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Geodesics on a K3 surface near the orbifold limit

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Abstract
This article studies Kummer K3 surfaces close to the orbifold limit. We improve upon estimates for the Calabi–Yau metrics due to Kobayashi. As an application, we study stable closed geodesics. We use the metric estimates to show how there are generally restrictions on the existence of such geodesics. We also show how there can exist stable, closed geodesics in some highly symmetric circumstances due to hyperkähler identities.

Keywords Calabi–Yau · Complex Monge–Ampère · Closed geodesics · Hyperkähler

Mathematics Subject Classification Primary 53C21; Secondary 53C22 · 53C26

1 Introduction

Einstein metrics are interesting objects both from a physics and from a geometry perspective. By postulating enough symmetry, non-compact, examples have been found in both Lorentzian and Riemannian geometry. For instance the solutions due to Schwarzschild [63], Eguchi–Hanson [25], Calabi [17], Gibbons–Hawking [31] and Kronheimer [44, 45], to name just a few. These are so-called gravitational instantons. On compact manifolds, very few examples are known explicitly. An idea dating back to Page [59] and Gibbons–Pope [33] is to desingularize certain orbifolds by blowing up the singular points and gluing in gravitational instantons. This procedure produces a family of almost-Einstein metrics with concentrated curvature. By an implicit function argument [28, 39, 40], by Twistor methods [50, 67, 68], or by solving the Monge–Ampère equation [21, 43, 47], one can perturb the given metric to an Einstein metric. Since the original metric was close to solving the Einstein equation, one could hope that the metric is close to the solution of the Einstein equation. This turns out to be the case in several instances. This in turn allows one to study the geometry of the unknown Einstein metric by studying its known approximation.

As the size of the exceptional divisor in the blow-up tends to 0, both the patchwork metric and the Einstein metric degenerate to an orbifold metric. In fact, work by Bando, Kasue, and Nakajima [7, 10, 57] and Anderson [2] shows that a converse is true; a sequence of compact 4-manifolds and Einstein metrics with volume, diameter, and Euler characteristic bounds

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have convergent subsequences. The limit spaces have at worst isolated orbifold singularities. The orbifold limits are the added points of a compactification of the moduli space of Einstein metrics [2, 10, 57]. This makes understanding the orbifold limit more important.

In this paper, we will study the oldest and best-known example, namely the Kummer construction of a K3 surface. In this case, one can go beyond the general convergence statements of Nakajima and Anderson, and describe in more detail exactly how the Einstein metric degenerates. This programme was initially carried out by Todorov and Kobayashi [43, 47]. We have chosen to go through the arguments of Kobayashi in great detail in the hope that the present paper can serve as an introduction to this fascinating topic.

As an application, we study stable, closed geodesics on Ricci-flat Kummer K3 surfaces close to the orbifold limit. The lengths of closed geodesics on a Riemannian manifold is a much studied geometric quantity (see [20, 22–24, 37] for some highlights). For generic metrics it has been shown [62, Corollary 2] that there are infinitely many geometrically distinct, closed geodesics. The same holds for an arbitrary metric if one imposes mild topological assumptions (see [32, Theorem 4] and [71, Theorem (2nd)]). The topological conditions are fulfilled by all compact Calabi–Yau manifolds. We propose to restrict attention to the lengths of *stable*, closed geodesics.

Definition 1.1 Let (M, g) be a Riemannian manifold. A closed geodesic $\gamma : \mathbb{S}^1 \to M$ is said to be **stable** if $\delta^2 E_\gamma \geq 0$. Written out, this means

$$\delta^2 E_\gamma(\xi, \xi) = \int_{\mathbb{S}^1} |\nabla_{\dot\gamma}\xi|^2 - \langle R(\dot\gamma, \xi)\xi, \dot\gamma, \rangle \, dt \geq 0 \tag{1.1}$$

for all vector fields ξ along γ, where $R(U, V):=\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]}$ is the Riemann tensor.

Consider the number $\mathcal{N}(L)$ of stable, closed geodesics of length at most L, where one counts families of geodesics as a single geodesic. If the manifold (M, g) is compact and real analytic then it is a consequence of [65, Proposition 1.2] that $\mathcal{N}(L)$ is finite for any $L \geq 0$. For stable geodesics one has the following trichotomy based on curvature.

Theorem 1.2 [53, 56] *Let (M, g) be a compact, connected, real-analytic Riemannian manifold of real dimension n. Let $\mathcal{N}(L)$ be as above. Then, we have the following asymptotic behaviour as $L \to \infty$.*

- *If $Ric \geq \kappa(n-1)$ for some $\kappa > 0$, then $\mathcal{N}(L)$ is constant for $L > \frac{\pi}{\sqrt{\kappa}}$.*
- *If the sectional curvature vanishes, then $\mathcal{N}(L) \sim c(M)L^n$ for some constant $c(M) > 0$ depending on the manifold.*
- *If the sectional curvature is negative everywhere, then $\mathcal{N}(L) \sim \frac{e^{c(M)L}}{c(M)L}$ for some constant $c(M) > 0$ depending on the manifold.*

The hardest part of the above statement is the negative curvature case, which is [53], [54, Equation 6.87]. The positive case follows by the proof of the Bonnet–Myers theorem, and the flat case is a direct computation on a flat torus.

Theorem 1.2 is an example of a comparison geometry, and a natural question is whether or not one can replace sectional curvature by Ricci curvature. This motivates the following question.

Question 1 *Let (M, g) be a compact, connected, Ricci-flat manifold of dimension n. Is it true that $\mathcal{N}(L) \sim c(M)L^n$ for some constant $c(M) > 0$?*

Remark 1.3 We do not know what happens for $Ric < 0$. Our guess is that this condition is too weak, seeing how (in the light of [48, Theorem A]) the condition $Ric < 0$ gives absolutely no information about the underlying manifold in dimensions $n \geq 3$.

If one additionally assumes that the manifold is Kähler, then P. Gao and M. Douglas have put forward physics-based arguments in [30] for why the answer to the above question should be yes.

Conjecture 1 [30] *Any compact, Ricci-flat Calabi–Yau manifold*[1] $(X, \tilde{g})$ *has stable, closed, non-constant geodesics. In fact, if the manifold is of real dimension n then* $\mathcal{N}(L) \sim C(X)L^n$ *for some constant* $C(X) > 0$.

When Douglas and Gao published their work, there were no examples of a single stable, closed geodesic on a Ricci-flat, compact Calabi–Yau manifold. They, however, suggest as a starting point to investigate the conjecture on a Kummer K3 surface (the construction will be recalled in Sect. 2). In this article, we follow their advice and derive some constraints on stable, closed geodesics on Kummer K3 surfaces. Additionally, we show that the Riemann curvature tensor vanishes at certain points if the K3 surface has enough symmetry. Roughly speaking, a Kummer K3 surface is the minimal resolution of the orbifold $\mathbb{T}^4/\{\pm 1\}$ equipped with a Kähler metric g which is Eguchi–Hanson near any blown-up point, flat far away from the exceptional divisor, and a gluing of these two in between. We shall refer to this metric g as the patchwork metric. By the Calabi–Yau theorem, 2.2, there exists a unique Ricci-flat metric $\tilde{g}$ in the Kähler class of g. What we show is then the following.

Theorem 1.4 *Let X be a Kummer K3 surface with metrics g and $\tilde{g}$ as above. Then, when the exceptional divisor $E \subset X$ has small enough volume, there is an open set $U \subset X$ with $E \subset U \subset X$ such that no stable, closed geodesic (with respect to either g or $\tilde{g}$) on X ever enters U.*

Theorem 1.5 *Assume the set-up of Theorem 1.4. Let $U_i \subset X$ be a suitable neighbourhood around a single component E_i of the exceptional divisor. Then, when the volume of E is small enough there are no stable, closed geodesics which stay completely inside U_i.*

See Theorems 3.1 and 3.4 for the detailed statements.

Theorem 1.6 *Let X be the Kummer K3 surface associated with the torus $T = \mathbb{C}^2/\Gamma$ where $\Gamma := (\mathbb{Z}\{1, i\})^2 \subset \mathbb{C}^2$. Let g be the patchwork metric and let $\tilde{g}$ denote the unique Ricci-flat metric in the Kähler class of g. Then there are totally geodesic tori $\mathbb{T}^2 \subset X$ and points $p \in \mathbb{T}^2$ where the Riemann tensor of $(X, \tilde{g})$ vanishes. Furthermore, if the minima of the curvature of $\mathbb{T}^2$ are local minima of the holomorphic sectional curvature of X, the tori are flat.*

See Theorem 5.4 for a more detailed statement and a precise description of how some of these tori look like. Theorem 5.5 is the precise statement of the second half of the theorem.

To our knowledge, the only previous work on stable geodesics on compact, Ricci-flat Calabi–Yau manifolds are the articles [13, 14, 30]. What Bourguignon and Yau prove in [14] is the following, a result we will need later.

[1] Our definition of a Calabi–Yau manifold is a Kähler manifold (X, g) with vanishing first Chern class $c_1(X) = 0$ and trivial first cohomology group, $H^1(X; \mathbb{R}) = 0$.

Theorem 1.7 [14] *Assume* $(X, \tilde{g})$ *is a hyperkähler manifold of real dimension* 4. *Assume* $\gamma : \mathbb{S}^1 \to X$ *is a non-constant geodesic. Then,* γ *is stable if and only if the entire Riemann curvature tensor vanishes along* γ.

Theorem 1.7 is both a clear-cut criterion and a major hurdle for stability. A priori, it is not clear that a Ricci-flat space has a single point with vanishing Riemann tensor. Indeed, the Eguchi–Hanson space of [17, 25] has a Ricci-flat metric with nowhere vanishing Riemann tensor. The Eguchi–Hanson Kähler potential is given by (2.1), and the norm of the curvature tensor in (2.4). This is a non-compact manifold, so it does not contradict the above conjecture. In the presence of symmetries, Theorem 1.6 tells us that the Riemann tensor vanishes at certain points on a Kummer K3 surface. We do not know of other results of this kind. In particular, we do not know what happens on an arbitrary K3 surface.

The layout of the paper is as follows. We recall the Kummer construction, including the patchwork metric and metric estimates, in Sect. 2. Section 3 deals with the no-go results Theorems 1.4 and 1.5. Section 4 starts by studying what one can say about the curvature of hyperkähler 4-manifolds in the presence of symmetry, before specializing to a particular Kummer K3 surface in Sect. 5. To improve the flow, we have relegated some of the computations of the derivatives of the curvature to "Appendix A". In Sect. 6, we reprove the metric estimates of R. Kobayashi and correct some of the methods. We also include a short discussion of other approaches one could try to deduce Kobayashi's estimates.

We end the introduction by listing some general obstructions to studying the conjecture of [30].

- There is no explicitly known, non-flat, Ricci-flat metric on a compact Riemannian manifold. The stability of geodesics is, however, very dependent on the details of the metric (e.g. Theorem 1.7).
- In the case of a K3 surface, Theorem 1.7 has as corollary that any stable, closed geodesic has nullity equal to 3, and the linearized Poincaré map has all eigenvalues equal to 1. This says that stable, closed geodesics on K3 surfaces are very degenerate critical points of the energy function, making them harder to study using Morse–Bott-type methods.
- The Ricci-flatness condition on a Calabi–Yau manifold can be though of as specifying the volume to be "Euclidean" (Eq. (2.6) is the precise meaning of this). Deriving statements about the length spectrum can as such be seen as asking for length information when given information about the volume.
- A compact Calabi–Yau manifold with Ricci-flat metric always has finite isometry group (a fact due to S. Bochner—see [60, Theorem 1.5, p. 167] for instance). This makes it challenging to construct geodesics as fixed point sets of isometries.
- The fundamental group of a compact Calabi–Yau manifold is always finite. So, unlike in the case of a flat torus, one cannot realize enough stable, closed geodesics as non-trivial homotopy classes to fulfil the conjecture. Indeed, on a non-simply connected, compact manifold there are always closed geodesics which minimize the energy in their homotopy class, and are as such **strictly stable**, meaning all variations of the energy function are nonnegative, and not just the second variation.
- In Bourguignon [13, Théorème 2.9] proves that if one has a hyperkähler 4-manifold with a strictly stable, closed, non-constant geodesic then the manifold is flat. In particular, K3 surfaces never have *strictly* stable, closed geodesics.

Some words about the notation. Local expressions for Kähler metrics g on manifolds X of complex dimension n will be treated as Hermitian $n \times n$ matrices. Determinants and traces of Hermitian matrices are with respect to the complex matrices, not the associated real matrices.

The **complex Hessian** of f will be denoted by $\partial\overline{\partial} f$, where we allow ourselves to sometimes think of this as a (1, 1)-form and sometimes as a Hermitian matrix, i.e. the components of the (1, 1)-form. So

$$(\partial\overline{\partial} f)_{\mu\overline{\nu}} := \partial_\mu \partial_{\overline{\nu}} f := \frac{\partial f}{\partial z^\mu \partial \overline{z}^\nu}.$$

Tensor norms of complex tensors and the **Laplacian** will also be defined using the Hermitian matrices as follows.

$$\begin{aligned}
\Delta f &:= g^{\overline{\nu}\mu} \partial_\mu \partial_{\overline{\nu}} f = \mathrm{Tr}(g^{-1}\partial\overline{\partial} f) =: \mathrm{Tr}_g(\partial\overline{\partial} f),\\
|\nabla f|_g^2 &:= g^{\overline{\nu}\mu} \partial_\mu f \overline{\partial_\nu f},\\
|\partial\overline{\partial} f|_g^2 &:= g^{\overline{\nu}\mu} g^{\overline{\beta}\alpha} (\partial_\mu \partial_{\overline{\beta}} f)(\partial_\alpha \partial_{\overline{\nu}} f) = \mathrm{Tr}(g^{-1}(\partial\overline{\partial} f) g^{-1}(\partial\overline{\partial} f)).
\end{aligned}$$

Similar definitions hold for higher rank tensors. The complex Laplacian acting on functions coincides (up to a constant scaling) with the real Laplacian. The complex Hessian does *not* coincide with the real Hessian. Indeed, in complex dimension 1 we have $\partial\overline{\partial} f = g\Delta f$.

For the **real Hessian**, we write $D^2 f$. Its pointwise norm is

$$|D^2 f|_g^2 := \mathrm{Tr}\left(g_{\mathbb{R}}^{-1} D^2 f g_{\mathbb{R}}^{-1} D^2 f\right),$$

where $g_{\mathbb{R}}$ is the symmetric $2n \times 2n$-matrix associated with g. The Hölder semi-norm of $D^2 f$ is defined as.[2]

$$|D^2 f|_\alpha = \sup_{x,y} \frac{|D^2 f(x) - D^2 f(y)|_g}{d(x, y)^\alpha},$$

where the supremum is over all $x \in X$ and all $y \neq x$ contained in normal coordinate charts centred at x, and the tensor $D^2 f(y)$ means the tensor at x obtained by parallel transport along the radial geodesic between x and y.

2 The Kummer construction

The Kummer construction is a well-known construction which associates with any 4-torus $T \cong \mathbb{C}^2/\Gamma$ a K3 surface X. The idea of the patchwork metric which we put on X comes from [33, 59]. To our knowledge, Kobayashi was the first to write out the details of this metric in [43]. This is our main source on the Kummer construction. There is a twistorial discussion in [50], but they do not have any explicit metric estimates. Another possible reference is [21]. See also [51, Chapter 5] for more details.

Let us first give an algebraic-geometric description of the Kummer construction. This is a standard result and can be found in [8, p. 224] for instance. Let $\Gamma \subset \mathbb{C}^2$ be a non-degenerate lattice. Let $T := \mathbb{C}^2/\Gamma$ be the associated 4-torus. Let $\mu_2 := \{\pm 1\}$ act diagonally on $\mathbb{C}^2$. Then, this induces an action on T with precisely 16 fixed points. The quotient $Y := T/\mu_2$ is a complex space with singular set $Sing(Y) = \Gamma/2\Gamma$. The singular points are A_1-singularities. Blowing up each of these singular points once leads to a non-singular space X along with a blow-down map $\pi : X \to Y$. This is the minimal resolution of X and is called the **Kummer K3 surface associated with the torus** T.

[2] See [49, p. 44].

Near any of the fixed points, the singular space Y looks like $\mathbb{C}^2/\mu_2$, and the resolution of this can be identified with $\mathcal{O}_{\mathbb{CP}^1}(-2) = T^*\mathbb{CP}^1$. We like to think of $\mathcal{O}_{\mathbb{CP}^1}(-2)$ as

$$\mathcal{O}_{\mathbb{CP}^1}(-2) = \left\{((z, w), (\xi : \varsigma)) \,\middle|\, z\varsigma^2 = w\xi^2\right\} \subset \mathbb{C}^2 \times \mathbb{CP}^1,$$

and the map $(\mathbb{C}^2\backslash\{0\})/\mu_2 \to \mathcal{O}_{\mathbb{CP}^1}(-2)\backslash\mathbb{CP}^1$ can be given as $[(z, w)] \mapsto ((z^2, w^2), (z : w))$. The blow-down map $\mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{C}^2/\mu_2$ inverting the above map away from the zero section will be discussed in the proof of Lemma 6.9. The resolution $\mathcal{O}_{\mathbb{CP}^1}(-2)$ can be equipped with the Eguchi–Hanson metric, which was discovered in [25] and generalized in [17]. See [52] for a detailed discussion about these metrics and the resolutions. The Eguchi–Hanson Kähler potential is given in (2.1).

To describe the metric we put on X, we need to have a look at what happens near a blown-up point. This is done for us in [43, pp. 293–297]. Let z be the coordinates on $\mathbb{C}^2$, and define $u{:=}|z|^2_{\mathbb{C}^2}{:=}|z_1|^2 + |z_2|^2$. Choose some $a > 0$ and let $f_{\mathrm{Euc}}, f_a \colon (\mathbb{C}^2\backslash\{0\})/\mu_2 \to \mathbb{R}$ be the **Euclidean Kähler potential** and **Eguchi–Hanson Kähler potential**, respectively, namely

$$\begin{aligned} f_{\mathrm{Euc}}(z) &:= u \\ f_a(z) &:= \sqrt{a^2 + u^2} - a \cdot \operatorname{arsinh}\left(\frac{a}{u}\right). \end{aligned} \tag{2.1}$$

Here $\operatorname{arsinh}(x){:=}\log\left(x + \sqrt{1 + x^2}\right)$ denotes the inverse function of sinh. The Eguchi–Hanson Kähler potential (2.1) is not regular down to $u = 0$, but the Eguchi–Hanson metric $g_{\mathrm{EH}}{:=}\partial\overline{\partial} f_a$ does extend across the zero-section $\mathbb{CP}^1 \subset \mathcal{O}_{\mathbb{CP}^1}(-2)$ to give a complete metric, as is seen by choosing suitable coordinates. Here, we are identifying $(\mathbb{C}^2\backslash\{0\})/\mu_2 \cong \mathcal{O}_{\mathbb{CP}^1}(-2)\backslash\mathbb{CP}^1$. We postpone this computation until Lemma 6.9 in Sect. 6.

Let $0 < \delta \ll 1$ be some fixed number and let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth cut-off function with the following properties.

- $\chi(u) = 1$ for $u \leq 1$
- $\chi(u) = 0$ for $u \geq 1 + \delta$.

Then

$$\Phi_a(z) := f_{\mathrm{Euc}}(z) + \chi(u(z))(f_a(z) - f_{\mathrm{Euc}}(z)) \tag{2.2}$$

defines a spherically symmetric Kähler potential on $(\mathbb{C}^2\backslash\{0\})/\mu_2$ for all values of a small enough. We shall write $\Phi_a(z){=:}\varphi_a(u(z))$. Furthermore, on the complement of any orbiball $V{:=}(\mathbb{C}^2\backslash B_R(0))/\mu_2$ we may write

$$\Phi_a(z) = u + a^2\xi_a(z) \tag{2.3}$$

for some function $\xi_a \colon V \to \mathbb{R}$ which is regular as $a \to 0$. For later use, we also record the norm squared of the Riemann tensor of the Eguchi–Hanson metric,

$$|\mathrm{Riem}_{g_{\mathrm{EH}}}|^2_{g_{\mathrm{EH}}} = \frac{24a^4}{(a^2 + u^2)^3}. \tag{2.4}$$

This is an L^2-function with most of its mass concentrated near $u = 0$; hence, the patchwork metric is a metric of concentrated curvature.

We want to define a Kähler metric on all of X whose Kähler potential is given by (2.2) close to the exceptional divisor and flat far away from it. Let $\pi : X \to Y$ be the blow-down map as above. Let $Sing(Y) = \cup_{i=1}^{16}\{p_i\}$ denote the singular points of Y and let $E{:=}\cup_{i=1}^{16} E_i{:=}\cup_{i=1}^{16} \pi^{-1}(\{p_i\})$ be the **exceptional divisor** of X. Fix a number $0 < \delta \ll 1$, and choose numbers a_i, $1 \leq i \leq 16$, such that $0 < a_i \ll 1$. Then, there exists a Kähler metric g on X with the following properties. Each component $E_i \subset X$ has a neighbourhood $U_i \subset X$ such that $U_i := Bl_0(B_{1+2\delta}(0)/\mu_2)$ and $E_i = \mathbb{CP}^1$. By scaling X if necessary, we may assume $U_i \cap U_j = \emptyset$ for $i \neq j$. On $Bl_0(B_{1+2\delta}(0)/\mu_2)$ the Kähler potential of g is given by (2.2) with parameter a_i. In particular g is equal to the Eguchi–Hanson metric with potential (2.1) on $Bl_0(B_1(0)/\mu_2)$ and $g_{|E_i} = a_i g_{FS}$ where g_{FS} is the Fubini–Study metric on $\mathbb{CP}^1$. On any of the necks $N_i := (B_{1+\delta}(0)\backslash B_1(0))/\mu_2$ the metric is not Ricci-flat. Outside of all the sets U_i the metric g is flat.

The metric g will be called the **patchwork metric**. We will follow [43] and write $|a|^2 := \sum_{i=1}^{16} a_i^2$. The limit $|a| \to 0$ is called the **orbifold limit**, and $|a|^2$ being small is what we mean by being close to the orbifold limit.

Remark 2.1 The cohomology-class of the patchwork metric does not depend on the specific choice of χ. If ξ is another cut-off function, then the difference of the resulting Kähler forms will locally look like $i\partial\overline{\partial}((\chi-\xi)(f_{\text{Euc}} - f_{a_i}))$, which is the differential of a globally defined function (one simply extends by 0 to the whole manifold). Hence, the Ricci-flat metric in the next theorem does not depend on the choice of χ.

The patchwork metric is not Ricci-flat due to the neck regions, hence is not the final metric we want to put on X. The celebrated Calabi–Yau theorem of [15, 16, 72] provides the solution to this problem.

Theorem 2.2 [72, Theorem 2] *Assume (X, g) is a Kähler manifold of complex dimension n, and assume the first Chern class vanishes, $c_1(X) = 0$. Let ω be the Kähler form associated with g, and let $\psi : X \to \mathbb{R}$ be a function such that $Ric_g = \partial\overline{\partial}\psi$. Define the constant $A > 0$ by*

$$A := \frac{\int_X \omega^n}{\int_X e^{\psi}\omega^n}. \tag{2.5}$$

Then, there is a unique function $\phi\colon X \to \mathbb{R}$ subject to the normalization $\int_X \phi\, \omega^n = 0$ such that $\tilde{\omega} := \omega + i\partial\overline{\partial}\phi$ is a Kähler form satisfying the ***Monge–Ampère*** *equation*

$$\tilde{\omega}^n = A e^{\psi}\omega^n. \tag{2.6}$$

The Kähler metric $\tilde{g}$ associated with $\tilde{\omega}$ is Ricci-flat.

When we additionally assume $H^1(X;\mathbb{R}) = 0$, then X has a finite fundamental group. This follows from Cheeger–Gromoll splitting theorem, [18] and the Calabi–Yau [72, Theorem 2]. Let $\tilde{X}$ denote the universal cover of X. Then $H^1(\tilde{X};\mathbb{Z}) = 0$, and $c_1(X) = 0 \implies c_1(\tilde{X}) = 0$. So $\tilde{X}$ admits a nowhere zero holomorphic n-form η. We normalize this so that

$$\exp(\psi)\frac{\omega^n}{n!} = i^{n^2}\eta \wedge \overline{\eta}.$$

In terms of this n-form, we may write

$$A = \frac{\int_{\tilde{X}} \omega^n}{\int_{\tilde{X}} e^{\psi}\omega^n} = \frac{\int_{\tilde{X}} \omega^n/n!}{\int_{\tilde{X}} i^{n^2}\eta \wedge \overline{\eta}}. \tag{2.7}$$

The Calabi–Yau theorem applies to any Kummer K3 surface. K3 surface are already simply connected, so we do not need to pass to a universal cover. The only work one has to do is to decide on what η is and then use this to compute the constant A in (2.7). We will formulate this as a little lemma and supply a proof since [43] does not compute A. Let η be the nowhere vanishing holomorphic 2-form on X induced from

$$\eta = dz_1 \wedge dz_2$$

on $\mathbb{C}^2$. For (measurable) subsets $V \subset X$, we write

$$\mathrm{Vol}_{\mathrm{Euc}}(V) = \frac{1}{4}\int_V \eta \wedge \overline{\eta}$$

and

$$\mathrm{Vol}_g(V) = \frac{1}{4}\int_V d\mathrm{Vol}_g = \frac{1}{4}\int_V \det(g)\eta \wedge \overline{\eta}.$$

Remark 2.3 Note that the volume forms here differ from the real versions by a factor of 4; hence, we divide the integrals by 4 to compensate. To see this, note that if $\eta = (dx_1 + i dy_1) \wedge (dx_2 + i dy_2)$, then $\eta \wedge \overline{\eta} = 4 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. One could alternatively define the Kähler form as $\omega = \frac{i}{2} g_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu$ and the holomorphic volume form to be $\eta = \frac{1}{2} dz \wedge dw$. This is a conventional choice, but we will drop these extra factors of $\frac{1}{2}$ to simplify several formulas.

Definition 2.4 Let X be a Kummer K3 surface with nowhere vanishing holomorphic volume form η. A pair z, w of locally defined coordinates on X such that $\eta = dz \wedge dw$ are called **holomorphic Darboux coordinates**.

Remark 2.5 If $\omega = i g_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu$ is the Kähler form in holomorphic Darboux coordinates, then

$$d\mathrm{Vol}_g = \frac{\omega^2}{2} = \det(g)\eta \wedge \overline{\eta}.$$

Lemma 2.6 *Let (X, g) be a Kummer K3 surface with patchwork metric g as described above. Then, the constant A given by (2.7) takes the value*

$$A = 1 - |a|^2 \frac{\pi^2}{2 Vol_{\mathrm{Euc}}(T)} \tag{2.8}$$

Proof By (2.7) we have

$$A = \frac{\int_X \omega^2}{2\int_X \eta \wedge \overline{\eta}}.$$

The denominator is simply

$$2\int_X \eta \wedge \overline{\eta} = \int_T \eta \wedge \overline{\eta} = 4\mathrm{Vol}_{\mathrm{Euc}}(T),$$

where we have used $\int_X = \int_Y = \frac{1}{2}\int_T$. Let $N = \cup_{i=1}^{16} N_i$ denote the union of all the neck regions of X. We may decompose the integral in the numerator into an integral over $X \backslash N$ and N, and use that $\omega^2 = 2\eta \wedge \overline{\eta}$ on $X \backslash N$ since this is satisfied by both the Euclidean metric and the Eguchi–Hanson metric. Thus,

$$\int_X \omega^2 = \int_{X \backslash N} \omega^2 + \int_N \omega^2 = 2\int_{X\backslash N} \eta \wedge \overline{\eta} + \int_N \omega^2 = 2\int_X \eta \wedge \overline{\eta} + \int_{X \backslash N} (\omega^2 - 2\eta \wedge \overline{\eta}). \tag{2.9}$$

It thus only remains to compute

$$\int_N \omega^2 = 8\mathrm{Vol}_g(N) = 8\sum_{i=1}^{16} \mathrm{Vol}_g(N_i).$$

Working on a single neck N_i, the Kähler potential of g is given by (2.2). For any spherically symmetric Kähler potential in 2 complex dimensions $F(z) = \Psi(u(z))$, we have $\det(\partial\overline{\partial}F) = (\Psi'(u)) \cdot (u\Psi'(u))'$, as one can check directly, where $u = |z|^2_{\mathbb{C}^2}$ as before. Using spherical coordinates with $r^2 = u$ (and hence also $r^3 dr = \frac{udu}{2}$), and recalling that $\Phi_a(z) = \varphi_a(u(z))$, we therefore have

$$\begin{aligned}\mathrm{Vol}_g(N_i) &= \int_{u=1}^{u=1+\delta} \int_{\mathbb{RP}^3} \det(\partial\overline{\partial}\Phi_{a_i})\, d\mathrm{Vol}_{\mathbb{RP}^3} \frac{udu}{2} \\ &= \frac{\mathrm{Vol}(\mathbb{RP}^3)}{2} \int_{u=1}^{u=1+\delta} (u\varphi'_{a_i}(u))(u\varphi'_{a_i}(u))'\, du = \frac{\mathrm{Vol}(\mathbb{RP}^3)}{4} \left(u\varphi'_{a_i}(u)\right)^2 \Big|_{u=1}^{u=1+\delta}.\end{aligned}$$

By (2.2) and the fact that the cut-off function χ satisfies $\chi(u = 1+\delta) = 0 = \chi'(u=1) = \chi'(u = 1+\delta)$ and $\chi(u=1) = 1$, we find $\varphi'_{a_i}(u=1) = \sqrt{1+a_i^2}$ and $\varphi_{a_i}(u = 1+\delta) = 1$, hence

$$\mathrm{Vol}_g(N_i) = \frac{\mathrm{Vol}(\mathbb{RP}^3)}{4} \left((1+\delta)^2 - (1+a_i^2)\right) = \mathrm{Vol}_{\mathrm{Euc}}(N_i) - \frac{\pi^2 a_i^2}{4}.$$

This shows

$$\int_{N_i} \omega^2 = 8\mathrm{Vol}_g(N_i) = 8\mathrm{Vol}_{\mathrm{Euc}}(N_i) - 2\pi^2 a_i^2 = 2\int_{N_i} \eta \wedge \overline{\eta} - 2\pi^2 a_i^2.$$

Summing over $1 \leq i \leq 16$ and inserting back into (2.9) gives (2.8). □

Remark 2.7 The fact that the expression (2.8) becomes negative for large enough values of $|a|^2$ shows why one had to restrict to small values of $|a|^2$ when gluing the metrics. The problem is traceable to the fact that (2.2) ceases to be plurisubharmonic for large values of $|a|^2$.

We also remark that (2.8) is independent of the choice of cut-off since $\int_X \omega^2$ only depends on the Kähler class and η is metric-independent. As such, the positivity of (2.8) puts a strict upper bound on $|a|$.

2.1 Estimates

The metrics g and $\tilde{g}$ on a Kummer K3 surface are related by the (nonlinear) elliptic PDE (2.6), so one could hope to get estimates on $g - \tilde{g}$ using Moser iteration and a maximum principle due to [72]. This works as long as the curvature is sufficiently concentrated and as long as no component E_i is shrunk a lot faster than the rest. Concretely, we make the following assumption.

Assumption 2.8 Let

$$r_a := \frac{\max_i a_i}{\min_i a_i}$$

be the **ratio** between the largest and smallest component of the exceptional divisor E. We assume there is a constant $C \geq 1$ independent of a such that

$$r_a \leq C$$

for all values $a = (a_1, \ldots, a_{16})$ under consideration.

The estimates we need were obtained by Kobayashi [43], and the results are as follows.

Theorem 2.9 [43, Equations 46–48] *Assume X is a Kummer K3 surface with Kähler form ω associated with the patchwork metric g and $\tilde{\omega} = \omega + i\partial\bar{\partial}\phi$ satisfies (2.6) with $\int_X \phi\, \omega^2 = 0$. Let $U \subset X$ be any open set such that $E \subset U$. Then, there are constants $C_k > 0$, $k \geq 0$, depending on U but not on the parameters a_i such that for all small enough values of $|a|$ we have*

$$\|\phi\|_{C^k(X\setminus U, g)} \leq C_k |a|^2. \tag{2.10}$$

Moreover, there is a constant $C > 0$, independent of the parameters a_i such that for all small enough values of $|a|$ we have

$$\|\phi\|_{C^k(X,g)} \leq C|a|^{2-\frac{k}{2}} \tag{2.11}$$

Kobayashi only states (2.11) for even k's. The odd cases are handled by the Gagliardo–Nierenberg interpolation inequality (see for instance [3, Theorem 3.70]).

These estimates will be crucial in Sect. 3 to prove Theorems 1.4 and 1.5.

Remark 2.10 In words, (2.10) says that when the exceptional divisor E is small, g is close to $\tilde{g}$ as long as one stays away from E. The second bound (2.11) gives weaker estimates valid also near E. The C^k-estimates of ϕ translate into C^{k-2}-estimates for the metric $\tilde{g}$. The C^4-estimates of ϕ are sometimes called estimates at the level of curvature.

We will reprove Theorem 2.9 in Sect. 6. Our arguments are roughly the same as Kobayashi's, with some added detail and some corrections.

2.2 Isometries

Even though the Ricci-flat metric $\tilde{g}$ is not known explicitly, it will inherit the isometries of g.

Proposition 2.11 *Assume the set-up of Theorem 2.2. Let $\tilde{g}$ be the Ricci-flat Kähler metric in the Kähler class of g. Assume $F: X \to X$ is a (anti-) holomorphic isometry of g. Then, F is also an isometry of $\tilde{g}$.*

Proof The isometry F preserves the Monge–Ampère Eq. (2.6) and the Kähler form ω up to sign. By the uniqueness of the solution, it also has to preserve $\tilde{\omega}$ up to sign. The details are as follows.

Let ω and $\tilde{\omega}$ be the Kähler forms of g and $\tilde{g}$, respectively. Let $\epsilon = +1$ if F is holomorphic and $\epsilon = -1$ if F is anti-holomorphic. Then, $F^*\omega = \epsilon\omega$. Since F is an isometry, we also have $\psi \circ F = \psi$ and

$$A = \frac{\int_X \omega^n}{\int_X e^{\psi}\omega^n} = \frac{\int_X F^*(\omega^n)}{\int_X F^*\left(e^{\psi}\omega^n\right)}.$$

Applying F^* to the Monge–Ampère Eq. (2.6) gives us

$$(F^*\tilde{\omega})^n = F^*(\tilde{\omega}^n) = F^*\left(Ae^{\psi}\omega^n\right) = Ae^{\psi\circ F}(F^*\omega)^n = \epsilon^n Ae^{\psi}\omega^n = \epsilon^n\tilde{\omega}^n.$$

So $\epsilon F^*\tilde{\omega}$ solves the Monge–Ampère equation. Since F is (anti-) holomorphic, we find

$$F^*(\tilde{\omega}) = \epsilon\left(\omega + i\partial\bar{\partial}\phi \circ F\right).$$

So both ϕ and $\phi \circ F$ solve the equation subject to the normalization

$$\int_X \phi\, \omega^n = \int_X (\phi \circ F) F^* \omega^n = 0.$$

By the uniqueness, we that conclude $\phi = \phi \circ F$ and F is an isometry of $\tilde{g}$. $\square$

Remark 2.12 Proposition 2.11 is known to the experts. In [1, Proposition 2.2], a very similar statement is proven for projective Calabi–Yau manifolds, $\iota \colon X \to \mathbb{CP}^N$ when one takes $\omega = \iota^*(\omega_{FS})$, i.e. the metric is induced by the Fubini–Study metric. Their arguments are essentially the above ones.

Proposition 2.13 *Assume X is a Kummer K3 surface with patchwork metric as described above. Assume all the components of the exceptional divisor have the same size, meaning $a_i = a_j$ for $1 \leq i, j \leq 16$. Let $\tilde{g}$ be the unique Ricci-flat metric in the Kähler class of g. Assume $F^{\mathbb{C}} \colon \mathbb{C}^2 \to \mathbb{C}^2$ is an affine map $F^{\mathbb{C}}(z) = Bz + b$ with $B \in U(2)$ such that $B\Gamma = \Gamma$ and $b \in \frac{1}{2}\Gamma$. Then $F^{\mathbb{C}}$ induces an isometry $F \colon X \to X$ of both g and $\tilde{g}$.*

Similarly, if $\tau^{\mathbb{C}} \colon \mathbb{C}^2 \to \mathbb{C}^2$ denotes the complex conjugation map and $\tau^{\mathbb{C}}(\Gamma) = \Gamma$, then $\tau^{\mathbb{C}}$ induces an isometry $\tau \colon X \to X$ of both g and $\tilde{g}$.

Maps of these forms are the only (anti-) holomorphic isometries of (X, g).

Proof The argument that one gets an induced map $F \colon X \to X$ goes as follows. Since $F^{\mathbb{C}}(z + \Gamma) = F^{\mathbb{C}}(z) + \Gamma$ holds for all $z \in \mathbb{C}^2$ we get an induced map $F^T \colon T \to T$. The requirement $b \in \frac{1}{2}\Gamma$ implies $F^T(-z) = -F^T(z)$, so we get a well-defined map $F^Y \colon Y \to Y$. Since this came from an affine map, it extends to the blow-up, and the argument, when written out, looks like this. To extend to the blow-up, it suffices to see what happens locally. Since blowing up commutes with taking the quotient, i.e. $Bl_0(\mathbb{C}^2/\mu_2) = Bl_0(\mathbb{C}^2)/\mu_2$, it suffices to see what happens when blowing up points in $\mathbb{C}^2$. In this case, we extend a given affine map $F^{\mathbb{C}} \colon \mathbb{C}^2 \to \mathbb{C}^2$ to a map $F^{Bl} \colon Bl_p(\mathbb{C}^2) \to Bl_{F(p)}(\mathbb{C}^2)$ by sending a pair $(q, \ell) \in \mathbb{C}^2 \times \mathbb{CP}^1$ to $(F(q), F(\ell))$, which makes sense since affine maps map lines to lines. This gives us our required map $F \colon X \to X$. The same line of arguments works for anti-holomorphic maps.

To see that the induced map is an isometry of g, we split X into two different kinds of regions; the flat region and the neck+Eguchi–Hanson regions (the sets called U_i in the construction above). In the flat region, there is nothing to show. In one of the patches U_i, we have a metric whose Kähler potential is given by (2.2), and this potential is preserved by maps of the above form. Completely analogous arguments work for the map τ.

We may therefore apply Proposition 2.11.

To see that the above maps are the only (anti-) holomorphic isometries of (X, g), one can argue using the curvature as follows. The neck regions are not Ricci-flat, whereas the complement is. So an isometry has to map the neck regions to neck regions. The Euclidean region is flat, and the Eguchi–Hanson patches have nowhere vanishing curvature. So these regions cannot be interchanged either. Letting $U = \cup_i U_i$ denote all the non-flat regions, we therefore have a map $f \colon X \backslash U \to X \backslash U$, and $F := \pi \circ f \circ \pi^{-1} \colon Y \backslash \pi(U) \to Y \backslash \pi(U)$. We will show that this lifts to an isometry of the flat torus (with some open set removed), hence has to have the above form. The orbifold $Y = T/\mu_2$ is simply connected, and by the Seifert–Van Kampen theorem, [35, Theorem 1.20], $Y \backslash \pi(U)$ is as well. Let $P \colon T \to Y$ denote the quotient map and let $V := P^{-1}(\pi(U)) \subset T$. Away from the singular points of Y, P is a covering map. By [35, Proposition 1.33], F lifts to an isometry $\tilde{F} \colon T \backslash V \to T \backslash V$. Hence, $\tilde{F}$ takes the above affine form. So f agrees with an isometry induced from an affine mapping of $\mathbb{C}^2$ on $X \backslash U$; hence, the two isometries agree everywhere. $\square$

Remark 2.14 We do not know if the isometry group of $(X, \tilde{g})$ is bigger than that of (X, g). For toy models, this can easily be the case. Consider $(\mathbb{C}^n/\Gamma, g_0)$ with any Kähler metric g_0 without non-trivial isometries. Use Theorem 2.2 to find the Ricci-flat metric g in the Kähler class of g_0. Then, g is the flat metric and thus has an infinite isometry group, even though the isometry group of g_0 is trivial.

2.3 Homothety of a Kummer K3 surface

Both the Euclidean metric and the Eguchi–Hanson Kähler potentials are homogeneous in simultaneous scaling of the coordinate z and the parameter a. This property will then be inherited by the solution ϕ. Here are the details.

Let $\alpha > 0$ and consider the homothety $S_\alpha^{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$S_\alpha^{\mathbb{C}}(z) = \alpha z.$$

Let $\Gamma \subset \mathbb{C}^2$ be any non-degenerate lattice and let $\Gamma_\alpha = S_\alpha^{\mathbb{C}}(\Gamma)$ be the scaled lattice. Denote by X and X_α the Kummer K3 surface associated with the lattice Γ and Γ_α, respectively. Let g_a denote the patchwork metric on X with parameter $a = (a_1, \ldots, a_{16})$, and denote by $\tilde{g}_a$ the Ricci-flat metric in the Kähler class of g_a. These are denoted by g and $\tilde{g}$, respectively, for most of the paper. On X_α, we put the same kind of patchwork metric with locally defined Kähler potential (2.2), but we modify the cut-off function to be

$$\chi_\alpha(z) = \chi\left(\frac{z}{\alpha}\right) = \chi(S_\alpha^{-1}(z)).$$

We denote this Kähler potential by Φ_a^α.

Lemma 2.15 *The map $S_\alpha^{\mathbb{C}}$ induces an isometry*

$$S_\alpha : (X, \alpha^2 g_a) \to (X_\alpha, g_{\alpha^2 a})$$

and

$$S_\alpha : (X, \alpha^2 \tilde{g}_a) \to (X_\alpha, \tilde{g}_{\alpha^2 a}).$$

Proof That we get induced maps can be argued as in the proof Proposition 2.13. The important properties are $S_\alpha^{\mathbb{C}}(z) = -S_\alpha^{\mathbb{C}}(-z)$ and that $S_\alpha^{\mathbb{C}}$ is affine.

One easily checks that the Eguchi–Hanson potential (2.1) satisfies

$$f_{\alpha^2 a}(\alpha z) = \alpha^2 f_a(z).$$

The same goes for the Euclidean potential. Hence, in an Eguchi–Hanson and neck region,

$$\Phi_{\alpha^2 a}^\alpha(\alpha z) = \alpha^2 \Phi_a(z).$$

We are abusing notation a bit, seeing how the left-hand side is locally defined on X_α and the right-hand side is locally defined on X, but one could think of the equality as happening on a punctured ball in $\mathbb{C}^2$. Away from the Eguchi–Hanson and neck regions, the isometry is clear. This shows the claimed isometry with respect to the patchwork metric.

For the isometry of the Ricci-flat metric, we argue as in the proof of Proposition 2.11. It is also clear that Ae^ψ is the same for (X, g_a) and $(X_\alpha, g_{\alpha^2 a})$. Hence,[3]

$$(\alpha^2(\omega_a + i\partial\overline{\partial}\phi_a))^2(z) = Ae^\psi(\alpha^2\omega_a)^2(z) = Ae^\psi(\omega_{\alpha^2 a})^2(S_\alpha(z)) = (\omega_{\alpha^2 a} + i\partial\overline{\partial}\phi_{\alpha^2 a})^2(S_\alpha(z)),$$

[3] Here, we are indulging in some abuse of notation. $\phi_a : X \to \mathbb{R}$, whereas $\phi_{\alpha^2 a} : X_\alpha \to \mathbb{R}$. We also write simply ψ even though it depends on a and appears once as a function $\psi : X \to \mathbb{R}$ and once as a function $\psi : X_\alpha \to \mathbb{R}$.

which shows that both $\alpha^2\phi_a(z)$ and $\phi_{\alpha^2 a}(S_\alpha(z))$ solve the Monge–Ampère equation. By uniqueness, $\alpha^2\phi_a(z) = \phi_{\alpha^2 a}(S_\alpha(z))$ and S_α is an isometry between the Ricci-flat metrics as well. □

The homothety S_α clearly maps $\Gamma/2\Gamma$ to $\Gamma_\alpha/2\Gamma_\alpha$, hence maps the exceptional divisor of X to the exceptional divisor of X_α. In fact, $(S_\alpha)_{|E}$ is α-independent. So Lemma 2.15 says in particular

$$(\alpha^2\phi_a)_{|E} = (\phi_{\alpha^2 a})_{|E}.$$

Here, we are abusing notation again, since the right-hand side E is a subset of X_α.

3 Closed geodesics: No–Go theorems

There are two main results about geodesics on a Kummer K3 surface in this section. Theorem 3.1 constrains stable, closed geodesics to stay away from the exceptional divisor E, whereas Theorem 3.4 says that closed, stable geodesics cannot stay inside an Eguchi–Hanson patch.

Theorem 3.1 *Let (X, g) be a Kummer K3 as constructed above. Let $\tilde{g}$ be the Ricci-flat metric in the Kähler class of g. Then, for each value of $|a|$ small enough, there is an a-dependent open set $V_a \subset X$ with $E \subset V_a$ such that no stable, closed geodesic (with respect to either g or $\tilde{g}$) in X ever enters V_a.*

Proof The idea is to use the estimates of [43] to show that the curvature of $\tilde{g}$ doesn't vanish anywhere near E. Then, we appeal to Theorem 1.7 to get our conclusion for geodesics with respect to $\tilde{g}$. Here are the details.

Let $p \in E_i \subset E$ and pick holomorphic normal coordinates with respect to g at p, meaning $g_{\mu\overline{\nu}}(p) = \delta_{\mu\nu}$, $g_{\mu\overline{\nu},\alpha}(p) = 0$, and $g_{\mu\overline{\nu},\alpha\overline{\beta}}(p) = -R_{\mu\overline{\nu}\alpha\overline{\beta}}(p)$ hold, where R is the Riemann curvature tensor of g. Write $\phi_{\mu\overline{\nu}}(p) := \frac{\partial^2\phi}{\partial z^\mu \partial \overline{z}^\nu}(p)$ and so on for more indices. Introduce the 2×2-matrix h via $h := \tilde{g}^{-1}(p) - g^{-1}(p) = \tilde{g}^{-1}(p) - \mathbb{1}$. By (2.11) for $k = 2$ we know that $\phi_{\mu\overline{\nu}}(p) \in \mathcal{O}(|a|)$, hence also $h \in \mathcal{O}(|a|)$, as is seen by writing $\tilde{g}^{-1}(p) = (\mathbb{1} + \partial\overline{\partial}\phi(p))^{-1} = \mathbb{1} + \sum_{k=1}^{\infty}(-1)^k(\partial\overline{\partial}\phi(p))^k$. We may use a standard formula for the Riemann tensor associated with a Kähler metric (see [72, Eq. 1.14] for instance) namely

$$\tilde{R}_{\mu\overline{\nu}\alpha\overline{\beta}} = -\frac{\partial^2 \tilde{g}_{\mu\overline{\nu}}}{\partial z^\alpha \partial \overline{z}^\beta} + \tilde{g}^{\overline{\lambda}\sigma} \frac{\partial \tilde{g}_{\mu\overline{\lambda}}}{\partial z^\alpha} \frac{\partial \tilde{g}_{\sigma\overline{\nu}}}{\partial \overline{z}^\beta}$$

to write the Riemann tensor $\tilde{R}$ of $\tilde{g}$ in the above coordinates as

$$\tilde{R}_{\mu\overline{\nu}\alpha\overline{\beta}}(p) = R_{\mu\overline{\nu}\alpha\overline{\beta}}(p) - \phi_{\mu\overline{\nu}\alpha\overline{\beta}}(p) + (\delta^{\lambda\sigma} + h^{\overline{\lambda}\sigma})\phi_{\mu\alpha\overline{\lambda}}(p)\phi_{\overline{\nu}\overline{\beta}\sigma}(p).$$

By (2.11) for $k = 3$ and $k = 4$ and the above bound on h we may estimate this as

$$R_{\mu\overline{\nu}\alpha\overline{\beta}}(p) - C \leq \tilde{R}_{\mu\overline{\nu}\alpha\overline{\beta}}(p) \leq R_{\mu\overline{\nu}\alpha\overline{\beta}}(p) + C \tag{3.1}$$

for some $|a|$-independent constant $C > 0$. Let $V \in T_pE_i \subset T_pX$ be such that $|V|_{\tilde{g}} = 1$. Then (2.11) for $k = 2$ gives $1 - \tilde{C}|a| \leq |V|_g^2 \leq 1 + \tilde{C}|a|$ for some constant $\tilde{C} > 0$. Inserting this into (3.1) we find

$$\mathrm{Sect}_g(p)(1 - \tilde{C}|a|)^2 - C \leq \mathrm{Sect}_{\tilde{g}}(p) \leq (1 + \tilde{C}|a|)^2 \mathrm{Sect}_g(p) + C, \tag{3.2}$$

where $\mathrm{Sect}_g(p)$ and $\mathrm{Sect}_{\tilde{g}}(p)$ denote the holomorphic sectional curvatures of g and $\tilde{g}$, respectively, evaluated on T_pE_i. Since we are on E_i we may use that $g_{|E_i} = a_i g_{FS}$, FS being short for Fubini–Study, to write $\mathrm{Sect}_g(p) = \frac{2}{a_i}$. Using the ratio $r_a = \frac{\max_{1\le i\le 16} a_i}{\min_{1\le i\le 16} a_i}$, we bound this as $\frac{2}{|a|} \le \mathrm{Sect}_g(p) \le \frac{4r_a}{|a|}$; hence, (3.2) finally becomes

$$\frac{2}{|a|} - C_1 + C_2|a| \le \mathrm{Sect}_{\tilde{g}}(p) \le 4r_a\left(\frac{1}{|a|} + C_3 + C_4|a|\right)$$

for $|a|$-independent constants $C_i > 0$. This shows that $\mathrm{Sect}_{\tilde{g}}(p) > 0$ for all $|a|$ small enough.

To see that there are no closed, stable geodesics with respect to g one can argue in a couple of ways. The fastest is probably to appeal to Theorem 1.7, which applies since the Eguchi–Hanson patch has a hyperkähler metric whose Riemann curvature tensor doesn't vanish in any point (see [25, Equation 2.28] or [51, Equation 4.8]). Another argument is to first see directly (see [70, Theorem 5.3], [51, Chapter 4.5]), or [52, Theorem 8] that the only closed, non-constant geodesics in Eguchi–Hanson space are the ones contained in the $\mathbb{CP}^1$. Hence, they are closed geodesics in $(\mathbb{CP}^1, g_{FS})$, all of which are unstable. □

Remark 3.2 That the components E_i of the exceptional divisor have induced metrics with positive curvature (even for the Ricci-flat metric) can be seen in numerical solutions like [38, Figure 3].

The next result says that any geodesic with respect to $\tilde{g}$ is close to being a geodesic with respect to g.

Theorem 3.3 *Assume the same set-up as in Theorem 3.1. Then, for all values of $|a|$ small enough there is a constant $C > 0$ independent of $|a|$ such that if $\gamma\colon (-\epsilon, \epsilon) \to X$ is a geodesic with respect to $\tilde{g}$ we have*

$$|D_t^g \dot{\gamma}(t)|_g \le C|a|^{\frac{1}{2}}|\dot{\gamma}|_g^2, \tag{3.3}$$

where $D_t^g = \nabla_{\dot{\gamma}}$ is the covariant derivative associated with g.

Proof Let Γ and $\tilde{\Gamma}$ denote the Christoffel symbols in some coordinates of g and $\tilde{g}$, respectively. Define the tensor Ψ by

$$\Psi^\lambda_{\mu\alpha} := \tilde{\Gamma}^\lambda_{\mu\alpha} - \Gamma^\lambda_{\mu\alpha}. \tag{3.4}$$

Then

$$\Psi^\lambda_{\mu\alpha} = \tilde{g}^{\overline{\sigma}\lambda}\left(\tilde{g}_{\alpha,\overline{\sigma},\mu} - \Gamma^\rho_{\alpha\mu}\tilde{g}_{\rho\overline{\sigma}}\right) = \tilde{g}^{\overline{\sigma}\lambda}\nabla_\mu \tilde{g}_{\alpha\overline{\sigma}}, \tag{3.5}$$

where ∇ denotes the covariant derivative associated with g. Let $\gamma\colon (-\epsilon, \epsilon) \to X$ be a geodesic with respect to $\tilde{g}$. Then,

$$|D_t^g\dot{\gamma}|_g = \left|D_t^g\dot{\gamma} - D_t^{\tilde{g}}\dot{\gamma}\right|_g = |\Psi(\dot{\gamma}, \dot{\gamma})|_g\,. \tag{3.6}$$

This is the equation which will give us (3.3) after we estimate $|\Psi(\dot{\gamma}, \dot{\gamma})|$. To this effect, we claim there exists $C > 0$ independent of $|a|$ such that $\|\Psi\|_{C^0(X,g)} \le C|a|^{\frac{1}{2}}$, and this will follow by Kobayashi's estimates. Let $p \in X$ and, like in the proof of Theorem 3.1, choose holomorphic normal coordinates at p with respect to g, meaning $g_{\mu\overline{\nu}}(p) = \delta_{\mu\nu}$ etc. In these coordinates, we may write

$$\Psi(p)^\lambda_{\mu\alpha} = (\delta^{\sigma\lambda} + h^{\overline{\sigma}\lambda})\phi(p)_{\mu\alpha\overline{\sigma}},$$

where h and $\phi(p)_{\mu\alpha\overline{\sigma}}$ are as in the proof of Theorem 3.1. In that proof, we also saw that $h \in \mathcal{O}(|a|)$. From (2.11), with $k = 3$ we have $|\phi(p)_{\mu\alpha\overline{\sigma}}| \leq \|\phi\|_{C^3(X,g)} \leq C_1|a|^{\frac{1}{2}}$. Altogether, we thus find

$$|\Psi(p)^{\lambda}_{\mu\alpha}| \leq C_1|a|^{\frac{1}{2}}(1 + C_2|a|) \leq C|a|^{\frac{1}{2}}.$$

At the point p, still in holomorphic normal coordinates, we thus compute

$$|\Psi|^2_g(p) = \sum_{\mu,\alpha,\lambda=1}^{2} \left|\Psi(p)^{\lambda}_{\mu\alpha}\right|^2 \leq \tilde{C}|a|.$$

The left-hand side is independent of coordinate system, and since p was arbitrary this proves the claim. □

Theorem 3.4 *Assume the same set-up as in Theorem 3.1. Then, for all values of $|a|$ small enough, no stable, closed geodesic with respect to $\tilde{g}$ can stay completely within an Eguchi–Hanson patch U_i.*

Proof Assume there is a closed, stable geodesic γ with respect to $\tilde{g}$ in an Eguchi–Hanson patch U_i. By Theorem 3.1, we may assume that γ stays away from $E_i \cong \mathbb{CP}^1$. Consider the distance squared $d(t)$ between $\gamma(t)$ and $\mathbb{CP}^1$. This distance is realized by geodesics as follows. Let $\rho_s(t)$ denote the family of radial[4] geodesics with respect to the metric g connecting $\gamma(t)$ and $\mathbb{CP}^1$, meaning $\rho_0(t) \in \mathbb{CP}^1$ for all t, $\rho_1(t) = \gamma(t)$, and $D^g_s \partial_s \rho_s(t) = 0$ for all $t \in \mathbb{S}^1$ and $s \in (0, 1)$. Then

$$d(t) = \int_0^1 |\partial_s \rho_s(t)|^2_g \, ds.$$

The function d needs to have a maximum, meaning there is some T such that $\dot{d}(T) = 0$ and $\ddot{d}(T) \leq 0$. We shall show that (3.3) implies $\ddot{d}(T) > 0$, hence forcing a contradiction.

We start by computing $\dot{d}(t)$. Note that $D^g_t \partial_s = D^g_s \partial_t$ (see for instance [19, Lemma 3.4]). Since $s \mapsto \rho_s(t)$ is a geodesic, $D^g_s \partial_s \rho_s(t) = 0$. So[5]

$$\begin{aligned}
\dot{d}(t) &= 2\mathrm{Re} \int_0^1 \left\langle D^g_t \partial_s \rho_s(t), \partial_s \rho_s(t)\right\rangle_g \, ds \\
&= 2\mathrm{Re} \int_0^1 \left\langle D^g_s \partial_t \rho_s(t), \partial_s \rho_s(t)\right\rangle_g \, ds \\
&= 2\mathrm{Re} \int_0^1 \partial_s \left(\langle \partial_t \rho_s(t), \partial_s \rho_s(t)\rangle_g\right) \, ds \\
&= 2\mathrm{Re} \left(\langle \partial_t \rho_s(t), \partial_s \rho_s(t)\rangle_g\right) \Big|_{s=0}^{s=1}.
\end{aligned}$$

The lower limit vanishes for all t since $\partial_s \rho_s(t)_{|s=0}$ is normal to $E_i \cong \mathbb{CP}^1$, whereas $\partial_t \rho_s(t)_{|s=0}$ is tangential. Differentiating this again we find

$$\ddot{d}(t) = 2\mathrm{Re} \left(\left\langle D^g_t \partial_t \rho_s(t), \partial_s \rho_s(t)\right\rangle_g + \left\langle \partial_t \rho_s(t), D^g_t \partial_s \rho_s(t)\right\rangle_g\right) \Big|_{s=1}.$$

[4] For $s > 0$, $\rho_s(t) \in \mathcal{O}_{\mathbb{CP}^1}(-2)\backslash\mathbb{CP}^1 \cong (\mathbb{C}^2\backslash\{0\})/\mu_2$. Writing $[z(t)]$ for the image of $\gamma(t)$ under this identification, $\rho_s(t)$ takes the form $\rho_s(t) = \theta(s, t)[z(t)]$ for some function θ, hence the name radial geodesic.

[5] In this section, we are thinking of the metric as a *hermitian* metric, whose real part is the corresponding Riemannian metric. This is to make better use of the complex coordinates on $\mathbb{C}^2$.

To say something more about these expressions we need to recall some basic facts about the Eguchi–Hanson metric g. The region we are interested in is $(B_1(0)\backslash\{0\})/\mu_2$ with Kähler potential given by (2.1). We use $z=(z_1,z_2)$ as coordinates and write $u=|z|^2_{\mathbb{C}^2}$ as before. The Kähler metric associated with (2.1) reads

$$\langle U,V\rangle_g=\sqrt{1+\frac{a_i^2}{u^2}}\left(\langle U,V\rangle_{\mathbb{C}^2}-\frac{a_i^2}{a_i^2+u^2}\frac{\langle U,z\rangle_{\mathbb{C}^2}\langle z,V\rangle_{\mathbb{C}^2}}{u}\right), \tag{3.7}$$

where $\langle U,V\rangle_{\mathbb{C}^2}:=\overline{U_1}V_1+\overline{U_2}V_2$ is the Euclidean inner product. From this, it follows that

$$\langle z,V\rangle_g=\frac{u}{\sqrt{a_i^2+u^2}}\langle z,V\rangle_{\mathbb{C}^2} \tag{3.8}$$

holds for any $V\in\mathbb{C}^2$.

Using the formula (see for instance [5, Equation 4.39]) $\Gamma^\lambda_{\mu\alpha}=\frac{\partial g_{\mu\overline{\nu}}}{\partial z^\alpha}g^{\overline{\nu}\lambda}$ we also find

$$\Gamma^\lambda_{\mu\alpha}=-\frac{a_i^2}{u(a_i^2+u^2)}\left(\overline{z}_\mu\delta^\lambda{}_\alpha+\overline{z}_\alpha\delta^\lambda{}_\mu-3\frac{\overline{z}_\alpha\overline{z}_\mu}{u}z^\lambda\right). \tag{3.9}$$

We can locally write $\rho_s(t)=\theta(s,t)z(t)$ for some function θ satisfying (amongst others[6]) $\theta(1,t)=1$ and $\partial_s\theta>0$ for all t. In particular $\partial_t\rho_{|s=1}=\dot{z}(t)$. Inserting this into our above expressions for $\dot{d}(t)$, we find

$$\begin{aligned}\dot{d}(t)&=2\mathrm{Re}\left((\partial_t\theta)(\partial_s\theta)|z(t)|_g^2+\theta(\partial_s\theta)\langle z(t),\dot{z}(t)\rangle_g\right)\Big|_{s=1}\\&=2(\partial_s\theta)(1,t)\mathrm{Re}\langle z(t),\dot{z}(t)\rangle_g.\end{aligned}$$

We deduce that $\dot{d}(T)=0\iff\mathrm{Re}\langle z,\dot{z}\rangle_g(T)=0$, which by (3.8) happens if and only if $\mathrm{Re}\langle z,\dot{z}\rangle_{\mathbb{C}^2}(T)=0$. We similarly find

$$\left\langle D^g_t\partial_t\rho_s(t),\partial_s\rho_s(t)\right\rangle_g\Big|_{s=1}=\partial_s\theta(1,t)\left\langle D^g_t\dot{z}(t),z(t)\right\rangle_g.$$

The term $\left\langle\partial_t\rho_s(t),D^g_t\partial_s\rho_s(t)\right\rangle_g\Big|_{s=1}$ needs a bit more work. We have

$$\begin{aligned}D^g_t\partial_s\rho^\lambda_s&=D^g_t(\partial_s\theta z)^\lambda=\partial_t(\partial_s\theta z)^\lambda+(\partial_s\theta)\Gamma^\lambda_{\mu\alpha}\dot{z}^\mu z^\alpha\\&=(\partial_t\partial_s\theta)z^\lambda+(\partial_s\theta)\dot{z}^\lambda+(\partial_s\theta)\Gamma^\lambda_{\mu\alpha}\dot{z}^\mu z^\alpha.\end{aligned}$$

Here, the indices are raised using the Euclidean metric; $z_\mu=z^\mu$ and so on. Using (3.9) we have

$$\Gamma^\lambda_{\mu\alpha}\dot{z}^\mu z^\alpha=-\frac{a_i^2}{a_i^2+u^2}\dot{z}^\lambda+\frac{2a_i^2}{u(a_i^2+u^2)}\langle z,\dot{z}\rangle_{\mathbb{C}^2}z^\lambda\overset{(3.8)}{=}-\frac{a_i^2}{a_i^2+u^2}\dot{z}^\lambda+\frac{2a_i^2}{\sqrt{a_i^2+u^2}}\langle z,\dot{z}\rangle_g z^\lambda.$$

[6] One can find θ more or less explicitly. Since $\rho_s(t)$ is supposed to be a radial geodesic, the geodesic equation has to be fulfilled, so using (3.9) and (3.8) we see that $\partial_s^2\theta+\frac{a_i^2}{(a_i^2+u^2\theta^4)\theta}(\partial_s\theta)^2=0$ is the ODE satisfied by θ. This has $\partial_s\theta(s,t)=\frac{\sqrt{d(t)}(a_i^2+\theta(s,t)^4u(t)^2)^{\frac{1}{4}}}{\theta(s,t)u(t)}$ as a first integral, where we can write $\sqrt{d(t)}=\int_0^1\frac{su(t)}{(a_i^2+s^4u(t)^2)^{\frac{1}{4}}}ds$. We shall not need these explicit expressions, however.

Inserting this, we find

$$D_t^g \partial_s \rho_s^\lambda = (\partial_t \partial_s \theta) z^\lambda + (\partial_s \theta) \frac{u^2}{a_i^2 + u^2} \dot{z}^\lambda + (\partial_s \theta) \frac{2a_i^2}{\sqrt{a_i^2 + u^2}} \langle z, \dot{z} \rangle_g z^\lambda.$$

Hence,

$$\begin{aligned} &\left\langle \partial_t \rho_s(t), D_t^g \partial_s \rho_s(t) \right\rangle_g \Big|_{s=1} = \left\langle \dot{z}(t), D_t^g \partial_s \rho_s(t) \right\rangle_g \\ &\quad = (\partial_t \partial_s \theta) \langle \dot{z}(t), z(t) \rangle_g + (\partial_s \theta) \left(\frac{2a_i^2}{\sqrt{a_i^2 + u(t)^2}} | \langle z(t), \dot{z}(t) \rangle_g |^2 + \frac{u(t)^2}{a_i^2 + u(t)^2} |\dot{z}(t)|_g^2 \right). \end{aligned}$$

Taking the real part and setting $t = T$ removes the term $(\partial_t \partial_s \theta) \langle \dot{z}(t), z(t) \rangle_g$ since $\operatorname{Re} \langle \dot{z}(T), z(T) \rangle_g = 0$.

We thus conclude

$$\ddot{d}(T) = 2(\partial_s \theta(1, T)) \left(\operatorname{Re} \left\langle z, D_t^g \dot{z} \right\rangle_g + \frac{u^2}{a_i^2 + u^2} |\dot{z}|_g^2 + \frac{2a_i^2}{\sqrt{a_i^2 + u^2}} | \langle z, \dot{z} \rangle_g |^2 \right).$$

We estimate this from below by dropping the nonnegative term $\frac{2a_i^2}{\sqrt{a_i^2+u^2}} | \langle z, \dot{z} \rangle_g |^2$ (recalling that $\partial_s \theta > 0$) and observing that

$$\operatorname{Re} \left\langle z, D_t^g \dot{z} \right\rangle_g \geq -|z|_g |D_t^g \dot{z}|_g \overset{(3.3)}{\geq} -|z|_g C |a|^{\frac{1}{2}} |\dot{z}|_g^2 \overset{(3.8)}{=} -C \frac{u^2}{\sqrt{a_i^2 + u^2}} |a|^{\frac{1}{2}} |\dot{z}|_g^2.$$

From this, we deduce

$$\ddot{d}(T) \geq \frac{2u^2(\partial_s \theta)}{\sqrt{a_i^2 + u^2}} |\dot{z}|_g^2 \left(-C|a|^{\frac{1}{2}} + \frac{1}{\sqrt{a_i^2 + u^2}} \right) \geq \frac{u^2(\partial_s \theta)}{\sqrt{a_i^2 + u^2}} |\dot{z}|_g^2 \left(-C|a|^{\frac{1}{2}} + \frac{1}{\sqrt{a_i^2 + 1}} \right).$$

By choosing $|a|$ small enough we have $\ddot{d}(T) > 0$ and d therefore does not have a maximum. This is the desired contradiction. □

Remark 3.5 The strategy of the above proof is similar to [70, Theorem 5.3] and [52, Theorem 8]. Enlarging slightly to include the neck region in the consideration would not change anything, since a similar argument would go through, where one compares with the Euclidean metric instead of the Eguchi–Hanson metric. See [51, Theorem 7.3] for details.

Let us stress that the above theorem forbids a stable, closed geodesic from *staying* inside of U_i. It does not rule out a geodesic entering and leaving U_i and closing up somewhere else in X. This is a possibility envisioned in [30, pp. 12–13], and we do not rule this out, but would note the restrictions imposed upon such a geodesic by Theorems 3.1 and 3.3.

4 Curvature of Hyperkähler 4-manifolds

In this section, we analyse the Riemann curvature tensor of a Hyperkähler 4-manifold. Most of the statements rely on having enough isometries. We need some preliminaries first. We will start by recalling some facts from Riemannian geometry.

Lemma 4.1 *Assume (M, g) is a Riemannian manifold with Levi–Civita connection ∇. Assume $F: M \to M$ is an isometry. Then,*

$$F_*(\nabla_U V) = \nabla_{F_* U} F_* V \tag{4.1}$$

holds for all tangent vector fields $U, V \in \Gamma(TM)$. In particular, if R denotes the Riemann curvature tensor, then

$$\langle R(V, W)U, Z\rangle_p = \langle R(F_* V, F_* W)F_* U, F_* Z\rangle_{F(p)} \tag{4.2}$$

holds for all $p \in M$ and all $U, V, W, Z \in T_p M$.

Proof Define ∇' by $\nabla'_U V := F_*^{-1}\left(\nabla_{F_* U} F_* V\right)$ and verify that ∇' is a torsion-free metric connection, hence $\nabla' = \nabla$. □

4.1 Curvature of hyperkähler 4-manifolds

We next turn to some facts about hyperkähler manifolds in real dimension four and at the same time establish some notation.

Lemma 4.2 *Let $(X, \tilde{g})$ be a Ricci-flat Kähler manifold of real dimension 4 with complex structure I and nowhere vanishing holomorphic 2-form η. Then, there exist complex structures J, K such that $IJ = K$. These complex structures are metric compatible, meaning $\nabla J = J\nabla$ and $\nabla K = K\nabla$.*

In fact, one can write down such complex structures explicitly, following [17, p. 287] or [9, p. 6]. Let z, w be holomorphic Darboux coordinates such that the metric $\tilde{g}$ takes the matrix form

$$\tilde{g} = \begin{pmatrix} \tilde{g}_{z\bar{z}} & \tilde{g}_{z\overline{w}} \\ \tilde{g}_{w\bar{z}} & \tilde{g}_{w\overline{w}} \end{pmatrix}$$

with

$$\det(\tilde{g}) = A$$

being some constant. Then, we may define a complex structure by

$$J\frac{\partial}{\partial z} = \frac{1}{\sqrt{A}}\left(-\tilde{g}_{z\overline{w}}\frac{\partial}{\partial \bar{z}} + \tilde{g}_{z\bar{z}}\frac{\partial}{\partial \overline{w}}\right) \tag{4.3}$$

and

$$J\frac{\partial}{\partial w} = \frac{1}{\sqrt{A}}\left(-\tilde{g}_{w\overline{w}}\frac{\partial}{\partial \bar{z}} + \tilde{g}_{w\bar{z}}\frac{\partial}{\partial \overline{w}}\right). \tag{4.4}$$

For any $p \in X$ and $V \in T_p M$ we introduce the notation

$$\sigma_{IJ}(V) := \langle R(V, IV)JV, V\rangle$$

and similarly for $\sigma_{IK}(V), \sigma_{JK}(V), \sigma_{II}(V)$ etc. These satisfy $\sigma_{IJ}(V) = \sigma_{JI}(V)$ and so on.

The Ricci-flatness condition[7] reads

$$\sigma_{II}(V) + \sigma_{JJ}(V) + \sigma_{KK}(V) = 0 \tag{4.5}$$

[7] Once one knows that there exists 3 metric-compatible, mutually orthogonal complex structures I, J, K, then (4.5) is actually a direct consequence of the Bianchi identity.

for any $V \in T_pM$.

The significance of the above σ's is that they determine the Riemann curvature tensor of X.

Lemma 4.3 *Let $(X, \tilde{g})$ be a hyperkähler manifold of real dimension 4. Fix some $V \in T_pX\setminus\{0\}$ for some $p \in X$. Then, the holomorphic sectional curvature $W \mapsto \sigma_{II}(W)$ at p defines a quartic on T_pX with coefficients given by $\sigma_{II}(V), \sigma_{JJ}(V), \sigma_{IJ}(V), \sigma_{IK}(V)$, and $\sigma_{JK}(V)$.*

Proof Let $W = \alpha V + \beta IV + \mu JV + \nu KV \in T_pX$ be any tangent vector. Then, by iteratively using identities like $\langle R(Iv, Jw)u, t\rangle = \left\langle R(I^2v, IJw)u, t\right\rangle = -\langle R(v, Kw)u, t\rangle$ for any $t, u, v, w \in T_pX$ and the standard Riemann tensor symmetries, one arrives at

$$\begin{aligned}\langle R(W, IW)IW, W\rangle = &\left(\alpha^2+\beta^2-\mu^2-\nu^2\right)^2\sigma_{II}(V) + 4(\beta\mu-\alpha\nu)^2\sigma_{JJ}(V)\\ &+4(\alpha\mu+\beta\nu)^2\sigma_{KK}(V) + 4(\beta\mu-\alpha\nu)(\alpha^2+\beta^2-\mu^2-\nu^2)\sigma_{IJ}(V)\\ &+4(\alpha\mu+\beta\nu)(\alpha^2+\beta^2-\mu^2-\nu^2)\sigma_{IK}(V)\\ &+8(\alpha\mu+\beta\nu)(\beta\mu-\alpha\nu)\sigma_{JK}(V).\end{aligned} \tag{4.6}$$

Using (4.5) one can solve away one of the σ's, $\sigma_{KK}(V) = -\sigma_{II}(V) - \sigma_{JJ}(V)$, say. So in particular, the holomorphic sectional curvature of W is completely determined by $\sigma_{II}(V), \sigma_{JJ}(V), \sigma_{IJ}(V), \sigma_{IK}(V)$, and $\sigma_{JK}(V)$. □

Remark 4.4 A neat way of writing (4.6) is to use the following coordinates on $\mathbb{S}^3$. Assuming $|W| = |V|$, we can write

$$\alpha + i\beta = \cos\left(\frac{\theta}{2}\right)e^{\frac{i}{2}(\psi+\phi)}$$
$$\mu + i\nu = \sin\left(\frac{\theta}{2}\right)e^{\frac{i}{2}(\psi-\phi)}$$

where

$$0 \le \theta < \pi$$
$$0 \le \phi \le 2\pi$$
$$0 \le \psi \le 4\pi$$

as in [25, Eq. 2.4]. Then

$$\begin{aligned}\langle R(W, IW)IW, W\rangle = &\cos^2(\theta)\sigma_{II}(V) + \sin^2(\theta)\sin^2(\phi)\sigma_{JJ}(V)\\ &+\sin^2(\theta)\cos^2(\phi)\sigma_{KK}(V) + \sin(2\theta)\sin(\phi)\sigma_{IJ}(V)\\ &+\sin(2\theta)\cos(\phi)\sigma_{IK}(V) + \sin^2(\theta)\sin(2\phi)\sigma_{JK}(V).\end{aligned} \tag{4.7}$$

We note the absence of ψ in (4.7), which corresponds to the $U(1)$-symmetry of the holomorphic sectional curvature.

The next result shows how the presence of symmetry can drastically reduce the available degrees of freedom in the Riemann tensor.

Theorem 4.5 *Let $(X, \tilde{g})$ be a hyperkähler manifold of real dimension 4. Assume $F\colon X \to X$ is a holomorphic isometry of order 4 with positive-dimensional fixed point set $M := Fix(F)$. Then, $\sigma_{IJ}(V) = \sigma_{IK}(V) = \sigma_{JK}(V) = 0$ and $\sigma_{JJ}(V) = \sigma_{KK}(V)$ for any $V \in T_pM \subset T_pX$ and $p \in M$.*

Proof Let us start by seeing how F_* acts in a single tangent space. Let $p \in M$, and $V \in T_pM$. Then, V is necessarily fixed by F_*. Since F is holomorphic, M is a complex submanifold, and IV is fixed by F_* as well. Since $p = F(p)$, everything will be taking place at p and we shall be omitting references to the point everywhere. Consider now $F_*(JV)$. Using that V is fixed, that F is an isometry, and that $JV \perp V$ we deduce $\langle V, F_*(JV)\rangle = \langle F_*V, F_*(JV)\rangle = \langle V, JV\rangle = 0$. Similarly, $\langle IV, F_*(JV)\rangle = 0$. It thus follows that there exists $\theta \in [0, 2\pi)$ such that $F_*(JV) = (\cos(\theta) + I\sin(\theta))JV$. A direct computation shows $F_*^4(JV) = (\cos(4\theta) + I\sin(4\theta))JV$. By assumption $F_*^4(JV) = JV$, so $\cos(4\theta) = 1$, $\sin(4\theta) = 0$. The solutions $\theta = 0$ and $\theta = \pi$ can be ruled out as these would make F an order 1 or 2 isometry, respectively (see [60, Theorem 1.1, p. 137] for instance). We conclude that $F_*(JV) = \pm KV$ and $F_*(KV) = \mp JV$. In words; F_* fixes T_pM and rotates the orthogonal complement of $T_pM \subset T_pX$ by 90°.

This already provides us with plenty of information due to Lemma 4.1. Recalling the notation of Lemma 4.2, we next aim to show that for any $p \in M$ and $V \in T_pX$ the following holds

$$\sigma_{IJ}(V) = \sigma_{IK}(V) = \sigma_{JK}(V) = 0 \tag{4.8}$$

and

$$\sigma_{JJ}(V) = \sigma_{KK}(V). \tag{4.9}$$

To see this, we use (4.2) and the above knowledge of how F_* acts

$$\begin{aligned}\sigma_{JK}(V) &= \langle R(V, JV)KV, V\rangle \\ &\overset{(4.2)}{=} \langle R(F_*V, F_*(JV))F_*(KV), F_*V\rangle \\ &= -\langle R(V, KV)JV, V\rangle \\ &= -\sigma_{KJ}(V).\end{aligned}$$

But $\sigma_{JK}(V) = \sigma_{KJ}(V)$, so $\sigma_{JK}(V) = 0$. Similarly, one computes $\sigma_{IJ}(V) = \pm\sigma_{IK}(V)$ and $\sigma_{IK}(V) = \mp\sigma_{IJ}(V)$, which combine to say $\sigma_{IJ}(V) = -\sigma_{IJ}(V)$ and $\sigma_{IK}(V) = -\sigma_{IK}(V)$. The same kind of computation also gives $\sigma_{JJ}(V) = \sigma_{KK}(V)$. □

The proposition can actually be generalized to isometries of other orders.

Theorem 4.6 *Let $(X, \tilde{g})$ be a hyperkähler manifold of real dimension 4. Assume $F: X \to X$ is a holomorphic isometry of order $k \geq 2$ with positive-dimensional fixed point set $M := Fix(F)$. Then $\sigma_{IJ}(V) = \sigma_{IK}(V) = \sigma_{JK}(V) = 0$ for any $V \in T_pM \subset T_pX$ and $p \in M$. If $k > 2$, then $\sigma_{JJ}(V) = \sigma_{KK}(V)$ and $\sigma_{JK}(V) = 0$ as well.*

Proof One argues as above that for $p \in M$ and $V \in T_pM$, F_* acts by rotating the orthogonal complement by $\theta = \pm\frac{2\pi}{k}$. One then computes as above that

$$\sigma_{IJ}(V) = \cos(\theta)\sigma_{IJ}(V) + \sin(\theta)\sigma_{IK}(V)$$

and

$$\sigma_{IK}(V) = \cos(\theta)\sigma_{IK}(V) - \sin(\theta)\sigma_{IJ}(V).$$

This means $(\sigma_{IJ}, \sigma_{IK})$ is fixed by a non-trivial rotation, hence has to be 0. Using this, we further compute

$$\sigma_{JJ}(V) = \cos^2(\theta)\sigma_{JJ}(V) + \sin^2(\theta)\sigma_{KK}(V)$$

When $\sin(\theta) \neq 0$, this implies $\sigma_{JJ}(V) = \sigma_{KK}(V)$. Similarly, one computes

$$\sigma_{JK}(V) = \cos(2\theta)\sigma_{JK}(V) + \sin(\theta)\cos(\theta)(\sigma_{KK}(V) - \sigma_{JJ}(V)),$$

hence $\sigma_{JK}(V) = 0$. □

Remark 4.7 One can prove the same theorem for Ricci-flat Kähler manifolds of complex dimension 2, i.e. without using the hyperkähler structure. Choosing local coordinates z_1, z_2 near a point $p \in M$ such that M locally takes the form $\{z_2 = 0\}$, the action of f_* is given by multiplying $\frac{\partial}{\partial z_2}$ by a k'th root of unity while leaving $\frac{\partial}{\partial z_1}$ invariant. Assuming $k > 2$, the only invariant components of the Riemann tensor are then the ones with an equal number of 2- and $\overline{2}$-indices. The independent components are then $R_{1\overline{1}1\overline{1}}$, $R_{1\overline{1}2\overline{2}}$, $R_{1\overline{2}2\overline{1}}$, $R_{2\overline{2}2\overline{2}}$. The Ricci-flatness gives us 3 additional equations,

$$0 = R_{\mu\overline{\nu}} = R_{\mu\overline{\nu}1\overline{1}}g^{\overline{1}1} + R_{\mu\overline{\nu}1\overline{2}}g^{\overline{2}1} + R_{\mu\overline{\nu}2\overline{1}}g^{\overline{1}2} + R_{\mu\overline{\nu}2\overline{2}}g^{\overline{2}2}.$$

This completely determines the Riemann tensor in terms of $R_{1\overline{1}1\overline{1}}$.

When $k = 2$, the invariant components are the ones where the total number of 2- and $\overline{2}$ indices is even. So we get one more independent component, namely $R_{1\overline{2}1\overline{2}}$. The Riemann tensor then has 2 degrees of freedom instead of 1.

Corollary 4.8 *Assume the set-up of Theorem 4.6 with $k > 2$. Then the Riemann curvature tensor of X at $p \in M \subset X$ is uniquely determined by the Gauss curvature of M at p. In fact, if $\mathcal{K}\colon M \to \mathbb{R}$ is the Gauss curvature, $V \in T_pM \subset T_pX$ is a unit vector, and $W = \alpha V + \beta IV + \mu JV + \nu KV \in T_pX$ is arbitrary, then*

$$\langle R(W, IW)IW, W\rangle = \left((\alpha^2 + \beta^2)^2 + (\mu^2 + \nu^2)^2 - 4(\alpha^2 + \beta^2)(\mu^2 + \nu^2)\right)\mathcal{K}(p)$$

Proof By the Gauss–Codazzi theorem (see [19, Theorem 2.5] for instance) and the fact that M is totally geodesic, $\sigma_{II}(V)$ equals the holomorphic sectional curvature (of the induced metric) of M, and the holomorphic sectional curvature exactly equals the Gauss curvature of a Riemann surface. The expression then follows from (4.6). □

Remark 4.9 Yet another corollary of the above result is that if the fixed point set has a connected component which is a torus, then there has to be points where the curvature of M vanishes (simply by the Gauss–Bonnet theorem), hence points where the entire curvature tensor of X vanishes. The zero set of the curvature on a Riemann surface does not have to be made up of geodesics, however. Indeed, just compute what happens for a torus embedded in $\mathbb{R}^3$, via $(\theta, \phi) \mapsto (((R + r\cos(\theta))\cos(\phi), (R + r\cos(\theta))\sin(\phi), r\sin(\theta))$. One finds that in this case, $\mathcal{K} = 0$ on the two circles $(R\cos(\phi), R\sin(\phi), \pm r)$, neither of which are geodesics.

Remark 4.10 In the coordinates of Remark 4.4, the sectional curvature reads

$$\langle R(W, IW)IW, W\rangle = \frac{1}{4}\left(3\cos(2\theta) - 1\right)\mathcal{K}$$

4.2 Derivatives of the curvature

To proceed, we will formulate some results about the derivative the holomorphic sectional curvature on a hyperkähler manifold. We will write ∇ (instead of $\tilde{\nabla}$) for the covariant derivative associated with $\tilde{g}$. We only refer to a single (hyperkähler) metric in this section, so the risk of confusion is minimal.

Proposition 4.11 *Let $(X, \tilde{g})$ be as in Theorem 4.5. Let V be a unit vector field defined in some open neighbourhood U of M with the property that $V_p \in T_pM$ for all $p \in M$. Then, for any critical point of the function $\sigma_{II}(V)\colon U \to \mathbb{R}$ lying in M, we have*

$$\begin{aligned}\Delta \langle R(V, IV)IV, V\rangle &:= (\nabla_V^2 + \nabla_{IV}^2 + \nabla_{JV}^2 + \nabla_{KV}^2) \langle R(V, IV)IV, V\rangle \\ &= -6\sigma_{II}(V)^2 - 24|\nabla_{JV} V|^2\sigma_{II}(V).\end{aligned} \tag{4.10}$$

Proof Let W be any vector field. Then

$$\nabla_W \langle R(V, IV)IV, V\rangle = \langle \nabla_W R(V, IV)IV, V\rangle + 4 \langle R(\nabla_W V, IV)IV, V\rangle$$

and

$$\begin{aligned}\nabla_W^2 \langle R(V, IV)IV, V\rangle = &\left\langle \nabla_W^2 R(V, IV)IV, V\right\rangle + 8 \langle \nabla_W R(\nabla_W V, IV)IV, V\rangle \\ &+ 4\left\langle R(\nabla_W^2 V, IV)IV, V\right\rangle + 4 \langle R(\nabla_W V, I\nabla_W V)IV, V\rangle \\ &+ 8 \langle R(\nabla_W V, IV)IV, \nabla_W V\rangle .\end{aligned}$$

The terms $\nabla_W R$ and $\nabla_W^2 R$ denote covariant derivatives of R as a 4-tensor, and we will deal with these terms last using the second Bianchi identity. Since $|V| = 1$, $\langle \nabla_W V, V\rangle = 0$ so there are real functions α, μ, ν depending on W such that

$$\nabla_W V = \alpha IV + \mu JV + \nu KV.$$

A quick computation reveals

$$\nabla_W^2 V = -(\alpha^2 + \mu^2 + \nu^2)V + (\nabla_W\alpha)IV + (\nabla_W\mu)JV + (\nabla_W\nu)KV.$$

We compute each of the above terms on the right-hand side, using the shorthand $\sigma_{II}(V)$ and so on introduced above.

$$\begin{aligned}\langle \nabla_W R(\nabla_W V, IV)IV, V\rangle &= \mu \langle \nabla_W R(V, KV)IV, V\rangle - \nu \langle \nabla_W R(V, JV)IV, V\rangle ,\\ \left\langle R(\nabla_W^2 V, IV)IV, V\right\rangle &= -(\alpha^2 + \mu^2 + \nu^2)\sigma_{II}(V) + (\nabla_W\mu)\sigma_{IK}(V) - (\nabla_W\nu)\sigma_{IJ}(V),\\ \langle R(\nabla_W V, I\nabla_W V)IV, V\rangle &= (\alpha^2 - \mu^2 - \nu^2)\sigma_{II}(V) + 2\alpha\mu\sigma_{IJ}(V) + 2\alpha\nu\sigma_{IK}(V),\\ \langle R(\nabla_W V, IV)IV, \nabla_W V\rangle &= \mu^2\sigma_{KK}(V) + \nu^2\sigma_{JJ}(V) - 2\mu\nu\sigma_{JK}(V).\end{aligned}$$

So

$$\begin{aligned}\nabla_W^2 \langle R(V, IV)IV, V\rangle =&\left\langle \nabla_W^2 R(V, IV)IV, V\right\rangle \\ &+ 8(\mu \langle \nabla_W R(V, KV)IV, V\rangle - \nu \langle \nabla_W R(V, JV)IV, V\rangle) \\ &+ 8 \left(\mu^2\sigma_{KK}(V) + \nu^2\sigma_{JJ}(V) - (\mu^2 + \nu^2)\sigma_{II}(V)\right) \\ &+ 4 ((\nabla_W\mu) + 2\alpha\nu) \sigma_{IK}(V) \\ &+ 4 (-(\nabla_W\nu) + 2\alpha\mu) \sigma_{IJ}(V) \\ &- 4\mu\nu\sigma_{JK}(V).\end{aligned}$$

Evaluating this at a critical point in the fixed point set, $p \in M$, simplifies the expression vastly. By Theorem 4.5, $\sigma_{IJ} = \sigma_{JK} = \sigma_{IK} = 0$ and $\sigma_{JJ} = \sigma_{KK} = -\frac{1}{2}\sigma_{II}$. Furthermore, being a critical point implies $\nabla_W\sigma_{II} = 0$ for all vector fields W, hence

$$\begin{aligned}0 = \nabla_W\sigma_{II} &= \langle \nabla_W R(V, IV)IV, V\rangle + 4 \langle R(\mu JV + \nu KV, IV)IV, V\rangle \\ &= \langle \nabla_W R(V, IV)IV, V\rangle + 4\mu\sigma_{IK} - 4\nu\sigma_{IJ} = \langle \nabla_W R(V, IV)IV, V\rangle .\end{aligned}$$

This will imply that also $\langle \nabla_W R(V, KV)IV, V\rangle = 0 = \langle \nabla_W R(V, JV)IV, V\rangle$, as we now demonstrate case by case. When $W = V$ or IV, the result follows by the same argument using isometries as in the proof of Theorem 4.5. The key being that the expression contains an odd number of JV- and KV-factors. When $W = JV$, we use the second Bianchi identity to say

$$\langle \nabla_{JV} R(V, KV)IV, V\rangle = -\langle \nabla_V R(KV, JV)IV, V\rangle - \langle \nabla_{KV} R(JV, V)IV, V\rangle .$$

On the last term, we apply the isometry sending $JV \mapsto \pm KV$, $KV \mapsto \mp JV$ to argue

$$\langle \nabla_{JV} R(V, KV)IV, V\rangle = -\frac{1}{2}\langle \nabla_V R(KV, JV)IV, V\rangle = -\frac{1}{2}\langle \nabla_V R(V, IV)IV, V\rangle .$$

The right-hand side vanishes at a critical point. When $W = KV$, we use the Bianchi identity to say

$$\langle \nabla_{KV} R(V, KV)IV, V\rangle = -\langle \nabla_{IV} R(V, KV)V, KV\rangle - \langle \nabla_V R(V, KV)KV, IV\rangle .$$

Writing $\nabla_V V = \alpha IV$ and $\nabla_{IV} V = \beta IV$ for some functions $\alpha, \beta\colon M \to \mathbb{R}$, we find

$$\begin{aligned} -\langle \nabla_{IV} R(V, KV)V, KV\rangle &= \nabla_{IV}\sigma_{KK} - 4\langle R(\nabla_{IV} V, KV)KV, V\rangle \\ &= -\frac{1}{2}\nabla_{IV}\sigma_{II} - 4\beta\sigma_{JK}. \end{aligned}$$

The last term vanishes on M, and the first term is 0 at a critical point. This shows that $\langle \nabla_W R(V, KV)IV, V\rangle = 0$ at a critical point $p \in M$. Using the isometry mapping JV to $\pm KV$ shows that $\langle \nabla_W R(V, JV)IV, V\rangle = 0$ as well.

All in all, this shows

$$\nabla_W^2 \sigma_{II} = \left\langle \nabla_W^2 R(V, IV)IV, V\right\rangle - 12(\mu^2 + \nu^2)\sigma_{II}$$

at any critical point $p \in M$. We recall $\mu = \langle JV, \nabla_W V\rangle$, $\nu = \langle KV, \nabla_W V\rangle$. Using the order 4 isometry shows

$$\begin{aligned} 0 = \langle JV, \nabla_V V\rangle &= \langle JV, \nabla_{IV} V\rangle , \\ \langle JV, \nabla_{JV} V\rangle &= \langle KV, \nabla_{KV} V\rangle , \end{aligned}$$

and

$$\langle KV, \nabla_{JV} V\rangle = -\langle JV, \nabla_{KV} V\rangle .$$

We therefore find

$$\Delta\sigma_{II} = \langle \Delta R(V, IV)IV, V\rangle - 24|\nabla_{JV} V|^2\sigma_{II}.$$

Using Proposition A.4 from Appendix A for ΔR, we finally find

$$\Delta\sigma_{II} = -6\sigma_{II}^2 - 24|\nabla_{JV} V|^2\sigma_{II}$$

at a critical point $p \in M$. □

Remark 4.12 A key fact used is that the Laplacian acting on the Riemann tensor gives something proportional to the square of the Riemann tensor. (A.4) is the precise statement. Something similar is true on an arbitrary Ricci-flat Kähler manifold without assuming it to be hyperkähler. The result is (see [66, Eq. 3.108])

$$-\frac{1}{2}\Delta R_{\mu\overline{\nu}\alpha\overline{\beta}} = R_{\mu\overline{\nu}\lambda\overline{\sigma}}R^{\overline{\sigma}\lambda}{}_{\alpha\overline{\beta}} + R_{\mu\overline{\sigma}\lambda\overline{\beta}}R^{\overline{\sigma}}{}_{\overline{\nu}\alpha}{}^{\lambda} - R_{\mu\overline{\lambda}\alpha\overline{\sigma}}R^{\overline{\lambda}}{}_{\overline{\nu}}{}^{\overline{\sigma}}{}_{\overline{\beta}}.$$

The second thing used in the above proof is that the order 4 symmetry restricts the degrees of freedom of the Riemann tensor, allowing us to get a simple expression for the Laplacian.

5 A special K3 surface

From now on we will specialize to a special Kummer K3 surface. The precise assumptions and notation is as follows.

Notation 5.1 Let $\Lambda:=\mathbb{Z}\{1, i\} \subset \mathbb{C}$ and let (X, g) denote the Kummer K3 surface associated with the torus with lattice $\Gamma:=\Lambda \oplus \Lambda$. We will also assume that all components of the exceptional divisor have the same volume, $a_i = a_j$ for $1 \leq i, j \leq 16$.

We write $Q\colon \mathbb{C}^2 \to Y$ for the quotient map (quotienting with respect to both Γ and μ_2) and $\pi : X \to Y$ for the blow-down map, with inverse $\pi^{-1}\colon Y\backslash Sing(Y) \to X\backslash E$.

For any suitable affine map $F^{\mathbb{C}}\colon \mathbb{C}^2 \to \mathbb{C}^2$, $F^{\mathbb{C}}(\mathbf{z}) = B\mathbf{z} + \mathbf{b}$ we denote the induced map (using Proposition 2.13) by $F\colon X \to X$. These maps will all have the property that they map E to E, so give rise to isometries (using the same name for the restrictions) $F\colon X\backslash E \to X\backslash E$.

These restricted maps satisfy $\pi \circ F \circ \pi^{-1} \circ Q = Q \circ F^{\mathbb{C}}$, a fact which will be used implicitly.

For $M^{\mathbb{C}} \subset \mathbb{C}^2$ with $Q(M^{\mathbb{C}}) \subset Y\backslash Sing(Y)$, we write $M:=\pi^{-1} \circ Q(M^{\mathbb{C}}) \subset X$.

We start by describing the isometries induced by affine maps.

Proposition 5.2 *The holomorphic isometries of (X, g) are induced by the affine maps of the form*

$$F^{\mathbb{C}}(\mathbf{z}) = B\mathbf{z} + \mathbf{b}$$

where

$$B \in U(2) \cap GL(2, \mathbb{Z}[i]) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mu_4 \right\}$$

with $\mu_4 := \{\pm 1, \pm i\}$, and $\mathbf{b} \in \left(\frac{1}{2}\Lambda\right)^2$. Any anti-holomorphic isometry is induced by one of the above forms composed with $\tau^{\mathbb{C}}$ for $\tau^{\mathbb{C}}(\mathbf{z}):=\bar{\mathbf{z}}$. There are 512 distinct isometries. All of them are isometries of $(X, \tilde{g})$ as well. The maximum order of these isometries is 8.

These isometries play a central role in the arguments to come. We will single out some of them.

Notation 5.3 We consider the following map $\mathbb{C}^2 \to \mathbb{C}^2$

$$f^{\mathbb{C}} \begin{pmatrix} z \\ w \end{pmatrix} := \begin{pmatrix} z \\ iw + \frac{1}{2} \end{pmatrix}, \tag{5.1}$$

along with the induced map $f\colon X \to X$. This map is an isometry of (X, g) of order 4.

We also consider the following subset:

$$M^{\mathbb{C}} := \left\{ \begin{pmatrix} z \\ \frac{1+i}{4} \end{pmatrix} \,\middle|\, z \in \mathbb{C} \right\} \tag{5.2}$$

The corresponding set in X is denoted by $M = \pi^{-1} \circ Q(M^{\mathbb{C}})$. Note that $Q(M^{\mathbb{C}})$ lands in $Y\backslash Sing(Y)$ due to the shift by an element not in $\frac{1}{2}\Gamma$, so π^{-1} is well defined.

Theorem 5.4 *The submanifold $M \subset X$ described above is a totally geodesic torus. There are points $p \in M$ such that the Riemann tensor of $(X, \tilde{g})$ vanishes at p.*

Proof The image $Q(M^{\mathbb{C}}) \subset Y$ lands in the smooth part due to the shift by $\frac{1+i}{4} \notin \frac{1}{2}\Gamma$. Clearly $Q(M^{\mathbb{C}})$ is a torus, and π is biholomorphic when restricted, $\pi : X \backslash E \xrightarrow{\cong} Y \backslash Sing(Y)$, hence $M \subset X$ is a torus.

Furthermore, M is fixed by the order 4 isometry f. So by Theorem 4.5, the Riemann tensor of $(X, \tilde{g})$ at $p \in M$ is uniquely determined by the Gauss curvature of M at p. But any torus has points of vanishing Gauss curvature due to the Gauss–Bonnet theorem; hence, there are points where the curvature of $(X, \tilde{g})$ vanishes. □

The next proposition indicates that the torus in Theorem 5.4 could be flat. The proposition contains a non-trivial assumption, however. Assume $M \subset X$ is a torus which is fixed by an order 4 isometry f. Let V be a unit vector field on M. By an extension of V to a neighbourhood $M \subset U \subset X$, we mean a unit vector field $V : U \to TX$ such that

- V restricts to a tangent vector field on M, $V_{|M} : M \to TM$;
- $\nabla_W V$ restricts to a tangent vector field on M, $(\nabla_W V)_{|M} : M \to TM$, for all vector fields $W : U \to TX$.

A concrete example would be that if (z, w) are local coordinates with M locally being given by $\{w = 0\}$, and $V = h\frac{\partial}{\partial z}$ with $h{:=}\frac{1}{|\partial_z|_{\tilde{g}}}$. Then (4.3) along with $\nabla_{\partial_{\bar{z}}}\partial_z = 0 = \nabla_{\partial_{\overline{w}}}\partial_z$ say $\nabla_{JV} V \parallel V$, and similarly for $\nabla_{KV} V$.

Theorem 5.5 *Let $M \subset (X, \tilde{g})$ be a torus which is fixed by an order 4 holomorphic isometry. Let V be a unit vector tangent field of M, extended to a neighbourhood U of M. If the function $\sigma_{II}(V) : U \to \mathbb{R}$ has its minima on M, then M is flat.*

Proof Proposition 4.11 gives us an expression for $\Delta\sigma_{II}$ at critical points. We will argue that the last term drops out. Using the order 4 isometry f, we see that $\nabla_{JV} V$ has to be proportional to a combination of JV and KV when restricted to M. Hence, $\nabla_{JV} V = 0$ on M. Proposition 4.11 therefore says

$$\Delta\sigma_{II} = -6\sigma_{II}^2$$

for any critical point on M. Hence, a minimum is possible if and only if $\sigma_{II} = 0$ at the minimum. But $\int_M \sigma_{II} \, d\mathrm{Vol}_{\tilde{g}_{|M}} = 0$ by the Gauss–Bonnet theorem, and $\sigma_{II} = 0$ everywhere on M as a consequence. □

Remark 5.6 Any critical point for $(\sigma_{II})_{|M}$ is also a critical point for σ_{II} due to the order 4 isometry. Indeed, $\nabla_{JV}\sigma_{II} = -\nabla_{JV}\sigma_{II}$ and similarly for ∇_{KV}. So $\nabla\sigma_{II} = 0$ if and only if $\nabla_V\sigma_{II} = \nabla_{IV}\sigma_{II} = 0$. Hence, there are critical points of σ_{II} on M. What is not clear (hence the assumption in the above result) is that these critical points are minima.

We end by pointing out that the lattice $\Gamma = \Lambda \oplus \Lambda$ with $\Lambda = \mathbb{Z}\{1, i\}$ is not the only possible choice leading to a result like Theorem 5.4. Another possibility is to choose $\zeta{:=}\exp\left(\frac{2\pi i}{3}\right)$, $\Lambda{:=}\mathbb{Z}\{1, \zeta\}$ and $\Gamma = \Lambda \oplus \Lambda$. The affine map

$$f^{\mathbb{C}}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z \\ \zeta w + \frac{1+\zeta}{2} \end{pmatrix}$$

induces an order 3 holomorphic isometry $f : X \to X$. The set

$$M^{\mathbb{C}}{:=}\left\{\begin{pmatrix} z \\ \frac{1+2\zeta}{6} \end{pmatrix} \,\middle|\, z \in \mathbb{C}\right\}$$

maps to a torus $M \subset X$ which is a connected component of the fixed point set of f. Theorem 5.4 therefore applies to this M.

6 Proof of Kobayashi's estimates

Here, we will go through large parts of the proof of Kobayashi's estimates. Most of the key arguments are due to [43], but there are two minor mistakes we correct. The first is that Kobayashi seems to assume the constant A from (2.7) is 1. One can of course absorb A into the definition of ψ, (6.1), but this will cause ψ to be different from 0 outside of the neck regions. The fact that A is not 1, but somewhat smaller, see (2.8), slightly changes the proof of Case 1 of Lemma 6.5. The second mistake is that Kobayashi refers to standard theory for the real Monge–Ampère equation to deduce Hölder bounds. One instead has to use the corresponding complex theory, which we do in Proposition 6.8.

Additionally, we clarify the choice of coordinates used by introducing holomorphic Darboux coordinates, and give a detailed analysis of a suitable coordinate system near a connected component $E_i \cong \mathbb{CP}^1$ of the exceptional divisor E.

Our notation is as before; (X, g) denotes a Kummer K3 surface with patchwork metric depending on 16 parameters a_i and $|a|^2 = \sum_i a_i^2$. The constant A is defined in (2.7) and takes the value given in (2.8). We introduce the function $\psi\colon X \to \mathbb{R}$ via

$$e^{\psi} = \frac{2\eta \wedge \overline{\eta}}{\omega^2}. \tag{6.1}$$

This can either be interpreted as the Radon–Nikodym derivative, or simply the proportionality function which must exist between the two top forms $\eta\wedge\overline{\eta}$ and $\omega^2/2$. In holomorphic Darboux coordinates, i.e. where $\eta = dz_1 \wedge dz_2$, we have

$$\psi = -\ln\det(g),$$

and (6.1) is a way of making this function globally well defined. As we prove in Appendix B, (B.6), we have

$$\|\psi\|_{C^0(X)} \leq C|a|^2. \tag{6.2}$$

for some constant $C > 0$. The argument is that $\psi = 0$ outside of the neck regions, and in the necks one can see that the Euclidean and Eguchi–Hanson metrics differ by terms of order a^2. With this notation out of the way, we may write the Monge–Ampère equation as

$$\tilde{\omega}^2 := (\omega + i\partial\overline{\partial}\phi)^2 = 2A\eta \wedge \overline{\eta} = Ae^{\psi}\omega^2. \tag{6.3}$$

6.1 C^0-estimates

The C^0 estimate of Kobayashi is as follows.

Proposition 6.1 [43] *Assume ϕ is the solution to (6.3) subject to the normalization*

$$\int_X \phi\, \omega^2 = 0.$$

Then, there is a constant $C > 0$ such that for all values of $|a|$ small enough, we have

$$\|\phi\|_{C^0(X)} \leq C|a|^2 \tag{6.4}$$

We will forego a full proof, which proceeds via Moser iteration and is well explained in [43]. We would nevertheless like to sketch parts of it to point out some important features.

Sketch of proof From the Monge–Ampère Eq. (6.3), we have

$$(1 - Ae^{\psi})\omega^2 = \omega^2 - \tilde{\omega}^2 = -i\partial\overline{\partial}\phi \wedge (\tilde{\omega} + \omega). \tag{6.5}$$

Multiplying this by $\phi|\phi|^{2(p-1)}$ for $p \geq 1$, integrating by parts, and dropping a nonnegative term leads to

$$\int_X |d|\phi|^p|^2\omega^2 \leq Cp\int_X |1 - A^{\psi}||\phi|^{2p-1}\omega^2 \leq C|a|^2 p\int_X |\phi|^{2p-1}\omega^2, \tag{6.6}$$

where we have also used (6.2) and (2.8). This is the key estimate we need. For $p = 1$, one can use the Hölder inequality on the right-hand side to get an L^2-norm of ϕ. Replacing the left-hand side by $\|\phi\|^2_{L^2(X,g)}$ via the Poincaré inequality (B.3), we get $\|\phi\|_{L^2(X,g)} \leq C|a|^2$. Going back to (6.6), we can use the Sobolev inequality (B.4) to replace the gradient norm by a higher L^p-norm. Concretely, one finds

$$\|\phi\|^{2p}_{L^{4p}(X,g)} \leq C\left(2p|a|^2 + \|\phi\|_{L^{2p}(X,g)}\right)\|\phi\|^{2p-1}_{L^{2p}(X,g)}. \tag{6.7}$$

Iterating this, starting at $p = 1$, will lead to the required L^∞-bounds. □

Remark 6.2 There are a couple of important aspects of this estimate we would like to point out. Firstly, the factor $|a|^2$ appears as $|1 - Ae^{\psi}|$, which essentially measures the Ricci curvature of the patchwork metric in the neck regions. A better approximation of the Ricci-flat metric would then have given us a smaller C^0-bound on the correction, which would have propagated into the C^k-bounds.

Secondly, the fact that the Patchwork metric has bounded volume, diameter and Ricci curvature as $a \to 0$ gives us uniform Poincaré- and Sobolev constants, which ensures that our constants stay a-independent. Details can be found in Appendix B.

6.2 C^2-estimates

The C^2-estimates follow from the Monge–Ampère equation as soon as we have estimates on $\Delta\phi$. To see this, we recall the function

$$\exp(\psi) = \frac{2\eta \wedge \overline{\eta}}{\omega^2}.$$

In holomorphic Darboux coordinates, the Monge–Ampère Eq. (6.3) reads

$$\det\left(g + \partial\overline{\partial}\phi\right) = A,$$

where A is defined by (2.7). The determinant of the complex 2-matrix we write as

$$\det\left(g + \partial\overline{\partial}\phi\right) = \det(g)\det\left(1 + g^{-1}\partial\overline{\partial}\phi\right) = \exp(-\psi)\left(1 + \mathrm{tr}(g^{-1}\partial\overline{\partial}\phi) + \det(g^{-1}\partial\overline{\partial}\phi)\right).$$

Now $\mathrm{tr}(g^{-1}\partial\overline{\partial}\phi) = \Delta\phi$ and $\mathrm{tr}(g^{-1}\partial\overline{\partial}\phi g^{-1}\partial\overline{\partial}\phi) = |\partial\overline{\partial}\phi|^2_g$ per definition, and

$$\begin{aligned}\det(g^{-1}\partial\overline{\partial}\phi) &= \frac{1}{2}\left(\mathrm{tr}(g^{-1}\partial\overline{\partial}\phi)^2 - \mathrm{tr}(g^{-1}\partial\overline{\partial}\phi g^{-1}\partial\overline{\partial}\phi)\right)\\ &= \frac{1}{2}\left((\Delta\phi)^2 - |\partial\overline{\partial}\phi|^2_g\right).\end{aligned}$$

With this, we may write the Monge–Ampère equation as

$$2(A\exp(\psi)-1)=2\Delta\phi+(\Delta\phi)^2-|\partial\overline{\partial}\phi|_g^2. \tag{6.8}$$

We note how (6.8) is independent of the choice of coordinates.

This tells us that a bound on $\Delta\phi$ directly translates into a bound on $\partial\overline{\partial}\phi$. So we set about bounding $\Delta\phi$ as in [43, 72]. Let $r_a=\frac{\max_i a_i}{\min_i a_i}$. Then, we will prove there is a constant $C>0$ independent of $|a|$ such that

$$-C|a|^2\le\Delta\phi\le Cr_a|a| \tag{6.9}$$

holds for any point in X. The trick to be employed in proving (6.9) is essentially a maximum principle. We will consider the function $F(x){:=}\exp(-C\phi(x))\mathrm{Tr}_g(\tilde{g})(x)=\exp(-C\phi(x))(2+\Delta\phi)$ for some positive constant C (to be determined later). The function F has a maximum due to the compactness of X, and at a maximum we have $\tilde{\Delta}F\le 0$, where we have introduced $\tilde{\Delta}F=\mathrm{tr}(\tilde{g}^{-1}\partial\overline{\partial}F)=\tilde{g}^{\overline{\nu}\mu}\partial_\mu\partial_{\overline{\nu}}F$. Below, there will be an extensive computation deriving a lower bound on $\tilde{\Delta}F$. This lower bound at a single point will essentially establish (a stronger version of) (6.9) at a single point, which then gets translated into a proof of (6.9) at an arbitrary point.

The proof is long, and is subdivided into several steps. We start by recalling a important estimate from [72], which essentially comes from computing two derivatives of the Monge–Ampère equation.

Proposition 6.3 [72, Equation 2.22] *Choose holomorphic normal coordinates at p and diagonalize*[8] *$\partial\overline{\partial}\phi(p)$, meaning $g_{\mu\bar{\nu}}(p)=\delta_{\mu\bar{\nu}}$, $g_{\mu\bar{\nu},\alpha}(p)=0$, and $\phi_{\mu\bar{\nu}}(p)=\delta_{\mu\bar{\nu}}\phi_{\mu\bar{\mu}}(p)$. Then, the following inequality holds for any positive real number C at the single point p.*

$$\begin{aligned}&\exp(C\phi)\tilde{\Delta}\Big(\exp(-C\phi)Tr_g(\tilde{g})\Big)\\&\quad\ge\Delta\psi-4R_{1\bar{1}2\bar{2}}-2CTr_g(\tilde{g})+\big(C+R_{1\bar{1}2\bar{2}}\big)\frac{Tr_g(\tilde{g})^2}{\det(\tilde{g})}.\end{aligned} \tag{6.10}$$

Lemma 6.4 *Let $K:X\to\mathbb{R}_{\ge0}$, $K{:=}|Riem|_g^2$ denote the Kretchsmann scalar for the patchwork metric g. Introduce*

$$R_a{:=}\left\|\sqrt{K}\right\|_{L^\infty(X,g)}. \tag{6.11}$$

Then, there are constants $C_1,C_2>0$ (independent of a) such that for all small enough values of $|a|$, we have the bounds

$$C_1\frac{r_a}{|a|}\le R_a\le C_2\frac{r_a}{|a|}. \tag{6.12}$$

In particular, in the special coordinates of Proposition 6.3, we have

$$R_{1\bar{1}2\bar{2}}(p)\le\sqrt{K}(p)\le C\frac{r_a}{|a|} \tag{6.13}$$

for an arbitrary point $p\in X$, and

$$R_{1\bar{1}2\bar{2}}(p)\le\sqrt{K}(p)\le C|a|^2 \tag{6.14}$$

if p is in a neck region.

[8] See [66, Prop. 3.1.1] for instance for a proof that these coordinates exist.

Proof In the Euclidean region, we have $K = 0$. In a neck region N_i, (2.3) implies there is a constant $C > 0$ such that $K \leq Ca_i^4$ for small enough values of a_i. In an Eguchi–Hanson patch U_i, (2.4) says $K(z) = \frac{24a_i^4}{(a_i^2+u^2)^3}$ with $u = |z|^2_{\mathbb{C}^2}$. All in all, we find

$$R_a^2 = \max_{1\leq i\leq 16} \frac{24}{a_i^2},$$

from which it follows that[9]

$$2\sqrt{6}\frac{r_a}{|a|} \leq R_a \leq 32\sqrt{6}\frac{r_a}{|a|}.$$

□

Lemma 6.5 *Let $x_m \in X$ denote a maximum of the function $F(x) = \exp(-2R_a\phi(x))Tr_g(\tilde{g})(x)$. Then, there is a constant $C > 0$ such that for all small enough values of $|a|$, we have*

$$\Delta\phi(x_m) \leq C|a|^2. \tag{6.15}$$

Proof At the single point x_m we introduce holomorphic normal coordinates as before such that $g_{\mu\bar{\nu}}(x_m) = \delta_{\mu\bar{\nu}}$, $g_{\mu\bar{\nu},\alpha}(x_m) = 0$, and $\phi_{\mu\bar{\nu}}(x_m) = \delta_{\mu\bar{\nu}}\phi_{\mu\bar{\mu}}(x_m)$. In these coordinates, we set

$$k(x_m) := R_{1\bar{1}2\bar{2}}(x_m)/R_a. \tag{6.16}$$

Note that per definition, $|k(x_m)| \leq 1$.

At the maximum of $\exp(-2R_a\phi)\mathrm{Tr}_g(\tilde{g})$, the left-hand side of (6.10) with $C = 2R_a$ has to be non-positive. With our choice of notation, we may write this as

$$0 \geq \frac{\Delta\psi(x_m)}{R_a} - 4k(x_m) - 4\mathrm{Tr}_g(\tilde{g}) + (2 + k(x_m))\frac{\mathrm{Tr}_g(\tilde{g})^2}{\det(\tilde{g})}.$$

Complete a square in $\mathrm{Tr}_g(\tilde{g})$ to arrive at

$$\begin{aligned} &\frac{4\det(\tilde{g})^2}{(2+k(x_m))^2} - \frac{\det(\tilde{g})}{R_a(2+k(x_m))}(\Delta\psi(x_m) - 4k(x_m)R_a) \\ &\geq \left(\mathrm{Tr}_g(\tilde{g}) - \frac{2\det(\tilde{g})}{2+k(x_m)}\right)^2. \end{aligned} \tag{6.17}$$

There are now two possible cases. Either x_m lies inside a neck region N_i or it lies outside of all the neck regions.

Case 1—x_m lies outside of the neck regions:

Outside of the necks, $\psi(x_m) = \Delta\psi(x_m) = 0$ by (6.1) by construction of g. Inserting this into (6.17) gives

$$4\frac{\det(\tilde{g})^2}{(2+k(x_m))^2} + 4\frac{k(x_m)\det(\tilde{g})}{2+k(x_m)} \geq \left(\mathrm{Tr}_g(\tilde{g}) - \frac{2\det(\tilde{g})}{2+k(x_m)}\right)^2. \tag{6.18}$$

The Monge–Ampère equation (6.3) says in holomorphic normal coordinates for g that $\det(\tilde{g}) = Ae^{\psi} = 1 - \Upsilon|a|^2 < 1$, where $\Upsilon > 0$ is some positive constant which was

[9] The argument written out is this. We have $\max_i \frac{1}{a_i} = \frac{r_a}{\max_i a_i}$. Then, one uses the comparison between the max-norm and Euclidean norm; $\max_i a_i \leq |a| \leq 16 \max_i a_i$.

computed in (2.8). Hence, one may overestimate the first term on the left-hand side of (6.18) by

$$4\frac{\det(\tilde g)^2}{(2+k(x_m))^2} < 4\frac{\det(\tilde g)}{(2+k(x_m))^2}. \tag{6.19}$$

Inserting this back into (6.18) and recognizing a square allows one to conclude

$$4\det(\tilde g)\frac{(k(x_m)+1)^2}{(k(x_m)+2)^2} \geq \left(\mathrm{Tr}_g(\tilde g) - \frac{\det(\tilde g)}{2+k(x_m)}\right)^2.$$

Taking a square root on both sides here and using $\det(\tilde g) < 1$ twice, one can conclude that

$$\begin{aligned}\frac{2(k(x_m)+1)}{k(x_m)+2} &> \sqrt{\det(\tilde g)}\frac{2(k(x_m)+1)}{k(x_m)+2} \\ &\geq \mathrm{Tr}_g(\tilde g) - \frac{2\det(\tilde g)}{2+k(x_m)} > \mathrm{Tr}_g(\tilde g) - \frac{2}{2+k(x_m)},\end{aligned}$$

or

$$2 = \frac{2(k(x_m)+2)}{k(x_m)+2} \geq \mathrm{Tr}_g(\tilde g).$$

This proves that

$$\Delta\phi(x_m) \leq 0$$

when x_m lies outside of the neck regions.

Case 2—x_m lies inside the neck regions:

When the maximum x_m lies in a neck region, we return to (6.17) and complete the square on the left-hand side. Suppressing the point x_m from the notation, the result is

$$\begin{aligned}&\frac{4\det(\tilde g)^2}{(2+k)^2} - \frac{\det(\tilde g)}{R_a(2+k)}(\Delta\psi - 4kR_a) \\ &= \left(\frac{2\det(\tilde g}{2+k} - \frac{1}{4R_a}(\Delta\psi - 4kR_a)\right)^2 - \frac{1}{16R_a^2}(\Delta\psi - 4kR_a)^2 \\ &\leq \left(\frac{2\det(\tilde g}{2+k} - \frac{1}{4R_a}(\Delta\psi - 4kR_a)\right)^2.\end{aligned} \tag{6.20}$$

Hence, (6.17) says

$$\mathrm{Tr}_g(\tilde g) \leq \frac{4\det(\tilde g)}{2+k} - \frac{1}{4R_a}(\Delta\psi - 4kR_a)$$

or

$$\Delta\phi(x_m) \leq \frac{4\det(\tilde g)}{2+k} + k - 2 - \frac{\Delta\psi}{4R_a} = \frac{4(\det(\tilde g)-1)+k^2}{2+k} - \frac{\Delta\psi}{4R_a}$$

Since we are in a neck region, the curvature k is bounded, $|k| \leq C\frac{|a|^3}{r_a}$. This follows by (B.5). In normal coordinates for g, the Monge–Ampère equation reads $\det(\tilde g) = Ae^{\psi}$, hence $|\det(\tilde g) - 1| \leq C|a|^2$ by (2.8) and (6.2). From (B.6), it also follows that $|\Delta\psi| \leq C|a|^2$, and so

$$\Delta\phi(x_m) \leq C|a|^2$$

for all $|a|$ small enough. This proves (6.15). □

Proof of Equation (6.9) To get an upper bound of $\nabla\phi$ at an arbitrary point, we can do as follows, where the first inequality is the definition of x_m.

$$\begin{aligned}
\mathrm{Tr}_g(\tilde{g}) &= e^{2R_a\phi(x)}\left(e^{-2R_a\phi(x)}\mathrm{Tr}_g(\tilde{g})\right) \\
&\leq e^{2R_a\phi(x)}\left(e^{-2R_a\phi(x_m)}(\mathrm{Tr}_g(\tilde{g})(x_m))\right) \\
&\overset{(6.15)}{\leq} e^{2R_a(\phi(x)-\phi(x_m))}(2+C|a|^2) \\
&\overset{(6.4),(6.12)}{\leq} e^{Cr_a|a|}(2+C|a|^2) \\
&\leq 2+\tilde{C}r_a|a|.
\end{aligned}$$

This proves the upper bound in (6.9).

The lower bound in (6.9) is considerably easier. Let α, β denote the eigenvalues of $g^{-1}\tilde{g}$ in some local coordinates. Then

$$\alpha+\beta=\mathrm{Tr}_g(\tilde{g}), \tag{6.21}$$

and by the inequality of arithmetic and geometric means,

$$\frac{\mathrm{Tr}_g(\tilde{g})}{2}=\frac{\alpha+\beta}{2}\geq\sqrt{\alpha\beta}=\sqrt{\det(g^{-1}\tilde{g})}=\sqrt{\frac{\det(\tilde{g})}{\det(g)}}=\sqrt{Ae^{\psi}}, \tag{6.22}$$

where we have inserted the Monge–Ampère Eq. (6.3), in the final step. By (2.8) and (6.2), one can find a constant $C>0$ such that

$$Ae^{\psi}\geq(1-C|a|^2)^2,$$

for sufficiently small values of $|a|$. This can be inserted into (6.22) to establish

$$2+\Delta\phi=\mathrm{Tr}_g(\tilde{g})\geq 2\sqrt{Ae^{\psi}}\geq 2(1-C|a|^2),$$

which proves the lower bound in (6.9). □

Corollary 6.6 *There is a constant $C>0$ independent of $|a|$ such that*

$$|\partial\overline{\partial}\phi|_g^2\leq Cr_a|a|. \tag{6.23}$$

hold everywhere on X.

Proof From (2.8), (6.2), (6.8), and (6.9), it follows that

$$|\partial\overline{\partial}\phi|_g^2\leq 2Cr_a|a|+C^2r_a^2|a|^2+2(1-A\exp(\psi))\leq\tilde{C}r_a|a|.$$

□

Remark 6.7 The appearance of r_a in the C^2-estimate is a consequence of using Yau's maximum principle, where the maximal holomorphic sectional curvature appears from a double derivative of the Monge–Ampère equation. This factor of r_a then propagates into the higher-order estimates. We do not know if this is reflected in the actual behaviour of the solution, or if it is an artefact of the proof. We assume r_a is uniformly bounded in a (Assumption 2.8).

It would in general be interesting (and useful for studying the Kähler–Ricci flow on singular manifolds) if one can derive Yau's C^2-bounds without using the maximum of the sectional curvature.

6.3 Hölder regularity

From the above bound on the complex Hessian $\partial\overline{\partial}\phi$, we will follow Siu [64] and Błocki [11, 12], to derive Hölder bounds on the real Hessian $D^2\phi$. Since the real and complex Hessians differ, the corresponding real and complex Monge–Ampère equations differ. Hence, one cannot simply apply the real theory directly, as [43, p. 302] does. The methods go back to Evans [26, 27], Krylov [46] and Trudinger [69].

Proposition 6.8 *There are constants $C > 0$ and $0 < \alpha < 1$ which do not depend on a such that*

$$|\phi|_{C^{2,\alpha}(X,g)} \leq C.$$

holds for all values of a small enough.

The strategy is the following. Locally, we may write $\tilde{g} = \partial\overline{\partial}\tilde{\phi}$ for some Kähler potential $\tilde{\phi}$. In suitable coordinates, the Monge–Ampère equation reads $\det(\partial\overline{\partial}\tilde{\phi}) = const.$, and by taking derivatives of this equation, we get an elliptic equation we can analyse using a local Harnack inequality. By combining bounds on sub- and supersolutions, we get a bound on the oscillation of $\tilde{\phi}$, which leads to the Hölder bound on $\tilde{\phi}$. Since $\tilde{g} = g + \partial\overline{\partial}\phi$, we may choose $\tilde{\phi} = \Phi + \phi$, where Φ is a Kähler potential for g (e.g. (2.2)). We will show that Φ is in $C^{2,\alpha}$ uniformly in the same local coordinates as for $\tilde{\phi}$. Hence, ϕ will be uniformly bounded as well. We divide the proof into 7 steps.
Step 1—Bounds on the patchwork metric: We first prove that we can cover X by coordinate charts in such a way that the Monge–Ampère equation becomes simple and the eigenvalues of the patchwork metric are under control away from the exceptional divisor.

Lemma 6.9 *For all $|a|$ small enough, there is an a-independent finite cover V_i of holomorphic Darboux coordinate charts*[10] *of X and a constant $C > 0$ such that the eigenvalues of the metrics g and $\tilde{g}$ in the local coordinates lie in the interval between $C^{-1}\frac{|a|}{r_a}$ and $C\frac{r_a}{|a|}$.*

For any compact set $K \subset X \backslash E$, there is an a-independent constant C_K such that the eigenvalues of g and $\tilde{g}$ in the local coordinate charts $K \cap V_i$ lie between C_K^{-1} and C_K.

Proof We start with the second part. As long as one stays away from the exceptional divisor, the patchwork metric g can locally be written

$$g = g_{\mathrm{Euc}} + |a|^2 h,$$

where h is bounded with bounded derivatives. Cover the compact set K with finitely many such coordinate charts to deduce the statement for g. The statement for $\tilde{g}$ follows by Corollary 6.6 as we next show. Let v be an eigenvector for $\tilde{g}$. Then,

$$|\lambda v| \leq |gv| + |(\partial\overline{\partial}\phi)g^{-1}gv| \leq C_K|v|(1 + C\sqrt{|a|}),$$

similarly for the lower bound.

To estimate the eigenvalues also near the exceptional divisor, we take a careful look at the Eguchi–Hanson metric. The Eguchi–Hanson metric on $\mathbb{C}^2\backslash\{0\}$ reads

$$g_{\mathrm{EH}} = \sqrt{1 + \frac{a^2}{u^2}}\left(\mathbb{1} - \frac{a^2}{a^2+u^2}\frac{\overline{z}\otimes z}{u}\right),$$

[10] If z, w are coordinates on V_i, then $\eta = dz \wedge dw$.

where $u = |z|^2$ is the Euclidean distance squared and we are writing a instead of a_i. The eigenvalues in these coordinates are $\frac{\sqrt{a^2+u^2}}{u}$ and $\frac{u}{\sqrt{u^2+a^2}}$, hence are not bounded. We therefore need to choose different coordinates. The metric extends to a complete metric on the total space of the cotangent bundle of $\mathbb{CP}^1$, $\mathcal{O}_{\mathbb{CP}^1}(-2)$, and we will first use coordinate patches on this total space. Recall that

$$\mathcal{O}_{\mathbb{CP}^1}(-2) = \left\{((z,w),(\xi:\varsigma)) \mid z\varsigma^2 = w\xi^2\right\} \subset \mathbb{C}^2 \times \mathbb{CP}^1.$$

Working on the coordinate chart $\{\xi \neq 0\}$ and writing $\zeta := \frac{\varsigma}{\xi}$, we have $w = \zeta^2 z$. On this coordinate patch, we introduce the map

$$f_1 \colon \mathcal{O}_{\mathbb{CP}^1}(-2) \cap \{\xi \neq 0\} \to \mathbb{C}^2/\mu_2$$

$$f_1((z,\zeta^2 z),(1:\zeta)) = \left[(\sqrt{z}, \zeta\sqrt{z})\right].$$

Here, the brackets on the right-hand side mean the μ_2-orbit. This map is well defined. The map f_2 is similarly defined on the set $\varsigma \neq 0$, and $f_1 = f_2$ on the overlap $\{\xi \neq 0 \neq \varsigma\}$. Hence, there is a blow-down map $f \colon \mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{C}^2/\mu_2$. See [52] for more details. Pulling back the Eguchi–Hanson line element using f_1 then yields

$$\begin{aligned} ds^2_{\mathrm{EH}} = \frac{1}{\sqrt{a^2+u^2}}\Bigg(&\frac{1}{4}(1+|\zeta|^2)^2|dz|^2 + \frac{a^2+(1+|\zeta|^2)u^2}{(1+|\zeta|^2)^2}|d\zeta|^2 \\ &+(1+|\zeta|^2)\mathrm{Re}(\bar{z}\zeta\, dz\, d\bar{\zeta})\Bigg), \end{aligned} \tag{6.24}$$

where $u = |z|(1+|\zeta|^2)$. Completely analogous expressions will be found on the other set $\{\varsigma \neq 0\}$, so we do not write these out.

This removes the divergence in the metric at $z = 0 \iff u = 0$. Indeed, we simply have

$$ds^2_{EH,z=0} = \frac{(1+|\zeta|^2)^2}{4a}|dz|^2 + \frac{a}{(1+|\zeta|^2)^2}|d\zeta|^2.$$

The eigenvalues of this metric are clearly uniformly bounded by $\frac{C}{a}$ and Ca on the set $|\zeta| \leq 1$ and $u \leq 1$. For $|\zeta| \geq 1$, we make one final change of coordinates, writing

$$z = \frac{y}{\zeta^2} \quad \& \quad \zeta = -\frac{1}{\upsilon}.$$

In these coordinates, still at $u = 0$, we find

$$ds^2_{\mathrm{EH}} = \frac{(1+|\upsilon|^2)^2}{4a}|dy|^2 + \frac{a}{(1+|\upsilon|^2)^2}|d\upsilon|^2.$$

These components are again uniformly bounded by Ca and $\frac{C}{a}$ for all $|\upsilon| \leq 1$. These estimates were for a single component of the exceptional divisor with parameter a_i, so the upper bound has to be modified to $C\frac{1}{a_i} \leq C\frac{r_a}{|a|}$ and the lower bound to $C|a| \leq \frac{\max_i a_i}{\min_i a_i}\min_i a_i \leq r_a a_i$. Combined with the estimates away from E, we the statement about the eigenvalues of g follow. The bounds on $\tilde{g}$ follow from Corollary 6.6 exactly as before.

To see that the above coordinates are (almost) holomorphic Darboux coordinates, we look at what happens to the holomorphic volume form $\eta = dz_1 \wedge dz_2$ under these coordinate transformations. With the above coordinates, $z_1 = \sqrt{z}$, $z_2 = \zeta\sqrt{z}$, $z = \frac{y}{\zeta^2}$ and $\zeta = -\frac{1}{\upsilon}$, we find

$$2dz_1 \wedge dz_2 = dz \wedge d\zeta = dy \wedge d\upsilon.$$

So simply multiplying z and y by a factor of 2 gives us holomorphic Darboux coordinates. □

The holomorphic Darboux coordinates are not suited for local analysis near the exceptional divisor since the ratio of the eigenvalues of g is unbounded as $a \to 0$. This can be remedied by rescaling the fibre coordinate z or y in the above proof.

Lemma 6.10 *Near a component E_i of the exceptional divisor, we may choose finitely many coordinate patches and coordinates (z_a, ζ) such that*

$$C^{-1}a_i \le g \le Ca_i$$

and

$$\eta = a_i \cdot dz_a \wedge d\zeta$$

*in these coordinates. We shall refer to the above coordinates as **rescaled holomorphic Darboux coordinates**.*

Proof We return to the expression (6.24). By a change of scale, $z_a := \frac{z}{2a}$ and $u_a := 2|z_a|(1 + |\zeta|^2)$, (6.24) becomes

$$ds^2_{\mathrm{EH}} = \frac{a}{\sqrt{1+u_a^2}}\Bigg((1+|\zeta|^2)^2|dz_a|^2 + \frac{\left(1+u_a^2(1+|\zeta|^2)\right)}{(1+|\zeta|^2)^2}|d\zeta|^2 + 4(1+|\zeta|^2)\mathrm{Re}\left(\overline{z_a}\zeta\, dz_a d\overline{\zeta}\right)\Bigg). \tag{6.25}$$

This expression has eigenvalues bounded by Ca and $C^{-1}a$ for all $u \le a$ (i.e. $u_a \le 1$) and $|\zeta| \le 1$. For $|\zeta| \ge 1$, we use $y_a := \zeta^2 z_a$ and $\upsilon = -1/\zeta$ as before to get the same bounds Ca and $C^{-1}a$. □

We will from now on be working locally on an open subset $V_j \subset X$. By Lemmas 6.9 and 6.10, we may choose this open set to have (rescaled) holomorphic Darboux coordinates. In particular, V_j may be taken biholomorphic to a Euclidean ball B_{2R} in $\mathbb{C}^2$ centred on the origin. The Ricci-flat metric $\tilde{g}$ satisfied the Monge–Ampère equation, which in these coordinates simply reads

$$\det(\tilde{g}) = const.,$$

where the constant is A $(a_i^2 A)$. We may assume there exists a locally defined Kähler potential $\tilde{\phi}\colon V_j \to \mathbb{R}$ with $\partial\overline{\partial}\tilde{\phi} = \tilde{g}$. We will write $\tilde{\phi}$ instead of $\tilde{\phi}_j$ not to clutter the notation.
Step 2—A local Harnack inequality: The key analysis result will be the following.

Proposition 6.11 [64, p. 102] *Let g be a Kähler metric on B_{2R}, the ball of radius $2R$ centred on $0 \in \mathbb{C}^n$. Let $q > n$. Then, there exists a $p > 0$ and $C > 0$ such that if $g^{\overline{\nu}\mu}\partial_\mu\partial_{\overline{\nu}}v \le \theta$ and $v > 0$ on B_{2R}, then*

$$R^{-2n/p}\,\|v\|_{L^p(B_R,g)} \le C\left(\inf_{B_R} v + R^{2(q-n)/q}\,\|\theta\|_{L^q(B_{2R},g)}\right). \tag{6.26}$$

The constant C depends on n, $diam(B_r, g)$, $Vol(B_r, g)$ and the constant in the Sobolev inequality

$$\|f\|^2_{L^{2n/(n-1)}(B_{2R},g)} \leq C_{Sob}\left(\|\nabla f\|^2_{L^2(B_{2R},g)} + \|f\|^2_{L^2(B_{2R},g)}\right)$$

for all compactly supported f.

We refer to [64, pp. 107–112] for a proof.

Corollary 6.12 *Let X be a Kummer K3 surface with Ricci-flat metric $\tilde{g}$. Let $V_j \cong B_{2R}$ be a holomorphic Darboux coordinate patch. Then, the constant C in (6.26) can be chosen independently of a.*

Proof The volume bound follows from the Monge–Ampère equation directly, which prescribes the volume form of $\tilde{g}$. The diameter bound for g is argued in the proof of Proposition B.2 in Appendix B. The diameter bound for $\tilde{g}$ follows from this and the bound on $\partial\overline{\partial}\phi$, (6.23). The Sobolev constant can be controlled as long as one has upper and lower bounds on the volume and diameter and a lower bound on the Ricci curvature - see Theorem B.1 in Appendix B. The Ricci curvature vanishes for $\tilde{g}$; hence, we have a uniform Sobolev constant. □

Step 3—Harnack inequality for supersolutions: The Monge–Ampère equation reads

$$\det(\tilde{g}) = const.$$

in (rescaled) holomorphic Darboux coordinates. Let $\zeta \in \mathbb{C}^2$ with $|\zeta| = 1$ be arbitrary. Differentiating the logarithm of the equation and using the Jacobi formula yields

$$\mathrm{Tr}(\tilde{g}^{-1}\partial\overline{\partial}\tilde{\phi}_\zeta) = 0$$

and

$$\mathrm{Tr}(\tilde{g}^{-1}\partial\overline{\partial}\tilde{\phi}_{\zeta\overline{\zeta}}) - \mathrm{Tr}(\tilde{g}^{-1}\partial\overline{\partial}\tilde{\phi}_{\overline{\zeta}}\tilde{g}^{-1}\partial\overline{\partial}\tilde{\phi}_\zeta) = 0.$$

The first term on the left-hand side is per definition $\tilde{\Delta}\tilde{\phi}_{\zeta\overline{\zeta}}$. The second term is the tensor norm of $\partial\overline{\partial}\tilde{\phi}_\zeta$, hence can be dropped to give the inequality

$$\tilde{\Delta}\tilde{\phi}_{\zeta\overline{\zeta}} \geq 0.$$

This allows us to apply the Harnack inequality to the function

$$v_{sup} := \sup_{B_{2R}} \tilde{\phi}_{\zeta\overline{\zeta}} - \tilde{\phi}_{\zeta\overline{\zeta}} \geq 0$$

to deduce

$$R^{-4/p}\left\|v_{sup}\right\|_{L^p(B_R,\tilde{g})} \leq C \inf_{B_R} v_{sup}. \tag{6.27}$$

The constant C on the right-hand side can be chosen to be independent of a, as discussed in step 2.

Step 4—Harnack inequality for subsolutions: We need two linear algebra results. The first goes as follows.

Lemma 6.13 [29, Lemme 1] *Let $\mathcal{H}_+$ denote all $n \times n$ Hermitian matrices with positive eigenvalues. Let $A \in \mathcal{H}_+$. Then,*

$$\det(A)^{1/n} = \frac{1}{n}\inf\{tr(AB) \mid B \in \mathcal{H}_+, \ \det(B) = 1\}.$$

Let $x, y \in B_{2R}$. We will specify x later, and y will be integrated. Let $B = \kappa \tilde{g}^{-1}(y)$, where $\kappa = \sqrt{A}$ ($\kappa = a_i \sqrt{A}$) when we are in (rescaled) holomorphic Darboux coordinates. Then, the Monge–Ampère equation says $\det(B) = 1$, and the lemma, implies

$$\sqrt{\kappa} = \sqrt{\det(\tilde{g}(x))} \leq \frac{1}{2}\mathrm{tr}(B\tilde{g}(x)).$$

On the other hand, we trivially have

$$\mathrm{tr}(B\tilde{g}(y)) = 2\sqrt{\kappa},$$

so

$$\mathrm{tr}(B(\tilde{g}(y) - \tilde{g}(x)) \leq 0. \tag{6.28}$$

To proceed, we need the second linear algebra result.

Lemma 6.14 [64, p. 103], [12, Lemma 5.17] *For $0 < \lambda < \Lambda < \infty$, let $S(\lambda, \Lambda)$ denote the set of Hermitian $n \times n$-matrices with eigenvalues in the interval $[\lambda, \Lambda]$. Then, one can find unit vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{C}^n$ and $0 < \lambda_* < \Lambda_* < \infty$ depending only on n, λ, and Λ such that every $H \in S(\lambda, \Lambda)$ can be written*

$$H = \sum_{k=1}^{N} \beta_k \zeta_k \otimes \overline{\zeta_k}$$

with $\beta_k \in [\lambda_*, \Lambda_*]$.

Remark 6.15 The proof yields $\lambda_* < \lambda/N$ and $\Lambda_* > \Lambda$, but can otherwise be chosen arbitrarily. We may also assume that the finite set of vectors contains an orthonormal basis.

With this at hand, we find locally defined functions β_k with $\lambda_* \leq \beta_k \leq \Lambda_*$ such that

$$B = \sqrt{\kappa}\tilde{g}^{-1}(y) = \sum_{k=1}^{N} \beta_k(y)\zeta_k \otimes \overline{\zeta_k}.$$

Hence,

$$\mathrm{tr}(B(\tilde{g}(y) - \tilde{g}(x)) = \sum_{k=1}^{N} \beta_k(y)\left(\tilde{\phi}_{\zeta_k\overline{\zeta_k}}(y) - \tilde{\phi}_{\zeta_k\overline{\zeta_k}}(x)\right).$$

Let

$$M_{k,R} := \sup_{B_R} \tilde{\phi}_{\zeta_k\overline{\zeta_k}}$$
$$m_{k,R} := \inf_{B_R} \tilde{\phi}_{\zeta_k\overline{\zeta_k}},$$

and introduce the oscillation

$$\mathrm{osc}(R) := \sum_{k=1}^{N}(M_{k,R} - m_{k,R}).$$

We also introduce the shorthand

$$w_k := \tilde{\phi}_{\zeta_k\overline{\zeta_k}}.$$

Let $\ell \in \{1, \ldots, N\}$ be arbitrary. Then, the Harnack inequality tells us

$$\begin{aligned} R^{-4/p} \left\| \sum_{k \neq \ell} \left(M_{k,2R} - w_k \right) \right\|_{L^p(B_R)} &\le R^{-4/p} \sum_{k \neq \ell} \left\| M_{k,2R} - w_k \right\|_{L^p(B_R)} \\ &\le C \left(\sum_{k \neq \ell} (M_{k,2R} - M_{k,R}) \right). \end{aligned}$$

The last sum can be estimated a bit. Since $M_{k,2R} - M_{k,R} \ge 0$, we can include the term $k = \ell$ on the right-hand side. We further have

$$M_{k,2R} - M_{k,R} \le M_{k,2R} - M_{k,R} + (m_{k,R} - m_{k,2R}) = (M_{k,2R} - m_{k,2R}) - (M_{k,R} - m_{k,R}).$$

So

$$R^{-4/p} \left\| \sum_{k \neq \ell} \left(M_{k,2R} - w_k \right) \right\|_{L^p(B_R)} \le C(\operatorname{osc}(2R) - \operatorname{osc}(R)). \tag{6.29}$$

From (6.28) and $\lambda_* \le \beta_k \le \Lambda_*$, we find

$$\lambda_* |w_\ell(y) - w_\ell(x)| \le \Lambda_* \sum_{k \neq \ell} |M_{k,2R} - w_k(y)|.$$

Taking averaged L^p-norms here and using (6.29) gives us

$$R^{-4/p} \left\| w_\ell - w_\ell(x) \right\|_{L^p(B_R)} \le \frac{\Lambda_*}{\lambda_*} C(\operatorname{osc}(2R) - \operatorname{osc}(R)),$$

where all the integrals are with respect to y. For any $\epsilon > 0$, we can find $x \in B_{2R}$ such that $w_\ell(x) = m_{\ell,2R} + \epsilon$. Using this on the left-hand side, we deduce

$$R^{-4/p} \left\| w_\ell - m_{\ell,2R} \right\|_{L^p(B_R)} \le \frac{\Lambda_*}{\lambda_*} C(\operatorname{osc}(2R) - \operatorname{osc}(R)) + \operatorname{Vol}(B_1)\epsilon.$$

Since ϵ was arbitrary, we can send it to 0 and deduce

$$R^{-4/p} \left\| w_\ell - m_{\ell,2R} \right\|_{L^p(B_R)} \le \frac{\Lambda_*}{\lambda_*} C(\operatorname{osc}(2R) - \operatorname{osc}(R)) \tag{6.30}$$

for any $\ell \in \{1, \ldots, N\}$.

Step 5—Combining both Harnack estimates: Let $\ell \in \{1, \ldots, N\}$. Then, we have

$$\begin{aligned} \operatorname{Vol}(B_R)^{1/p} (M_{\ell,2R} - m_{\ell,2R}) &= \left\| M_{\ell,2R} - m_{\ell,2R} \right\|_{L^p(B_R)} \\ &\le \left\| w_\ell - m_{\ell,2R} \right\|_{L^p(B_R)} + \left\| M_{\ell,2R} - w_\ell \right\|_{L^p(B_R)}. \end{aligned}$$

Multiplying both sides by $R^{-4/p}$ and using both (6.27) and (6.30) yields

$$M_{\ell,2R} - m_{\ell,2R} \le C \frac{\Lambda_*}{\lambda_*} \left(\operatorname{osc}(2R) - \operatorname{osc}(R) \right).$$

Summing over ℓ gives us

$$\operatorname{osc}(R) \le \delta \operatorname{osc}(2R), \tag{6.31}$$

where $\delta := 1 - \frac{\lambda_*}{\Lambda_* C N}$.

The inequality (6.31) gives us the Hölder regularity by [34, Lemma 8.23]. Indeed, let $r < R$. Choose $m > 0$ so that

$$2^{-m}R \le r \le 2^{-m+1}R,$$

i.e.

$$m \ge \frac{\log\left(\frac{R}{r}\right)}{\log(2)}.$$

Then, (6.31) and the monotonicity of osc say

$$\operatorname{osc}(r) \le \operatorname{osc}(2^{-m+1}R) \le \delta^{m-1}\operatorname{osc}(R) \le \frac{1}{\delta}\cdot\delta^{\frac{\log\left(\frac{R}{r}\right)}{\log(2)}}\operatorname{osc}(R) = \frac{1}{\delta}\left(\frac{r}{R}\right)^{-\frac{\log(\delta)}{\log(2)}}\operatorname{osc}(R).$$

The oscillation $\operatorname{osc}(R)$ can be bounded by bounding $|\tilde{\phi}_{\zeta\overline{\zeta}}|$ for arbitrary ζ. For $\zeta \in \mathbb{C}^2$, there are $\mu, \nu \in \mathbb{C}$ such that $\zeta = \mu z + \nu w$, hence

$$\tilde{\phi}_{\zeta\overline{\zeta}} = |\mu|^2\tilde{\phi}_{z\overline{z}} + 2\mathrm{Re}(\mu\overline{\nu}\tilde{\phi}_{z\overline{w}}) + |\nu|^2\tilde{\phi}_{w\overline{w}} = \left\langle \begin{pmatrix}\mu\\ \nu\end{pmatrix}, \tilde{g}\begin{pmatrix}\mu\\ \nu\end{pmatrix}\right\rangle.$$

The right-hand side can be bounded by a constant (depending on ζ) and the largest eigenvalue of $\tilde{g}$. Hence, by Lemmas 6.9 and 6.10,

$$|\tilde{\phi}_{\zeta\overline{\zeta}}| \le C\Lambda,$$

and

$$\operatorname{osc}(r) \le C\Lambda\left(\frac{r}{R}\right)^{\alpha} \tag{6.32}$$

with

$$\alpha = -\frac{\log(1-\frac{\lambda_*}{C\Lambda_*})}{\log(2)}.$$

To finish, we have to bound α. We distinguish between being near and away from the exceptional divisor. Lemma 6.9 yields uniform bounds on the eigenvalues of $\tilde{g}$ away from the exceptional divisor. So $\frac{\lambda_*}{\Lambda_*}$ is uniformly bounded, and thus $\alpha < 1$ uniformly.

Near the exceptional divisor, the eigenvalues λ and Λ are poorly behaved in the coordinates (6.24), so we use the rescaled coordinates (6.25). In these coordinates, $\frac{\lambda}{\Lambda}$ is uniformly bounded by Lemma 6.10 as long as $u \le a_i$. This can be achieved by rescaling $R \mapsto \sqrt{a_i}R$. By Lemma 6.10 and Corollary 6.6, we have $\Lambda \le C\sqrt{|a|}$. Hence,

$$\operatorname{osc}(r) \le C\left(\frac{r}{R}\right)^{\alpha}|a|^{\frac{1-\alpha}{2}}$$

with $\alpha < 1$ uniformly. So we get Hölder estimates also near the exceptional divisor.
Step 6—Hölder bounds on g:

Lemma 6.16 *Let $\{V_i\}$ be the cover of (rescaled) holomorphic Darboux coordinate neighbourhoods of Lemmas 6.9 and 6.10. Let $\Phi_j : V_i \to \mathbb{R}$ be Kähler potentials for the patchwork metric g. Then, there are uniform constants $C > 0$ and $0 < \alpha \le 1$ such that*

$$\|\Phi_j\|_{C^{2,\alpha}(V_i,g)} \le C.$$

Proof For the Euclidean region, this is clear. On an annulus K around a component of the exceptional divisor E_i, the potential (2.1) can be written

$$f_{\text{EH}} = f_{\text{Euc}} + a_i^2 \xi_K$$

for some smooth function ξ_K which is regular as $a_i \to 0$. In the neck region, the patchwork potential can thus be written

$$\Phi = f_{\text{Euc}} + a_i^2 \chi \xi_K$$

where χ is a smooth cut-off function as described in Sect. 2. These expressions then give the required bound.

Near the exceptional divisor, one could compute the real Hessian of the Eguchi–Hanson potential (2.1) directly and compare. But it is probably easier to just repeat steps 3–5, since the potential Φ satisfies

$$\det(\partial\overline{\partial}\Phi) = const.$$

where the constant is 1 (a_i^2) in (rescaled) holomorphic Darboux coordinates. □

Step 7—Hölder bounds on ϕ: In a coordinate patch V_i, we write

$$\tilde{\phi} = \Phi_i + \phi,$$

, hence

$$|D^2\phi|_g^2 \le |D^2\tilde{\phi}|_g^2 + |D^2\Phi|_g^2 \le C|D^2\tilde{\phi}|_{\tilde{g}}^2 + C \le \tilde{C},$$

where we have used Corollary 6.6 to compare the g- and $\tilde{g}$-norm.

6.4 C^1 estimates

The C^1 estimates of ϕ follow from a general result in Riemannian geometry.

Lemma 6.17 *Let (M, g) be a compact, connected Riemannian manifold with diameter $d{:=}diam(M, g)$. Then, there exists a monotonically increasing function $\alpha\colon [0, \infty) \to \mathbb{R}$ such that*

$$\sup_{p\in M} |\nabla f|_g^2 \le \alpha(d) \sup_{p\in M} |D^2 f|_g^2 \tag{6.33}$$

holds for any $f \in C^2(M; \mathbb{R})$.

Proof The idea is simply to integrate the double derivative along a curve. Here are the details.

For an $f \in C^2(M; \mathbb{R})$, let $q \in M$ denote a critical point. This point exists due to the compactness of M. Let $p \in M$ be any point. For $\epsilon > 0$, let $\gamma\colon [0, T]$ denote a unit speed curve with $\gamma(0) = q$, $\gamma(T) = p$, and $T = L(\gamma) \le d(p, q) + \epsilon$. Let $\xi\colon [0, T] \to \mathbb{R}$ denote the function

$$\xi(t){:=}|\nabla f|_g^2(\gamma(t)).$$

This is differentiable, and

$$\xi'(t) \le 2|D^2 f \nabla f|_g \le |D^2 f|_g^2 + |\nabla f|_g^2,$$

where we have used the unit speed condition and the inequality between arithmetic and geometric mean. This differential inequality tells us

$$\frac{d}{dt} e^{-t} \xi(t) \leq e^{-t} |D^2 f|_g^2(\gamma(t)),$$

which integrates to

$$e^{-T} \xi(T) \leq \int_0^T e^{-t} |D^2 f|_g^2(\gamma(t))\, dt \leq T \sup_{p \in M} |D^2 f|_g^2(p).$$

This then yields

$$\xi(T) \leq (d+\epsilon) e^{d+\epsilon} \sup_{p \in M} |D^2 f|_g^2(p).$$

Doing the same construction for other points tells us

$$\sup_{p \in M} |\nabla f|_g^2(p) \leq (d+\epsilon) e^{d+\epsilon} \sup_{p \in M} |D^2 f|_g^2(p).$$

Letting $\epsilon \to 0$ gives the required bound with $\alpha(d) = de^d$. □

Remark 6.18 The above-constructed α is not sharp. Indeed, if there are no critical points for f between $\gamma(0) = q$ and $\gamma(T) = p$, then the function $\xi(t) = |\nabla f|_g(\gamma(t))$ is differentiable, and repeating the above argument yields $\xi'(t) \leq |D^2 f|_g(\gamma(t))$, hence

$$\sup_{p \in M} |\nabla f|_g \leq d \sup_{p \in M} |D^2 f|_g.$$

Since Morse functions are dense in the C^2-topology, we can approximate an arbitrary function with a function with isolated critical points. By shifting the curve γ a bit, we can make sure there are no critical points between q and p, and use the above estimate. All in all one finds $\alpha(d) = d^2$ as a better constant.

6.5 Higher-order estimates

Kobayashi's approach to proving the higher-order derivatives is worth sketching here. For $t \in [0, 1]$, consider the equation

$$\left(\omega + i\partial\overline{\partial}\phi_t\right)^2 = \left(1 + t\left(Ae^{\psi} - 1\right)\right)\omega^2 \tag{6.34}$$

subject to

$$\int_X \phi_t\, \omega^2 = 0.$$

Here A, ψ and ω are as before. This is what one considers for the continuity method to show that the Monge–Ampère equation has a solution. Computing the t-derivative of (6.34) yields

$$\tilde{\Delta}_t \left(\frac{\partial \phi_t}{\partial t}\right) = \frac{Ae^{\psi} - 1}{1 + t(Ae^{\psi} - 1)}, \tag{6.35}$$

where we have introduced

$$\tilde{\Delta}_t := \mathrm{Tr}\left(\left(g + \partial\overline{\partial}\phi_t\right)^{-1} \partial\overline{\partial}\right).$$

Then, (2.8) and (6.2) tell us that the right-hand side is uniformly bounded from above and below,

$$\left| \frac{Ae^{\psi} - 1}{1 + t(Ae^{\psi} - 1)} \right| \leq C|a|^2.$$

Indeed, writing $\varpi := \frac{Ae^{\psi}-1}{1+t(Ae^{\psi}-1)}$, we have

$$\|\varpi\|_{C^k(X,g)} \leq C_k|a|^2$$

for all $k \geq 0$. Repeating the proofs[11] of the $C^{2.\alpha}$-bounds for ϕ_t instead of ϕ gives us uniform Hölder bounds on $\partial\overline{\partial}\phi_t$. Staying away from the exceptional divisor, we can write $g = g_{\mathrm{Euc}} + \mathcal{O}(|a|^2)$. So (6.35) is a Poisson equation with Hölder continuous coefficients and Hölder continuous right-hand side. So Schauder estimates, [34, Theorem 6.2] from elliptic theory gives local $C^{k,\alpha}$-bounds on $\partial_t\phi_t$, hence also on $\phi = \phi_1$ by integrating and using $\phi_0 = 0$.

Near the exceptional divisor, one has to use coordinates like (6.24) or (6.25). The last coordinates require a rescaling, hence the drop in powers of a in [43, equations 47, 48].

6.6 Some alternative proof strategies

There are by now several other routes one could attempt do deduce Kobayashi's estimates. We would like to mention some which were suggested by the referee as interesting venues for further research. The first would be to use Donaldson's bound on the correction in $W^{5,2}(X, g)$, [21]. One could try to estimate the Sobolev constant in the embedding $W^{5,2}(X, g) \to C^{2,\alpha}(X, g)$, which would allow one to circumvent most of Kobayashi's arguments. This Sobolev constant will probably not be uniformly bounded since the sectional curvature and injectivity radius of (X, g) are not uniformly controlled, but if one can get a good enough control of it in terms of either of these two quantities, this should suffice. The second route would be to try to use estimates from the Joyce construction. If one starts out with (X, g) as the almost Ricci-flat Kummer K3 of the paper and consider the product $X \times \mathbb{T}^3$, then one can write down an almost G_2 structure on this. After perturbing using [41, Theorem 11.6.1], one gets an actual G_2-structure. This is a Ricci-flat metric on $X \times \mathbb{T}^3$, hence is an isometric product due to the Cheeger splitting theorem. The G_2-perturbation of Joyce has thus perturbed the patchwork metric into a Ricci-flat metric (circumventing the Monge–Ampère equation). The precise estimates in Joyce on this perturbation are, however, too weak for the purposes of this paper. If one could strengthen these estimates like in the recent work by Daniel Platt [61], then one should be able to deduce estimates like Kobayashi's using the Joyce construction. A third idea would be to build on more general theory of desingularizing Einstein orbifolds like the recent work by Ozuch [58, Theorem 4.6]. They deal with the problem of the injectivity radius being uncontrolled by introducing weights in their function spaces which counteract this. It would still entail some work to deduce Kobayashi's unweighted estimates from this more general framework, but it should be doable.

[11] There is a slight added difficulty with the Hölder bound. The Monge–Ampère equation for ϕ_t in holomorphic Darboux coordinates reads $\det(g + \partial\overline{\partial}\phi_t) = (1-t)e^{-\psi} + tA$, so the right-hand side is not constant in the neck region when $t \neq 1$. The same proof goes through, however, with some harmless additional terms appearing in (6.31). We refer to [64, pp. 100–107] for the necessary modifications.

7 Discussion and outlook

7.1 Flatness of the tori

In the author's unpublished PhD thesis [51], there is an argument that a certain special Lagrangian torus L, which can be found by similar arguments as in Theorem 5.4 using an anti-holomorphic isometry of rank 2 of is flat. The argument was that a special Lagrangian submanifold has a prescribed volume form

$$\mathrm{Vol}_L = \sqrt{A}\mathrm{Re}(\eta)_{|L} = \sqrt{A}g(\cdot, J_0\cdot),$$

where J_0 is a complex structure on the flat torus T which is orthogonal to I and A is the constant in (2.8). But L is Kähler with the Kähler form $\mathrm{Vol}_L = \omega_J = \tilde{g}(\cdot, J\cdot)$. Hence,

$$\tilde{g}(\cdot, J\cdot) = \sqrt{A}g(\cdot, J_0\cdot).$$

Using the isometries again [51], argues $J_{|L} = J_0$, hence $\tilde{g}_{|L} = g_{|L}$ and the torus is flat. The proof of $J_{|L} = J_0$ is sadly wrong, however.

7.2 Reducing the amount of symmetry

Theorem 5.4 only works for very special tori. A natural questions is what happens when one perturbs the underlying lattice Γ or the sizes a_i. We do now know the answer to this, but would remark that closed geodesics need not behave nicely under metric perturbations. The oldest example of this is due to Morse [55, Chapter IX, Theorem 4.1], where he studies ellipsoids which are almost spherical. What he finds is that there are closed geodesics which become infinitely long when the ellipsoid is deformed to a round sphere. There is a modern proof in [42], and generalizations by Ballmann [4] and Bangert [6]. What this tells us is that one cannot in general expect the closed geodesics of a perturbed metric to be perturbations of the original closed geodesics.

We do now know if the situation is improved when considering hyperkähler deformations, and this is a question we hope to return to in the future.

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Appendix A: Laplacian of the Riemann tensor

Here, we present part of the computation underlying Proposition 4.11. We start with a couple of lemmas.

Lemma A.1 *Assume $(X, \tilde{g})$ is a hyperkähler 4-manifold. Let V be any (locally defined) tangent vector field. Then,*

$$\begin{aligned}\langle \Delta R(V, IV)IV, V\rangle = {} & \langle [\nabla_{JV}, \nabla_V]R(V, KV)IV, V\rangle + \langle [\nabla_{JV}, \nabla_{IV}]R(V, JV)IV, V\rangle \\ & + \langle [\nabla_V, \nabla_{KV}]R(V, JV)IV, V\rangle \\ & + \langle [\nabla_{KV}, \nabla_{IV}]R(V, KV)IV, V\rangle\,, \end{aligned} \tag{A.1}$$

where

$$\Delta R = (\nabla_V^2 + \nabla_{IV}^2 + \nabla_{JV}^2 + \nabla_{KV}^2)R$$

is the Laplacian acting on 4-tensors and all the commutators on the right-hand side are acting on the 4-tensor R.

Proof This follows from using the second Bianchi identity twice along with some hyperkähler identities. By the second Bianchi identity and $IJ = K$, we find

$$\nabla_{JV} R(V, IV) = -\nabla_V R(IV, JV) - \nabla_{IV} R(JV, V) = \nabla_V R(V, KV) + \nabla_{IV} R(V, JV).$$

Hence,

$$\begin{aligned}\nabla_{JV}^2 R(V, IV) &= \nabla_{JV}\nabla_V R(V, KV) + \nabla_{JV}\nabla_{IV} R(V, JV) \\ &= [\nabla_{JV}, \nabla_V]R(V, KV) + \nabla_V\nabla_{JV} R(V, KV) \\ &\quad + [\nabla_{JV}, \nabla_{IV}]R(V, JV) + \nabla_{IV}\nabla_{JV} R(V, JV). \end{aligned} \tag{A.2}$$

To the second term, we apply the second Bianchi identity again to get

$$\begin{aligned}\nabla_V\nabla_{JV} R(V, KV) &= -\nabla_V^2 R(KV, JV) - \nabla_V\nabla_{KV} R(JV, V) \\ &= -\nabla_V^2 R(V, IV) + \nabla_V\nabla_{KV} R(V, JV). \end{aligned}$$

The fourth term we handle similarly, using the second Bianchi identity in the last two components to write

$$\begin{aligned}\langle \nabla_{IV}\nabla_{JV} R(V, JV)IV, V\rangle &= -\left\langle \nabla_{IV}^2 R(V, JV)V, JV\right\rangle - \langle \nabla_{IV}\nabla_V R(V, JV)JV, IV\rangle \\ &= \left\langle \nabla_{IV}^2 R(V, JV)JV, V\right\rangle + \langle \nabla_{IV}\nabla_V R(V, JV)KV, V\rangle\,. \end{aligned}$$

Hence, we have

$$\begin{aligned}\left\langle \nabla_{JV}^2 R(V, IV)IV, V\right\rangle = {} & \langle [\nabla_{JV}, \nabla_V]R(V, KV)IV, V\rangle \\ & + \langle [\nabla_{JV}, \nabla_{IV}]R(V, JV)IV, V\rangle \\ & - \left\langle \nabla_V^2 R(V, IV)IV, V\right\rangle + \langle \nabla_V\nabla_{KV} R(V, JV)IV, V\rangle \\ & + \left\langle \nabla_{IV}^2 R(V, JV)JV, V\right\rangle + \langle \nabla_{IV}\nabla_V R(V, JV)KV, V\rangle\,. \end{aligned}$$

Performing the same computation with KV instead of JV, we find

$$\begin{aligned}\left\langle\nabla^2_{KV}R(V,IV)IV,V\right\rangle =& \langle[\nabla_V,\nabla_{KV}]R(V,JV)IV,V\rangle\\ &+\langle[\nabla_{KV},\nabla_{IV}]R(V,KV)IV,V\rangle\\ &-\left\langle\nabla^2_V R(V,IV)IV,V\right\rangle-\langle\nabla_V\nabla_{JV}R(V,KV)IV,V\rangle\\ &+\left\langle\nabla^2_{IV}R(V,KV)KV,V\right\rangle-\langle\nabla_{IV}\nabla_V R(V,KV)JV,V\rangle.\end{aligned}$$

Using the Bianchi identity again, we rewrite

$$-\langle\nabla_V\nabla_{JV}R(V,KV)IV,V\rangle=\left\langle\nabla^2_V R(KV,JV)IV,V\right\rangle+\langle\nabla_V\nabla_{KV}R(JV,V)IV,V\rangle,$$

so

$$\begin{aligned}\left\langle\nabla^2_{KV}R(V,IV)IV,V\right\rangle =& \langle[\nabla_V,\nabla_{KV}]R(V,JV)IV,V\rangle\\ &+\langle[\nabla_{KV},\nabla_{IV}]R(V,KV)IV,V\rangle\\ &+\langle\nabla_V\nabla_{KV}R(JV,V)IV,V\rangle\\ &+\left\langle\nabla^2_{IV}R(V,KV)KV,V\right\rangle-\langle\nabla_{IV}\nabla_V R(V,KV)JV,V\rangle.\end{aligned}$$

Adding these two expressions and using the Ricci-flatness (equivalently, the first Bianchi identity) to write

$$\left\langle\nabla^2_{IV}R(V,JV)JV,V\right\rangle+\left\langle\nabla^2_{IV}R(V,KV)KV,V\right\rangle=-\left\langle\nabla^2_{IV}R(V,IV)IV,V\right\rangle,$$

we arrive at

$$\begin{aligned}\langle\Delta R(V,IV)IV,V\rangle =& \langle[\nabla_{JV},\nabla_V]R(V,KV)IV,V\rangle+\langle[\nabla_{JV},\nabla_{IV}]R(V,JV)IV,V\rangle\\ &+\langle[\nabla_V,\nabla_{KV}]R(V,JV)IV,V\rangle+\langle[\nabla_{KV},\nabla_{IV}]R(V,KV)IV,V\rangle\end{aligned}$$

as announced. □

Lemma A.2 *Assume $(X,\tilde g)$ is a hyperkähler 4-manifold. Let U, V, W be any tangent vector fields and write $(I_1,I_2,I_3)=(I,J,K)$ for the complex structures satisfying $IJ=K$. Then,*

$$\begin{aligned}\left\langle[\nabla_W,\nabla_U]R(V,I_iV)I_jV,V\right\rangle =& \left\langle\nabla_{[W,U]}R(V,I_iV)I_jV,V\right\rangle\\ &-2\left\langle R(R(W,U)V,I_iV)I_jV,V\right\rangle\\ &-2\left\langle R(V,I_iV)I_jV,R(W,U)V\right\rangle\end{aligned}\qquad(A.3)$$

Proof This follows from the definitions and is quite standard. We start by computing

$$\begin{aligned}\nabla_U\left\langle R(V,I_iV)I_jV,V\right\rangle =& \left\langle\nabla_U R(V,I_iV)I_jV,V\right\rangle\\ &+2\left\langle R(\nabla_UV,I_iV)I_jV,V\right\rangle\\ &+2\left\langle R(V,I_iV)I_jV,\nabla_UV\right\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla_W \nabla_U \left\langle R(V, I_i V) I_j V, V\right\rangle = &\left\langle \nabla_W \nabla_U R(V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle \nabla_U R(\nabla_W V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle \nabla_U R(V, I_i V) I_j V, \nabla_W V\right\rangle \\ &+ 2\left\langle \nabla_W R(\nabla_U V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R(\nabla_W \nabla_U V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R(\nabla_U V, I_i \nabla_W V) I_j V, V\right\rangle \\ &+ 4\left\langle R(\nabla_U V, I_i V) I_j V, \nabla_W V\right\rangle \\ &+ 2\left\langle \nabla_W R(V, I_i V) I_j V, \nabla_U V\right\rangle \\ &+ 4\left\langle R(\nabla_W V, I_i V) I_j V, \nabla_U V\right\rangle \\ &+ 2\left\langle R(V, I_i V) I_j \nabla_W V, \nabla_U V\right\rangle \\ &+ 2\left\langle R(V, I_i V) I_j V, \nabla_W \nabla_U V\right\rangle .\end{aligned}$$

Subtracting the same expression with U and W swapped, we get

$$\begin{aligned}\nabla_{[W,U]} \left\langle R(V, I_i V) I_j V, V\right\rangle = &\left\langle [\nabla_W, \nabla_U] R(V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R([\nabla_W, \nabla_U] V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R(V, I_i V) I_j V, [\nabla_W, \nabla_U] V\right\rangle .\end{aligned}$$

The left-hand side can be written as

$$\begin{aligned}\nabla_{[W,U]} \left\langle R(V, I_i V) I_j V, V\right\rangle = &\left\langle \nabla_{[W,U]} R(V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R(\nabla_{[W,U]} V, I_i V) I_j V, V\right\rangle \\ &+ 2\left\langle R(V, I_i V) I_j V, \nabla_{[W,U]} V\right\rangle .\end{aligned}$$

On the right-hand side, we use the definition of the curvature tensor to write

$$[\nabla_W, \nabla_U] V = R(W, U) V + \nabla_{[W,U]} V.$$

This yields the claimed formula. □

From now on, we assume V is a (locally defined) unit vector field. We recall that $\sigma_{IJ} = \sigma_{IJ}(V)$, etc., are defined so that

$$R(V, IV)V = -\sigma_{II} IV - \sigma_{IJ} JV - \sigma_{IK} KV$$

and so on for $R(V, JV)$ and $R(V, KV)$. Using this and the previous two lemmas, we arrive at

Proposition A.3 *Assume $(X, \tilde{g})$ is a hyperkähler 4-manifold. Let V be any locally defined unit tangent vector field. Then,*

$$\begin{aligned}\langle \Delta R(V, IV) IV, V\rangle = &\left\langle \nabla_{[JV, IV]} R(V, JV) IV, V\right\rangle \\ &+ \left\langle \nabla_{[JV, V]} R(V, KV) IV, V\right\rangle \\ &+ \left\langle \nabla_{[V, KV]} R(V, JV) IV, V\right\rangle \\ &+ \left\langle \nabla_{[KV, IV]} R(V, KV) IV, V\right\rangle \\ &- 4(\sigma_{II}^2 + 2\sigma_{JJ}\sigma_{KK} + \sigma_{IJ}^2 + \sigma_{IK}^2 - 2\sigma_{JK}^2) \qquad \text{(A.4)}\end{aligned}$$

Proof Lemma A.2 says

$$\begin{aligned}\langle [\nabla_{JV}, \nabla_V] R(V, KV)IV, V\rangle = &\left\langle \nabla_{[JV,V]} R(V, KV)IV, V\right\rangle \\ &- 2\,\langle R(R(JV, V)V, KV)IV, V\rangle \\ &- 2\,\langle R(V, KV)IV, R(JV, V)V\rangle\,.\end{aligned}$$

Using $R(JV, V)V = \sigma_{IJ} IV + \sigma_{JJ} JV + \sigma_{JK} KV$, we find

$$\begin{aligned}\langle R(R(JV, V)V, KV)IV, V\rangle &= \sigma_{IJ}\,\langle R(IV, KV)IV, V\rangle + \sigma_{JJ}\,\langle R(JV, KV)IV, V\rangle \\ &= \sigma_{IJ}^2 - \sigma_{II}\sigma_{JJ}.\end{aligned}$$

and

$$\begin{aligned}\langle R(V, KV)IV, R(JV, V)V\rangle &= \sigma_{JJ}\,\langle R(V, KV)IV, JV\rangle + \sigma_{JK}\,\langle R(V, KV)IV, KV\rangle \\ &= \sigma_{JJ}\sigma_{KK} - \sigma_{JK}^2.\end{aligned}$$

Hence,

$$\begin{aligned}\langle [\nabla_{JV}, \nabla_V] R(V, KV)IV, V\rangle = &\left\langle \nabla_{[JV,V]} R(V, KV)IV, V\right\rangle \\ &+ 2\left(\sigma_{JJ}\sigma_{II} + \sigma_{JK}^2 - \sigma_{JJ}\sigma_{KK} - \sigma_{IJ}^2\right).\end{aligned}$$

Similar computations yield

$$\begin{aligned}\langle [\nabla_{JV}, \nabla_{IV}] R(V, JV)IV, V\rangle = &\left\langle \nabla_{[JV,IV]} R(V, JV)IV, V\right\rangle \\ &+ 2\left(\sigma_{II}\sigma_{KK} + \sigma_{JK}^2 - \sigma_{JJ}\sigma_{KK} - \sigma_{IK}^2\right), \\ \langle [\nabla_{V}, \nabla_{KV}] R(V, JV)IV, V\rangle = &\left\langle \nabla_{[V,KV]} R(V, JV)IV, V\right\rangle \\ &+ 2\left(\sigma_{II}\sigma_{KK} + \sigma_{JK}^2 - \sigma_{JJ}\sigma_{KK} - \sigma_{IK}^2\right),\end{aligned}$$

and

$$\begin{aligned}\langle [\nabla_{KV}, \nabla_{IV}] R(V, KV)IV, V\rangle = &\left\langle \nabla_{[KV,IV]} R(V, KV)IV, V\right\rangle \\ &+ 2\left(\sigma_{JJ}\sigma_{II} + \sigma_{JK}^2 - \sigma_{JJ}\sigma_{KK} - \sigma_{IJ}^2\right).\end{aligned}$$

Adding up these four contributions, writing $\sigma_{KK} + \sigma_{JJ} = -\sigma_{II}$, and using Lemma A.1 gives (A.4). □

Along the fixed point set M of an order 4 holomorphic isometry $f\colon X \to X$, we may simplify greatly.

Proposition A.4 *Assume the set-up of Proposition 4.11. Let* $\alpha := \langle IV, \nabla_V V\rangle$ *and* $\beta := \langle IV, \nabla_{IV} V\rangle$. *Then,*

$$\langle \Delta R(V, IV)IV, V\rangle = (\alpha + \beta)\,\nabla_V \sigma_{II}(V) - \alpha \nabla_{IV}\sigma_{II}(V) - 6\sigma_{II}^2 \tag{A.5}$$

holds for any point on M.

Proof Let $f\colon X \to X$ denote the isometry with M as fixed point set. Due to $f_*(JV) = \pm KV$, we find

$$\begin{aligned}\langle \Delta R(V, IV)IV, V\rangle = &\, 2\left\langle \nabla_{[JV,IV]} R(V, JV)IV, V\right\rangle \\ &+ 2\left\langle \nabla_{[JV,V]} R(V, KV)IV, V\right\rangle \\ &- 6\sigma_{II}^2 \end{aligned}\tag{A.6}$$

along M. So we analyse the commutators. Since V is unit speed, we have $\langle V, \nabla_W V\rangle = 0$ for any W. Since f_* fixes V, IV and rotates JV, KV, there have to be functions $\alpha, \beta, \mu, \nu \colon M \to \mathbb{R}$ such that

$$\begin{aligned}\nabla_V V &= \alpha IV\\ \nabla_{IV} V &= \beta IV,\\ \nabla_{JV} V &= \mu JV + \nu KV,\\ \nabla_{KV} V &= \mu KV - \nu JV.\end{aligned}$$

Using the formula $[W, U] = \nabla_W U - \nabla_U W$, we thus arrive at

$$\begin{aligned}[JV, IV] &= -\nu JV + (\mu + \beta)KV,\\ [JV, V] &= \mu JV + (\alpha + \nu)KV.\end{aligned}$$

The second Bianchi identity tells us

$$\langle \nabla_{JV} R(V, JV)IV, V\rangle = \langle \nabla_{IV} R(V, JV)JV, V\rangle - \langle \nabla_V R(V, JV)JV, V\rangle$$

and

$$\langle \nabla_{KV} R(V, JV)IV, V\rangle = \langle \nabla_V R(V, IV)IV, V\rangle + \langle \nabla_{JV} R(V, KV)IV, V\rangle .$$

Applying the isometry f to these further gives

$$\langle \nabla_{JV} R(V, JV)IV, V\rangle = \langle \nabla_{KV} R(V, KV)IV, V\rangle$$

and

$$\langle \nabla_{KV} R(V, JV)IV, V\rangle = - \langle \nabla_{JV} R(V, KV)IV, V\rangle ,$$

hence

$$2 \langle \nabla_{KV} R(V, JV)IV, V\rangle = \langle \nabla_V R(V, IV)IV, V\rangle .$$

We have

$$\langle \nabla_{IV} R(V, JV)JV, V\rangle = \nabla_{IV}\sigma_{JJ} - 4 \langle R(\nabla_{IV} V, JV)JV, V\rangle = \nabla_{IV}\sigma_{JJ} + 4\beta\sigma_{JK},$$

so $\sigma_{JJ} + \sigma_{KK} = -\sigma_{II}$ along with $\sigma_{JJ} = \sigma_{KK}$ tell us

$$2 \langle \nabla_{IV} R(V, JV)JV, V\rangle = -\nabla_{IV}\sigma_{II}$$

on M. Similarly,

$$2 \langle \nabla_V R(V, JV)JV, V\rangle = -\nabla_V\sigma_{II}.$$

So

$$2 \langle \nabla_{JV} R(V, JV)IV, V\rangle = \nabla_V\sigma_{II} - \nabla_{IV}\sigma_{II} = 2 \langle \nabla_{KV} R(V, KV)IV, V\rangle$$

and

$$2 \langle \nabla_{KV} R(V, JV)IV, V\rangle = \nabla_V\sigma_{II} = -2 \langle \nabla_{JV} R(V, KV)IV, V\rangle$$

Inserting these into (A.6) then results in

$$\begin{aligned}\langle \Delta R(V, IV)IV, V\rangle = &-2\nu \langle \nabla_{JV} R(V, JV)IV, V\rangle + 2(\mu + \beta) \langle \nabla_{KV} R(V, JV)IV, V\rangle\\ &+ 2\mu \langle \nabla_{JV} R(V, KV)IV, V\rangle + 2(\alpha + \nu) \langle \nabla_{KV} R(V, KV)IV, V\rangle\\ = &(\alpha + \beta)\nabla_V\sigma_{II}(V) - \alpha\nabla_{IV}\sigma_{II}(V) - 6\sigma_{II}^2\end{aligned}$$

For any point on M. □

Remark A.5 We note how the right-hand side of (A.5) is expressed solely in quantities computable on M, even though the Laplacian on the left-hand side involves derivatives normal to M.

Appendix B: Appendix—parameter independence

In this appendix, we prove the parameter independence of the Poincaré and Sobolev inequalities used in Kobayashi's estimates. This will be a consequence of the following well-known result. One potential source with a proof is [36, Theorem 5.3], where the Poincaré inequality gets discussed as part of the proof.

Theorem B.1 *Let (M, g_λ) is a compact manifold of dimension $m \geq 3$ with a family of Riemannian metrics g_λ. Assume there is a λ-independent constant $C > 0$ giving*

- *a lower bound on the volume,*

$$C \leq Vol_{g_\lambda}(M),$$

- *an upper bound on the diameter*

$$diam_{g_\lambda}(M) \leq C,$$

- *and a lower bound on the Ricci curvature*

$$-Cg_\lambda \leq Ric_{g_\lambda}.$$

Then, there are λ-independent constants $C_P, C_S > 0$ depending on the dimension m such that the following Poincaré and Sobolev inequalities hold for all $f \in H^1(M, g_\lambda) = \{f \in L^2(M, g_\lambda) \,\big|\, |df|_{g_\lambda} \in L^2(M, g_\lambda)\}$:

$$\|f - f_{av}\|^2_{L^2(M,g_\lambda)} \leq C_P \, \|df\|^2_{L^2(M,g_\lambda)} \tag{B.1}$$

and

$$\|df\|^2_{L^2(M,g_\lambda)} \geq C_S \left(\|f\|^2_{L^{\frac{2m}{m-2}}(M,g_\lambda)} - Vol_{g_\lambda}(M)^{-\frac{2}{m}} \, \|f\|^2_{L^2(M,g_\lambda)} \right), \tag{B.2}$$

where

$$f_{av} := \frac{1}{Vol_{g_\lambda}(M)} \int_M f \, dVol_{g_\lambda}$$

is the average.

Theorem B.1 applies to the Kummer construction.

Proposition B.2 *Let (X, g) denote a Kummer K3 with patchwork metric g and Kähler form ω. Then, for all $|a|$ small enough, there is a constant $C > 0$ independent of $|a|$ such that*

$$\int_X f^2 \omega^2 \leq C \int_X |df|^2_g \, \omega^2 \tag{B.3}$$

holds for all $f \in H^1(X, g)$ with

$$\int_X f \, \omega^2 = 0$$

and

$$\|df\|^2_{L^2(X,g)} \geq C\left(\|f\|^2_{L^4(X,g)} - Vol_g(X)^{-1/2}\,\|f\|^2_{L^2(X,g)}\right) \tag{B.4}$$

holds for all $f \in H^1(X,g)$.

Proof That the diameter is bounded can be seen as follows. The distance between suitably close pairs of points in the Euclidean region is independent of a. For pairs of point p, q in the Eguchi–Hanson region, one can use the triangle inequality and compare with radial geodesics going from p to the zero section and out again to q. The radial distance is bounded by the Euclidean distance, as one sees by computing the length of the curve $z(t) = tp$,

$$d_{\mathrm{EH}}(0,p)^2 = \int_0^1 g_{\mathrm{EH}}(\dot z, \dot z)\,dt = |p|^4_{\mathbb{C}^2}\int_0^1 \frac{t^2}{\sqrt{a_i^2 + t^4|p|^4_{\mathbb{C}^2}}}\,dt \leq |p|^2_{\mathbb{C}^2} = d_{\mathrm{Euc}}(0,p)^2.$$

In the neck region, the distance is close to Euclidean by (2.3). So the distance between any two points in X can be bounded (using the triangle inequality several times) uniformly by a-independent quantities.

The volume form is Euclidean ($\omega^2/2 = \eta\wedge\overline{\eta}$) outside of the neck regions where the gluing takes place. The Ricci curvature vanishes outside of the necks. So it only remains to bound the volume form and Ricci curvature in the neck regions. However, on any compact subset of the complement of the zero section in Eguchi–Hanson space, $K \subset \mathcal{O}_{\mathbb{CP}^1}(-2)\backslash\mathbb{CP}^1$, it follows from the explicit form of the Kähler potential (2.1) that we can find a smooth function ξ_K such that

$$f_{\mathrm{EH}} = f_{\mathrm{Euc}} + a^2\xi_K.$$

Furthermore, the function ξ_K is regular as $a \to 0$. From this, it follows that the potential for the patchwork metric in the neck regions reads

$$\Phi_a = f_{\mathrm{Euc}} + |a|^2\chi\xi_K,$$

and the patchwork metric reads

$$g = g_{\mathrm{Euc}} + |a|^2 h \tag{B.5}$$

for some uniformly bounded h with uniformly bounded derivatives. This then tells us

$$\det(g) = 1 + |a|^2\mathrm{tr}(h) + |a|^4\det(h) \tag{B.6}$$

has the form $\det(g) = 1 + |a|^2 \cdot$ bounded and

$$\mathrm{Ric} = -\partial\overline{\partial}\ln\det(g) = |a|^2\tilde{\xi}$$

for some uniformly bounded $\tilde{\xi}$. All in all, there is an $|a|$-independent constant $C > 0$ such that

$$(1 - C|a|^2)\eta\wedge\overline{\eta} \leq \omega^2 \leq (1 + C|a|^2)\eta\wedge\overline{\eta}$$

and

$$|\mathrm{Ric}|_g \leq C|a|^2$$

hold on all of X. With these, we get $|a|$-independent bounds on the volume of X and the Ricci curvature of g for all small enough values of $|a|$. □

References

1. Alekseevsky, D.V., Graev, M.M.: Calabi–Yau metric on the Fermat surface. Isometries and totally geodesic submanifolds. J. Geom. Phys. **7**(1), 21–43 (1990)
2. Anderson, M.T.: Ricci curvature bounds and Einstein metrics on compact manifolds. J. Am. Math. Soc. **2**(3), 455–490 (1989)
3. Aubin, T.: Nonlinear Analysis on Manifolds. Monge–Ampère Equations. Springer, Berlin (1982)
4. Ballmann, W.: On the lengths of closed geodesics on convex surfaces. Invent. Math. **71**, 593–597 (1983)
5. Ballmann, W.: Lectures on Kähler Manifolds. EMS (2006)
6. Bangert, V.: On the lengths of closed geodesics on almost round spheres. Math. Z. **191**, 549–558 (1986)
7. Bando, S.: Bubbling out of Einstein manifolds. Tohoku Math. J. (2) **42**(2), 205–216 (1990)
8. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: Compact Complex Surfaces, 2nd edn. Springer, Berlin (2004)
9. Bielawski, R.: Complete hyperkähler $4n$-manifolds with a local tri-hamiltonian $\mathbb{R}^n$-action. Math. Ann. **314**, 505–528 (1999)
10. Bando, S., Kasue, A., Nakajima, H.: On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth. Invent. Math. **97**, 313–349 (1989)
11. Błocki, Z.: Interior regularity of the complex Monge–Ampère equation in convex domains. Duke Math. J. **105**(1), 167–181 (2000)
12. Błocki, Z.: Chapter 5—The Calabi–Yau theorem. In: Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics. Springer (2012)
13. Bourguignon, J.-P.: Sur les géodésiques fermées des variétés quaternionniennes de dimension 4. Math. Ann. **221**(2), 153–165 (1976)
14. Bourguignon, J.-P., Yau, S.-T.: Sur les métriques riemanniens à courbure de Ricci nulle sur le quotient d'une surface K3. C. R. Acad. Sci. Paris, tome 277, Ser. A 1175–1177 (1973)
15. Calabi, E.: The space of Kähler metrics. Proc. Int. Congr. Math. Amsterdam **2**, 206–207 (1954)
16. Calabi, E.: On Kähler manifolds with vanishing canonical class. In: Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz, pp. 78–89. Princeton University Press (1957)
17. Calabi, E.: Metriques kähleriennes et fibres holomorphes. Annales scientifique de l'E.N.S. 4e serie, tome **12**(2) (1979)
18. Cheeger, J., Gromoll, D.: The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differ. Geom. **6**, 119–128 (1971)
19. Carmo, M.P.: Riemannian Geometry. Birkhäuser, Basel (1992)
20. Duistermaat, H., Guillemin, V.: The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. **29**, 39–80 (1975)
21. Donaldson, S. K.: Calabi–Yau metrics on Kummer surfaces as a model gluing problem. In: Advances in Geometric Analysis, Advanced Lectures in Mathematics (ALM), vol. 21, pp. 109–118. International Press, Somerville, MA (2012)
22. de Verdière, Y.C.: Spectre du laplacien et longueurs des géodésiques périodiques i. Compos. Math. **27**, 83–106 (1973)
23. de Verdière, Y.C.: Spectre du laplacien et longueurs des géodésiques périodiques ii. Compos. Math. **27**, 159–184 (1973)
24. de Verdière, Y.C.: Spectrum of the Laplace operator and periodic geodesics: thirty years after. Ann. Inst. Fourier **57**(7), 2429–2463 (2007)
25. Eguchi, T., Hanson, A.J.: Selfdual solutions to Euclidean Gravity. Ann. Phys. **120**, 82–105 (1979)
26. Evans, L.C.: Classical solutions of fully nonlinear, convex, second order elliptic equations. Commun. Pure Appl. Math. **25**, 333–363 (1982)
27. Evans, L.C.: Classical solutions of the Hamilton–Jacobi–Bellman equation for uniformly elliptic operators. Trans. Am. Math. Soc. **275**, 245–255 (1983)
28. Foscolo, L.: ALF gravitational instantons and collapsing Ricci-flat metrics on the K3 surface. J. Differ. Geom. **112**(1), 79–120 (2019)
29. Gaveau, B.: Méthodes de contrôle optimal en analyse complexe. I. Résolution d'équations de Monge-Ampère. J. Funct. Anal. **25**(4), 391–411 (1977)
30. Gao, P., Douglas, M.R.: Geodesics on Calabi–Yau manifolds and winding states in non-linear sigma models. Front. Phys. **1** (2013)
31. Gibbons, G.W., Hawking, S.: Gravitational multi-instantons. Phys. Lett. B **78**(4), 430–432 (1978)
32. Gromoll, D., Meyer, W.: Periodic geodesics on compact Riemannian manifolds. J. Differ. Geom. **3**(3–4), 493–510 (1969)
33. Gibbons, G.W., Pope, C.N.: The positive action conjecture and asymptotically Euclidean metrics in quantum gravity. Commun. Math. Phys. **66**(3), 267–290 (1979)

34. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of the Second Order (reprint of the 1998 Edition). Springer, Berlin (2001)
35. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2009)
36. Hebey, E.: Nonlinear analysis on manifolds: Sobolev space and inequalities. In: AMS and Courant Institute of Mathematical Sciences (2000)
37. Huber, H.: Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. Math. Ann. **138**, 1–26 (1959)
38. Headrick, M., Wiseman, T.: Numerical Ricci-flat metrics on K3. Class. Quantum Gravity **22**, 4931–4960 (2005)
39. Joyce, D.: Compact 8-manifolds with holonomy Spin(7). Invent. Math. **123**, 507–552 (1996)
40. Joyce, D.: Compact Riemannian 7-manifolds with holonomy G2. J. Differ. Geom. **43**(2), (1996)
41. Joyce, D.: Compact Manifolds with Special Holonomy. Oxford University Press, Oxford (2000)
42. Klingenberg, W.P.A.: Riemannian Geometry (Second Revised Edition). Walter de Gruyter, Berlin (1995)
43. Kobayashi, R.: Moduli of Einstein metrics on a K3 surface and degeneration of type 1. Adv. Stud. Pure Math. **18–11**, 257–311 (1990)
44. Kronheimer, P.B.: A Torelli-type theorem for gravitational instantons. J. Differ. Geom. **29**, 685–697 (1989)
45. Kronheimer, P.B.: The construction of ALE spaces as hyper-Kähler quotients. J. Differ. Geom. **29**, 665–683 (1989)
46. Krylov, N.V.: Boundedly inhomogeneous elliptic and parabolic equations. Izv. Akad.Nauk SSSR **46**, 487–523 (1982). English translation: Math. USSRIzv. **20**, 459–492 (1983)
47. Kobayashi, R., Todorov, A.N.: Polarized period map for generalized K3 surfaces and the moduli of Einstein metrics. Tohoku Math. J. (2) **39**(3), 341–363 (1987)
48. Lohkamp, J.: Metrics of Negative Ricci Curvature. Ann. Math. (2) **140**(3), 655–683 (1994)
49. Lee, J.M., Parker, T.H.: The Yamabe problem. Bull. Am. Math. Soc. (N.S.) **17**(1), 37–91 (1987)
50. LeBrun, C., Singer, M.: A kummer-type construction of self-dual 4-manifolds. Math. Ann. **300**, 165–180 (1994)
51. Lye, J.O.: Stable Geodesics on a K3 Surface. Ph.D. Thesis, Albert-Ludwigs Universität Freiburg, (2019). Unpublished thesis
52. Lye, J. O.: A detailed look at the Calabi–Eguchi–Hanson spaces (2022). arXiv:2201.07295
53. Margulis, G.A.: Applications of ergodic theory to the investigation of manifolds of negative curvature. Funct. Anal. Appl. **3**, 335–336 (1969)
54. Margulis, G.A.: On Some Aspects of the Theory of Anosov Systems. Springer, Berlin (2000)
55. Morse, M.: The Calculus of Variations in the Large. AMS Coloquium Publications, Rhode Island (1934)
56. Myers, S.B.: Riemannian manifolds with positive mean curvature. Duke Math. J. **8**(2), 401–404 (1941)
57. Nakajima, H.: Hausdorff convergence of Einstein 4-manifolds. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35**(2), 411–424 (1988)
58. Ozuch, T.: Noncollapsed degeneration of Einstein 4-manifolds II. Geom. Topol. **26**, 1529–1634 (2022)
59. Page, D.N.: A physical picture of the K3 gravitational instanton. Phys. Lett. B **80**(1–2), 55–57 (1978)
60. Petersen, P.: Riemannian Geometry. Springer, Berlin (1998)
61. Platt, D.: Improved estimates for G_2-structures on the Generalised Kummer Construction (2022). arXiv:2011.00482v3
62. Rademacher, H.-B.: On the average indices of closed geodesics. J. Differ. Geom. **29**(1), 65–83 (1989)
63. Schwarzschild, K.: Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **7**, 189–196 (1916)
64. Siu, Y.-T.: Lectures on Hermitian–Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics. Birkhäuser, Berlin (1987)
65. Schmidt, B., Sutton, C.J.: Two remarks on the length spectrum of a Riemannian manifold. Proc. Am. Math. Soc. **139**(11), 4113–4119 (2011)
66. Song, J., Weinkove, B.: An Introduction to the Kähler-Ricci Flow. Springer, Berlin (2013)
67. Topiwala, P.: A new proof of the existence of Kähler–Einstein metrics on K3. I. Invent. Math. **89**(2), 425–448 (1987)
68. Topiwala, P.: A new proof of the existence of Kähler–Einstein metrics on K3. II. Invent. Math. **89**(2), 449–454 (1987)
69. Trudinger, N.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. Trans. Am. Math. Soc. **278**, 751–769 (1983)
70. Tsai, C.-J., Wang, M.-T.: Mean curvature flows in manifolds of special holonomy. J. Differ. Geom. **108**(3), 531–569 (2018)
71. Vigué-Poirrier, M., Sullivan, D.: The homology theory of the closed geodesic problem. J. Differ. Geom. **11**(4), 633–644 (1976)

72. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I. Commun. Pure Appl. Math. **31**(3), 339–411 (1978)