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Goal oriented error control for stationary incompressible flow coupled to a heat equation

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In this work, we apply goal oriented error estimation to a stationary Navier-Stokes benchmark problem coupled with the heat equation. Furthermore, we compare three different methods for the sensitivity weight recovery.

1 Introduction

In many applications (e.g., multiphysics problems) as for instance in wave guide writing laser materials processing within the excellence cluster PhoenixD[1], not the entire solution is of primary concern, but some specific quantity of interest. In laser wave guide writing, light (Maxwell equations), material responses (incompressible Navier-Stokes), and heat developments interact. In this work, we concentrate on the later, namely, coupling stationary incompressible Navier-Stokes with a heat equation. For goal-oriented error control, the dual weighted resiudal method [7] is one possibility to estimate the arising errors when computing these quantites of interest. An extensive framework for nonlinear problems, multiple goal functionals, and balancing discretization and nonlinear iteration errors was designed in [2, 3] and which forms the basis of the current study.

2 The goal and model problem

Let U be a Banach space. The goal is to find $J(u)$ such that u solves $A(u)(v) = 0$ for all $v \in U$, where $A : U \times U \mapsto \mathbb{R}$ is semilinear and $J : U \mapsto \mathbb{R}$ is nonlinear. As model problem we consider the stationary Navier-Stokes benchmark problem NS2D-1[2]; see also [5]. Additionally we extend the benchmark problem by an additional equation for the temperature. The computational domain Ω is given by $\Omega := (0, H) \times (0, 2.2) \setminus \mathcal{B}$, where $H = 0.41$, and $\mathcal{B} := \{x \in \mathbb{R}^2 : \|x-(0.2, 0.2)\| \leq \frac{1}{20}\}$. Finally the problem reads as: Find $\mathbf{u} := (u, p, T) \in U := [H^1(\Omega)]^2 \times L^2(\Omega) \times H^1(\Omega)$ such that

$$
\begin{aligned}
-\nu\Delta u + (u\cdot\nabla)u - \nabla p &= 0 && \text{in } \Omega, & u &= 0 && \text{on } \Gamma_{\text{no-slip}}, & T &= 5 && \text{on } \Gamma_{\text{inflow}},\\
\nabla\cdot u &= 0 && \text{in } \Omega, & u &= \hat{u} && \text{on } \Gamma_{\text{inflow}}, & T &= 100 && \text{on } \partial\mathcal{B},\\
-\Delta T + u\cdot\nabla T &= 0 && \text{in } \Omega, & \nu\frac{\partial u}{\partial \vec{n}} - p\cdot\vec{n} &= 0 && \text{on } \Gamma_{\text{outflow}}, & \frac{\partial T}{\partial n} &= 0 && \text{on } \Gamma_{N,T},
\end{aligned}
$$

where $\Gamma_{inflow} := \partial\Omega \cap (\{0\} \times \mathbb{R})$, $\Gamma_{outflow} := \partial\Omega \cap (\{2.2\} \times \mathbb{R})$, $\Gamma_{no-slip} := \partial\Omega \setminus (\Gamma_{\text{inflow}} \cup \Gamma_{\text{outflow}})$, $\Gamma_{N,T} := \partial\Omega \setminus (\Gamma_{\text{inflow}} \cup \partial\mathcal{B})$, $\nu = 10^{-3}$ and $\hat{u}(x, y) := 1.2y(H - y)/H^2$. The weak form implicitly generates our operator A. Here, the goal J is to find the lift which is defined as $J(\mathbf{u}) := 500 \int_{\partial\mathcal{B}} \left[\nu\frac{\partial u}{\partial \vec{n}} - p\vec{n}\right] \cdot \vec{e}_2 \; \mathrm{d}s_{(x,y)}$ where $\vec{e}_2 = (0, 1)$.

3 The error estimator

The error estimator is based on the dual weighted residual method. Therefore, we have to consider the primal problem: Find $u \in U$ such that $A(u)(v) = 0$ for all $v \in U$, and the adjoint problem: Find $z \in U$ such that $A'(u)(z)(v) = J'(u)(v)$ for all $v \in U$. Here all derivaties are Fréchet derivates with respect to u. Moreover, we use two different finite element spaces U_h and $U_h^{(2)}$ for U. Furthermore, we assume that we have two approximations $\tilde{u}_h^{(2)}$ and $\tilde{z}_h^{(2)}$ on $U_h^{(2)}$ as well as two approximations $\tilde{u}$ and $\tilde{z}$ on U_h. The error estimator we will use here is then given by

$$\tilde{\eta}^{(2)} := \frac{1}{2}\rho(\tilde{u})(\tilde{z}_h^{(2)} - \tilde{z}) + \frac{1}{2}\rho^*(\tilde{u}, \tilde{z})(\tilde{u}_h^{(2)} - \tilde{u}) - \rho(\tilde{u})(\tilde{z}) - \rho(\tilde{u}_h^{(2)})(\frac{\tilde{z}_h^{(2)} + \tilde{z}}{2}) + \frac{1}{2}\rho^*(\tilde{u}_h^{(2)}, \tilde{z}_h^{(2)})(\tilde{u}_h^{(2)} - \tilde{u}) + \tilde{\mathcal{R}}^{(3)(2)},$$

where $\rho(\tilde{u})(\cdot) := -A(\tilde{u})(\cdot)$ and $\rho^*(\tilde{u}, \tilde{z})(\cdot) := J'(\tilde{u}) - A'(\tilde{u})(\cdot, \tilde{z})$. For the definition of $\tilde{\mathcal{R}}^{(3)(2)}$ we refer to [2]. This estimator should estimate the error in J and is motivated in [2, 4]. Furthermore, it is also shown that this error estimator is efficient and reliable if $|J(\tilde{u}_h^{(2)}) - J(u)| < c|J(\tilde{u}) - J(u)|$ with $c < 1$.

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[2] http://www.featflow.de/en/benchmarks/cfdbenchmarking/flow/dfg_benchmark1_re20.html

4 Numerical example

In our numerical example, the model problem and our error estimator are combined using the same algorithms as in [2]. Furthermore, we compare the results to the one from the original benchmark problem as well as three different choices how to get the approximations on the space U_h^2. The three different choices are using *int* (interpolation), *full* (full solves), and *new* (combination of interpolation and full solves). For more information on those, we refer to [2]. The discretization was done using continuous biquadratic (biquartic) finite elements for the velocity and bilinear (biquadratic) elements for p and T on U_h ($U_h^{(2)}$). Since the velocity does not depend on the temperature we use the reference value presented in [6]. The implementation is based on the finite element library deal.II [1]. The solution for the temperature is visualized in Fig. 1.

Fig. 1: Solution of temperature variable T.

Fig. 2: Relative errors $\frac{|J(\tilde{u})-J(u)|}{|J(u)|}$ and effectivity indices: $I_{eff} := \frac{|\tilde{\eta}^{(2)}|}{|J(\tilde{u})-J(u)|}$.

In Figure 2(left) the relative errors of the three different methods are visualized. We see that there is just a minor difference between the results. Since *int* and *new* both have less computational cost, we suggest these methods. However, we emphasize, that this has a strong dependence on the chosen example. The figure on right hand side gives us information, about how well the error is estimated. Since the aim is to have $I_{eff} = 1$ we prefer *new* and *full*. If we compare the results of this example with the results in [2] for the original benchmark problem, we do not see any difference. In fact the meshes, and errors are identical. Of course when we compute the temperature we need additional degrees of freedom.

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