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Solvable conjugacy class graph of groups

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In this paper we introduce the graph $\Gamma_{sc}(G)$ associated with a group G, called the solvable conjugacy class graph (abbreviated as SCC-graph), whose vertices are the nontrivial conjugacy classes of G and two distinct conjugacy classes C, D are adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y\rangle$ is solvable.

We discuss the connectivity, girth, clique number, and several other properties of the SCC-graph. One of our results asserts that there are only finitely many finite groups whose SCC-graph has given clique number d, and we find explicitly the list of such groups with $d = 2$. We pose some problems on the relation of the SCC-graph to the solvable graph and to the NCC-graph, which we cannot solve.

1. Introduction

Various graphs have been defined on finite groups, using various properties of groups, and studied to understand the interplay between groups and graphs. Commutativity is one of such graph defining properties on groups. The first graph arising from commutativity is the commuting graph [6] which is the complement of the non-commuting graph [19]. The commuting graph of a group G is a graph whose vertices are the nontrivial elements of G and two distinct vertices x, y are adjacent if $\langle x, y\rangle$ is abelian.

Another way of defining graphs on G is given by considering the conjugacy classes as the vertex set and adjacency is defined using properties of conjugacy classes [3]. We write x^G to denote the conjugacy class $\{gxg^{-1} : g \in G\}$ of $x \in G$. Mixing the concepts of commutativity and conjugacy class, in 2009 Herzog et al. [14] have introduced commuting conjugacy class graph (abbreviated as CCC-graph) of G as a graph whose vertex set is the set of nontrivial conjugacy classes of G and two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is abelian for some $x' \in x^G$ and $y' \in y^G$.

Extending the notion of CCC-graph, in 2017 Mohammadian and Erfanian [18] introduced the nilpotent conjugacy class graph (abbreviated as NCC-graph) of a group. Its vertex set is the set of nontrivial conjugacy classes of G, and two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is nilpotent for some $x' \in x^G$ and $y' \in y^G$. Note that the CCC-graph is a spanning subgraph of the NCC-graph of G.

In this paper, we further extend the notions of CCC-graph and NCC-graph and introduce the solvable conjugacy class graph (abbreviated as SCC-graph) of G. The SCC-graph of a group G is a simple undirected graph, denoted by $\Gamma_{sc}(G)$, with

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vertex set $\{x^G : 1 \neq x \in G\}$ and two distinct vertices x^G and y^G are adjacent if there exist two elements $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is solvable. It is clear that the NCC-graph is a spanning subgraph of the SCC-graph of G.

The aim of this paper is to discuss the connectivity, girth, clique number, and some properties of SCC-graph.

For each of these conjugacy class graphs, we sometimes consider a variant where the vertex set is G, rather than the set of conjugacy classes in G, with the adjacency rules as described. So, if two conjugacy classes are equal or adjacent in the conjugacy class version, then all pairs of vertices in those classes are adjacent in the graph on G. This change leaves several properties, such as connectedness and diameter, unchanged. The reason for doing this is that we want to compare the solvable conjugacy class graph with the usual solvable graph (see [4]) on G, which is easier if the vertex sets are the same. We will call the version of the SCC-graph with vertex set G the *expanded SCC-graph* on G. The expanded CCC-graph and NCC-graph are defined analogously.

We write $V(\Gamma)$ to denote the vertex set of a graph Γ. We write $u \sim v$ to denote that the vertices u, v are adjacent.

The distance between two vertices u and v of Γ is denoted by $d(u, v)$. Recall that the diameter of a graph is the maximum distance between its vertices. If $u = v$ then we write $d(u, v) = 0$. The solvabilizer of x, denoted by $\mathrm{Sol}_G(x)$, is the set given by $\{y \in G : \langle x, y \rangle \text{ is solvable}\}$.

We write $\mathrm{Sol}(G) = \{x \in G : \langle x, y \rangle \text{ is solvable for all } y \in G\}$. Clearly, $\mathrm{Sol}(G) = \bigcap_{u \in G} \mathrm{Sol}_G(u)$ and $Z(G) \subseteq \mathrm{Sol}(G)$. Also, if G is finite then $\mathrm{Sol}(G)$ is the solvable radical of G (see [12]).

We note that the second and third authors, together with Arunkumar and Selvaganesh, have considered graphs defined by combining a graph on the group (such as the commuting graph) with an equivalence class (such as conjugacy) in [1]. However, the SCC-graph is not considered in that paper.

2. Properties of the SCC-graph

We begin with a simple observation. Let a and b be two elements of G such that a^G and b^G are joined in the SCC-graph of G. This means that there exist $a' \in a^G$ and $b' \in b^G$ such that $\langle a', b' \rangle$ is solvable. Without loss of generality, we can assume that $a' = a$. For suppose that $(a')^h = a$. Then $\langle a', b' \rangle^h = \langle a, (b')^h \rangle$ is solvable, since it is a conjugate of (and hence isomorphic to) $\langle a', b' \rangle$.

Theorem 2.1. *Let G be a finite group. Then the SCC-graph of G is complete if and only if G is solvable.*

Proof. If G is solvable, $\langle x, y \rangle$ is also solvable for all $x, y \in G$. In particular, if a^G, b^G are two vertices of $\Gamma_{sc}(G)$ and $x \in a^G$, $y \in b^G$ then $\langle x, y \rangle$ is solvable. Therefore, a^G and b^G are adjacent. Hence, $\Gamma_{sc}(G)$ is a complete graph.

Conversely, suppose that $\Gamma_{sc}(G)$ is complete. Then, by the observation at the end of the last section, for every $a, b \in G$, there is a conjugate b' of b such that $\langle a, b' \rangle$ is solvable. By [11, Theorem A], we conclude that G is solvable. $\square$

Theorem 2.2. *Let G be a finite solvable group. Then G has complete expanded NCC-graph if and only if G is nilpotent.*

Proof. The reverse implication is clear. We prove the contrapositive of the forward implication. Moreover, it is easy to see that, if the expanded NCC-graph of G is complete, then the same is true of G/N for any normal subgroup N of G. So we assume that G has expanded NCC-graph complete but is not nilpotent, and is minimal with respect to these properties. Thus, any proper quotient of G is nilpotent.

Suppose G is not nilpotent. Then there is a prime p such that $N = O_p(G) \neq 1$. Since G/N is nilpotent and $O_p(G/N) = 1$, N is a Sylow p-subgroup of G, so $|N|$ and $|G/N|$ are coprime. By the Schur–Zassenhaus Theorem [21, Theorem 10.30], N has a complement $H \cong G/N$. Since G is not nilpotent, H acts non-trivially on N. Hall's extension of the Burnside basis theorem ([7], see [21, Theorem 11.12, Corollary 11.13]) now implies that H acts faithfully on $N/\Phi(N)$, where $\Phi(N)$ is the Frattini subgroup of N. If $\Phi(N) \neq 1$, we have a contradiction to the fact that $G/\Phi(N)$ is nilpotent. So N is an elementary abelian p-group.

Let $x \in G$ be a p'-element such that $1 \neq xN \in Z(G/N)$. Then there exists $y \in N$ such that $xy \neq yx$. It follows that $\langle x, y \rangle$ is not nilpotent. Now for every $g \in G$ we have $gxg^{-1} \equiv x \pmod N$. Hence, also $\langle gxg^{-1}, y \rangle$ is not nilpotent. Therefore, x and y are not adjacent in the expanded NCC-graph. Contradiction. $\square$

At this point we record several questions about SCC-graphs, which we have not been able to answer, together with some comments.

Problem 2.3. Given a finite group G, describe the set of vertices of the expanded SCC-graph of G which are joined to all others.

We observe that, in the solvable graph, the set of dominant vertices is just the solvable radical of G, by the result of [12]. Hence the set of dominant vertices in the expanded SCC-graph contains the solvable radical. However, it can be larger. Consider the simple groups $\mathrm{PSL}(2, 2^d)$, with $d \geq 2$. Each group has a unique conjugacy class of involutions, and every element of the group is mapped to its inverse by conjugation by some involution. So, for any element $a \in G$, there is an involution $b \in G$ so that $\langle a, b\rangle$ is dihedral, and hence solvable. Thus the involutions are dominant vertices. On the other hand, G is non-abelian simple, so its solvable radical is trivial. The sporadic Janko group J_1 also has this property.

Problem 2.4.

(a) For which finite groups G is the expanded SCC-graph of G equal to the solvable graph of G?
(b) For which finite non-solvable groups G is the expanded SCC-graph of G equal to the expanded NCC-graph of G?

It is known that

- the expanded CCC-graph is equal to the commuting graph if and only if G is a 2-Engel group (that is, satisfies the commutator identity $[x, y, y] = 1$ for all $x, y \in G$ [1, Theorem 2.2]);
- the solvable graph is equal to the nilpotent graph if and only if G is nilpotent [8, Proposition 11.1(b)].

It may be that the answers to these two questions are "G is solvable" and "G is nilpotent" respectively.

We cannot solve Problem 2.4, but we have made some progress on part (b). We give here a reduction which shows that it suffices to consider almost simple groups other than symmetric and alternating groups.

Suppose that G is a non-solvable group such that the expanded SCC-graph and NCC-graph coincide. Let $S \trianglelefteq G$ be the solvable radical. If $xS, yS \in G/S$ are adjacent in the expanded SCC-graph of G/S, then there exists $g \in G$ such that $H := \langle x, gyg^{-1}\rangle$ is solvable. By Theorem 2.2, there exists $h \in H$ such that $\langle x, hgyg^{-1}h^{-1}\rangle$ is nilpotent. Then also $\langle xS, hgy(hg)^{-1}S\rangle$ is nilpotent and xS is adjacent to yS in the expanded NCC-graph of G/N. Hence, we may assume that $S = 1$. Now let N be a minimal normal subgroup of G. Then $N = T_1 \times \cdots \times T_n$ for isomorphic non-abelian simple groups $T_1, \dots, T_n$. Let $T := \{(t, \dots, t)\} \leq N$ be a diagonal subgroup. Then elements in T are conjugate in G if and only if they are conjugate by automorphisms of T induced by its normaliser in G. So in order to derive a contradiction we may replace G by a subgroup of $\mathrm{Aut}(T)$ containing T: that is, we assume that G is an almost simple group.

Let $A_n \leq G \leq S_n$ and let p be the largest prime $\leq n$. By Bertrand's Postulate, $n < 2p$. Let $x \in A_n$ be a p-cycle and $y \in N_{A_n}(\langle x\rangle)$ a disjoint product of a $(p-1)$-cycle and a transposition such that y generates $\mathrm{Aut}(\langle x\rangle)$. Clearly x and y are adjacent in the expanded SCC-graph. Suppose that there exists $g \in G$ such that $\langle x, gyg^{-1}\rangle$ is nilpotent. Since x and y have coprime orders, it follows that x commutes with gyg^{-1}. On the other hand, y and gyg^{-1} have the same cycle type. Hence, the cycles of gyg^{-1} must be disjoint to the p-cyclic x. This is impossible since $p + (p-1) + 2 > 2p > n$.

We give here another open problem, which is loosely related to the above problems.

Problem 2.5. For which finite graphs Γ is there a finite group G such that Γ is isomorphic to an induced subgraph of $\Gamma_{sc}(G)$?

We note that every finite graph can be embedded in the solvable graph of some finite group (see [8, p. 93]), but this construction does not descend to the solvable conjugacy class graph.

Next we turn to the questions of connectedness and diameter. The girth will be discussed in the next section. We adopt the convention that vertices in different components have infinite distance. We begin with a simple observation.

Proposition 2.6. *Let G be a non-solvable group such that it has an element of order pq, where p, q are primes. If $p \neq q$ then $\mathrm{girth}(\Gamma_{sc}(G)) = 3$ and hence $\Gamma_{sc}(G)$ is not a tree.*

Proof. Let $a \in G$ be an element of order pq. If $p \neq q$ then $o(a^q) = p$ and $o(a^p) = q$. Also, $\langle a, a^q\rangle$, $\langle a^q, a^p\rangle$ and $\langle a^p, a\rangle$ are abelian groups. Since a^G, $(a^q)^G$ and $(a^p)^G$ are distinct, we have the following triangle

$$a \sim a^q \sim a^p \sim a$$

in $\Gamma_{sc}(G)$. Therefore, $\mathrm{girth}(\Gamma_{sc}(G)) = 3$ and hence $\Gamma_{sc}(G)$ is not a tree. □

Proposition 2.7. *Let $x \in G \setminus \{1\}$ and $a, b \in \mathrm{Sol}_G(x) \setminus \{1\}$. Then a^G and b^G are connected and $d(a^G, b^G) \leq 2$. In particular, if $\mathrm{Sol}(G) \neq \{1\}$ then $\Gamma_{sc}(G)$ is connected and $\mathrm{diam}(\Gamma_{sc}(G)) \leq 2$.*

Proof. Since $a, b \in \mathrm{Sol}_G(x) \setminus \{1\}$, $\langle a, x\rangle$ and $\langle x, b\rangle$ are solvable. Therefore, $d(a^G, x^G) \leq 1$ and $d(x^G, b^G) \leq 1$. Hence, the result follows.

If $\mathrm{Sol}(G) \neq \{1\}$ then there exists an element $z \in G$ such that $z \neq 1$ and $z \in \mathrm{Sol}(G)$. Therefore, $z \in \mathrm{Sol}_G(w)$ for all $w \in G \setminus \{1\}$. Let u^G and v^G be any two vertices of $\Gamma_{sc}(G)$. Then $u, v \in \mathrm{Sol}_G(z) \setminus \{1\}$. Therefore, by the first part it follows that $d(u^G, v^G) \leq 2$. Hence, $\mathrm{diam}(\Gamma_{sc}(G)) \leq 2$. □

Remark 2.8. For any two distinct vertices $x^G, y^G \in V(\Gamma_{sc}(G))$, $x^G \sim y^G$ if and only if $\mathrm{Sol}_G(gxg^{-1}) \cap y^G \neq \emptyset$ for all $g \in G$. Also, x^G is an isolated vertex if and only if $\mathrm{Sol}_G(gxg^{-1}) \subseteq x^G \cup \{1\}$ for all $g \in G$.

Theorem 2.9. *If G, H are arbitrary nontrivial groups then the graph $\Gamma_{sc}(G \times H)$ is connected and $\mathrm{diam}(\Gamma_{sc}(G \times H)) \le 3$. In particular, $\Gamma_{sc}(G \times G)$ is connected and $\mathrm{diam}(\Gamma_{sc}(G \times G)) \le 3$. Further, $\mathrm{diam}(\Gamma_{sc}(G \times G)) = 3$ if and only if $\mathrm{diam}(\Gamma_{sc}(G)) \ge 3$ (possibly infinite).*

Proof. Let (x, y) and (u, v) be two nontrivial elements of $G \times H$. Without any loss we may assume that $x \neq 1_G$ and $v \neq 1_H$, where 1_G and 1_H are identity elements of G and H respectively; then

$$(x, y)^{G\times H} \sim (x, 1_H)^{G\times H} \sim (1_G, v)^{G\times H} \sim (u, v)^{G\times H}.$$

This shows that $\mathrm{diam}(\Gamma_{sc}(G \times H)) \le 3$. Putting $H = G$, it follows that $\mathrm{diam}(\Gamma_{sc}(G \times G)) \le 3$.

Suppose that $\mathrm{diam}(\Gamma_{sc}(G)) \le 2$. Let $(x, y), (u, v)$ be two vertices in $\Gamma_{sc}(G \times G)$. If $x = 1_G$, then there is a path of length 2 between these vertices, namely $(1_G, y) \sim (u, 1_H) \sim (u, v)$; so we may suppose that $x, u \neq 1_G$. Then there exist $a \in G \setminus \{1_G\}$ such that $x^G \sim a^G \sim u^G$. Therefore $\langle x, a^g\rangle$ and $\langle a^h, u\rangle$ are solvable for some $g, h \in G$. We have $\langle (x, y), (a, 1_H)^{(g,1)}\rangle = \langle x, a^g\rangle \times \langle y\rangle$ and $\langle (a, 1_H)^{(h,1)}, (u, v)\rangle = \langle a^h, u\rangle \times \langle v\rangle$ are solvable; so $d((x, y), (u, v)) \le 2$.

Conversely, suppose that $\mathrm{diam}(\Gamma_{sc}(G)) \ge 3$, and take two elements x and u of $G \setminus \{1_G\}$ whose distance is at least 3. If there were a path $(x, x)^{G\times G} \sim (a, b)^{G\times G} \sim (u, u)^{G\times G}$, there would be paths $x^G \sim a^G \sim u^G$ and $x^G \sim b^G \sim y^G$ in G; the only possibility would be $a = b = 1_G$, which is excluded. So $\mathrm{diam}(\Gamma_{sc}(G \times G)) \ge 3$, and since we have the reverse inequality we must in fact have equality. □

A dominating set of a graph Γ is a subset S of $V(\Gamma)$ such that every vertex in $V(\Gamma) \setminus S$ is adjacent to at least one vertex in S. The domination number of Γ, denoted by $\lambda(\Gamma)$, is the minimum cardinality of dominating sets of Γ.

Proposition 2.10. *Let G be a non-solvable group. Then $\lambda(\Gamma_{sc}(G)) = 1$ if $|\mathrm{Sol}(G)| \neq 1$.*

Proof. Let x be a nontrivial element in $\mathrm{Sol}(G)$. Then $x^G \in V(\Gamma_{sc}(G))$. Let $y^G \in V(\Gamma_{sc}(G)) \setminus \{x^G\}$ be an arbitrary vertex. Then $\langle x, y\rangle$ is solvable. Therefore, x^G and y^G are adjacent. Hence, $\{x^G\}$ is a dominating set of $\Gamma_{sc}(G)$ and so $\lambda(\Gamma_{sc}(G)) = 1$. □

3. Clique number

The *clique number* $\omega(\Gamma)$ of a graph Γ is the number of vertices in the largest complete subgraph of Γ. In this section we investigate the clique number of the SCC-graphs of finite groups. The main theorem of this section is that there are only finitely many finite groups whose SCC-graph has a given clique number.

We begin with a theorem of Landau [17]. Let $k(G)$ denote the number of conjugacy classes of the group G.

Proposition 3.1. *For any positive integer m, there are only finitely many finite groups which have $k(G) = m$.*

Proof. Let $x_1, \dots, x_m$ be conjugacy class representatives, and let $n_i = |C_G(x_i)|$ for $i = 1, \dots, m$. Then $|x_i^G| = |G|/n_i$; so

$$\sum_{i=1}^{m} \frac{1}{n_i} = 1.$$

Now there are only finitely many expressions of 1 as a sum of m fractions with unit numerator (this is “folklore”, but is not a difficult exercise). Moreover, the largest value of n_i is $|C_G(1)| = |G|$. □

Now we can deal with solvable groups.

Theorem 3.2. *There are only finitely many solvable groups G for which $\Gamma_{sc}(G)$ has given clique number d.*

Proof. By Theorem 2.1, if G is solvable then $\Gamma_{sc}(G)$ is complete, so its clique number is $k(G) - 1$ (since the identity is omitted from the graph); now Proposition 3.1 finishes the result. □

We now give a result which will be used several times.

Theorem 3.3. *Let G be a finite group. If G has an element of order $n = \Pi_{i=1}^{m} p_i^{k_i}$, where p_i's are distinct primes. Then $\Gamma_{sc}(G)$ has a clique of size $\Pi_{i=1}^{m}(k_i + 1) - 1$.*

Proof. Let $x \in G$ be an element of order n. Then $(x^r)^G \sim (x^s)^G$ for all proper divisors r, s of n. Since total number of proper divisors of $n = \Pi_{i=1}^{m} p_i^{k_i}$ is $\Pi_{i=1}^{m}(k_i + 1) - 1$, we get a clique in $\Gamma_{sc}(G)$ of size $\Pi_{i=1}^{m}(k_i + 1) - 1$. □

Next, we prove the main result for $d = 2$, in a stronger form.

Theorem 3.4. *With the exception of the cyclic groups of orders* 1, 2 *and* 3 *and the symmetric group of degree* 3*, every finite group* G *has the property that* $\Gamma_{sc}(G)$ *contains a triangle (that is, has girth* 3*).*

Proof. If G is solvable, then $k(G) = \omega(\Gamma_{sc}(G)) + 1$ (the extra 1 coming from the identity of G), so G has at most three conjugacy classes. The groups listed in the theorem are all those having this property.

So we may assume that G is non-solvable. If G has an element whose order is not a prime power, then some power (say g) of this element has order pq, where p and q are distinct primes. Then $\Gamma_{sc}(G)$ contains a clique of size 3, by Theorem 3.3.

So we may further assume that every element of G has prime power order.

These groups were first studied by Higman [15] in 1957; Suzuki [22] determined the simple groups with this property in 1965. Subsequently all such groups have been classified [5,13]. The story is somewhat tangled, perhaps due to the lack of a common name for the class. Subsequently two names were proposed; a group with this property is called a *CP group* by some authors, and an *EPPO group* by others. These groups have arisen in connection with other graphs defined on groups, including the Gruenberg–Kegel graph (or prime graph) and the power graph: see [9]. The result we require is that a non-solvable group in which every element has prime power order satisfies one of the following:

(a) G is one of A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,17)$, M_{10} or $\mathrm{PSL}(3,4)$;
(b) G has a normal subgroup N such that G/N is $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$ or $\mathrm{Sz}(32)$, and N is a direct sum of copies of the natural G/N-module over its field of definition.

Suppose first that we are in case (b). If we can find a triangle in the solvable conjugacy class group of G/N, then it lifts to a triangle in $\Gamma_{sc}(G)$. So it is enough to add the four possibilities for G/N to the list of groups in case (a).

In $\mathrm{Sz}(8)$, there are three conjugacy classes of elements of order 13, all represented in a cyclic subgroup of order 13, giving us a triangle. Similar arguments apply to $\mathrm{Sz}(32)$ (using an element of order 41), $\mathrm{PSL}(2,8)$ (order 7), and $\mathrm{PSL}(2,17)$ (order 3 and two classes of order 9). In $\mathrm{PSL}(2,4)$, a dihedral subgroup of order 10 meets two conjugacy classes of elements of order 5 and one class of involutions. A similar argument applies to A_6 (using a dihedral group of order 10), $\mathrm{PSL}(2,7)$ (using a non-abelian group of order 21) $\mathrm{PSL}(3,4)$ (a non-abelian group of order 21) and M_{10} (a quaternion group of order 8 meets two conjugacy classes of elements of order 4 and one class of involutions). All this information is easily obtained from the $\mathbb{ATLAS}$ *of Finite Groups* [10]. □

Now we come to the main result of the section.

Theorem 3.5. *For any positive integer* d*, there are only finitely many finite groups* G *such that* $\omega(\Gamma_{sc}(G)) = d$*.*

This theorem can be regarded as a strengthening of Landau's result.

Proof. We assume that there are groups G of arbitrarily large order such that $\Gamma_{sc}(G)$ has clique number at most d, and aim for a contradiction. We proceed in a number of steps.

Step 1. By Theorem 3.2, we can assume that G is non-solvable.

Step 2. We can assume that the solvable radical of G is trivial. For suppose that $\mathrm{Sol}(G) \neq 1$ and that $|G/\mathrm{Sol}(G)| = m$, and suppose we know that m is bounded by a function of d. Then $\mathrm{Sol}(G)$ contains non-identity elements from at most d conjugacy classes of G, since these classes form a clique in $\Gamma_{sc}(G)$. Since each such class splits into at most m conjugacy classes in $\mathrm{Sol}(G)$, we see that $\mathrm{Sol}(G)$ has at most dm non-trivial conjugacy classes, and hence has order bounded by a function of d and m. So if m is also bounded by a function of d, then $|G|$ is bounded by a function of d, as required.

Step 3. Let G be a group with clique number bounded by d, and let S be the socle of G (the product of the minimal normal subgroups). Then S is a product of non-abelian simple groups. We can assume that the number of factors is bounded. For if we choose one non-identity element from each factor, the chosen elements generate an abelian group; and elements of this group which have different numbers of non-identity coordinates are not conjugate in G.

Step 4. $C_G(S) = 1$. For $C_G(S)$ is a normal subgroup of G, and so contains a minimal normal subgroup, say M. But then $M \leq S \cap C_G(S) = Z(S)$, contradicting the fact that S is a product of centreless groups.

Step 5. It follows that G acts faithfully on S by conjugation. Elements of G permute the factors. Since their number is bounded, we can assume that G fixes all the factors; so the socle is simple, and G is almost simple.

Step 6. Now we invoke the Classification of Finite Simple Groups. We can assume that G is sufficiently large that its socle is not a sporadic group. So there are three cases:

Case 1: S is alternating, so $G = A_n$ or S_n. Let H be the subgroup of G with $\lfloor n/2 \rfloor$ orbits of size 2 (and one fixed point if n is odd). Then H is abelian, and elements of H may be products of any even number of transpositions up to $2\lfloor n/4 \rfloor$. But elements with different numbers of transpositions are non-conjugate, so $\Gamma_{sc}(G)$ contains a clique of size $\lfloor n/4 \rfloor$. Thus n is bounded.

Case 2: G is classical of large rank.
Suppose first that the socle of G is $\mathrm{PSL}(n,q)$ for large n. Then dropping to a subgroup of index at most 2, G is the quotient of a subgroup of $\mathrm{P\Gamma L}(n,q)$ by the subgroup of scalar matrices. The diagonal matrices with determinant 1 mod scalars form an abelian group of rank at least $n-2$, and elements with different numbers of non-1 diagonal entries (greater than $n/2$) are pairwise non-conjugate, giving a large clique.
Now suppose that G is symplectic, unitary, or orthogonal. Then a cover of G acts on its natural module; this module is an orthogonal direct sum of r hyperbolic planes and an isotropic space (see [23]), where r is the Witt index (and is at least $(n-2)/2$, where n is the dimension of the module). So the cover of G contains the direct product of r copies of the 2-dimensional classical group, and the same contradiction is obtained.

Case 3: G is of Lie type over a large field $\mathrm{GF}(q)$. In this case, G contains a subgroup C of index at most 2 in the multiplicative group of the field (inside a subgroup of Lie rank 1, which is either $\mathrm{PSL}(2,q)$ or $\mathrm{Sz}(q)$). Every conjugate of a generator g of C is the image of g or g^{-1} under a field automorphism. So, if $q=p^r$ with p prime, then a conjugacy class of G contains at most $2r$ generators. But altogether there are at least $\phi((q-1)/2)$ generators, where ϕ is Euler's function. The ratio of these two numbers tends to infinity with q. So a clique of size larger than d can be found if q is sufficiently large. $\square$

An alternative proof of the theorem runs as follows. Using arguments as above, we reduce to the case where G is a simple group. Then we apply a recent result of Hung and Yang [16], asserting that the number of prime divisors of a finite simple group is bounded above by a (quartic) function of the maximum number of prime divisors of an element order. If the clique number of the SCC-graph is bounded, then the number of prime divisors of an element order is bounded, and hence the number of prime divisors of $|G|$ is bounded.

Next we claim that the prime divisors are bounded. Let $p_1,\dots,p_s$ be the prime divisors of $|G|$. We show by induction on i that p_i is bounded in terms of d. Since the solvable radical is trivial, we may assume that $p_1=2$. Now let $i>1$ and take an element x in G of order $p=p_i$; let $C=\langle x\rangle$. The $p-1$ generators of C can lie in at most d distinct conjugacy classes. Hence the cyclic group $N_G(C)/C_G(C)$ has order at least $(p-1)/d$ (since an element of G conjugating a generator of C to another must normalize C). On the other hand, every prime divisor of the order of $N_G(C)/C_G(C)$ divides $p-1$ and therefore lies in $\{p_1,\dots,p_{i-1}\}$. Moreover, the exponents of the Sylow p_j-subgroups are bounded by p_j^d. So by induction, $|N_G(C)/C_G(C)|$ is bounded in terms of d. Consequently, p is bounded in terms of d as well.

Now Theorem 5.4 of Babai, Goodman and Pyber [2] gives the result.

We have not attempted to write down an explicit function bounding $|G|$ in terms of the clique number of $\Gamma_{sc}(G)$.

Corollary 3.6. *Given g, there are only finitely many finite groups G for which $\Gamma_{sc}(G)$ can be embedded in a surface of genus g.*

This holds because the genus of an embedding of the complete graph K_n is an unbounded function of n.

4. Distance in SCC-graph for locally finite group

A locally finite group is a group for which every finitely generated subgroup is finite. An element of a group is said to be a p-element if the order of the element is a power of p, where p is a prime. In this section we obtain some results on distance between two vertices of $\Gamma_{sc}(G)$ for some locally finite groups, analogous to certain results in [14,18].

Proposition 4.1. *Let G be a locally finite group. If $x, y\in G\setminus\{1\}$ are p-elements, where p is a prime, then $d(x^G, y^G)\le 1$.*

Proof. Since G is a locally finite group and $x,y\in G\setminus\{1\}$ are p-elements, the subgroup $\langle x,y\rangle$ is finite. Let P be a Sylow p-subgroup of $\langle x,y\rangle$ containing x. Then $y^g=gyg^{-1}\in P$ for some $g\in G$ since all the Sylow p-subgroups are conjugate. Therefore, $\langle x,y^g\rangle$ is solvable and so $d(x^G,y^G)\le 1$. $\square$

Proposition 4.2. *Let G be a locally finite group. If $x,y\in G$ are of non-coprime orders, then $d(x^G,y^G)\le 3$. If either x or y is of prime order then $d(x^G,y^G)\le 2$.*

Proof. Let $o(x)=pm$ and $o(y)=pn$, where p is a prime and m,n are positive integers. Then x^m and y^n are nontrivial p-elements of G. Therefore, by Proposition 4.1, we have

$$d((x^m)^G,(y^n)^G)\le 1.$$

Clearly, $d(x^G,(x^m)^G)\le 1$ and $d((y^n)^G,y^G)\le 1$. Therefore, if $x^G\ne y^G$ then $x^G\sim(x^m)^G\sim(y^n)^G\sim y^G$ is a path from x^G to y^G. Hence, $d(x^G,y^G)\le 3$.

Suppose that $o(x)=pm$ and $o(y)=p$. Then x^m and y are nontrivial p-elements of G. Therefore, by Proposition 4.1, we have

$$d((x^m)^G, y^G) \leq 1.$$

Thus $x^G \sim (x^m)^G \sim y^G$ is a path from x^G to y^G. Hence, $d(x^G, y^G) \leq 2$. $\square$

Proposition 4.3. *Let G be a locally finite group and $x, y \in G$. Suppose p and q are prime divisors of $o(x)$ and $o(y)$, respectively, and that G has an element of order pq. Then*

(a) *$d(x^G, y^G) \leq 5$, and moreover $d(x^G, y^G) \leq 4$ if either x or y is of prime power order.*
(b) *If either a Sylow p-subgroup or a Sylow q-subgroup of G is a cyclic or generalized quaternion finite group, then $d(x^G, y^G) \leq 4$. Moreover, $d(x^G, y^G) \leq 3$ if either x or y is of prime order.*
(c) *If both Sylow p-subgroup and Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups, then $d(x^G, y^G) \leq 3$. Moreover, $d(x^G, y^G) \leq 2$ if either x or y is of prime order.*

Proof. Let $o(x) = pm$ and $o(y) = qn$ for some positive integers m, n. Let $a \in G$ be an element of order pq. Then $o(a^q) = p$ and $o(a^p) = q$. Also, a^p commutes with a^q.

(a) We have

$$d(x^G, (x^m)^G) \leq 1,\ d((a^q)^G, (a^p)^G) = 1,\ \text{and}\ d((y^n)^G, y^G) \leq 1.$$

Since $o(x^m) = o(a^q) = o(y^n) = p$, by Proposition 4.1, we have

$$d((x^m)^G, (a^q)^G) \leq 1 \text{ and } d((a^p)^G, (y^n)^G) \leq 1.$$

Therefore, $d(x^G, y^G) \leq 5$.
If $o(x) = p^s$ for some positive integer s then, by Proposition 4.1, we have $d(x^G, (a^q)^G) \leq 1$. Similarly, if $o(y) = q^t$ for some positive integer t then $d(y^G, (a^p)^G) \leq 1$. Therefore, $d(x^G, y^G) \leq 4$.

(b) Without any loss of generality assume that Sylow p-subgroup of G is either a cyclic group or a generalized quaternion finite group. Let P and Q be two Sylow p-subgroups of G containing x^m and a^q respectively. Since P is finite, by [20, Theorem 14.3.4], Q is also finite and $P = gQg^{-1}$ for some $g \in G$ and so $ga^qg^{-1} \in P$. Therefore, $\langle x^m \rangle$ and $\langle ga^qg^{-1} \rangle$ are subgroups of P having order p. Since P is cyclic or a generalized quaternion group, by [20, Theorem 5.3.6], we have $\langle x^m \rangle = \langle ga^qg^{-1} \rangle$. Therefore, $ga^qg^{-1} = (x^m)^i$ for some integer i and so $\langle x, ga^qg^{-1} \rangle = \langle x, (x^m)^i \rangle = \langle x \rangle$. Hence $d(x^G, (a^q)^G) \leq 1$. We also have

$$d((a^q)^G, (a^p)^G) = 1,\ d((a^p)^G, (y^n)^G) \leq 1,\ \text{and}\ d((y^n)^G, y^G) \leq 1.$$

Thus $d(x^G, y^G) \leq 4$.
If $o(x) = p$ then $\langle x \rangle = \langle ga^qg^{-1} \rangle$. Therefore, $x = ga^{qt}g^{-1}$ for some integer t. We have $x^G = (a^{qt})^G$ and so $\langle a^{qt}, a^p \rangle$ is abelian. Hence, $d(x^G, (a^p)^G) \leq 1$ and so $d(x^G, y^G) \leq 3$.

(c) If both Sylow p-subgroup and Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups, then proceeding as part (b) we get

$$d(x^G, (a^q)^G) \leq 1,\ d((a^q)^G, (a^p)^G) = 1,\ \text{and}\ d((a^p)^G, y^G) \leq 1.$$

Therefore, $d(x^G, y^G) \leq 3$.
If $o(x) = p$ then proceeding as in part (b), we have $d(x^G, (a^p)^G) \leq 1$ and so $d(x^G, y^G) \leq 2$. $\square$

We conclude this section with the following consequence.

Theorem 4.4. *Let G be a finite group. Let H and K be two subgroups of G such that H is normal in G, $G = HK$ and $\Gamma_{sc}(H), \Gamma_{sc}(K)$ are connected. If there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that h^G and x^G are connected in $\Gamma_{sc}(G)$, then $\Gamma_{sc}(G)$ is connected.*

Proof. Let $a, b \in G$ such that a^G and b^G are two distinct vertices in $\Gamma_{sc}(G)$.

If $a, b \in H$ then there exists a path from a^H to b^H, since $\Gamma_{sc}(H)$ is connected. Hence, a^G and b^G are connected. Let $a \notin H$ and $o(a) = n$. Let $f : G/H \to K/(H \cap K)$ be an isomorphism and $f(aH) = x(H \cap K)$, where $x \in K$. Then $x^n(H \cap K) = f(a^nH) = H \cap K$ and so $x^n \in H \cap K$. Let $d = \gcd(o(a), |K|)$. Then there exist integers r, s such that

$$x^d = x^{nr+|K|s} = (x^n)^r.(x^{|K|})^s \in H \cap K.$$

Therefore, $d > 1$. Let p be a prime divisor of d. Then there exists an element $k_1 \in K$ such that $\gcd(o(a), o(k_1)) \neq 1$. Therefore, by Proposition 4.2, there is a path from a^G to k_1^G. Similarly, if $b \notin H$ then there exists an element $k_2 \in K$ such that there is a path from k_2^G to b^G. We have $k_1^G = k_2^G$ or there is a path from k_1^K to k_2^K, since $\Gamma_{sc}(K)$ is connected. Therefore, $k_1^G = k_2^G$ or there is a path from k_1^G to k_2^G. Thus a^G and b^G are connected. If $b \in H$ then, by given conditions, there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that there is a path from x^G to h^G and a path from h^G to b^G (since $\Gamma_{sc}(H)$ is connected). Since $x \notin H$, proceeding as above we get a path from x^G to k_3^G for some $k_3 \in K$ and hence a path from a^G to x^G. Thus we get a path from a^G to b^G. Hence, $\Gamma_{sc}(G)$ is connected. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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