like in* (2.2), *H_f is given by*

$$H_f = \sum_k \left(\frac{\partial f}{\partial_{\xi_k}} \frac{\partial}{\partial_{x_k}} - \frac{\partial f}{\partial_{x_k}} \frac{\partial}{\partial_{\xi_k}} \right). \tag{2.9}$$

We want to adress that not only the symplectic form w is conserved along the flow of a Hamiltonian vector field H_f, but also a quite large class of functions. This is an important fact, as we will be interested in the transport of certain symbols along the flow of Hamiltonian vector fields later. Let us give an overview about conserved quantities along Hamiltonian flows:

Proposition 2.3: *Let $f : M \to \mathbb{C}$ be a function on a symplectic manifold (M, w). Then the following quantities are conserved along the flow of the Hamiltonian vector field H_f:*

(i) *The symplectic form w,*
(ii) *The function f itself which generates the flow and*
(iii) *Any function g, which is an analytic function in f, i.e.*

$$g = \sum_k a_k f^k. \tag{2.10}$$

Proof: (i): Was already mentioned in *Proposition 2.2.*
(ii): By definition, the derivative of f along H_f is $df(H_f) = w(H_f, H_f)$. But this is zero, because w is a two form. Therefore f is conserved along the flow of H_f.
(iii): We first show $df^k(H_f) = 0$. We have $df^k(H_f) = w(H_{f^k}, H_f)$. Let us calculate H_{f^k}:

$$H_{f^k} = \sum_j \left(\partial_{\xi_j}(f^k)\partial_{x_j} - \partial_{x_j}(f^k)\partial_{\xi_j}\right) = \sum_j \left(kf^{k-1}\partial_{\xi_j} f\partial_{x_j} - kf^{k-1}\partial_{x_j} f\partial_{\xi_j}\right) \tag{2.11}$$

$$= kf^{k-1}\sum_j \left(\partial_{\xi_j} f\partial_{x_j} - \partial_{x_j} f\partial_{\xi_j}\right) = kf^{k-1}H_f. \tag{2.12}$$

Inserting this into $df^k(H_f)$ leads to $df^k(H_f) = w(kf^{k-1}H_f, H_f) = kf^{k-1}w(H_f, H_f) = 0$. Now, because w and the Hamiltonian field are linear any linear combination of the f^k is also conserved. But this actually means that any analytic function like in (2.10) is conserved. *q.e.d.*

Of course, not every function g on the cotangent bundle of a manifold is an analytic function of another function f. However, we can still consider its derivative $dg(H_f)$ even if it is not zero. It turns out, that this directional derivative defines another very useful structure in symplectic geometry.

Definition 2.3: *Given two functions $f : M \to \mathbb{C}$ and $g : M \to \mathbb{C}$ we define the Poisson bracket $\{f, g\}$ of f and g as*

$$\{f, g\} := w(H_f, H_g) = df(H_g). \tag{2.13}$$

Lemma 2.2: *The Poisson bracket defined in Definition 2.3 has the following properties:*

(i) *It is anticommutative, $\{f, g\} = -\{g, f\}$,*
(ii) *Bilinearity, i.e. $\{f, g\}$ is linear in both f and g,*
(iii) *It fulfills the Leibniz rule $\{fg, h\} = \{f, h\}g + f\{g, h\}$,*
(iv) *Jacobi identity: There is $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ and*
(v) *$H_{\{f,g\}} = [H_f, H_g]$ with the Lie-bracket $[X, Y] = X \circ Y - Y \circ X$.*

The reader may check the properties described in *Lemma 2.2* either in local coordinates (2.9) or by the abstract definitions. We want to just make use of these facts to derive useful information from the Poisson bracket:

Remarks 2.2:

• From the definition (2.13) and the anticommutativity (i) in *Lemma 2.2*, we see $df(H_g) = -dg(H_f)$. Therefore, if the Poisson bracket $\{f, g\}$ vanishes, f is constant along H_g and g is constant along H_f.
• As long as f and g Poisson commute, $\{f, g\} = 0$, the corresponding Hamiltonian vector fields H_f and H_g also commute by (v). This means that the flows $\Phi_{H_f,t}$ and $\Phi_{H_g,t}$ commute locally, see for example [18].

2.1.3 Canonical Transformations and Compositions of Symplectomorphisms

We have now learned about two types of symplectomorphisms, the general ones and the symplectomorphisms generated by a Hamiltonian flow. Before comparing these two types, we want to define another type of transformation, which will be the most interesting in the chapters to come. Let us from now on assume that the symplectic manifold we are working on is always the cotangent bundle of some manifold, so we write T^*M instead of M from now on.

Definition 2.4: *A symplectomorphism $\alpha : T^*M\backslash\{0\} \to T^*M\backslash\{0\}$ is called canonic or a canonical transformation, if it is positivley one-homogenous in the fibre variable, which means that if (x, ξ) is mapped to $\alpha(x, \xi) = (y, \eta)$, $(x, \lambda\xi)$ is mapped to $(y, \lambda\eta), \lambda > 0$.*

There are several easy examples how to generate canonical transformations. One possibility is to generate it with the help of a one-homogenous function f: If a function $f : T^*(M) \to \mathbb{R}$ is one-homogenous in the fibre, $f(x, \lambda\xi) = \lambda f(x, \xi)$ then so are the ∂_{ξ_k} components of the Hamiltonian field H_f. This part of the vector field describes the movement inside the fibre, and it turns out that the corresponding flow is also compatible with the homogeneity:

Proposition 2.4: *Let $f : T^*M \to \mathbb{R}$ be a function, which is positively one-homogenous in the fibre variable ξ. Then the flow along its Hamiltonian vector field H_f is a canonical transformation.*

Another classical example are the lifted diffeomorphisms. Given any diffeomorphism $g : M \to M$ the inverse transpose of the differential D_g, the so called co-differential, is a (pointwise) linear map on the fibres of T^*M. This produces a canonical transformation via

Proposition 2.5: *If $g : M \to M$ is a diffeomorphism on the base manifold M, the* *lifted map*

$$\alpha(x,\xi) := (g(x), (Dg)_x^{-T}(\xi)) \tag{2.14}$$

is a canonical transformation.

Proof: The homogeneity follows directly from the linearity of the co-differential $(Dg)^{-T}$. Remembering the fact that in Darboux coordinates w has the form (2.2), we can write in local coordinates

$$w((v_x, v_\xi), (z_x, z_\xi)) = \langle v_x, z_\xi \rangle - \langle v_\xi, z_x \rangle, \tag{2.15}$$

for any tangential vectors (v_x, v_ξ) and (z_x, z_ξ) in the tangent space of T^*M. Now, the map $D\alpha$, which has to preserve the values of (2.15) can be induced by α,

$$D\alpha(v_x, v_\xi) = (Dg(v_x), (Dg)^{-T}(v_\xi)). \tag{2.16}$$

We then calculate

$$w(D\alpha(v_x, v_\xi), D\alpha(z_x, z_\xi)) = w((D_g(v_x), (Dg)^{-T}(v_\xi)), (Dg(z_x), (Dg)^{-T}(z_\xi))) \tag{2.17}$$

$$= \langle Dg(v_x), (Dg)^{-T}(z_\xi) \rangle - \langle (Dg)^{-T}(v_\xi), Dg(z_x) \rangle = \langle v_x, z_\xi \rangle - \langle v_\xi, z_x \rangle, \tag{2.18}$$

where we used, that the transpose is equal to the adjoint in the coordinates. The last equality means, that the symplectic form is invariant. *q.e.d.*

After presenting the relevant types of symplectic transformations, we want to state, how these types are connected to each other and whether they form a group.

Proposition 2.6: *For two arbitrary symplectomorphisms C_1, C_2 on the symplectic manifold T^*M their compositions $C_1 \circ C_2$ and $C_2 \circ C_1$ are again symplectic.*

Proof: Let C_1 and C_2 be symplectic. Furthermore, let v and z be vectors in $T_p(T^*M)$. We have to check, that

$$w(D(C_1 \circ C_2)(p)(v), D(C_1 \circ C_2)(p)(z)) = w(v, z). \tag{2.19}$$

First, there is the chain rule. We may write $D(C_1 \circ C_2)(p) = (DC_1)(C_2(p)) \cdot (DC_2)(p)$. Therefore we have

$$w(D(C_1 \circ C_2)(p)(v), D(C_1 \circ C_2)(p)(z)) \tag{2.20}$$

$$= w((DC_1)(C_2(p)) \cdot (DC_2)(p)(v), (DC_1)(C_2(p)) \cdot (DC_2)(p)(z)). \tag{2.21}$$

If we write $DC_2(p)(v) = \tilde{v}$ and $DC_2(p)(z) = \tilde{z}$, the object of interest is equal to

$$w((DC_1)(C_2(p))(\tilde{v}), (DC_1)(C_2(p))(\tilde{z})). \tag{2.22}$$

But then, we know that C_1 is symplectic, from which we see that the expression in (2.22) is equal to $w(\tilde{v}, \tilde{z}) = w(v, z)$ and the last equality is true due to the fact, that C_2 is also assumed to be symplectic. *q.e.d.*

Proposition 2.7: *If C_1 and C_2 are canonical, so are their compositions $C_1 \circ C_2$ and $C_2 \circ C_1$.*

Proof: Canonical transformations C_1 and C_2 are symplectic by definition. By *Proposition 2.6*, their compositions are also symplectic. It remains to prove that the compositions are one-homogenous in the fibre. Set $C_2(x, \xi) = (y, \eta)$ and $(C_1 \circ C_2)(x, \xi) = (z, \beta)$, which implies $C_1(y, \eta) = (z, \beta)$. Now consider $(C_1 \circ C_2)(x, \lambda\xi)$:

$$(C_1 \circ C_2)(x, \lambda\xi) = C_1(C_2(x, \lambda\xi)) = C_1((y, \lambda\eta)) = (z, \lambda\beta). \tag{2.23}$$

But this means, that the composition is one homogenous. *q.e.d.*

Concerning compositions of symplectomorphisms, we want to give one last statement, without proving it:

Proposition 2.8: *The composition of the flows $\Phi_{H_f,t}$ and $\Phi_{H_g,t}$ of two Hamiltonian vector fields H_f and H_g is again a flow of a Hamiltonian vector field (possibly time dependent).*

At last, before talking about the necessary objects concerning Fourier integral operators, let us show, how general symplectic flows and Hamiltonian flows are connected to each other:

2.1.4 Local Triviality

Proposition 2.9: *Let $\Phi_{X,t}$ be the flow of some vector field X. If the flow is symplectic, then for each point $p \in T^*M$ there exists a neighborhood V, such that $\Phi_{X,t} = \Phi_{H_f,t}$ with the Hamiltonian vector field H_f of some function f.*

Proof: The proof directly follows from the Poincaré lemma: If $\Phi_{X,t}$ is symplectic, $w(X, \cdot)$ is closed by *Proposition 2.1.* The Poincarè lemma states that locally every closed form is also exact. But then, locally $w(X, \cdot)$ is exact, which means that $X = H_f$. *q.e.d.*

Corollary 2.1: *If the first de Rham cohomology $H^1(M)$ is trivial, every symplectic flow*

$\Phi_{X,t}$ *is Hamiltonian.*

Proof: A trivial first de Rham cohomology $H^1(M)$ means that every closed form on M is also exact. Note that the de Rham cohomology $H^k(M)$ is isomorphic to that of the cotangent bundle $H^k(T^*M)$ for any k: We can identify the base manifold M with the zero section $M_0 = \{(x,0), x \in M, 0 \in T^*M\}$, which is a subset of the cotangent bundle. The cotangent bundle can then be deformed into the zero section M_0 with the help of the homotopy $F_t(x,\xi) = (x,(1-t)\xi), t \in [0,1]$. Therefore T^*M is homotopy equivalent to M and this means that it has the same de Rham cohomology like M (this holds actually for every vector bundle E over M). But then $H^1(T^*M)$ is trivial and therefore every closed form on T^*M is exact, which leads to the fact that every symplectic flow is Hamiltonian. *q.e.d.*

2.1.5 Lagrangian Submanifolds

Until now we have only defined and explained the purely geometrical structures appearing in symplectic geometry. Of course, we want to find connections between these geometrical structures and the corresponding analytical objects like operators on some function space, for example $L^2(M)$. In order to make a nice transition to the analytical part of symplectic geometry, we define another geometric object, which we will connect afterwards to the corresponding analytical operator:

Definition 2.5: *A submanifold S of any symplectic manifold N is called a Lagrangian submanifold, if dim(S)= dim(N)/2 and $w_{|S} = 0$.*

Remark 2.3: There are no submanifolds S with greater dimension than $\dim(N)/2$, such that the restriction of w to S vanishes. Any submanifold of arbitrary dimension where the symplectic form vanishes are called *isotropic* submanifolds. In this sense a Lagrangian submanifold is a maximal isotropic submanifold.

Proposition 2.10: *For any canonical transformation C on the contangent bundle $T^*M\backslash\{0\}$, one may consider the twisted graph of C,*

$$\Lambda = \{(x,\xi),(y,-\eta),(x,\xi) = C(y,\eta)\}. \tag{2.24}$$

Then Λ is a Lagrangian submanifold of $(T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$ with respect to the induced symplectic form $w_2 := w \oplus w$ on $(T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$.

Proof: First, for any symplectic form w on $(T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$, two vectors, on which w acts on are of the form

$$((\delta x,\delta\xi),(\delta y,\delta\eta),(\tilde{\delta x},\tilde{\delta\xi}),(\tilde{\delta y},\tilde{\delta\eta})). \tag{2.25}$$

If we choose $w = w_2$, this results in

$$w_2((\delta x, \delta\xi), (\delta y, \delta\eta), (\tilde{\delta x}, \tilde{\delta\xi}), (\tilde{\delta y}, \tilde{\delta\eta})) \tag{2.26}$$

$$= w((\delta x, \delta\xi), (\tilde{\delta x}, \tilde{\delta\xi})) + w((\delta y, \delta\eta), (\tilde{\delta y}, \tilde{\delta\eta})). \tag{2.27}$$

Remember that any canonical transformation C is a diffeomorphism, thus it has an invertible differential DC. We therefore have $(\delta x, \delta\xi) = ((DC)(\delta y, \delta\eta))$ and $(\tilde{\delta x}, \tilde{\delta\xi}) = ((DC)(\tilde{\delta y}, \tilde{\delta\eta}))$ as long as the points are taken out of the graph of C. Then we can write (2.26) as

$$w((\delta x, \delta\xi), (\tilde{\delta x}, \tilde{\delta\xi})) + w((\delta y, \delta\eta), (\tilde{\delta y}, \tilde{\delta\eta})) = w((DC)(\delta y, \delta\eta), (DC)(\tilde{\delta y}, \tilde{\delta\eta})) \tag{2.28}$$

$$+w((\delta y, \delta\eta), (\tilde{\delta y}, \tilde{\delta\eta})) \tag{2.29}$$

$$= w((\delta y, \delta\eta), (\tilde{\delta y}, \tilde{\delta\eta})) + w((\delta y, \delta\eta), (\tilde{\delta y}, \tilde{\delta\eta})), \tag{2.30}$$

where we used that C is a symplectomorphism by assumption. Note that (2.30) will be the result for the usual graph of C. Since Λ is a twisted graph, we get an additional minus sign in the first term of (2.30) when restricting w_2 to $T\Lambda$, which means that in fact w is vanishing on $T\Lambda$. Therefore Λ is isotropic. It has maximal dimension $\dim(\Lambda) = \dim(T^*M)$, because it is the graph of a transformation. *q.e.d.*

From *Proposition 2.10* we see that canonical transformations always define Lagrangian submanifolds. Not every Lagrangian submanifold S is given by a canonical transformation, but it is alwas possible to express S locally by a generating function ϕ, as long as S is conic. This function ϕ builds the bridge between symplectic geometry and the analysis of Fourier integral operators.

2.2 Fourier Integral Operators

2.2.1 Motivation and Definition

In the last chapter, we learned that any pseudodifferential operator A may be expressed as

$$(Au)(x) := \int e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi, \tag{2.31}$$

with the symbol a of A. In this definition one can directly see that any pseudodifferential operator is fully defined by the values of its symbol a. The function $\exp(i(x-y)\xi)$ always stays the same, for all pseudodifferential operators. However, it is possible to define operators, which are even more general than pseudodifferential operators by changing this factor: Instead of A defined as above, we want to work with operators like

$$(A_\phi u)(x) := \int e^{i\phi(x,y,\xi)} a(x, y, \xi) u(y) dy d\xi. \tag{2.32}$$

As a motivating example, let us consider the classical wave equation together with given initial values:

$$(i)\ (\partial_t^2 + \Delta)u = 0, \quad (ii)\ u(0) = g_0, \quad (iii)\ (\partial_t u)(0) = g_1. \tag{2.33}$$

If we seek for solutions in the Schwartz space $S(\mathbb{R}^n)$ we may use the Fourier transform to generate a solution in terms of the initial values g_0 and g_1. More precisely, by taking the Fourier transform of ((2.33), (i)) only with respect to the position variables, we get

$$\partial_t^2 \hat{u}(\xi, t) = -|\xi|^2 \hat{u}(\xi), \tag{2.34}$$

if Δ is the standard Laplacian $\Delta = -\sum_j \partial_{x_j}^2$. Interpreting ξ as a parameter, this is just an ordinary differential equation with the general solution

$$\hat{u}(\xi, t) = Ce^{i|\xi|t} + De^{-i|\xi|t}. \tag{2.35}$$

We can also transform the initial conditions, which results in $\hat{u}(0) = \hat{g}_0$ and $\partial_t \hat{u}(0) = \hat{g}_1$. These transformed initial conditions give rise to conditions on the coefficients C and D, namely

$$(1)\ C + D = \hat{g}_0, \quad (2)\ (C - D) = \frac{-i\hat{g}_1}{|\xi|}. \tag{2.36}$$

After solving this system for C and D, the unique solution to the Cauchy problem (2.32) is given by $\mathcal{F}^{-1}(\hat{u}(\xi, t))$ with

$$\hat{u}(\xi, t) = \frac{\hat{g}_0|\xi| - i\hat{g}_1}{2|\xi|} e^{i|\xi|t} + \frac{\hat{g}_0 + i\hat{g}_1}{2|\xi|} e^{-i|\xi|t}. \tag{2.37}$$

If we plug (2.37) into the inverse Fourier transform and rearrange the resulting terms, we can write the original solution as

$$u(x,t) = \frac{1}{8\pi^2} \left(\int e^{i(\xi(x-y)+|\xi|t)} g_0(y) dy d\xi + \int e^{i(\xi(x-y)-|\xi|t)} g_0(y) dy d\xi \right) \tag{2.38}$$

$$\left(-i \int e^{i(\xi(x-y)+|\xi|t)} \frac{1}{|\xi|} g_1(y) dy d\xi + i \int e^{i(\xi(x-y)-|\xi|t)} \frac{1}{|\xi|} g_1(y) dy d\xi \right). \tag{2.39}$$

The representation $(2.38) - (2.39)$ is expressed via four integrals of the desired form (2.32) with $\phi(x, y, \xi) = \xi(x - y) \pm |\xi|t$ and $a_0 = 1$ and $a_{-1} = 1/|\xi|$. This motivates the following definition for a Fourier integral operator

Definition 2.5: *For any function $\phi(x, y, \xi) = \phi_0(x, \xi) - \xi y$, where ϕ_0 is a one homogenous function in the fibre variable ξ, and some $a \in S^m$ we define a corresponding Fourier integral operator A_ϕ via*

$$(A_\phi u)(x) := \int e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) dy d\xi. \tag{2.40}$$

We call ϕ the phase and a the amplitude of the Fourier integral operator A_ϕ.

Remark 2.4: In the foregoing example of the wave equation there were operators $\phi(x, y, t, \xi)$ involved which also depend on time. As there is no integration in t in *Definition 2.5* we may regard such operators as parameter dependent operators of the form (2.40). We will see in the chapter about hyperbolic systems that any scalar hyperbolic equation is solved by a time dependent Fourier integral operator.

2.2.2 Compositions and Examples

While the operators of pseudoddifferential type form an algebra, the case of Fourier integral operators is much more difficult. For two Fourier integral operators A_{ϕ_1} and B_{ϕ_2} with possibly different phase functions and amplitudes, it is neither clear whether the sum nor the product is again of Fourier integral type. The approach of finding a way how to check, whether two Fourier integral operators may be composed with each other, is purely geometrical: One can associate a Lagrangian submanifold of $(T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$ to each Fourier integral operator and then it remains to check, whether these Lagrangian submanifolds behave well with respect to each other. The theorem which clarifies this correspondence is given by

Theorem 2.1: (see [3], Theorem 21.2.16) *Let $S \subset (T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$ be a conic Lagrangian submanifold. Then, locally near a point $((x_0, \xi_0), (y_0, \eta_0)) \in (T^*M\backslash\{0\}) \times (T^*M\backslash\{0\})$ the submanifold S can be represented via a phase function $\phi(x, y, \xi)$ which is homogenous of degree 1 in ξ and non-degenerate such that*

$$S = \{((x, \partial_x\phi), (y, -\partial_y\phi)) : \partial_\xi\phi = 0\}. \tag{2.41}$$

Theorem 2.1 is extremely important, as it shows a direct (local) correspondence between Lagrangian submanifolds and Fourier integral operators: Any Fourier integral operator A_ϕ with a phase function ϕ can be associated with the submanifold S in (2.41) defined by ϕ and vice versa, any Lagrangian submanifold S generates (locally) a Fourier integral operator. We want to give an example in order to see how the class of pseudodifferential operators corresponds to a subclass of Lagrangian submanifolds:

Example 2.2: Let A_ϕ be a Fourier integral operator with $\phi(x, y, \xi) = (x - y)\xi$. Then A_ϕ is a pseudodifferential operator and one may write $A_\phi := A$. If we want to build the Lagrangian submanifold in (2.41) and denote it by $S := \Lambda_\phi$, first of all there is $\partial_\xi\phi = 0 \;\Leftrightarrow\; x = y$. Then Λ_ϕ is the submanifold

$$\Lambda_\phi = \{(x,\xi),(y,\xi) : x = y\} = \{(x,\xi),(x,\xi)\} \in (T^*M\backslash\{0\}) \times (T^*M\backslash\{0\}). \tag{2.42}$$

Therefore all pseudodifferential operators are Fourier integral operators, which are associated to the graph of the identity map. Of course, the identity is a special example of a canonical transformation. We want to give another example of the representation of a canonical transformation as a Fourier integral operator.

Example 2.3: Assume that $g : M \to M$ is any given diffeomorphism on the base manifold M. A natural way to represent this diffeomorphism on the function space $L^2(M)$ is

$$(A_{\phi_g}u)(x) := u(g^{-1}(x)). \tag{2.43}$$

We want to show that A_{ϕ_g} can be written as a Fourier integral operator and we want to determine the associated Lagrangian submanifold Λ_{ϕ_g}. Using $u(g^{-1}(x)) = \mathcal{F}^{-1}(\mathcal{F}(u(g^{-1}(x))))$ there is

$$u(g^{-1}(x)) = \left(\frac{1}{2\pi}\right)^n \int e^{i(g^{-1}(x)-y)\xi} u(y)dyd\xi. \tag{2.44}$$

This is a Fourier integral operator with phase $\phi_g(x,y,\xi) = (g^{-1}(x) - y)\xi$ and amplitude $a = 1$. By (2.41) Λ_{ϕ_g} becomes

$$\Lambda_{\phi_g} = \{(x, \partial_x g^{-1}(x)\xi),(y,\xi) : y = g^{-1}(x)\} = \{(g(y),(Dg)_y^{-T}(\xi),(y,\xi)\}. \tag{2.45}$$

The last equality shows that the map $u(x) \mapsto u(g^{-1}(x))$ is associated to the lift of g, which we have already seen in *Proposition 2.5*.

We want to give a theorem, which tells us under which conditions on the corresponding Lagrangian submanifolds Λ_{ϕ_1} and Λ_{ϕ_2} two operators A_{ϕ_1} and B_{ϕ_2} can be composed to another Fourier integral operator C_{ϕ_3}. We will only state the theorem without proving it. For more details, see ([4], *Theorem 2.4.1*).

Theorem 2.2: *Let A_{ϕ_1} and B_{ϕ_2} be two Fourier integral operators on a compact manifold M. If Λ_{ϕ_1} and Λ_{ϕ_2} denote the corresponding Lagrangian submanifolds, consider the conditions*

$$(C_1)\ \eta \neq 0,\ \text{if}\ (x,\xi,y,\eta) \in \Lambda_{\phi_1}\ \text{or}\ (y,\eta,z,\xi) \in \Lambda_{\phi_2} \tag{2.46}$$

$$(C_2)\ \xi \neq 0\ \text{or}\ \zeta \neq 0\ \text{if}\ (x,\xi,y,\eta) \in \Lambda_{\phi_1}\ \text{and}\ (y,\eta,z,\zeta) \in \Lambda_{\phi_2} \tag{2.47}$$

$$(C_3)\ \Lambda_{\phi_1} \times \Lambda_{\phi_2}\ \text{intersects}\ T^*(M)\backslash\{0\} \times (diag(T^*(M)\backslash\{0\})) \times T^*(M)\backslash\{0\}\ \text{transversally}. \tag{2.48}$$

As long as all the conditions $(C_1)-(C_3)$ hold, the composition $A_{\phi_1} \circ A_{\phi_2}$ is well defined and up to a smoothing error equal to a Fourier integral operator C_{ϕ_3}. Moreover, if $\Lambda_{\phi_1} = \Lambda_{C_1}$ is the graph of a canonical transformation C_1 and $\Lambda_{\phi_2} = \Lambda_{C_2}$ belongs to a canonical trans-

formation C_2, the composition $A_{\phi_1} \circ B_{\phi_2}$ is associated to $\Lambda_{C_1 \circ C_2}$.

Remark 2.5: *Theorem 2.2* tells us that Fourier integral operators associated to canonical transformations may be seen as (up to smoothing errors) representations of the group of canonical transformations. This fact will be used later to more extent.

Kumano-go has established many theorems about the composition between pseudodifferential and Fourier operators in his book about pseudodifferential operators and he has given formulas for the asymptotic expansion of the amplitude and phase of the composed operators (see [1], *Chapter 10*). In this work, we will deal a lot with G-operators, which may be seen as abstract compositions of operators of pseudodifferential type and Fourier integral operators. Therefore we do not really need the composition formulas of *Kumano-go*. However, there is one special case of such a composition, which is important if one wants to see and use that the G-operators actually form an algebra. This is the also quite well known theorem of Egorov:

Theorem 2.3: (*Egorov*) *Assume that P is a given pseudodifferential operator of order m and that A_ϕ is an invertible Fourier integral operator. Then the conjugation $A_\phi P A_\Phi^{-1}$ is (up to a lower order error) again an order m pseudodifferential operator. If the Lagrangian submanifold Λ_ϕ associated to A_ϕ is the graph of a canonical transformation C, the symbol of the conjugated operator $A_\phi P A_\Phi^{-1} := P^{C^{-1}}$ is*

$$\sigma(P^{C^{-1}})(x, \xi) = \sigma(P)(C^{-1}(x, \xi)). \tag{2.49}$$

Remark 2.6: The theorem of Egorov is still true for Shubin type operators on $\mathbb{R}^n$, as long as A_ϕ is a Fourier integral operator defined by a phase and amplitude of Shubin type.

2.3 G-Operators

2.3.1 The Algebra of G-Operators

The theorem of Egorov creates the idea of considering operators of the form PA_ϕ with a pseudo P and an invertible Fourier integral operator A_ϕ: If we try to compose an operator PA_ϕ with another operator QB_ψ, we can write the product as

$$PA_\phi QB_\psi = PA_\phi QA_\phi^{-1} A_\phi B_\psi = PQ^{C^{-1}} A_\phi B_\psi \tag{2.50}$$

using *Theorem 2.3* for the definition of $Q^{C^{-1}}$ ((2.50) holds modulo operators of lower order, which can be neglected concerning Fredholm theory).Now, if one furthermore assumes that A_ϕ and B_ψ are the representations Φ of some group elements of a group G,

$A_\phi = \Phi_{g_1}$, $B_\psi = \Phi_{g_2}$, then we can compose them and get $PA_\phi QB_\psi = PQ^{C^{-1}}\Phi_{g_1\circ g_2}$. This motivates us to consider classes of operators, which are abstract compositions of pseudos and Fourier integral operators:

Definition 2.6: *Let $\mathcal{A}$ be the algebra of pseudodifferential operators. If G is some group with a representation $\Phi : G \to \mathcal{L}(H^s(M))$, such that the algebra $\mathcal{A}$ is invariant under conjugation by the representation, $\mathcal{A} = \Phi_g^{-1}\mathcal{A}\Phi_g$, for all $g \in G$, we can define operators of the form $D_g = \sum_g A_g\Phi_g$. Operators of this type form an algebra, which we call the algebra of G-type operators.*

Corollary 2.2: *Denote by $A \times_\Phi G$ the set of G-operators on a manifold M, where Φ_g is a representation as a Fourier integral operator for every g and g is associated to a canonical transformation C_g. Then the following properties hold:*

$$(i)\ \Phi_g A_g = A_g^{C_g^{-1}}\Phi_g. \tag{2.51}$$

$$(ii)\ (A_{g_1}B_{g_2})^{C_{g_3}^{-1}} = A_{g_1}^{C_{g_3}^{-1}} B_{g_2}^{C_{g_3}^{-1}}. \tag{2.52}$$

$$(iii)\ A^{C_g} = \Phi_g^{-1}A_g\Phi_g. \tag{2.53}$$

$$(iv)\ \text{If the representation } \Phi \text{ is unitary, } (A_g\Phi_g)^\dagger = A_g^C\Phi_{g^{-1}}. \tag{2.54}$$

$$(v)\ (\Phi_g^{-1}A_g\Phi_g)^\dagger = \Phi_g^{-1}A_g^\dagger\Phi_g \text{ for any unitary } \Phi. \tag{2.55}$$

proof: (i) : $\Phi_g A_g = \Phi_g A_g \Phi_g^{-1}\Phi_g = A_g^{C^{-1}}\Phi_g$.
(ii) : $(A_{g_1}B_{g_2})^{C_{g_3}^{-1}} = \Phi_{g_3}(A_{g_1}B_{g_2})\Phi_{g_3}^{-1} = (\Phi_{g_3}A_{g_1}\Phi_{g_3}^{-1})(\Phi_{g_3}B_{g_2}\Phi_{g_3}^{-1}) = A_{g_1}^{C_{g_3}^{-1}} B_{g_2}^{C_{g_3}^{-1}}$.
(iii) : Let us start with $A_g^{C_g^{-1}} = \Phi_g A_g \Phi_g^{-1}$. If we multiply both sides with Φ_g from the right and Φ_g^{-1} from the left, this is equivalent to $A_g = \Phi_g^{-1}A_g^{C_g^{-1}}\Phi_g$. Because Φ is a representation, this is just the conjugation with the representation of the inverse element g^{-1}. As the symbol of this conjugation of $A_g^{C_g^{-1}}$ has to be equal to that of A_g itself, conjugation with Φ_g^{-1} must evaluate symbols on the flow C_g.
(iv) : Assume that Φ is unitary. Then we have

$$(A_g\Phi_g)^\dagger = \Phi_g^\dagger A_g^\dagger = \Phi_g^{-1}A_g^\dagger = \Phi_g^{-1}A_g^\dagger\Phi_g\Phi_g^{-1} = A_g^{\dagger C_g}\Phi_{g^{-1}}, \tag{2.56}$$

where we used the result from c) for the inverse conjugation.
(v) : $(\Phi_g^{-1}A_g\Phi_g)^\dagger = \Phi_g^\dagger A_g^\dagger\Phi_g^{-\dagger} = \Phi_g^{-1}A_g^\dagger\Phi_g$. *q.e.d.*

2.3.2 Trajectory Symbol and its Invertibility

Of course the set of all G-operators forms an extension of the set of pseudodifferential operators. The Fredholm property of a pseudodifferential operator, which is one of the main objects of this thesis and will be discussed in the next chapter, reflects in the invertibility of its symbol. This lets us guess that there must also be some useful definition of a symbol of a G-operator, such that the Fredholm property is equivalent to invertibility of that symbol. We want to define that so called "trajectory" symbol now and recall some facts about it, and we will state the equivalence to the Fredholm property in the next chapter.

Definition 2.7: *Consider a G-operator of the form $D = \sum_g A_g \Phi_g$. Furthermore, let us assume that the pseudodifferential coefficients A_g are zero order operators acting on $L^2(M)$ whith some manifold M and that the representation Φ is unitary modulo compact errors, i.e. $\Phi_g^\dagger = \Phi_g^{-1} + K_g$ with compact K_g for all group elements $g \in G$. The trajectory symbol of D is the operator valued symbol*

$$\sigma(D) = \sum_g \sigma(A_g)\tau_g : l^2(G) \to l^2(G) \tag{2.57}$$

defined by the actions

$$(\sigma(A_g)(x,\xi)f)(h) = \sigma(A_g)(h(x,\xi))f(h), \quad (\tau_g f)(h) = f(g^{-1}h), \tag{2.58}$$

for any function $f \in l^2(G)$.

Remark 2.7: Similar to the notion of a principal symbol of pseudodifferential operators, one may define a principal symbol of a G-operator by simply replacing $\sigma(A_g)$ in (2.57) by $\sigma_p(A_g)$.

Lemma 2.3: *The trajectory symbol defined in Definition 2.7 fulfills:*

(a)*: The trajectrory symbol is linear,*
(b) : $\sigma(\Phi_g^{-1}) = \tau_g^{-1}$,
(c) : *The principal symbol is multiplicative:* $\sigma_p(D_1D_2) = \sigma_p(D_1)\sigma_p(D_2)$
(d) : $\sigma_p(A_g^{C_g^{-1}}) = \tau_g \sigma_p(A_g)\tau_g^{-1}$.

proof: (a): Follows directly from the definition.
(b): Since Φ is a representation, there is $\sigma(\Phi_g^{-1}) = \sigma(\Phi_{g^{-1}}) = \tau_{g^{-1}}$. As the action of $\tau_{g^{-1}}$ is given by $(\tau_{g^{-1}}f)(h) = f(gh)$, it is clearly the inverse operator to τ_g.
(c): We only need to show the statement for $D_1 = A_g\Phi_g$ and $D_2 = B_{\tilde{g}}\Phi_{\tilde{g}}$, the corresponding statement for sums follows from the linearity (a). On the one hand, we have

$$\sigma_p(D_1D_2) = \sigma_p(A_g\Phi_g B_{\tilde{g}}\Phi_{\tilde{g}}) = \sigma_p(A_g\Phi_g B_{\tilde{g}}\Phi_g^{-1}\Phi_g\Phi_{\tilde{g}}) = \sigma_p(A_g B_{\tilde{g}}^{C_g^{-1}}\Phi_{g\tilde{g}}) \tag{2.59}$$

$$= \sigma_p(A_g B_{\tilde{g}}^{C_g^{-1}})\tau_{g\tilde{g}} = \sigma_p(A_g)\sigma_p(B_{\tilde{g}}^{C_g^{-1}})\tau_{g\tilde{g}}. \tag{2.60}$$

Note that we have used in (2.60), that the principal symbol of pseudodifferential operators itself is multiplicative. Let us calculate the action of (2.60):

$$(\sigma_p(D_1D_2)f)(h) = (\sigma_p(A_g)\sigma_p(B_{\tilde{g}}^{C_g^{-1}})\tau_{g\tilde{g}}f)(h) = \sigma_p(A_g)\sigma_p(B_{\tilde{g}}^{C_g^{-1}})f((g\tilde{g})^{-1}h) \tag{2.61}$$

$$= \sigma_p(A_g)\sigma_p(B_{\tilde{g}}^{C_g^{-1}})(h(x,\xi))f((g\tilde{g})^{-1}h) \tag{2.62}$$

$$= \sigma_p(A_g)(h(x,\xi))\sigma_p(B_{\tilde{g}})(g^{-1}h(x,\xi))f((g\tilde{g})^{-1}h). \tag{2.63}$$

On the other hand,

$$\sigma_p(D_1)\sigma_p(D_2) = \sigma_p(A_g\Phi_g)\sigma_p(B_{\tilde{g}}\Phi_{\tilde{g}}) = \sigma_p(A_g)\tau_g\sigma_p(B_{\tilde{g}})\tau_{\tilde{g}}. \tag{2.64}$$

If we let this act on some function $f \in l^2(G)$ the result is

$$(\sigma_p(A_g)\tau_g\sigma_p(B_{\tilde{g}})\tau_{\tilde{g}}f)(h) = (\sigma_p(A_g)\tau_g\sigma_p(B_{\tilde{g}}))f(\tilde{g}^{-1}h) \tag{2.65}$$

$$= \sigma_p(A_g)\tau_g(\sigma_p(B_{\tilde{g}})(h(x,\xi))f(\tilde{g}^{-1}h)) \tag{2.66}$$

$$= \sigma_p(A_g)(h(x,\xi))\sigma_p(B_{\tilde{g}})(g^{-1}h(x,\xi))f(\tilde{g}^{-1}g^{-1}h) \tag{2.67}$$

$$= \sigma_p(A_g)(h(x,\xi))\sigma_p(B_{\tilde{g}})(g^{-1}h(x,\xi))f((g\tilde{g})^{-1}h). \tag{2.68}$$

We see that both actions (2.63) and (2.68) coincide, meaning that the operators are the same.
(d) :

$$\sigma_p(A_g^{C_g^{-1}}) = \sigma_p(\Phi_g A_g \Phi_g^{-1}) = \sigma_p(\Phi_g)\sigma_p(A_g)\sigma_p(\Phi_g^{-1}) = \tau_g\sigma_p(A_g)\tau_g^{-1}, \tag{2.69}$$

where we made use of both (b) and (c). *q.e.d.*

Remark 2.8: Note that the multiplicativity of the principal symbol σ_p also extends to matrices with G-operators as entries, just as in the case of pseudodifferential operators.

For pseudodifferential operators the invertibility of the principal symbol is just equivalent to $\sigma_p(A)$ being non zero for the scalar case, or $\det(\sigma_p(A)) \neq 0$ in the case of a system of operators. In the case of G-operators invertibility is harder to check even in the scalar case, as we have to invert an operator-valued symbol, taking values on $l^2(G)$. The invertibility conditions as described in the work of *Antonevich* and *Lebedev*, see ([5], *Theorem 4.6.5*), make use of ergodic measures. We want to explain this concept and then state the conditions for the invertibility..

Definition 2.8: *Let (X, μ_g) be a space with a probability measure μ_g and $g : X \to X$ a diffeomorphism on X. If $\mu_g(g^{-1}(\Omega)\Delta\Omega) = 0$ (where $A\Delta B$ denotes the symmetric difference of A and B) implies $\mu_g(\Omega) = 0$ or $\mu_g(\Omega) = 1$ for every measurable subset $\Omega \subset X$, μ_g is called an ergodic measure with respect to the diffeomorphism g.*

The invertibility conditions, which will be generated by the problems in the later chapters can in some cases be reduced to the invertibility of a single operator of the type

$\sigma_p(D) = a + b\tau_g$. For such operators, the invertibility conditions are

Theorem 2.4: *Let $D = A\Phi_{g_1} + B\Phi_{g_2}$ be a two-term G-operator, where A, B are zero order operators and Φ is an (modulo compact) unitary representation of a group G. The principal symbol $\sigma_p(D) = a + b\tau_g$ is invertible iff one of the following conditions is true:*

$$(*)\ a(x,\xi) \neq 0\ \forall (x,\xi) \in S^*(M) \text{ and } \int_{S^*(M)} \ln(|a|)d\mu_g > \int_{S^*(M)} \ln(|b|)d\mu_g \tag{2.70}$$

or

$$(**)\ b(x,\xi) \neq 0\ \forall (x,\xi) \in S^*(M) \text{ and } \int_{S^*(M)} \ln(|b|)d\mu_g > \int_{S^*(M)} \ln(|a|)d\mu_g \tag{2.71}$$

for all ergodic measures μ_g with respect to g.

Remark 2.9: The integrals in (2.70) and (2.71) are called the *geometric means* of a and b. From *Theorem 2.4* we can directly deduce that any G-operator symbol $\sigma_p(D) = a \pm a\tau_g$ can never be invertible, because the geometric means of the two symbols would be the same, no matter which ergodic measure is used for the calculation. However, such operators often have a left inverse, as we will see later.

3 Fredholm Operators

The main goal of this work lies in the derivation of the conditions, under which a certain hyperbolic system induces Fredholm operators. This chapter shall serve the purpose to state the general facts and theorems about abstract Fredholm theory and Fredholm theory involving pseudodifferential or $G-$type operators. In particular the Fredholm theory of matrices of operators will be considered. A good summary concerning the abstract Fredholm theory can be found in ([3], *Chapter 19.1*).

3.1 The General Fredholm Theory

3.1.1 Definitions and General Properties

Definition 3.1: *Let $A : X \to Y$ be a bounded operator between two Banach spaces X and Y. A is called a Fredholm operator, as long as the numbers $a_1 := dim(ker(A))$ and $a_2 := dim(Y/Im(A)) := dim(coker(A))$ are both finite. In this case we set*

$$\text{ind}(A) := a_1 - a_2 \tag{3.1}$$

and call this number the index of A.

Let us first state a few well known facts about Fredholm operators. On finite dimensional spaces, surely every linear operator A is a Fredholm operator. The interesting operators arise when one considers infinite dimensional spaces X and Y. In this case, *Definition 3.1* means, that the operator A is injective and surjective up to finite dimensional errors. As full injectivity and surjectivity would reflect in the existence of a left- and right inverse, Fredholm operators can be seen as operators wich have left and right inverses up to finite dimensional errors:

Theorem 3.1: *An operator $A : X \to Y$ is Fredholm, iff there exist operators $F_{A,L}$ and $F_{A,R} : Y \to X$ such that*

$$F_{A,L}A = 1_X + K_{A,L}, \quad AF_{A,R} = 1_Y + K_{A,R} \tag{3.2}$$

with finite dimensional operators $K_{A,L}$ and $K_{A,R}$. We call $F_{A,L/R}$ a left/right parametrix to A.

Another important fact about Fredholm operators is the closedness of their range:

Theorem 3.2: *Given any Fredholm operator $A : X \to Y$ between to Banach spaces X and Y the range Im(A) is closed.*

A natural and intuitive choice for the error terms $K_{A,L}$ and $K_{A,R}$ would be the projections onto the kernel of A ($K_{A,L}$) and onto some complement of the image of A ($K_{A,R}$).

The finite dimensionality condition for a_1 and a_2 in *Definition 3.1* guarantees that these projections are finite dimensional. Note that on Hilbert spaces X, Y the orthogonal complement of the image of A is isomorphic to the quotient $Y/\mathrm{Im}(A)$ and that the projections can be always chosen as orthogonal projections. While *Theorem 3.1* is an intuitive fact, there is also a non intuitive but very remarkable statement about the error terms in (3.2):

Lemma 3.1 (Atkinson-Nikolskii): *Consider any operator $A : X \to Y$. Then it suffices to find $F_{A,L}$ and $F_{A,R}$ such that the error terms $K_{A,L}$ and $K_{A,R}$ from Theorem 3.1 are compact operators, and A is already a Fredholm operator.*

We want to remind that compact operators in general need not to be operators with finite dimensional range. *Lemma 3.1* tells us that if one wants to test whether an operator A has a left and right inverse up to finite dimensional errors (3.2), i.e. if A is of Fredholm type, it actually suffices to check whether A is invertible up to compact errors. Before stating some other useful theorems, we want to make a few remarks, which are easy to see from the facts already mentioned:

Remarks:

• If $A : X \to Y$ is an (bounded) operator between two Hilbert spaces, the cokernel coker(A) is isomorphic to the kernel of the adjoint $A^\dagger$. Given an inner product $\langle \cdot, \cdot \rangle_Y$ on the target space Y the cokernel is isomorphic to the orthogonal complement of $\mathrm{Im}(A)$. This means, that given any element $g \in \mathrm{coker}(A)$, we have

$$\langle Af, g \rangle_Y = 0, \ \forall f \in X. \tag{3.3}$$

But this means

$$\langle f, A^\dagger g \rangle_X = 0, \ \forall f \in X \tag{3.4}$$

and thus $A^\dagger g = 0$, meaning that g lies in the kernel of $A^\dagger$.

• We can see from the first remark that for any Fredholm operator A acting between Hilbert spaces, its adjoint $A^\dagger$ is also a Fredholm operator: There is $\dim(\ker(A^\dagger)) = \dim(\mathrm{coker}(A))$ and $\dim(\mathrm{coker}(A^\dagger)) = \dim(\ker((A^\dagger)^\dagger)) = \dim(\ker(A))$ and therefore $\mathrm{ind}(A^\dagger)$ = - $\mathrm{ind}(A)$.

• It is also possible to define the notion of a Fredholm operator for unbounded (densely defined) operators: If $A : \mathcal{D}(X) \subset X \to Y$ is an unbounded operator between Banach spaces X and Y, in addition to $\dim(\mathrm{Ker}(A)) < \infty$ and $\dim(\mathrm{coker}(A)) < \infty$ we requiere that A is a closed operator. For example, the Laplacian $\Delta : H^2(M) \subset L^2(M) \to L^2(M)$, $\Delta :=$ div $\circ$ grad is an unbounded Fredholm operator, as long as the manifold M is compact. Note that on compact manifolds the kernel of Δ consists only of the constant functions on M and is therefore one dimensional. Because it is essentially self adjoint, it has index zero.

• We can use *Lemma 3.1* and the fact, that pseudodifferential operators of negative order are compact on a compact manifold M, to construct a parametrix for any elliptic pseudodifferential operator $A : H^s(M) \to H^{s-m}(M)$ of order m. We remember that the elliptic operators are operators with invertible principal symbol (outside $\xi = 0$). Then we define F to be the pseudodifferential operator of order $-m$, with symbol $\sigma(F) = \sigma_p(A)^{-1}$. Then by construction (and using the multiplicativity of the principal symbol) the principal symbol of FA and AF is 1, thus $1 - FA$ and $1 - AF$ are both operators of negative order, and therefore compact. The resulting conclusion is that any elliptic pseudodifferential operator on a compact manifold is a Fredholm operator. (Note that F is both a left and right parametrix to A. In the case of pseudodifferential operators $A : H^s(M) \to H^{s-m}(M)$ we therefore do not need to distinguish between left and right parametrices and we may sometimes simply write A^p for a parametrix to A.)

How do Fredholm operators behave under perturbation by other operators? It is a fact from functional analysis that the set of invertible operators is an open set, which means, that for any invertible $A : X \to Y$ one can always find a normsmall (with respect to the operator norm) nonvanishing K such that $A + K$ is still invertible. A similar result holds for the set of Fredholm operators:

Theorem 3.3: *Let $A : X \to Y$ be a Fredholm operator. Then, if $K : X \to Y$ is compact or if it has an operator norm $||K||$ which is sufficiently small, $A + K$ is still Fredholm (which means that the set of Fredholm operators is open) with the same index. Moreover, the index is constant on connected components of the set of all Fredholm operators.*

Theorem 3.3 is extremely important, because it tells us that compact or norm small errors may always be neglected if one is interested in the Fredholm theory of an operator. This means that concerning Fredholm theory, we can work with equivalence classes $[A]_\sim$ of an operator A where all equivalent operators to A are those, that differ from A by a compact error. Also, operators which have a sufficiently small distance from A may be treated as equivalent operators. Before going into the details of index theory, we want to present a few further simple facts, which are proven easily with the given definitions:

Proposition 3.1: *Let $A : X \to Y$ be a Fredholm operator. Then the following facts hold:*

a) If $F_{A,L/R}$ is a left/right parametrix to A, so is $F_{A,L/R} + K$, for any compact operator K.
b) If $F_{A,L/R}$ is a left/right parametrix to A, it is also a left/right parametrix to the operator $A + K$ for any compact K.
c) The difference of two left/right parametrices $F_{A,L_1/R_1}$ and $F_{A,L_2/R_2}$ is always compact.

Proof: a): For any left parametrix $F_{A,L}$ of A, $F_{A,L}A = 1 + K_{A,L}$ with compact $K_{A,L}$ and

$$(F_{A,L} + K)A = F_{A,L}A + KA = 1 + K_{A,L} + KA. \tag{3.5}$$

for any compact operator K. Now, the space $\mathcal{K}(X, X)$ of compact operators is an (two

sided) ideal of the space $\mathcal{L}(X,X)$ of bounded linear operators, and a subspace. This means that if we set $\tilde{K} := K_{A,L} + KA$ this is again a compact operator, which means that $F_{A,L} + K$ is a left parametrix. The proof for a right parametrix is analogous.
b): Similar to a) we calculate

$$F_{A,L}(A+K) = F_{A,L}A + F_{A,L}K = 1 + K_{A,L} + F_{A,L}K = 1 + \tilde{K}, \tilde{K} = K_{A,L} + F_{A,L}K. \quad (3.6)$$

c): First, we see that the difference $F_{A,L_1} - F_{A,L_2}$ applied to A is always compact:

$$(F_{A,L_1} - F_{A,L_2})A = F_{A,L_1}A - F_{A,L_2}A = (1 + K_{A,L_1}) - (1 + K_{A,L_2}) = K_{A,L_1} - K_{A,L_2}. \quad (3.7)$$

$$:= \tilde{K}. \quad (3.8)$$

Applying a multiplication by any right parametrix from the right, this becomes

$$(F_{A,L_1} - F_{A,L_2})AF_{A,R} = \tilde{K}F_{A,R} \;\Leftrightarrow\; (F_{A,L_1} - F_{A,L_2})(1 + K_{A,R}) = \tilde{K}F_{A,R} \quad (3.9)$$

$$\Leftrightarrow (F_{A,L_1} - F_{A,L_1}) = \tilde{K}F_{A,R} - (F_{A,L_1} - F_{A,L_1})K_{A,R} := \hat{K} \quad (3.10)$$

with a compact operator $\hat{K}$. *q.e.d.*

It turns out that it is often useful to work with products of Fredholm operators. As, for example, a lot of properties of matrices can be understood better after one diagonalizes them, i.e. considering $T^{-1}AT$ instead of A, it is also true that a lot of properties of Fredholm operators can be seen by using products of operators. In fact, products of Fredholm operators are Fredholm again, and for the index of the product we have

Proposition 3.2: *As long as two operators $A : X \to Y, B : Y \to Z$ are Fredholm, their product BA is also Fredholm. Moreover,*

$$\mathrm{ind}(BA) = \mathrm{ind}(A) + \mathrm{ind}(B). \quad (3.11)$$

There are several things, which directly follow from *Proposition 3.2.* We want to give an overview of this things in a corollary:

Corollary 3.1: *Assume that $A : X \to Y$ is any Fredholm operator. Then the following statements are true:*

a) If F_A is a left and right parametrix to A its index is $ind(F_A) = -ind(A)$.
b) Let B be an invertible operator. Then $B^{-1}A$ and AB^{-1} have both the same index as A.
c) Assuming that $A : H^s(M) \to H^{s-m}(M)$ is an elliptic pseudodifferential operator of order m, then its index is independent of s. Moreover, the index only depends on the principal part of A.

Proof: a): On the one hand, ind($F_A A$) = ind ($1 + K_A$) = ind (1)= 0, because 1 is an invertible operator and has therefore trivial kernel and cokernel. On the other hand, using *Proposition 3.2*, ind($F_A A$)= ind(F_A) + ind(A), which means ind(F_A) = - ind(A). Note that F_A is itself a Fredholm operator, because A is its left and right parametrix.
b): Any invertible operator is injective and surjective, thus such a B has trivial kernel ond cokernel, and therefore ind(B) = ind(B^{-1})=0. b) then follows from the product formula (3.11).
c) Instead of A, let us consider the operator $A_0 := \Lambda^{s-m} A \Lambda^{-s} : L^2(M) \to L^2(M)$ with $\Lambda = \sqrt{\Delta + 1}$. Because Λ is an invertible operator, we get $\mathrm{ind}(A_0) = \mathrm{ind}(A)$ with b). Let us first show that the index is independent of lower order terms. For that consider some perturbation $A + \delta A$, $\delta A \in \mathrm{OPS}^{m-1}$ of the operator A. We can compute

$$\mathrm{ind}(A + \delta A) = \mathrm{ind}(\Lambda^{s-m}(A + \delta A)\Lambda^{-s}) = \mathrm{ind}(\Lambda^{s-m} A \Lambda^{-s} + \Lambda^{s-m}(\delta A)\Lambda^{-s}). \tag{3.12}$$

By the composition rule for pseudodifferential operators we have that $\Lambda^{s-m}(\delta A)\Lambda^{-s}$ $\in \mathrm{OPS}^{s-m+m-1-s} = \mathrm{OPS}^{-1}$. As long as the considered manifold is compact, this is a compact operator, which can be neglected. Therefore

$$\mathrm{ind}(A + \delta A) = \mathrm{ind}(\Lambda^{s-m} A \Lambda^{-s}) = \mathrm{ind}(A), \tag{3.13}$$

meaning that the index only depends on the principal part of the operator. Now, from *Corollary 1.1* we know $\Lambda^{s-m} A \Lambda^{-s} = \Lambda^{s-m}(\Lambda^{-s} A + \tilde{A})$ with $\tilde{A} \in \mathrm{OPS}^{m-s-1}$. Therefore

$$\Lambda^{s-m} A \Lambda^{-s} = \Lambda^{s-m}\Lambda^{-s} A + \Lambda^{s-m}\tilde{A} = \Lambda^{-m} A + B, \tag{3.14}$$

where $B = \Lambda^{s-m}\tilde{A}$ is of order -1. This means that the principal part of A_0 is independent of the parameter s. As already shown one can deduce that both the Fredholm property and the index must also be independent of s, and the same holds for the operator A. *q.e.d.*

3.1.2 Index Formulas

Now we have seen some important facts about general Fredholm operators and how they and their parametrices behave under perturbation. The next logical question would be, if there is any practical way to calculate the index of a Fredholm operator. One can get an intuitive idea for an index formula, if one considers (3.2) again and assumes that the parametrices $F_{A,L}$ and $F_{A,R}$ are chosen good enough, such that $K_{A,L}$ and $K_{A,R}$ are indeed the projections onto the kernel and cokernel of the operator. Because projections have eigenvalues $\lambda_j \in \{1, 0\}$ it follows that dim(ker(A)) = Tr($K_{A,L}$) and dim(coker(A)) = Tr($K_{A,R}$) or equivalently

$$\operatorname{ind}(A) = \operatorname{Tr}(F_{A,L}A - 1) - \operatorname{Tr}(AF_{A,R} - 1). \tag{3.15}$$

If we remember that any traceclass operator is automatically compact, we can already guess that (3.15) might be a formula for the index of A as long as the involved operators are trace class. It turns out that there is an even more general theorem involving powers of the operators in (3.15):

Theorem 3.4: *Let $A : X \to Y$ be a Fredholm operator. If there exists some $N > 0$ with the property that $(F_{A,L}A - 1)^N$ and $(AF_{A,R} - 1)^N$ are both trace class, the index of A is given by*

$$\operatorname{ind}(A) = \operatorname{Tr}((F_{A,L}A - 1)^N) - \operatorname{Tr}((AF_{A,R} - 1)^N). \tag{3.16}$$

Remark 3.2: As already discussed, if we consider an elliptic operator $A : H^s(M) \to H^{s-m}(M)$ of order m, we can choose any F_A with $\sigma_p(F_A) = \sigma_p(A)^{-1}$ and it will be a left and right parametrix for A as long as M is a compact manifold. The differences $F_A A - 1$ and $AF_A - 1$ are then of order -1. If we consider $M = \mathbb{R}^n$ instead of a compact manifold and if A is an operator of Shubin type, the F_A will be still a parametrix. Moreover, we can use the fact that for $N > 2n$ $(F_A A - 1)^N$ and $(AF_A - 1)^N$ are of order less than $-2n$ and therefore trace class, as long as A is a Shubin operator. This means that in the Shubin case we can explicitly find a parametrix F_A and a number N, such that the index formula (3.16) is computable.

The index formula (3.16) in *Theorem 3.4* holds for general operators on Banach spaces, as long as the operators, of which the trace is taken of, are trace class. For the special case of pseudodifferential operators, there is the famous Atiya-Singer Index formula which expresses the index by the Chern character of a certain bundle associated to the operator A:

Theorem 3.5:(*Atiyah-Singer*) *Any elliptic pseudodifferential operator A acting between two vector bundles E and F over the closed manifold M is a Fredholm operator. Its index is given by*

$$\int_{T^*M} \operatorname{ch}(A)\operatorname{Td}(M). \tag{3.17}$$

Remark 3.3: We want to clarify the objects used in (3.17), so that we can use them better later. Both the Chern character $\operatorname{ch}(A)$ and the Todd class $\operatorname{Td}(M)$ are characteristic classes, which are determinants or traces of analytic functions of $\Omega/(2\pi i)$ with the curvature form Ω of some vector bundle. If one considers the (complexified) tangent bundle $T_{\mathbb{C}}M$, the Todd class is calculated via inserting the curvature form Ω into the function

$$f(z) = \frac{z}{1 - e^{-z}} \tag{3.18}$$

and taking the determinant of the result. Any analytic function $f(\Omega)$ of the curvature form is interpreted as a matrix with two forms as entries and the determinant is calculated by taking sums of wedge products of the components. Note that as the manifold M is assumed to be of finite dimension, the power series $f(\Omega)$ converges, because any term with order greater than n vanishes. Moreover, the Todd class has an expansion

$$\mathrm{Td}(TM) = 1 + \sum_k \mathcal{T}_k \tag{3.19}$$

with $4k$-forms $\mathcal{T}_k$. For any bundle E with curvature form Ω_E the Chern character is given by

$$\mathrm{ch}(E) = \mathrm{Tr}(e^{\Omega/2\pi i}). \tag{3.20}$$

In the case of the elliptic operator A, one first associates a virtual bundle to A, which is a tuple of two vector bundles together with an isomorphism between the components, and then calculates the Chern character of that virtual bundle: Consider the pullbacks π^*E and π^*F, where π is the natural projection from the tangent bundle TM to the base manifold M. As we assume the operator A to be elliptic, the principal symbol $\sigma_p(A)$ is an isomorphism between these two pullbacks. If ∂_E and ∂_F are two connections on E and F, we define the $\sigma_p(A)$ dependent connection $\partial_{\sigma_p(A),E} := \partial_E + \sigma_p(A)^{-1}\partial\sigma_p(A)$. Then $\mathrm{ch}(A)$ is the Chern character of the virtual bundle $(\pi^*E, \pi^*F, \sigma_p(A))$,

$$\mathrm{ch}(A) = \mathrm{Tr}(e^{\Omega_A}) - \mathrm{Tr}(e^{\Omega_F}) \tag{3.21}$$

where Ω_A is the curvature form associated to $\partial_{\sigma_p(A)}$. As one can already guess, the Atiyah-Singer index formula (3.17) is formulated in an abstract way, and there are more practical representations of (3.17), which are better computable. Fedosov has expressed (3.17) directly via the principal symbol of the operator A which is a much more practical representation:

Theorem 3.6: (*Fedosov*, see ([16])) *Let A be a pseudodifferential operator which acts on the vector valued functions on the closed manifold M, i.e the vector bundle E is trivial. Then, as long A is elliptic, it has the index*

$$\mathrm{ind}(A) = \sum_j c_j \int_{S^*(M)} [\mathrm{Tr}((\sigma_p(A)^{-1} d\sigma_p(A))^{2j-1})\mathrm{Td}(M)]_{top}, \tag{3.22}$$

where the coefficients c_j are given by

$$c_j = \left(\frac{1}{2\pi i}\right)^j \frac{(j-1)!}{(2j-1)!}. \tag{3.23}$$

By S^*M we denote the cosphere bundle of M, which has the dimension 2n-1 and $[\cdot]_{top}$ denotes the top degree form of order 2n-1.

The index formula of Fedosov is indeed very useful: As it is directly expressed via differential forms in $\sigma_p(A)$, there are a few things which can be seen easily just by using simple facts about the nature of differential forms. We can directly derive a corollary for scalar operators from (3.22):

Corollary 3.2: *Assume that M is a closed manifold of dimension greater than 1. If A is a scalar elliptic operator acting on scalar functions in $H^s(M)$, then the index of A is zero.*

Proof: If we assume A to be a scalar operator, $\sigma_p(A)^{-1}d\sigma_p(A)$ is a one-form. Since any power greater than 1 of a one-form vanishes, only the term with $j = 1$ in (3.22) survives. Moreover, if the manifold M has dimension greater than one, the dimension of the cosphere bundle $S^*(M)$, $2n-1$, is also greater than 1 and therefore the one-form $\sigma_p(A)^{-1}d\sigma_p(A)$ has to be paired with one of the Todd-class forms $\mathcal{T}_k$ in such a way, that the product is a $2n-1$ form in order to get a nonvanishing integral. However, because Td(M) consists of $4k$ forms over the base manifold M, it only has differentials dx_j in the coordinates of the base manifold. Since $\sigma_p(A)^{-1}d\sigma_p(A)$ is a one form, it has at most differentials in the fibre variable ξ of order 1. But then the whole integrand in (3.22) has at most order one differentials in the fibre variable ξ. For $\dim(M) \geq 3$ the fibres in $S^*(M)$ are at least two dimensional, which means that the integration over the fibres in (3.22) will make the integrand vanish. If we consider $\dim(M) = 2$, the Todd class is trivial. In this case, (3.22) simplifies to

$$\operatorname{ind}(A) = \frac{1}{2\pi i}\int_{S^*(M)} \sigma_p(A)^{-1}d\sigma_p(A) \tag{3.24}$$

and this is also zero, because $\dim(S^*(M)) = 3$, which is greater than the order of the differential 1-form $\sigma_p(A)^{-1}d\sigma_p(A)$. q.e.d.

Remark 3.4: Note that there are manifolds, where the whole Todd class is trivial, $\mathrm{Td}(M) = 1$. (For example all manifolds with dimension $\dim(M) \leq 3$ have trivial Todd class). If M has dimension 1, we can express the index of a scalar operator with the help of winding numbers:

$$\dim(M) = 1 \;\Rightarrow\; \operatorname{ind}(A) = \frac{1}{2\pi i}\int_{S^*(M)} \sigma_p(A)^{-1}d\sigma_p(A) \tag{3.25}$$

$$= \frac{1}{2\pi i}\int_M \sigma_p(A)^{-1}(x,\xi=1)d\sigma_p(A) - \frac{1}{2\pi i}\int_M \sigma_p(A)^{-1}(x,\xi=-1)d\sigma_p(A). \tag{3.26}$$

Assuming that M is a closed curve in the complex plane parametrized by $\gamma : [0,T] \to \mathbb{C}$ the two expressions in (3.26) are just the winding numbers of $\sigma_p \circ \gamma$ along the zero point for $\xi = 1$ and $\xi = -1$.

Remark 3.5: We want to remind the reader that the Fedosov index formula (3.22) is also true on $\mathbb{R}^n$ when using Shubin type operators. Since $\mathbb{R}^n$ is a flat manifold, $\Omega = 0$, its Todd class is trivial, and therefore the Fedosov index formula on $\mathbb{R}^n$ becomes

$$\mathrm{ind}(A) = C_n \int_{S^*(M)} \mathrm{Tr}((\sigma_p(A)^{-1} d\sigma_p(A))^{2n-1}). \tag{3.27}$$

3.1.3 Fredholm Theory of G-Operators

Before coming to the part of this chapter, where we will investigate the properties of systems of operators, we want to state a few facts about the Fredholm theory involving G-operators. The Fredholm property of G-operators is in general much more complicated, as inverting the trajectory symbol is more complicated than inverting a usual pseudodifferential symbol. Moreover, the invertibility of the symbol of a G-operator is not always equivalent to its Fredholm property. For such a statement one needs a few assumptions on the group G itself. In that manner, we want to introduce the definition of amenability in order to have a symbolic calculus, where the invertibility is equivalent to Fredholm property of the corresponding operators:

Definition 3.2: *Let G be a discrete group. G is called amenable, if there exists a finitely additive measure μ on G, satisfying*

$$(i)\ \mu(G) = 1, \quad (ii)\ \mu(gH) = \mu(H),\ \forall H \subset G. \tag{3.28}$$

Examples:

- If G is any abelian group, it is amenable. In particular $\mathbb{Z}$ is amenable.
- Any group containing the free group over two generators as a subgroup is not amenable.

Later in the main chapters of this work, we will consider discrete groups, which are generated by a finite set of Hamiltonian flows. We will see that we have to impose conditions on the flows, such that the generated group does not contain free subgroups. Now, consider a group G with an action on some manifold M. We want to state conditions guaranteeing that the invertitibility of trajectory symbols is equivalent to the Fredholm property of G-operators (for further details, see ([8]):

Theorem 3.7: *Consider a group G with an action on the manifold M. If the action is topologically free, and if G is amenable, any G-operator $D = \sum_g A_g \Phi_g : L^2(M) \to L^2(M)$ with order zero pseudodifferential coefficients A_g and a (modulo compact) unitary representation Φ is Fredholm, iff its trajectory symbol $\sigma_p(D) = \sum_g \sigma_p(A_g)\tau_g : l^2(G) \to l^2(G)$ is invertible for every $(x, \xi) \in S^*M$.*

A natural example for a group G with an action on the manifold M is the group of diffeomorphisms on M. Any diffeomorphism $g \in G$ can be lifted to a canonical transformation on the cotangent bundle T^*M as follows: Define the transformation α on the cotangent bundle via

$$\alpha(x, \xi) := (g(x), Dg^{-T}(x)\xi), \tag{3.29}$$

with Dg^{-T} being the inverse transpose of the differential of g. For any diffeomorphism g, α is automatically a canonical transformation. We can choose the representation Φ_g to be a Fourier integral operator, which is associated to this canonical transformation α. For the resulting G-operators, which may be seen as G-operators associated to shifts, there is a generalization of the Fedosov index formula. We can formulate it in the following way:

Theorem 3.8: (see [14], Theorem 1.5) *Let G be a group of diffeomorphisms on M and assume that these diffeomorphisms act topologically freely and isometrically on the manifold M. If G is in addition amenable, the index of a G-operator $D : H^s(M) \to H^{s-m}(M)$ with invertible principal trajectorial symbol is*

$$\mathrm{ind}(D) = \sum_j c_j \int_{S^*(M\times\mathbb{R}/G)} \mathrm{Tr}_\epsilon[(\sigma_p(\mathcal{D})^{-1} d\sigma_p(\mathcal{D}))^{2j-1}\mathrm{Td}(M)]_{top} \tag{3.30}$$

where $S^*(M \times \mathbb{R}/G)$ is the cosphere bundle of the orbifold $(M \times \mathbb{R})/G$ with the action $g \cdot (x, t) := (g(x), t + 1), (x, t) \in M \times \mathbb{R}, g \in G$. By $\mathcal{D}$ we denote the external product of the natural extension of D to the cylinder $M \times \mathbb{R}$ with the index 1 operator $A = \partial_t + t$, see ([14]) for details.

Sometimes it is possible to express the index of a G-operator through the indices of usual pseudodifferential operators. This is for example the case, if one considers two term operators with scalar coefficients:

Lemma 3.2: *Let A_1 and A_2 be two scalar pseudodifferential operators and Φ_g a Fourier integral operator associated to some group element g of a group G. The index of the two term operator*

$$F = A_1 + A_2\Phi_g : H^s(M) \to H^{s-m}(M) \tag{3.31}$$

is given by $\mathrm{ind}(F) = \mathrm{ind}(A_{max})$*, as long as F is of Fredholm type. By A_{max} we denote the pseudodifferential operator in* (3.31) *with the greater geometric mean (see Theorem 2.4, Remark 2.9).*

Proof: The statement follows directly from *Theorem 2.4*: If F is Fredholm, either A_1 or A_2 is Fredholm (this can be seen using (2.70) – (2.71): The principal symbols of either A_1 or A_2 need to be invertible, which means that they are Fredholm). In the case where A_1 is Fredholm (and the geometric mean of A_1 is greater than the geometric mean of A_2), consider $A_1 + sA_2\Phi_g,\ s \in [0, 1]$. This is a homotopy of Fredholm operators and by *Theorem 3.3* F and A_1 must have the same index. In the case that A_2 is Fredholm, one may consider the homotopy $sA_1 + A_2\Phi_g$ and use that $A_2\Phi_g$ has the same index as A_2. *q.e.d.*

Remark 3.6: With $S^*(M\times\mathbb{R}/G)$ in (3.30) we denote the cosphere bundle of the quotient $(M\times\mathbb{R})/G$. Note that this quotient is in general not a manifold. The action defined in *Theorem 3.8* acts freely and properly, which guarantees that the quotient has indeed the structure of a manifold. The formula (3.30) looks just the same as the usual Fedosov formula, but we should mention that Fedosov has proven his formula only for bundles with finite rank. In the G-operator case $\sigma_p(\mathcal{D})$ is an operator on $l^2(G)$ for every point in the quotient space $(M\times\mathbb{R})/G$. Then we can define the vector bundle ϵ over $(M\times\mathbb{R})/G$ with infinite rank and fibres $l^2(G)$. So (3.30) is somehow an infinite dimensional version of the classical Fedosov index theorem.

The operators of interest in this work are usually not of scalar type. Like in the case of matrices, systems of operators have nice properties when they have a special form, like being triangular. We want to deal only with systems of operators in the second part of this chapter and derive some useful theorems about them.

3.2 The Fredholm Theory for Systems of Operators

3.2.1 Triangular Matrix Systems

First, we want to consider matrices of operators which are arbitraryly large. Afterwards we will investigate the special case of 2×2 matrices, which are of particular interest.

Theorem 3.9: *Consider a matrix $\mathcal{A}$ of operators, such that any component $\mathcal{A}_{ij}$ is an operator between Hilbert spaces. If the matrix $\mathcal{A}$ is diagonal,*

$$\mathcal{A}=\begin{pmatrix} A_1 & 0 & \cdots & 0 & 0\\ 0 & A_2 & \cdots & 0 & 0\\ \vdots & & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & A_n \end{pmatrix} \tag{3.32}$$

then it is Fredholm, iff all the diagonal entries A_k are Fredholm. The index of $\mathcal{A}$ is given by

$$\operatorname{ind}(\mathcal{A})=\sum_k \operatorname{ind}(A_k). \tag{3.33}$$

Proof:
"$\Rightarrow$": Consider the matrix in (3.32) and assume that one of the nonzero entries A_k is not a Fredholm operator. Then either the kernel or the cokernel of A_k is infinite dimensional. If the kernel is infinite dimensional, there is a sequence of infinitely many linear independent elements e_j^k which are in the kernel of A_k. But then the vectors v_j^k with components $v_j^k=(0,0,...,e_j^k,0,...,0)$ are in the kernel of $\mathcal{A}$ for every j, meaning that $\mathcal{A}$ itself has infinite dimensional kernel and thus can not be Fredholm. If A_k has infinite dimensional

cokernel, the adjoint $A_k^\dagger$ has infinite dimensional kernel. The same argument as for the case of an infinite dimensional kernel of A_k shows that $\mathcal{A}^\dagger$ must have infinite dimensional kernel in this case. But this again means that $\mathcal{A}$ has infinite dimensional cokernel. Again, $\mathcal{A}$ can not be a Fredholm operator.

"$\Leftarrow$": If all the diagonal entries A_k are Fredholm, they have parametrices A_k^p with $A_k A_k^p = 1 + K_{k,R}$ and $A_k^p A_k = 1 + K_{k,L}$. Defining

$$\mathcal{A}^p := \begin{pmatrix} A_1^p & 0 & \cdots & 0 & 0 \\ 0 & A_2^p & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & A_n^p \end{pmatrix} \tag{3.34}$$

we have $\mathcal{A}\mathcal{A}^p = diag(1 + K_{k,R})$ and $\mathcal{A}^p\mathcal{A} = diag(1 + K_{k,L})$. This is equivalent to the fact that $\mathcal{A}$ has a left and right parametrix, thus it is Fredholm. It remains to prove, that the index formula (3.33) is true. For that, we can make use of the general trace formula (3.16): Choose A_k^p as the parametrices for A_k in such a way, that the corresponding error terms $(1 - A_k^p A_k)$ and $(1 - A_k A_k^p)$ are of trace class. Then $(1 - \mathcal{A}^p\mathcal{A})$ and $(1 - \mathcal{A}\mathcal{A}^p)$ are of trace class with the compact errors $K_{k,L}$ and $K_{k,R}$ on the diagonal (see equation (3.2)). Taking the trace only involves the traces of the diagonal components, therefore the off diagonal components do not matter. After both traces $\mathrm{Tr}(1 - \mathcal{A}^p\mathcal{A})$ and $\mathrm{Tr}(1 - \mathcal{A}\mathcal{A}^p)$ are taken, the corresponding terms can be rearranged to get (3.33). (q.e.d.)

Remark 3.7: Note that the "$\Leftarrow$" direction of the proof of *Theorem 3.7* can also be proven in the case of a triangular matrix $\mathcal{A}$: For that one takes a similar ansatz for the parametrix $\mathcal{A}^p$, where the diagonal entries are chosen also to be A_k^p. If $\mathcal{A}$ is lower triangular, this will also hold for $\mathcal{A}^p$. The off diagonal components of $\mathcal{A}^p$ can be constructed by using the components A_k^p and the lower diagonal components of $\mathcal{A}$.

The result presented in *Theorem 3.8* allows us to express the index of "trigonalizable" operators in terms of their diagonal components. We state in a corollary what we exactly mean by that:

Corollary 3.3: *Let a matrix of operators $\mathcal{A}$ be of Fredholm type. Furthermore, assume that there exists a Fredholm operator T with a parametrix T^p, such that*

$$T^p\mathcal{A}T = D + S, \tag{3.35}$$

with a compact operator S and a diagonal matrix D. Then the formula

$$\mathrm{ind}(\mathcal{A}) = \sum_k \mathrm{ind}(D_k) \tag{3.36}$$

is true, where the D_k are the diagonal entries of D.

Proof: By the product formula for Fredholm operators, the index of $\mathcal{A}$ is equal to the index of $T^p\mathcal{A}T$, since T and its parametrix T^p have indices with an opposite sign. This means, $\text{ind}(\mathcal{A}) = \text{ind}(D+S) = \text{ind}(D)$, because S is assumed to be compact. By *Theorem 3.9*, the index can be expressed via the sum of the indices of the diagonal entries. *q.e.d.*

3.2.2 2×2 Systems

The foregoing theorems were stated for general matrices of operators. As we will consider the second order scalar wave equation later, 2×2 systems representing the order reduction of such equations are of special interest. There are very useful decompositions for matrices of arbitrary operators, which have the size 2×2. We want to present this decompositions to conclude this chapter:

Theorem 3.10: *Let $\mathcal{A}$ be the 2×2 matrix*

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.37}$$

of operators a, b, c, d. As long as the first entry a is a Fredholm operator, the matrix $\mathcal{A}$ has the decomposition

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ ca^p & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^p b \end{pmatrix} \begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix} + K, \tag{3.38}$$

with a compact matrix K and a parametrix a^p of a.

Proof: If we assume a^p to be a left- and right parametrix to a, then $aa^p = 1 + K_R$ and $a^p a = 1 + K_L$ with compact operators K_R and K_L. Let us compute the product of the three matrices in (3.38):

$$\begin{pmatrix} 1 & 0 \\ ca^p & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^p b \end{pmatrix} \begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ ca^p a & d - ca^p b \end{pmatrix} \begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix} \tag{3.39}$$

$$= \begin{pmatrix} a & (1+K_R)b \\ c(1+K_L) & c(1+K_L)a^p b + d - ca^p b \end{pmatrix} = \begin{pmatrix} a & b + K_R b \\ c + cK_L & ca^p b + cK_L a^p b + d - ca^p b \end{pmatrix} \tag{3.40}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + K, \tag{3.41}$$

with the compact matrix

$$K = \begin{pmatrix} 0 & K_R b \\ cK_L & cK_L a^p b \end{pmatrix} \tag{3.42}$$

and this is equivalent to the statement of the theorem. *q.e.d.*

Corollary 3.5: *If the a in the foregoing theorem is not only Fredholm but even invertible, we have the decomposition* (3.38) *with* $K = 0$.

Proof: For an invertible a, we chose $a^p = a^{-1}$ and the errors K_L and K_R are both zero, therefore K from (3.42) is also zero. *q.e.d.*

The decomposition (3.38) can be used to express the index of a 2×2 system of operators through the indices of the entries a, b, c and d. If we make use of *Theorem 3.9*, we come to the result

Proposition 3.4: *Let* $\mathcal{A}$ *be an operator of the form* (3.37), *with Fredholm a. Then the index of* $\mathcal{A}$ *is*

$$\operatorname{ind}(\mathcal{A}) = \operatorname{ind}(a) + \operatorname{ind}(d - ca^p b). \tag{3.43}$$

Proof: First we observe that the first and third matrix in (3.47) is invertible, with inverses

$$\begin{pmatrix} 1 & 0 \\ ca^p & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -ca^p & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a^p b \\ 0 & 1 \end{pmatrix}. \tag{3.44}$$

The invertibility of these operators means that their index is zero. But then the index of $\mathcal{A}$ simplifies to

$$\operatorname{ind}(\mathcal{A}) = \operatorname{ind}\left(\begin{pmatrix} 1 & 0 \\ ca^p & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d - ca^p b \end{pmatrix}\begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix} + K\right) \tag{3.45}$$

$$= \operatorname{ind}\left(\begin{pmatrix} 1 & 0 \\ ca^p & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d - ca^p b \end{pmatrix}\begin{pmatrix} 1 & a^p b \\ 0 & 1 \end{pmatrix}\right) \tag{3.46}$$

$$= \operatorname{ind}\left(\begin{pmatrix} a & 0 \\ 0 & d - ca^p b \end{pmatrix}\right), \tag{3.47}$$

since the index of the both invertible matrices in (3.46) is zero. Now, this last matrix is diagonal. We can therefore apply *Theorem 3.9* and get the desired result. *q.e.d.*

4 Solution Theory of Hyperbolic Systems

In the Chapters 1-3 we developed the mathematical tools neccessary to describe hyperbolic systems and their solution theory. We will now apply the important results from the first chapters and show that solution operators to hyperbolic systems of a certain type may be written out in terms of Fourier integrals and G-operators.

4.1 Globally Hyperbolic Manifolds

Equations of hyperbolic type are usually stated on globally hyperbolic manifolds. Therefore, before we can explain what we understand under hyperbolic systems, we first give a definition of global hyperbolicity:

Definition 4.1: *Let (M, g) be a Lorentzian manifold, i.e. g_p defines a pointwise isomorphism between T_pM and T_p^*M, which has a representation as a symmetric invertible matrix with Lorentz signature $(-, +, +..., +)$ in local coordinates. A curve $\gamma : I \subset \mathbb{R} \to M$ is called timelike, if $||\dot{\gamma}(s)||_g^2 := g(\dot{\gamma}, \dot{\gamma}) < 0$. We call a subset $\Sigma \subset M$ a Cauchy hypersurface, as long as every inextendible timelike geodesic in M intersects Σ exactly once. M is a globally hyperbolic manifold, if it posseses such a subset Σ.*

A manifold which is globally hyperbolic, has many advantages in contrast to some arbitrary generic manifold. One of the good properties is that topologically it is fully determined by the Cauchy surface Σ: Any globally hyperbolic manifold may be seen as the cylinder $\Sigma \times \mathbb{R}$ for any given Cauchy hypersurface Σ. Let us write down this fact rigorously in the form of a proposition:

Proposition 4.1: (see [6], Theorem 1.2) *Let (M, g) be a globally hyperbolic manifold and Σ a given Cauchy hypersurface. Then any other Cauchy hypersurface $\tilde{\Sigma}$ is homeomorphic to Σ and the manifold M itself is homeomorphic to $\Sigma \times \mathbb{R}$. If Σ is a C^k-submanifold of M, any two Cauchy hypersurfaces are C^k-diffeomorphic to each other and M is C^k-diffeomorphic to $\Sigma \times \mathbb{R}$. Moreover, if (x, t) are the coordinates on M interpreted as a cylinder, i.e. $x \in \Sigma$, $t \in \mathbb{R}$, the metric g takes the form*

$$g = -N(t)^2 dt^2 + g_t, \tag{4.1}$$

where g_t is a Riemannian metric on the Cauchy surface Σ_t which is identified with $\Sigma \times \{t\}$.

Example 4.1: As the coordinate $t \in \mathbb{R}$ of a globally hyperbolic manifold may be interpreted as the time flowing, natural examples of globally hyperbolic manifolds are of course globally hyperbolic spacetimes, which are of dimension 4. For the flat Minkowski space M_{Mink} we have $\Sigma = \mathbb{R}^3$ and $g = -dt^2 + g_0$ with $g_t = g_0$ being the flat metric on $\mathbb{R}^3$.

Example 4.2: Any FLRW spacetime, which is a solution to the Einstein equations on cosmologically large scales, is a globally hyperbolic manifold. Depending on the curvature parameter $k \in \{-1, 0, 1\}$ the Cauchy surfaces Σ_t are diffeomorphic to either the hyperbolic space $\mathbb{H}^3$ ($k = -1$), the sphere $\mathbb{S}^3$ ($k = 1$) or the flat space $\mathbb{R}^3$ ($k = 0$). The FLRW metric is given by

$$g = -dt^2 + h(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{4.2}$$

with the surface element $d\Omega$. From now on, we will speak of the coordinate t as being the time coordinate of the globally hyperbolic manifold M.

4.2 Types of Hyperbolicity and Example, Order Reduction

Given the geometrical setting where we want to work on, let us now come to the definition of hyperbolicity in relation to a system of equations. First define the space

$$L^2([-T,T], H^s(M)) := \left\{ u \in \Gamma(E) : u(t) \in H^s(\Sigma_t), \int_{[-T,T]} ||u(t)||^2_{H^s(\Sigma_t)} dt < \infty \right\} \tag{4.3}$$

for a section u in some vector bundle E over M. If we choose any fixed time t_0 as the initial time, Σ_0 may be seen as the boundary of $\Sigma \times \mathbb{R}_+ \subset M$. A system of equations is said to be of hyperbolic type, if the boundary value problem, where the boundary data is given at Σ_0, is well posed:

Definition 4.2: *Given a globally hyperbolic manifold* (M, g)*, one may consider the system of equations*

$$\partial_t u = Lu + f, \;\; f \in L^2([-T,T], H^s(M)) \tag{4.4}$$

with a matrix $L = L(t, x, \partial_x)$ *with first order pseudodifferential operators as entries. The system* (4.4) *is called hyperbolic, if for given initial data* $u(0) = g_0 \in H^s(\Sigma_0)$ *there is a unique solution* $u \in L^2([-T,T], H^s(M))$ *of* (4.4).

Remark 4.1: When explicitly constructing a solution to (4.4) later on, we will restrict to the case $f = 0$. Because $f = 0$ is in $L^2(\mathbb{R}, H^s(M))$ we are also searching for solutions in this space.

One might ask, which conditions on the matrix L are sufficient for the hyperbolicity. There are different types of hyperbolic systems. Let us mention the most important ones:

Definition 4.3: *Consider the system* (4.4)*. We call the system*

(*a*) *Symmetrizable, if for* $|\xi| \geq 1$ *there exists a matrix* $R \in \mathrm{OPS}^0$ *with positive defi-*

nite principal symbol, such that $(RL + (RL)^\dagger) \in \mathrm{OPS}^0$,
(b) *symmetric, if the* R *in* (a) *is the identity* $R = 1$,
(c) *strictly hyperbolic, if* $\sigma_p(L)$ *has purely imaginary and distinct eigenvalues for* $\xi \neq 0$.

Proposition 4.2: (see, [7], Chapter IV, Theorem 3.2)

(a) *For any symmetrizable system, the initial value problem*

$$\partial_t u = Lu + f, \quad u(0) = g_0, \; g_0 \in H^s(M), f \in L^2([-T,T], H^s(M)) \tag{4.5}$$

is uniquely solvable with a solution $u \in L^2([-T,T], H^s(M))$.
(b) *Every strictly hyperbolic system is symmetrizable.*

As we can see from *Proposition 4.2*, the class of symmetrizable systems is the largest of the three classes presented in *Definition 4.3*, as it contains both other classes. Let us give an example, from which one can see that even the class of strictly hyperbolic systems is already quite large:

Example 4.3: Let us consider the wave equation on $\Sigma \times \mathbb{R}$ for any Riemmanian manifold (Σ, g). If g^{kj} are the components of the inverse metric tensor, the principal part of the Laplace Beltrami operator $\Delta = \mathrm{div} \circ \mathrm{grad}_g$ is given by

$$\Delta_p = -g^{kj} \partial_{x_k} \partial_{x_j}. \tag{4.6}$$

Now let us take a look at the equation $\partial_t^2 u = -\Delta u$: This is a second order scalar equation. We first need to make an order reduction, such that the equation takes the form of (4.4). For that, let us introduce the new variables

$$u_1 := \Lambda u, \quad u_2 := \partial_t u, \quad \Lambda = \sqrt{\Delta + 1} \in \mathrm{OPS}^1. \tag{4.7}$$

If we take the derivative with respect to time of both variables u_1, u_2 this results in

$$\partial_t u_1 = \partial_t \Lambda u = \Lambda \partial_t u = \Lambda u_2, \quad \partial_t u_2 = \partial_t^2 u = \Delta u = \Delta \Lambda^{-1} u_1, \tag{4.8}$$

or equivalently, written down in matrix form

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \\ -\Delta\Lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.9}$$

Since $\Lambda \in \mathrm{OPS}^1$, it follows $\Lambda^{-1} \in \mathrm{OPS}^{-1}$ and $\Delta\Lambda^{-1} \in \mathrm{OPS}^1$ by the composition formula from Chapter 2. This means that we can identify the matrix in (4.9) with the L from *Definition 4.2* Now, let us come to the principal symbol of that L. Because the principal symbol is multiplicative, we have $\sigma_p(\Lambda^{-1}) = 1/\sigma_p(\Lambda)$. Furthermore, $\sigma_p(\Delta\Lambda^{-1}) = \sigma_p(\Delta)\sigma_p(\Lambda)^{-1}$ such that the principal symbol of L is given by

$$\sigma_p(L) = \begin{pmatrix} 0 & \sigma_p(\Lambda) \\ -\sigma_p(\Delta)\sigma_p(\Lambda)^{-1} & 0 \end{pmatrix} \tag{4.10}$$

The characteristic polynomial of this matrix is $\chi_L(\lambda) = \lambda^2 + \sigma_p(\Delta)$, leading to the eigenvalues

$$\lambda_{1/2} = \pm i\sqrt{\sigma_p(\Delta)} = \pm i\sqrt{g^{kj}\xi_k\xi_j}. \tag{4.11}$$

Because the metric g is positive definite, one can see from (4.11) that both eigenvalues are purely imaginary and distinct, meaning that the wave equation is equivalent to a strictly hyperbolic system.

Remark 4.2: Recall that in classical courses on PDE, a scalar hyperbolic equation of second order is a partial differential equation, such that the matrix A defining the principal part of the equation has one positive and $n-1$ negative eigenvalues. All equations of the form $\partial_t^2 u + Pu = 0$ for an elliptic differential operator of second order fall into this class of equations. Note that if we search for solutions, which are two times differentiable, the matrix corresponding to the elliptic part P will be a symmetric positive definite matrix. One then may always choose the metric g on the manifold M, such that the principal parts of Δ and P coincide. Therefore all scalar equations $\partial_t^2 u + Pu = 0$ with an elliptic P will be equivalent to a strictly hyperbolic system.

Remark 4.3: The technique of order reduction performed in *Example 4.3* can of course also be done for higher order equations. If one works with a m-th order scalar equation of the form

$$\left(\partial_t^m - \sum_{j=0}^{m-1} A_{m-j}\partial_t^j\right) u = 0 \tag{4.12}$$

with operators A_{m-j} of order $m-j$, the new variables

$$u_j = \partial_t^{j-1}\Lambda^{m-j}u,\ j = 1, ..., m \tag{4.13}$$

transform (4.12) into the system

$$\partial_t \begin{pmatrix} u_1 \\ \\ \vdots \\ \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & \Lambda & 0 & & 0 \\ & 0 & \Lambda & & \\ \vdots & & & & \vdots \\ & & & & \Lambda \\ b_1 & b_2 & b_3 & & b_m \end{pmatrix} \begin{pmatrix} u_1 \\ \\ \vdots \\ \\ u_m \end{pmatrix}, \tag{4.14}$$

where $b_j = A_{m-j+1}\Lambda^{j-m}$ (see for example *Taylor*, [7], *Chapter IV, Equation 1.3*). Then the eigenvalues of $\sigma_p(L)$ with L being the matrix in (4.14) are connected to $\sigma_p(P), P =$

$\partial_t^m - \sum_{j=0}^{m-1} A_{m-j}\partial_t^j$ via $\lambda_j = i\tau_j$ with the zeros τ_j of the equation $\sigma_p(P) = 0$. From this we see that any higher order equation (4.12) is equivalent to a strictly hyperbolic system, as long as $\sigma_p(P)$ has only real and distinct zeros.

4.3 Diagonalization of Strictly Hyperbolic Systems

The class of equations, which is the most important in this work, is the class of strictly hyperbolic systems. The reason for that is that - under mild additional assumptions on the distance between two differerent eigenvalues of $\sigma_p(L)$ - it is possible to express the solution operator to (4.4) for $f = 0$ with the help of Fourier integral operators and G-operators in an easy way. Let us explain some of the advantages of strictly hyperbolic systems and then develop the explicit solution theory.

Lemma 4.1: *Let $\partial_t u = Lu$ be a strictly hyperbolic system, with $L \in \mathrm{OPS}^1$. Furthermore, let $\sigma(T)$ be the matrix that diagonalizes $\sigma_p(L)$, i.e. $\sigma(T)^{-1}\sigma_p(L)\sigma(T)$ is a diagonal matrix. As long as $T := \mathrm{op}(\sigma(T))$ is an invertible operator, the transformed variable $v := T^{-1}u$ satisfies*

$$\partial_t v = Dv + Sv, \tag{4.15}$$

with a diagonal first order matrix D and $S \in \mathrm{OPS}^0$. The entries of the diagonal operator D are operators $d_j = \mathrm{Op}(\lambda_{p,j})$ where $\lambda_{p,j}$ are the principal parts ofthe eigenvalues of $\sigma_p(L)$.

Proof: The property of $\sigma_p(L)$ having n different eigenvalues leads to the fact that $\sigma_p(L)$ is a diagonalizable matrix (for $\xi \neq 0$). Denoting with $\sigma(T)$ the transformation, which diagonalizes $\sigma_p(L)$, we can calculate the principal symbol of $T^{-1}LT$:

$$\sigma_p(T^{-1}LT) = \sigma_p(T^{-1})\sigma_p(L)\sigma_p(T) = \sigma_p(T)^{-1}\sigma_p(L)\sigma_p(T) \tag{4.16}$$

(since T^{-1} is a parametrix to T we used $\sigma_p(T^{-1}) = \sigma_p(T)^{-1}$, see the remarks at the beginning of *Chapter 3*). Let us first prove that the right hand side of (4.16) is equal to the principal part of $\sigma(D)$, where $\sigma(D) = \sigma(T)^{-1}\sigma_p(L)\sigma(T)$. For that, decompose $\sigma(T)$ as $\sigma(T) = \sigma_p(T) + \sigma_r(T)$ where $\sigma_p(T)$ is the principal part. Then we see

$$\sigma(T)^{-1} = (\sigma_p(T) + \sigma_r(T))^{-1} = \left(\sigma_p(T)\left(1 + \frac{\sigma_r(T)}{\sigma_p(T)}\right)\right)^{-1} = \sigma_p(T)^{-1}\left(1 + \frac{\sigma_r(T)}{\sigma_p(T)}\right)^{-1}. \tag{4.17}$$

Because the rest term $\sigma_r(T)$ is of lower order than $\sigma_p(T)$, the quotient $\sigma_r(T)/\sigma_p(T)$ will have norm lower than 1 asymptotically. We can thus use the geometrical series to derive that the principal part of (4.17) is given by $\sigma_p(T)^{-1}$. Writing $\sigma(T)^{-1} = \sigma_p(T)^{-1} + \tilde{\sigma}_r(T)$

one can express $\sigma(D)$ as

$$\sigma(D) = \sigma(T)^{-1}\sigma_p(L)\sigma(T) = (\sigma_p(T)^{-1} + \tilde{\sigma}_r(T))\sigma_p(L)(\sigma_p(T) + \sigma_r(T)) \tag{4.18}$$

$$= \sigma_p(T)^{-1}\sigma_p(L)\sigma_p(T) + \sigma_0(L, T). \tag{4.19}$$

Note that we used $\sigma_0(L, T)$ as a notation for all the lower order terms involved. From (4.18) we deduce $\sigma_p(D) = \sigma_p(T)^{-1}\sigma_p(L)\sigma_p(T)$. Coming back to (4.16) we conclude $\sigma_p(T^{-1}LT) = \sigma_p(D)$, which means that $T^{-1}LT$ can be written as $T^{-1}LT = D + R$ with $R \in \mathrm{OPS}^0$ and D as described in the theorem. Now let us compute the derivative of the variable $v = T^{-1}u$:

$$\partial_t v = \partial_t(T^{-1}u) = \dot{T}^{-1}u + T^{-1}\dot{u} = \dot{T}^{-1}Tv + T^{-1}\dot{u} = \dot{T}^{-1}Tv + T^{-1}Lu. \tag{4.20}$$

When one multiplies $T^{-1}LT = D + R$ by T^{-1} from the right this gives $T^{-1}L = DT^{-1} + RT^{-1}$. This can be inserted into (4.19) to express the calculation fully in terms of v:

$$\partial_t v = \dot{T}^{-1}Tv + (D + R)T^{-1}u = \dot{T}^{-1}Tv + Dv + Rv = Dv + (\dot{T}^{-1}T + R)v. \tag{4.21}$$

Note that if T has any order m, both T^{-1} and its derivative will have the opposite order $-m$, therefore the rest term $(\dot{T}^{-1}T + R)$ in (4.21) must be of order zero. q.e.d.

Lemma 4.1 tells us that strictly hyperbolic systems may be diagonalized up to an error of order zero. This is helpful, since diagonal systems would only involve solving a set of scalar equations. For the wave equation $\partial_t^2 u = \Delta u$, the only error term of order zero is the term arising from the lower order part of L:

Example 4.4: Consider again the wave equation $\partial_t^2 u = -\Delta u$. As already discussed, the corresponding first order matrix is

$$L = \begin{pmatrix} 0 & \Lambda \\ -\Delta\Lambda^{-1} & 0 \end{pmatrix}. \tag{4.22}$$

We also already calculated the eigenvalues of $\sigma_p(L)$ to be $\lambda_{1/2} = \pm i\sqrt{\sigma_p(\Delta)}$. The corresponding operators $d_{1/2}$ are $d_{1/2} = \pm i\sqrt{\Delta}$. This means that we can transform the wave equation to the almost diagonal system

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{\Delta} & 0 \\ 0 & -i\sqrt{\Delta} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{4.23}$$

4.4 Solution Operators

We want to make use of the property of strictly hyperbolic sytems to be almost diagonalizable to construct a solution theory for them. For that, let us first define, what we understand by a solution operator:

Definition 4.4: *Consider any hyperbolic equation $\partial_t u = Lu$. A solution operator associated to this equation is a family $\Phi_{s,t}$ of operators such that $u(t') := \Phi_{t',s}u(s)$ solves $\partial_t u = Lu$ on the interval $I = [s,t)$. Furthermore, we want $\Phi_{s,s} = 1, \forall s \in \mathbb{R}$.*

Remark 4.4:

• Solution operators to hyperbolic equations may be composed as long as the second parameter of the first operator and the first parameter of the second operator coincide: For two operators $\Phi_{s,t}$ and $\Phi_{t,\tilde{s}}$ the composition is given by $\Phi_{s,t} \circ \Phi_{t,\tilde{s}} = \Phi_{s,\tilde{s}}$. From this we derive $\Phi_{s,t}^{-1} = \Phi_{t,s}$.
• One can transfer the partial differential equation $\partial_t u = Lu$ to an equation for the solution operator: Writing $u(t) = \Phi_{s,t}u(s)$ we get

$$(\partial_t u)(t) = (\partial_t \Phi_{s,t})u(s) \overset{!}{=} (L\Phi_{s,t})u(s) \tag{4.24}$$

which means that $\Phi_{s,t}$ itself has to fulfill the equation $\partial_t \Phi_{s,t} = L\Phi_{s,t}$, which is an operator valued equation.
• Note that for hyperbolic systems a solution operator always exists: By definition, a system is hyperbolic, as long as the initial value problem is uniquely solvable. This actually means that there is an isomorphism between any function $u(s)$ on Σ_s and the kernel of the operator $D = \partial_t - L$.
• For $s = 0$ we may simply write $\Phi_{s,t} = \Phi_t$.

Proposition 4.3: *Let $\partial_t u = Lu$ and $\partial_t v = Jv$ be two scalar strictly hyperbolic equations with solution operators Φ_t and Ψ_t. Then the composition $\Phi_t\Psi_t$ is also a solution operator to a strictly hyperbolic equation. It solves*

$$\partial_t w = (L + J^{\alpha_t^{-1}})w \tag{4.25}$$

with α_t being the Hamiltonian flow along $H_{\sigma_p(L)}$.

Proof: Using the Leibniz rule, the derivative of the product$\Phi_t\Psi_t$ calculates to

$$\partial_t(\Phi_t\Psi_t) = \dot{\Phi}_t\Psi_t + \Phi_t\dot{\Psi}_t = L\Phi_t\Psi_t + \Phi_t J\Psi_t \tag{4.26}$$

$$= L\Phi_t\Psi_t + (\Phi_t J\Phi_t^{-1})\Phi_t\Psi_t = (L + J^{\alpha_t^{-1}})\Phi_t\Psi_t. \tag{4.27}$$

Note that we used Egorov´s theorem in the calculation, which is possible because of J being a scalar operator. Now, a scalar equation will be strictly hyperbolic, as long as L

is of the form $L = i\tilde{L}$ with an operator $\tilde{L}$ with real principal symbol. If the L and J in this example are of this type, so is the sum $(L + J^{\alpha_t^{-1}})$. Therefore the resulting equation (4.24) is again strictly hyperbolic. *q.e.d.*

Remark 4.5: Equations of the form $\partial_t u = iLu$ are not only interesting when dealing with equations of scalar type: As long as the operator L is self adjoint, the solution operator Φ_t will be unitary, $\Phi_t^{-1} = \Phi_t^\dagger$. This can be seen easily by checking that the norm of $u(t)$ is conserved:

$$\partial_t ||u(t)||^2 = \partial_t \langle u, u\rangle = \langle \dot{u}, u\rangle + \langle u, \dot{u}\rangle = \langle iLu, u\rangle + \langle u, iLu\rangle \tag{4.28}$$

$$= i\langle Lu, u\rangle - i\overline{\langle Lu, u\rangle} = -2\mathrm{Im}(\langle Lu, u\rangle) = 0, \tag{4.29}$$

since L is assumed to be self adjoint, which means $\langle Lu, u\rangle \in \mathbb{R}$. Thus the norm is conserved by the equation and therefore the solution operator must be unitary.

In the following chapters of this work we will be confronted with terms of the form $A\Phi_t^{-1}B\Phi_t$. In a lot of calculations it will be useful, if the operators A or B in such an expression commute with the solution operator Φ_t.

Proposition 4.4: *Consider a hyperbolic system $\partial_t u = Lu$. Any time independent operator A which commutes with $L = L(t)$ for all times t, commutes with the solution operator $\Phi_{t,s}$.*

Proof: Let u be a solution of $\partial_t u = Lu$. Instead of u, consider the variable $v := Au$. The variable v is also a solution of the hyperbolic equation $\partial_t v = Lv$ since

$$\partial_t v = \partial_t(Au) = A\partial_t u = ALu = LAu = Lv, \tag{4.30}$$

where we used the commutativity of A and $L = L(t)$ in the last step. For any given time t_0 the value of the solution v on the Cauchy-hypersurface Σ_0 is given by $v(t_0) = Au(t_0)$. Now consider a third variable, $w(t) := \Phi_{t,s}Au(s)$ where we have fixed s. Again, w solves $\partial_t w = Lw$ because

$$\partial_t w = \partial_t(\Phi_{t,s}Au(s)) = (\partial_t\Phi_{t,s})Au(s) = L\Phi_{t,s}Au(s) = Lw. \tag{4.31}$$

The initial value of w is given by $w(t_0) = \Phi_{s,s}Au(s) = Au(s)$ by $\Phi_{s,s} = 1$. We recognize that both variables v and w are solutions of to the same hyperbolic equation with the same initial values. By the definition of a hyperbolic system, we must have $w = v$. Since the original variable $u = u(t)$ may always be written as $u(t) = \Phi_{t,s}u(s)$, we have the equality $A\Phi_{t,s}u(s) = \Phi_{t,s}Au(s)$ which is true for any arbitrary initial value $u(s)$. This statement is equivalent to $A\Phi_{t,s} = \Phi_{t,s}A$. *q.e.d.*

We want to note that the time independence of A in *Proposition 4.4* is in general necessary for the statement of the proposition: For a time dependent operator $A = A(t)$ the time derivative in (4.30) and (4.31) produces extra terms due to the Leibniz rule, so that the variables v and w from the proof do not fulfill the same equation anymore. However, it is possible to find conditions under which a proof similar to that of *Proposition 4.4* can be written down even for the case of a time dependent A:

Proposition 4.5: *Let $\partial_t u = Lu$ define a hyperbolic system and let $A = A(t)$ be a time dependent operator, which can be defined on every Cauchy-hypersurface Σ_s for all times t. If A is invertible and if the operator $\dot{A}A^{-1}$ is invariant under conjugation by the solution operator, i.e.*

$$\Phi_{t,s}\dot{A}A^{-1}\Phi_{s,t} = \dot{A}A^{-1}, \tag{4.32}$$

then $A(t)$ commutes with $\Phi_{t,s}$, as long as $A(t)$ commutes with $L(t)$ for all t and as long as the equation $\partial_t u = (L + \dot{A}A^{-1})u$ is hyperbolic.

Proof: We try to define variables v and w in the same way as in the proof of *Proposition 4.4* and search for the conditions leading to the same evolution equations for both variables. Therefore, let $v = A(t)\Phi_{t,s}u(s)$ and $w = \Phi_{t,s}A(t)u(s)$. These two variables satisfy the equations

$$\partial_t v = \partial_t(A(t)\Phi_{t,s}u(s)) = \dot{A}\Phi_{t,s}u(s) + A\dot{\Phi}_{t,s}u(s) = \dot{A}\Phi_{t,s}u(s) + AL\Phi_{t,s}u(s) \tag{4.33}$$

$$= \dot{A}\Phi_{t,s}u(s) + LA\Phi_{t,s}u(s) = \dot{A}\Phi_{t,s}u(s) + Lv. \tag{4.34}$$

For a noninvertible A, the first term in (4.34) can not be expressed in terms of v. Therefore we assume that A^{-1} exists. Then, (4.33) – (4.34) is equivalent to

$$\partial_t v = (L + \dot{A}A^{-1})v. \tag{4.35}$$

Now let us calculate the corresponding equation for w:

$$\partial_t w = \partial_t(\Phi_{t,s}A(t)u(s)) = \dot{\Phi}_{t,s}Au(s) + \Phi_{t,s}\dot{A}u(s) = L\Phi_{t,s}Au(s) + \Phi_{t,s}\dot{A}u(s) \tag{4.36}$$

$$= Lw + \Phi_{t,s}\dot{A}A^{-1}\Phi_{s,t}w = (L + \Phi_{t,s}\dot{A}A^{-1}\Phi_{s,t})w. \tag{4.37}$$

Because both w and v have the same initial value at the time s, the assumptions made in *Proposition 4.5* together with (4.37) lead to the commutativity of $A(t)$ and $\Phi_{t,s}$. *q.e.d.*

Remark 4.6: One of the prerequisites in *Proposition 4.5* is that the matrix $K = L + \dot{A}A^{-1}$ has to define a hyperbolic system, where L is already assumed to be hyperbolic. We want to remark that for a symmetrizable L and for an order zero invertible operator A K will be always symmetrizable and therefore still hyperbolic: The requirement $K + K^\dagger \in \mathrm{OPS}^0$ for symmetrizable systems $\partial_t u = Ku$ only depends on the principal part of K since $K + K^\dagger$ will be of order zero iff the principal part (=order 1 part) of K is anti-self adjoint, i.e. $K_1^\dagger = -K_1$. When we assume A to be of order zero, this will be also true for its invers and for the product $\dot{A}A^{-1}$. But then $K = L + \dot{A}A^{-1}$ is a lower order perturbation of L, which means that K will be symmetrizable hyperbolic as long as L itself is symmetrizable.

Concerning condition (4.32) one could ask, whether this condition takes a simpler form in the case of a scalar equation $\partial_t u = Lu$: This can be motivated by *Egorov´s Theorem*, because for scalar operators (4.32) simply means that the operator $\dot{A}A^{-1}$ must be invariant under the Hamiltonian flow associated to L. In fact, one can derive a proposition in the scalar case, which is directly connected to these ideas:

Proposition 4.6: *Let $L \in \mathrm{OPS}^1$ define the scalar hyperbolic equation $\partial_t u = Lu$. Then, up to a lower order error, an (possibly time dependent) operator A commutes with the solution operator $\Phi_{s,t}$ to $\partial_t u = Lu$, as long as $A(t)$ commutes with $L(t)$ for all t.*

Proof: Assume that $A(t)$ commutes with $L(t)$ for all times t and t, i.e.

$$A(t)L(t) - L(t)A(t) = 0 \quad \forall t. \tag{4.38}$$

By multiplying the desired equation $A(t)\Phi_{s,t} = \Phi_{s,t}A(t)$ from the left with $\Phi_{s,t}^{-1} = \Phi_{t,s}$, *Proposition 4.6* would be true iff

$$\Phi_{t,s}A(t)\Phi_{s,t} = \Phi_{t,s}\Phi_{s,t}A(t) = A(t), \tag{4.39}$$

where the above equation has to be read modulo lower order errors. Using a version of Egorov´s theorem, which can be found in ([7], p.147-150) we see that (4.39) is true as long as the principal symbol of $A(t)$ is invariant under the Hamiltonian flow along the vector field $H_{\sigma_p(L)}$, which is the Hamiltonian vector field associated to $\sigma_p(L)$. Taking the symbol on both sides of (4.38) leads to $\sigma(A(t)L(t) - L(t)A(t)) = 0$. Note that both products $A(t)L(t)$ and $L(t)A(t)$ are operators of order $m+1$ and by *Remark 1.2* the principal symbol $\sigma_p(A(t)L(t) - L(t)A(t))$ has to be zero. We can use (1.15) to deduce that the order m part of the symbol $\sigma(A(t)L(t) - L(t)A(t))$ must be equal to

$$\sigma_m(A(t)L(t) - L(t)A(t)) = H_{\sigma_p(L)}\sigma_p(A)(t). \tag{4.40}$$

Since we want the commutator $[A(t), L(t)]$ to vanish completely, (4.40) has also to be zero. But this means that the derivative of $\sigma_p(A)(t)$ along the vector field $H_{\sigma_p(L)}$ vanishes for all times, which means that $\sigma_p(A)(t)$ is conserved along the integral curves of this vector field. *q.e.d.*

Let us take a closer look on scalar strictly hyperbolic equations like in *Proposition 4.3:* As already indicated in *Chapter* 3, where we solved the wave equation by using the Fourier transform, we would expect the solution operators to scalar equation of the form $\partial_t u = iLu$ to be of Fourier integral type. It turns out that, up to a compact error this is the case. We want to only state the solution theorem here, further details on the construction may be read in *Taylor*, ([7], *Chapter VIII, Theorem 3.1*).

Theorem 4.1: *Consider the scalar hyperbolic initial value problem*

$$(i)\ \partial_t u = iLu, \quad (ii)\ u(0) = g_0 \in H^s(\Sigma_0) \tag{4.41}$$

for $L \in \mathrm{OPS}^1$ *with real principal symbol* $\sigma_p(L)$. *There exists a time interval* $I = [0, r]$ *such that the solution to* (i)-(ii) *is given by*

$$u(t) = \int e^{i\phi(t,x,\xi)} a(t,x,\xi)\hat{g}_0(\xi) d\xi \tag{4.42}$$

up to a smooth error. The phase ϕ *and amplitude* a *of the Fourier integral operator in* (4.38) *are determined by the conditions*

$$\partial_t \phi(t,x,\xi) = \sigma_p(L)(t,x,\nabla_x \phi(t,x,\xi)), \quad \phi(0) = x\xi \tag{4.43}$$

$$\left(\partial_t - \sum_k \frac{\partial \sigma_p(L)}{\partial_{\xi_k}} \frac{\partial}{\partial_{x_k}}\right) a - \left(iL_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!} \sigma_p(L)^{(\alpha)} \phi_\alpha\right) a = 0, \quad a(0) = 1, \tag{4.44}$$

where L_1 *and* L_0 *are the first and zero order parts of* L. *Conditions* (4.42)-(4.43) *are called the eikonal equation and transport equation. Note that* $\phi \in S^1$ *and* $a \in S^0$ *may be chosen to be of principal type,* $a = a_0$.

4.5 Some Remarks, Examples and Special Solution Theory

Remark 4.7: Remember Egorov´s theorem from the section about Fourier integral operators: The conjugation of any pseudo A of order m by a Fourier integral operator Φ is again a pseudo of order m. The principal symbol of the new operator is the principal symbol of A evaluated on the time $t = 1$ inverse canonical transformation associated to Φ. We see from (4.42) that solution operators to scalar hyperbolic equations are Fourier integral operators. Therefore one might ask, what would happen, if one considers conjugation of a pseudo A by such a solution operator. It turns out that the canonical transformation associated to (4.42) is the Hamiltonian flow along the Hamiltonian vector field generated by $\sigma_p(L)$. But this means that conjugations of operators $A \in \mathrm{OPS}^m$ by the solution operators to scalar hyperbolic equations depend only on the principal part of the equation.

Remark 4.8: If $(4.41), (i)$ is an equation involving Shubin operators, one can take the Fourier integral ansatz (4.42) with ϕ and a being of Shubin type. One then can derive the same conditions for ϕ and a like in *Theorem 4.1*.

Remark 4.9: In principle (4.41) is a special case of the general hyperbolic initial value problem (4.4) with $f = 0$. Because $f = 0 \in L^2(\mathbb{R}, H^s(M))$ the *exact* solution to (4.41) should be defined for all $t \in \mathbb{R}$. So, why did we define the compact interval $I = [0, r]$, on which the approximate solution has the form given by (4.42)-(4.44)? The problem is that in general the eikonal equation (4.43) is only solvable for small t. Therefore the construction of an approximate solution via *Theorem 4.1* might only work for a finite time length $\hat{t}$, but this time length is at least as long as the existence interval of the eikonal equation (4.43).

Let us come back to *Lemma 4.1.* For $S = 0$ we would need to solve n decoupled scalar equations, which can be done just as explained in *Theorem 4.1* for every component, resulting in a diagonal solution operator. The problem is that $S \in \mathrm{OPS}^0$ is in general not of negative order. From (4.44) we derive that the zero order part of L is involved in the construction of the solution operator of scalar equations, which means that the matrix S may also influence the solution of a system like (4.15). We want to take advantage of the solution theory for systems similar to (4.15) developed by *Kumano-go*, see ([1], *Chapter 10, p.313-327*) . It will help us to see that for compact manifolds we in fact need to consider only the diagonal part D in (4.15) as long as the entries of D behave in a certain way:

Theorem 4.2: *Let $\partial_t u = Lu$ define a hyperbolic system, which is equivalent to the system $\partial_t v = Dv + Sv$ by some change of variables $v := T^{-1}u$. If the eigenvalues λ_j of $\sigma_p(L)$ are imaginary with constant multiplicities and also satisfy the distance condition*

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq C|\xi|, \quad (t,x,\xi) \in [0,T] \times M \times \{|\xi| \geq R\} \tag{4.45}$$

for some constants $C > 0$ and $R > 0$, the equation $\partial_t v = Dv + Sv$ has a solution operator which can be written as

$$\Phi_t = N\Phi_D Q, \;\; N = 1 + \sum_{\nu=1}^{\infty} N^\nu, N^\nu \in \mathrm{OPS}^{-\nu} \tag{4.46}$$

where Q is a parametrix to N and Φ_D is the solution operator to the diagonal system $\partial_t w = Dw$.

Remark 4.10: The case of a strictly hyperbolic system being equivalent to a system like in (4.15) is a special case of *Theorem 4.2* where the multiplicity of each eigenvalue is given by 1. For systems where the eigenvalues λ_j have multiplicities $k_j > 1$ the diagonal operator Φ_D from (4.46) will consist of blocks with block size equal to the multiplicity of each eigenvalue. In this case (multiplicity of eigenvalues greater 1 but constant) each block of size k_j is made of k_j identical entries solving $\partial_t u_j = \lambda_j u_j$. An example for an operator L, where $\sigma_p(L)$ has constant multiplicities which can be greater than one, is given by the Dirac operator on Σ_t, $L(t) = \mathcal{D}(t)$, see ([10]). The case of non constant multiplicities is much more complicated. For further details on the solution theory of systems with non constant multiplicities, see ([1], *Chapter 10, p.360-370)*

5 Projections

In this chapter we want to collect some useful properties and theorems about projections in general. The main theorems in this thesis will be statements about the Fredholm conditions for operators, which arise from nonlocal boundary conditions of hyperbolic equations and since these boundary conditions are given by pseudodifferential projections, we will use the results and facts stated in this chapter later.

5.1 Properties of General Projections

Let $(V, \langle\cdot,\cdot\rangle)$ be any finite or infinite dimensional vector space with an inner product. A projection is a map $P : V \to V$ with $P^2 = P$. The following three terms are important and will be used from this point on:

Definition 5.1: *Let $P : V \to V$ be a projection. We call P*

(i) pseudodifferential, if $V = \mathcal{H}$ is a Hilbert space on which a pseudodifferential calculus can be defined and if $P \in OPS^0$,
(ii) orthogonal, if $P^\dagger = P$ and
(iii) complementary to another projection Q, if $Im(P) + Im(Q) = V$.

Let us now collect some facts concerning projections which fulfill one or more of the three conditions $(i) - (iii)$. First of all, given any projection P there always exists a complementary projection, namely for example $Q = 1 - P$. If $Q = 1 - P$ we call it the *strong complementary* projection to P. This definition is motivated by the fact that Q is complementary to P but has even more useful properties:

Proposition 5.1: *Let $P : V \to V$ be a projection. Then $Q = 1 - P$ is a complementary projection and we have $Ker(Q) = Im(P), Im(Q) = Ker(P)$.*

Proof: First of all, $Q^2 = (1-P)(1-P) = 1 - P - P + P^2 = 1 - 2P + P = 1 - P$, thus Q is a projection. We obviously have $v = Pv + (1 - P)v, \forall v \in V$ meaning $\mathrm{Im}(P + Q) = V$ and since $\mathrm{Im}(P + Q) \subset \mathrm{Im}(P) + \mathrm{Im}(Q)$ we get that Q is complementary. Moreover, let be $v \in \mathrm{Ker}(Q)$. This means $(1 - P)v = 0$ which is equivalent to $v = Pv$ thus v being in the image of P. On the other hand, for any arbitrary vector v, there is $QPv = (1 - P)Pv = (P - P^2)v = (P - P)v = 0$, meaning that the vector Pv which is in the image of P, is also contained in the kernel of Q. The last statement $\mathrm{Im}(Q) = \mathrm{Ker}(P)$ follows similarly. *q.e.d.*

The proof for the statement $\mathrm{Ker}(Q) = \mathrm{Im}(P)$ uses the fact $QP = 0$. Although it seems to be intuitive, it is in general not true that $QP = 0$ implies the orthogonality of the images $\mathrm{Im}(Q)$ and $\mathrm{Im}(P)$ with respect to the inner product $\langle\cdot,\cdot\rangle$. In order to characterize orthogonality of images, we need the orthogonality in condition (ii):

Proposition 5.2: *Let P_1 and P_2 be two orthogonal projections, $P_1^\dagger = P_1$ and $P_2^\dagger = P_2$. Then $P_1P_2 = 0$ iff $Im(P_1) \perp Im(P_2)$.*

Proof: Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Because $P_1P_2 = 0$, we have

$$0 = \langle P_1P_2v, w\rangle = \langle P_2v, P_1^\dagger w\rangle = \langle P_2v, P_1w\rangle \ \forall (v, w) \in V \times V \tag{5.1}$$

which means $\mathrm{Im}(P_1) \perp \mathrm{Im}(P_2)$. Reading (5.1) from right to left, we see that $\mathrm{Im}(P_1) \perp \mathrm{Im}(P_2)$ also implies $P_1P_2 = 0$. *q.e.d.*

Thus we can call two orthogonal projections P_1, P_2 *orthogonal to each other* as long as $P_1P_2 = 0$ or $P_2P_1 = 0$. What can we say about two projections, if both equations $P_1P_2 = 0$ and $P_2P_1 = 0$ hold? The following proposition allows a nice decomposition of any vector $v \in V$ into its parts lying in the images of P_1 and P_2:

Proposition 5.3: *Consider two projections P_1, P_2, which are not necessary orthogonal projections but satisfy $P_1P_2 = P_2P_1 = 0$ and are complementary. Then any vector $v \in V$ has the decomposition*

$$v = P_1v + P_2v, \tag{5.2}$$

meaning that in this case P_2 is a strong complementary projection to P_1.
Proof: The complementarity of P_1 and P_2 allows the decomposition $v = P_1\tilde{v} + P_2\tilde{w}$. Applying P_1 on both sides of this equation leads to

$$P_1v = P_1^2\tilde{v} + P_1P_2\tilde{w} \ \Leftrightarrow \ P_1v = P_1\tilde{v}. \tag{5.3}$$

Similarly, applying P_2 generates $P_2v = P_2\tilde{w}$ and altogether we get $v = P_1\tilde{v} + P_2\tilde{w} = P_1v + P_2v$. *q.e.d.*

The most important operators in the main chapter of this work will be of the form $\Phi_1 A \Phi_1^{-1}$ with a pseudodifferential matrix A and some automorphism Φ_1 on a Hilbert space. These operators arise from boundary conditions given by certain projections P_1 and P_2 and their conjugations $\Phi_1 P_1 \Phi_1^{-1}$ and $\Phi_1 P_2 \Phi_2^{-1}$. Now we want to state some properties of these conjugations:

Proposition 5.4: *Given any projection $P : V \to V$ and an automorphism $\Phi : V \to V$ the operator $\Phi P \Phi^{-1}$ is again a projection.*

proof: We compute

$$(\Phi P \Phi^{-1})^2 = \Phi P(\Phi^{-1}\Phi)P\Phi^{-1} = \Phi P^2 \Phi^{-1} = \Phi P \Phi^{-1}, \tag{5.4}$$

thus $\Phi P \Phi^{-1}$ is a projection as long as P is. *q.e.d.*

Moreover, if the endomorphism Φ is unitary, we have the following corollary:

Corollary 5.1:
(i) If the endomorphism Φ from Proposition 5.4 is not only invertible but even unitary, $\Phi^\dagger = \Phi^{-1}$, it preserves orthogonality of images, thus for P_1, P_2 being two projections orthogonal to each other (meaning $Im(P_1) \perp Im(P_2)$ but P_1, P_2 not necessarily have to be orthogonal themselves) their conjugations $\Phi P_1 \Phi^{-1}$ and $\Phi P_2 \Phi^{-1}$ are projections which are still orthogonal to each other.
(ii) If Φ is unitary and P is an orthogonal projection, $\Phi P \Phi^{-1}$ is also an orthogonal projection.

Proof: (i) That the conjugations $\Phi P_{1/2} \Phi^{-1}$ are projections has already been proven in *Proposition 5.4*. Now assume that P_1 and P_2 have orthogonal images, thus $\langle P_1 v, P_2 w\rangle = 0$, $\forall (v,w) \in V \times V$. The inner products of the conjugated images are given by

$$\langle \Phi P_1 \Phi^{-1} v, \Phi P_2 \Phi^{-1} w\rangle = \langle P_1 \Phi^{-1} v, \Phi^\dagger \Phi P_2 \Phi^{-1} w\rangle = \langle P_1 \Phi^{-1} v, \Phi^{-1} \Phi P_2 \Phi^{-1} w\rangle \tag{5.5}$$

$$= \langle P_1 \Phi^{-1} v, P_2 \Phi^{-1} w\rangle = 0, \tag{5.6}$$

since P_1 and P_2 are assumed to be orthogonal to each other.
(ii) Let $P = P^\dagger$ be an orthogonal projection and Φ be unitary. Then there is

$$(\Phi P \Phi^{-1})^\dagger = (\Phi^{-1})^\dagger P^\dagger \Phi^\dagger = \Phi P \Phi^{-1}, \tag{5.7}$$

meaning that $\Phi P \Phi^{-1}$ is still orthogonal. *q.e.d.*

5.2 Properties of Pseudodifferential Projections

Before stating the main theorem of this section which will be used to derive necessary conditions for some operators in the main section of this work to be Fredholm, we want to adress a few facts about projections fulfilling (i) in *Definition 5.1*, namely pseudodifferential projections. Since pseudodifferential operators of order zero on a compact manifold are uniquely characterized by their principal symbol, it is helpful to know some properties of the symbol of a pseudodifferential projection. Some of them are the following:

Proposition 5.5: *Assume that P is some pseudodifferential projection, acting on sections in a vector bundle E with finite rank over a manifold M. Then its principal symbol $\sigma_p(P) = \sigma_p(P)(x,\xi)$ is a matrix valued projection for each $(x,\xi) \in T^*M$.*

Proof: Because for any two pseudodifferential operators A_1, A_2 the principal symbol is multiplicative, $\sigma_p(A_1 A_2) = \sigma_p(A_1)\sigma_p(A_2)$ we have

$$\sigma_p(P) = \sigma_p(PP) = \sigma_p(P)\sigma_p(P) = \sigma_p(P)^2, \ \forall (x,\xi) \in TM^*. \quad q.e.d. \tag{5.8}$$

Remark 5.2: If P is acting on scalar functions, equation (5.8) is an equation for a scalar symbol and therefore there must be $\sigma(P)(x,\xi) \in \{0,1\}, \forall (x,\xi) \in T^*M$. This means that interesting pseudodifferential projections only arise if the rank of the vector bundle E is greater than 1.

Now that we know that the principal symbol of a pseudodifferential projection is always a projection we can ask, whether the converse statement is also true. It turns out that the converse statement in fact holds up to a compact error:

Proposition 5.6: *Let $\sigma_p(P)$ be the principal symbol of a pseudodifferential operator $P \in \mathrm{OPS}^0$ acting on L^2-sections of a vector bundle E over a compact manifold M. If $\sigma_p(P)$ is a projection, $\sigma_p(P)^2 = \sigma_p(P)$, then P itself must be a projection up to a compact error, meaning $P^2 = P + K$, with a compact operator K.*

Proof: Because P is of order zero, so is its square P^2. Using the multiplicity of the principal symbol, $\sigma_p(P)^2 = \sigma_p(P)$ is equivalent to $\sigma_p(P^2 - P) = 0$. But then $P^2 - P$ is actually not of order zero but even of order -1, $P^2 - P = K \in \mathrm{OPS}^{-1}$. Since such an operator is compact on a compact manifold, the above statement is true. *q.e.d.*

At last we want to give a corollary which tells the conditions under which two given pseudodifferential projections P_1 and P_2 have the property that their sum $(P_1 + P_2)$ or their product P_1P_2 is also a pseudodifferential projection, neglecting compact perturbations:

Corollary 5.2: *For two arbitrary pseudodifferential projections P_1, P_2 of order zero the following statements hold:*
(i) The products P_1P_2 and P_2P_1 are pseudodifferential projections up to a compact error, as long as $\sigma_p(P_1)\sigma_p(P_2) = \sigma_p(P_2)\sigma_p(P_1)$,
(ii) If $\sigma_p(P_1)\sigma_p(P_2) = -\sigma_p(P_2)\sigma_p(P_1)$, the sum (P_1+P_2) is a pseudodifferential projection up to a compact error.

Proof: (i) First of all, P_1P_2 and P_2P_1 are pseudodifferential operators of order zero by the composition theorem for pseudodifferential operators. Let us take a look at the symbols, assuming $\sigma_p(P_1)\sigma_p(P_2) = \sigma_p(P_2)\sigma_p(P_1)$:

$$\sigma_p(P_1P_2)^2 = \sigma_p(P_1P_2)\sigma_p(P_1P_2) = \sigma_p(P_1)(\sigma_p(P_2)\sigma_p(P_1))\sigma_p(P_2) \tag{5.9}$$

$$= \sigma_p(P_1)(\sigma_p(P_1)\sigma_p(P_2))\sigma_p(P_2) \tag{5.10}$$

$$= \sigma_p(P_1)^2\sigma_p(P_2)^2 = \sigma_p(P_1)\sigma_p(P_2) = \sigma_p(P_1P_2). \tag{5.11}$$

Therefore $\sigma_p(P_1P_2)$ is a projection and using *Proposition 5.6* we get that P_1P_2 itself has

to be a projection up to a compact error. The proof for $P_2 P_1$ is analogous.
(ii) Let the symbols anticommute, $\sigma_p(P_1)\sigma_p(P_2) = -\sigma_p(P_2)\sigma_p(P_1)$. Then:

$$\sigma_p(P_1 + P_2)^2 = (\sigma_p(P_1) + \sigma_p(P_2))^2 = (\sigma_p(P_1) + \sigma_p(P_2))(\sigma_p(P_1) + \sigma_p(P_2)) \tag{5.12}$$

$$= \sigma_p(P_1)^2 + \sigma_p(P_1)\sigma_p(P_2) + \sigma_p(P_2)\sigma_p(P_1) + \sigma_p(P_2)^2 = \sigma_p(P_1)^2 + \sigma_p(P_2)^2 \tag{5.13}$$

$$= \sigma_p(P_1) + \sigma_p(P_2) = \sigma_p(P_1 + P_2), \tag{5.14}$$

which means that $\sigma_p(P_1 + P_2)$ is a projection. Again, it follows from *Proposition 5.6* that $P_1 + P_2$ is a projection up to a compact perturbation. *q.e.d.*

5.3 Important Theorems

After stating these useful facts about pseudodifferential projections and projections in general, we now want to prove a theorem which will be useful later to derive how certain projections have to be connected in order that some operator associated with this projections can be of Fredholm type. Let P_1, P_2 be to arbitrary projections. We consider the two operators

$$\mathcal{D} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} : V \to V \times V, \quad D(v) = \begin{pmatrix} P_1 v \\ P_2 v \end{pmatrix}, \tag{5.15}$$

$$\mathcal{P} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} : V \times V \to V \times V. \tag{5.16}$$

Note that we have $Im(\mathcal{P}) = \mathrm{Im}(P_1) \times \mathrm{Im}(P_2)$ and that $\mathcal{P}$ is a projection on the vector space $V \times V$. Obviously, there is always $\mathrm{Im}(\mathcal{D}) \subset \mathrm{Im}(\mathcal{P})$. We want to investigate under which conditions we have equality of the two images, or at least equality up to a finite dimensional error. The answer to this question is given by the two following theorems:

Theorem 5.1: *Let $P_1, P_2 : V \to V$ be two arbitrary projections. Then we have*

$$Im(\mathcal{D}) = Im(\mathcal{P}) \quad \Leftrightarrow \quad Ker(P_1) + Ker(P_2) = V. \tag{5.17}$$

Proof: $\Leftarrow$: $\mathrm{Ker}(P_1) + \mathrm{Ker}(P_2) = V$ means that we can write $v = v_1 + v_2$ with $v_1 \in \mathrm{Ker}(P_1)$ and $v_2 \in \mathrm{Ker}(P_2)$ for any $v \in v$. So, consider $(v, w) \in V \times V$ and let $v = v_1 + v_2$, $w = w_1 + w_2$ be the corresponding decompositions. Since $\mathrm{Im}(\mathcal{D}) \subset \mathrm{Im}(\mathcal{P})$ is a trivial statement which is always true, we only need to show $\mathrm{Im}(\mathcal{P}) \subset \mathrm{Im}(\mathcal{D})$. If a point is lying in $\mathrm{Im}(\mathcal{P})$ we may use the decompositions above and write

$$\begin{pmatrix} P_1 v \\ P_2 w \end{pmatrix} = \begin{pmatrix} P_1(v_1 + v_2) \\ P_2(w_1 + w_2) \end{pmatrix} = \begin{pmatrix} P_1 v_2 \\ P_2 w_1 \end{pmatrix}. \tag{5.18}$$

Now define the vector $x := w_1 + v_2$. Then we have $P_1x = P_1v_2$ and $P_2x = P_2w_1$. But this actually means that

$$\begin{pmatrix} P_1v \\ P_2w \end{pmatrix} = \begin{pmatrix} P_1x \\ P_2x \end{pmatrix} \tag{5.19}$$

and therefore $\mathrm{Im}(\mathcal{P}) \subset \mathrm{Im}(\mathcal{D})$.
$\Rightarrow$: Assume that $\mathrm{Ker}(P_1) + \mathrm{Ker}(P_2) \neq V$. Then there exists some vector $v \in V$ such that for any $v_1 \in \mathrm{Ker}(P_1)$ and $v_2 \in \mathrm{Ker}(P_2)$ $v \neq v_1 + v_2$. Next consider

$$\begin{pmatrix} P_1v \\ P_2 \cdot (2v) \end{pmatrix} \in \mathrm{Im}(\mathcal{P}). \tag{5.20}$$

We want to assume that $\mathrm{Im}(\mathcal{D}) = \mathrm{Im}(\mathcal{P})$, although $\mathrm{Ker}(P_1) + \mathrm{Ker}(P_2) \neq V$. Then there exists an $x \in v$ with $P_1x = P_1v$ and $P_2x = P_2 \cdot (2v)$. These two equations are equivalent to $P_1(x-v) = 0$ and $P_2(x-2v) = 0$, meaning that $(x-v) \in \mathrm{Ker}(P_1)$ and $(x-2v) \in \mathrm{Ker}(P_2)$. But then $v = (x-v) - (x-2v)$ and v would be written as a sum of elements in the kernels of P_1 and P_2. This is a contradiction, so we have $\mathrm{Im}(\mathcal{D}) \neq \mathrm{Im}(\mathcal{P})$. *q.e.d.*

Remark 5.3: For the proof of *Theorem 5.1* we have not used the fact that P_1, P_2 are projections. So in fact *Theorem 5.1* is also valid for arbitrary linear maps $A_1, A_2 : V \to V$. The reason why we stated the theorem in such a form is because in the main chapter of this thesis we will need a similar theorem, which is a version of *Theorem 5.1* with finite dimensional errors. And in order to prove this finite dimensional error version, we need the defining property of projections:

Theorem 5.2: *Given two arbitrary projections $P_1, P_2 : V \to V$ it is possible to find two finite dimensional subspaces F and J, such that:*

$$\mathrm{Im}(\mathcal{P}) = \mathrm{Im}(\mathcal{D}) + F \tag{5.21}$$

is equivalent to

$$\mathrm{Ker}(P_1) + \mathrm{Ker}(P_2) + J = V. \tag{5.22}$$

Proof: Up to some modifications with finite dimensional spaces, the proof of *Theorem 5.2* is similar to the proof of *Theorem 5.1*.
$\Leftarrow$: Assume one can write $v = v_1 + v_2 + v_j$, with $v_1 \in \mathrm{Ker}(P_1), v_2 \in \mathrm{Ker}(P_2), v_j \in J$ and $\dim(J) < \infty$. Then

$$\begin{pmatrix} P_1v \\ P_2w \end{pmatrix} = \begin{pmatrix} P_1(v_1 + v_2 + v_j) \\ P_2(w_1 + w_2 + w_j) \end{pmatrix} = \begin{pmatrix} P_1v_2 + P_1v_j \\ P_2w_1 + P_2w_j \end{pmatrix}. \tag{5.23}$$

Again, as in the proof of *Theorem 5.1*, we can chose $x := w_1 + v_2$ and again $P_1x = P_1v_2$ and $P_2x = P_2w_1$. This and equation (5.21) combined give

$$\begin{pmatrix} P_1 v \\ P_2 w \end{pmatrix} = \begin{pmatrix} P_1 x \\ P_2 x \end{pmatrix} + \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}, \tag{5.24}$$

with $\tilde{v} = P_1 v_j, \tilde{w} = P_2 w_j$. We have $(\tilde{v}, \tilde{w}) \in \mathrm{Im}(P_1 |_J) \times \mathrm{Im}(P_2 |_J)$ and since J is assumed to be finite dimensional, the vector $(\tilde{v}, \tilde{w})$ also lies in a finite dimensional space.
$\Rightarrow$: Let $v \in V$ be any vector. Assuming that we can write

$$\begin{pmatrix} P_1 v \\ P_2 \cdot (2v) \end{pmatrix} = \begin{pmatrix} P_1 x \\ P_2 x \end{pmatrix} + \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}, \ (\tilde{v}, \tilde{w}) \in F. \tag{5.25}$$

Read componentwise the corresponding equations are

$$(i) \ P_1 v = P_1 x + \tilde{v}, \quad (ii) \ P_2 \cdot (2v) = P_2 x + \tilde{w}. \tag{5.26}$$

We apply P_1 on both sides of (i) and P_2 on both sides of (ii) und using $P_1^2 = P_1, P_2^2 = P_2$ (here we are using that P_1, P_2 are projections) we arrive at

$$P_1(v - x - \tilde{v}) = 0, \quad P_2(2v - x - \tilde{w}) = 0. \tag{5.27}$$

But this again means $(v - x - \tilde{v}) \in \mathrm{Ker}(P_1)$ and $(2v - x - \tilde{w}) \in \mathrm{Ker}(P_2)$. Combining this gives

$$v = (2v - x - \tilde{w}) - (v - x - \tilde{v}) + (\tilde{w} - \tilde{v}). \tag{5.28}$$

If one now defines the map $(1, -1) : V \times V \to V, \ (1, -1)(v, w) = v - w$, we immediately see that v can be written as $v = v_1 + v_2 + z$, with $v_1 \in \mathrm{Ker}(P_1), \ v_2 \in \mathrm{Ker}(P_2)$ and $z \in (1, -1) |_F$ and we have $\dim((1, -1) |_F) < \infty$. *q.e.d.*

Remark 5.4: Remembering *Proposition 5.1* we could also have written *Theorem 5.2* in the form

$$Im(\mathcal{P}) = Im(\mathcal{D}) + F \quad \Leftrightarrow \quad Im(1 - P_1) + Im(1 - P_2) + J = V, \tag{5.29}$$

which will turn out to be more useful later.

At last we want to explain how the Fredholm property of some pseudodifferential (or G-type) operator A is connected to ellipticity when A is an operator acting between the images of two projections P_0 and P_1. For more details about this topic, see *Seiler*, ([9], *Chapter 2 - Chapter 3*).

Theorem 5.3: *Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be two Hilbert spaces and $A : Im(P_0) \to Im(P_1)$ be a G-operator acting between the images of two projections $P_0 : \mathcal{H}_0 \to \mathcal{H}_0$ and $P_1 : \mathcal{H}_1 \to \mathcal{H}_1$. Then A is Fredholm iff there exists a G-operator A^p with*

$$\sigma_p(A^p)\sigma_p(A) = \sigma_p(P_0), \quad \sigma_p(A)\sigma_P(A^p) = \sigma_p(P_1). \tag{5.30}$$

6 Compact and Small Perturbations

When dealing with Fredholm operators and its index theory, compact perturbations of certain objects often do not change the index or Fredholm property of the operator in question. As already stated in the section about Fredholm operators, for any Fredholm operator A and a compact operator K, the operator $A + K$ is again Fredholm with the same index. However, if the operator A arises from other operators containing pseudodifferential projections or solution operators to hyperbolic equations, it is not always clear that compact perturbations of these operators result in only compact perturbations of the operator A itself. The same holds for norm small perturbations. In this section we want to give a few examples where certain properties of operators, for example as being unitary, are invariant under perturbations by compact or norm small errors. We will need these examples later to see how certain Fredholm operators have an invariant index exactly under this perturbations.

6.1 Types of Perturbations

As a first simple example, let us see how a projection operator on a Hilbert space changes when adding a compact error:

Proposition 6.1: *Let $P : \mathcal{H} \to \mathcal{H}$ be a projection on a Hilbert space $\mathcal{H}$. If $K : \mathcal{H} \to \mathcal{H}$ is a compact operator, $(P + K)$ is a projection up to a compact error, meaning $(P + K)^2 = (P + K) + \tilde{K}$.*

Proof: Calculating the square of $P + K$ results in

$$(P+K)^2 = (P+K)(P+K) = P^2 + PK + KP + K^2 = P + PK + KP + K^2 \quad (6.1)$$

$$= (P+K) + (PK + KP + K^2 - K) = (P+K) + \tilde{K} \quad (6.2)$$

when we set $\tilde{K} = PK + KP + K^2 - K$. Note that $\tilde{K}$ is in fact a compact operator since the set of compact operators $K(\mathcal{H})$ is a Banach ideal. *q.e.d*

Remark 6.1: We could have replaced the compact operator K in *Proposition 6.1* by a norm small operator, and an analogous statement like *Proposition 6.1* would hold: Given any norm small operator ϵA we would have $(P + \epsilon A)^2 = (P + \epsilon A) + \epsilon\tilde{A}$.

The solution operators to hyperbolic equations are often unitary and unitarity is a very useful property since inverses can be calculated more easily. Requiring that for a unitary operator Φ the compact perturbed operator $\Phi + K$ is unitary again is a difficult task, because in general $\Phi + K$ does not have to be even invertible. But it turns out that $(\Phi + K)^\dagger$ is at least a parametrix for $\Phi + K$:

Proposition 6.2: *Consider a unitary operator $\Phi : \mathcal{H} \to \mathcal{H}$, $\Phi^\dagger = \Phi^{-1}$ and a compact operator K. Then $\Phi + K$ is unitary up to a compact error, in that sense, that $(\Phi + K)^\dagger$ is a left and a right parametrix for $\Phi + K$.*

proof: We compute both products, the product with $(\Phi + K)^\dagger$ from left and from right:

$$(\Phi + K)(\Phi + K)^\dagger = (\Phi + K)(\Phi^\dagger + K^\dagger) = (\Phi + K)(\Phi^{-1} + K^\dagger) \tag{6.3}$$

$$= \Phi\Phi^{-1} + \Phi K^\dagger + K\Phi^{-1} + KK^\dagger = 1 + \Phi K^\dagger + K\Phi^{-1} + KK^\dagger = 1 + K_R, \tag{6.4}$$

with $K_R = \Phi K^\dagger + K\Phi^{-1} + KK^\dagger$ being compact. Therefore $(\Phi + K)^\dagger$ is a right parametrix for $\Phi + K$. Similarly,

$$(\Phi + K)^\dagger(\Phi + K) = 1 + \Phi^{-1}K + K^\dagger\Phi + K^\dagger K = 1 + K_L, \tag{6.5}$$

thus it is also a left parametrix. *q.e.d.*

For norm small errors, we can use that if ϵ is small enough, $\Phi + \epsilon A$ is still invertible for a unitary operator Φ, because the set of invertible operators is open. In this case we get an even better statetement than *Proposition 6.2*:

Proposition 6.3: *Assume that Φ is any unitary operator on a Hilbert space $\mathcal{H}$ and $\Phi + \epsilon A$ is a norm small perturbation of Φ, where the parameter ϵ is chosen such small that $\Phi + \epsilon A$ is still invertible. Then $\Phi + \epsilon A$ is unitary up to a norm small error, meaning $(\Phi + \epsilon A)^{-1} = (\Phi + \epsilon A)^\dagger + \epsilon\tilde{A}$.*

Proof: We have

$$(\Phi + \epsilon A)(\Phi + \epsilon A)^\dagger = (\Phi + \epsilon A)(\Phi^\dagger + \epsilon A^\dagger) = (\Phi + \epsilon A)(\Phi^{-1} + \epsilon A^\dagger) \tag{6.6}$$

$$= \Phi\Phi^{-1} + \epsilon\Phi A^\dagger + \epsilon A\Phi^{-1} + \epsilon^2 AA^\dagger = 1 + \epsilon(\Phi A^\dagger + A\Phi^{-1} + \epsilon AA^\dagger). \tag{6.7}$$

Now, since $(\Phi + \epsilon A)$ is assumed to be still invertible, we can insert $1 = (\Phi + \epsilon A)(\Phi + \epsilon A)^{-1}$ into equation (6.7):

$$(\Phi + \epsilon A)(\Phi + \epsilon A)^\dagger = (\Phi + \epsilon A)(\Phi + \epsilon A)^{-1} + \epsilon(\Phi A^\dagger + A\Phi^{-1} + \epsilon AA^\dagger). \tag{6.8}$$

Multiplying this by $(\Phi + \epsilon A)^{-1}$ from the left the adjoint becomes

$$(\Phi + \epsilon A)^\dagger = (\Phi + \epsilon A)^{-1} + \epsilon(\Phi + \epsilon A)^{-1}(\Phi A^\dagger + A\Phi^{-1} + \epsilon AA^\dagger) \tag{6.9}$$

$$= (\Phi + \epsilon A)^{-1} + \epsilon\tilde{A}. \quad \textit{q.e.d.} \tag{6.10}$$

6.2 Theorems about (almost) Complementarity

Later on we will face some situations, where sums of the images of projections, which are almost complementary, are considered. Let us state three theorems concerning this topic:

Theorem 6.1: *Let P_1, P_2 be strongly complementary projections on a Hilbert space $\mathcal{H}$, $P_2 = 1 - P_1$. Then, for any compact operator K, there exists a finite dimensional space J with*

$$Im(P_1) + Im(P_2 + K) + J = \mathcal{H}. \tag{6.11}$$

Proof: Let us write down the sum $\mathrm{Im}(P_1) + \mathrm{Im}(P_2 + K)$:

$$\mathrm{Im}(P_1) + \mathrm{Im}(P_2 + K) = \{P_1 v + (P_2 + K)w, (v, w) \in V \times V\} \tag{6.12}$$

$$= \{P_1 v + P_2 w + Kw, (v, w) \in V \times V\} \tag{6.13}$$

$$= \{P_1 v + P_2 v + Kv, v \in V\} \cup \{P_1 v + P_2 w + Kw, w \neq v\} = \{v + Kv, v \in v\} \cup U, \tag{6.14}$$

where we used $P_1 v+P_2 v = v$ and set $U = \{P_1 v+P_2 w+Kw, w \neq v\}$. Furthermore we have $\{v+Kv, v \in v\} = Im(1+K)$ and therefore $\mathrm{Im}(1+K) \subset (\mathrm{Im}(P_1)+\mathrm{Im}(P_2+K))$. Now $1+K$ is a Fredholm operator with index 0. This means that the orthogonal complement of its image, $\mathrm{Im}(1+K)^\perp$ is finite dimensional. Because $\mathrm{Im}(P_1)+\mathrm{Im}(P_2+K)$ contains $\mathrm{Im}(1+K)$, its orthogonal complement can not be larger, thus $\dim((\mathrm{Im}(P_1) + \mathrm{Im}(P_2 + K))^\perp) < \infty$. We can therefore just choose this orthogonal complement as the space J in (6.11). *q.e.d.*

Of course *Theorem 6.1* becomes a lot more complicated, when the projections P_1, P_2 are more arbitrary and not necessarily strong complementary. However, if they are at least complementary, $\mathrm{Im}(P_1)+\mathrm{Im}(P_2) = \mathcal{H}$ we can give two more theorems under certain assumptions:

Theorem 6.2: *Let P_1 and P_2 be two complementary projections, $Im(P_1)+Im(P_2) = \mathcal{H}$. If K is a finite dimensional selfadjoint perturbation of P_2 (note that finite dimensional is stronger than compact!), $dim(Im(K)) < \infty, K^\dagger = K$, then*

$$Im(P_1) + Im(P_2 + K) + Im(P_2) \mid_{Im(K)} = \mathcal{H}. \tag{6.15}$$

Proof: It suffices to prove $\mathrm{Im}(P_2) \subset (\mathrm{Im}(P_2 + K) + \mathrm{Im}(P_2) \mid_{\mathrm{Im}(K)})$, because P_1 and P_2 are assumed to be complementary. In general we always have a decompostion $\mathcal{H} = \mathrm{Im}(K) + \mathrm{Im}(K)^\perp$. Furthermore we have $\mathrm{Im}(K)^\perp \cong \mathrm{Ker}(K^\dagger)$. If now $K = K^\dagger$ is selfadjoint, one can decompose $\mathcal{H}$ into $\mathcal{H} = \mathrm{Im}(K) + \mathrm{Ker}(K)$ Then, if $v = v_1 + v_2$ is the representation of v in this decomposition of $\mathcal{H}$, then

$$\mathrm{Im}(P_2 + K) = \{P_2 v + Kv, v \in V\} = \{P_1(v_1 + v_2) + K(v_1 + v_2)\} \tag{6.16}$$

$$= \{P_2 v_1 + P_2 v_2 + K v_2, v_1 \in \mathrm{Im}(K), v_2 \in \mathrm{Ker}(K)\} \tag{6.17}$$
$$= \{(P_2 + K) v_1 + P_2 v_2, v_1 \in \mathrm{Im}(K), v_2 \in \mathrm{Ker}(K)\} \tag{6.18}$$
$$= \mathrm{Im}(P_2 + K) \mid_{\mathrm{Im}(K)} + \mathrm{Im}(P_2) \mid_{\mathrm{Ker}(K)} . \tag{6.19}$$

It directly follows that $\mathrm{Im}(P_2 + K) + \mathrm{Im}(P_2) \mid_{\mathrm{Im}(K)} = \mathrm{Im}(P_2) + \mathrm{Im}(P_2 + K) \mid_{\mathrm{Im}(K)}$ and this contains $\mathrm{Im}(P_2)$. *q.e.d.*

At last we want to give a theorem which uses the idea that $Im(P_2 + K)$ is somehow connected with $Im(P_2, K)$ while $Im(P_2) + Im(K)$ is connected to $Im(P_2) \times Im(K)$. Thinking with this connections in mind it is intuitive that if P_2 and K are connected to each other via the kernel condition in *Theorem 5.1* from the projections section, equation (6.11) should also hold. The next theorem confirms this:

Theorem 6.3: *Assume that P_1 and P_2 are complementary projections and K is finite dimensional, but not necessary self adjoint. If in addition $Ker(P_2) + Ker(K) + F = \mathcal{H}$, $dim(F) < \infty$, equation* (6.11) *holds and J may be chosen as $J = Im(P_2) \mid_{Im(K^\dagger)}$.*

Proof: First we see that since K is finite dimensional, so is $K^\dagger$ and as the images of finite dimensional operators are closed subspaces, we may again take the decomposition $\mathcal{H} = \mathrm{Im}(K^\dagger) + \mathrm{Ker}(K)$ just as in the proof of *Theorem 6.2*. Then, assuming $\mathrm{Ker}(P_2) + \mathrm{Ker}(K) + F = \mathcal{H}$, $\mathrm{Im}(P_2 + K)$ becomes

$$\mathrm{Im}(P_2 + K) = \{P_2(v_1 + v_2 + v_F) + K(v_1 + v_2 + v_F), v_1 \in \mathrm{Ker}(P_2), v_2 \in \mathrm{Ker}(K), v_F \in F\} \tag{6.20}$$
$$= \{P_2 v_2 + P_2 v_F + K v_1 + K v_F\} = \{P_2 v_2 + K v_1 + (P_2 + K) v_F\} \tag{6.21}$$
$$= \mathrm{Im}(P_2) \mid_{\mathrm{Ker}(K)} + \mathrm{Im}(K) \mid_{\mathrm{Ker}(P_2)} + \mathrm{Im}(P_2 + K) \mid_F . \tag{6.22}$$

Adding $\mathrm{Im}(P_2) \mid_{\mathrm{Im}(K^\dagger)}$ results in $\mathrm{Im}(P_2) \subset \mathrm{Im}(P_2 + K) + \mathrm{Im}(P_2) \mid_{\mathrm{Im}(K^\dagger)}$ and therefore equation (6.11) is fulfilled. *q.e.d.*

7 Main Problem and its Fredholm Conditions

Now that we have explained the basics of the calculus needed for facing the main problem of this work in the last few chapters, we can now state the main problem and develop its Fredholm- and index theory.

7.1 General Setting and First Observations

Consider a Lorentzian globally hyperbolic manifold (M, g) with compact Cauchy hypersurfaces. Furthermore, let Σ_0 (time $t = 0$) and Σ_1 (time $t = 1$) be two different Cauchy hypersurfaces. Let E be a finite rank vector bundle over M and $L^2(\Sigma_0)$ and $L^2(\Sigma_1)$ denote the L^2 spaces of sections of E on the corresponding Cauchy hypersurfaces. Assume that $L = L(t, x, \partial_x)$ is a first order pseudodifferentail operator (continous in t) on $L^2(M)$ defining a symmetrizable hyperbolic system $\partial_t u = Lu$ and $P_{\pm,0/1}, \mathcal{A}_{0/1}$ are zero order pseudodifferential operators on $L^2(\Sigma_0)$ and $L^2(\Sigma_1)$ respectively. Furthermore we want $P_{+,0/1}$ and $P_{-,0/1}$ to be complementary pseudodifferential projections at each time $t \in \{0, 1\}$, $L^2(\Sigma_{0/1}) = \mathrm{Im}(P_{+,0/1}) + \mathrm{Im}(P_{-,0/1})$. Given these ingredients we want to investigate the index theory of the nonlocal problem

$$(i)\partial_t u = Lu, \quad (ii)P_{+,0}\mathcal{A}_0 u(0) = g_0 \in \mathrm{Im}(P_{+,0}), \quad (iii)P_{-,1}\mathcal{A}_1 u(1) = g_1 \in \mathrm{Im}(P_{-,1}), \quad (7.1)$$

with $u \in L^2(M)_{0,1}$. The aim of this chapter will be to develop Fredholm conditions for (7.1), first for the general case and then after taking some further assumptions on L or the projections $P_{\pm,0/1}$ into account. Without any further assumptions, we have the following first observation:

Theorem 7.1: *The problem* (7.1) *is Fredholm, meaning that the solution space is finite dimensional and that only finitely many linear conditions on g_0 and g_1 are needed in order to guarantee solutions, if and only if the operator*

$$\mathcal{D} = \begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ \Phi_1^{-1} P_{-,1}\mathcal{A}_1\Phi_1 \end{pmatrix} : L^2(\Sigma_0) \to \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1}P_{-,1}\Phi_1) \subset L^2(\Sigma_0) \times L^2(\Sigma_0) \quad (7.2)$$

is Fredholm.

Proof: Because the equation $\partial_t u = Lu$ is symmetrizable, there exists an invertible family of solution operators Φ_t and every solution of (7.1), (i) evaluated at time t may be written as $u(t) = \Phi_t u(0) := \Phi(t)u_0$. If we insert this into (7.1)(iii), we get the system

$$\begin{pmatrix} P_{+,0}\mathcal{A}_0 u_0 \\ P_{-,1}\mathcal{A}_1\Phi_1 u_0 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}. \quad (7.3)$$

Multiplying the second row of (7.3) by the inverse solution operator Φ_1^{-1} and using that

$(\Phi_1^{-1} g_1) \in Im(\Phi_1^{-1} P_{-,1} \Phi_1)$ if $g_1 \in Im(P_{-,1})$, (7.3) is equivalent to

$$\begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ \Phi_1^{-1} P_{-,1}\mathcal{A}_1\Phi_1 \end{pmatrix} u_0 = \begin{pmatrix} g_0 \\ \Phi_1^{-1} g_1 \end{pmatrix} \in \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1). \tag{7.4}$$

Because (7.1) is Fredholm iff (7.4) is Fredholm, we have reduced the Fredholm property of (7.1) to that of the operator $\mathcal{D}$. *q.e.d.*

Necessary Conditions:

Before establishing conditions, which are equivalent to the Fredholm property of $\mathcal{D}$, we want to find out, which conditions are necessary. In fact there are choices for the projections $P_{\pm,0/1}$ such that $\mathcal{D}$ is never Fredholm, no matter how one chooses $\mathcal{A}_0$ and $\mathcal{A}_1$ and similarly, there are choices for $\mathcal{A}_0$ and $\mathcal{A}_1$, such that for any arbitrary combination of projections $\mathcal{D}$ is never a Fredholm operator. These necessary conditions are directly related to *Theorem 5.1* and *Theorem 5.2* from the projections section and take this form:

Theorem 7.2: *The operator $\mathcal{D}$ from Theorem 7.1 can only be Fredholm, if there exist finite dimensional subspaces J and F of $L^2(\Sigma_0)$ with*

$$Ker(P_{+,0}) + Ker(\Phi_1^{-1} P_{-,1}\Phi_1) + J = L^2(\Sigma_0), \tag{7.5}$$

and

$$Im(\mathcal{A}_0^\dagger) + Im(\Phi_1^\dagger \mathcal{A}_1^\dagger \Phi_1^{-\dagger}) + F = L^2(\Sigma_0). \tag{7.6}$$

Proof: First, from *Proposition 5.4*, we see that $\Phi_1^{-1} P_{-,1}\Phi_1$ is a projection. Then the image of $\mathcal{D}$ in equation (7.2) is a subset of the image of

$$\tilde{\mathcal{D}} = \begin{pmatrix} P_{+,0} \\ \Phi_1 P_{-,1}\Phi_1 \end{pmatrix} : L^2(\Sigma_0) \to \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1). \tag{7.7}$$

This means that the cokernel of $\mathcal{D}$ has at least the dimension of the cokernel of $\tilde{\mathcal{D}}$ and therefore it is necessary that $\dim(\mathrm{Coker}(\tilde{\mathcal{D}})) < \infty$. If we now use *Theorem 5.2*, setting $P_1 = P_{+,0}$ and $P_2 = \Phi_1^{-1} P_{-,1}\Phi_1$ we see that this is equivalent to condition (7.5). Now, in order to see condition (7.6), we notice that $\mathrm{Ker}(\mathcal{D}) \cong \mathrm{Coker}(\mathcal{D}^\dagger)$. If we want $\mathcal{D}$ to be a Fredholm operator, it needs to have finite dimensional kernel, thus $\dim(\mathrm{Coker}(\mathcal{D}^\dagger)) < \infty$ is necessary. Note that $\mathcal{D}$ is a bounded operator on $L^2(\Sigma_0)$ and $\mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1) = \mathrm{Im}(\mathcal{P})$,

$$\mathcal{P} = \begin{pmatrix} P_{+,0} & 0 \\ 0 & \Phi_1^{-1} P_{-,1}\Phi_1 \end{pmatrix} \tag{7.8}$$

being the image of a projection $\mathcal{P} : L^2(\Sigma_0) \times L^2(\Sigma_0) \to L^2(\Sigma_0) \times L^2(\Sigma_0)$ is a closed subspace of $L^2(\Sigma_0) \times L^2(\Sigma_0)$. Therefore we can define the adjoint of $\mathcal{D}$ to act as an operator

$$\mathcal{D}^\dagger : \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1} \Phi_1) \to L^2(\Sigma_0). \tag{7.9}$$

Then $\mathcal{D}^\dagger$ is defined via $\langle \mathcal{D}v, (w,z)\rangle_{L^2(\Sigma_0)\times L^2(\Sigma_0)} = \langle v, \mathcal{D}^\dagger(w,z)\rangle_{L^2(\Sigma_0)}$, $v \in L^2(\Sigma_0), (w,z) \in \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1)$. By using the definition of the inner product on $L^2(\Sigma_0)\times L^2(\Sigma_0)$ one can easily see

$$\mathcal{D}^\dagger(w,z) = \mathcal{A}_0^\dagger P_{+,0}^\dagger w + \Phi_1^\dagger \mathcal{A}_1^\dagger P_{-,1}^\dagger \Phi_1^{-\dagger} z,\ (w,z) \in \mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1}\Phi_1). \tag{7.10}$$

Furthermore we can set $w = P_{+,0}\tilde{w}$ and $z = \Phi_1^{-1} P_{-,1}\Phi_1 \tilde{z}$ such that the image of $\mathcal{D}^\dagger$ takes the form

$$\mathrm{Im}(\mathcal{D}^\dagger) = \{\mathcal{A}_0^\dagger P_{+,0}^\dagger P_{+,0}\tilde{w} + \Phi_1^\dagger \mathcal{A}_1^\dagger P_{-,1}^\dagger \Phi_1^{-\dagger}\Phi_1^{-1} P_{-,1}\Phi_1 \tilde{z}\} \tag{7.11}$$

$$= \mathrm{Im}(\mathcal{A}_0^\dagger P_{+,0}^\dagger P_{+,0}) + \mathrm{Im}(\Phi_1^\dagger \mathcal{A}_1^\dagger P_{-,1}^\dagger \Phi_1^{-\dagger}\Phi_1^{-1} P_{-,1}\Phi_1). \tag{7.12}$$

Now, the sum of the images in (7.12) is certainly contained in $\mathrm{Im}(\mathcal{A}_0^\dagger) + \mathrm{Im}(\Phi_1^\dagger \mathcal{A}_1^\dagger \Phi_1^{-\dagger})$. Thus the codimension of this sum needs to be finite dimensional, and this is equivalent to condition (7.6). *q.e.d.*

7.2 Some Examples

Now that we know about the necessary conditions for $\mathcal{D}$ to be Fredholm, we want to give some special cases for the choices of the projections $P_{\pm,0/1}$ and operators $\mathcal{A}_0, \mathcal{A}_1$, where this conditions are satisfied.

Example 7.1: Let K be a finite rank selfadjoint operator and let $\mathrm{Ker}(P_{+,0}) + \mathrm{Ker}(P_{-,0}) = L^2(\Sigma_0)$. Assume furthermore that the solution operator Φ is unitary. Then the choice $P_{-,1} = \Phi_1 P_{-,0} \Phi_1^{-1} + K$ and $\mathcal{A}_0 = \mathcal{A}_1 = 1$ satisfies (7.5) and (7.6). (7.6) is trivial, as for $\mathcal{A}_0 = \mathcal{A}_1 = 1$ (7.6) would be equivalent to $\mathrm{Im}(1) + \mathrm{Im}(1) + F = L^2(\Sigma_0)$, which is obviously true. Let us show the validity of (7.5). First, we have

$$P_{-,1} = \Phi_1 P_{-,0}\Phi_1^{-1} + K \quad \Leftrightarrow \quad \Phi_1^{-1} P_{-,1}\Phi_1 = P_{-,0} + \Phi_1^{-1} K \Phi_1. \tag{7.13}$$

We know from *Proposition 5.4* that $\Phi_1^{-1} P_{-,1}\Phi_1$ is again a projection and we also see that the unitarity of Φ conserves the self adjointness of K, i.e. $(\Phi_1^{-1} K \Phi_1)^\dagger = (\Phi_1^{-1} K\Phi_1)$. But then

$$\mathrm{Ker}(P_{+,0}) + \mathrm{Ker}(\Phi_1^{-1} P_{-,1}\Phi_1) = \mathrm{Ker}(P_{+,0}) + \mathrm{Ker}(P_{-,0} + \Phi_1^{-1} K\Phi_1) \tag{7.14}$$

$$= \mathrm{Im}(1 - P_{+,0}) + \mathrm{Im}((1 - P_{-,0}) - \Phi_1^{-1} K \Phi_1). \tag{7.15}$$

The assumption $\mathrm{Ker}(P_{+,0}) + \mathrm{Ker}(P_{-,0}) = L^2(\Sigma_0)$ is equivalent to the statement that $(1 - P_{+,0})$ and $(1 - P_{-,0})$ are complementary projections. Remembering *Theorem 6.2* one can deduce that the orthogonal complement to (7.15) has to be of finite dimension.

Example 7.2: If we again choose $\mathcal{A}_0 = \mathcal{A}_1 = 1$ and if $P_{+,0} = 1 - P_{-,0}$, $P_{+,1} = 1 - P_{-,1}$ we can set $P_{+,1} = \Phi_1 P_{+,0} \Phi_1^{-1} + K$ for any compact operator K (not necessarily finite dimensional) and (7.5) and (7.6) are fulfilled by *Theorem 6.1*: Similarly to the previous example we calculate

$$\mathrm{Ker}(P_{+,0}) + \mathrm{Ker}(\Phi_1^{-1} P_{-,1} \Phi_1) = \mathrm{Im}(P_{-,0}) + \mathrm{Im}(P_{+,0} - \Phi_1^{-1} K \Phi_1). \tag{7.16}$$

Because $P_{+,0}$ and $P_{-,0}$ are strong complements of each other and $\Phi_1^{-1} K \Phi_1$ is compact, we can apply *Theorem 6.1* and (7.5) will be true. For example, the Dirac equation in *Bär-Strohmaiers* work *An Index Theorem for Lorentzian Manifolds with compact spacelike Cauchy Boundary* fulfills these requirements (see [10], *Chapter 2*). Note that the projections onto the positive and negative eigenfunctions of the Dirac operator at one selected time $t = t_0$ are in fact strong complements. In order to see $P_{+,1} = \Phi_1 P_{+,0} \Phi_1^{-1} + K$, we use the same notation like *Bär-Strohmaier*: Let

$$\Phi_1 = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix} \tag{7.17}$$

be the representation of the solution operator in the positive and negative eigenvalue basis defined by $P_{\pm,0/1}$. Then we have

$$P_{+,1} - \Phi_1 P_{+,0} \Phi_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_{++}^\dagger & Q_{-+}^\dagger \\ Q_{+-}^\dagger & Q_{--}^\dagger \end{pmatrix} \tag{7.18}$$

using the unitarity of the time evolution. Further simplification results in

$$P_{+,1} - \Phi_1 P_{+,0} \Phi_1^{-1} = \begin{pmatrix} Q_{-+} Q_{-+}^\dagger & -Q_{++} Q_{-+}^\dagger \\ Q_{-+} Q_{++}^\dagger & -Q_{-+} Q_{-+}^\dagger \end{pmatrix}. \tag{7.19}$$

Moreover, we may use that Q_{-+} is a compact operator, which was already shown by *Bär-Strohmaier*. Then all the entries in (7.17) are in fact compact.

Usually one wants to connect the Fredholm theory of operators to the invertibility of a symbol. The principal symbol of the operator $\mathcal{D}$ in (7.2) however is a non square matrix and therefore not invertible. Because $\mathcal{D}$ has an image lying in $\mathrm{Im}(P_{+,0}) \times \mathrm{Im}(\Phi_1^{-1} P_{-,1} \Phi_1)$ which is a projection space we have to interpret the invertibility of the principal symbol in the sense of *Theorem 5.3*. Note that the relevant projections in our case are $P_0 = 1$ and $P_1 = \mathcal{P}$,

$$\mathcal{P} = \begin{pmatrix} P_{+,0} & 0 \\ 0 & \Phi_1^{-1} P_{-,1} \Phi_1 \end{pmatrix}. \tag{7.20}$$

Proposition 7.1: *The operator $\mathcal{D}$ from Theorem 7.1 is Fredholm, iff there exist two Fredholm operators $\mathcal{D}_1^p, \mathcal{D}_2^p$ with*

$$\mathcal{D}_1^p P_{+,0}\mathcal{A}_0 + \mathcal{D}_2^p \Phi_1^{-1} P_{-,1}\mathcal{A}_1\Phi_1 = 1 + K, \tag{7.21}$$

$$\begin{pmatrix} P_{+,0}\mathcal{A}_0\mathcal{D}_1^p & P_{+,0}\mathcal{A}_0\mathcal{D}_2^p \\ \Phi_1^{-1}P_{-,1}\mathcal{A}_1\Phi_1\mathcal{D}_1^p & \Phi_1^{-1}P_{-,1}\mathcal{A}_1\Phi_1\mathcal{D}_2^p \end{pmatrix} = \begin{pmatrix} P_{+,0} & 0 \\ 0 & \Phi_1^{-1}P_{-,1}\Phi_1 \end{pmatrix} + \tilde{K} \tag{7.22}$$

where K and $\tilde{K}$ are compact operators.

Proof: From *Theorem 5.3* we see that $\mathcal{D}$ must have parametrices as an operator between projection spaces. The corresponding projections are given by $P_0 = 1$, $P_1 = \mathcal{P}$. This means that there must exist an operator $\mathcal{D}^p$ with components $\mathcal{D}_1^p$ and $\mathcal{D}_2^p$ such that

$$\mathcal{D}^p\mathcal{D} = 1 + K, \quad \mathcal{D}\mathcal{D}^p = \mathcal{P} + \tilde{K}. \tag{7.23}$$

Written out componentwise this is equivalent to (7.21) and (7.22). *q.e.d.*

Example 7.3: For some special cases, it is actually easy to guess a parametrix $\mathcal{D}^p$ to $\mathcal{D}$. For example, choose $\mathcal{A}_1 = \mathcal{A}_0 = 1$. In this case we make the Ansatz $\mathcal{D}_1^p = P_{+,0}$, $\mathcal{D}_2^p = P_{-,0}$. From this and (7.18) we can extract $P_{+,0} + P_{-,0}\Phi_1^{-1}P_{-,1}\Phi_1 = 1 + K$. Under the additional assumption $P_{-,0} = 1 - P_{+,0}$ we have

$$P_{-,0}\Phi_1^{-1}P_{-,1}\Phi_1 = P_{-,0} + K. \tag{7.24}$$

Let us say, that $P_{-,1}$ has been chosen in a way that it fulfills (7.24). The components in the upper row of (7.22) give true equations automatically, just by the Ansatz. The last component of the left matrix in (7.22) is $\Phi_1^{-1}P_{-,1}\Phi_1 P_{-,0}$ and it has to be equal to $\Phi_1^{-1}P_{-,1}\Phi_1$ (mod K). If we assume this to be true, the last condition $\Phi_1^{-1}P_{-,1}\Phi_1\mathcal{D}_1^p = 0$ is always true (mod K). Moreover, let Φ be a unitary time evolution operator and let the projections be chosen as orthogonal projections. Then $\Phi_1^{-1}P_{-,1}\Phi_1 P_{-,0}$ is the adjoint of the left side of (7.24) and must therefore be equal to $P_{-,0}$ up to a compact error. But then by (7.22) we have $P_{-,0} = \Phi_1^{-1}P_{-,1}\Phi_1$ (mod K). If the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are assumed to be arbitrary Fredholm operators with parametrices $\mathcal{A}_0^p$ and $\mathcal{A}_1^p$, which commute with $P_{+,0}$ and the solution operator Φ_1, the same procedure can be done with the parametrix given by the Ansatz $\mathcal{D}_1^p = P_{+,0}\mathcal{A}_0^p$ and $\mathcal{D}_2^p = P_{-,0}\mathcal{A}_1^p$. Let us summarize these facts in the form of a proposition:

Proposition 7.2: *Consider problem* (7.1) *and assume that the projections $P_{-,0}$ and $P_{-,1}$ are orthogonal and that they are connectet to each other via the flow of the solution, $P_{-,1} = \Phi_1 P_{-,0}\Phi_1^{-1} + \hat{K}$ with a compcat operator $\hat{K}$. Furthermore let $P_{+,0} = 1 - P_{-,0}$ and let L define a system of equations with unitary time evolution Φ. Then, as long as the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are both Fredholm and commute with $P_{+,0}$ and the time evolution Φ_1, the parametrix to $\mathcal{D}$ is given by*

$$\mathcal{D}^p = (P_{+,0}\mathcal{A}_0^p, P_{-,0}\mathcal{A}_1^p). \tag{7.25}$$

7.3 Reduction to the $\mathcal{D}_{L^2}$-Case

The condition $P_{-,0} = \Phi_1^{-1}P_{-,1}\Phi_1 + \hat{K}$ was already apparent in *Example 7.1* where we presented the case considered by *Bär-Strohmaier*. It seems that this condition is a good choice for (7.5) in that sense that it produces a Fredholm operator $\mathcal{D}$ with good properties like having a very simple parametrix like in *Proposition 7.2*. Let us investigate this condition further in order to see, how it can be used to simplify the Fredholm theory of $\mathcal{D}$ by a great amount:

Theorem 7.3: *Let $P_{+,0} = 1 - P_{-,0}$, $P_{+,1} = 1 - P_{-,1}$ and $P_{-,1} = \Phi_1 P_{-,0}\Phi_1^{-1} + K$ with a compact operator K. The operator $\mathcal{D}$ in* (7.2) *is Fredholm iff*

$$\mathcal{D}_{L^2} := P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 : L^2(\Sigma_0) \to L^2(\Sigma_0) \tag{7.26}$$

is Fredholm.

Proof: First let us simply plug in $P_{-,1} = \Phi_1 P_{-,0}\Phi_1^{-1} + K$ into equation (7.2). Then $\mathcal{D}$ gets the form

$$\mathcal{D} = \begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ \Phi_1^{-1}(\Phi_1 P_{-,0}\Phi_1^{-1} + K)\mathcal{A}_1\Phi_1 \end{pmatrix} = \begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 + \Phi_1^{-1}K\mathcal{A}_1\Phi_1 \end{pmatrix} \tag{7.27}$$

$$= \begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Phi_1^{-1}K\mathcal{A}_1\Phi_1 \end{pmatrix} : L^2(\Sigma_0) \to Im(P_{+,0}) \times Im(\Phi_1^{-1}P_{-,1}\Phi_1). \tag{7.28}$$

Note that $L^2(\Sigma_0) = Im(P_{+,0}) \oplus Im(P_{-,0})$ is a direct sum, since we assumed that $P_{+,0} = 1 - P_{-,0}$ and $P_{-,0}$ are strong complements. Any vector containing components of spaces whose sum is direct is isomorphic to the sum of its components. This means that there exists an isomorphism such that

$$\begin{pmatrix} P_{+,0}\mathcal{A}_0 \\ P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 \end{pmatrix} \cong (P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1) : L^2(\Sigma_0) \to L^2(\Sigma_0.) \tag{7.29}$$

Now the column

$$\begin{pmatrix} 0 \\ \Phi_1^{-1}K\mathcal{A}_1\Phi_1 \end{pmatrix} \tag{7.30}$$

consists of compact operators. This means that the Fredholm and index theory for the operator in (7.28) is equal to that of the first summand which is equal to the desired $\mathcal{D}_{L^2}$ by (7.29). *q.e.d.*

Proposition 7.3: *Consider* (7.1) *together with all conditions made in Theorem 7.3. If both operators* $\mathcal{A}_0$ *and* $\mathcal{A}_1$ *are Fredholm and the commutators* $[P_{+,0}, \mathcal{A}_{0/1}]$ *and* $[P_{+,0}, L_t]$ *are both zero (modulo compact) for all* t*,* $\mathcal{D}_{L^2}$ *is Fredholm. A parametrix is given by*

$$\mathcal{D}_{L^2}^p = P_{+,0}\mathcal{A}_0^p + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1 \tag{7.31}$$

with parametrices $\mathcal{A}_0^p$ *and* $\mathcal{A}_1^p$ *of* $\mathcal{A}_0$ *and* $\mathcal{A}_1$*. If additionally these parametrices are chosen in such a way that* $1-\mathcal{A}_{0/1}\mathcal{A}_{0/1}^p$ *and* $1-\mathcal{A}_{0/1}^p\mathcal{A}_{0/1}$ *are all trace-class, the index of* $\mathcal{D}_{L^2}$ *is*

$$\mathrm{Ind}(\mathcal{D}_{L^2}) = \mathrm{Ind}(P_{+,0}\mathcal{A}_0P_{+,0}) + \mathrm{Ind}(P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0}) \tag{7.32}$$

where the operators

$$P_{+,0}\mathcal{A}_0P_{+,0} : Im(P_{+,0}) \to Im(P_{+,0}), \quad P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0} : Im(P_{-,0}) \to Im(P_{-,0}) \tag{7.33}$$

are regarded as operators between projection spaces.

Proof: Assume that $\mathcal{A}_0$ and $\mathcal{A}_1$ are both Fredholm with parametrices $\mathcal{A}_0^p$ and $\mathcal{A}_1^p$. We try $P_{+,0}\mathcal{A}_0^p + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1$ as a parametrix for D_{L^2}:

$$(P_{+,0}\mathcal{A}_0^p + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1)(P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1) \tag{7.34}$$

$$= P_{+,0}\mathcal{A}_0^pP_{+,0}\mathcal{A}_0 + P_{+,0}\mathcal{A}_0^pP_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1. \tag{7.35}$$

From (7.35) we see that the commutator conditions $[P_{+,0}, \mathcal{A}_{0/1}] = [P_{+,0}, L_t] = 0$ will be sufficient to make $D_{L^2}^p = P_{+,0}\mathcal{A}_0^p + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\Phi_1$ a parametrix for D_{L^2}: First, if $P_{+,0}$ commutes with $\mathcal{A}_0$ and $\mathcal{A}_1$ so does $P_{-,0} = 1 - P_{+,0}$. From *Proposition 4.4* in the chapter about hyperbolic systems, we know that $[P_{+,0}, L_t] = 0\ \forall t$ means that $P_{+,0}$ and $P_{-,0}$ also commute with the solution operator Φ_1 and its inverse. With this we can arrange the factors in the second and third term in (7.35) in such a way that we have a product between $P_{+,0}$ and $P_{-,0}$, which makes this terms vanish. The remaining two terms give

$$D_{L^2}^pD_{L^2} = P_{+,0}\mathcal{A}_0^p\mathcal{A}_0 + P_{-,0}\Phi_1^{-1}\mathcal{A}_1^p\mathcal{A}_1\Phi_1 \tag{7.36}$$

$$= P_{+,0}(1 + K_{0,L}) + P_{-,0}\Phi_1^{-1}(1 + K_{1,L})\Phi_1 \tag{7.37}$$

with compact operators $K_{0,L}$ and $K_{1,L}$. We want the last line to be equal to the identity operator up to a compact error. Using $P_{+,0} + P_{-,0} = 1$ this is true and we have

$$D_{L^2}^pD_{L^2} = 1 + P_{+,0}K_{0,L} + P_{-,0}\Phi_1^{-1}K_{1,L}\Phi_1 \tag{7.38}$$

(The proof that $D_{L^2}^p$ is also a right parametrix is analogous). Now let the parametrices be choosen such that $1 - \mathcal{A}^p_{0/1}\mathcal{A}_{0/}$ and $1 - \mathcal{A}_{0/1}\mathcal{A}^p_{0/1}$ are trace class. This means that the corresponding compact operators $K_{0,L/R}$ and $K_{1,L/R}$ are trace class. Because the set of trace class operators is an ideal, $P_{+,0}K_{0,L/R}$ and $P_{-,0}\Phi_1^{-1}K_{L/R}$ must also be trace class. With this the index of D_{L^2} calculates to

$$\operatorname{ind}(D_{L^2}) = \operatorname{Tr}(1 - D^p_{L^2}D^2_L) - \operatorname{Tr}(1 - D_{L^2}D^p_{L^2}) \tag{7.39}$$

$$= \operatorname{Tr}(P_{+,0}K_{0,R}) + \operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,R}\Phi_1) - \operatorname{Tr}(P_{+,0}K_{0,L}) - \operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,L}\Phi_1) \tag{7.40}$$

$$= (\operatorname{Tr}(P_{+,0}K_{0,R}) - \operatorname{Tr}(P_{+,0}K_{0,L})) + (\operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,R}\Phi_1) - \operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,L}\Phi_1)). \tag{7.41}$$

Let us relate (7.41) to the indices of the desired operators in (7.33). If we regard $P_{+,0}\mathcal{A}_0P_{+,0}$ and $P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0}$ as operators between projection spaces, $P_{+,0}\mathcal{A}^p_0P_{+,0}$ and $P_{-,0}\Phi_1^{-1}\mathcal{A}^p_1\Phi_1P_{-,0}$ are parametrices, where the error terms are of trace class. With this (7.32) becomes

$$\operatorname{ind}(P_{+,0}\mathcal{A}_0P_{+,0}) + \operatorname{ind}(P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0}) \tag{7.42}$$

$$= \operatorname{Tr}(P_{+,0} - P_{+,0}\mathcal{A}^p_0P_{+,0}P_{+,0}\mathcal{A}_0P_{+,0}) - \operatorname{Tr}(P_{+,0} - P_{+,0}\mathcal{A}_0P_{+,0}P_{+,0}\mathcal{A}^p_0P_{+,0}) \tag{7.43}$$

$$+\operatorname{Tr}(P_{-,0} - P_{-,0}\Phi_1^{-1}\mathcal{A}^p_1\Phi_1P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0}) - \operatorname{Tr}(P_{-,0} - P_{-,0}\Phi_1^{-1}\mathcal{A}_1\Phi_1P_{-,0}\Phi_1^{-1}\mathcal{A}^p_1\Phi_1P_{-,0}) \tag{7.44}$$

$$= \operatorname{Tr}(P_{+,0} - P_{+,0}(K_{0,L} + 1)P_{+,0}) - \operatorname{Tr}(P_{+,0} - P_{+,0}(K_{0,R} + 1)P_{+,0}) \tag{7.45}$$

$$+\operatorname{Tr}(P_{-,0} - P_{-,0}\Phi_1^{-1}(1 + K_{1,L})\Phi_1P_{-,0}) - \operatorname{Tr}(P_{-,0} - P_{-,0}\Phi_1^{-1}(1 + K_{1,R})\Phi_1P_{-,0}) \tag{7.46}$$

$$= -\operatorname{Tr}(P_{+,0}K_{0,L}P_{+,0}) + \operatorname{Tr}(P_{+,0}K_{0,R}P_{+,0}) \tag{7.47}$$

$$-\operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,L}\Phi_1P_{-,0}) + \operatorname{Tr}(P_{-,0}\Phi_1^{-1}K_{1,R}\Phi_1P_{-,0}). \tag{7.48}$$

If we use the cyclicity of the trace in the last line, we directly see that (7.47) – (7.48) is equal to (7.41). *q.e.d.*

Remark 7.3: In contrast to the general operator $\mathcal{D}$, $\mathcal{D}_{L^2}$ has the advantage of being regarded as an operator from $L^2(\Sigma_0)$ to $L^2(\Sigma_0)$. In this case both projections P_0 and P_1 from *Theorem 5.3* are equal to the identity and the Fredholm property of $\mathcal{D}_{L^2}$ is equivalent to the usual ellipticity condition (without using the theory of operators acting between projection spaces). If $\mathcal{D}_{L^2}$ consists only of G-operators with an amenable group G which acts topologically freely, this implies that $\mathcal{D}_{L^2}$ is Fredholm iff its principal trajectorial symbol is invertible. If we want the solution operator to be built of G-operators, we only need to impose that the system $\partial_t u = Lu$ may be transformed to an almost diagonal system $\partial_t u = Du + Su, S \in \mathrm{OPS}^0$ and that the eigenvalue distance condition from *Theorem 4.2* is valid. For example, as already shown in *Chapter 4*, all strictly hyperbolic systems satisfying the eigenvalue distance condition will fall into this class. Then we only need a condition, which is sufficient for the amenability of G. Let us write down a collection of the assumptions we make for our system $\partial_t u = Lu$, such that the Fredholm theory of (7.1) may be reduced to the invertibility of a symbol:

7.4 The $\mathcal{D}_{L^2}$-Case with Eigenvalue Distance Condition

Assumptions 7.1:

From now on, we make the following assumptions about the system (7.1): First, we want that all assumptions made in *Theorem 7.3* hold, such that $\mathcal{D}$ can be reduced to $\mathcal{D}_{L^2}$. Then,

$$(D)\ L = D + S,\ S \in \mathrm{OPS}^0,\quad |\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq \alpha|\xi|,\ t \in \mathbb{R}, |\xi| \geq M \tag{7.49}$$

for a diagonal operator $D \in \mathrm{OPS}^1$ with the diagonal entries id_j of constant multiplicity such that $\lambda_j = \sigma_p(d_j)$ is real and some positive constant α. Furthermore, we want the d_j to be self-adjoint up to negative order perturbations,

$$(S)\ d_j = \tilde{d}_j + k_j,\ \tilde{d}_j = \tilde{d}_j^*,\ k_j \in \mathrm{OPS}^m, m < 0. \tag{7.50}$$

Let α_k be the time $t = 1$ Hamilton flows along λ_k. At last we impose that the group G generated by the $(n^2 - n)/2$ elements $\alpha_i^{-1} \circ \alpha_j$ is an amenable group.

Remark 7.4: From (D) we conclude that the solution operator Φ may be expressed as $\Phi = N\Phi_D Q$ up to a compact error. This is due to *Theorem 4.2*. Condition (S) ensures that the solution operator to the diagonal system $\partial_t u = Du$ is unitary up to a compact error. An argument to understand this is the fact that operators of negative order are not appearing in the eikonal or transport equation in *Theorem 4.1*, and because $\partial_t u = Du$ is a diagonal system it can be reduced to a set of scalar hyperbolic equations. This means that we can replace $d_j = \tilde{d}_j + k_j$ by $\tilde{d}_j$ in (7.50) and we will still have the same solution operator (up to a compact error), since the k_j which are of negativ order will not influence the equations (4.43) − (4.44). But the solution operator to $\partial_t u = Du$ with a selfadjoint D is unitary. The amenability of the group G generated by the flows $\alpha_i^{-1} \circ \alpha_j$ will ensure that the Fredholm property of the problem is equivalent to the invertibility of a principal symbol.

Now we want to show that $\mathcal{D}_{L^2}$ may in fact be expressed with the help of G-operators. For that we use the following lemma:

Lemma 7.1: *Let Φ_D be the solution operator to $\partial_t u = Du$ with an operator D like in* (7.49). *Denoting by $\alpha_{j,1}$ the $t = 1$ Hamiltonian flows along H_{λ_j}, the conjugation of any operator $A : L^2(\Sigma_0) \to L^2(\Sigma_0)$ by the solution operator $\Phi_{D,1}$ is then given by*

$$(\Phi_{D,1}^{-1} A \Phi_{D,1})_{ij} = A_{ij}^{\alpha_i} \Phi_{D,1,\alpha_i^{-1} \circ \alpha_j} \tag{7.51}$$

as long as the components A_{ij} of A are pseudodifferential operators.

Proof: Since Φ_D is diagonal, the components of $\Phi_{D,1}^{-1} A \Phi_{D,1}$ are just given by $(\Phi_{D,1}^{-1} A \Phi_{D,1})_{ij}$ $= \Phi_{D,1,\alpha_i}^{-1} A_{ij} \Phi_{D,1,\alpha_j}$. Now we can insert a 1, in order to see that this is a G-operator:

$$\Phi^{-1}_{D,1,\alpha_i} A_{ij} \Phi_{D,1\alpha_j} = \Phi^{-1}_{D,1,\alpha_i} A_{ij} (\Phi_{D,1,\alpha_i} \Phi^{-1}_{D,1,\alpha_i}) \Phi_{D,1,\alpha_j} = (\Phi^{-1}_{D,1,\alpha_i} A_{ij} \Phi_{D,1,\alpha_i}) \Phi^{-1}_{D,1,\alpha_i} \Phi_{D,1,\alpha_j} \quad (7.52)$$

$$= A^{\alpha_i}_{ij} \Phi^{-1}_{D,1,\alpha_i} \Phi_{D,1,\alpha_j}. \quad (7.53)$$

For the last equality we used Egorov´s theorem and the fact that each $\Phi_{D,1,\alpha_i}$ is a Fourier integral operator which is associated to a canonical transformation equal to the Hamiltonian flow along α_i. *q.e.d.*

From *Lemma 7.1* we conclude that every component in $\mathcal{D}_{L^2}$ would be a G-operator with a simple representation like (7.51) if we could replace the Φ_1 in $\mathcal{D}_{L^2}$ by $\Phi_{D,1}$. Let us argue that for the Fredholm theory of $\mathcal{D}_{L^2}$ in fact Φ_1 can be replaced by $\Phi_{D,1}$:

Proposition 7.4: *Consider* (7.1) *together with the condition* $P_{-,1} = \Phi_1 P_{-,0} \Phi_1^{-1} + K$ *such that the Fredholm theory of* $\mathcal{D}$ *is reduced to that of* $\mathcal{D}_{L^2}$*. Then* $\mathcal{D}_{L^2}$ *is Fredholm iff*

$$\mathcal{D}_{L^2,D} = P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi^{-1}_{D,1}\mathcal{A}_1\Phi_{D,1} \quad (7.54)$$

is Fredholm, as long as condition (D) *from Assumptions 7.1 holds. Moreover, the principal trajectorial symbol* $\sigma_p(\mathcal{D}_{L^2,D})$ *depends only on the principal part of* D*.*

Proof: Condition (D) in *Assumptions 1* guarantees the existence of an operator

$$N = 1 + \sum_{\nu=1}^{\infty} N^\nu, \; N^\nu \in \mathrm{OPS}^{-\nu} \quad (7.55)$$

with a parametrix Q, such that $\Phi_1 = N\Phi_{D,1}Q$. Because N has principal symbol 1, so has Q and we may write

$$Q = 1 + \sum_{\nu=1}^{\infty} Q^\nu, \; Q^\nu \in \mathrm{OPS}^{-\nu}. \quad (7.56)$$

As negative order operators are compact on compact manifolds, we may write $N = 1 + N^-$, $Q = 1 + Q^-$ with N^-, Q^- being compact. But then we can write

$$\Phi_1 = N\Phi_{D,1}Q = (1+N^-)\Phi_{D,1}(1+Q^-) = \Phi_{D,1} + \Phi_{D,1}Q^- + N^{-1}\Phi_{D,1} + N^-\Phi_{D,1}Q^- \quad (7.57)$$

$$= \Phi_{D,1} + K_{N,Q} \quad (7.58)$$

where $K_{N,Q}$ is also compact. Similarly, the conjugation $\Phi_1^{-1}\mathcal{A}_1\Phi_1$ in $\mathcal{D}_{L^2}$ is equal to the conjugation $\Phi^{-1}_{D,1}\mathcal{A}_1\Phi_{D,1}$ up to a compact error. This means that the index theory of $\mathcal{D}_{L^2}$ is equivalent to the index theory of $\mathcal{D}_{L^2,D}$. Concerning the trajectorial symbol, we can use *Lemma 7.1*: $\sigma_p(\Phi^{-1}_{D,1}\mathcal{A}_1\Phi_{D,1})$ will be equal to the trajectorial symbol of operators like (7.51) componentwise. Because there is $\sigma_p(A^{\alpha_i}_{ij}\Phi_{D,1,\alpha_i^{-1}\circ\alpha_j}) = (\sigma_p(A_{ij})\circ\alpha_i)\tau_{\alpha_i^{-1}\circ\alpha_j}$ and the Hamiltonian flow α_i is generated by the principal part of d_i, there is no dependence on lower order parts of D. *q.e.d.*

7.5 Summary

Let us close this chapter by giving a summarizing theorem of the results:

Theorem 7.4: *Consider the Fredholm theory of problem* (7.1)*. In any case (without any further assumptions) conditions* (7.5) *and* (7.6) *are necessary for the Fredholm property. A neccesary and sufficient condition for the Fredholm property is given by* (7.21) – (7.22) *in Proposition 7.1. If the projections are connected by* $P_{-,1} = \Phi_1 P_{-,0} \Phi_1^{-1} + K$ *with a compact operator* K *and all the assumptions from Assumptions 7.1 hold, the problem is Fredholm iff the operator* $\mathcal{D}_{L^2,D} : L^2(\Sigma_0) \to L^2(\Sigma_0)$ *(*(7.54)*) is Fredholm. In this case the Fredholm property only depends on the principal part of the equation* $\partial_t u = Lu$ *(as shown in the proof of Proposition 7.4). Moreover, the Fredholm property is equivalent to the invertibility of* $\sigma_p(\mathcal{D}_{L^2,D})$ *which is the principal trajectorial symbol of* $\mathcal{D}_{L^2,D}$ *regarded as a G-operator. As long as the projection* $P_{+,0}$ *commutes with* $\mathcal{A}_0$*,* $\mathcal{A}_1$ *and* L *for all times* t*, the index of the problem is given by* (7.32)*.*

Remark 7.5: All statements about problem (7.1) arising from *Assumptions 1* are also true for the case $\Sigma_0 = \mathbb{R}^n$ if one uses the algebra of Shubin type operators: For statements like *Proposition 7.4* we used that operators of negative order are automatically compact, and this is still true in the Shubin case.

Remark 7.6: There exists a condition involving Poisson brackets, which guarantees the amenability of the group G generated by the Hamiltonian flows along H_{λ_j}, as long as H_{λ_j} is sufficiently smooth: For example, if for all different times $(t, \tilde{t})$ the Poisson brackets $\{\lambda_j(t, x, \xi), \lambda_l(\tilde{t}, x, \xi)\}$ are equal to zero, the corresponding time $t = 1$ flows α_j and α_k are commutative. To see this, one may take a look at the second remark in *Remark 2.2*: $\{\lambda_j(t, x, \xi), \lambda_k(\tilde{t}, x, \xi)\} = 0$ implies that for all fixed t and fixed $\tilde{t}$ the corresponding integral curves $\alpha_{j,s}$ and $\alpha_{k,\tilde{s}}$ of timelength s and $\tilde{s}$ would be commutative, as long as s and $\tilde{s}$ are chosen small enough. For both time $t = 1$ flows $\alpha_{j,1}$ and $\alpha_{k,1}$ we have the approximate decompositions

$$\alpha_{j,1} \approx \alpha_{j,\epsilon}(0) \circ \alpha_{j,\epsilon}(\epsilon) \circ ... \circ \alpha_{j,\epsilon}(l\epsilon) \circ ... \circ \alpha_{j,\epsilon}(1-\epsilon), \tag{7.59}$$

$$\alpha_{k,1} \approx \alpha_{k,\epsilon}(0) \circ \alpha_{k,\epsilon}(\epsilon) \circ ... \circ \alpha_{k,\epsilon}(l\epsilon) \circ ... \circ \alpha_{k,\epsilon}(1-\epsilon) \tag{7.60}$$

where $\alpha_{j/k,\epsilon}(l\epsilon)$ are the time ϵ flows along $\lambda_{j/k}(l\epsilon)$. Note that (7.59) – (7.60) is a direct consequence of Euler´s method if one expresses the Hamiltonian vector fields H_{λ_j} and the corresponding flows in local coordinates. For further detail, see ([17]).The smaller we choose the value, the better the approximations (7.59) and (7.60) get. But no matter how fine the approximations are chosen, two approximations of $\alpha_{j,1}$ and $\alpha_{k,1}$ like in (7.59) – (7.60) will always be commutative, because their building blocks $\alpha_{j/k,\epsilon}(l)$ are commutative by the Poisson commutativity condition. But then the exact flows $\alpha_{j,1}$ and $\alpha_{k,1}$ will also commute. In this case the Hamiltonian flows $\alpha_{j,1}$ generate an abelian group. But this means that the group G generated by $\alpha_i^{-1} \circ \alpha_j$ (see *Assumptions 7.1*) will also be abelian and therefore amenable.

8 Fredholm Conditions for Second Order Problems

After discussing the Fredholm conditions for general first order symmatrizable and in particular for strictly hyperbolic systems in the last section, we want to explicitly calculate the Fredholm conditions for those first order systems, which arise from an order reduction of a special class of second order equations. For example, one could consider the wave equation as a second order hyperbolic equation and ask for solutions, given the value of the wave at time $t = 0$ and some combination of the wave and its velocity at time $t = 1$. Problems of this form may then be reduced to a system, which is a special case of equation (7.1) from the last chapter and the Fredholm conditions stated in *Chapter 7* may be applied. We will state the general Fredholm conditions for second order hyperbolic equations with nonlocal boundary conditions and investigate some examples, as the one of the case *Savin, Boltachev* considered.

8.1 Time Dependent Second Order Wave Equations

8.1.1 Dependence on Lower Order Parts and Order Reduction

The general setting of a globally hyperbolic manifold used in the last section includes most models for our universe modeled as a 4-dimensional space time. If only large scales matter for the model of the universe, one can use FLRW models for the universe, meaning that the Lorentzian metric g takes the simple form

$$g = -dt^2 + h(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{8.1}$$

where $h(t)$ is (taking the current curvature radius equal to 1) the scale factor and $k \in \{-1, 0, 1\}$ the curvature parameter of our universe. In other words, if g_t is the Riemmanian metric on the Cauchy-hypersurface Σ_t, then $g_t = h(t)g_0, h > 0$. We want to find out, under which assumptions and to which extent any second order hyperbolic nonlocal problem concerning a hyperbolic equation $\partial_t^2 u = Pu$ with an elliptic differential operator $P = a^{ij}\partial_{x_i}\partial_{x_j} + b_j\partial_{x_j} + c$ is reducible to one of the systems discussed in *Chapter 7* and to which extent this system only depends on $\sqrt{\Delta}$. We will derive the Fredholm conditions for second order nonlocal problems in the general case and write also down the simplifications arising from the special case of an FLRW metric (8.1). As the principal part of Δ is given by $g^{ij}\partial_{x_i}\partial_{x_j}$ one could just choose the metric tensor part to be equal to $a^{ij}\partial_{x_i}\partial_{x_j}$ and therefore the principal part of any second order hyperbolic equation is always given by the Laplacian. However, concerning the first order part of the operator it could differ. Let us show that for the Fredholm theory only the principal part of P matters, so one may restrict to the Laplacian always. For that, we first define the general second order hyperbolic nonlocal problem:

Problem 8.1 *Consider the second order nonlocal problem*

$$(i)\partial_t^2 u = Pu, \quad ((A_0 + B_0\partial_t)u)(0) = g_0 \in L^2(\Sigma_0), \quad (iii)((A_1 + B_1\partial_t)u)(1) = g_1 \in L^2(\Sigma_1), \tag{8.2}$$

with first order operators A_0, A_1, zero order operators B_0, B_1 and an elliptic differential operator $P = j(t)a^{ij}\partial_{x_i}\partial_{x_j}$ defined by a symmetric positive definite matrix a^{ij}. We want to find out, for which choices of A_0, A_1 and B_0, B_1 this problem has the Fredholm property, i.e. there are finitely many independent solutions $u \in H^1(M)$ and only finitely many conditions on g_0 and g_1 needed to guarantee solvability.

Lemma 8.1: *The Fredholm property of Problem 8.1 depends only on the principal part of P, which means that P might be replaced by the Laplacian Δ.*

Proof: From *Chapter* 4, (4.23) we know that $(8.2), (i)$ is equivalent to the system

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{P} & 0 \\ 0 & -i\sqrt{P} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \; J \in \mathrm{OPS}^0. \tag{8.3}$$

This is a strictly hyperbolic system and it has the form (D) from (7.49). Note that the condition

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq \alpha|\xi| \tag{8.4}$$

reduces to the condition that P has to be a uniformly elliptic operator. This is always the case on a compact manifold, which means that (8.3) indeed fulfills all necessary conditions. We want to show that $(8.2), (ii) - (iii)$ may be rewritten in a form which resembles $(7.1), (ii) - (iii)$ in *Chapter 7*. For that let us start with calculating the transformation matrix T. The matrix L of first order operators, which can be diagonalized to the system (8.3) is given by

$$L = \begin{pmatrix} 0 & \Lambda \\ -P\Lambda^{-1} & 0 \end{pmatrix}. \tag{8.5}$$

Because we may choose the principal symbol of P to coincide with that of the Laplacian, we have

$$\sigma_p(P\Lambda^{-1}) = \frac{\sigma_p(\Delta)}{\sigma_p(\Lambda)} = \frac{\sigma_p(\Delta)}{\sqrt{\sigma_p(1+\Delta)}} = \frac{\sigma_p(\Delta)}{\sqrt{\sigma_p(\Delta)}} = \sqrt{\sigma_p(\Delta)} \tag{8.6}$$

as long as the Laplacian in $\Lambda = \sqrt{1+\Delta}$ is the same time dependent operator as the Laplace-Beltrami operator defined by the metric on Σ_t. Formally the order reducing transformation $u_1 = \Lambda u$, $u_2 = \partial_t u$ would result in a system defined by the operator

$$\tilde{L} = \begin{pmatrix} 0 & \Lambda \\ -P\Lambda^{-1} & 0 \end{pmatrix} + \begin{pmatrix} \dot{\Lambda}\Lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{8.7}$$

if one chooses Λ to be time dependent. However, the difference between $\tilde{L}$ and L turns out to be of order zero, and the conjugation of this extra term with an operator T diagonalizing the principal part of (8.5) may be absorbed into the zero order error J in (8.3). On the symbol level we therefore need to diagonalize the matrix

$$\sigma_p(L) = \begin{pmatrix} 0 & \sqrt{\sigma_p(\Lambda)} \\ -\sqrt{\sigma_p(\Lambda)} & 0 \end{pmatrix}. \tag{8.8}$$

The eigenvalues of this matrix are $x_{1/2} = \pm i\sqrt{\sigma_p(\Delta)}$. This results in the following equations for the components η_i of the corresponding eigenvectors:

$$\mp i\sqrt{\sigma_p(\Delta)}\eta_1 + \sqrt{\sigma_p(\Delta)}\eta_2 = 0. \tag{8.9}$$

Normalizing the eigenvectors which arise from this equation results in the transformation matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{8.10}$$

This turns out to be a very simple matrix with constant components as entries. With $u = Tv$ the components of u become

$$u_1 = \frac{1}{\sqrt{2}}(v_1 + v_2), \quad u_2 = \frac{i}{\sqrt{2}}(v_1 - v_2). \tag{8.11}$$

Now let us rewrite $(8.2), (ii) - (iii)$ in terms of the coordinates (v_1, v_2):

$$((A_0 + B_0\partial_t)u)(0) = A_0\Lambda^{-1}u_1(0) + B_0u_2(0) \tag{8.12}$$

$$= \frac{1}{\sqrt{2}}A_0\Lambda^{-1}(v_1(0) + v_2(0)) + \frac{i}{\sqrt{2}}B_0(v_1(0) - v_2(0)) \tag{8.13}$$

$$= \frac{1}{\sqrt{2}}(A_0\Lambda^{-1} + iB_0)v_1(0) + \frac{1}{\sqrt{2}}(A_0\Lambda^{-1} - iB_0)v_2(0) = g_0, \tag{8.14}$$

$$((A_1 + B_1\partial_t)u)(1) = \frac{1}{\sqrt{2}}A_1\Lambda^{-1}(v_1(1) + v_2(1)) + \frac{i}{\sqrt{2}}B_1(v_1(1) - v_2(1)) \overset{!}{=} g_1. \tag{8.15}$$

$v_1(1)$ and $v_2(1)$ can be expressed in terms of $v_1(0)$ and $v_2(0)$ with the help of the solution operator: We know that the solution to (8.3) is the solution operator Φ_D of the diagonal part plus a compact error. If we try two write $(8.2), (ii) - (iii)$ in the form $P_{\pm,0/1}\mathcal{A}_{0/1}v(0/1) = \tilde{g}(0/1)$ compact errors in the time evolution will produce only compact errors in this condition and the corresponding operator $\mathcal{D}_{L^2}$. Therefore it suffices to let only Φ_D be involved in the calculations. Proceeding with the left hand side of (8.15) we get

$$... = \frac{1}{\sqrt{2}} A_1 \Lambda^{-1} (\Phi_{11} v_1(0) + \Phi_{12} v_2(0)) + \frac{i}{\sqrt{2}} B_1 (\Phi_{11} v_1(0) - \Phi_{12} v_2(0)) \tag{8.16}$$

$$= \frac{1}{\sqrt{2}} (A_1 \Lambda^{-1} + iB_1) \Phi_{11} v_1(0) + \frac{1}{\sqrt{2}} (A_1 \Lambda^{-1} - iB_1) \Phi_{12} v_2(0) \overset{!}{=} g_1. \tag{8.17}$$

Note that $\Phi_{t1/2}$ are the diagonal components of Φ_D, solving $\dot{\Phi}_{t1/2} = \pm i\sqrt{\Delta}\Phi_{t1/2}$. Summarizing, we can define the four matrices

$$P_{+,0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{-,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8.18}$$

$$\mathcal{A}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}}(A_0 \Lambda^{-1} + iB_0) & \frac{1}{\sqrt{2}}(A_0 \Lambda^{-1} - iB_0) \\ 0 & 0 \end{pmatrix}, \tag{8.19}$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}}(A_1 \Lambda^{-1} + iB_1) & \frac{1}{\sqrt{2}}(A_1 \Lambda^{-1} - iB_1) \end{pmatrix}. \tag{8.20}$$

to arrive at

$$(i^*) \partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{P} & 0 \\ 0 & -i\sqrt{P} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ (ii^*) P_{+,0} \mathcal{A}_0 v(0) = \tilde{g}_0, \ (iii^*) P_{-,1} \mathcal{A}_1 v(1) = \tilde{g}_1 \tag{8.21}$$

with

$$\tilde{g}_0 = \begin{pmatrix} g_0 \\ 0 \end{pmatrix}, \quad \tilde{g}_1 = \begin{pmatrix} 0 \\ g_1 \end{pmatrix}. \tag{8.22}$$

The Fredholm property of the system (8.21) is equivalent to that (8.2), but the advantage of (8.20) lies in the fact that it is written down exactly in the form of (7.1) so we may use the Fredholm theory developed in the foregoing chapter. We notice that up to a compact operator $1 - P_{+,0} = P_{-,0} = \Phi_{D1}^{-1} P_{-,1} \Phi_{D1} = \Phi_1^{-1} P_{-,1} \Phi_1$ meaning that we can use $\mathcal{D}_{L^2}$ from *Theorem 7.3* in order to investigate the Fredholm property of (8.2). Using *Theorem 7.4* we know that the Fredholm conditions of such a problem only depend on the principal part of the matrix D in (8.3). Since

$$\sqrt{P} = \text{op}(\sigma_p(\sqrt{P}) + \sigma_r(\sqrt{P})) = \text{op}(\sqrt{\sigma_p(P)}) + \text{op}(\sigma_r(\sqrt{P})) = \text{op}(\sqrt{\sigma_p(\Delta)}) + \text{op}(\sigma_r(\sqrt{P})) \tag{8.23}$$

the principal part of $\sqrt{P}$ is determined by the Laplacian. *q.e.d.*

8.1.2 The General Fredholm Conditions

In the proof of *Lemma 8.1* we already did most of the work to derive the Fredholm conditions of (8.2). Let us use the matrices $\mathcal{A}_{0/1}$ to derive the Fredholm conditions with the help of the associated operator $\mathcal{D}_{L^2}$:

Theorem 8.1: *Problem 1 is Fredholm iff the matrix of G-operators*

$$\mathcal{D}_{L^2} = \frac{1}{\sqrt{2}} \begin{pmatrix} (A_0\Lambda^{-1} + iB_0) & (A_0\Lambda^{-1} - iB_0) \\ (A_1\Lambda^{-1} + iB_1)^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} & (A_1\Lambda^{-1} - iB_1)^{\alpha_2} \end{pmatrix} \tag{8.24}$$

is Fredholm as an operator acting on $L^2(\Sigma_0) \times L^2(\Sigma_0)$ *(*$\Phi_{\alpha_2^{-1}\circ\alpha_1} = \Phi_{12}^{-1} \circ \Phi_{11}$ *is regarded as the representation of the flow* $\alpha_2^{-1} \circ \alpha_1$*, where* α_1 *and* α_2 *are the time* $t = 1$ *Hamiltonian flows along* $H_{\lambda_{1/2}}, \lambda_{1/2} = \pm\sigma_p(\sqrt{\Delta})$ *). The relevant group G is isomorphic to the group* $\mathbb{Z}$ *in this case. As long as the group of powers of the Hamiltonian flow* $\alpha_2^{-1} \circ \alpha_1$ *acts topologically freely on* $S^*(\Sigma_0)$*,* $\mathcal{D}_{L^2}$ *is Fredholm iff the trajectorial symbol* $\sigma_p(\mathcal{D}_{L^2})$ *is invertible.*

Proof: From *Chapter 7* we know that (8.20) is Fredholm iff

$$\mathcal{D}_{L^2} = P_{+,0}\mathcal{A}_0 + P_{-,0}\Phi_{D1}^{-1}\mathcal{A}_1\Phi_{D1} \tag{8.25}$$

is a Fredholm operator. Using the matrices from (8.18) – (8.20) we see that $\mathcal{D}_{L^2}$ is of the form (8.24). The invertibility of the principal symbol $\sigma_p(\mathcal{D}_{L^2})$ is equivalent to the Fredholm condition, as long as the group G is amenable and acts topologically freelly and as long as the solution operators Φ_g are unitary operators (mod K). Note that there is only one Fourier integral operator contained in the components of the $\mathcal{D}_{L^2}$ from (8.24). Therefore we can expect that a parametrix of $\mathcal{D}_{L^2}$ has components lying in the algebra generated by pseudodifferential operators and powers of the operator $\Phi_{\alpha_2^{-1}\circ\alpha_1}$. But this means that G is actually the group $\mathbb{Z}$, which is an amenable group. The self adjointness of the Laplacian Δ assures that the solution operators in the matrix Φ_D are unitary. If the assumption about the topologically free action holds, the invertibility of the principal trajectorial symbol is equivalent to the Fredholm property. *q.e.d.*

The matrix D_{L^2} in (8.24) seems to have a simple form, as there is only one G-operator involved in the lower corner and the other three operators are of pseudodifferential type. However, even for an operator of this form it is hard to state the general Fredholm conditions in terms of the components. Let us show that even 2×2 matrix operators involving only one entry with a G-operator are already much more complicated than a 2×2 matrix with only pseudodifferential entries:

Example 8.1: If we had to derive the Fredholm conditions for some operator

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8.26}$$

where A, B, C, D are just pseudodifferential operators, the invertibility of the principal symbol $\sigma_p(F)$ (which is equivalent to the Fredholm property) can be easily stated as

$$\det(\sigma_p(F)) = \sigma_p(A)\sigma_p(D) - \sigma_p(C)\sigma_p(D) \neq 0 \quad \xi \neq 0. \tag{8.27}$$

Of course, if one instead is interested in operators of the form

$$\mathcal{F} = \begin{pmatrix} A & B \\ C\Phi & D \end{pmatrix} \tag{8.28}$$

with pseudodifferential operators A, B, C, D and a Fourier integral operator Φ associated to some group element of an amenable group one could still define something which might represent a determinant of the principal symbol:

$$\det(\sigma_p(\mathcal{F})) := \sigma_p(A)\sigma_p(D) - \sigma_p(C)\tau\sigma_p(B) = \sigma_p(A)\sigma_p(D) - \sigma_p(C)\tau\sigma_p(B)\tau^{-1}\tau \tag{8.29}$$

$$= \sigma_p(A)\sigma_p(D) - \sigma_p(C)\sigma_p(\Phi B\Phi^{-1})\tau = \sigma_p(A)\sigma_p(D) - \sigma_p(C)\sigma_p(B)^{\alpha^{-1}}\tau, \tag{8.30}$$

$$\sigma_p(\mathcal{F}) = \begin{pmatrix} \sigma_p(A) & \sigma_p(B) \\ \sigma_p(C)\tau & \sigma_p(D) \end{pmatrix}. \tag{8.31}$$

For a 2×2 matrix $\sigma_p(F)$, where the entries are pseudodifferential symbols, the inverse principal symbol would be

$$\sigma_p(F)^{-1} = \frac{1}{\det(\sigma_p(F))} \begin{pmatrix} \sigma_p(D) & -\sigma_p(B) \\ -\sigma_p(C) & \sigma_p(A) \end{pmatrix}. \tag{8.32}$$

Let us see whether the naive ansatz

$$\sigma_P(\mathcal{F})^{-1} := \frac{1}{\det(\sigma_p(\mathcal{F}))} \begin{pmatrix} \sigma_p(D) & -\sigma_p(D) \\ -\sigma_p(C)\tau & \sigma_p(A) \end{pmatrix} \tag{8.33}$$

produces the inverse of $\sigma_p(\mathcal{F})$, assuming that the invers of the determinant in (8.32) exists. Calculating the product $\sigma_p(\mathcal{F})\sigma_p(\mathcal{F})^{-1}$ results in

$$\begin{pmatrix} \sigma_p(A) & \sigma_p(B) \\ \sigma_p(C)\tau & \sigma_p(D) \end{pmatrix} \frac{1}{\det(\sigma_p(\mathcal{F}))} \begin{pmatrix} \sigma_p(D) & -\sigma_p(B) \\ -\sigma_p(C)\tau & \sigma_p(A) \end{pmatrix} \tag{8.34}$$

$$= \det(\sigma_p(\mathcal{F}))^{-1} \begin{pmatrix} \sigma_p(A)\sigma_p(D) - \sigma_p(B)\sigma_p(C)\tau & -\sigma_p(A)\sigma_p(B) + \sigma_p(B)\sigma_p(A) \\ (\sigma_p(C)\sigma_p(D)^{\alpha^{-1}} - \sigma_p(D)\sigma_p(C))\tau & -\sigma_p(C)\sigma_p(B)^{\alpha^{-1}}\tau + \sigma_p(D)\sigma_p(A) \end{pmatrix}, \tag{8.35}$$

where

$$\det(\sigma_p(\mathcal{F}))^{-1} = (\sigma_p(A)\sigma_p(D) - \sigma_p(C)\sigma_p(B)^{\alpha^{-1}}\tau)^{-1}. \tag{8.36}$$

Since the pseudodifferential entries of $\mathcal{F}$ are of scalar type, we may use commutativity of the corresponding symbols to see that the right column of (8.35) is equal to the right column of the identity matrix without further assumptions. However, if we want the first column to coincide with that of the identity matrix, we need invariance of the principal symbols of B and D under the flow, $\sigma_p(B)^{\alpha^{-1}} = \sigma_p(B)$, $\sigma_p(D)^{\alpha^{-1}} = \sigma_p(D)$. Furthermore, we need to check to which extent $\sigma_p(\mathcal{F})^{-1}$ is a left inverse to $\sigma_p(\mathcal{F})$ as the flow invariance of $\sigma_p(B)$ and $\sigma_p(D)$ will only guarantee that it is a right inverse. The product $\sigma_p(\mathcal{F})^{-1}\sigma_p(\mathcal{F})$ gives

$$\frac{1}{\det(\sigma_p(\mathcal{F}))}\begin{pmatrix} \sigma_p(D) & -\sigma_p(B) \\ -\sigma_p(C)\tau & \sigma_p(A) \end{pmatrix}\begin{pmatrix} \sigma_p(A) & \sigma_p(B) \\ \sigma_p(C)\tau & \sigma_p(D) \end{pmatrix} \tag{8.37}$$

$$= \det(\sigma_p(\mathcal{F}))^{-1}\begin{pmatrix} \sigma_p(D)\sigma_p(A) - \sigma_p(B)\sigma_p(C)\tau & \sigma_p(D)\sigma_p(B) - \sigma_p(B)\sigma_p(D) \\ (-\sigma_p(C)\sigma_p(A)^{\alpha^{-1}} + \sigma_p(A)\sigma_p(C))\tau & -\sigma_p(C)\sigma_p(B)^{\alpha^{-1}}\tau + \sigma_p(A)\sigma_p(B) \end{pmatrix} \tag{8.38}$$

Again, the second column of the resulting matrix is automatically identical to that of the identity. But from the second entry in the first column we see that even if we already assumed the flow invariance of the principal parts of B and D, there has to be additional invariance of the principal part of A. In other words, if we want to construct an inverse for the principal symbol of $\mathcal{F}$ in a similar way as we would do it for an usual matrix of pseudos we have to assume that three out of the four pseudodifferential operators A, B, C and D have flow invariant principal part - this is a very hard restriction. The conditions $\sigma_p(A)^{\alpha^{-1}} = \sigma_p(A)$ and $\sigma_p(B)^{\alpha^{-1}} = \sigma_p(B)$ alone would lead to $\sigma_p(A_0)^{\alpha_1} = \sigma_p(A_0)$ and $\sigma_p(B_0)^{\alpha_1} = \sigma_p(B_0)$ in the case $\mathcal{F} = \mathcal{D}_{L^2}$. Therefore we have shown that in the general case of *Problem 8.1* where there is no flow invariance of the operators involved in (8.2), it is hard to find Fredholm conditions and construct a parametrix for $\mathcal{D}_{L^2}$.

8.1.3 Fredholm Conditions of Special Examples

We will investigate a few special examples of $\mathcal{D}_{L^2}$ where the Fredholm conditions can be derived explicitely as conditions on the components.

Theorem 8.2: *If the entry $A_0\Lambda^{-1} + iB_0$ is a Fredholm operator, the operator $\mathcal{D}_{L^2}$ is Fredholm iff the operator*

$$F = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}((A_0\Lambda^{-1} + iB_0)^p(A_0\Lambda^{-1} - iB_0))^{\alpha_1^{-1}\circ\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} \tag{8.39}$$

$$: L^2(\Sigma_0) \to L^2(\Sigma_0) \tag{8.40}$$

is Fredholm. In this case, the index of $\mathcal{D}_{L^2}$ is given by

$$\operatorname{ind}(\mathcal{D}_{L^2}) = \operatorname{ind}(A_0\Lambda^{-1} + iB_0) + \operatorname{ind}(F). \tag{8.41}$$

Proof: The theorem follows directly from *Theorem 3.10* from the section about Fredholm operators. The operator $d - ca^p b$ from *Theorem 3.10* to which the Fredholm property of a 2×2 system with Fredholm type first entry can be reduced is here equal to the operator

$$G = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1}(A_0\Lambda^{-1} + iB_0)^p(A_0\Lambda^{-1} - iB_0). \tag{8.42}$$

Then we realize that for an operator of the form $A\Phi_g B$ we have

$$A\Phi_g B = A(\Phi_g B \Phi_g^{-1})\Phi_g = AB^{g^{-1}}\Phi_g \tag{8.43}$$

and we use this identity to show $F = G$. *q.e.d.*

Remark 8.1: A closer look on (8.41) shows that the index of $\mathcal{D}_{L^2}$ under the assumption that the first entry is Fredholm will be always equal to zero for manifolds Σ_0 with dimension greater than one: The first index in (8.41) is the index of a scalar pseudodifferential operator, which is zero. Concerning the index of F, we can use *Lemma 3.2* to see that this index will also be equal to the index of some scalar pseudodifferential operator, which will also vanish.

If one is only interested in conditions on the operators $A_{0/1}, B_{0/1}$ such that problem (8.2) has a solution space depending on finitely many parameters, it is not necessary that the whole problem is Fredholm. For this purpose it would be sufficient that $\mathcal{D}_{L^2}$ has a finite dimensional kernel. Therefore it is also interesting to investigate some conditions, under which the problem is only semifredholm.

Proposition 8.1: *Consider one of the following special cases of Problem 1 (under the assumption that $A_0\Lambda^{-1} + iB_0$) is Fredholm:*

$(*)$ $B_0 = 0$ *and either* $A_1 = 0$*,* B_1 *Fredholm or* $B_1 = 0$*,* A_1 *Fredholm.*

$(**)$ $A_0 = 0$ *and either* $A_1 = 0$*,* B_1 *Fredholm or* $B_1 = 0$*,* A_1 *Fredholm.*

Both of the cases $(*)$ and $(**)$ are never Fredholm, but always semifredholm, as for the corresponding problems a left parametrix may be constructed.

Proof: $(*)$: When $B_0 = 0$, the operator F from (8.39) simplifies to

$$F = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}((A_0\Lambda^{-1})^p(A_0\Lambda^{-1}))^{\alpha_1^{-1}\circ\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} \tag{8.44}$$

$$= (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}(1+K)^{\alpha_1^{-1}\circ\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1}, \tag{8.45}$$

with K being the compact remainder resulting from the multiplication of $A_0\Lambda^{-1}$ with its parametrix. Performing the conjugation by $\Phi_{\alpha_1^{-1}\circ\alpha_2}$ will leave the identity operator invariant and produce a new compact operator $\tilde{K}$ out of K. This will create an operator F which differs from

$$\tilde{F} = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} - (A_1\Lambda^{-1} + iB_1)^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} \tag{8.46}$$

by some compact operator. Therefore it suffices to refer to $\tilde{F}$ concerning the Fredholm theory of the problem. Now, the principal symbol of $\tilde{F}$ is

$$\sigma_p(\tilde{F}) = \sigma_p(A_1\Lambda^{-1} - iB_1)^{\alpha_2} - \sigma_p(A_1\Lambda^{-1} + iB_1)^{\alpha_2}\tau_{\alpha_2^{-1}\circ\alpha_1}. \tag{8.47}$$

Such a scalar trajectory symbol is invertible, as long as one of the coefficients in (8.47) is invertible and for all ergodic measures μ the geometric means

$$\int_{S^*\Sigma_0} \ln(|\sigma_p(A_1\Lambda^{-1} - iB_1)^{\alpha_2}|)d\mu \neq \int_{S^*\Sigma_0} \ln(|\sigma_p(A_1\Lambda^{-1} + iB_1)^{\alpha_2}|)d\mu. \tag{8.48}$$

(see *Theorem 2.4*) But as long as $A_1 = 0$ or $B_1 = 0$, this can never be the case, because the geometric means will be the same, no matter which ergodic measure is used. This is equivalent to the fact that the operator $\tilde{F}$ is never a Fredholm operator. Now set $B_1 = 0$ and assume that A_1 is a Fredholm operator. For the construction of the left parametrix, let us again consider the original operator $\mathcal{D}_{L^2}$. Its principal symbol is given by

$$\sigma_p(\mathcal{D}_{L^2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_p(A_0\Lambda^{-1}) & \sigma_p(A_0\Lambda^{-1}) \\ \sigma_p(A_1\Lambda^{-1})^{\alpha_2}\tau_{\alpha_2^{-1}\circ\alpha_1} & \sigma_p(A_1\Lambda^{-1})^{\alpha_2} \end{pmatrix} \tag{8.49}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_p(A_0\Lambda^{-1})\tau_{\alpha_1^{-1}\circ\alpha_2} & 0 \\ \sigma_p(A_1\Lambda^{-1})^{\alpha_2} & \sigma_p(A_1\Lambda^{-1})^{\alpha_2} \end{pmatrix} \begin{pmatrix} \tau_{\alpha_2^{-1}\circ\alpha_1} & \tau_{\alpha_2^{-1}\circ\alpha_1} \\ 0 & 1 - \tau_{\alpha_2^{-1}\circ\alpha_1} \end{pmatrix}. \tag{8.50}$$

The product of the two triangular matrices in (8.50) will have a left inverse if both of the matrices have one. Since the matrices are triangular, we only need to check invertibility of the diagonal entries. For the first matrix in (8.50), both diagonal entries are invertible, which follows from the assumption of the Fredholm properties of $A_0\Lambda^{-1} + iB_0$ and A_1. The diagonal entries in the second diagonal matrix are $\tau_{\alpha_2^{-1}\circ\alpha_1}$, which is invertible, and $1 - \tau_{\alpha_2^{-1}\circ\alpha_1}$. Let us find an argument, why $1 - \tau_{\alpha_2^{-1}\circ\alpha_1}$ has a left inverse. Any function $f \in l^2(G)$ in the kernel of this operator must satisfy

$$((1 - \tau_{\alpha_2^{-1}\circ\alpha_1})f)(g) = 0, \;\; g \in G. \tag{8.51}$$

We may write $g = (j, k)$ for any group element, because every $g \in G$ is just a composition of integer powers of α_1 and α_2. Note that since the flows α_1 and α_2 commute the group which they generate is isomorphic to $\mathbb{Z}^2$ with $(j, k) \sim (k, j)$ Then (8.51) is equivalent to

$$f(j, k) = f(j + 1, k - 1) \;\forall (j, k) \in \mathbb{Z}^2_\sim \tag{8.52}$$

This would actually mean that there are countable infitely many points with the same value of f. Such a function can only be in $l^2(G)$ if it is identical to the null function, $f = 0$. Therefore the kernel of $(1 - \tau_{\alpha_2^{-1}\circ\alpha_1})$ is trivial meaning that there exists a left inverse for this operator. But then there also exist a left inverse for both matrices in (8.50) which results in the whole problem represented by $\mathcal{D}_{L^2}$ having a left parametrix. For the case $A_1 = 0$ we have

$$\sigma_p(\mathcal{D}_{L^2}) = \begin{pmatrix} \sigma_p(A_0\Lambda^{-1}) & \sigma_p(A_0\Lambda^{-1}) \\ i\sigma_p(B_1)^{\alpha_2}\tau_{\alpha_2^{-1}\circ\alpha_1} & -i\sigma_p(B_1)^{\alpha_2} \end{pmatrix} \tag{8.53}$$

$$= \begin{pmatrix} \sigma_p(A_0\Lambda^{-1})\tau_{\alpha_1^{-1}\circ\alpha_2} & 0 \\ i\sigma_p(B_1)^{\alpha_2} & i\sigma_p(B_1)^{\alpha_2} \end{pmatrix} \begin{pmatrix} \tau_{\alpha_2^{-1}\circ\alpha_1} & \tau_{\alpha_2^{-1}\circ\alpha_1} \\ 0 & -1 - \tau_{\alpha_2^{-1}\circ\alpha_1} \end{pmatrix}. \tag{8.54}$$

With the same argumentation as above one can show that the matrix product in (8.54) has a left inverse as long as B_1 is Fredholm.

(∗∗) For the case $A_0 = 0$, the proof is basically analogous. The operator corresponding to (8.37) is given by

$$\hat{F} = (A_1\Lambda^{-1} - iB_1)^{\alpha_2} + (A_1\Lambda^{-1} + iB_1)^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1}. \tag{8.55}$$

Again, for $A_1 = 0$ or $B_1 = 0$ the geometric means will be the same, which shows that $\mathcal{D}_{L^2}$ is never Fredholm. The corresponding decompositions of $\mathcal{D}_{L^2}$ are given by

$$A_1 = 0: \quad \sigma_p(\mathcal{D}_{L^2}) = \begin{pmatrix} i\sigma_p(B_0) & -i\sigma_p(B_0) \\ i\sigma_p(B_1)^{\alpha_2}\tau_{\alpha_2^{-1}\circ\alpha_1} & -i\sigma_p(B_1)^{\alpha_2} \end{pmatrix} \tag{8.56}$$

$$= \begin{pmatrix} i\sigma_p(B_0)\tau_{\alpha_1^{-1}\circ\alpha_2} & 0 \\ i\sigma_p(B_1)^{\alpha_2} & i\sigma_p(B_1)^{\alpha_2} \end{pmatrix} \begin{pmatrix} \tau_{\alpha_2^{-1}\circ\alpha_1} & -\tau_{\alpha_2^{-1}\circ\alpha_1} \\ 0 & -1 + \tau_{\alpha_2^{-1}\circ\alpha_1} \end{pmatrix}, \tag{8.57}$$

$$B_1 = 0: \quad \sigma_p(\mathcal{D}_{L^2}) = \begin{pmatrix} i\sigma_p(B_0) & -i\sigma_p(B_0) \\ \sigma_p(A_1\Lambda^{-1})^{\alpha_2}\tau_{\alpha_2^{-1}\circ\alpha_1} & \sigma_p(A_1\Lambda^{-1})^{\alpha_2} \end{pmatrix} \tag{8.58}$$

$$= \begin{pmatrix} i\sigma_p(B_0)\tau_{\alpha_1^{-1}\circ\alpha_2} & 0 \\ \sigma_p(A_1\Lambda^{-1})^{\alpha_2} & \sigma_p(A_1\Lambda^{-1})^{\alpha_2} \end{pmatrix} \begin{pmatrix} \tau_{\alpha_2^{-1}\circ\alpha_1} & -\tau_{\alpha_2^{-1}\circ\alpha_1} \\ 0 & 1 + \tau_{\alpha_2^{-1}\circ\alpha_1} \end{pmatrix}. \tag{8.59}$$

Like in (8.51) one can show that $\mp 1 + \tau_{\alpha_2^{-1}\circ\alpha_1}$ has trivial kernel. The existence of a left parametrix will follow from this just as in (∗). *q.e.d.*

Let us give one more example of Fredholm conditions arising from the assumption that $A_0\Lambda^{-1} + iB_0$ is of Fredholm type. For some usual pseudodifferential operator P perturbations of lower order do not change the Fredholm property. We want to see to which extent this is also true for the current problem by investigating the case where the operators at time $t = 1$ are lower order perturbations of the time $t = 0$ operators, i.e. $A_1 = \Phi_{1/2,1}A_0\Phi_{1/2,1}^{-1} + K_0$, $K_0 \in \mathrm{OPS}^0$, $B_1 = \Phi_{1/2,1}B_0\Phi_{1/2,1}^{-1} + K_{-,1}$, $K_{-,1} \in \mathrm{OPS}^{-1}$.

Proposition 8.2: *Let A_0 and B_0 be given operators such that $A_0\Lambda^{-1} + iB_0$ is a Fredholm operator. Moreover, assume that Λ commutes with both solution operators $\Phi_{1,1}$ and $\Phi_{1,2}$. Consider the following cases for the definition of A_1 and A_0:*

$$(i)\ A_1 = \Phi_{1,1}A_0\Phi_{1,1}^{-1} + K_0,\ K_0 \in \mathrm{OPS}^0 \quad B_1 = \Phi_{1,1}B_0\Phi_{1,1}^{-1} + K_{-,1},\ K_{-,1} \in \mathrm{OPS}^{-1} \tag{8.60}$$

$$(ii)\ A_1 = \Phi_{1,2}A_0\Phi_{1,2}^{-1} + K_0,\ K_0 \in \mathrm{OPS}^0 \quad B_1 = \Phi_{1,2}B_0\Phi_{1,2}^{-1} + K_{-,1},\ K_{-,1} \in \mathrm{OPS}^{-1}. \tag{8.61}$$

It holds that (i) is never Fredholm and (ii) is Fredholm as long as $A_0\Lambda^{-1} - iB_0$ is a Fredholm operator and

$$\int_{S^*\Sigma_0} \ln(|\sigma_p(A_0\Lambda^{-1} - iB_0)|) d\mu_G \neq \int_{S^*\Sigma_0} \ln(|\sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2}|) d\mu_g \tag{8.62}$$

for all ergodic measures μ_g.

Proof: Since $A_0\Lambda^{-1} + iB_0$ is assumed to be Fredholm, we may again reduce the Fredholm property of $\mathcal{D}_{L^2}$ to that of the operator F ((8.39)). The principal symbol of the operator $(A_1\Lambda^{-1} + iB_1)^{\alpha_2}$ appearing in F and under the assumption (8.60) becomes

$$\sigma_p(A_1\Lambda^{-1} + iB_1)^{\alpha_2} = \sigma_p(\Phi_{1,2}^{-1}(\Phi_{1,1}A_0\Phi_{1,1}^{-1}\Lambda^{-1} + K_0\Lambda^{-1} + i\Phi_{1,1}B_0\Phi_{1,1}^{-1} + K_{-,1})\Phi_{1,2}) \tag{8.63}$$

$$= \sigma_p(\Phi_{1,2}^{-1}\Phi_{1,1}A_0\Lambda^{-1}\Phi_{1,1}^{-1}\Phi_{1,2} + i\Phi_{1,2}^{-1}\Phi_{1,1}B_0\Phi_{1,1}^{-1}\Phi_{1,2}) = \sigma_p(A_0\Lambda^{-1} + iB_0)^{\alpha_1^{-1}\circ\alpha_2} \tag{8.64}$$

where we used the commutativity of Λ with the solution operator and the fact that terms involving $K_{-,1}$ and $K_0\Lambda^{-1}$ do not contribute to the principal symbol. Now this may be used to calculate the principal symbol of the operator F:

$$\sigma_p(F) = \sigma_p(A_1\Lambda^{-1} - iB_1)^{\alpha_2} - \sigma_p(A_1\Lambda^{-1} + iB_1)^{\alpha_2} \frac{\sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2}}{\sigma_p(A_0\Lambda^{-1} + iB_0)^{\alpha_1^{-1}\circ\alpha_2}} \tau_{\alpha_1^{-1}\circ\alpha_2} \tag{8.65}$$

$$= \sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2} - \sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2} \tau_{\alpha_1^{-1}\circ\alpha_2}. \tag{8.66}$$

Note that we used (8.64) and a corresponding version for $\sigma_p(A_1\Lambda^{-1} - iB_1)^{\alpha_2}$ to simplify (8.65). The resulting operator in (8.66) is again an operator, where the geometric means of the coefficients are identical, so the problem under the assumption (8.60) can not be Fredholm. Calculating the principal symbol of $A_1\Lambda^{-1} + iB_1$ under the assumption (8.61) results in

$$\sigma_p(A_1\Lambda^{-1} + iB_1)^{\alpha_2} = \sigma_p(\Phi_{1,1}(\Phi_{1,2}A_0\Phi_{1,2}^{-1}\Lambda^{-1} + K_0\Lambda^{-1} + i\Phi_{1,2}B_0\Phi_{1,2}^{-1} + K_{-,1})\Phi_{1,1}^{-1}) \tag{8.67}$$

$$= \sigma_p(A_0\Lambda^{-1} + iB_0). \tag{8.68}$$

This generates an operator F with principal symbol

$$\sigma_p(F) = \sigma_p(A_1\Lambda^{-1} - iB_1)^{\alpha_2} - \sigma_p(A_0\Lambda^{-1} + iB_0) \frac{\sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2}}{\sigma_p(A_0\Lambda^{-1} + iB_0)} \tau_{\alpha_2^{-1}\circ\alpha_1}. \tag{8.69}$$

$$= \sigma_p(A_0\Lambda^{-1} - iB_0) - \sigma_p(A_0\Lambda^{-1} - iB_0)^{\alpha_1^{-1}\circ\alpha_2} \tau_{\alpha_2^{-1}\circ\alpha_1}. \tag{8.70}$$

If we now compare the geometric means of the coefficients in order to get the Fredholm conditions, we get condition (8.62).

q.e.d.

8.2 Simplifications in the FLRW Case and for the Time Independent Equation

All the Fredholm conditions we derived until now are valid not only for the wave equation on a FLRW space but for the general wave equation on a globally hyperbolic manifold. There are some simplifications which arise in the FLRW case concerning the flow evaluation of certain operators in the Fredholm conditions. Let us state shortly in which sense the Fredholm conditions already derived are becoming easier in the FLRW case.
First off all, we want to show that in the FLRW case $\Phi_{1,2}^{-1} = \Phi_{1,1}$.

Proposition 8.3: *Consider a globally hyperbolic manifold on which the metric tensor is given by the FLRW metric* (8.1)*. Then the two scalar solution operators* $\Phi_{1,1}$ *and* $\Phi_{1,2}$ *are inverses of each other.*

Proof: The equations

$$\dot{\Phi}_{t,1/2} = \pm i\sqrt{\Delta_1}\Phi_{t,1/2} \tag{8.71}$$

with the principal part Δ_1 of Δ can be solved exactly in the FLRW case: From (8.1) we know that the principal part of the Laplace-Beltrami operator Δ associated to the metric g_t is

$$\Delta_1 = g^{ij}\partial_{x_i}\partial_{x_j} = hg^{ij}(0)\partial_{x_i}\partial_{x_j} = h\Delta_{0,1} \tag{8.72}$$

with Δ_0 being the Laplacian associated to g_0. Then the equations we need to solve are

$$\dot{\Phi}_{1/2,t} = \pm i\sqrt{h}\sqrt{\Delta_{0,1}}\Phi_{1/2,t} \tag{8.73}$$

(with initial conditions $\Phi_{1/2,0} = 1$). It is not hard to guess that the solutions are

$$\Phi_{1/2,t} = e^{\pm i\sqrt{\Delta_{0,1}}\int_0^t h(s)ds}. \tag{8.74}$$

We realize $\Phi_{2,t} = \Phi_{1,t}^{-1}$. *q.e.d.*

Proposition 8.4: *In the FLRW case, we have the following simplifications of the Fredholm conditions already derived:*

(+) The Fourier integral operator in $\mathcal{D}_{L^2}$ from Theorem 1 simplifies to

$$\Phi_{\alpha_2^{-1}\circ\alpha_1} = \Phi_{\alpha_1}^2 = \Phi_{1,1}^2. \tag{8.75}$$

(++) The Fredholm conditions stated in Theorem 8.2, Proposition 8.1 and Proposition 8.2 simplify in that sense that every term of the form C^{α_2} may be replaced by $C^{\alpha_1^{-1}}$ and every term if the form $D^{\alpha_1^{-1}\circ\alpha_2}$ may be replaced by $D^{\alpha_1^{-2}}$.

Proof: (+) Using *Proposition 8.3* we directly see

$$\Phi_{\alpha_2^{-1}\circ\alpha_1} = \Phi_{\alpha_2}^{-1}\Phi_{\alpha_1} = \Phi_{\alpha_1}^2. \tag{8.76}$$

(++) For any operator C we have

$$\sigma_p(C)^{\alpha_2} = \sigma_p(\Phi_{1,2}^{-1}C\Phi_{1,2}) = \sigma_p(\Phi_{1,1}C\Phi_{1,1}^{-1}) = \sigma_p(C)^{\alpha_1^{-1}}. \tag{8.77}$$

Moreover it follows $\sigma_p(C)^{\alpha_1^{-1}\circ\alpha_2} = \sigma_p(C)^{\alpha_1^{-2}}$. *q.e.d.*

Example 8.2: Let us show that the Fredholm conditions (8.39) reduce to the conditions *Savin* and *Boltachev* discovered in [11], when we investigate the corresponding special case of (8.2). First, in the paper of *Savin, Boltachev* the wave equation was time independent. The resulting solution operators $\Phi_{1,t}$ and $\Phi_{2,t}$ of the diagonal system $\partial_t u = Du$ are then just

$$\Phi_{1,t} = e^{i\sqrt{\Delta}t}, \quad \Phi_{2,t} = e^{-i\sqrt{\Delta}t}. \tag{8.78}$$

Moreover, the Hamilton fields $H_{\sqrt{\Delta}}$ and $H_{-\sqrt{\Delta}} = -H_{\sqrt{\Delta}}$ are both time independent, which makes the corresponding flows α_1 and α_2 inverse to each other. This means that the group G generated by the Hamiltonian flows is just $\mathbb{Z}$. The operator $\Phi_{\alpha_2^{-1}\circ\alpha_1}$ then equals $\Phi_{\alpha_1^2}$ with corresponding shift operator $\tau_{\alpha_1^2} = \tau_1^2$ on $l^2(\mathbb{Z})$. The boundary conditions considered by *Savin, Boltachev* are

$$u(0) = g_0, \quad A_1 u(1) + B_1(\partial_t u)(1) = g_1 \tag{8.79}$$

so we have $A_0 = 1$, $B_0 = 0$. Inserting this into (8.39) results in

$$F = (A_1\Lambda^{-1} - iB_1)^{\alpha_1^{-1}} - (A_1\Lambda^{-1} + iB_1)^{\alpha_1^{-1}}\Phi_{\alpha_1^2}. \tag{8.80}$$

The operator, to which the Fredholm property reduces in the work of *Savin* and *Boltachev* was

$$D = (-iA_1\sqrt{\Delta}^{-1} + B_1)e^{i\sqrt{\Delta}} + (iA_1\sqrt{\Delta}^{-1} + B_1)e^{-i\sqrt{\Delta}}. \tag{8.81}$$

Multiplying (8.79) by $-ie^{i\sqrt{\Delta}}$ from the right gives

$$D(-i)e^{i\sqrt{\Delta}} = -(A_1\sqrt{\Delta}^{-1} + iB_1)e^{2i\sqrt{\Delta}} + (A_1\sqrt{\Delta}^{-1} - iB_1). \tag{8.82}$$

Note that because the factor $(-i)e^{i\sqrt{\Delta}}$ is invertible, the Fredholm property of (8.81) is equivalent to that of (8.82). As already used before we remember that the principal part of Λ^{-1} is equal to the principal part of $\sqrt{\Delta}^{-1}$. We already know that the Fredholm property only depends on the principal parts of operators. But then (8.82) and the operator F from (8.80) have practically the same principal parts, with the only difference that the coefficients $(A_1\Lambda^{-1} - iB_1)$ and $(A_1\Lambda^{-1} + iB_1)$ are evaluated at the flow α_1^{-1} in F, whereas there is no flow evaluation in the coefficients of the operator in (8.82). However, if we state the Fredholm conditions with the help of the geometric means, (8.80) will be Fredholm if and only if (8.82) is Fredholm and the flow will make no difference. Therefore we have shown that the Fredholm conditions found by *Savin* and *Boltachev* are indeed a special case of *Theorem 8.2*.

8.3 Fredholm Conditions without Order Reduction

8.3.1 The General Second Order Fredholm Conditions

The foregoing procedure to derive the Fredholm conditions for *Problem 8.1* can be summarized as follows: We took the orignal second order equation and the corresponding first order boundary data, converted it to a first order system of equations with zero order boundary data and applied the theory of the foregoing chapter. It is a justified question, whether it would also be possible to find a solution operator to the original second order equation (8.2), (i), and reduce the Fredholm property of *Problem 8.1* to an operator which is directly associated to the second order equation. We want to show in this last section of this work that this is indeed possible and that by assuming the Fredholm property of A_0 itself instead of the Fredholm property of $A_0\Lambda^{-1} + iB_0$ one may get Fredholm conditions in a similar style like those in *Theorem 8.2*.

Theorem 8.3: *Alternatively to Theorem 8.1, Problem 8.1 is Fredholm iff the operator*

$$\mathcal{D}_2 = \begin{pmatrix} A_0 & B_0 \\ A_1\Psi_1 + B_1\dot{\Psi}_t(1) & A_1\Psi_{\partial 1} + B_1\dot{\Psi}_{\partial t}(1) \end{pmatrix} : H^1(\Sigma_0) \times L^2(\Sigma_0) \to L^2(\Sigma_0) \times L^2(\Sigma_1) \tag{8.83}$$

is Fredholm, where the operators Ψ_t and $\Psi_{\partial t}$ together form the solution operator of the second order equation $\partial_t^2 u = \Delta u$ in the sense that

$$u(t) = \Psi_t g_0 + \Psi_{\partial t} g_1 \tag{8.84}$$

solves the second order Cauchy problem

$$\partial_t^2 u = \Delta u, \quad u(0) = g_0, \quad (\partial_t u)(0) = g_1. \tag{8.85}$$

Proof: It is known that the Cauchy problem for the wave equation $\partial_t^2 u = \Delta u$ is well posed, i.e. for given initial data $u(0)$ and $(\partial_t u)(0)$ there exists a unique solution. Since this solution is linear in the initial data, there must be two operators Ψ_t and Ψ_{∂_t} such that

$$u(t) = \Psi_t u(0) + \Psi_{\partial_t}(\partial_t u)(0). \tag{8.86}$$

and $\Psi_t(0) = 1$, $\Psi_{\partial_t}(0) = 0$. If we are searching for solutions $u \in H^1(M)$, there has to be $\Psi_t : H^1(\Sigma_0) \to H^1(\Sigma_t)$, $\Psi_{\partial_t} : L^2(\Sigma_0) \to H^1(\Sigma_t)$. If we want to write conditions $(8.2), (ii) - (iii)$ with the help of a matrix acting on $u(0)$ and $(\partial_t u)(0)$, the first row of (8.83) is just the matrix representation of $(8.2), (ii)$. For $(2), (iii)$ we may simply use (8.84):

$$A_1 u(1) + B_1(\partial_t u)(1) = A_1(\Psi_1 u(0) + \Psi_{\partial_1}(\partial_t u)(0)) + B_1(\dot{\Psi}_t(1)u(0) + \dot{\Psi}_{\partial_t}(1)(\partial_t u)(0)) \tag{8.87}$$

$$= (A_1\Psi_1 + B_1\dot{\Psi}_t(1))u(0) + (A_1\Psi_{\partial_1} + B_1\dot{\Psi}_{\partial_t}(1))(\partial_t u)(0). \tag{8.88}$$

(8.86) generates the second row of the matrix in (8.81). *q.e.d.*

8.3.2 Second Order Solution Operators

Example 8.3: As long as the Laplacian Δ is time independent, the explicit dependence of the solution operators Ψ_t and Ψ_{∂_t} on Δ can be found quite easily. For that we make the ansatz

$$u(t) = e^{i\sqrt{\Delta}t}\mu_1(x) + e^{-i\sqrt{\Delta}t}\mu_2(x) \tag{8.89}$$

in order to solve (8.85). We want to express $\mu_1(x)$ and $\mu_2(x)$ as functions of g_1 and g_2. By simply inserting the initial conditions we get

$$u(0) = \mu_1(x) + \mu_2(x) = g_0, \quad (\partial_t u)(0) = i\sqrt{\Delta}(\mu_1(x) - \mu_2(x)) = g_1. \tag{8.90}$$

Because the first order operator $\sqrt{\Delta}$ is a Fredholm operator, we can multiply the second condition in (8.90) by a parametrix $\sqrt{\Delta}^p$ and solve (8.90) for μ_1 and μ_2 up to an error lying in the image of a compact operator. This yields

$$\mu_1(x) = \frac{g_0 - i\sqrt{\Delta}^p g_1}{2}, \quad \mu_2(x) = \frac{g_0 + i\sqrt{\Delta}^p g_1}{2}. \tag{8.91}$$

With (8.91) we can now express the solution u in terms of g_0 and g_1:

$$u = e^{i\sqrt{\Delta}t}\left(\frac{g_0}{2} - \frac{i\sqrt{\Delta}^p g_1}{2}\right) + e^{-i\sqrt{\Delta}t}\left(\frac{i\sqrt{\Delta}^p g_1}{2} + \frac{g_0}{2}\right) \tag{8.92}$$

$$= \frac{1}{2}\left(e^{i\sqrt{\Delta}t} + e^{-i\sqrt{\Delta}t}\right) g_0 + \frac{i\sqrt{\Delta}^p}{2}\left(e^{-i\sqrt{\Delta}t} - e^{i\sqrt{\Delta}t}\right) g_1. \tag{8.93}$$

Using the last equality we can directly identify Ψ_t and Ψ_{∂_t} as

$$\Psi_t = \frac{1}{2}\left(e^{i\sqrt{\Delta}t} + e^{-i\sqrt{\Delta}t}\right), \quad \Psi_{\partial_t} = \frac{i\sqrt{\Delta}^p}{2}\left(e^{-i\sqrt{\Delta}t} - e^{i\sqrt{\Delta}t}\right). \tag{8.94}$$

After deriving the explicit expressions for Ψ_t and Ψ_{∂_t} for the time independent case in the foregoing example, we want to compare these expressions to what we get in the case of a time dependent Laplacian. It turns out that in the time dependent case the solution operators Ψ_t and Ψ_{∂_t} have basically the same form like in (8.94) (up to compact errors and some modifications with Λ) and are obtained from (8.94) by simply replacing $\exp(\pm i\sqrt{\Delta}t)$ with the more general solution operators $\Phi_{t,1/2}$:

Proposition 8.3: *If one considers the equation $\partial_t^2 u = \Delta u$ with a Laplace-Beltrami operator Δ associated to a general time dependent metric $g = g(t)$, the solution operators Ψ_t and Ψ_{∂_t} of the Cauchy-problem* (8.85) *are*

$$\Psi_t = \frac{1}{2}\Lambda^{-1}(t)(\Phi_{1,t} + \Phi_{2,t})\Lambda(0) + K_t, \quad \Psi_{\partial_t} = \frac{i}{2}\Lambda^{-1}(t)(\Phi_{2,t} - \Phi_{1,t}) + K_{\partial_t} \tag{8.95}$$

with compact operators K_t, K_{∂_t}, the solution operators $\Phi_{1/2,t}$ fulfilling $\dot{\Phi}_{1/2,t} = \pm i\sqrt{\Delta}\Phi_{1/2,t}$ and $\Lambda^{-1}(1), \Lambda(0)$ being the evaluations of the operators at the corresponding times.

Proof: We will make use of the previously performed order reduction and diagonalization (see *Lemma 8.1*) and transform everything back to the original function u in order to express the solution to $\partial_t^2 u = \Delta u$ via $\Phi_{1/2,t}$. First we have

$$u(t) = \Lambda^{-1}(t)u_1(t) = \frac{1}{\sqrt{2}}\Lambda^{-1}(t)(v_1(t) + v_2(t)). \tag{8.96}$$

Now we know from (8.3) and *Theorem 4.2* that $v_1(t) = \Phi_{1,t}v_1(0) + \sum_j K_{1j}v_j(0)$, $v_2(t) = \Phi_{2,t}v_2(0) + \sum_j K_{2,j}v_j(0)$ with K_{ij} being the components of a compact matrix K. Furthermore, with the transformation matrix T from (8.10) we may express $v_1(0)$ and $v_2(0)$ by $u_1(0)$ and $u_2(0)$ via

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Leftrightarrow \quad v_1(0) = \frac{1}{\sqrt{2}}(u_1(0) - iu_2(0)), \; v_2(0) = \frac{1}{\sqrt{2}}(u_1(0) + iu_2(0)). \tag{8.97}$$

Proceeding with the calculation of u we get

$$u = \frac{1}{2}\Lambda^{-1}(t)(\Phi_{1,t}(u_1(0) - iu_2(0)) + \Phi_{2,t}(u_1(0) + iu_2(0)) + \tilde{K}u_1(0) + \hat{K}u_2(0)) \tag{8.98}$$

where $\tilde{K}$ and $\hat{K}$ are compact operators which come from calculating products of the operators K_{ij} with the components of T^{-1}. If we rearrange the terms and use $u_1(0) = \Lambda(0)u(0)$, $u_2(0) = (\partial_t u)(0)$ we arrive at

$$u(t) = \frac{1}{2}\Lambda^{-1}(t)(\Phi_{1,t} + \Phi_{2,t})\Lambda(0)u(0) + \frac{i}{2}\Lambda^{-1}(t)(\Phi_{2,t} - \Phi_{1,t})(\partial_t u)(0) + \tilde{K}\Lambda^{-1}(t)u(0) + \hat{K}(\partial_t u)(0). \tag{8.99}$$

As the two operators $\tilde{K}\Lambda^{-1}$ and $\hat{K}$ are both compact, we can set $\tilde{K}\Lambda^{-1} = K_t$, $\hat{K} = K_{\partial_t}$ and get the desired form of the solution operators stated in (8.93). *q.e.d.*

After we expressed the solution operators Ψ_t and Ψ_{∂_t} in terms of $\Phi_{1/2,t}$ we may use these expressions to get Fredholm conditions for *Problem 1* directly from the operator $\mathcal{D}_2$ and not from the order reduced operator $\mathcal{D}_{L^2}$.

Theorem 8.4: *The operator $\mathcal{D}_2$ from Theorem 8.3 is Fredholm, iff the operator*

$$\mathcal{D}_\Phi = \begin{pmatrix} \overbrace{\frac{1}{2}\Lambda^{-1}(1)(A_1 + iB_1\sqrt{\Delta})\Phi_{1,1}\Lambda(0) + \frac{1}{2}\Lambda^{-1}(1)(A_1 - iB_1\sqrt{\Delta})\Phi_{2,1}\Lambda(0)}^{A_0} & \overbrace{\frac{1}{2}\Lambda^{-1}(B_1\sqrt{\Delta} - iA_1)\Phi_{1,1} + \frac{1}{2}\Lambda^{-1}(iA_1 + B_1\sqrt{\Delta})\Phi_{2,1}}^{B_0} \end{pmatrix} \tag{8.100}$$

$$: H^1(\Sigma_0) \times L^2(\Sigma_0) \to L^2(\Sigma_0) \times L^2(\Sigma_0) \tag{8.101}$$

is Fredholm.

Proof: Let us first calculate the derivatives $\dot{\Psi}_t$ and $\dot{\Psi}_{\partial_t}$ needed for the operator $\mathcal{D}_2$:

$$\dot{\Psi}_t = \frac{1}{2}(\dot{\Lambda}^{-1}(\Phi_{1,t} + \Phi_{2,t}) + \Lambda^{-1}(\dot{\Phi}_{1,t} + \dot{\Phi}_{2,t}))\Lambda(0) + \dot{K}_t \tag{8.102}$$

$$= \frac{1}{2}((\dot{\Lambda}^{-1} + i\Lambda^{-1}\sqrt{\Delta})\Phi_{1,t} + (\dot{\Lambda}^{-1} - i\Lambda^{-1}\sqrt{\Delta})\Phi_{2,t})\Lambda(0) + \dot{K}_t, \tag{8.103}$$

$$\dot{\Psi}_{\partial_t} = \frac{i}{2}\dot{\Lambda}^{-1}(\Phi_{2,t} - \Phi_{1,t}) + \frac{i}{2}\Lambda^{-1}(\dot{\Phi}_{2,t} - \dot{\Phi}_{1,t}) \tag{8.104}$$

$$= \left(\frac{1}{2}\Lambda^{-1}\sqrt{\Delta} - \frac{i}{2}\dot{\Lambda}^{-1}\right)\Phi_{1,t} + \left(\frac{1}{2}\Lambda^{-1}\sqrt{\Delta} + \frac{i}{2}\dot{\Lambda}^{-1}\right)\Phi_{2,t} + \dot{K}_{\partial_t}. \tag{8.105}$$

Note that as we are interested in the Fredholm theory of $\mathcal{D}_2$, if we insert (8.103) and (8.105) into (8.83) only operators with the highest order will matter. Thus we can neglect the compact terms $\dot{K}_t$, $\dot{K}_{\partial_t}$ as well as all terms involving the time derivative $\dot{\Lambda}^{-1}$:

These will appear in $\mathcal{D}_2$ only as products with B_1, which is a zero order operator and since $\dot{\Lambda}^{-1}$ is of order -1, so will be all products with B_1. All other operators involved in the components of $\mathcal{D}_2$ are of order zero. If we now plug (8.103) and (8.105) into (8.83) ommiting the $\dot{K}_t$, $\dot{K}_{\partial_t}$ and all $\dot{\Lambda}^{-1}$ terms, we arrive precisely at the operator D_Φ from (8.98). *q.e.d.*

8.3.3 Special Case of the Second Order System

With *Theorem 8.4* we can connect *Problem 8.1* to a matrix of $G-$operators, which is more related to the original second order equation $\partial_t u = \Delta u$ rather than to its order reduced system of equations. While we had to assume the Fredholm property of $A_0\Lambda^{-1} + iB_0$ for the Fredholm conditions from *Theorem 8.2*, the operator $\mathcal{D}_\Phi$ gives us alternative Fredholm conditions if we assume A_0 to be Fredholm:

Theorem 8.5: *As an alternative to Theorem 8.2, we may assume that A_0 is a Fredholm operator. Then Problem 8.1 is Fredholm as long as*

$$F_2 = (B_1\sqrt{\Delta}(1+i(\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}})+A_1((\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}}-i))+(A_1(i-(\Lambda(0)A_0^pB_0)^{\alpha_2^{-1}})+B_1\sqrt{\Delta}(1-i(\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}}))\Phi_{\alpha_2\circ\alpha_1^{-1}} \tag{8.106}$$

$$: L^2(\Sigma_0) \to L^2(\Sigma_0) \tag{8.107}$$

is Fredholm.

Proof: Just as in the proof of *Theorem 8.2* we make use of *Theorem 3.10* and apply it to $\mathcal{D}_\Phi$. This reduces the Fredholm property of $\mathcal{D}_\Phi$ to that of

$$\tilde{F}_2 = \frac{1}{2}\Lambda^{-1}(B_1\sqrt{\Delta}(1+i(\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}})+A_1((\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}}-i))\Phi_{1,1}+\frac{1}{2}\Lambda^{-1}(A_1(i-(\Lambda(0)A_0^pB_0)^{\alpha_2^{-1}})+B_1\sqrt{\Delta}(1-i(\Lambda(0)A_0^pB_0)^{\alpha_1^{-1}}))\Phi_{2,1}. \tag{8.108}$$

Since multiplication with invertible operators does not change Fredholm property we can multiply $\tilde{F}_2$ by 2Λ from the left and by $\Phi_{1,1}^{-1}$ from the right. This results in the Fredholm property of $\mathcal{D}_\Phi$ being reduced to that of F_2. *q.e.d.*

8.4 Index Formula Simplifications

The index formula (8.41) expresses the index of the 2×2 matrix operator $\mathcal{D}_{L^2}$ in terms of the indices of its components. However, although the first term $\mathrm{ind}(A_0\Lambda^{-1} + iB_0)$ can be computed with the help of known index formulas like the Fedosov index formula, the second term $\mathrm{ind}(F)$ is the index of a two term G operator and therefore one has to compute geometric means in order to know which of the two coefficients in (8.39) will be the operator relevant for the index (see *Lemma 3.2*). Let us discuss two cases, where one can express (8.41) in an easy way.

First Case:

Investigating the matrix $\mathcal{D}_{L^2}$ from (8.24) we see that it becomes triangular if the upper right entry vanishes. In this case we could simply express the index as the sum of the indices of two pseudodifferential operators, and such indices may be computed with the use of Fedosovs index formula.

Proposition 8.4: *Assume* $\sigma_p(A_0\Lambda^{-1}) = i\sigma_p(B_0)$*. In this case the index of* $\mathcal{D}_{L^2}$ *is given by the integral*

$$\operatorname{ind}(\mathcal{D}_{L^2}) = \frac{1}{4\pi}\int_{S^*\Sigma_0} \sigma_P(B_0)^{-1} d\sigma_p(B_0) + \frac{1}{2\pi i}\int_{S^*\Sigma_0} \sigma_p(A_1\Lambda^{-1} - iB_1)^{-1} d\sigma_p(A_1\Lambda^{-1} - iB_1), \quad (8.109)$$

as long as B_0 *and* $A_1\Lambda^{-1} - iB_1$ *are of Fredholm type and* Σ_0 *is one-dimensional. If* $dim(\Sigma_0) > 1$*, the index is equal to zero.*

Proof: First we want to remind that the index just as the Fredholm property itself does only depend on the principal part of the operator $\mathcal{D}_{L^2}$. Therefore we do not need the entry $A_0\Lambda^{-1} - iB_0$ to fully vanish, we only need a vanishing principal symbol. This explains the assumption $\sigma_p(A_0\Lambda^{-1}) = i\sigma_p(B_0)$. Denoting by P_0 the principal part of an order zero operator, the principal part of $\mathcal{D}_{L^2}$ then becomes

$$\mathcal{D}_{L^2,0} = \begin{pmatrix} (A_0\Lambda^{-1} + iB_0)_0 & 0 \\ (A_1\Lambda^{-1} + iB_1)_0^{\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1} & (A_1\Lambda^{-1} - iB_1)_0^{\alpha_2} \end{pmatrix}. \quad (8.110)$$

As we already know, the Fredholm property of such a triangular operator will be given as long as the diagonal entries are Fredholm themselves. In this case, all diagonal entries are usual pseudodifferential operators. Therefore we have

$$\operatorname{ind}(\mathcal{D}_{L^2}) = \operatorname{ind}(A_0\Lambda^{-1} + iB_0) + \operatorname{ind}((A_1\Lambda^{-1} - iB_1)^{\alpha_2}) \quad (8.111)$$

as long as the corresponding operators are Fredholm. Fortunalety both operators involved are pseudodifferential, so we may use Fedosovs formula. Moreover, since

$$(A_1\Lambda^{-1} - iB_1)^{\alpha_2} = \Phi_2^{-1}(A_1\Lambda^{-1} - iB_1)\Phi_2 \quad (8.112)$$

this operator has the same index as $(A_1\Lambda^{-1} - iB_1)$, because left or right multiplication by invertible operators does not change the index. This means that we may omit the flow α_2 in (8.111). $\sigma_p(A_0\Lambda^{-1}) = i\sigma_p(B_0)$ means that the principal symbol of the first entry in (8.110) becomes $2i\sigma_p(B_0)$. Then we remember *Corollary 3.2* from the chapter about Fredholm operators: Since both B_0 and $A_1\Lambda^{-1} - iB_1$ are of scalar type, the index of $\mathcal{D}_{L^2}$ will be zero for all manifolds with dimension greater than one. If the dimension of Σ_0 is one, we use formula (3.25) from *Chapter 3* together with $\sigma_p(A_0\Lambda^{-1} + iB_0) = 2i\sigma_p(B_0)$ and arrive at the desired formula (8.109). *q.e.d.*

Second case:

We have already shown that as long as $A_0\Lambda^{-1} + iB_0$ is of Fredholm type, the Fredholmness of $\mathcal{D}_{L^2}$ can be reduced to that of the operator F in (8.39). This operator has a trajectorial symbol of the form $\sigma_p(F) = a + b\tau$, which is really hard to invert in general. But since operators of the form $b\tau$ are very simple to invert, we want to investigate the case $a = 0$, i.e. $\sigma_p((A_1\Lambda^{-1} - iB_1)^{\alpha_1}) = 0$.

Proposition 8.5: *Let $A_0\Lambda^{-1}+iB_0$ be a Fredholm operator. Furthermore, let $\sigma_p(A_1\Lambda^{-1}) = i\sigma_p(B_1)$. Define the operator J by*

$$J = 2iB_1^{\alpha_2}((A_0\Lambda^{-1} + iB_0)^p)(A_0\Lambda^{-1} - iB_0))^{\alpha_1^{-1}\circ\alpha_2}. \tag{8.113}$$

As long as B_1 and $(A_0\Lambda^{-1} - iB_0)$ are both Fredholm, so is J and one may consider $JJ^p - 1 = K_R$ and $J^pJ - 1 = K_L$, $J^p = op(\sigma_p(J)^{-1})$. For N large enough, we get

$$\mathrm{ind}(\mathcal{D}_{L^2}) = \mathrm{ind}(A_0\Lambda^{-1} + iB_0) + \mathrm{ind}(J). \tag{8.114}$$

which, as a sum of to indices of scalar operators, is again zero for dim(M)>1.

Proof: For operators A_1 and B_1 with $\sigma_p(A_1\Lambda^{-1}) = i\sigma_p(B_0)$, the Fredholm property of the operator F from (8.39) is equivalent to the Fredholm property of

$$\tilde{F} = -(A_1\Lambda^{-1} + iB_1)^{\alpha_2}((A_0\Lambda^{-1} + iB_0)^p(A_0\Lambda^{-1} - iB_0))^{\alpha_1^{-1}\circ\alpha_2}\Phi_{\alpha_2^{-1}\circ\alpha_1}. \tag{8.115}$$

Note that the assumption $\sigma_p(A_1\Lambda^{-1}) = i\sigma_p(B_1)$ means that the zero order parts of $A_1\Lambda^{-1}$ and iB_1 are identical. We can therefore expand the first operator in the product (8.115) as

$$(A_1\Lambda^{-1} + iB_1)^{\alpha_2} = 2iB_1^{\alpha_2} + K_-^{\alpha_2} \tag{8.116}$$

where $K_-^{\alpha_2}$ is the conjugation of a negative order operator by the operator $\Phi_{\alpha_2^{-1}}$. Thus $\tilde{F}$ can be written as the sum of $-J\Phi_{\alpha_2^{-1}\circ\alpha_1}$ and a negative order operator, which is compact and therefore not relevant concerning the Fredholm theory of $\mathcal{D}_{L^2}$. By inserting $\tilde{F}$ into (8.41) we are left with

$$\mathrm{ind}(\mathcal{D}_{L^2}) = \mathrm{ind}(A_0\Lambda^{-1} + iB_0) + \mathrm{ind}(-J\Phi_{\alpha_2^{-1}\circ\alpha_1}). \tag{8.117}$$

Since multiplication with invertible operators do not change the index, we get (8.114).
q.e.d.

Remark 8.2: Both cases $\sigma_p(A_0\Lambda^{-1}) = i\sigma_p(B_0)$ and $\sigma_p(A_1\Lambda^{-1}) = i\sigma_p(B_1)$ have a certian Φ independence in common: Although the general operator $\mathcal{D}_{L^2}$ and its general Fredholm conditions depend on the flow defined by $\Phi_{\alpha_2^{-1}\circ\alpha_1}$ we see in (8.111) that under the assumption $\sigma_p(A_0\Lambda^{-1}) = i\sigma_p(B_0)$ (with Fredholm B_0) neither the index nor the Fredholm property actually depend on the Hamiltonian flow, as we can simply omit it in (8.111) (as already discussed in the proof). The same holds for the index of J in (8.113):

Since J is a product of three Fredholm operators, its index will be equal to the sum of the indices of these three operators, and apearing flows in these indices can again be omitted.

8.5 Further Possible Considerations

The theory from *Chapter* 7 led to the fact that only one G-operator is involved in the operator $\mathcal{D}_{L^2}$ from (8.24). Of course from a purely mathematical point of view one could instead consider more general 2×2 operators of the type

$$\begin{pmatrix} a\Phi_{11} + b\Phi_{12} & c\Phi_{11} + d\Phi_{12} \\ e\Phi_{11} + f\Phi_{12} & g\Phi_{11} + h\Phi_{12} \end{pmatrix}. \tag{8.118}$$

The corresponding operator F from *Theorem 8.2* would be more complicated as a G operator, as it would not be in the form $F = a + b\Phi$ anymore. However, a consideration of the FLRW case would be still beneficial: For the case of a FLRW metric (8.1) we have $\pm\sigma_p(\sqrt{\Delta}) = \pm\sqrt{h}\sqrt{\sigma_p(\Delta_0)}$. By *Theorem 8.1* these functions are defining the flows α_1 and α_2. For the poisson brackets we calculate

$$\{\sigma_p(\sqrt{\Delta})(t), -\sigma_p(\sqrt{\Delta})(\tilde{t})\} = -\sqrt{h(t)h(\tilde{t})}\{\sigma_p(\sqrt{\Delta_0}), \sigma_p(\sqrt{\Delta_0})\} = 0,\ \forall(t,\tilde{t}). \tag{8.119}$$

By *Remark 7.6* we conclude that the flows α_1 and α_2 are commutative in the FLRW case (note that the symbold $\pm\sigma_p(\sqrt{\Delta})$ are smooth outside the zero section, thus the integral curves can be approximated by the use of Euler´s method just like in *Remark 7.6*). Therefore the group G associated to the G-operator (8.118) would be again amenable, which leds us use the trajectory symbol as a criterion for the Fredholm property.

A general interesting consideration would be problems, where the operators $A_{0/1}$ and $B_{0/1}$ in (8.2) are given, but the metric tensor g_t defining the principal part of the equation $\partial_t^2 u = \Delta u$ is unknown. A search for the Fredholm conditions of the operator F from *Theorem 8.2* would then be a search for sutiable flows making this operator Fredholm. Since the flows $H_{\pm\sigma_p(\sqrt{\Delta})}$ are geodesic flows, this question would be related to the geometry of the underlying manifold. One could also regard such a problem as a problem of control theory: Given the initial data $A_0 u(0) + B_0(\partial_t u)(0) = g_0$ and the final data $A_1 u(1) + B_1(\partial_t u)(1) = g_1$, one would ask for the time dependent equation $\partial_t u = \Delta(t)u$, which connects these two data.

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