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On the renormalization group fixed point of the two-dimensional Ising model at criticality

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We analyze the renormalization group fixed point of the two-dimensional Ising model at criticality. In contrast with expectations from tensor network renormalization (TNR), we show that a simple, explicit analytic description of this fixed point using operator-algebraic renormalization (OAR) is possible. Specifically, the fixed point is characterized in terms of spin-spin correlation functions. Explicit error bounds for the approximation of continuum correlation functions are given.

The statistical mechanics of classical lattice systems continue to present fascinating and remarkable physics. The stochastic geometry exhibited by models as fundamental and elementary as the Ising model¹ exhibits a beautiful structure whose active study persists to the current day². Most intriguing here is the critical phenomena of the model as it approaches a phase transition³. Applications of the Ising model and its generalisations range from superconductivity⁴, fault-tolerant quantum computation⁵, high energy physics⁶, to genetics^{7,8} and the social sciences⁹ and beyond. The two-dimensional case of the Ising model is one of the most well-studied systems in statistical physics, with nearly 80 years of history dating back at least to 1944, with the celebrated work of Lars Onsager¹⁰, who solved the the model on a square lattice in the absence of external magnetic field. This solution is the cornerstone of much of modern statistical physics, and thereby the Ising model has become the benchmark for analytic and numerical methods alike.

During the past decade tensor networks¹¹ have risen to prominence as a powerful tool to study complex systems. These have a rich history originating in the works of Kadanoff¹² and Wilson¹³, the density matrix renormalization group¹⁴, and branching out into a multitude of methods with a wide variety of applications from 2D systems through to models with anyonic excitations. One fascinating area of such works applies modern tensor-network techniques to classical models of statistical physics. This was arguably revolutionized by the tensor renormalization group (TRG) of Levin-Nave¹⁵ having a wide range of applications^{16–19}, which has been refined in various forms, in particular to deal with entanglement of local degrees of freedom such as tensor entanglement-filtering renormalization group²⁰, high-order tensor renormalization group^{21–26}, tensor network renormalization (with or without positivity)^{27–29}. Here impressive numerical results suggest the general applicability of the TRG, and relatives such as tensor network renormalization, as a general purpose method for investigating partition functions of classical lattice models. Although the TRG does flow to a fixed point off criticality – i.e., an infinite bond dimension is required to express the fixed-point tensor – it is still useful for the study of critical phenomena. The goal of explicitly computing fixed-point tensors for critical systems—closely related to the approximation of continuum limits—is still an outstanding challenge for tensor-network methods.

The desire for an explicit RG capable of describing the continuum limit of lattice discretizations of quantum field theories has led to the recent development of operator algebraic renormalization (OAR)^{30–37}. This emerging RG method is closely related to tensor network methods such as the multi-scale entanglement renormalization ansatz (MERA)^{38,39}, and has enjoyed notable recent successes in the computational and analytic approximation of a variety of quantum field theories, from conformal field theories to higher-dimensional models. It is an intriguing open question to determine whether OAR is applicable in the context of classical criticality and, if so, whether it can furnish any information about the fixed-point tensor at phase transitions.

In this Letter we demonstrate that OAR is capable of exactly representing critical points of classical lattice models. To do this we generalize OAR to apply to partition functions of classical lattice models and analytically compute the action of the OAR group on the transfer operator of the 2D Ising model. We obtain thereby an explicit and analytic representation of the fixed-point tensor. In accordance with expectations arising in previous TRG studies we find that this tensor requires an infinite bond dimension.

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Basics of 2d Ising

The two-dimensional anisotropic Ising model on a $N \times M$ square lattice with periodic boundary conditions can be naturally formulated as a tensor network (see Fig. 1), i.e. its canonical partition,

$$Z_{MN} = \sum_{\{\sigma\}, \{\mu\}} \prod_{k=-N}^{N-1} \prod_{j=-M}^{M-1} A_{\mu_j, k, \mu_{j+1}, k, \sigma_j, k, \sigma_{j+1}, k}, \tag{1}$$

is given in terms of the tensor $A_{\mu\mu'\sigma\sigma'} = \delta_{\mu,\sigma} e^{K_1\mu\mu'} e^{K_2\sigma\sigma'}$ with $\mu, \mu', \sigma, \sigma' \in \{\pm 1\}$ as well as horizontal and vertical coupling constants K_1, K_2 .

Spin-spin and other correlation functions are conveniently expressed using the horizontal transfer matrix V_M (see Fig. 2) naturally given in the $\sigma^{(3)}$ -basis^{40,41}:

$$\langle e_\sigma, V_M e_{\sigma'} \rangle = \sum_{\{\mu\}} \prod_{j=-M}^{M-1} A_{\mu_j, \mu_{j+1}, \sigma_j, \sigma'_j} = \prod_{j=-M}^{M-1} e^{K_1\sigma_j\sigma_{j+1} + K_2\sigma_j\sigma'_j}, \tag{2}$$

where $e_\sigma = \otimes_{j=-M}^M e_{\sigma_j}$, $\sigma_j^{(3)} e_{\sigma_j} = \sigma_j e_{\sigma_j}$. As an operator on the Hilbert space $\mathcal{H}_M = \otimes_{j=-M}^M \mathbb{C}^2$, associated with each row of the lattice, the transfer matrix V_M takes the form,

$$V_M = C(K_2)^{\frac{M}{2}} e^{K_1 \sum_{j=-M}^{M-1} \sigma_j^{(3)} \sigma_{j+1}^{(3)}} e^{K_2 \sum_{j=-M}^{M-1} \sigma_j^{(1)}}, \tag{3}$$

where $\tanh(K_2^*) = e^{-2K_2}$ and $C(K_2) = 2 \sinh(2K_2)$, which decomposes into operators associated with vertical couplings, $V_M^{(1)} = (2 \sinh(2K_2))^{\frac{M}{2}} e^{K_2^* \sum_{j=-M}^{M-1} \sigma_j^{(1)}}$, and horizontal coupling respectively $V_M^{(3)} = e^{K_1 \sum_{j=-M}^{M-1} \sigma_j^{(3)} \sigma_{j+1}^{(3)}}$. While the partition function is given by the trace of the horizontal transfer matrix, $Z_{MN} = \text{tr}(V_M)$, the correlation functions are more naturally expressed using the symmetrized transfer matrix $V_M^{(sym)} = (V_M^{(3)})^{\frac{1}{2}} V_M^{(1)} (V_M^{(3)})^{\frac{1}{2}}$ resulting in:

$$\langle \sigma_{j_1, k_1} \dots \sigma_{j_n, k_n} \rangle = \frac{1}{Z_{MN}} \text{tr} \left((V_M^{(sym)})^N \sigma_{j_1 k_1}^{(3)} \dots \sigma_{j_n k_n}^{(3)} \right), \tag{4}$$

where $\sigma_{jk}^{(3)} = (V_M^{(sym)})^k \sigma_j^{(3)} (V_M^{(sym)})^{-k}$.

OAR for 2d Ising. Exploiting the operator-algebraic structure of the transfer matrix formulation, we can apply OAR to analyze the large-scale behavior of correlation functions: $V_M^{(sym)}$ is a positive, trace-class operator on \mathcal{H}_M inducing a quasi-free Gibbs state, $\rho_{MN} = \frac{1}{Z_{MN}} (V_M^{(sym)})^N$, on the quantum spin chains given in terms of the Pauli algebra $\mathcal{P}_M = \otimes_{j=-M}^{M-1} M_2(\mathbb{C})$. By the Jordan-Wigner transform, $a_j = (\prod_{-M \leq l < j} \sigma_l^{(1)})^{\frac{1}{2}} (\sigma_j^{(3)} + i\sigma_j^{(2)})$, the latter is isomorphic to the algebra of complex fermions $\mathfrak{A}_M = \mathfrak{A}_{\text{CAR}}(\mathfrak{h}_M)$ with one-particle Hilbert space $\mathfrak{h}_M = \ell^2(\Lambda_M)$, $\Lambda_M = \{-M, \dots, M-1\}$ (The boundary conditions for \mathfrak{h}_M are chosen such that $V_M^{(3)}$ is an exponential of quadratic

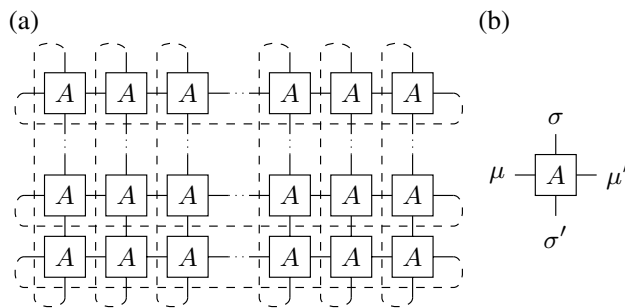


Figure 1. Illustration of the partition function Z_{MN} in (a) as a two-dimensional tensor network built from the local tensor A in (b). Dashed lines indicate contractions due to periodic boundary conditions.

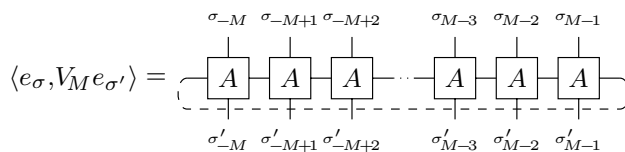


Figure 2. Illustration of the Horizontal transfer matrix V_M associated with the tensor A .

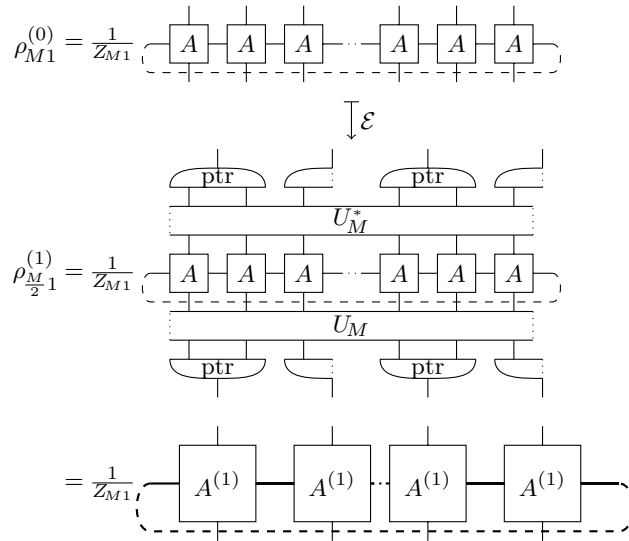


Figure 3. Illustration of a single renormalization group step in OAR applied to the transfer matrix V_M (indices are suppressed, note that $Z_{M1} = Z_{\frac{M}{2}1}^{(1)}$). The disentangler U_M can be decomposed into 2-local operations which defines the renormalized tensor $A^{(1)}$ with increased horizontal bond dimension by the results of Ref.⁴⁴ combined with singular value decomposition.

expressions in annihilation and creation operators⁴⁰. For finite M , the eigenstate corresponding to the largest eigenvalue of $V_M^{(sym)}$ is obtained with anti-periodic boundary conditions for \mathfrak{h}_M). We define the renormalization group transformation (A trace-preserving quantum channel), $\mathcal{E} : \mathcal{S}_{2M} \rightarrow \mathcal{S}_M$, that coarse grains states on the chain of twice the length, \mathcal{S}_{2M} , to those on the given length, \mathcal{S}_M , by its dual quantum channel, $\alpha : \mathfrak{A}_M \rightarrow \mathfrak{A}_{2M}$:

$$\text{tr}(\mathcal{E}(\rho)A) = \text{tr}(\rho \alpha(A)), \quad \rho \in \mathcal{S}_{2M}, A \in \mathfrak{A}_M. \tag{5}$$

The dual quantum channel is naturally given by an isometry³³, $R : \mathfrak{h}_M \rightarrow \mathfrak{h}_{2M}$:

$$\alpha(a(\xi)) = a(R(\xi)), \quad R(\xi)_{j'} = \sum_{j=-M}^{M-1} \xi_j \sum_{n \in \mathbb{Z}} h_n \delta_{2j, j'-n} \tag{6}$$

for $\xi \in \mathfrak{h}_M$ and $a(\xi) = \sum_{j=-M}^{M-1} \bar{\xi}_j a_j$. The coefficients h_n are given by the low-pass filter of a real, orthonormal, compactly supported scaling function $s \in C^r(\mathbb{R})$, satisfying the scaling equation $s(x) = \sum_{n \in \mathbb{Z}} h_n 2^{\frac{1}{2}} s(2x - n)$ (appropriately periodized to comply with the boundary conditions)⁴². The renormalization group transformation takes a particularly simple form in momentum space,

$$R(\hat{\xi})_{\theta'} = 2^{\frac{1}{2}} m_0(\theta') \hat{\xi}_{2\theta'}, \quad m_0(\theta') = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-i\theta' n}, \tag{7}$$

where $\hat{\xi}_\theta = \sum_{j=-M}^{M-1} e^{-i\theta j} \xi_j$ for $\theta \in \frac{\pi}{M} \{-M, \dots, M-1\}$. In this way, we realize (discrete) renormalization group flow lines within the state space \mathcal{S}_M by,

$$\rho_{MN}^{(m)} = \mathcal{E}^m(\rho_{(2^m M)N}), \quad 0 \leq m < \infty, \tag{8}$$

using the Gibbs state ρ_{MN} as an input. On the Pauli algebra \mathcal{P}_M , the coarse graining takes the form: $\mathcal{E}(\cdot) = \text{ptr}(U_M^*(\cdot)U_M)$, where ptr is the partial trace with respect to the natural embedding $\mathcal{H}_{\frac{M}{2}} \subset \mathcal{H}_M$, and U_M is a unitary parametrized by the low-pass filter h_n which coincides with the wavelet disentangler in Refs.^{43,44} (see Supplementary Material for further details). Fig. 3 illustrates how (8) can be interpreted in terms of TNR which is dual to the construction of a MERA.

Infinite volume formulation

We can avoid additional complications in the discussion of the renormalization group fixed point due to boundary conditions, necessary for the algebras $\mathcal{P}_M, \mathfrak{A}_M$ at finite M and N by passing to an infinite volume formulation, i.e. $M, N \rightarrow \infty$: First, we observe that imposing the asymptotic scaling conditions,

$$K_1 \sim \beta t^{(3)} N^{-1}, \quad K_2^* \sim \beta t^{(1)} N^{-1}, \tag{9}$$

for $\beta, t^{(3)}, t^{(1)} > 0$ for $N \rightarrow \infty$, provides a Gibbs state, $\rho_{MN} \xrightarrow{N \rightarrow \infty} \rho_M = \frac{1}{Z_M} e^{-\beta H_M}$, of the transverse-field Ising Hamiltonian at inverse temperature β as a consequence of Trotter's product formula,

$$H_M = - \sum_{j=-M}^{M-1} (t^{(3)} \sigma_j^{(3)} \sigma_{j+1}^{(3)} + t^{(1)} \sigma_j^{(1)}). \tag{10}$$

Second, we note that the definition of the dual quantum channels $\alpha : \mathfrak{A}_M \rightarrow \mathfrak{A}_{2M}$ is compatible with taking the infinite volume limit, $\lim_{M \rightarrow \infty} \mathfrak{A}_M = \cup_M \mathfrak{A}_M = \mathfrak{A}$, in the sense of quasi-local algebras⁴⁵, which leads to a description of the limit $M \rightarrow \infty$ in terms of the fermion algebra, $\mathfrak{A} = \mathfrak{A}_{\text{CAR}}(\mathfrak{h})$, with one-particle space $\mathfrak{h} = \ell^2(\mathbb{Z})$, and the renormalization group transformation, $\alpha : \mathfrak{A} \rightarrow \mathfrak{A}$, defined by the analogue of (6). The dynamics on \mathfrak{A} is determined by the Hamiltonian H , formally given by (10) for $M \rightarrow \infty$, which is still well-defined as a derivation on strictly local elements of \mathfrak{A} . In this limit, the Gibbs states ρ_M provide quasi-free KMS-states $\omega_\beta : \mathfrak{A} \rightarrow \mathbb{C}$ determined by the two-point function:

$$\omega_\beta(a(\xi)a^\dagger(\eta)) = \langle \xi, C_\beta^{(1)} \eta \rangle, \quad \omega_\beta(a^\dagger(\xi)a^\dagger(\eta)) = \langle \bar{\xi}, C_\beta^{(2)} \eta \rangle. \tag{11}$$

The covariance operators $C_\beta^{(1)}, C_\beta^{(2)}$ have momentum-space kernels,

$$C_\beta^{(1)}(\theta, \theta') = \pi \left(1 + \Re \left(\frac{z_\theta}{|z_\theta|} \right) \tanh(\beta |z_\theta|) \right) \delta(\theta - \theta'), \quad C_\beta^{(2)}(\theta, \theta') = -i\pi \Im \left(\frac{z_\theta}{|z_\theta|} \right) \tanh(\beta |z_\theta|) \delta(\theta - \theta'), \tag{12}$$

where $z_\theta = t^{(1)} - e^{i\theta} t^{(3)}$. The expressions remain meaningful in the limit $\beta \rightarrow \infty$ providing a ground state of H on \mathfrak{A} . Evaluating the renormalization group flow (8) results in sequences of renormalized states $\omega_\beta^{(m)} = \omega_\beta \circ \alpha^m$ which are quasi-free by construction and, thus, determined by their two-point functions:

$$\begin{aligned} \omega_\beta^{(m)}(a(\xi)a^\dagger(\eta)) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \left(1 + \Re \left(\frac{z_\theta}{|z_\theta|} \right) \tanh(\beta |z_\theta|) \right) 2^m \left(\prod_{n=0}^{m-1} |m_0(2^n \theta)|^2 \right) \widehat{\xi}_{2^m \theta} \widehat{\eta}_{2^m \theta}, \\ \omega_\beta^{(m)}(a^\dagger(\xi)a^\dagger(\eta)) &= -\frac{i}{4\pi} \int_{-\pi}^{\pi} d\theta \Im \left(\frac{z_\theta}{|z_\theta|} \right) \tanh(\beta |z_\theta|) 2^m \left(\prod_{n=0}^{m-1} |m_0(2^n \theta)|^2 \right) \widehat{\xi}_{-2^m \theta} \widehat{\eta}_{2^m \theta}. \end{aligned} \tag{13}$$

Fixed points and admissible scaling limits are determined by analyzing the convergence of (13) for $m \rightarrow \infty$ under suitable renormalization conditions imposed on the couplings $t^{(1)}, t^{(3)}$ and the inverse temperature β .

The fixed point at criticality

In the quantum spin-chain formulation, the critical line corresponds to equal couplings $t^{(3)} = t^{(1)} = t$ in the Hamiltonian (10) in the limit $\beta \rightarrow \infty$, which is equivalent to $K_1 \approx K_2^*$ (at large $N \gg 1$ by (9)), i.e. the critical line of the two-dimensional Ising model given by the tensor A corresponding to the well-know critical coupling $K = K_1 = K_2 = \frac{1}{2} \ln(1 + \sqrt{2})$ in the isotropic case. In view of (13), we have $z_\theta = t(1 - e^{i\theta})$ and $|z_\theta|^2 = 4t^2 \sin^2(\frac{1}{2}\theta)$. Using the change of variables $k = 2^m \theta$ and noting that $\tanh(\beta |z_\theta|) \xrightarrow{\beta \rightarrow \infty} 1 - \delta_{\theta,0}$, we find:

$$\begin{aligned} \omega_\infty^{(m)}(a(\xi)a^\dagger(\eta)) &= \frac{1}{4\pi} \int_{-2^m \pi}^{2^m \pi} dk \left(1 + \frac{1 - \cos(2^{-m}k)}{2|\sin(\frac{1}{2}2^{-m}k)|} \prod_{n=1}^m |m_0(2^{-n}k)|^2 \right) \widehat{\xi}_k \widehat{\eta}_k \xrightarrow{m \rightarrow \infty} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk |\hat{s}(k)|^2 \widehat{\xi}_k \widehat{\eta}_k, \\ \omega_\infty^{(m)}(a^\dagger(\xi)a^\dagger(\eta)) &= \frac{i}{4\pi} \int_{-2^m \pi}^{2^m \pi} dk \left(\frac{\sin(2^{-m}k)}{2|\sin(\frac{1}{2}2^{-m}k)|} \prod_{n=1}^m |m_0(2^{-n}k)|^2 \right) \widehat{\xi}_{-k} \widehat{\eta}_k \xrightarrow{m \rightarrow \infty} \frac{i}{4\pi} \int_{-\infty}^{\infty} dk \text{sign}(k) |\hat{s}(k)|^2 \widehat{\xi}_{-k} \widehat{\eta}_k, \end{aligned} \tag{14}$$

by Lebesgue's dominated convergence theorem applied to $\prod_{n=1}^m m_0(2^{-n}k) \xrightarrow{m \rightarrow \infty} \hat{s}(k)$ ⁴², see also Ref.³³, Lem. 3.7 for an adapted decay estimate for m_0 . By passing to the self-dual chiral Majorana fields, $\psi_{\pm j} = e^{\pm i\frac{\pi}{4}} a_j + e^{\mp i\frac{\pi}{4}} a_j^\dagger$, we recognize that the limits in (14) are the vacuum two-point functions of the $c = \frac{1}{2}$ free-fermion conformal field theories (CFTs) of the two chiral halves of the critical Ising fixed point:

$$\omega(\psi_\pm(\xi * s)\psi_\pm(\eta * s)^*) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{1}{2} (1 \pm \text{sign}(k)) |\hat{s}(k)|^2 \widehat{\xi}_k \widehat{\eta}_k, \quad \omega(\psi_\pm(\xi * s)\psi_\mp(\eta * s)^*) = 0, \tag{15}$$

where $(\xi * s)(x) = \sum_{j \in \mathbb{Z}} \xi_j s(x - j)$ for $\xi \in \mathfrak{h}$. We directly infer from (15) that the scaling function s controls the resolution at which the CFT is probed.

Error bounds on fermions correlations

It is an immediate consequence of the construction that explicit error bounds on the approximation of dynamical fermionic n -point functions of the scaling limit state ω can be derived using the methods of^{33,34}:

$$|\omega^{(m)}(a_{t_1}^{(+)}(\xi_1) \dots a_{t_n}^{(+)}(\xi_n)) - \omega(a_{t_1}^{(+)}(\xi_1) \dots a_{t_n}^{(+)}(\xi_n))| \leq \delta, \tag{16}$$

given a set of one-particle vectors $\xi_1, \dots, \xi_n \in \mathfrak{h}$ and effective lattice times $t_1^{(0)}, \dots, t_n^{(0)}$ as well as continuum times t_1, \dots, t_n (referring to the effective dynamics H after rescaling m -times and the massless free-fermion dynamics in the scaling limit respectively). In particular, we find,

$$\delta = \delta(m, T) \leq 2^{-m} C_T, \tag{17}$$

for $|t_1|, \dots, |t_n| \in [0, T]$, large effective lattice times $t_i^{(0)} \sim 2^m t_i$, and some constant $C_T > 0$ otherwise only depending on one-particle norms of ξ_1, \dots, ξ_n and the scaling function s . In general, the error in (16) can only be small for large effective lattice times just as the equal-time correlation approximate their continuum counterparts at large distances, as seen from (15). But, here an exponential separation of effective lattice and continuum times is not necessary at the expense of a slower decay of the error δ (see Supplementary Material).

Instability of the fixed point at criticality

The question of stability of fixed points in the framework of TNR has been of interest recently⁴⁶. Although, we cannot address this question for OAR in full detail in this Letter, we can make the following observation: In the space of quasi-free (initial) states characterized by covariance operators $C = C_\beta(t^{(1)}, t^{(3)})$ in the sense of (11), it is an immediate consequence of (13) that the critical state given by $C_{crit.} = C_{\beta=\infty}(t, t)$ is unstable, because $\frac{z_\theta}{|z_\theta|} \xrightarrow{m \rightarrow \infty} \text{sign}(\lambda)$ and $|z_\theta| \xrightarrow{m \rightarrow \infty} t^{(1)}|\lambda|$ for $\theta = 2^{-m}k$ and $\lambda = 1 - \frac{t^{(3)}}{t^{(1)}} \in [-\infty, 0) \cup (0, 1]$ (non-critical). In particular, at $\beta = \infty$, the states are driven towards: (1) the disorder fixed point $\lambda = 1$ ($t^{(1)} = \text{const.}, t^{(3)} = 0$) for $\lambda \in (0, 1]$, or (2) the order fixed point $\lambda = -\infty$ ($t^{(1)} = 0, t^{(3)} = \text{const.}$) for $\lambda \in [-\infty, 0)$. By a similar reasoning that led to (14), the disorder fixed point is given by the Fock state with respect to a, a^\dagger while the order fixed point is given by the anti-Fock state (resulting from an equal weight mixture of the two extremal ground states of H_M at $t^{(1)} = 0$).

Spin-spin correlations

The correspondence between quasi-free states on \mathfrak{A} and even states on the infinite-volume Pauli algebra $\mathcal{P} = \otimes_{j \in \mathbb{Z}} M_2(\mathbb{C})$ allows for a characterization of the critical fixed point in terms of spin-spin correlation functions,

$$\omega(\sigma_{j_1}^{(3)} \dots \sigma_{j_{2n}}^{(3)}) = \omega\left(\prod_{k=1}^n \prod_{l_k=j_{2k-1}}^{j_{2k}-1} \Psi(0, i\delta_{l_k}) \Psi(\delta_{l_k+1}, 0)\right), \quad \omega(\sigma_{j_1}^{(3)} \dots \sigma_{j_{2n+1}}^{(3)}) = 0, \tag{18}$$

for $j_1 \leq \dots \leq j_{2n} \leq j_{2n+1}$, where $\Psi(\xi, \eta) = a(\xi - i\eta) + a^\dagger(\overline{\xi + i\eta})$, $\xi, \eta \in l^2(\mathbb{Z})$, is Araki's self-dual field⁴⁵. These correlation functions are precisely the scaling limits of the Ising correlation functions (4) at criticality $K_1 = K_2^*$ for $k_1, \dots, k_n = 0$. The quasi-free structure of ω allows for the evaluation of (18) in terms of a Pfaffian, which further reduces to well-known Toeplitz determinant^{40,47} with the crucial difference that scaling-limit two-point function is given by (14). The real-time, analytic continuations of the critical Ising correlation functions with $k_1, \dots, k_n \neq 0$ can be obtained from (14) by means of the scaling limit of the time-evolution of H_M (see Supplementary Material for a sketch).

Other scaling limits

Inspecting (13) it is straightforward to construct massive and finite-temperature scaling limits: If we impose the renormalization conditions $\lambda = 1 - \frac{t^{(3)}}{t^{(1)}} \sim 2^{-m}\mu_0 > 0$ and $\beta \sim 2^m\beta_0$ for arbitrary $\mu_0 \geq 0$ and $\beta > 0$, we will obtain the equilibrium state at temperature β_0 of a free fermion quantum field of mass m_0 :

$$\begin{aligned} \omega_{\mu_0, \beta_0}(a(\xi * s)a^\dagger(\eta * s)) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \left(1 + \frac{\mu_0}{\omega_{\mu_0}(k)} \tanh(\beta_0 t \omega_{\mu_0}(k))\right) |\hat{s}(k)|^2 \overline{\hat{\xi}_k} \hat{\eta}_k, \\ \omega_{\mu_0, \beta_0}(a^\dagger(\xi * s)a^\dagger(\eta * s)) &= \frac{i}{4\pi} \int_{-\infty}^{\infty} dk \frac{k}{\omega_{\mu_0}(k)} \tanh(\beta_0 t \omega_{\mu_0}(k)) |\hat{s}(k)|^2 \hat{\xi}_{-k} \hat{\eta}_k \end{aligned} \tag{19}$$

where $\omega_{\mu_0}(k)^2 = \mu_0^2 + k^2$ is the massive continuum dispersion relation, $(\xi * s)(x) = \sum_{j \in \mathbb{Z}} \xi_j s(x - j)$ for $\xi \in \mathfrak{h}$, and $t^{(3)} \xrightarrow{m \rightarrow \infty} t$. As before, the scaling function s controls the resolution at which the continuum quantum field is probed.

We have presented an explicit description of the critical fixed point of the two-dimensional classical Ising model using OAR which may be understood as a Wilson-Kadanoff RG scheme dual to tensor-network methods. In particular, if OAR is applied to density matrices given in terms of transfer matrices of classical lattice systems, it is operationally dual to a (thermal) MERA derived from TNR³⁹. Our explicit representation of the critical fixed point relies on an implementation of OAR using wavelet methods that was previously introduced in Refs.³²⁻³⁵, and the duality with TNR is manifestly exhibited by the unitary defining the coarse-graining channel \mathcal{E} (see Supplementary Material), which directly corresponds to the exact disentangler of Evenbly and White for the ground state of the Ising quantum chain^{43,44}. In our construction of the scaling limit, a particularly important role is played by the scaling function associated with a given low-pass filter, as this function controls the resolution at which the fixed-point tensor is probed at unit scale – either in terms of fermionic or spin-spin correlation functions. As a direct consequence of this feature we explicitly observe a universal large-scale behavior independent of the specific choice of scaling functions. Another important advantage of our method over other

approaches such as the exact MERA is the provision of explicit, provable error bounds on the approximation of correlation functions for sufficiently regular scaling functions that are independent of the design problem of Hilbert-pair wavelets⁴⁸. Such error bounds allow for a direct understanding of the simulation of QFTs/CFTs by quantum computers³⁴. We have exhibited a direct correspondence of the critical fixed point with the vacuum (or Neveu-Schwarz) sector of the Ising CFT with an explicit formula for the two-point functions of the self-dual chiral Majorana field (see (15)). By our method, it is possible to obtain fixed points corresponding to other sectors, e.g., the Ramond sector, by working in a finite-volume setting including different, e.g., anti-periodic, boundary conditions, which will be discussed elsewhere. In addition, we are planning to further clarify the relation of our construction of the scaling limit of the Ising model with previously known results about the Ising QFT/CFT – specifically via spin-spin correlation functions^{41,49–51} and the explicit construction of the spin field operator⁵².

Data availability

All data generated or analyzed during this study are included in this published article [and its supplementary information files].

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