Research Article

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Coupling of Finite and Boundary Elements for Singularly Nonlinear Transmission and Contact Problems

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Abstract: This article discusses the well-posedness and error analysis of the coupling of finite and boundary elements for interface problems in nonlinear elasticity. It concerns p-Laplacian-type Hencky materials with an unbounded stress-strain relation, as they arise in the modelling of ice sheets, non-Newtonian fluids or porous media. We propose a functional analytic framework for the numerical analysis and obtain a priori and a posteriori error estimates for Galerkin approximations to the resulting boundary/domain variational inequality.

Keywords: Boundary Element Method, Nonlinear Elasticity, FE-BE Coupling, A Priori Error Estimates, A Posteriori Error Estimates

MSC 2010: 65N30, 65N38, 74M15, 65K15

Dedicated to Professor W. L. Wendland on his 85th birthday.

1 Introduction

Adaptive finite element/boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of uniformly elliptic interface problems in computational mechanics [18, 27]. In recent years, strongly nonlinear materials with an unbounded relationship between stress and strain have been of interest, from nonlinear diffusion and image processing, porous media and filtration to the modelling of glaciers and non-Newtonian fluids, such as in Herschel–Bulkley models [2–4, 17, 26]. Mathematically, the p-Laplacian or related nonlinear Lamé operators provide examples of simple model problems, for which the relation between stress and strain follows a power law; more realistic examples include Carreau-type laws [6]. Variational inequalities arise from the boundary conditions and at interfaces, from simplified friction laws, or from total variation regularization in mathematical imaging [1, 11, 15, 22]

This article provides the theoretical framework for the adaptive coupling of finite element and boundary elements for such materials in a surrounding linearly elastic medium.

At the interface, transmission conditions correspond to a continuous deformation and continuous normal stresses. Contact conditions like friction and nonpenetration [23] allow for sliding and the opening of gaps between the materials. They significantly complicate the numerical analysis and computations as they give rise to a variational inequality with a closed, convex set K of admissible test and trial functions and a non-differentiable functional for the frictional energy. The challenges of the contact constraints do not only involve the formulation, but also affect the convergence of numerical methods: typically, the solution is of reduced regularity at the interface between contact and non-contact. As the location of the interface is a priori unknown,

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special meshes like geometrically graded ones cannot be used to resolve the singularity. A posteriori error estimates and the resulting adaptive mesh refinement procedures give rise to efficient numerical methods.

1.1 Formulation of the Problem

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded Lipschitz domain, $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ its complement. For $p \in (1, \infty)$, we look for a solution $(u, u_c) \in (W^{1,p}(\Omega))^n \times (W^{1,2}_{\mathrm{loc}}(\Omega^c))^n$ of

$$-\operatorname{div} A'(\varepsilon(u)) = f \quad \text{in } \Omega, \tag{1.1a}$$

$$-\mu\Delta u_c - (\lambda + \mu) \operatorname{grad} \operatorname{div} u_c = 0 \quad \text{in } \Omega^c, \tag{1.1b}$$

$$A'(\varepsilon(u))\nu - T_x^* u_c = t_0 \quad \text{on } \partial\Omega, \tag{1.1c}$$

$$u - u_c = u_0 \quad \text{on } \Gamma_t. \tag{1.1d}$$

Here $\mu > 0$, $\lambda > -\mu$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i)$, and $A' \colon L^p(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}} \to L^{p'}(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}}$ describes the nonlinear material in Ω, e.g. $A'(x) = |x|^{p-2}x$ for p-Laplacian-like materials or $A'(x) = (|x|^{1-\delta}(1 + |x|^2)^{\frac{\delta}{2}})^{p-2}x$ for Carreau laws, $\delta \in [0, 1]$.

The interface $\partial\Omega = \overline{\Gamma_t \sqcup \Gamma_s}$, $\Gamma_t \neq \emptyset$, is decomposed into open subsets, with ν the unit outer normal. Further, T_x^* denotes the natural conormal derivative $2\mu\partial_\nu + \lambda\nu \operatorname{div} + \mu\nu \times \operatorname{curl}$.

The transmission conditions on Γ_t are complemented by contact conditions on Γ_s, given in terms of the normal and tangential components of u, $u_n = \nu \cdot u$ and $u_t = u - u_n\nu$, and the stress, $\sigma_n(u) = -\nu A'(\varepsilon(u))\nu$ and $\sigma_t(u) = -A'(\varepsilon(u))\nu - \sigma_n(u)\nu$, respectively, as well as the friction threshold $0 \leq \mathcal{F} \in L^\infty(\Gamma_s)$,

$$\sigma_n(u) \leq 0, \quad u_{0,n} + u_{c,n} - u_n \leq 0, \quad \sigma_n(u)(u_{0,n} + u_{c,n} - u_n) = 0,$$
$$|\sigma_t(u)| \leq \mathcal{F}, \quad \sigma_t(u)(u_{0,t} + u_{c,t} - u_t) + \mathcal{F}|u_{0,t} + u_{c,t} - u_t| = 0.$$

Finally, u_c is required to satisfy a radiation condition. In two dimensions ($n = 2$), this is given by

$$u_c(x), \quad \operatorname{grad} u_c(x) = \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty,$$

while in three dimensions ($n = 3$), it takes the form

$$u(x) = \mathcal{O}(|x|^{-1}), \quad \operatorname{grad} u(x) = \mathcal{O}(|x|^{-2}) \quad \text{as } |x| \to \infty.$$

The data are required to belong to the following spaces:

$$f \in (L^{p'}(\Omega))^n, \quad u_0 \in (W^{\frac{1}{2},2}(\partial\Omega))^n, \quad t_0 \in (W^{-\frac{1}{2},2}(\partial\Omega))^n, \quad 0 \leq \mathcal{F} \in L^\infty(\Gamma_s).$$

In Theorem 1, we show that problem (1.1) admits a unique weak solution $(u, u_c) \in W^{1,p}(\Omega)^n \times W^{1,2}_{\mathrm{loc}}(\Omega^c)^n$ for the material laws of interest in this article, equations (3.1)–(3.2) below, provided that for $n = 2$ the compatibility condition $\int_\Omega f e_j + \langle t_0, e_j\rangle = 0$ $(j = 1, 2)$ holds. Here, e_1, e_2 is the standard basis of $\mathbb{R}^2$.

To solve problem (1.1) numerically, we first use the Poincaré–Steklov operator S for the Lamé equation in Ω^c to reduce the exterior problem to $\partial\Omega$. A conforming Galerkin approximation is used to discretize the resulting domain/boundary variational inequality in a suitable Banach space.

1.2 Main Results

We identify a suitable functional analytic framework for the numerical analysis of finite element/boundary element methods for (1.1). In particular, we obtain a priori and a posteriori error estimates for Galerkin approximations to the equivalent boundary/domain variational inequality. The a posteriori estimate complements recent estimates obtained for mixed finite element formulations of friction problems in linear elasticity and gives rise to adaptive mesh refinement procedures.

The analysis of the strongly nonlinear problem (1.1) poses analytical difficulties, especially for the range $p \in (1,2)$ of relevance to applications. In this case, the bilinear form on $\partial\Omega$ for the boundary element method fails to be continuous in natural function spaces related to the nonlinear operator in Ω.

A key technical insight concerns the role of the Banach space

$$X^p = \{(u,v) \in (W^{1,p}(\Omega))^n \times (\tilde{W}^{1-\frac{1}{r},r}(\Gamma_s))^n : u|_{\partial\Omega} + v \in W^{\frac{1}{2},2}(\partial\Omega)^n\},$$

where $r = \min\{p,2\}$. The space X^p is motivated by Costabel's analysis of layer potentials in [10], but we provide a direct approach.

We especially focus on the well-posedness and a sharp error analysis of the friction problem when $p \in (1,2)$. As a key result, Theorem 4 gives a sharp a posteriori estimate for the error of Galerkin approximations to the variational inequality. It complements recent results for mixed finite element formulations of friction problems [19, 21, 24] and is new even in the elliptic case.

Theorem 4 still involves the uncomputable exact operator S. In Theorem 5, we obtain a computable a posteriori error estimate involving the layer potential operators, in a nontrivial extension of the proof of Theorem 4.

The existence of a unique X^p-solution is shown in Theorem 1, and Theorem 3 gives an a priori estimate for Galerkin approximations. As an example of the added difficulty when $p \in (1,2)$, the variational inequality no longer splits into an equality on Ω and an inequality on $\partial\Omega$, unless the artificial regularity assumption $u|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega)^2$ is imposed.

The mathematical differences between $p < 2$ and $p \geq 2$ are not artificial. They reflect the different physical behavior: while pseudoplastic materials like ice or molasses ($p < 2$) get stiffer and stiffer under a smaller stress, possibly infinitely so, the opposite happens in the dilatant case like a thick emulsion of sand and water ($p > 2$). For scalar problems, our previous work [14] obtained weaker estimates for the p-Laplacian contact problem in the technically easier case $p \geq 2$, as well as for interface problems involving the double-well potential [13]. Fast stabilized hp boundary element methods for contact are developed in [5]. For linear and nonlinear elliptic problems, coupled finite and boundary elements have long been studied; see e.g. [9, 25].

Notation. In this article, C denotes a positive constant which may take different values from line to line. We write $f \lesssim g$ if there exists a constant $C > 0$ with $f \leq Cg$.

2 Preliminaries

2.1 Sobolev Spaces

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Set $p' = \frac{p}{p-1}$ whenever $p \in (1,\infty)$. We will also denote $r = \min\{p,2\}$ and $q = \max\{p,2\}$.

Before analyzing a variational formulation of (1.1), we recall some properties of L^p-Sobolev spaces on Ω.

Definition 1. We define the following properties.

(a) The Sobolev spaces $W^{k,p}_{(0)}(\Omega)$, $k \in \mathbb{N}_0$, are the completion of $C^\infty_{(c)}(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} = \|u\|_p + \sum_{|\gamma|=k} \|\partial^\gamma u\|_p.$$

The second term in the norm will be denoted by $|u|_{W^{k,p}(\Omega)} = |u|_{k,p}$. Let

$$W^{-k,p'}_0(\Omega) = (W^{k,p}(\Omega))' \quad \text{and} \quad W^{-k,p'}(\Omega) = (W^{k,p}_0(\Omega))'.$$

(b) $W^{1-\frac{1}{p},p}(\partial\Omega)$ denotes the space of traces of $W^{1,p}(\Omega)$-functions on the boundary. It coincides with the Besov space $B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)$ as obtained by real interpolation of Sobolev spaces [28, 29], and one may define $W^{s,p}(\partial\Omega) = B^s_{p,p}(\partial\Omega)$ for $s \in (-1,1)$.

(c) For Lipschitz $\Gamma_0 \subset \partial\Omega$ open, we define the subspace of supported distributions

$$\tilde{W}^{s,p}(\Gamma_0) = \{u \in W^{s,p}(\partial\Omega) : \operatorname{supp} u \subset \Gamma_0\}$$

and the space of extensible distributions $W^{s,p}(\Gamma_0) = W^{s,p}(\partial\Omega)/\tilde{W}^{s,p}(\partial\Omega \setminus \overline{\Gamma_0})$ with the quotient norm.

Remark. Note the following.
(a) $(W^{s,p}(\partial\Omega))' = W^{-s,p'}(\partial\Omega)$ and $W^{s,2}(\partial\Omega) = H^s(\partial\Omega)$.
(b) $W^{s,2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and $\|u\|_{W^{s,p}(\Omega)} \le \|u\|_{W^{s,2}(\Omega)}$ for $1 < p \le 2$.
(c) The Poincaré–Steklov operator S of the Lamé operator on Ω^c is continuous between $(W^{\sigma+\frac{1}{2},2}(\partial\Omega))^n$ and $(W^{\sigma-\frac{1}{2},2}(\partial\Omega))^n$, $\sigma \in (-\frac{1}{2}, \frac{1}{2})$.
(d) Points (a) to (c) imply that the quadratic form $\langle Su, u\rangle$ associated to S is well-defined on $(W^{1-\frac{1}{p},p}(\partial\Omega))^n$ if $p \ge 2$. If $\partial\Omega$ is smooth, S is an elliptic pseudodifferential operator on $\partial\Omega$ of order 1, and the form $\langle Su, u\rangle$ is unbounded for $p < 2$.

Parts (a) and (b) of this remark are well-known properties of Sobolev spaces [28, 29]. Part (c) follows from [10].

2.2 Boundary Integral Operators

We recall the fundamental solutions for the Lamé operator,

$$G(x,y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda+2\mu)}\left\{\log(|x-y|^{-1})\,\mathrm{Id} + \frac{\lambda+\mu}{\lambda+3\mu}\,\frac{(x-y)(x-y)^T}{|x-y|^2}\right\} \quad \text{in } \mathbb{R}^2,$$
$$G(x,y) = \frac{\lambda + 3\mu}{8\pi\mu(\lambda+2\mu)}\left\{\frac{1}{|x-y|}\,\mathrm{Id} + \frac{\lambda+\mu}{\lambda+3\mu}\,\frac{(x-y)(x-y)^T}{|x-y|^3}\right\} \quad \text{in } \mathbb{R}^3;$$

see [20, (2.2.2)]. They allow to define layer potentials on $\partial\Omega$ associated to the Lamé problem,

$$\mathcal{V}\phi(x) = 2\int_{\partial\Omega} G(x,x')\,\phi(x')\,dx',$$
$$\mathcal{K}\phi(x) = 2\int_{\partial\Omega} [T^*_{x'}G(x,x')]^T\phi(x')\,dx',$$
$$\mathcal{K}'\phi(x) = 2\int_{\partial\Omega} T^*_x G(x,x')\phi(x')\,dx',$$
$$\mathcal{W}\phi(x) = -2T^*_x\int_{\partial\Omega} [T^*_{x'}G(x,x')]^T\phi(x')\,dx'.$$

These operators extend from $C^\infty(\partial\Omega)^n$ to a bounded map $\left(\begin{smallmatrix}\mathcal{K} & \mathcal{V}\\ \mathcal{W} & \mathcal{K}'\end{smallmatrix}\right)$ on the Sobolev space $W^{\frac{1}{2},2}(\partial\Omega)^n \times W^{-\frac{1}{2},2}(\partial\Omega)^n$.

If $n = 3$ or if the capacity of $\partial\Omega$ is less than 1 (which can always be achieved by scaling), $\mathcal{V}$ and $\mathcal{W}$ are selfadjoint operators on $W^{-\frac{1}{2},2}(\partial\Omega)^n$, $\mathcal{V}$ is positive and $\mathcal{W}$ non-negative. The Steklov–Poincaré operator for the exterior Lamé problem can be expressed in terms of the layer potentials as

$$S = \frac{1}{2}(\mathcal{W} + (1-\mathcal{K}')\mathcal{V}^{-1}(1-\mathcal{K}))\colon W^{\frac{1}{2},2}(\partial\Omega)^n \subset W^{-\frac{1}{2},2}(\partial\Omega)^n \to W^{-\frac{1}{2},2}(\partial\Omega)^n$$

and defines a positive and selfadjoint operator. It satisfies

$$T^*_x u_c|_{\partial\Omega} = -S(u_c|_{\partial\Omega}) \tag{2.1}$$

for every solution u_c of the Lamé equation in Ω^c which satisfies the decay condition from Section 1.1 at infinity. Therefore, S gives rise to a coercive and symmetric bilinear form $\langle Su, u\rangle$ on $W^{\frac{1}{2},2}(\partial\Omega)^n$.

2.3 Korn's Inequality

Existence of a unique solution to (1.1) will be shown using Korn's inequality and coercivity.

Proposition 1. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $\Gamma_0 \subset \partial\Omega$ has positive $(n-1)$-dimensional measure, and $p \in (1,\infty)$. Then there is a constant $C > 0$ such that*

$$\|u\|_{W^{1,p}(\Omega)} \le C(\|\varepsilon(u)\|_{L^p(\Omega)} + \|u|_\Gamma\|_{L^1(\Gamma_0)}) \quad \textit{for all } u \in (W^{1,p}(\Omega))^n.$$

The proof follows the argument in the scalar case from [14, Proposition 2], using Korn's inequality instead of the scalar Poincaré–Friedrichs inequality.

3 Analysis of the Boundary/Domain Formulation

For $r = \min\{p, 2\}$, we consider the space

$$X^p = \{(u, v) \in (W^{1,p}(\Omega))^n \times (\tilde{W}^{1-\frac{1}{r},r}(\Gamma_s))^n \ : u|_{\partial\Omega} + v \in W^{\frac{1}{2},2}(\partial\Omega)^n\}$$

equipped with the norm

$$\|u, v\|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{\tilde{W}^{1-\frac{1}{r},r}(\Gamma_s)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

Note that $X^p = (W^{1,p}(\Omega))^n \times (\tilde{W}^{\frac{1}{2},2}(\Gamma_s))^n$ when $p \geq 2$ so that we recover a vector-valued variant of the Banach spaces considered in [14].

Lemma 1. *$(X^p, \|\cdot\|_{X^p})$ is a Banach space, and $\|u, v\|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}$ defines an equivalent norm on X^p.*

Proof. It is readily verified that $\|\cdot\|_{X^p}$ defines a norm on X^p. To show completeness, let $(u_j, v_j) \in X$ be a Cauchy sequence. Then (u_j, v_j) converges to a limit (u, v) in the Banach space $W^{1,p}(\Omega)^n \times \tilde{W}^{1-\frac{1}{r},r}(\Gamma_s)^n$. Also, $u_j|_{\partial\Omega} + v_j$ converges to a limit w in $W^{\frac{1}{2},2}(\partial\Omega)^n$. However, the continuity of the trace operator assures that $u_j|_{\partial\Omega} \to u|_{\partial\Omega}$ in $W^{1-\frac{1}{p},p}(\partial\Omega)^n$. Therefore in $W^{1-\frac{1}{p},p}(\partial\Omega)^n$, hence also in $W^{1-\frac{1}{r},r}(\partial\Omega)^n$, $u_j|_{\partial\Omega} + v_j$ converges both to $u|_{\partial\Omega} + v$ and to w. This means that $u|_{\partial\Omega} + v = w \in W^{\frac{1}{2},2}(\partial\Omega)^n$, or $(u, v) \in X^p$.

To see the equivalence of norms, note that $|u, v|_{X^p} \leq \|u, v\|_{X^p}$. On the other hand, the continuous inclusion of $W^{\frac{1}{2},2}(\partial\Omega)$ into $W^{1-\frac{1}{r},r}(\partial\Omega)$, of $W^{1-\frac{1}{p},p}(\partial\Omega)$ into $W^{1-\frac{1}{r},r}(\partial\Omega)$, and the continuity of the trace operator from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega)$ imply

$$\begin{aligned}\|u, v\|_{X^p} &\leq \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega}\|_{W^{1-\frac{1}{r},r}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{1-\frac{1}{r},r}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}\\ &\leq \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)}\\ &\lesssim \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} = |u, v|_{X^p}.\end{aligned}$$

The assertion follows. □

Recall the function $A' \colon L^p(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}} \to L^{p'}(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}}$ in the interior problem of (1.1). We assume that A' defines a bounded, continuous and uniformly monotone operator so that, in particular, for $p \in (1, 2)$,

$$\begin{aligned}\langle A'(x) - A'(y), x - y\rangle &\gtrsim (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2}\|x - y\|^2_{L^p(\Omega)},\\ \langle A'(x) - A'(y), z\rangle &\lesssim \|x - y\|^{p-1}_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.\end{aligned} \tag{3.1}$$

When $p \in [2, \infty)$, we require

$$\begin{aligned}\langle A'(x) - A'(y), x - y\rangle &\gtrsim \|x - y\|^p_{L^p(\Omega)},\\ \langle A'(x) - A'(y), z\rangle &\lesssim (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2}\|x - y\|_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.\end{aligned} \tag{3.2}$$

These assumptions are satisfied by materials of p-Laplacian-type [16].

The variational formulation of the contact problem (1.1) will be stated with the help of the functional

$$J(u, v) = \langle A(\varepsilon(u)), \varepsilon(u)\rangle + \frac{1}{2}\langle S(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v\rangle - L(u, v)$$

on X^p. Here, A is related to A' by $D_v\langle A(\varepsilon(u)), \varepsilon(u)\rangle = \langle A'(\varepsilon(u)), \varepsilon(v)\rangle$, where D_v denotes the Fréchet derivative in direction v. Specifically, $A(x) = \frac{1}{p}|x|^{p-2}x$ when $A'(x) = |x|^{p-2}x$. Further, $v = u_0 + u_c - u$,

$$j(v) = \int_{\Gamma_s} \mathcal{F}|v_t|, \quad \text{and} \quad L(u, v) = \int_{\Omega} fu + \langle t_0 + Su_0, u|_{\partial\Omega} + v\rangle.$$

This paper investigates the numerical approximation of the following nonsmooth variational problem over the closed convex subset

$$K = \{(u, v) \in X^p : v_n \leq 0 \text{ and if } n = 2, \text{ then } \langle Se_j, u|_{\partial\Omega} + v - u_0\rangle = 0 \text{ for } j = 1, 2\} \subset X^p.$$

Find $(\hat{u}, \hat{v}) \in K$ such that

$$J(\hat{u}, \hat{v}) + j(\hat{v}) = \min_{(u,v)\in K} J(u, v) + j(v). \tag{3.3}$$

Note that j is Lipschitz, but not differentiable.

Analogous to [14], one observes that problem (3.3) is equivalent to contact problem (1.1). The existence of a unique solution to the latter is therefore a consequence of the following theorem.

Theorem 1. *The following statements hold.*

(a) *Contact problem* (1.1) *is equivalent to problem* (3.3).

(b) *There exists a unique minimizer* $(\hat{u}, \hat{v}) \in K$ *in problem* (3.3).

The crucial ingredient in the proof is the following monotonicity estimate.

Lemma 2. *The operator DJ given by the variation of J is uniformly monotone on X^p. Let*

$$r = \min\{p, 2\}, \quad q = \max\{p, 2\} \quad \text{and} \quad C > 0.$$

Then, for every $(u_1, v_1), (u_2, v_2) \in X^p$ with $\|u_1, v_1\|_{X^p(\Omega)}, \|u_2, v_2\|_{X^p(\Omega)} < C$, there holds

$$\begin{aligned}\|u_2 - u_1, v_2 - v_1\|_{X^p}^q \lesssim_C\ & \langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1)\rangle \\ & + \langle S((u_2 - u_1)|_{\partial\Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\rangle \\ \lesssim_C\ & \|u_2 - u_1, v_2 - v_1\|_{X^p}^r.\end{aligned}$$

Proof. The upper bound is a consequence of estimates (3.1), (3.2) for the nonlinear operator and the boundedness of S from $W^{\frac{1}{2},2}(\partial\Omega)^n$ to $W^{-\frac{1}{2},2}(\partial\Omega)^n$. For $p \geq 2$, we refer to [14, Lemma 3] for the proof of an analogous lower estimate in a scalar p-Laplacian problem.

When $p < 2$, the monotonicity of A' and the coercivity of S imply

$$\begin{aligned}&\langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1)\rangle + \langle S((u_2 - u_1)|_{\partial\Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\rangle \\ &\quad\gtrsim \|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}^p + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 \\ &\quad\gtrsim \|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}^2 + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|_{\tilde{W}^{\frac{1}{2},2}(\Gamma_s)}^2 + \|u_2 - u_1\|_{W^{\frac{1}{2},2}(\Gamma_t)}^2 \\ &\qquad + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 \\ &\quad\gtrsim \|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}^2 + \delta\|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}^2 + \|u_2 - u_1\|_{W^{\frac{1}{2},2}(\Gamma_t)}^2 \\ &\qquad + \|(u_2 - u_1)|_{\partial\Omega} + v_2 - v_1\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2.\end{aligned} \tag{3.4}$$

In the last inequality, we use the continuous inclusion $\tilde{W}^{\frac{1}{2},2}(\Gamma_s) \subset \tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)$. Korn's inequality, Proposition 1, implies

$$\|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}^2 + \|u_2 - u_1\|_{W^{\frac{1}{2},2}(\Gamma_t)}^2 \gtrsim \|u_2 - u_1\|_{W^{1,p}(\Omega)}^2. \tag{3.5}$$

Further, note, from the triangle inequality, the convexity of $x \mapsto x^2$ as well as the continuity of the trace map from $W^{1,p}(\Omega)$ to $\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)$,

$$\begin{aligned}\|v_2 - v_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}^2 &\leq \big(\|(u_2 - u_1)|_{\Gamma_s} + v_2 - v_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)} + \|(u_2 - u_1)|_{\Gamma_s}\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}\big)^2 \\ &\leq 2\|(u_2 - u_1)|_{\Gamma_s} + v_2 - v_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}^2 + 2\|u_2 - u_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}^2 \\ &\leq 2\|(u_2 - u_1)|_{\Gamma_s} + v_2 - v_1\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)}^2 + 2C'\|u_2 - u_1\|_{W^{1,p}(\Omega)}^2.\end{aligned} \tag{3.6}$$

The asserted estimate follows from (3.4), (3.5) and (3.6), after choosing $\delta > 0$ in (3.4) sufficiently small.

Uniform monotonicity on all of X^p is shown similarly, but for large $\|\varepsilon(u_2 - u_1)\|_{L^p(\Omega)}$, the exponent 2 in the lower bound has to be replaced by p. □

Proof (of Theorem 1). By definition of J, we may choose an infimizing sequence $(u_j, v_j) \in K$. Coercivity and the continuity of L imply that

$$\begin{aligned}\infty > \max_j J(u_j, v_j) &\gtrsim \|\varepsilon(u_j)\|_{L^p(\Omega)^p} + \|u_j|_{\partial\Omega} + v_j\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + j(v_j) + L(u_j, v_j) \\ &\gtrsim \|\varepsilon(u_j)\|_{L^p(\Omega)^p} + \|u_j|_{\partial\Omega} + v_j\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + j(v_j) - C\|u_j, v_j\|_{X^p}.\end{aligned}$$

Therefore, there exists $C' < \infty$ such that $\|u_j, v_j\|_{X^p} < C'$ and $\|u_j|_{\partial\Omega} + v_j\|_{W^{\frac{1}{2},2}(\partial\Omega)} < C'$. As the ball of radius $2C'$ in X is weakly sequentially compact, (u_j, v_j) converges weakly to an element $(\hat{u}, \hat{v}) \in X^p$ after passing to a subsequence. Similarly, extracting another subsequence if necessary, we may assume that also $u_j|_{\partial\Omega} + v_j$ converges weakly to $\hat{w} \in W^{\frac{1}{2},2}(\partial\Omega)$. Since $v_j = (u_j|_{\partial\Omega} + v_j) - u_j|_{\partial\Omega}$ and $W^{\frac{1}{2},2}(\partial\Omega) \subset W^{1-\frac{1}{p},p}(\partial\Omega)$, the uniqueness of limits in $W^{1-\frac{1}{p},p}(\partial\Omega)$ assures $\hat{v} = \hat{w} - \hat{u}|_{\partial\Omega}$. As K is closed, we deduce that $(\hat{u}, \hat{v}) \in K \cap X'$.

To show that $J(\hat{u}, \hat{v}) = \inf_K J$, it suffices to prove

$$J(\hat{u}, \hat{v}) \leq \lim_{j\to\infty} J(u_j, v_j) = \inf_K J.$$

Note that both $u \mapsto \langle A(\varepsilon(u)), \varepsilon(u)\rangle$ and $w \mapsto \langle Sw, w\rangle$ are weakly lower semicontinuous on $W^{1,p}(\Omega)$, $W^{\frac{1}{2},2}(\partial\Omega)$, respectively, so that

$$\langle A(\varepsilon(\hat{u})), \varepsilon(\hat{u})\rangle \leq \liminf_{j\to\infty} \langle A(\varepsilon(u_j)), \varepsilon(u_j)\rangle,$$
$$\langle S\hat{w}, \hat{w}\rangle \leq \liminf_{j\to\infty} \langle S(u_j|_{\partial\Omega} + v_j), u_j|_{\partial\Omega} + v_j\rangle.$$

As the inclusion of $\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)$ into $L^1(\Gamma_s)$ is compact, $v_j \to \hat{v}$ in $L^1(\Gamma_s)$. Hence $j(\hat{v}) = \lim_{j\to\infty} j(v_j)$. Finally, $L(\hat{u}, \hat{v}) = \lim_{j\to\infty} L(u_j, v_j)$ because L is a continuous linear functional on X. We conclude

$$\begin{aligned}\lim_{j\to\infty} J(u_j, v_j) &\geq \liminf_{j\to\infty} \langle A(\varepsilon(u_j)), \varepsilon(u_j)\rangle + \liminf_{j\to\infty} \langle S(u_j|_{\partial\Omega} + v_j), u_j|_{\partial\Omega} + v_j\rangle + \lim_{j\to\infty} j(v_j) - \lim_{j\to\infty} L(u_j, v_j)\\ &\geq \langle A(\varepsilon(\hat{u})), \varepsilon(\hat{u})\rangle + \langle S\hat{w}, \hat{w}\rangle + j(\hat{v}) - L(\hat{u}, \hat{v})\\ &= \langle A(\varepsilon(\hat{u})), \varepsilon(\hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), \hat{u}|_{\partial\Omega} + \hat{v}\rangle + j(\hat{v}) - L(\hat{u}, \hat{v}) = J(\hat{u}, \hat{v}).\end{aligned}$$

Since (u_j, v_j) was an infimizing sequence, $(\hat{u}, \hat{v})$ is a minimum of J.

Concerning uniqueness, let $(\hat{u}_1, \hat{v}_1), (\hat{u}_2, \hat{v}_2) \in K$ be two distinct minimizers of J. As a sum of convex summands, J is convex and therefore must be constant on the line segment

$$\{(\hat{u}_1 + t(\hat{u}_2 - \hat{u}_1), \hat{v}_1 + t(\hat{v}_2 - \hat{v}_1) : t \in [0, 1])\}$$

in X^p. Thus

$$\langle DJ(\hat{u}_2, \hat{v}_2) - DJ(\hat{u}_1, \hat{v}_1), (\hat{u}_2 - \hat{u}_1, \hat{v}_2 - \hat{v}_1)\rangle = 0,$$

or for our particular J,

$$0 = \langle A'(\varepsilon(\hat{u}_2)) - A'(\varepsilon(\hat{u}_1)), \varepsilon(\hat{u}_2) - \varepsilon(\hat{u}_1)\rangle + \langle S((\hat{u}_2 - \hat{u}_1)|_{\partial\Omega} + \hat{v}_2 - \hat{v}_1), (\hat{u}_2 - \hat{u}_1)|_{\partial\Omega} + \hat{v}_2 - \hat{v}_1\rangle.$$

The conclusion follows from Lemma 2. □

4 Discretization and A Priori Error Analysis

For simplicity, we assume that Ω is a polygonal domain (if $n = 2$) or a polyhedral domain (if $n = 3$).

Let $\{\mathcal{T}_h\}_{h\in I}$ be a regular triangulation of Ω into disjoint open regular triangles ($n = 2$) or tetrahedra ($n = 3$) T such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$. Each element has at most one edge or face, respectively, on $\partial\Omega$, and the closures of any two of them share at most a single vertex, edge or face. Let h_T denote the diameter of $T \in \mathcal{T}_h$ and ρ_T the diameter of the largest inscribed ball. We assume that $1 \leq \max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} \leq R$ independent of h and that $h = \max_{T\in\mathcal{T}_h} h_T$. The set of edges and faces of the triangles and tetrahedra, respectively, in $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$. Associated to $\mathcal{T}_h$ is the space $W_h^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ of functions whose restrictions to every $T \in \mathcal{T}_h$ are linear.

The boundary $\partial\Omega$ is triangulated by $\{l \in \mathcal{E}_h : l \subset \partial\Omega\}$. For $r = \min\{p, 2\}$, $W_h^{1-\frac{1}{r},r}(\partial\Omega)$ denotes the corresponding space of continuous, piecewise linear functions, and $\tilde{W}_h^{1-\frac{1}{r},r}(\Gamma_s)$ the subspace of those supported on Γ_s. Finally, $W_h^{-\frac{1}{2},2}(\partial\Omega) \subset W^{-\frac{1}{2},2}(\partial\Omega)$ is the space of piecewise constant functions, and

$$X_h^p = W_h^{1,p}(\Omega) \times W_h^{1-\frac{1}{r},r}(\partial\Omega) \subset X^p.$$

We denote the canonical inclusion maps by $i_h\colon W_h^{1,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, $j_h\colon \tilde{W}_h^{1-\frac{1}{r},r}(\Gamma_s) \hookrightarrow \tilde{W}^{1-\frac{1}{r},r}(\Gamma_s)$ and $k_h\colon W_h^{-\frac{1}{2},2}(\partial\Omega) \hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$.

The discrete problem involves the discretized functional

$$J_h(u_h, v_h) = \langle A(\varepsilon(u_h)), \varepsilon(u_h)\rangle + \frac{1}{2}\langle S_h(u_h|_{\partial\Omega} + v_h), u_h|_{\partial\Omega} + v_h\rangle - L_h(u_h, v_h)$$

on X_h^p. Here

$$S_h = \frac{1}{2}(\mathcal{W} + (1-\mathcal{K}')k_h(k_h^*\mathcal{V}k_h)^{-1}k_h^*(1-\mathcal{K})) \quad \text{and} \quad L_h(u_h, v_h) = \int_\Omega fu_h + \langle t_0 + S_h u_0, u_h|_{\partial\Omega} + v_h\rangle.$$

The approximate Steklov–Poincaré operator S_h is coercive uniformly in h, i.e. $\langle S_h u_h, u_h\rangle \geq \alpha_S \|u_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}$ with α_S independent of h, provided h is sufficiently small [27]. Therefore, as in the previous section, we obtain the following result.

Theorem 2. *The discrete minimization problem*

$$J_h(\hat{u}_h, \hat{v}_h) + j(\hat{v}_h) = \min_{(u_h,v_h)\in K\cap X_h^p} J_h(u_h, v_h) + j(v_h) \tag{4.1}$$

admits a unique minimizer.

Our Galerkin method for the numerical approximation relies on an equivalent reformulation of the continuous and discretized minimization problems (3.3), (4.1) as variational inequalities: find $(\hat{u}, \hat{v}) \in K$ such that

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u - \hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v}\rangle + j(v) - j(\hat{v}) \geq L(u - \hat{u}, v - \hat{v}) \tag{4.2}$$

for all $(u, v) \in K$.

The discretized variant reads as follows: find $(\hat{u}_h, \hat{v}_h) \in K \cap X_h^p$ such that

$$\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h)\rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle + j(v_h) - j(\hat{v}_h) \geq L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \tag{4.3}$$

for all $(u_h, v_h) \in K \cap X_h^p$.

As these variational inequalities are equivalent to the minimization problems (3.3), (4.1), Theorems 1 and 2 assure that they admit unique solutions.

Theorem 3. *Let* $(\hat{u}, \hat{v}) \in K$ *the solution to the variational inequality* (4.2) *and* $(\hat{u}_h, \hat{v}_h) \in K \cap X_h^p$ *the solution to the discretization* (4.3).

(a) *The following a priori estimate holds with* $q = \max\{p, 2\}$*:*

$$\begin{aligned}\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|^q_{X^p} \lesssim \inf_{(u_h,v_h)\in K\cap X_h^p} &\{\|\varepsilon(\hat{u} - u_h)\|_{L^p(\Omega)} + \|(\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{v} - v_h\|_{L^1(\Gamma_s)}\}\\ &+ \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(\mathcal{V}^{-1}(1-\mathcal{K})(\hat{u} + \hat{v} - u_0), W_h^{-\frac{1}{2},2}(\partial\Omega))^2.\end{aligned}$$

(b) *If* $\hat{v} \in \tilde{W}^{\frac{1}{2},2}(\Gamma_s)^n$*, e.g. for* $p \geq 2$ *or* $\Gamma_s = \emptyset$*, the estimate can be improved to*

$$\begin{aligned}\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|^q_{X^p} \lesssim \inf_{(u_h,v_h)\in K\cap X_h^p} &\{\|\varepsilon(\hat{u} - u_h)\|^\beta_{L^p(\Omega)} + \|(\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{v} - v_h\|_{L^1(\Gamma_s)}\}\\ &+ \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(\mathcal{V}^{-1}(1-\mathcal{K})(\hat{u} + \hat{v} - u_0), W_h^{-\frac{1}{2},2}(\partial\Omega))^2.\end{aligned}$$

Here $\beta = \frac{2}{3-p}$ *for* $p < 2$*, while* $\beta = p' = \frac{p}{p-1}$ *for* $p \geq 2$.

Proof. Adding the continuous variational inequality (4.2) with $(u, v) = (\hat{u}_h, \hat{v}_h)$ and the discrete variational inequality (4.3), we see that

$$\begin{aligned}0 \leq\ &\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}\rangle\\ &+ j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v})\\ &+ \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) - \varepsilon(\hat{u}_h)\rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle\\ &+ j(v_h) - j(\hat{v}_h) - L_h(u_h - \hat{u}_h, v_h - \hat{v}_h).\end{aligned}$$

Hence

$$\begin{aligned}
&\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(\hat{u}_h)\rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\
&\quad \le \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(\hat{u}_h)\rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\
&\qquad + \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}\rangle \\
&\qquad + j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v}) \\
&\qquad + \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) - \varepsilon(\hat{u}_h)\rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle \\
&\qquad + j(v_h) - j(\hat{v}_h) - L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \\
&\quad = \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) - \varepsilon(\hat{u})\rangle + \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u})|_{\partial\Omega} + v_h - \hat{v}\rangle \\
&\qquad + j(v_h) - j(\hat{v}) - L(u_h - \hat{u}, v_h - \hat{v}) - (L_h - L)(u_h - \hat{u}_h, v_h - \hat{v}_h) \\
&\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle \\
&\quad = \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(u_h)\rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle \\
&\qquad + \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h) - \varepsilon(\hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u})|_{\partial\Omega} + v_h - \hat{v}\rangle - L(u_h - \hat{u}, v_h - \hat{v}) \\
&\qquad + j(v_h) - j(\hat{v}) - (L_h - L)(u_h - \hat{u}_h, v_h - \hat{v}_h) \\
&\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle. \qquad (4.4)
\end{aligned}$$

Let $p < 2$. To bound $\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}) - \varepsilon(u_h)\rangle$, we use estimate (3.1) and Young's inequality for any $\delta > 0$ as follows:

$$\begin{aligned}
\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle &\lesssim \|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)}^{p-1} \|\varepsilon(\hat{u} - u_h)\|_{L^p(\Omega)} \\
&\lesssim \delta^{\frac{2}{p-1}} \|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)}^2 + \delta^{-\frac{2}{3-p}} \|\varepsilon(\hat{u} - u_h)\|_{L^p(\Omega)}^{\frac{2}{3-p}}.
\end{aligned}$$

On the other hand, for $p \ge 2$, the upper bound (3.2) yields

$$\begin{aligned}
\langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle &\lesssim \|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)} \|\varepsilon(\hat{u} - u_h)\|_{L^p(\Omega)} \\
&\lesssim \delta^{p} \|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)}^{p} + \delta^{-p'} \|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)}^{p'}.
\end{aligned}$$

As for the second term, we use the boundedness of S from $W^{\frac{1}{2},2}(\partial\Omega)^n$ to $W^{-\frac{1}{2},2}(\partial\Omega)^n$ to estimate

$$\begin{aligned}
&\langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle \\
&\quad \lesssim \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)} \|(\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\|_{W^{\frac{1}{2},2}(\partial\Omega)} \\
&\quad \lesssim \delta \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \delta^{-1} \|(\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2.
\end{aligned}$$

Without further assumptions on $\hat{v}$, using the Cauchy–Schwarz inequality, we estimate the second line of (4.4) by a multiple of

$$\|\varepsilon(u_h - \hat{u})\|_{L^p(\Omega)} + \|(u_h - \hat{u})|_{\partial\Omega} + v_h - \hat{v})\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

For part (b), where $\hat{v} \in \tilde{W}^{\frac{1}{2},2}(\Gamma_s)$, one may use the variational inequality for an improved estimate: substituting $(u, v) = (u_h, \hat{v})$ and $(u, v) = (2\hat{u} - u_h, \hat{v})$ into the variational inequality on X^p, we obtain

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h) - \varepsilon(\hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u})|_{\partial\Omega}\rangle = L(u_h - \hat{u}, 0).$$

With this, the second line of (4.4) reduces to $\langle S(\hat{u}|_{\partial\Omega} + \hat{v}), v_h - \hat{v}\rangle + L(0, \hat{v} - v_h)$, i.e. to

$$-\langle t_0 - S(\hat{u}|_{\partial\Omega} + \hat{v} - u_0), v_h - \hat{v}\rangle = -\langle A'(\varepsilon(\hat{u})) \cdot \nu, v_h - \hat{v}\rangle \le \|\mathcal{F}\|_{L^\infty(\Gamma_s)} \|v_{h,n} - \hat{v}_n\|_{L^1(\Gamma_s)}.$$

Here we have used (2.1) with $u_c|_{\partial\Omega} = \hat{u}|_{\partial\Omega} + \hat{v} - u_0$, as well as the transmission condition in (1.1c) on $\partial\Omega$. For the third line of (4.4),

$$j(v_h) - j(\hat{v}) = \int_{\Gamma_s} \mathcal{F}(|v_{h,t}| - |\hat{v}_t|) \le \int_{\Gamma_s} \mathcal{F}(|v_{h,t} - \hat{v}_t|) \le \|\mathcal{F}\|_{L^\infty(\Gamma_s)} \|v_{h,t} - \hat{v}_t\|_{L^1(\Gamma_s)}.$$

Finally, the last line of (4.4) simplifies as follows:

$$\begin{aligned}
&- (L_h - L)(u_h - \hat{u}_h, v_h - \hat{v}_h) + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle \\
&\quad = \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \delta^{-1}\|(S_h - S)(\hat u_h|_{\partial\Omega} + \hat v_h - u_0)\|^2_{W^{-\frac12,2}(\partial\Omega)} + \delta\|(u_h - \hat u_h)|_{\partial\Omega} + v_h - \hat v_h\|^2_{W^{\frac12,2}(\partial\Omega)}\\
&\lesssim \delta^{-1}\|(S_h - S)(\hat u_h|_{\partial\Omega} + \hat v_h - u_0)\|^2_{W^{-\frac12,2}(\partial\Omega)}\\
&\qquad + \delta\|(u_h - \hat u)|_{\partial\Omega} + v_h - \hat v\|^2_{W^{\frac12,2}(\partial\Omega)} + \delta\|(\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\|^2_{W^{\frac12,2}(\partial\Omega)}.
\end{aligned}
$$

The term involving $S_h - S$ is known to be bounded by (see [8])

$$
\operatorname{dist}_{W^{-\frac12,2}(\partial\Omega)}(\mathcal{V}^{-1}(1-\mathcal{K})(\hat u + \hat v - u_0), W_h^{-\frac12,2}(\partial\Omega))^2.
$$

To sum up, for general $\hat v$, we obtain, for $\alpha = \frac{p}{p-1}$, $\beta = \frac{2}{3-p}$ $(p < 2)$ or $\alpha = p$, $\beta = p'$ $(p \geq 2)$ and $q = \max\{p, 2\}$,

$$
\begin{aligned}
&\langle A'(\varepsilon(\hat u)) - A'(\varepsilon(\hat u_h)), \varepsilon(\hat u) - \varepsilon(\hat u_h)\rangle + \langle S((\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h), (\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\rangle\\
&\quad\lesssim \delta^{\alpha}\|\varepsilon(\hat u - \hat u_h)\|^q_{L^p(\Omega)} + \delta\|(\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\|^2_{W^{\frac12,2}(\partial\Omega)} + \delta^{-\beta}\|\varepsilon(\hat u - u_h)\|^{\beta}_{L^p(\Omega)}\\
&\qquad + \|\varepsilon(u_h - \hat u)\|_{L^p(\Omega)} + \|(u_h - \hat u)|_{\partial\Omega} + v_h - \hat v\rangle\|_{W^{\frac12,2}(\partial\Omega)}\\
&\qquad + \delta^{-1}\|(\hat u - u_h)|_{\partial\Omega} + \hat v - v_h\|^2_{W^{\frac12,2}(\partial\Omega)} + \|v_h - \hat v\|_{L^1(\Gamma_s)} + \delta\|(u_h - \hat u)|_{\partial\Omega} + v_h - \hat v\|^2_{W^{\frac12,2}(\partial\Omega)}\\
&\qquad + \delta^{-1}\operatorname{dist}_{W^{-\frac12,2}(\partial\Omega)}(\mathcal{V}^{-1}(1-\mathcal{K})(\hat u + \hat v - u_0), W_h^{-\frac12,2}(\partial\Omega))^2.
\end{aligned}
$$

The lowest exponents dominate.

When $\hat v \in \tilde W^{\frac12,2}(\Gamma_s)^n$, the estimates for (4.4) yield

$$
\begin{aligned}
&\langle A'(\varepsilon(\hat u)) - A'(\varepsilon(\hat u_h)), \varepsilon(\hat u) - \varepsilon(\hat u_h)\rangle + \langle S((\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h), (\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\rangle\\
&\quad\lesssim \delta^{\alpha}\|\varepsilon(\hat u - \hat u_h)\|^q_{L^p(\Omega)} + \delta\|(\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\|^2_{W^{\frac12,2}(\partial\Omega)} + \delta^{-\beta}\|\varepsilon(\hat u - u_h)\|^{\beta}_{L^p(\Omega)}\\
&\qquad + \delta^{-1}\|(\hat u - u_h)|_{\partial\Omega} + \hat v - v_h\|^2_{W^{\frac12,2}(\partial\Omega)} + \|v_h - \hat v\|_{L^1(\Gamma_s)} + \delta\|(u_h - \hat u)|_{\partial\Omega} + v_h - \hat v\|^2_{W^{\frac12,2}(\partial\Omega)}\\
&\qquad + \delta^{-1}\operatorname{dist}_{W^{-\frac12,2}(\partial\Omega)}(\mathcal{V}^{-1}(1-\mathcal{K})(\hat u + \hat v - u_0), W_h^{-\frac12,2}(\partial\Omega))^2.
\end{aligned}
$$

Note that, as in Lemma 2, the monotonicity of A' and the coercivity of S allow to bound the left-hand side from below by $\|\varepsilon(\hat u - \hat u_h)\|^q_{L^p(\Omega)} + \|(\hat u - \hat u_h)|_{\partial\Omega} + \hat v - \hat v_h\|^2_{W^{\frac12,2}(\partial\Omega)}$.

Choosing $\delta > 0$ sufficiently small, the claimed estimates follow. □

Remark. Theorem 3 proves convergence of the proposed FE-BE coupling procedure for quasi-uniform grid refinements. However, generic weak solutions to the contact problem (1.1) only belong to X^p and not to any higher-order Sobolev space. Therefore, the convergence can be arbitrarily slow as the grid size h tends to 0.

5 A Posteriori Analysis I

In this section, we establish an a posteriori error estimate for the Galerkin solutions obtained from the variational inequality (4.3). A refinement of the argument, given in Section 6, will lead to a fully computable upper bound for the error in Theorem 5.

Theorem 4. *Let $r = \min\{p, 2\}$ and $q = \max\{p, 2\}$. Let $(\hat u, \hat v) \in K$ be the solution to the variational inequality* (4.2) *and $(\hat u_h, \hat v_h) \in K \cap X_h^p$ the solution to the discretization* (4.3). *The following a posteriori estimate holds:*

$$
\begin{aligned}
\|\hat u - \hat u_h, \hat v - \hat v_h\|^q_{X^p} \lesssim{}& \Big(\sum_{T\subset\Omega} h_T^{p'}\|f + \operatorname{div} A'(\varepsilon(\hat u_h))\|^{p'}_{L^{p'}(T)}\Big)^{q'/p'} + \Big(\sum_{E\subset\Omega} h_E\|[A'(\varepsilon(\hat u_h))\nu]\|^{p'}_{L^{p'}(E)}\Big)^{q'/p'}\\
&+ \|t_0 + S_h(u_0 - \hat u_h|_{\partial\Omega} + \hat v_h) - A'(\varepsilon(\hat u_h))\nu\|^{q'}_{W^{1-\frac1r,r'}(\partial\Omega)}\\
&+ \int_{\Gamma_s}\{\mathcal{F}|\hat v_{h,t}| + \sigma_t(\hat u_h)\hat v_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat u_h)\hat v_{h,n})_+\\
&+ \|\sigma_n(\hat u_h)_+\|_{W^{-1+\frac1r,r'}(\Gamma_s)} + \|(|\sigma_t(\hat u_h)| - \mathcal{F})_+\|_{W^{-1+\frac1r,r'}(\Gamma_s)}\\
&+ \|(S_h - S)(\hat u_h|_{\partial\Omega} + \hat v_h - u_0)\|^2_{\tilde W^{1-\frac1r,r'}(\Gamma_s)}.
\end{aligned}
$$

For the proof of Theorem 4, we consider the variational inequality (4.3) for $v_h = \hat{v}_h$ and with $u_h \mapsto u_h$ and $u_h \mapsto 2\hat{u}_h - u_h$, respectively. Problem (4.3) then splits into an interior equation and an inequality on the boundary: for all $(u_h, v_h) \in K \cap X_h^p$,

$$\begin{aligned}\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h)\rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), u_h|_{\partial\Omega}\rangle &= \int_\Omega f u_h + \langle t_0 + S_h u_0, u_h\rangle = L_h(u_h, 0),\\ \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), v_h - \hat{v}_h\rangle + j(v_h) - j(\hat{v}_h) &\geq \langle t_0 + S u_0, v_h - \hat{v}_h\rangle = L_h(0, v_h - \hat{v}_h).\end{aligned} \tag{5.1}$$

For the continuous inequality, we only get a weaker assertion because $u|_{\partial\Omega} + v$ needs to be in $W^{\frac{1}{2},2}(\partial\Omega)$. Choosing $u = \hat{u} + \hat{u}_h - u_h$, $v = \hat{v} + \hat{v}_h - v_h$ for any $(u_h, v_h) \in X_h^p$ with $v_h \leq \hat{v} + \hat{v}_h$ transforms (4.2) into the estimate

$$\begin{aligned}&\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h)\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle\\ &\quad\leq j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h).\end{aligned} \tag{5.2}$$

In combination with the following coercivity estimate, we may start to derive an a posteriori estimate:

$$\begin{aligned}&\|\varepsilon(\hat{u} - \hat{u}_h)\|^q_{L^p(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}\\ &\quad\lesssim \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle + \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h)\rangle\\ &\qquad + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle\\ &\qquad + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle.\end{aligned} \tag{5.3}$$

We consider the second and fourth term on the right-hand side of (5.3),

$$\begin{aligned}&\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h)\rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h)\rangle\\ &\quad + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle.\end{aligned}$$

Applying the equality in (5.1) to

$$\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h)\rangle + \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle$$

and inequality (5.2) to

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h)\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle,$$

we estimate their sum by

$$\begin{aligned}&- L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h)\\ &\quad - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), v_h - \hat{v}_h\rangle + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle.\end{aligned}$$

For the terms

$$\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u} - u_h)\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle,$$

we use the variational inequality (4.2) with $(u, v) = (u_h, v_h)$ to conclude from (5.3) the following:

$$\begin{aligned}&\|\varepsilon(\hat{u} - \hat{u}_h)\|^q_{L^p(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}\\ &\quad\lesssim L(\hat{u} - u_h, \hat{v} - v_h) + j(v_h) - j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle\\ &\qquad - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), v_h - \hat{v}_h\rangle\\ &\qquad - L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h)\\ &\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle\\ &\quad= \int_\Omega f(\hat{u} - u_h) + \langle t_0 + S u_0, (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle + j(\hat{v} + \hat{v}_h - v_h) + j(v_h) - 2j(\hat{v})\\ &\qquad - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle\\ &\qquad - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), v_h - \hat{v}_h\rangle - \langle (S_h - S)u_0, (u_h - \hat{u}_h)|_{\partial\Omega}\rangle + \langle t_0 + S u_0, v_h - \hat{v}_h\rangle\\ &\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle\\ &\quad= \int_\Omega f(\hat{u} - u_h) + j(\hat{v} + \hat{v}_h - v_h) + j(v_h) - 2j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - u_h)\rangle\\ &\qquad + \langle t_0 - S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle\\ &\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle.\end{aligned}$$

Integrating by parts, we obtain

$$\int_\Omega f(\hat u - u_h) - \langle A'(\varepsilon(\hat u_h)), \varepsilon(\hat u - u_h)\rangle = \int_\Omega (f + \operatorname{div} A'(\varepsilon(\hat u_h)))(\hat u - u_h) - \sum_{E\subset\Omega}\int_E [A'(\varepsilon(\hat u_h))\nu](\hat u - u_h)|_{\partial\Omega} - \langle A'(\varepsilon(\hat u_h))\nu, (\hat u - u_h)|_{\partial\Omega}\rangle_{\partial\Omega}.$$

The first term is estimated as usual for $u_h = \hat u_h + \Pi_h(\hat u - \hat u_h)$ using the Hölder inequality and the properties of the interpolation operator Π_h (see e.g. [7]),

$$\int_\Omega (f + \operatorname{div} A'(\varepsilon(\hat u_h)))(\hat u - u_h) \lesssim \|\hat u - \hat u_h\|_{W^{1,p}(\Omega)}\Big(\sum_{T\subset\Omega} h_T^{p'}\|f + \operatorname{div} A'(\varepsilon(\hat u_h))\|^{p'}_{L^{p'}(T)}\Big)^{1/p'},$$

with $p' = \frac{p}{p-1}$. Similarly,

$$\sum_{E\subset\Omega}\int_E [A'(\varepsilon(\hat u_h))\nu](\hat u - u_h)|_{\partial\Omega} \lesssim \|\hat u - \hat u_h\|_{W^{1,p}(\Omega)}\Big(\sum_{E\subset\Omega} h_E\|[A'(\varepsilon(\hat u_h))\nu]\|^{p'}_{L^{p'}(E)}\Big)^{1/p'}.$$

It remains to consider the boundary contributions. To do so, recall the strong formulation of the contact conditions in terms of $\sigma_n(u)$ and $\sigma_t(u)$ on Γ_s. In terms of $v = u_0 + u_c - u$, they are written as

$$\sigma_n(u) \le 0,\quad v_n \le 0,\quad \sigma_n(u)v_n = 0,$$
$$|\sigma_t(u)| \le \mathcal{F},\quad \sigma_t(u)v_t + \mathcal{F}|v_t| = 0.$$

Then, substituting $v_h = \hat v_h$, we obtain

$$j(\hat v + \hat v_h - v_h) = j(\hat v) = \int_{\Gamma_s}\mathcal{F}|\hat v_t| = -\langle\sigma_t(\hat u), \hat v_t\rangle = -\langle\sigma(\hat u), \hat v\rangle.$$

Also,

$$j(\hat v_h) - \langle A'(\varepsilon(\hat u_h))\nu, \hat v_h\rangle \le \int_{\Gamma_s}\{\mathcal{F}|\hat v_{h,t}| + \sigma_t(\hat u_h)\hat v_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat u_h)\hat v_{h,n})_+.$$

Together, the terms

$$\begin{aligned} &j(\hat v + \hat v_h - v_h) + j(v_h) - 2j(\hat v) - \langle A'(\varepsilon(\hat u_h))\nu, (\hat u - u_h)|_{\partial\Omega}\rangle_{\partial\Omega}\\ &\quad= -j(\hat v) + j(\hat v_h) - \langle A'(\varepsilon(\hat u_h))\nu, \hat v_h\rangle_{\partial\Omega} - \langle A'(\varepsilon(\hat u_h))\nu, (\hat u - u_h)|_{\partial\Omega} + \hat v - \hat v_h - \hat v\rangle_{\partial\Omega}\end{aligned}$$

are hence dominated by

$$\begin{aligned} &\langle\sigma(\hat u), \hat v\rangle + \int_{\Gamma_s}\{\mathcal{F}|\hat v_{h,t}| + \sigma_t(\hat u_h)\hat v_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat u_h)\hat v_{h,n})_+ - \langle A'(\varepsilon(\hat u_h))\nu, (\hat u - u_h)|_{\partial\Omega} + \hat v - \hat v_h - \hat v\rangle_{\partial\Omega}\\ &\quad= \int_{\Gamma_s}\{\mathcal{F}|\hat v_{h,t}| + \sigma_t(\hat u_h)\hat v_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat u_h)\hat v_{h,n})_+ - \langle A'(\varepsilon(\hat u_h))\nu, (\hat u - u_h)|_{\partial\Omega} + \hat v - \hat v_h\rangle_{\partial\Omega} + \langle\sigma(\hat u) - \sigma(\hat u_h), \hat v\rangle.\end{aligned}$$

We split the σ-term into tangential and normal parts

$$\langle\sigma(\hat u) - \sigma(\hat u_h), \hat v\rangle = \langle\sigma_n(\hat u) - \sigma_n(\hat u_h), \hat v_n\rangle + \langle\sigma_t(\hat u) - \sigma_t(\hat u_h), \hat v_t\rangle.$$

Using the contact conditions $\langle\sigma_n(\hat u), \hat v_n\rangle = 0$ and $v_n \le 0$, we estimate the normal part as follows ($r' = \frac{r}{r-1}$):

$$\langle\sigma_n(\hat u) - \sigma_n(\hat u_h), \hat v_n\rangle \le -\langle\sigma_n(\hat u_h), \hat v_n\rangle \le -\langle\sigma_n(\hat u_h)_+, \hat v_n\rangle \lesssim \|\sigma_n(\hat u_h)_+\|_{W^{-1+\frac{1}{r},r'}(\Gamma_s)}\|\hat v_n\|_{\tilde W^{1-\frac{1}{r},r}(\Gamma_s)}.$$

For the tangential contribution, involving the Tresca friction, we find it convenient to write $\sigma_t(\hat u) = -\zeta\mathcal{F}$ with $|\zeta| \le 1$ and $|v_t| = \zeta v_t$. Then

$$\begin{aligned}\langle\sigma_t(\hat u) - \sigma_t(\hat u_h), \hat v_t\rangle &= -\langle\zeta\mathcal{F}, \hat v_t\rangle - \langle\sigma_t(\hat u_h), \hat v_t\rangle = -\langle\mathcal{F}, |\hat v_t|\rangle - \langle\sigma_t(\hat u_h), \hat v_t\rangle\\ &\le \langle(|\sigma_t(\hat u_h)| - \mathcal{F})_+, |\hat v_t|\rangle \lesssim \|(|\sigma_t(\hat u_h)| - \mathcal{F})_+\|_{W^{-1+\frac{1}{r},r'}(\Gamma_s)}\||\hat v_t|\|_{\tilde W^{1-\frac{1}{r},r}(\Gamma_s)}\\ &\lesssim \|(|\sigma_t(\hat u_h)| - \mathcal{F})_+\|_{W^{-1+\frac{1}{r},r'}(\Gamma_s)}\|\hat v_t\|_{\tilde W^{1-\frac{1}{r},r}(\Gamma_s)}.\end{aligned}$$

We conclude

$$\begin{aligned}
&\|\varepsilon(\hat{u}-\hat{u}_h)\|^q_{L^p(\Omega)} + \|(\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^q_{W^{\frac{1}{2},2}(\partial\Omega)} \\
&\quad \lesssim \|\hat{u}-\hat{u}_h\|_{W^{1,p}(\Omega)}\Big(\sum_{T\subset\Omega} h_T^{p'}\|f + \operatorname{div} A'(\varepsilon(\hat{u}_h))\|^{p'}_{L^{p'}(T)}\Big)^{1/p'} \\
&\qquad + \|\hat{u}-\hat{u}_h\|_{W^{1,p}(\Omega)}\Big(\sum_{E\subset\Omega} h_E\|[A'(\varepsilon(\hat{u}_h))\nu]\|^{p'}_{L^{p'}(E)}\Big)^{1/p'} \\
&\qquad + \int_{\Gamma_s}\{\mathcal{F}|\hat{v}_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat{u}_h)\hat{v}_{h,n})_+ \\
&\qquad + \|t_0 + S_h(u_0 - \hat{u}_h|_{\partial\Omega} + \hat{v}_h) - A'(\varepsilon(\hat{u}_h))\nu\|^{q'}_{W^{1-\frac{1}{r},r'}(\partial\Omega)} \\
&\qquad + \|\sigma_n(\hat{u}_h)_+\|_{\tilde{W}^{1-\frac{1}{r},r'}(\Gamma_s)} + \|(|\sigma_t(\hat{u}_h)| - \mathcal{F})_+\|_{\tilde{W}^{-1+\frac{1}{r},r'}(\Gamma_s)} \\
&\qquad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle.
\end{aligned}$$

The assertion of Theorem 4 follows.

6 A Posteriori Analysis II

Note that the final term in the a posteriori estimate in Theorem 4 is not computable, and practical computations will directly formulate the operator S in terms of the layer potentials $\mathcal{V}, \mathcal{W}, \mathcal{K}, \mathcal{K}'$ (or equivalently, the Calderón projector). The main result of this section is an a posteriori error estimate, which contains only computable terms on the right-hand side. This is achieved by expressing the bilinear form directly in terms of the layer potentials, rather than S, and deducing coercivity of the bilinear form from the coercivity properties of $\mathcal{V}$ and $\mathcal{W}$.

We consider the space

$$Y^p = X^p \times W^{-\frac{1}{2},2}(\partial\Omega)^n,$$

equipped with the norm

$$\|u, v, \phi\|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_s)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}.$$

From Lemma 1, we conclude that $(Y^p, \|\cdot\|_{Y^p})$ is a Banach space and

$$|u, v, \phi|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|_{\partial\Omega} + v\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}$$

an equivalent norm on Y^p. We consider the discretization in finite-dimensional subspaces

$$Y_h^p = X_h^p \times W_h^{-\frac{1}{2},2}(\partial\Omega)^n$$

of Y^p. The following theorem will be obtained.

Theorem 5. *Let* $r = \min\{p, 2\}$ *and* $q = \max\{p, 2\}$. *Let* $(\hat{u}, \hat{v}, \hat{\phi}) \in Y^p$ *be the solution to the variational inequality* (6.2) *and* $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in K \cap Y_h^p$ *the solution of its discretization. Then, with* $F(a) = |A'(a)|^{1/2}|a|^{-1/2}a$, *the following a posteriori estimate holds:*

$$\begin{aligned}
\|\hat{u}-\hat{u}_h, \hat{v}-\hat{v}_h, \hat{\phi}-\hat{\phi}_h\|^q_{Y^p} &\lesssim \Big(\sum_{T\subset\Omega} h_T^{p'}\|f + \operatorname{div} A'(\varepsilon(\hat{u}_h))\|^{p'}_{L^{p'}(T)}\Big)^{q'/p'} + \sum_{E\subset\Omega} h_E\|[F(\varepsilon(u_h))]\|^2_{L^2(E)} \\
&\quad + \|t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}'-1)\hat{\phi}_h - A'(\varepsilon(\hat{u}_h))\nu\|^{q'}_{W^{1-\frac{1}{r},r'}(\partial\Omega)} \\
&\quad + \|\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\
&\quad + \int_{\Gamma_s}\{\mathcal{F}|\hat{v}_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}\} + \int_{\Gamma_s}(\sigma_n(\hat{u}_h)\hat{v}_{h,n})_+ \\
&\quad + \|\sigma_n(\hat{u}_h)_+\|_{W^{-1+\frac{1}{r},r'}(\Gamma_s)} + \|(|\sigma_t(\hat{u}_h)| - \mathcal{F})_+\|_{W^{-1+\frac{1}{r},r'}(\Gamma_s)}.
\end{aligned}$$

The proof of Theorem 5 relies on a coercivity estimate, which we now establish. We use a theoretical stabilization as in [12]: let $r_1, \dots, r_D$ be a basis of the space of rigid body motions, and consider their orthogonal projections $\xi_1, \dots, \xi_D$ onto $L^2(\partial\Omega)$. The arguments in [12, Lemma 4 and Proposition 5] show that $|u, v, \phi|_{Y^p}$ is equivalent to the norm

$$|u, v, \phi|^2_{Y^p,s} = \|\varepsilon(u)\|^2_{L^p(\Omega)} + \langle \mathcal{W}(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v\rangle + \langle \phi, \mathcal{V}\phi\rangle + \sum_{j=1}^{D} |\langle \xi_j, (1 - \mathcal{K})(u|_{\partial\Omega} + v) + \mathcal{V}\phi\rangle|^2. \tag{6.1}$$

On Y^p, we have the following equivalent formulation of the contact problem (1.1): find

$$(\hat{u}, \hat{v}, \hat{\phi}) \in K' = (K \cap X^p) \times W^{-\frac{1}{2},2}(\partial\Omega)^n$$

such that, for all $(u, v, \phi) \in K'$,

$$\begin{aligned}
\langle A'(\varepsilon(\hat{u})), \varepsilon(u)\rangle + \langle \mathcal{W}(\hat{u}|_{\partial\Omega} + \hat{v}) + (\mathcal{K}' - 1)\hat{\phi}, u|_{\partial\Omega}\rangle &= \int_\Omega fu + \langle t_0 + \mathcal{W}u_0, u\rangle, \\
\langle \mathcal{W}(\hat{u}|_{\partial\Omega} + \hat{v}) + (\mathcal{K}' - 1)\hat{\phi}, v - \hat{v}\rangle + j(v) - j(\hat{v}) &\geq \langle t_0 + \mathcal{W}u_0, v - \hat{v}\rangle, \\
\langle \phi, \mathcal{V}\hat{\phi} + (1 - \mathcal{K})(\hat{u}|_{\partial\Omega} + \hat{v})\rangle &= \langle \phi, (1 - \mathcal{K})u_0\rangle.
\end{aligned}$$

The latter can be written as

$$B(\hat{u}, \hat{v}, \hat{\phi}; u - \hat{u}, v - \hat{v}, \phi - \hat{\phi}) + j(v) - j(\hat{v}) \geq \Lambda(u - \hat{u}, v - \hat{v}, \phi - \hat{\phi}) \tag{6.2}$$

for all $(u, v, \phi) \in K'$, with

$$\begin{aligned}
B(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) &= \langle A'(\varepsilon(u)), \varepsilon(\tilde{u})\rangle + \langle \mathcal{W}(u|_{\partial\Omega} + v) + (\mathcal{K}' - 1)\phi, \tilde{u}|_{\partial\Omega} + \tilde{v}\rangle + \langle \tilde{\phi}, \mathcal{V}\phi + (1 - \mathcal{K})(u|_{\partial\Omega} + v)\rangle, \\
\Lambda(u, v, \phi) &= \langle t_0 + \mathcal{W}u_0, u|_{\partial\Omega} + v\rangle + \int_\Omega fu + \langle \phi, (1 - \mathcal{K})u_0\rangle.
\end{aligned}$$

The discretized problem is obtained by restricting to Y^p_h, and we denote its solution by $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h)$. We also consider a stabilized problem that, for all $(u_h, v_h, \phi_h) \in K' \cap Y^p_h$,

$$\tilde{B}(\hat{u}_{s,h}, \hat{v}_{s,h}, \hat{\phi}_{s,h}; u_h - \hat{u}_{s,h}, v_h - \hat{v}_{s,h}, \phi_h - \hat{\phi}_{s,h}) + j(v_h) - j(\hat{v}_{s,h}) \geq \tilde{\Lambda}(u_h - \hat{u}_{s,h}, v_h - \hat{v}_{s,h}, \phi_h - \hat{\phi}_{s,h}),$$

where

$$\begin{aligned}
\tilde{B}(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) &= B(u, v, \phi; \tilde{u}, \tilde{v}, \tilde{\phi}) + \sum_{j=1}^{D} \langle \xi_j, \mathcal{V}\phi + (1 - \mathcal{K})(u|_{\partial\Omega} + v)\rangle \langle \xi_j, \mathcal{V}\tilde{\phi} + (1 - \mathcal{K})(\tilde{u}|_{\partial\Omega} + \tilde{v})\rangle, \\
\tilde{\Lambda}(u, v, \phi) &= \Lambda(u, v, \phi) + \sum_{j=1}^{D} \langle \xi_j, (1 - \mathcal{K})u_0\rangle \langle \xi_j, \mathcal{V}\phi + (1 - \mathcal{K})(u|_{\partial\Omega} + v)\rangle,
\end{aligned}$$

respectively. Because the variational inequality (6.2) is an equality in ϕ, as in [12, Proposition 3], the solution to the stabilized and nonstabilized problems coincide, $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) = (\hat{u}_{s,h}, \hat{v}_{s,h}, \hat{\phi}_{s,h})$. However, the stabilized variational inequality is coercive in the stabilized norm (6.1),

$$\begin{aligned}
&\|\varepsilon(\hat{u} - \hat{u}_h)\|^q_{L^p(\Omega)} + \langle \mathcal{W}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\
&\qquad + \langle \mathcal{V}(\hat{\phi} - \hat{\phi}_h), \hat{\phi} - \hat{\phi}_h\rangle + \sum_{j=1}^{D} |\langle \xi_j, (1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h)\rangle|^2 \\
&\lesssim \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u} - \hat{u}_h)\rangle \\
&\qquad + \langle \mathcal{W}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + (\mathcal{K}' - 1)(\hat{\phi} - \hat{\phi}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\
&\qquad + \langle (1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h), \hat{\phi} - \hat{\phi}_h\rangle \\
&\qquad + \sum_{j=1}^{D} |\langle \xi_j, (1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h)\rangle|^2.
\end{aligned}$$

The key estimate for B is given by the following lemma.

Lemma 3. *For all* $(\hat{u}, \hat{v}, \hat{\phi}), (u, v, \phi) \in Y^p = X^p \times W^{-\frac{1}{2},2}(\partial\Omega)^2$ *such that* $\|\varepsilon(\hat{u})\|_{L^p(\Omega)}, \|\varepsilon(u)\|_{L^p(\Omega)} < C$, *we have*

$$\begin{aligned}\|\varepsilon(\hat{u} - u)\|^2_{L^p(\Omega)} &+ \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{\phi} - \phi\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)}\\ &\lesssim_C B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - u, \hat{v} - v, \eta) - B(u, v, \phi; \hat{u} - u, \hat{v} - v, \eta),\end{aligned}$$

where $2\eta = \hat{\phi} - \phi + \mathcal{V}^{-1}(1 - \mathcal{K})((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v)$.

Proof. This follows from the identity

$$\begin{aligned}&B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - u, \hat{v} - v, \eta) - B(u, v, \phi; \hat{u} - u, \hat{v} - v, \eta)\\ &\quad= \langle A'(\varepsilon(\hat{u})) - A'(\varepsilon(u)), \varepsilon(\hat{u}) - \varepsilon(u)\rangle + \frac{1}{2}\langle \mathcal{W}((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\rangle\\ &\qquad+ \langle S((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\rangle + \frac{1}{2}\langle \mathcal{V}(\hat{\phi} - \phi), \hat{\phi} - \phi\rangle.\end{aligned}$$

Also,

$$\|\eta\|_{W^{-\frac{1}{2},2}(\partial\Omega)} \lesssim \|\hat{\phi} - \phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)} + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$ □

Combining the above results with the estimates from Section 5, we now prove the a posteriori error estimate stated at the beginning of this section.

Proof of Theorem 5. We use the estimate from Lemma 3 with $(u, v, \phi) = (\hat{u}_h, \hat{v}_h, \hat{\phi}_h)$,

$$\begin{aligned}\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|^q_{Y^p} &\lesssim \|\varepsilon(\hat{u} - \hat{u}_h)\|^2_{L^p(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{\phi} - \hat{\phi}_h\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)}\\ &\lesssim_C B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \eta) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \eta),\end{aligned}$$

where $2\eta = \hat{\phi} - \hat{\phi}_h + \mathcal{V}^{-1}(1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h)$. Using first the variational inequality (6.2) and then its discretization to estimate the right-hand side, we obtain

$$\begin{aligned}\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|^q_{Y^p} &\lesssim \Lambda(\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \eta) + j(\hat{v}_h) - j(\hat{v})\\ &\qquad- B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \eta)\\ &\lesssim \Lambda(e - e_h, \tilde{e} - \tilde{e}_h, \eta - \eta_h) + j(\hat{v}_h + \tilde{e}_h) - j(\hat{v})\\ &\qquad- B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; e - e_h, \tilde{e} - \tilde{e}_h, \eta - \eta_h)\end{aligned}$$

for all $(e_h, \tilde{e}_h, \eta_h) \in Y^p_h$. Here we have set $e = \hat{u} - \hat{u}_h$ and $\tilde{e} = \hat{v} - \hat{v}_h$. The definitions of B and Λ imply

$$\begin{aligned}&\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|^q_{Y^p}\\ &\quad\lesssim \int_\Omega f(e - e_h) + \langle t_0 + \mathcal{W}u_0, (e - e_h)|_{\partial\Omega} + \tilde{e} - \tilde{e}_h\rangle + \langle \eta - \eta_h, (1 - \mathcal{K})u_0\rangle\\ &\qquad+ \int_{\Gamma_s} \mathcal{F}|\hat{v}_{h,t} + \tilde{e}_{h,t}| - \int_{\Gamma_s} \mathcal{F}|\hat{v}_{h,t} + \tilde{e}_t|\\ &\qquad- \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(e - e_h)\rangle - \langle \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h) + (\mathcal{K}' - 1)\hat{\phi}_h, (e - e_h)|_{\partial\Omega} + \tilde{e} - \tilde{e}_h\rangle\\ &\qquad- \langle \eta - \eta_h, \mathcal{V}\hat{\phi}_h + (1 - \mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h)\rangle.\end{aligned}$$

Now one may proceed as in the proof of Theorem 4 to obtain the a posteriori error estimate in Theorem 5. □

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