S. Rams, M. Schütt *Res Math Sci* (2021) 8:22
https://doi.org/10.1007/s40687-021-00262-7

Research in the Mathematical Sciences

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Twelve Rational curves on Enriques surfaces

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Abstract

Given $d \in \mathbb{N}$, we prove that any polarized Enriques surface (over any field k of characteristic $p \neq 2$ or with a smooth K3 cover) of degree greater than $12d^2$ contains at most 12 rational curves of degree at most d. For $d > 2$, we construct examples of Enriques surfaces of high degree that contain exactly 12 rational degree-d curves.

Keywords: Enriques surface, Rational curve, Polarization, Genus one fibration, Hyperbolic lattice, Parabolic lattice

Mathematics Subject Classification: 14J28, 14J27, 14C20

1 Introduction

It is known by the work of Barth–Peters, Nikulin, and Kondo that any Enriques surface Y over $\mathbb{C}$ contains infinitely many (possibly singular) rational curves. In fact, as soon as there is a single smooth rational curve, then there are infinitely many of them outside seven specific cases—the types I–VII due to Nikulin [17] and Kondo [12] which will also be relevant for our considerations. Here, we restrict the problem to (again, possibly singular) rational curves of small degree relative to a given polarization H of Y. For $d \in \mathbb{N}$, let

$$r_d := r_d(Y) := \# \left\{ \text{rational curves } C \subset Y \text{ with } \deg(C) = d = C \cdot H \right\}$$

and

$$S_d := r_1 + \cdots + r_d = \#\{\text{rational curves } C \subset Y \text{ with } \deg(C) \leq d\}.$$

We work over an algebraically closed field k, but our main result is almost independent of the characteristic in the sense that we only require that the K3 cover of the Enriques surface Y is smooth (i.e. we exclude the types of classical and supersingular Enriques surfaces in characteristic 2):

Theorem 1 *Let $d \in \mathbb{N}$. Assume that Y is an Enriques surface with smooth K3 cover. Fix a polarization H on Y such that $H^2 > 12d^2$. Then,*

$$S_d \leq 12.$$

The methods to prove this result build on those developed for K3 surfaces in [19], but remarkably, the result is substantially stronger in the sense that the bound $H^2 > 12d^2$

cannot be improved much (outside characteristics 2, 5 at least, see Proposition 21) for $d > 1$. Here, the often delicate geometry of Enriques surfaces acts very much to our advantage. We also show by engineering explicit surfaces that the bound is attained as follows:

Proposition 2 (i) *For any even $d \geq 6$ and any $h \geq 3d$, there is a $2h$-polarized Enriques surface with $r_d = 12$.*

(ii) *Assume that char$(k) \neq 2$. For any $d \geq 3$ and any $h \geq 9$ with $d \mid h$, there is a $2h$-polarized Enriques surface containing 12 smooth rational curves of degree d.*

Over fields of characteristic 2 the statement of Proposition 2 (ii) no longer holds, see Proposition 20.

The cases of smooth rational curves of degree 1 and 2, i.e. lines and conics, are more delicate than the above and will be treated elsewhere.

2 Set-up

Throughout this note, we consider polarized Enriques surfaces of degree $2h$, i.e. pairs (Y, H) where Y is an Enriques surface over k and H is a very ample divisor of square $H^2 = 2h$. If char$(k) = 2$, then we assume that Y has a smooth K3 cover X (these Enriques surfaces are often called singular). Then, the linear system $|H|$ defines an embedding

$$\varphi_{|H|}: \quad Y \hookrightarrow Y_H \subset \mathbb{P}^h,$$

which is an isomorphism onto its image.

It is well-known that each very ample divisor on an Enriques surface satisfies the inequality $H^2 \geq 10$. For char$(k) \neq 2$ the question whether a given divisor H on Y is very ample can be answered by the methods developed in [3]. The discussion of the general case can be found in [4]. For the convenience of the reader, we recall the (well-known) result we need below.

Criterion 3 ([4, Remark 2.4.19]) *Let H be a big and nef divisor on an Enriques surface Y. Then, H is very ample if and only if*

$$H \cdot D \geq 3 \text{ for every half-pencil } D \text{ of a genus 1 fibration on } Y \tag{2.1}$$

and $H \cdot E > 0$ for every (-2)-curve $E \subset Y$.

Proof By [4, Lemma 2.4.10] there exists a half-pencil D' of a genus 1 fibration on Y such that the equality $\Phi(H) = H \cdot D'$ holds. Thus, Criterion 3 is [4, Remark 2.4.19]. □

3 Basics

Given a polarized Enriques surface Y, we follow [19] and consider the set

$$\Gamma = \left\{\text{rational curves } C \subset Y \text{ of degree } C \cdot H \leq d\right\},$$

which we interpret as a graph without loops with (possibly multiple) edges corresponding to the intersection points of rational curves $C, C' \in \Gamma$. Here, each vertex $C \in \Gamma$ comes attached with two values, the degree $C \cdot H$ and the square C^2. Together the rational curves generate the formal group

$$M := \mathbb{Z}\Gamma \subset \text{Div}(Y),$$

equipped with the intersection pairing extending that on the vertices. Clearly, M may be degenerate (i.e. $\ker(M) \neq 0$). By the Hodge index theorem, we have the following three cases:

1. elliptic—$M \otimes \mathbb{R}$ is negative-definite ($\ker(M) = 0$);
2. parabolic—$M \otimes \mathbb{R}$ is negative semi-definite, but not elliptic ($\ker(M) \neq 0$);
3. hyperbolic—$M \otimes \mathbb{R}$ has a one-dimensional positive-definite subspace and none of greater dimension.

The elliptic case and the parabolic case can be studied exactly along the lines of [19, §4,5]—with the extra benefit that there are no quasi-elliptic fibrations complicating the analysis in small characteristic (cf. [4, Thm 4.9.3]). For brevity, we just sketch the arguments:

Lemma 4 (i) *If M is elliptic, then it is an orthogonal finite sum of Dynkin diagrams (ADE-type).*
(ii) *If M is parabolic, then it is an orthogonal finite sum of Dynkin diagrams and at least one isotropic vertex or extended Dynkin diagram ($\tilde{A}\tilde{D}\tilde{E}$-type).*

Proof (i) If M is elliptic, then $C^2 = -2$ for all $C \in \Gamma$. Moreover,

$$C \cdot C' \leq 1 \quad \forall\, C, C' \in \Gamma,$$

for otherwise $(C + C')^2 \geq 0$. Hence M is an orthogonal sum of Dynkin diagrams; since M embeds into the hyperbolic lattice $\mathrm{Num}(Y)$ of rank 10 (cf. [2]), the rank of M is bounded by $\rho(Y) - 1 = 9$. Thus, the orthogonal sum is finite as claimed (and there are only finitely many possible configurations).

(ii) If M is parabolic, then we directly obtain a constraint on self-intersection

$$C^2 \in \{0, -2\} \quad \forall\, C \in \Gamma. \tag{3.1}$$

Similarly, intersection numbers of pairs of curves are restricted as follows. For any isotropic $C \in \Gamma$, we have

$$C \cdot C' = 0 \quad \forall\, C' \in \Gamma, \tag{3.2}$$

for else $(2C + C')^2 > 0$. The same statements holds for the intersection of $C' \in \Gamma$ with any isotropic divisor D supported on Γ, and for any two isotropic divisors supported on Γ. For the same reason, the following inequality holds

$$C \cdot C' \leq 2 \quad \forall\, C, C' \in \Gamma \quad \text{with } C^2 = C'^2 = -2. \tag{3.3}$$

Here, equality makes $(C + C')$ isotropic and thus orthogonal to all other curves in Γ (so C, C' generate the extended Dynkin diagram $\tilde{A}_1$). It follows that Γ consists of pairwise disjoint isotropic curves (i.e. curves of arithmetic genus $p_a = 1$, a priori possibly infinite in number) and standard or extended Dynkin diagrams (again finite in number for rank reasons). □

Corollary 5 (i) *If M is elliptic, then $\#\Gamma = \mathrm{rank}\, M \leq 9$.*
(ii) *If M is parabolic, then $\#\Gamma \leq 12$.*

Proof (i) If M is elliptic, then $\#\Gamma = \mathrm{rank}(M) \leq 9$ as in the proof of Lemma 4(i).

(ii) If M is parabolic, then Lemma 4 implies the existence of a divisor D of Kodaira type supported on Γ, i.e. a nodal or cuspidal cubic or, in the case of an extended Dynkin diagram, a configuration of (-2) curves fitting a singular fibre of an elliptic fibration as classified by Kodaira [9–11] and Tate [26]. Then, $|D|$ or $|2D|$ induces a genus one fibration

$$Y \to \mathbb{P}^1 \tag{3.4}$$

with D as a (multiple) fibre. By construction, Γ is the set of rational fibre components of (3.4) that have degree at most d. The fibration formula for the Euler–Poincaré characteristic $e(Y) = 12$ yields the claimed bound. □

The corollary also provides us with a recipe for producing Enriques surfaces of arbitrary polarization with 12 rational curves of small degree by arranging for them to be fibre components of a suitable genus one fibration. This will be treated in Sects. 9 and 10.

4 Divisors of Kodaira type

Drawing closer to the proof of Theorem 1, we assume throughout this and the next four sections that

$$H^2 > 12d^2. \tag{4.1}$$

As in [19, §7], this implies that the restrictions (3.1)–(3.3) continue to hold true. Moreover, the analogue of (3.2),

$$D \cdot C' = 0 \quad \forall \text{ isotropic } C' \in \Gamma, \tag{4.2}$$

holds for any isotropic effective divisor D with $\deg(D) \leq 6d$ supported on Γ. To see this, just set $c := D \cdot C'$, $d' := \deg(C')$, $d'' := \deg(D)$, write down the Gram matrix of H, C', D, and compute its determinant

$$-c(cH^2 - 2d''d'),$$

which is negative for $c \geq 1$. But then the lattice generated by H, C', D cannot be hyperbolic, contradiction.

The following property will be instrumental for all arguments to follow.

Lemma 6 *If Γ is not elliptic, then it supports a divisor of Kodaira type.*

Proof If Γ contains a curve C with $C^2 = 0$, then we are done; so by (3.1), we may assume that Γ consists of (-2)-curves. Thus, we may take a maximal elliptic subconfiguration Γ' of Γ (an orthogonal sum of Dynkin diagrams) and attach any curve in $\Gamma \setminus \Gamma'$ to it. By assumption, the resulting configuration is no longer elliptic. We claim that it supports an isotropic vector. Indeed, using the restriction (3.3), verifying the claim amounts to a simple case-by-case analysis starting from any Dynkin diagram involved in Γ'. □

We continue by explaining how Lemma 6 combined with Corollary 5 gives way to an ineffective proof of Theorem 1 (i.e. for $H^2 \gg 0$—we will use this later for Proposition 20). This builds on the concept of instrinsic polarization in the terminology of [6]. For any hyperbolic subgraph $\Gamma' \subset \Gamma$, the instrinsic polarization is defined as the $\mathbb{Q}$-divisor $H_{\Gamma'} \in \mathbb{Q}\Gamma'$ determined (if it exists, otherwise we get a contradiction) by the degree conditions

$$C \cdot H_{\Gamma'} = C \cdot H \quad \forall C \in \Gamma'. \tag{4.3}$$

As in [19, Prop. 6.2], the polarization H on Y crucially satisfies the condition

$$H^2 \leq H_{\Gamma'}^2. \tag{4.4}$$

Lemma 7 *If* $\#\Gamma > 9$ *and* $H^2 \gg 0$, *then* Γ *is parabolic.*

Proof By Corollary 5 (i), Γ cannot be elliptic, so we assume it to be hyperbolic. By Lemma 6, Γ contains a parabolic subgraph Γ_0 which may be extended by a single curve $C \in \Gamma$ to a hyperbolic subgraph $\Gamma' \subset \Gamma$.

Presently, there are only finitely many possible configurations for Γ_0; by the above restrictions (3.1)–(3.3), the same holds for Γ'. Finally, by (4.4), the maximum of all self-intersections $H_{\Gamma'}^2$ gives an upper bound for H^2. □

Ineffective proof of Theorem 1 If $\#\Gamma > 9$ and $H^2 \gg 0$, then Γ is parabolic by Lemma 7. Thus, Corollary 5 (ii) shows that $\#\Gamma \leq 12$. □

Remark 8 To obtain a lower bound for H^2 to rule out Γ being hyperbolic, we will refine the idea that appeared already in the proof of Lemma 7 (see also [19]).

We assume that Γ is hyperbolic. We choose a basis Γ_0 of $\Gamma/\ker$ with Gram matrix G. The intrinsic polarization can be expressed as:

$$H_0 = G^{-1}\mathbf{d},$$

where the coordinates of $\mathbf{d}$ are the degrees of the elements of the basis Γ_0. Thus, estimating H_0^2 reduces to an optimization problem. But all degrees are positive, so we obtain an easy bound in terms of the entries g_{ij} of G^{-1} by

$$H_0^2 \leq \sum_{i,j} \max(0, g_{ij}) d^2. \tag{4.5}$$

This bound is attained when all g_{ij} are nonnegative by all curves in Γ_0 having degree d. In the presence of negative entries, the bound can be improved by arranging for a decomposition $G^{-1} = G_0 + G_+$, with G_+ negative semi-definite containing in its kernel the vector with all entries the same (cf. [19, Lemma 9.1]).

5 Hyperbolic case—preparations

In view of Corollary 5, it remains to study the case when M is hyperbolic in order to work out an effective proof of Theorem 1.

In this section, we maintain the assumption (4.1). As in [19], for a divisor $D = \sum_i n_i C_i$ of Kodaira type, we define the weight by the equality $\mathrm{wt}(D) = \sum_i n_i$. In particular, if D is supported on Γ, we have $\deg(D) \leq \mathrm{wt}(D)d$.

Lemma 9 *Assume that* Γ *is hyperbolic. Then,*

$$C^2 = -2 \quad \forall\, C \in \Gamma, \quad \textit{and} \quad C \cdot C' \leq 1 \quad \forall\, C, C' \in \Gamma. \tag{5.1}$$

In particular, there are no divisors of Kodaira type $\mathrm{I}_1, \mathrm{I}_2, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ *supported on* Γ. *Moreover, if a divisor of type* I_3 *or* I_4 *is supported on* Γ, *then it is a half-pencil of a genus one fibration such that* Γ *contains none of its multisections of index* > 2.

Proof Let D be a divisor of Kodaira type, such that all components of its support belong to Γ. Since Γ is hyperbolic, there is a curve $C \in \Gamma$ serving as a multisection of the genus one fibration (3.4).

We let $r := C \cdot D$; note that, for a general fibre F, we have $C \cdot F = r$ or $2r$, depending on whether D is a fibre or a half-pencil. Consider the sublattice

$$L = \langle C, D\rangle \subset \mathrm{Num}(Y).$$

We consider the intrinsic polarization

$$H_L \in L \otimes \mathbb{Q} \quad \text{determined by} \quad C \cdot H_L = C \cdot H, \;\; D \cdot H_L = D \cdot H,$$

that satisfies the inequality $H^2 \leq H_L^2$. Solving the above linear system for H_L, we use (4.1) to obtain the following inequalities:

$$12d^2 < H_L^2 \leq 2\deg(D)(r\deg(C) + \deg(D))/r^2 \leq 2\mathrm{wt}(D)(r + \mathrm{wt}(D))d^2/r^2. \tag{5.2}$$

This readily implies that $\mathrm{wt}(D) > 2$, ruling out fibres of Kodaira type $\mathrm{I}_1, \mathrm{I}_2, \mathrm{II}, \mathrm{III}$ to be supported on Γ. In particular, combined with (3.1), it shows that any curve in Γ is nodal (i.e. smooth rational). Furthermore, $C \cdot C' \neq 2$ for any $C, C' \in \Gamma$, so (5.1) follows from (3.3).

Finally, if D has weight at most 4 (or more generally degree at most $4d$), then (5.2) yields $r = 1$, making D a half-pencil and any multisection $C \in \Gamma$ a bisection. This applies to Kodaira types $\mathrm{I}_3, \mathrm{I}_4$ and rules out IV (since multiple fibres are multiplicative). □

In [12], Kondo pioneered a construction of (complex) Enriques surfaces by using suitable disjoint sections on elliptic K3 surfaces. This was later generalized in [7] as we shall use in Sect. 10. Here, we note the following useful consequence:

Corollary 10 *If Γ is hyperbolic and supports a divisor of Kodaira type* I_3 *or* I_4*, then Y arises from Kondo's construction.*

Proof We have just seen that any multisection in Γ is a bisection. This splits on the K3 cover into two disjoint sections. Hence, the reasoning from [20–22] (or [15]) applies to show that independent of the characteristic, Y arises from Kondo's construction. □

6 Hyperbolic case—reduction

The following lemma will substantially simplify our analysis of possible hyperbolic graphs Γ. As stated before, in this section, we continue to make the assumption (4.1), so we can apply the results from the previous two sections.

Lemma 11 *If Γ is hyperbolic, then it supports a divisor D of Kodaira type* I_n $(n \geq 3)$ *which is a half-pencil of a genus one fibration on Y. Moreover, one can choose D such that all multisections of* $|2D|$ *that belong to Γ are bisections.*

Proof If Γ supports a half-pencil, then it has the claimed type, so let us just assume to the contrary that Γ supports no half-pencil.

By Lemma 6, there is a divisor D of Kodaira type supported on Γ. Consider D together with a multisection $C \in \Gamma$ as in Sect. 5. Since D is not a half-pencil, we have $C \cdot D \geq 2$. Recall from (5.1) that $\Theta \cdot C \leq 1$ for any component Θ of D and the multisection C is nodal.

If D has type I_m^*, then either C meets at least two components of D, so there is a cycle (type I_n) supported on Γ, or it meets just a single double fibre component. But then the multisection meets the fibre component transversally, so C must be a bisection. Moreover, on the K3 cover C splits into two (disjoint) sections, none of which may meet the double component of an I_m^* fibre of the fibration induced on the K3 cover, contradiction.

The same reasoning applies to D of Kodaira type IV^*. Indeed, the intersection $D \cdot C$ must be even, so C cannot meet D in exactly one point on the triple component.

For type III^*, the multisection C could a priori also meet just the fourfold fibre component Θ, but then Γ would also support a divisor of type I_0^*, and we conclude as before. An analogous argument applies to type II^*.

Finally, if there is a divisor of type I_n supported on Γ which is not a half-pencil, then it connects with C through at least two distinct fibre components; this yields a cycle of length at most $\lfloor \frac{n}{2} \rfloor + 2$. Iterating this procedure as long we do not get a half-pencil, we obtain a divisor of type I_3 or I_4 supported on Γ. This is a half-pencil by Lemma 9, contradiction, which completes the proof of the first claim of the lemma.

Let D be a divisor of Kodaira type I_n $(3 \leq n \leq 9)$ supported on Γ such that $|2D|$ induces a genus one fibration. If Γ contained a multisection of $|2D|$ of index > 2, then the latter could be used to produce a cycle of type I_3 or I_4 with all components in Γ. By Lemma 9 Γ contains only bisections of the fibration such a cycle defines, so we can find a divisor D of Kodaira type I_n supported on Γ such that all multisections in Γ are in fact bisections of $|2D|$. □

7 Hyperbolic case with at most 3 bisections

Let $\#\Gamma > 12$. In this section, we rule out the possibility that Γ contains at most three bisections of the genus one fibration $|2D|$ (that exists by Lemma 11) as soon as (4.1) holds.

By Corollary 5, the graph Γ is hyperbolic. If there are at most three bisections of $|2D|$ in Γ, then there are at least ten fibre components supported on Γ. As soon as (4.1) holds, this turns out to be very restrictive, since fibres of type $\mathrm{I}_1, \mathrm{I}_2$ cannot be supported on Γ by Lemma 9 and types $\mathrm{I}_3, \mathrm{I}_4$ are automatically multiple.

Recall that the Jacobian fibration of $|2D|$ is a rational elliptic surface. Naturally, it shares the same singular fibres with Y, except that on Y, smooth or semi-stable fibres (Kodaira type $I_n, n \geq 0$) may come with multiplicity two. The classification of rational elliptic surfaces of [18] and Lemma 9 then give the six configurations 7.1–7.6 below, each with exactly three bisections supported on Γ.

Let $B \in \Gamma$ be a bisection of $|2D|$. Since B is a nodal curve, it splits into two disjoint sections O, P of the induced elliptic fibration on the K3 cover X and one can use the theory of Mordell–Weil lattices [25], and the fact that B meets no curve in Γ with multiplicity greater than one, to compute (an upper bound for) the height of P. We also use the classification of extremal rational elliptic surfaces in [13,14] and the geometry of Enriques surfaces with finite automorphism groups analysed in [12,15].

7.1 Two I_5

In this case, we have $h(P) \leq 4 - \frac{5}{2} - 2 \cdot \frac{4}{5} < 0$, contradiction.

7.2 I_4 and I_1^*

We obtain $h(P) \leq 4 - 2 - 2 \cdot 1 = 0$, so P necessarily is 2-torsion, and Y has Kondo's type II with exactly 12 smooth rational curves by [12,15], so $\#\Gamma \leq 12$, contradiction.

7.3 I_3 and IV*

Here, $h(P) = 4 - \frac{3}{2} - 2 \cdot \frac{4}{3} < 0$, contradiction.

7.4 Two I_3 and two A_2 embedding into I_3 each (outside characteristic 2, since then there can only be one multiple fibre, and characteristic 3, because the fibration ceases to exist there)

In this case $h(P) = 4 - 2 \cdot \frac{3}{2} = 1$, i.e. B meets the components of the unramified I_3 fibres not contained in Γ, since other configurations would cause $h(P) < 0$. It follows that the K3 cover has $\rho(X) > 20$, so $\rho(X) = 22$ by [1]. But then the lattice L generated by fibre components, torsion sections and P embeds into some supersingular K3 lattice $\Lambda_{p,\sigma}$ which is p-elementary. However, L is primitive (since else there would be a section Q of height $h(Q) = h(P)/9 = 1/9$ which is impossible with the given fibre types), and one computes that $L^\vee/L$ contains a subgroup isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. This prevents L from embedding into $\Lambda_{p,\sigma}$ (cf. [8, Thm. 6.1]).

7.5 Two I_4 and two A_1 embedding into I_2 each (outside characteristic 2, since this surface ceases to exist there)

We have $h(P) \leq 4 - 2 \cdot 2 = 0$, so P is 2-torsion and Y has Kondo's type III. This exists outside characteristics 2, 5 with exactly 20 smooth rational curves by [12,15]. Fixing two I_4 fibres and a bisection B to be contained in Γ, B determines the precise components $\Theta, \Theta' \in \Gamma$ of the I_2 fibres, since it meets the respective other component with multiplicity two (which is thus not contained in Γ by Lemma 9). But then, all other multisections are ruled out, again by Lemma 9, since each meets Θ or Θ' with multiplicity two, so $\#\Gamma = 11$, contradiction.

7.6 I_3, I_6 and A_1 embedding into I_2 (or into III in characteristic 3), (again outside characteristic 2, since there I_3 degenerates into IV which cannot be multiple)

In this case, we have $h(P) \leq 4 - \frac{3}{2} - 2 \cdot \frac{5}{6} = \frac{5}{6}$, so this case is not ruled out as the other ones. However, any other configuration would yield $h(P) < 0$, so the configuration is unique, and we deduce that Y is of Kondo's type VII.

Until now, we have shown the following lemma.

Lemma 12 *Let* $\#\Gamma > 12$ *and* $H^2 > 12d^2$. *If* Γ *contains at least ten components of singular fibres of the genus one fibration* $|2D|$ (*cf. Lemma* 11), *then it contains exactly ten components and they form the configuration* 7.6.

To complete the analysis of the case with three or less bisections contained in Γ, we refrain to an optimization approach. To this end, we assume that the ten fibre components form the configuration 7.6 and fix a $\mathbb{Q}$-basis of $\mathrm{Num}(Y)$ consisting of a bisection B, I_3 components, five components of I_6 (only one of which meets B), and the I_2 component off B.

Lemma 13 *If the above ten rational curves and the remaining component of* I_6 *are all in* Γ, *then* $H^2 < 12d^2$.

Proof The above curves generate $\mathrm{Num}(Y)$, so their degrees determine the polarization H, and there is no need to consider the intrinsic polarization. Computing an upper bound for

H^2 amounts to optimizing the quadratic form given by the inverse of the Gram matrix G over (the $\mathbb{Q}$-points of) the set $(0, 1]^{10}$ (taking into account that $\deg(I_6) = 2\deg(I_3)$). Thanks to the specific shape of G^{-1}, this can be broken down into two separate optimization problems—one in three variables and one in four variables. These are few enough variables to handle all the boundary components involved. Interlacing the two resulting partial upper bounds, one obtains that the quadratic form always evaluates less than 12 as desired. □

This completes the analysis of the case with three or less bisections contained in Γ.

Remark 14 We will show in Proposition 21 that, Lemma 13 is close to sharp in the sense that the above Enriques surface contains 13 smooth rational curves of degree $\leq d$ with respect to a polarization of degree $12d - 2$.

8 Proof of Theorem 1

In this section, we finally prove Theorem 1. The proof is preceded by several lemmas. At first, we reduce our analysis of Kodaira divisors to two fibre types only.

Lemma 15 *If* $H^2 > 12d^2$ *and* $\#\Gamma > 12$*, then* Γ *supports a divisor of type either* I_3 *or* I_4*.*

Proof By Lemma 11, there is a half-pencil of type I_n with all its components in Γ, such that Γ contains only fibre components and bisections of the genus one fibration it defines. Moreover, Lemma 12 combined with Lemma 13 shows that Γ supports at least four such bisections.

If two of the bisections meet the same fibre component, then either they are adjacent, so we get a I_3 divisor, or they are perpendicular and we get a I_0^* divisor which is met by some fibre component with multiplicity one, contradiction. Hence, all bisection meet different fibre components. If $n < 8$, then there are two bisections meeting adjacent fibre components. Thus, we get a I_4 divisor (if the bisections intersect) or a I_1^* divisor (if the bisections are disjoint), again intersected by some fibre component with multiplicity one. If $n = 8$ or 9, the same reasoning applies, but the divisors could also have types I_2^* (with the same contradiction) and I_5—which would be multiple, so we can revisit the case $n < 8$ to conclude the proof. □

Now, we are in the position to rule out the existence of a triangle (i.e. a I_3 fibre) in Γ.

Lemma 16 *If* $H^2 > 12d^2$ *and* $\#\Gamma > 12$*, then* Γ *supports no divisor of type* I_3*.*

Proof Assume to the contrary that Γ supports a triangle. Since $\#\Gamma > 12$, the graph Γ is hyperbolic, so we can apply Lemma 9 to show that the I_3-configuration is a half-pencil of a genus one fibration such that Γ contains only its fibre components and bisections. Moreover, by Lemmas 12, 13 Γ must contain at least four bisections. Thus, we can assume that Γ supports a divisor of type I_3 with four bisections. We claim that

$$H^2 \leq 11\frac{5}{6}d^2. \tag{8.1}$$

Indeed, the proof of (8.1) amounts to bounding the degree of the intrinsic polarization for some test (sub)configurations. The bound is obtained from the entries of the inverse of the Gram matrix of the (sub)configuration in question (see Remark 8). If there are two adjacent bisections, then the rank five lattice L generated by the I_3 fibre and these bisections gives

$$H_L^2 \leq \begin{cases} 9\frac{1}{6} & \text{if the bisections meet the same fibre component,} \\ 10\frac{3}{8} & \text{if the bisections meet different fibre components.} \end{cases}$$

Otherwise, all bisections are perpendicular, and a quick analysis gives the inequality (8.1) for the intrinsic polarization in each possible case. (In the case when each fibre component is met by at least one bisection, one can split off a negative semi-definite matrix as in Remark 8 to compensate for negative entries in G^{-1}.)

Obviously, (8.1) yields the desired contradiction and completes the proof. □

In particular, we have shown that Γ contains a quadrangle (i.e. an I_4 fibre) as soon as $\#\Gamma > 12$ and the degree of the polarization is high enough (i.e. (4.1) holds). Recall that by Lemma 9, the system $|2I_4|$ endows the Enriques surface Y with a genus one fibration.

Lemma 17 *If* $H^2 > 12d^2$ *and* $\#\Gamma > 12$*, then* Γ *supports at most* 4 *bisections of the fibration* $|2I_4|$.

Proof Observe that any two bisections meeting the same fibre component are disjoint, for otherwise they would form a I_3 divisor, which we have covered already in Lemma 16. Moreover, if there were more than two bisections meeting the same fibre component of the I_4, then we could build a I_0^* divisor intersected by some fibre component with multiplicity one, contradiction. Hence, each fibre component is met by at most two bisections.

If the fibration given by $|2I_4|$ has five bisections, then one can systematically go through all possible configurations to confirm the claim. In fact, for each configuration, one easily finds a suitable subconfiguration of rank exceeding 10. But this is the Picard number of any Enriques surface, regardless of the characteristic (see [2]), contradiction. □

After these preparations, we can finally complete the proof of the bound.

Proof of Theorem 1 In order to derive a contradiction, we assume that $H^2 > 12d^2$ and $\#\Gamma > 12$. Recall that Γ is hyperbolic by Corollary 5.

Lemmas 15, 16 yield that Γ contains an I_4 configuration. From Lemma 9, we infer that Γ consists of fibre components and bisections of $|2I_4|$. Lemma 17 implies that Γ contains at most four bisections.

If Γ contains at least 10 fibre components, then Lemma 12 combined with Lemma 13 leads to a contradiction. Thus, the assumption $\#\Gamma > 12$ implies that Γ contains exactly nine fibre components and exactly four bisections. The classification in [18] gives the following configurations:

8.1 $I_4 + I_5$

Together with one bisection these curves support a divisor of type I_3, leading back to Lemma 16, or of type I_2^* met by some fibre component with multiplicity one, contradiction.

8.2 $I_4 + I_0^*$

Here, each bisection connects with I_0^* to another I_4, so one of these has at least five bisections, which is impossible by Lemma 17.

8.3 $I_4 + A_3 + A_1 + A_1$

The root lattices could be realized inside a single I_1^* or inside $I_4+I_2+I_2$. In the former case, any bisection splits into section and two-torsion section on the K3 cover, so we derive Kondo's type II surfaces—with exactly 12 smooth rational curves by [12,15]. In the latter case, the given curves together with any bisection form a $\mathbb{Q}$-basis of $\mathrm{Num}(Y)$, and one can easily check that any configuration either leads back to a previous case or leaves no room for further smooth rational bisections fitting our scheme imposed by (5.1).

8.4 $I_4 + D_5$

This case either gives Kondo's family of type II again or leaves no room for more than one smooth rational bisection.

8.5 $I_4 + I_4 + A_1$

Here, any bisection splits into section and two-torsion section on the K3 cover, so the bisection does not meet the A_1 summand (which embeds into an I_2 fibre). Hence, all computations can be carried out in a lattice of rank at most 9. For each possible configuration, this quickly leads to a contradiction. □

9 Enriques surfaces with 12 rational curves of even degree

Proving Proposition 2 (i) amounts to the following:

Proposition 18 *Let $d > 2$ and $h \geq 3d$. Then, there is an Enriques surface of degree $2h$ containing 12 rational curves of degree $2d$.*

Proof Let Y be a general Enriques surface such that the following hold:

- it contains no smooth rational curve;
- each genus one fibration on Y has exactly 12 reduced singular fibres, all of which are nodal cubics;
- for each genus one fibration on Y both half-fibres are irreducible, smooth elliptic curves.

The existence of such a surface can be shown with help of the relation between deformations of unnodal Enriques surfaces and deformations of their Jacobians (see [16, Remark 5.6]).

Recall that there exist half-pencils E, E' of genus one fibrations on Y such that $E \cdot E' = 1$, (this follows from [5, Thm 6.1.10]; here the assumption that Y is unnodal is of importance only when $\mathrm{char}(k) = 2$).

As in the K3 case, we want to work with a sublattice

$$L = \begin{pmatrix} 0 & d \\ d & c \end{pmatrix} \hookrightarrow \mathrm{Num}(Y) \cong U + E_8$$

for some $c \in \{0, \ldots, 2d-2\}$.

Here, we set up L as follows: We consider $U \subset \mathrm{Num}(Y)$, that is generated by the half-pencils E, E' and pick $D' \in U^\perp$ with $D'^2 = c - 2d$. Consider the divisor

$$D = E + dE' + D' \quad \text{with} \quad D^2 = c.$$

By Riemann–Roch, D is effective (since $D \cdot E' = 1$ and $D^2 \geq 0$). Moreover, by assumption, all singular fibres of $|2E|$ have type I_1, so there are 12 in number, none of which is multiple.

Consider the divisor

$$H = NE + D \quad (N \in \mathbb{N}).$$

We claim that H is very ample if $N \geq 3$. Indeed, we have

- $H^2 = 2Nd + c \geq 6N$;
- $H \cdot C > 0$ for any curve $C \subset Y$, since C is multisection for $|2E|$ or $|2E'|$, so $H \cdot C \geq (E + E') \cdot C > 0$;
- $H \cdot E'' > 2$ for any half-pencil E'' on Y, since either $2E'' \sim 2E$ and $H \cdot E'' = H \cdot E = d > 2$ or E'' is a multisection of $|2E|$, so $H \cdot E'' \geq N$.

It follows from Criterion 3 that for any $h \geq 9$, we find N and c as above such that H is very ample with $H^2 = 2h$. By construction, (Y, H) contains rational 12 curves of degree $2d$ (the singular fibres of $|2E|$). □

10 Enriques surfaces with 12 smooth rational curves of fixed degree

In this section, we prove Proposition 2 (ii). This is equivalent to the following:

Proposition 19 *Assume that the characteristic of k differs from 2. Let $d > 2$ and $h \geq 9$ such that $d \mid h$. Then, there is an Enriques surface of degree $2h$ containing 12 smooth rational curves of degree d.*

Proof To prove the proposition, we aim at constructing a three-dimensional family of Enriques surfaces Y with a genus one fibration with six fibres of type I_2 and a rational bisection B with $B^2 = 2$ which meets each reducible fibre in both components.

10.1 Base change construction

Start with the rational elliptic surface S which will feature as Jacobian of Y. This takes the shape

$$S: \quad y^2 = x(x - f)(x - g), \quad f, g \in k[t], \ \deg(f) = \deg(g) = 2.$$

We want to apply the base change construction from [7], i.e. endow some quadratic base change of S (an elliptic K3 surface X) with a section P which is anti-invariant in $\mathrm{MW}(X)$ for the deck transformation ι of the double cover $X \dashrightarrow S$. Thus, we are led to work on the quadratic twist

$$S': \quad qy^2 = x(x - f)(x - g) \tag{10.1}$$

at some quadratic non-square polynomial $q \in k[t]$. We need to endow S' with a section P' of height 3 which meets all reducible fibres (6 I_2 and 2 I_0^*) non-trivially. That is, $P'.O' = 2$ with intersection points at another quadratic polynomial $w \in k[t]$. In the above affine model (10.1), the section P' takes the shape

$$P' = \left(\frac{fgr}{w^2}, \frac{fg(f-g)r'}{w^3}\right), \quad r, r' \in k[t], \ \deg(r) = \deg(r') = 2. \tag{10.2}$$

where the I_2 fibres are located at the zeroes of $fg(f - g)$ (so the y-coordinate of P' vanishes at all of them as should be). We now make the following convenient normalizations and choices:

- w is monic (by absorbing the top coefficient into the nominator);

- f is monic (by rescaling x, y)
- the zeroes of $f - g$ are $t = 0, \infty$ by Möbius transformation, i.e. the top and bottom coefficients agree:

$$f = t^2 + bt + c^2; \quad g = t^2 + b't + c^2.$$

- r is a square with zero at $t = -1$; the second choice amounts to a Möbius transformation again, while the first is one of a few natural choices on elliptic surfaces with two-torsion in MW, cf. [20, §8.1.4].

For P' to meet the node of I_2 fibres at $t = 0, \infty$ of the Weierstrass model (10.1) (so it meets the non-identity component of the Kodaira–Néron model), we read off the top coefficient of r and subsequently the bottom coefficient of w:

$$r = (t + 1)^2, \quad w = t^2 + d't + c.$$

This yields $r' = (t + 1)r''$, and it remains to choose the coefficients b, b', c, d' in such a way that substituting the x-coordinate of P' into the RHS of (10.1) gives a square r''^2 next to the other obvious squares. This amounts to computing the discriminant of a degree 4 polynomial and has, up to symmetry in b and b', the unique solution

$$d' = 2d - 1 - c, \quad b' = d^2 - 2c.$$

One then verifies that generically this exactly gives the configuration of singular fibres and section which we are aiming for.

10.2 Enriques quotient

The quadratic base change of S (and S') ramified at the zeroes of q generically gives the announced elliptic K3 surface X with 12 I_2 fibres and section P of height 6 pulling back from P'. By construction, P' is anti-invariant in $\mathrm{MW}(X)$ for the deck transformation ι, so composing ι with translation by P gives a fixed point free involution j (since P meets non-trivial two-torsion points at the fixed fibres, corresponding to the non-identity components of the I_0^* fibres on S'). Hence $Y = X/\langle j \rangle$ is an Enriques surface, and O, P map down to a rational bisection B which meets each I_2 fibre in both components and which has two nodes at the zeroes of w, i.e. $B^2 = 2$. (Note that $P.O = 4$ and $(P + O)^2 = 4$.)

10.3 Degree *d* smooth rational curves on *Y*

Consider the induced half-pencil E on Y. Instead of working with the bisection B, we note that the isotropic divisor

$$E' = B - E$$

is effective by Riemann–Roch (since $E \cdot E' = 1$). We claim that $|2E'|$ has no base locus, for otherwise there would be a (-2)-curve C such that

$$E' \cdot C < 0 \quad \text{and} \quad E'' = E' + (E' \cdot C)C > 0.$$

But then $E \cdot E' = 1$ implies, since E is nef, that $E \cdot C = 0$ or 1 and same with interchanged values for $E \cdot E''$. In the former case, we derive the contradiction

$$0 < B \cdot C = (E + E') \cdot C < 0.$$

In the latter case, we have $E \cdot E'' = 0$, so $E \equiv E''$ and C is a bisection of $|2E|$. But then, it splits on the K3 cover X into two disjoint sections always meeting opposite components of the I_2 fibres, and as in Sect. 7, the height of one section relative to the other returns $h = 4 - 12 \cdot \frac{1}{2} = -2$ which is impossible.

To conclude the proof of Proposition 19, consider the divisor

$$NE + dE'$$

which is easily verified, using Criterion 3, to be very ample for $N, d \geq 3$. This leads to the claimed range of polarizations, and to 12 smooth rational curves of degree d—the components of the six I_2 fibres. □

10.4 Odd degree curves in characteristic 2

We conclude this section with an indication why characteristic 2 can be rather special:

Proposition 20 *Let d be odd. Let Y be an Enriques surface with smooth K3 cover over a field of characteristic 2, endowed with a polarization of degree $H^2 \gg 0$. Then,*

$$r_d < 11.$$

Proof Assume first that $r_d = 12$. Then, Lemma 7 implies that Γ cannot be hyperbolic, so by Corollary 5, it is parabolic, and all curves in Γ form fibre components (of the same degree d) of a genus one fibration on Y. Necessarily, this is semi-stable, and all reduced fibres have the same degree, hence the same type I_{2n}. (If there were an odd number of components, then $H \cdot F$ would be odd for a general fibre F, but writing H as a $\mathbb{Z}$-linear combination of fibre components and multisections of course leads to $H.F$ being even.) Along the same lines, the support of any multiple fibre may only be smooth or of type I_n. This allows only for the following configurations:

$$6 \times I_2, \quad 5 \times I_2 + 2 \times I_1, \quad 2 \times I_4 + 2 \times I_2.$$

Note that the first configuration is used in Proposition 19—outside characteristic 2! Indeed, so far our argument has been independent of the characteristic. But then, the last two configurations involve two multiple fibres which is impossible in characteristic 2. Moreover, the first and the last configuration do not exist on a rational elliptic surface in characteristic 2 by [24, Thm. 8.9]. Hence, $r_d < 12$.

We continue by assuming that $r_d = 11$. As before, all curves in Γ are supported on the fibres of a genus one fibration, so there are two possibilities:

1. either all singular fibres are multiplicative, and exactly one of the 12 rational fibre components is not contained in Γ,
2. or exactly one singular fibre is additive (without wild ramification), and all rational fibre components are contained in Γ.

The second case is easy to rule out, since the only additive fibre types without wild ramification in characteristic 2 are IV, IV* by [23]. But then, the first has three fibre components, contradicting what we have seen above, while the second has degree $12d$ which is too large to be met by I_n fibres (supported on Γ) by inspection of the Euler–Poincaré characteristic.

In the first case, one easily checks that the multiple fibre, if singular, is supported on Γ, leaving the following configurations of 11 rational curves in Γ:

$$2 \times I_4 + I_2(\text{multiple}) + A_1, \quad 5 \times I_2 + I_1(\text{multiple}).$$

The first configuration has A_1 inside I_2, so this cannot exist as pointed out before. Similarly, the second configuration generally forces a 2-torsion section upon the Jacobian by [18].

But then, by [24, Cor. 8.32], there is an additive fibre in characteristic 2, ruling out this configuration as well. □

11 An Enriques surface with 13 smooth rational curves of small degree

In this section we demonstrate that the bound for H^2 in Theorem 1 cannot be improved beyond $H^2 > 12d^2 - 2$ as soon as $d > 1$. To this end, we prove the following:

Proposition 21 *Let Y be the Enriques surface of Kondo's type VII over a field of characteristic $\neq 2, 5$. Let $d \in \mathbb{N}$. Then, Y admits a polarization H of degree $H^2 = 12d^2 - 2$ and 12 smooth rational curves of degree d as well as one of degree 2.*

Proof As in Sect. 7.6, Y is endowed a genus one fibration with reducible fibres I_6, I_3 (ramified, corresponding to the half-pencil E) and I_2 (or III in characteristic 3). Fix a component Θ of the I_2 resp. III fibre and fix those three smooth rational bisections B_1, B_2, B_3 which do not meet Θ (and are pairwise disjoint). We postulate that

- B_1, B_2, B_3 and all components of the I_6 and I_3 fibres have degree d;
- $\deg(\Theta) = 2$.

These degrees determine the polarization H uniquely as

$$H = (3d - 2)E + d(B_1 + B_2 + B_3) + \Theta'$$

where Θ' is the component of the I_2 resp. III fibre other than Θ. We claim that H is very ample. To verify this by Criterion 3, one computes directly that

- $H^2 = 12d^2 - 2 \geq 10$;
- $H \cdot C > 0$ for any smooth rational curve on Y (the other degrees appearing are $4d - 2$ and $6d - 2$);
- $H \cdot C > 0$ for any other curve in Y, since this is either a multisection of $|2E|$, so $H \cdot C \geq (3d-2)E \cdot C \geq 3d-2$, or a (half)-fibre, so $H \cdot C = d(B_1 + B_2 + B_3) \cdot C \geq 3d$.

It remains to discuss the case where E' is a half-pencil $\not\equiv E$. Consider the half-pencils

$$E_i'' = B_i + \Theta' \quad \text{with bisections} \quad B_j, B_k \quad (\{i, j, k\} = \{1, 2, 3\}).$$

Then, either $E' \equiv E_i''$ for some i, so $H \cdot E' \geq (B_j + B_k) \cdot \Theta' = 4d$, or E' is a multisection for $|2E|$ and all $|2E_i''|$, so

$$H \cdot E' \geq ((3d - 2)E + B_i + \Theta') \cdot E' \geq 3d - 1, \tag{11.1}$$

with equality for each i if and only if $E' \cdot E = 1, E' \cdot \Theta' = 1$ and $E' \cdot B_i = 0$. Now Θ' meets each smooth rational curve on Y with even multiplicity by [12, Fig. 7.7], and the possible half-pencils are classified in [12, Table 2]:

- $E' \equiv I_2$ or I_3, so $E' \cdot \Theta' \in 2\mathbb{Z}$;
- $E' \equiv \frac{1}{2} I_5$ (for either of the two I_5 fibres) or $\frac{1}{2} I_9$. Since $E' \cdot B_i = 0$, each B_i is contained in some fibre of $|2E'|$. But then, one of the fibres contains two of the B_i, so $\Theta' \cdot E' \geq \Theta' \cdot (B_i + B_j)/2 = 2$.

In either case, (11.1) thus improves to $H \cdot E' \geq 3d \geq 3$. Hence Criterion 3 applies to show that H is very ample. By construction, Y contains the smooth rational curves of the given degrees. □

Authors' contributions
We thank the referee for helpful comments.

Funding Open Access funding enabled and organized by Projekt DEAL. Research partially supported by the National Science Centre, Poland, Opus Grant No. 2017/25/B/ST1/00853 (S. Rams).

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Received: 3 December 2020 Accepted: 18 March 2021 Published online: 12 April 2021

References

1. Artin, M.: Supersingular *K*3 surfaces. Ann. Sci. Éc. Norm. Sup. **4**(7), 543–568 (1974)
2. Bombieri, E., Mumford, D.: Enriques classification of surfaces in char. p, III. Invent. Math. **35**, 197–232 (1976)
3. Cossec, F.: Projective models of Enriques surfaces. Math. Ann. **2**65, 283–334 (1983)
4. Cossec, F., Dolgachev, I., Liedtke, Ch.: Enriques surfaces I. With an appendix by S. Kondō. manuscript. http://www.math.lsa.umich.edu/~idolga/ (2020)
5. Dolgachev, I., Kondō, S.: Enriques surfaces II., manuscript. http://www.math.lsa.umich.edu/~idolga/ (2020)
6. Degtyarev, A.: Lines on smooth polarized K3 surfaces. Discrete Comput. Geom. **62**, 601–648 (2019)
7. Hulek, K., Schütt, M.: Enriques surfaces and Jacobian elliptic surfaces. Math. Z. **268**, 1025–1056 (2011)
8. Katsura, T., Schütt, M.: Zariski K3 surfaces. Rev. Ibero. Mat. **36**, 869–894 (2020)
9. Kodaira, K.: On compact analytic surfaces I. Ann. Math. **71**, 111–152 (1960)
10. Kodaira, K.: On compact analytic surfaces II. Ann. Math. **77**, 563–626 (1963)
11. Kodaira, K.: On compact analytic surfaces III. Ann. Math. **78**, 1–40 (1963)
12. Kondō, S.: Enriques surfaces with finite automorphism groups. Jpn. J. Math. (N.S.) **12**, 191–282 (1986)
13. Lang, W.E.: Extremal rational elliptic surfaces in characteristic p. I. Beauville surfaces. Math. Z. **207**, 429–437 (1991)
14. Lang, W.E.: Extremal rational elliptic surfaces in characteristic p. II. Surfaces with three or fewer singular fibres. Ark. Mat. **32**, 423–448 (1994)
15. Martin, G.: Enriques surfaces with finite automorphism group in positive characteristic. Algebraic Geom. **6**, 592–649 (2019)
16. Martin, G.: Automorphisms of unnodal Enriques surfaces, preprint (2019). arXiv:1908.00049
17. Nikulin, V.: On a description of the automorphism groups of Enriques surfaces. Sov. Math. Dokl. **30**, 282–285 (1984)
18. Oguiso, K., Shioda, T.: The Mordell–Weil lattice of a rational elliptic surface. Comment. Math. Univ. St. Pauli **40**, 83–99 (1991)
19. Schütt, M., Rams, S.: 24 rational curves on K3 surfaces, preprint (2020). arXiv:1907.04182v2
20. Schütt, M.: Moduli of Gorenstein $\mathbb{Q}$-homology projective planes, preprint (2016). arXiv:1505.04163v2
21. Schütt, M.: $\mathbb{Q}_\ell$-cohomology projective planes from Enriques surfaces in odd characteristic, preprint (2017). arXiv:1611.03847v3
22. Schütt, M.: $\mathbb{Q}_\ell$-cohomology projective planes and Enriques surfaces in characteristic two. Épijournal de Géométrie Algébrique **3**, 24 (2019)
23. Schütt, M., Schweizer, A.: On the uniqueness of elliptic K3 surfaces with maximal singular fibre. Ann. Inst. Fourier **63**, 689–713 (2013)
24. Schütt, M., Shioda, T.: Mordell–Weil lattices, Erg. der Math. und ihrer Grenzgebiete, 3. Folge, Band 70. Springer, Berlin (2019)
25. Shioda, T.: On the Mordell–Weil lattices. Comment. Math. Univ. St. Pauli **39**, 211–240 (1990)
26. Tate, J.: Algorithm for determining the type of a singular fibre in an elliptic pencil. In: Modular Functions of one Variable IV (Antwerpen 1972), Lecture Notes in Mathematics, vol. 476, pp. 33–52 (1975)

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