Response Statistics and Failure Probability Determination of Nonlinear Stochastic Structural Dynamical Systems

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Erklärung

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Kurzfassung

Es werden neuartige Approximationstechniken für die Analyse und Bewertung nichtlinearer dynamischer Systeme im Bereich der stochastischen Dynamik vorgeschlagen. Die effiziente Bestimmung von Antwortstatistiken und Zuverlässigkeitsschätzungen für nichtlineare Systeme ist nach wie vor eine Herausforderung, insbesondere für Systeme mit singulären Matrizen oder fraktional abgeleiteten Termen. Diese Arbeit befasst sich mit den Herausforderungen von drei Hauptthemen.

Das erste Thema betrifft die Bestimmung der Antwortstatistiken nichtlinearer Systeme mit mehreren Freiheitsgraden und singulären Matrizen, die kombinierten deterministischen und stochastischen Erregungen ausgesetzt sind. Singuläre Matrizen können in den Bewegungsgleichungen technischer Systeme aus verschiedenen Gründen auftreten, z. B. aufgrund einer Modellierung mit redundanten Koordinaten oder aufgrund der Modellierung mit zusätzlichen Zwangsbedingungen. Außerdem ist es üblich, dass nichtlineare Systeme gleichzeitig stochastische und deterministische Erregungen erfahren.

In diesem Zusammenhang wird zunächst ein neuartiger Lösungsrahmen zur Bestimmung der Antwort solcher Systeme bei kombinierter deterministischer und stochastischer Anregung stationärer Art entwickelt. Dies wird durch die Anwendung der Methode des harmonischen Gleichgewichts und der verallgemeinerten statistischen Linearisierungsmethode erreicht. Es wird ein überbestimmtes Gleichungssystem erzeugt, welches mithilfe der Theorie verallgemeinerter inverser Matrizen gelöst wird.

Anschließend wird der entwickelte Rahmen auf Systeme ausgedehnt, die einer Mischung aus deterministischen und stochastischen Erregungen nichtstationärer Art unterliegen. Die verallgemeinerte statistische Linearisierungsmethode wird verwendet, um das nichtlineare Teilsystem zu behandeln, das einer nichtstationären stochastischen Anregung unterliegt. In Verbindung mit einer Zustandsraumformulierung führt dies zu einer Matrixdifferentialgleichung, die die stochastische Systemantwort beschreibt. Anschließend werden die entwickelten Gleichungen mit

numerischen Methoden gelöst.

Die Genauigkeit der vorgeschlagenen Techniken wurde durch die Anwendung auf mit redundanten Koordinaten modellierte nichtlineare strukturelle Systeme, sowie auf Systeme mit piezoelektrischen Vibrationsenergie-Harvestern demonstriert.

Das zweite Thema betrifft die normkonforme Analyse der stochastischen Dynamik nichtlinearer Struktursysteme mit fraktional abgeleiteten Elementen. Zunächst wird ein neuartiges Näherungsverfahren zur effizienten Bestimmung der Antwortspitzen nichtlinearer Struktursysteme mit fraktional abgeleiteten Elementen vorgeschlagen, die einer mit einem bestimmten seismischen Auslegungsspektrum kompatiblen Anregung ausgesetzt sind. Die vorgeschlagenen Methoden beinhalten die Ableitung eines evolutionären Erregungsleistungsspektrums, das dem Auslegungsspektrum in einem stochastischen Sinne entspricht. Der Spitzenwert wird durch die Verwendung äquivalenter linearer Elemente in Verbindung mit normgerechten Auslegungsspektren angenähert, was für Ingenieure in der Praxis von Vorteil ist. Es wurden nichtlineare Struktursysteme mit fraktional abgeleiteten Termen in den beschreibenden Bewegungsgleichungen betrachtet. Ein besonderes Merkmal ist die Verwendung von lokalisierten zeitabhängigen äquivalenten linearen Elementen, die den klassischen Ansätzen mit zeitinvarianten statistischen Linearisierungsmethoden überlegen sind.

Anschließend wird die Näherungsmethode erweitert, um eine stochastische inkrementelle dynamische Analyse für nichtlineare Struktursysteme mit fraktional abgeleiteten Elementen durchzuführen, die stochastischen Erregungen ausgesetzt sind, die mit modernen seismischen Bemessungsregeln abgestimmt sind. Die vorgeschlagene Methode stützt sich auf der Kombination von stochastischer Mittelwertbildung und statistischen Linearisierungsmethoden, was zu einem effizienten und umfassenden Weg zur Ermittlung der Wahrscheinlichkeitsdichtefunktion der Verschiebungsantwort führt. Anstelle der traditionellen Kurven wird eine stochastische inkrementelle dynamische Analyseoberfläche erzeugt, die zu zuverlässigen Statistiken höherer Ordnung der Systemantwort führt.

Abschließend wird das Problem der ersten Exkursionswahrscheinlichkeit nichtlinearer dynamischer Systeme bei ungenau definierten Gaußschen Lasten betrachtet. Dazu muss ein verschachteltes Doppelschleifenproblem gelöst werden, das im Allgemeinen ohne den Rückgriff auf Er-

satzmodellierungsverfahren nicht lösbar ist. Um diese Herausforderungen zu bewältigen, wird in dieser Arbeit zunächst ein verallgemeinerter Operator-Norm-Ansatz vorgeschlagen, der auf einer statistischen Linearisierungsmethode basiert. Seine Effizienz wird dadurch erreicht, dass die Doppelschleife durchbrochen wird und die Werte der epistemisch unsicheren Parameter, die Grenzen für die Versagenswahrscheinlichkeit darstellen, a priori bestimmt werden. Der vorgeschlagene Ansatz kann den Berechnungsaufwand erheblich verringern und eine zuverlässige Schätzung der Versagenswahrscheinlichkeit liefern.

Schlüsselwörter: Stochastische Dynamik, singuläre Matrizen, kombinierte Anregung, statistische Linearisierung, stochastische Mittelwertbildung, Vibrationsenergie-Harvester, Entwurfsspektrum, fraktional abgeleitete Elemente, inkrementelle dynamische Analyse, ungenaue Wahrscheinlichkeit, ersten Exkursionswahrscheinlichkeit, intervall-stochastischer Prozess.

Abstract

Novel approximation techniques are proposed for the analysis and evaluation of nonlinear dynamical systems in the field of stochastic dynamics. Efficient determination of response statistics and reliability estimates for nonlinear systems remains challenging, especially those with singular matrices or endowed with fractional derivative elements. This thesis addresses the challenges of three main topics.

The first topic relates to the determination of response statistics of multi-degree-of-freedom nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations. Notably, singular matrices can appear in the governing equations of motion of engineering systems for various reasons, such as due to a redundant coordinates modeling or due to modeling with additional constraint equations. Moreover, it is common for nonlinear systems to experience both stochastic and deterministic excitations simultaneously.

In this context, first, a novel solution framework is developed for determining the response of such systems subject to combined deterministic and stochastic excitation of the stationary kind. This is achieved by using the harmonic balance method and the generalized statistical linearization method. An over-determined system of equations is generated and solved by resorting to generalized matrix inverse theory.

Subsequently, the developed framework is appropriately extended to systems subject to a mixture of deterministic and stochastic excitations of the non-stationary kind. The generalized statistical linearization method is used to handle the nonlinear subsystem subject to non-stationary stochastic excitation, which, in conjunction with a state space formulation, forms a matrix differential equation governing the stochastic response. Then, the developed equations are solved by numerical methods.

The accuracy for the proposed techniques has been demonstrated by considering nonlinear structural systems with redundant coordinates modeling, as well as a piezoelectric vibration energy harvesting device have been employed in the relevant application part.

The second topic relates to code-compliant stochastic dynamic analysis of nonlinear structural systems with fractional derivative elements. First, a novel approximation method is proposed to efficiently determine the peak response of nonlinear structural systems with fractional derivative elements subject to excitation compatible with a given seismic design spectrum. The proposed methods involve deriving an excitation evolutionary power spectrum that matches the design spectrum in a stochastic sense. The peak response is approximated by utilizing equivalent linear elements, in conjunction with code-compliant design spectra, hopefully rendering it favorable to engineers of practice. Nonlinear structural systems endowed with fractional derivative terms in the governing equations of motion have been considered. A particular attribute pertains to utilizing localized time-dependent equivalent linear elements, which is superior to classical approaches utilizing standard time-invariant statistical linearization method.

Then, the approximation method is extended to perform stochastic incremental dynamical analysis for nonlinear structural systems with fractional derivative elements exposed to stochastic excitations aligned with contemporary aseismic codes. The proposed method is achieved by resorting to the combination of stochastic averaging and statistical linearization methods, resulting in an efficient and comprehensive way to obtain the response displacement probability density function. A stochastic incremental dynamical analysis surface is generated instead of the traditional curves, leading to a reliable higher order statistics of the system response.

Lastly, the problem of the first excursion probability of nonlinear dynamic systems subject to imprecisely defined stochastic Gaussian loads is considered. This involves solving a nested double-loop problem, generally intractable without resorting to surrogate modeling schemes. To overcome these challenges, this thesis first proposes a generalized operator norm framework based on statistical linearization method. Its efficiency is achieved by breaking the double loop and determining the values of the epistemic uncertain parameters that produce bounds on the probability of failure a priori. The proposed framework can significantly reduce the computational burden and provide a reliable estimate of the probability of failure.

Keywords: stochastic dynamics, singular matrices, combined excitation, statistical linearization, stochastic averaging, vibration energy harvester, design spectrum, fractional derivative elements, incremental dynamics analysis, imprecise probability, first-excursion probability, interval stochastic process.

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I dedicate this thesis to my parents

for their endless love

Contents

Chapter 1
Introduction

1.1 Motivation

Stochastic dynamics relates to the assessment of response statistics and reliability of structural or mechanical dynamical systems subject to uncertain conditions. In structural and mechanical engineering, many phenomena, attributed to the uncontrollable causation, are considered as the random processes, such as seismic ground motion, traveling vehicles over rough surfaces, turbulence due to strong winds, atmospheric turbulence and jet noise (e.g. Grigoriu, 2002; Roberts and Spanos, 2003; Li and Chen, 2009). The dynamical analysis of structural or mechanical systems subject to external excitations, modeled as the random processes, is known as random vibration analysis. In addition, the fundamental theory around stochastic dynamics deals with dynamical systems with uncertainty in model parameters, which can arise from various sources such as manufacturing tolerances, variations in material properties, and uncertainty in the geometry (e.g. Ibrahim, 2008). This leads to dynamical analysis with uncertainty structural parameters, known as stochastic structural analysis or stochastic finite element analysis (e.g. Li and Chen, 2009; Ghanem and Spanos, 2003). The significance of stochastic dynamics lies in the fact that most real physical systems are exposed to random dynamic environments during their lifespan. The randomness in the excitations or model parameters can result in stochastic system responses, potentially leading to failures in structural or mechanical systems (e.g. Roberts and Spanos, 2003).

It is generally recognized that Einstein's study of Brownian motion in 1905 (Einstein et al., 1905) is the origin of stochastic dynamics. Subsequently, in 1908, Langevin proposed a stochastic differential equation to describe the motion of Brownian particles close to a physical point of view (Langevin, 1908). Fokker and Planck later introduced the partial differential equation to model the Brownian motion of particles (Fokker, 1913; Planck, 1917), whose rigorous mathematical basis was further established by Kolmogorov (Kolmogoroff, 1931). The development of probability theory and relevant tools (such as spectral density), provided a solid mathematical foundation for the theory of stochastic dynamics (e.g. Wiener, 1930; Kolmogoroff, 1931; Rice, 1944). In the 1950s, random vibration was initially applied to structural or mechanical systems to solve the vibration problem of aircraft panels in the aerospace industry, basing on the principles of structural dynamics and probability theory (e.g. Crandall, 1958). Crandall is widely recognized as the person who introduces random vibration of mechanical systems to engineers. More details on the historical development of stochastic dynamics can be found in the literature review (Paez, 2006). Since then, there have been significant breakthroughs in the theory, technologies, experiments, and applications of stochastic dynamics, (e.g. Lin, 1967; Soong and Bogdanoff, 1974; Soong and Grigoriu, 1993; Newland, 1993; Lin, 1995; Elishakoff, 1999; Grigoriu, 2002; Roberts and Spanos, 2003; Lutes and Sarkani, 2004; Socha, 2007; Li and Chen, 2009).

The main objective of stochastic dynamics is to characterize the statistical properties of dynamical system responses, such as mean, variance, and probability density function (PDF), under uncertainty environments. These response statistics are subsequently utilized to perform reliability analysis, sensitivity analysis and optimization design, which facilitate engineers in realistic engineering design. Specifically, in structural and mechanical engineering fields, this is critical for ensuring structural safety and risk assessment, which often involves solving the first-passage problem by calculating the probability of the response reaching or crossing a predefined safety margin for the first time (e.g. Crandall et al., 1966).

A flourishing field of numerical and analytical techniques have been developed to determine response statistics and estimate reliability of stochastically excited dynamical systems. However, efficiently determining such response statistics and reliability estimates for nonlinear systems remains a challenging task, especially for some complex systems. In engineering, the inherent complexity of physical mechanism often causes more complex structural models, such as the presence of singular matrices in the governing equation of motion or the system modeling with fractional derivative elements. Thus, the attention has shifted towards modeling the stochastic dynamic behavior of these more complex systems (e.g. Jerez et al., 2022b). Note, in passing, that while it may seem intuitively evident that a system is complex, there is no universally accepted definition of complex systems. Further, the difficulty of analyzing complex systems arises not only from their physical modeling but also from the practical environment in which they operate, which involves more intricate excitation modeling, such as the combination of deterministic and stochastic excitation or imprecise stochastic loading.

In this thesis, an effort is made to address several challenges for nonlinear stochastic dynamical systems, especially for those with singular matrices and those endowed with fractional derivative elements. The thesis focuses, first, on determining the response statistics of nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations. Next, a novel framework is developed for approximating the peak response of hysteretic systems with fractional derivative elements subject to code-compliant seismic excitations. Finally, a highly efficient approach is proposed for bounding the first excursion probability of nonlinear systems under imprecise stochastic loading.

1.2 Systems with nonlinear or hysteretic behavior

1.2.1 Sources of nonlinearity

Almost all structural and mechanical systems in nature exhibit nonlinear behavior due to various factors. For instance, the nonlinear restoring force could be caused by the material nonlinearity and geometric nonlinearity (e.g. Bonet et al., 2021). Physical nonlinearity occurs when the relationship between the stress and stain does not satisfy the Hooke's law while geometric nonlinearity arises from the nonlinear relationship between the strain and displacement when a structure undergoes large displacements (e.g. Kerschen et al., 2006). In addition, structures may exhibit hysteretic behavior with a decrease in stiffness and/or strength under severe excitations. Thus, the restoring force is affected not only by the current state of the system, but also by its response history, resulting in energy dissipation. Then, nonlinear damping can arise from effects such as dry friction effects and hysteretic damping (e.g. Caughey and Vijayaraghavan, 1970; Sherif and Omar, 2004), while inertial nonlinearity is primarily induced by inerters used in vibration control (e.g. Smith, 2002; Marian and Giaralis, 2014). Furthermore, nonlinearity also arises from boundary conditions, such as free surfaces in fluids and contacts with rigid constraints (e.g. Tsai and Yue, 1996; Babitsky and Krupenin, 2001).

1.2.2 The importance of nonlinear stochastic dynamic analysis

As previously discussed, real-world systems commonly display nonlinear behavior during their lifetime, in contrast to the idealized linear assumption (e.g. Roberts and Spanos, 2003). Neglecting nonlinearity in system modeling can result in a particularly inadequate approximation of system behavior and inaccurate analysis, leading to potentially serious consequences. The consideration of nonlinearity becomes even more significant with increasing vibration amplitude, particularly in the context of random vibration, where the excitation in the stochastic sense has the probability to reach a large scale, leading to large response amplitudes (e.g. Iwan, 1974). Therefore, it is crucial to account for large excitations in the design phase, even if their probability is low (e.g. Roberts and Spanos, 2003). Specifically, in structural engineering, buildings and other structural systems commonly exhibit a hysterestic behavior under severe seismic excitations. A suitable stochastic description of seismic excitations is interwoven with an appropriate nonlinear model to ensure the accuracy of reliability assessment and design procedures for structural systems (e.g. Roberts and Spanos, 2003; Li and Chen, 2009). In addition, some special phenomena, such as chaotic vibration and self-excited vibration (e.g. Kansara et al., 2014), cannot be accurately assessed by resorting to the linear dynamics theory. Employing linear dynamics theory to evaluate the stochastic response of such intrinsically nonlinear systems will lead to errors (e.g. Roberts and Spanos, 2003).

1.2.3 System response determination: current solution treatments

In the field of stochastic dynamics, since the excitation of the considered nonlinear systems is described in a probabilistic manner, it is necessary to determine the response statistics (e.g. mean, mean square, power spectrum and probability density function) and then employ them to estimate the system reliability. However, this task is significantly more challenging, in comparison to the linear counterparts. Over the years, various techniques have been developed to tackle this issue.

The deterministic techniques, for computing the integration of the governing equations of motion of nonlinear systems, can be employed to accurately estimate the response statistics of stochastically excited nonlinear systems, by resorting to the Monte Carlo Simulation (MCS) method (e.g. Bird, 1981; Rubino and Tuffin, 2009; Rubinstein and Kroese, 2016). However, the MCS method involves simulating numerous samples and performing nonlinear response time history analysis for each sample. To achieve sufficient accuracy, a relatively large number of time history excitations are required in conjunction with the integration of the governing equations of motion to determine the response statistics. In this setting, despite its broad applicability, the demanding computational cost of MCS methods is prohibitive, particularly for systems with numerous DOFs and high nonlinearity. To mitigate this issue, advanced simulation methods based on MCS have recently been developed by utilizing efficient sampling methods, such as importance sampling (e.g. Melchers, 1989), Latin hypercube sampling (e.g. Stein, 1987), adaptive sampling (e.g. Bucher, 1988), descriptive sampling (e.g. Saliby, 1990), line sampling (e.g. Koutsourelakis et al., 2004), directional sampling (e.g. Nie and Ellingwood, 2004), and subset sampling (e.g. Au and Beck, 2001). The above-advanced simulation methods are much more efficient than the traditional MCS method. However, these numerical methods still involve nonlinear dynamical analysis for systems under a number of samples for stochastic excitation. This problem is further exacerbated in problems involving the repeated evaluation of stochastic dynamical systems, such as reliability-based design optimization formulations (e.g. Jerez et al., 2022c; Jensen et al., 2021; Jensen et al., 2022; Jerez et al., 2022a), in which the corresponding computational effort can become excessive or even prohibitive.

Alternatively, analytical/approximatation techniques have been developed for determining the response statistics of nonlinear structural systems, which avoid the significant computational demands, including statistical linearization methods (e.g. Caughey, 1963; Lutes, 1970; Roberts and Spanos, 2003; Socha, 2007; Mitseas et al., 2016b), perturbation methods (e.g. Crandall, 1963; Nayfeh, 2008; Holmes, 2012), equivalent nonlinear system methods (e.g. Caughey, 1986; Zhu and Yu, 1989; Cai and Lin, 1988), diffusion process theory (e.g. Yong and Lin, 1987; Scheurkogel and Elishakoff, 1988; Caughey, 2018), stochastic averaging techniques (e.g. Roberts and Spanos, 1986; Zhu, 1988; Spanos et al., 2018) and moment closure methods (e.g. Iyengar and Dash, 1978; Crandall, 1980; Kuehn, 2016).

Among these techniques, the statistical linearization method is considered as the most versatile and applicable for dealing with the nonlinear structural systems in the context of random vibration. Specifically, the original governing equations of motion of the nonlinear system are replaced by equivalent linear equations based on mean-square minimization criteria (e.g. Roberts and Spanos, 2003). Compared to other methods, which are often limited to systems with specific nonlinear models or forms of excitation, the statistical linearization method is widely used in engineering problems. However, the assumption of equivalent linear systems in the method may cause the response to deviate from the exact one. In this context, the stochastic averaging method combined with statistical linearization is proposed to accurately estimate the response statistics of the stochastic dynamical systems, based on the diffusion process theory. This is achieved by assuming a Markov process for the amplitude of the system response. (e.g. Zhu, 1988). In this thesis, the approximate semi-analytical techniques based on statistical linearization or stochastic averaging methods are developed for response and reliability analysis of nonlinear (hysteretic) systems.

1.3 Systems with singular matrices

Formulating the equation of motion for complex nonlinear systems can be challenging. In general, the minimum number of coordinates are utilized to formulate the equation of motion of dynamic structural or mechanical systems, resulting in the symmetric, positive-definite and non-singular mass matrices. However, due to the complexity of the system under consideration, the technique of utilizing the minimum number of coordinates to generalize the motion can be a non-trivial task for special case problems (Schutte and Udwadia, 2010). In this setting, Udwadia and his co-workers have developed a technique to efficiently model complex multi-body systems by utilizing redundant coordinates in the field of deterministic dynamics (Udwadia and Kalaba, 1992, 1996; Udwadia and Kalaba, 2000; Udwadia and Phohomsiri, 2006; Schutte and Udwadia, 2010; Udwadia and Wanichanon, 2013). Apart from redundant coordinate modeling, singular matrices can also arise from the absence of certain-order derivatives for multi-degree-of-freedom (MDOF) systems in the equation of motion, such as ill-conditioned systems (e.g. Maciejewski, 1990b), systems with hysteretic nonlinearity (e.g. Wen, 1976), energy harvesting devices (e.g. Adhikari et al., 2009), and translational motion of rigid bodies (e.g. Jr and Kurdila, 2006). Further, due to the presence of singular matrices, classical methodologies for determining the closed forms of the system response statistics are not applicable. Recently, several techniques have been developed for deriving the response statistics of specific problems in stochastic dynamics.

1.3.1 System modeling with redundant coordinates

The importance of system modeling and the ease of obtaining the equation of motion for the correctly modeled multi-body system, are reported in the literature (e.g. Kane and Levinson, 1980; Schiehlen, 1984). The use of redundant coordinates in complex multi-body mechanical systems can significantly reduce the laborious task of generating equations of motion (e.g. Udwadia and Kalaba, 1996).

Current approaches to system modeling involve casting the multi-body systems into tree topologies, which rapidly increases the effort of formulating the equation of motion for complex systems (Schutte and Udwadia, 2010). Apart from that, they also limit the flexibility in generating the equations of motion as they conceptualize a predefined modeling structure. Any change in the system constraints would require a redesign of the system motion within this structure. In addition, these approaches are problem-specific to determine the Lagrange multiplier and require constraints to be functionally independent. Determining the Lagrange multiplier and verifying the above requirement are not straightforward tasks and can be challenging, especially for systems with numerous DOFs and non-integrable constraints (Udwadia and Phohomsiri, 2006). It poses difficulties in explicitly determining the motion of complex systems.

Udwadia and co-workers have proposed a novel approach to modeling constrained mechanical systems, including those with holonomic and non-holonomic constraints (Udwadia and Kalaba, 1992, 1996; Udwadia and Kalaba, 2000; Udwadia and Phohomsiri, 2006; Schutte and Udwadia, 2010). This approach avoids the need for complex and cumbersome descriptions of such systems by using more coordinates than the minimum required. The approach involves decomposing the complex system into smaller subsystems, formulating their equations of motion separately, and then combining them using appropriate constraints to model the composite system in a straightforward and relatively simple manner. However, the use of constraints to ensure structural compatibility can introduce dependent coordinates into the system modeling, resulting in singular matrices in the equations of motion. The presence of singular matrices poses challenges to standard methodologies in subsequent analyses, as highlighted in previous studies (e.g. Lutes and Sarkani, 2004; Roberts and Spanos, 2003).

1.3.2 Systems with singular matrices: diverse applications

The presence of singular matrices in the equation of motion not only arises from redundant coordinate system modeling, but also appears in the applications of physical sciences and engineering.

An ill-conditioned algebraic system, characterized by a coefficient matrix that is nearly singular, is a common source of systems with singular matrices (Fragkoulis, 2017). This issue is prevalent in structural and mechanical engineering, such as earthquake and wind engineering, particularly in the inverse problem of identifying external forces (e.g. Turco, 2005; Reichel and Rodriguez, 2013). Moreover, ill-conditioned systems also arise in motion simulation of graphics objects (e.g. Maciejewski, 1990a; Terzopoulos and Witkin, 1988) and large-scale constrained mechanical systems (e.g. Mani et al., 1985). Typically, in handling such ill-conditioned algebraic systems, small terms in the coefficient matrices are ignored for efficiency by setting them to zero, leading to singular matrices in the governing equation of motion (e.g. Kawano et al., 2013).

Furthermore, the appearance of singular matrices may stem from mathematical formulation problems, such as the absence of derivatives of certain degrees of freedom. A general application is the category of systems that exhibit hysteretic nonlinear behavior. The popular models for describing hysteretic behavior are the bilinear model (e.g. Roberts and Spanos, 2003) and the Bouc-Wen model (Wen, 1976) due to their simplicity and versatility in approximating hysteretic patterns. Specifically, the restoring force in these models depends on both the instantaneous deformation and its past-time history, which requires an auxiliary first-order differential equation coupled with the second-order differential equation of the original system to construct the governing motion. This coupling of equations results in a singular mass matrix for the overall system modeling. Singular matrices can also appear in the equations of motion as a result of non-white stochastic excitations by a series of filters subject to white noise (e.g. Pasparakis et al., 2022a). In addition, non-viscously damped systems can also lead to singular matrices in the system equations of motion (e.g. Wagner and Adhikari, 2003; Woodhouse, 1998; Adhikari, 2013). Specifically, these systems are often described with an exponential damping model and analyzed by state space methods. This introduces internal variables with deficient rankings of the damping matrices, resulting in the appearance of singular matrices in the system equations (e.g. Adhikari and Wagner, 2004; Adhikari, 2013).

Singular matrices are also encountered in systems with mechanical and electrical subsystems. A characteristic category of applications are energy harvesting devices. Specifically, these devices typically include a mechanical system (e.g. cantilever beams), exhibiting vibrational behavior under external loads. Smart materials such as piezoelectric patches form the electrical subsystem of these devices, which convert the vibration energy into electrical current to power both themselves and other interconnected devices (e.g. Adhikari et al., 2009). The theory and applications of energy harvesting devices have rapidly been developed, as they offer an alternative to conventional batteries for powering low-energy consumption electronics. Examples of such applications include medical implants in the healthcare field (e.g. Shi et al., 2018), wildlife tracking and bio-logging devices (e.g. Reissman and Garcia, 2008), high-rise buildings to harness and dissipate the vibration energy (e.g. Xie et al., 2015), structural health monitoring (e.g. Park et al., 2008; Le et al., 2015). The interested reader is also directed to Priya and Inman, 2009; Safaei et al., 2019; Nechibvute et al., 2012; Selvan and Mohamed Ali, 2016 for additional applications. Due to the inherent coupling of electrical and mechanical systems, singular matrices naturally arise in the governing equations of motion for these devices (e.g. Adhikari et al., 2009).

1.3.3 System response determination: current solution treatments

In general, the classical approach for response determination of systems usually involves inverting matrices, which is straightforward for the system with non-singular matrices. Unfortunately, this approach is not applicable when dealing with systems with singular matrices. The popular methodologies of dealing with the singular matrices are problem-specific, such as optimization algebraic or so-called regularization methods. For instance, in complex constrained systems, considerable effort is often dedicated to computing or eliminating Lagrange multipliers to mitigate singular matrices (e.g. Mariti et al., 2011). Also, regularization methods have been explored in the literature to treat ill-posed inverse problems (e.g. Hansen, 1998; Tikhonov, 1963; Tikhonov et al., 1995). However, regardless of their case-specific defect, these methods are not robust, as they can be sensitive to small perturbations of the inputs, leading to inaccurate solutions. Given the broad range of applications involving singular matrices, in conjunction with the challenges and limitations associated with existing methods, there is a pressing need to develop appropriate techniques to deal with nonlinear systems with singular matrices in the field of stochastic dynamics. Consequently, it has been an active area of research.

Fragkoulis and his coworkers (Fragkoulis et al., 2014; Fragkoulis et al., 2016a,b; Fragkoulis et al., 2015, 2017) first proposed methods to treat dynamical systems with singular matrices subject to stochastic excitations by utilizing generalized matrix inverse methodology in the time domain. Since then, the topic of systems with singular matrices in the field of random vibration has gained significant attention, resulting in numerous literature on different applications. Specifically, Kougioumtzoglou et al., 2017 extended Fragkoulis's works to analyze linear and nonlinear stochastic dynamical systems with singular matrices in the frequency domain. Pantelous and Pirrotta, 2017 developed a modified modal analysis for systems with singular matrices, which offers a straightforward and simplified method of assessing the natural frequencies. In Pirrotta et al., 2019, a frequency domain treatment based on generalized modal analysis was developed to determine the response statistics of structural systems modeled via dependent coordinates. Pasparakis and his co-workers developed harmonic wavelets based methods to determine the time-frequency response of stochastic dynamical system with singular matrices (Pasparakis et al., 2019, 2021, 2022a,c,b). Pirrotta et al., 2021 proposed both time-domain and frequency-domain methods to determine the response of systems endowed with singular matrices and fractional derivative elements. Fragkoulis et al., 2022a; Fragkoulis et al., 2022b also proposed an asymptotic approximation methodology where the random eigenvalue problem was reformulated and solved for systems with singular random parameter matrices. These approaches make use of the theory of generalized matrix inverse theory, such as Moore-Penrose inverse theory. Simultaneously, researchers have also employed alternative tools, such as polynomial matrix theory (Antoniou et al., 2017) and Kronecker canonical forms of matrix pencils (Karageorgos et al., 2021), to formulate the equations of motion and determine the response of MDOF systems.

Although a substantial body of literature exists on determining the response of system with singular matrices in stochastic dynamics, most of the works are limited to systems under purely stationary stochastic excitations. However, many structural and mechanical systems endure more complex excitations, especially combined deterministic and stochastic excitations in practice. Thus, given the broad applications of singular matrices modeling, the need of analyzing nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations arises. This thesis presents two semi-analytical techniques for determining the response statistics of nonlinear systems with singular matrices subject to combined deterministic and stationary excitations, and combined deterministic and non-stationary excitations, respectively. The proposed techniques are demonstrated by relevant examples, especially their applications in energy harvester devices.

1.4 Systems endowed with fractional derivative elements

Classical continuous (or discrete) mechanics theory, based on the integer differential calculus, has been widely used for modeling and analyzing the dynamical system in random vibration. Nevertheless, recent advances in mathematical tools, such as fractional calculus that allows for non-integer order calculations, have been utilized to accurately model the behavior of media (e.g. Fragkoulis et al., 2019a). Due to its tremendous new features, such as memory and hereditary, fractional calculus has emerged as a rapidly developing theoretical or mathematical topic with applications in various fields of science and engineering. In particular, it has found significant use in mechanical and structural engineering to model the behavior of viscoelastic materials and in vibration control for structural systems (e.g. Bagley and Torvik, 1983a; Rüdinger, 2006). A brief introduction to fractional calculus, especially for fractional derivative, is given in this section.

1.4.1 Historical background

The origins of fractional calculus can be traced to the personal communication between Leibniz and de L'Hôpital in 1695, wherein the concept of a non-integer order derivative was discussed. Subsequently, several mathematicians such as Fourier, Laplace, Lacroix, and Euler also made contributions to the field during the first systematic study stage. Contributions to the definition of the fractional calculus up to the end of nineteenth century were made by mathematicians, including Liouville, Riemannn, Grunwald and Letnikov (as discussed in Ross, 1977). However, at that time fractional calculus was mainly considered as a purely mathematical tool without a clear geometric and physical interpretation.

In 1921, Nutting used a power-law time function with fractional order to model stress relaxation in materials based on experiment data (Nutting, 1921). This work marked the beginning of applying fractional calculus to describe the constitutive behavior of viscoelastic media, which was further explored in Gemant, 1936; Bosworth, 1946; Blair and Caffyn, 1949. In this setting, Bagley and Torvik firstly provided the physical interpretation of fractional calculus for the viscoelastic phenomenon (e.g. Bagley and Torvik, 1983a). Caputo introduced a new definition of fractional operators that offered advantages over previous definitions (Caputo, 1966, 1967) and the first monograph on the subject was completed in 1974 (Oldham and Spanier, 1974). Since then, fractional calculus has become a topic of particular interest, with significant breakthroughs documented in the literature over the past 50 years (e.g. Miller and Ross, 1993; Podlubny, 1999; Hilfer, 2000; Sabatier et al., 2007; Dalir and Bashour, 2010; Baleanu et al., 2012).

Three fundamental definitions of fractional derivatives are presented below. The first definition pertains to the discrete form of the α-order Grunwald-Letnikov fractional derivative for a variable x,

$$\mathrm{D}_{0,t}^{\alpha}x(t) = \lim_{N\to\infty}\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}x(t-jt/N), \tag{1.1}$$

where $\alpha \in \mathbf{R}$ is the fractional order; t, j and N represents the time and time step and the total number of time steps, respectively; $\Gamma(\cdot)$ denotes the Gamma function, and t/N is incremental time step. Secondly, the Riemann-Liouvile representation is given by,

$$\mathrm{D}_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \tag{1.2}$$

where n is an integer for which $n-1 \leq \alpha < n$ and τ denotes the time lag. Then, the Caputo representation is defined as

$$\mathrm{D}_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \tag{1.3}$$

where $x^{(n)}(\tau)$ denotes the n-order derivative of x. Note, in passing, that when $0 < \alpha < 1$, the Caputo definition reduces to

$$\mathrm{D}_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \tag{1.4}$$

From these definitions, it is evident that fractional derivatives take into account the past state of a system. Contrasting these definitions, it can be observed that the Grunwald representation provides a discrete version, whereas the remaining two definitions involve an integral function with a time lag τ. In addition, compared to Caputo's representation, Riemann-Liouville's definition cannot be utilized in the Laplace transformation and its deviation to an arbitrary constant is not zero, which is contrary to common sense. Therefore, Caputo's definition is more practical in engineering, and for this study, the definition in Eq. (1.4) will be employed.

The physical interpretation of integer-order derivatives and integrals is clear. In general, a function $x(t)$ represents the displacement of an object, with its first derivative $\dot{x}(t)$ corresponding to velocity and the second derivative $\ddot{x}(t)$ denoting acceleration. However, fractional calculus stems from a purely mathematical viewpoint and its physical interpretation is rather left unexplored in the literature. This limitation is probably due to that it has no obvious geometrical meaning. In the last decades, experiments via creep and relaxation tests have demonstrated that the viscoelastic behavior of various materials can be more accurately described by fractional derivatives (Podlubny, 1999; Di Paola et al., 2011). Specifically, viscoelastic materials exhibit a unique combination of properties of solids (elasticity) and liquids (viscosity). This results in materials that possess properties, such as the memory, which states the materials exhibit time-dependent strain. Fractional derivative modeling with the integral convolution operator can reflect the memory effect and frequency-dependent viscoelastic damping (e.g. Gorenflo et al., 2002; Mainardi, 1996; Koeller, 1984).

Fractional calculus has emerged as a powerful tool with diverse applications in various fields of physics and engineering, including modern economics (e.g. Tarasov, 2019), modeling viscoelastic behavior of polymer materials (e.g. Bagley and Torvik, 1983a; Sasso et al., 2011), non-local elasticity (e.g. Paola and Zingales, 2008; Failla et al., 2010; Failla et al., 2013), diffusion fluid mechanics problems (e.g. Kulish and Lage, 2002), biophysical systems (e.g. Hilfer, 2000). In the rapidly developing field of structural and mechanical engineering, in particular, several research efforts pertaining to seismic isolation and vibration control applications have demonstrated the successful use of fractional derivatives to model the behavior of fluid or solid viscoelastic dampers (e.g. Makris et al., 1990; Makris and Constantinou, 1992; Makris et al., 1993; Koh and Kelly, 1990; Papoulia and Kelly, 1997; Hwang and Wang, 1998; Lee and Tsai, 1994; Shen and Soong, 1995; Rüdinger, 2006; Di Matteo et al., 2014; Di Matteo et al., 2015).

1.4.2 System response determination: current solution treatments

The first attempt to develop response determination frameworks for dynamical systems with fractional derivative elements pertains to deterministic cases. The developed methods include Laplace transforms (e.g. Bagley and Torvik, 1983b, 1985), Fourier transforms (e.g. Gaul et al., 1989), numerical methods (e.g. Koh and Kelly, 1990; Yuan and Agrawal, 2002), eigenvector expansion (e.g. Suarez and Shokooh, 1997; Rossikhin and Shitikova, 2006), the averaging method (e.g. Wahi and Chatterjee, 2004; Shen et al., 2012), Adomian decomposition (e.g. Pálfalvi, 2010) and harmonic balance method (e.g. Rossikhin and Shitikova, 2000; Leung and Guo, 2011). The reader is referred to Padovan and Sawicki, 1998; Drăgănescu, 2006; Saha Ray et al., 2005 for additional information on the different methods.

A more comprehensive and thorough understanding of structural performance can be obtained by system analysis in stochastic dynamics, in comparison to the deterministic counterparts. Thus, there is a need to extend the aforementioned deterministic framework to determine the response of systems endowed with fractional derivative elements subject to stochastic excitations.arises. First, several MCS or numerical methodologies have been proposed to address the problem over the last decades. Specifically, Spanos and Zeldin, 1997 introduced abstract state-space representations to determine the response of such systems subject to stationary excitation. The time domain techniques have also been developed by using mathematical tools, such as the eigenvector expansion method and the Laplace-transform-based technique (e.g. Agrawal, 1999, 2002; Agrawal, 2001). In addition, Kun et al., 2003 proposed a Fourier-transform-based technique to obtain Duhamel integral-type expression for this aim.

However, the MCS or numerical methods require significant computational time to solve such systems, due to the numerical solution of the convolution integral associated with the fractional derivative operator and complex nonlinearity. Analytical or approximation techniques can serve as a remedy. Huang and Jin, 2009 developed a stochastic averaging method to analyze strongly nonlinear systems with light damping and fractional derivative elements. Due to the versatility of the statistical linearization and the accuracy of stochastic averaging method, several schemes, based on the combination of these two methods, have been successfully developed for determining the response statistics of nonlinear systems with fractional derivative elements. In detail, Kougioumtzoglou and Spanos, 2016 proposed a harmonic wavelet based approximate technique for determining the response evolutionary power spectrum. Galerkin scheme-based approaches were developed to determine the survival probability and the first-passage probability of such nonlinear/hysteretic systems (Spanos et al., 2016; Di Matteo et al., 2018). Fragkoulis et al., 2019b developed a framework to approximate the response statistics of nonlinear systems with fractional derivative elements subject to non-stationary excitation. Then, Fragkoulis and Kougioumtzoglou extended their previous work to estimate the survival probability (Fragkoulis and Kougioumtzoglou, 2023). The interested reader is also referred to Chen and Zhu, 2009; Chen et al., 2013; Zhang et al., 2021; Sun and Yang, 2020; Xiao et al., 2022; Luo et al., 2022, 2023; Fragkoulis et al., 2023, for additional information on this topic.

The focus of this thesis is to develop novel methodologies for engineers to apply the subject of stochastic dynamics and fractional calculus in practice, particularly in earthquake structural engineering, where more accurate system modeling with fractional derivative elements for viscoelastic materials is needed. In this context, this thesis aims to develop a novel approximate framework for the estimation of the peak response and the analysis of incremental dynamics analysis of nonlinear structural systems with fractional derivative elements subject to stochastic excitations compatible with a given seismic design spectrum. This framework builds upon the methodology proposed in Fragkoulis et al., 2019b and takes into account a comprehensive understanding of the controlled structural performance, aligning with the seismic design spectrum specified in current codes.

1.5 System subject to combined stochastic and deterministic excitation

1.5.1 Diverse engineering applications

Previous research in the field of random vibration mainly focuses on the response determination of stochastically excited nonlinear systems. However, many structural or mechanical systems in engineering practice potentially operate under more complicated excitation conditions, which are often encountered a combination of stochastic and deterministic excitation. For instance, in the case of energy harvesting devices with structural monitoring sensors, the deterministic excitation comes from the primary modes of the monitored structures, while the stochastic excitation arises from factors such as tire-road interaction or footfalls (e.g. Green et al., 2013; Zuo and Zhang, 2013; Dai and Harne, 2018). Similarly, systems subject to combined stochastic and deterministic excitations can also be found in applications pertaining to wind turbine blades vibration under turbulent flow (e.g. Namachchivaya, 1991; Zhu and Wu, 2004; Megerle et al., 2013), nonlinear vibration of beams and plates (e.g. Spanos and Malara, 2020), vibration of gear systems (e.g. Yang, 2013; Fang et al., 2018; Zhang and Spanos, 2020b) and ship roll motion (e.g. Ren et al., 2019).

1.5.2 System response determination: current solution treatments

Due to the nonlinear nature of the systems, the superposition principle cannot be applied, meaning that the response of such systems cannot be treated separately as a deterministic and a stochastic part and then simply added together. Various approximate analytical techniques have been developed over the last decades to address this problem.

In detail, focusing, first, on the study of oscillators subject to combined deterministic and stochastic excitations, a technique combining harmonic balance and stochastic averaging methods was developed in Haiwu et al., 2001 to determine the response of the Duffing oscillator. Then, the stochastic averaging method was utilized to analyze nonlinear oscillators subject to combined harmonic and white-noise excitations (Huang et al., 2000) and was later extended to bound the first-passage failure problem (Zhu and Wu, 2004). Further, the stochastic averaging method was also extended to treat the nonlinear oscillator with fractional derivative damping for determining the first passage problem (Chen and Zhu, 2009) and stochastic jump and bifurcation problem (Chen and Zhu, 2011). Moreover, Anh and his co-workers approached the problem of Duffing and Van der Pol nonlinear oscillators by resorting to a combination of averaging and equivalent linearization methods (Anh and Hieu, 2012; Anh et al., 2014). In addition, Zhu and Guo, 2015 employed a mix of harmonic balance method and Gaussian closure method to investigate the response of a Duffing oscillator under combined harmonic and random excitations. In a separate study, Kong and his co-workers utilized the harmonic balance and statistical linearization method to treat the hysteretic system with fractional derivative elements (Kong et al., 2022b), while Wei et al., 2021 employed a combination of the weighted-average method, stochastic averaging, and finite difference method to determine stationary responses and bifurcations for a nonlinear Markov jump system.

In 2019, the first study of nonlinear MDOF systems subject to combined deterministic and stochastic excitation was conducted by Spanos and his co-workers, wherein a novel technique was developed to address this issue by utilizing harmonic balance and statistical linearization methods(Spanos et al., 2019). Alternatively, this problem is also addressed by combining harmonic averaging and statistical linearization methods (Zhang and Spanos, 2020a). Further, the framework was extended to treat the classical Bouc-Wen hysteretic MODF systems (Kong and Spanos, 2021).

1.5.3 Aims and objectives

The techniques for systems under combined deterministic and stochastic excitations in section 1.5.2 concentrate on conventionally modeled nonlinear systems. To the best of the author's knowledge, almost no techniques are proposed for nonlinear systems with singular matrices. In this thesis, as discussed in section 1.3, singular matrices are prevalent in diverse applications, such as redundant coordinates modeling and systems modeling with additional constraint equation. Given the practical scenario of systems experiencing combined deterministic and stochastic excitations, the need of investigation of nonlinear systems with singular matrices under such excitations arises.

All techniques mentioned above focus primarily on nonlinear systems subject to combined deterministic and stochastic excitation of a stationary kind. However, due to the increasing demand for more accurate modeling of stochastic excitation in nature, researchers have turned their attention towards modeling stochastic processes describing excitations met in nature, exhibiting non-stationary characteristics (e.g. Mitseas et al., 2014c). Thus, the study of nonlinear systems subject to combined deterministic and non-stationary stochastic excitations naturally comes into the sight of researchers. Kong and co-workers in Han et al., 2022; Kong et al., 2022a developed a statistical linearization based framework combined with Lyapunov-like differential equations to address this issue. However, this framework is limited to conventionally modeled nonlinear systems and can not be applied to the nonlinear systems with singular matrices. Thus, this thesis also aims to develop a technique to determine the response statistics for nonlinear systems with singular matrices under combined deterministic and non-stationary excitations.

Further, the proposed techniques have also been utilized in the applications of energy harvester devices, considering its application to combined excitation in section 1.5 and its application in singular matrices modeling for the system consisting of mechanical and electricity sub-systems discussed in section 1.3.

1.6 Design Spectrum Analysis

1.6.1 Importance of inelastic seismic design

Considering the significant economic losses and fatalities resulting from structure collapses during earthquakes, the topic of seismic resistance of structures is of paramount importance in structural engineering. Current aseismic codes of practice for structures allow engineers to use the Uniform Hazard Spectrum (UHS) to evaluate the action of structures under different levels of intensity of ground motions (e.g. CEN, 2004). The UHS is derived in a probabilistic and empiricism manner by considering the peak response of linear viscously damped single-degree-of-freedom (SDOF) oscillators with different natural frequencies that exceed a pre-specified probability under a variety of ground motions, for a nominal critical viscous damping ratio (e.g. Chopra, 2007). It is a linear elastic response spectrum. However, in reality, structures are expected to exhibit ductile behavior under severe earthquakes, to achieve the goal of economically efficient design. This leads to the use of elastic response spectra to estimate inelastic displacements of structures.

1.6.2 Inelastic response determination: current solution treatments

A representative technique for determining the inelastic seismic response determination of structures involves performing nonlinear response history analysis (NRHA) for a majority of seismic acceleration time-histories. Specifically, seismic acceleration time histories are first selected from large databases with subjective preference and expertise and/or artificially generated according to the criteria whose average design spectrum matches the target UHS (e.g. Katsanos et al., 2010). These seismic accelerations should then be further scaled and modified to ensure the desired compatibility with the given UHS; the interested reader is referred to literature (e.g. Grigoriu, 2011; Haselton et al., 2012). Further, although current codes enable engineers to determine system response with a minimum number of ground motions, a number of ground motions are still needed to reduce the diversity and deviation of the peak response. Therefore, the NRHA for structures subject to seismic accelerations should be conducted in an MCS manner. In this way, both the requirements of efficient numerical methods of NRHA for complex structures and a amount of seismic accelerations render the technique computationally intensive.

In this setting, current seismic codes (e.g. Eurocode 8) favor the simplified methods for inelastic seismic design to reduce the computation cost. These methods estimate inelastic deformation during strong earthquake ground motions by using strength reduction factors R. These factors represent the ratio of elastic strength demand to inelastic strength demand (e.g. Miranda, 1993). The inelastic strength demand is estimated by the yield strength to ensure the ductility ratio demand μ, while the ductility ratio demand μ is the ratio of the peak inelastic relative displacement to its yield displacement. Taking into account factors such as system period T, soil conditions at the site, and the ductility demand, many studies have been conducted to build the $R-\mu-T$ relationship, mainly obtained by numerical integration of inelastic viscously damped oscillators for a large ensemble of ground motions (e.g. Mahin and Bertero, 1981; Miranda, 1993, 2000; Borzi et al., 2001; Miranda and Ruiz-García, 2002; Chopra and Chintanapakdee, 2004; Chopra, 2007; Karakostas et al., 2007; Silva et al., 2023). The $R-\mu-T$ relationship then helps to construct the inelastic seismic design spectra (e.g. Chopra and Chintanapakdee, 2004). However, due to the diversity of nonlinear/hysterestic characteristics, inherent properties of ground motions and sites, such an indirect approach leads to misjudgments of the actual building response (e.g. Liu et al., 2004).

Furthermore, stochastic dynamics remains complex and impractical for the majority of engineers to apply in the design of structures, despite the importance of stochastic modeling for ground motions. In this setting, the need arises to develop an approach that allows engineers to determine the response statistics or reliability assessment of nonlinear structures compatible with a given UHS in the field of random vibration. To this end, Giaralis and Spanos proposed a stochastic dynamics-based framework to obtain the seismic demand estimation of bilinear hysteretic SDOF oscillators (Giaralis and Spanos, 2010). In addition, Mitseas et al., 2017; Mitseas et al., 2018, 2019 proposed a novel statistical linearization-based framework to estimate the peak inelastic response of MDOF systems subject to code-compliant seismic excitation. These approaches are achieved without undertaking the NRHA method. This framework involves several steps: (I) the derivation of a stationary stochastic power spectrum representing the stationary excitation process via a given design response spectrum for a special damping ratio in code-compliant seismic structure design, (II) calculating effective linear properties (ELPs) for each DOF by the statistical linearization and decoupling method, (III) renewing the excitation spectrum by updating the damping ratios from the ELPs and repeating the steps (I) and (II) until a good convergence for damping ratios, (IV) estimating the peak response by these final ELPs in conjunction with the given UHS. Later, the framework was extended to model decomposition (Mitseas and Beer, 2019), fragility analysis (Mitseas and Beer, 2020), and first-excursion stochastic incremental dynamical analysis (Mitseas and Beer, 2021). Note, in passing, that the excitation acting on the structures in the framework is derived from a given response spectrum with a desired compatibility.

Nevertheless, previous literature (Giaralis and Spanos, 2010; Mitseas et al., 2018) has only focus on the strong part of the seismic excitation, thus, a stationary stochastic consideration was made. With the more accurate description of natural phenomena, the ground motions are modelled as a non-stationary process (e.g. Conte et al., 1992; Deodatis, 1996; Spanos et al., 2005; Cacciola, 2010; Yeh and Wen, 1990; Der Kiureghian and Crempien, 1989; Barbato and Conte, 2008). The non-stationary characteristic results from the variation of the intensity and frequency content of the ground motion with time. The intensity increases rapidly to a maximum in a few seconds and decreases slowly until it vanishes to background noise, and the frequency content shifts to lower frequencies with time increasing (e.g. Wang et al., 2002). In this setting, the need for transient modeling of earthquake ground motions arises. In structural engineering, to facilitate engineers in practice for the design of structures, the seismic excitation should be compatible with the given UHS in current codes according to certain spectrum-compatible criteria. This is achieved by Cacciola, who proposed a technique for generating spectrum compatible fully non-stationary earthquakes with consideration of the non-stationary frequency content of seismic assessment (Cacciola, 2010).

1.6.3 Incremental dynamical analysis methodology

The concept of performance-based engineering (PBE) is introduced to predict the performance of structures throughout their lifetime in a more intelligent and informed manner by taking into account uncertainties (e.g. Krawinkler, 1999; Barbato et al., 2013). This concept consists of four stages in earthquake engineering, namely hazard analysis, structural analysis, damage analysis, and loss analysis (Porter, 2003). Specifically, the excitations are parameterized by intensity measures (IMs) that are appropriate for different hazard levels, such as peak ground acceleration and spectral acceleration. Through structural analysis, the system response parameters, such as inter-story drift, are obtained and represented by Engineering Demand Parameters (EDPs). Then, the EDPs are used in a set of fragility functions to model the probability of different levels of structural damage. Further, the relationship between IMs and EDPs, along with damage measures, is used to estimate decision variables, such as fatalities, financial loss risk, and life cycle costs. The reader interested in PBE in earthquake engineering is referred to the literature (e.g. Zareian, 2006; Krawinkler and Miranda, 2004; Günay and Mosalam, 2013; Barbato and Tubaldi, 2013; Tubaldi et al., 2014; Mitseas et al., 2016a, 2014a,b, 2015).

Incremental dynamical analysis (IDA) is a widely recognized method of PBE in earthquake engineering that formulates the relationship between IMs and EDPs (e.g. Vamvatsikos and Cornell, 2002; Vamvatsikos and Fragiadakis, 2010). Specifically, IDA evaluates the performance of dynamical systems by subjecting them to various ground motions with different levels of seismic intensity in an MCS context. NRHA is performed for each and every scaled ground motion to obtain the structural response parameters. Thus, the IDA curves are achieved with the relationship between each level of seismic intensity and the corresponding response magnitude. However, the task of performing NRHA in the MCS fashion for each and every scaled ground motion is computationally demanding. Several efforts have been made to reduce the cost of IDA in MCS context, such as parallel methods (Vamvatsikos, 2011) and the efficient Latin hypercube sampling method (Vamvatsikos, 2014). In addition, IDA is also involved in the selection and scaling of ground motion accelerations. The selection of accelerations increases the computational cost to achieve sufficient accuracy with a large number of data sets, while the scaling acceleration remains a highly controversial issue (e.g. Grigoriu, 2011).

Alternatively, dos Santos et al., 2016 proposed an efficient stochastic IDA methodology for nonlinear/hysteretic oscillators by utilizing the stochastic averaging and statistical linearization method. A novel IDA surface is determined by utilizing the concepts of the nonlinear stochastic dynamics in a semi-analytical way, as opposed to a computationally expensive MCS method.

1.6.4 Aims and objectives

Considering the limitations of current methods for design spectrum analysis and the broad range of applications for nonlinear systems with fractional derivative elements discussed in section 1.4, this thesis aims to establish a framework for estimating the peak inelastic response of SDOF hysteretic systems with fractional derivative elements subject to excitation compatible with elastic response UHS. In addition, due to the widely acknowledged IDA for predicting the performance of structures, the method is extended to perform stochastic IDA for the hysteretic SDOF systems with fractional derivative elements subject to excitation compatible with the elastic response UHS.

1.7 First excursion probability of structural dynamics systems: imprecise probability

The inherent randomness of many phenomena leads to their characterization as stochastic processes. This inherent randomness is defined as aleatory uncertainty. However, due to a lack of knowledge, incomplete or even conflicting information and other epistemic sources of uncertainty (e.g. Smith, 2013), the stochastic process modeling can not be precisely described. This leads to the epistemic uncertainty. Thus, the aleatory uncertainty combined with epistemic uncertainty in the stochastic process hinders the application of uncertainty propagation of systems by the classical stochastic dynamics techniques. In this context, imprecise probability (e.g. Beer et al., 2013) may offer an appropriate tool to address this issue. This section, first, introduces the source of imprecise probability, followed by the current solution techniques. Then, the imprecise probability stochastic process is illustrated. Finally, the aim of this thesis, i.e., bounding the first excursion probability of nonlinear structures subject to imprecise stochastic excitations, is presented.

1.7.1 Sources of imprecise probability

Uncertainty can generally be categorized into two types, namely aleatory uncertainty and epistemic uncertainty (e.g. Smith, 2013). These uncertainties are commonly encountered in the field of structural dynamics due to the inherent randomness of a natural phenomenon and structural properties, such as wind and earthquakes, as well as a lack of knowledge or incomplete information (e.g. Smith, 2013; Beer et al., 2013). Natural phenomena are too complex to be precisely described, and various techniques have been developed to quantify them, such as power spectrum density. For instance, the Clough-Penzien spectrum (Clough and Penzien, 1975) and the

Davenport spectrum (Davenport, 1961) are commonly employed as seismic and wind models, respectively. Aleatory uncertainty may also arise in structural materials due to natural variations in the raw materials or the manufacturing process. For instance, the strength of concrete is varied from the mix design, the water-cement ratio, the curing process, and the quality of the raw materials, resulting in the uncertain in material properties. To address the issues of aleatory uncertainties, various frameworks have been developed based on the probability theory (e.g. Vanmarcke and Grigoriu, 1983; Shinozuka and Sato, 1967). However, sufficient information is needed to determine the parameters in the probabilistic models for simulating these uncertainties in time and space. in practice, the information may be unavailable or incomplete due to limited observations or high experimental costs. This leads to the epistemic uncertainty in the stochastic process model.

In this context, subjective probability density function approaches can serve as a remedy to model epistemic uncertainty in cases of subjective assumptions with sufficient justification. However, in many cases, this is considered as a questionable and problematic technique since it uses unjustified probability density functions to compensate for information limitations and incompleteness, leading to a false sense of reliability assessment (e.g. Flage et al., 2018). In engineering, formulating the desired mathematical models to accurately model phenomena without ignoring important information and/or introducing unwarranted assumptions remains a challenge (e.g. Beer et al., 2013).

Further, certain set theories, such as intervals and fuzzy sets (e.g. Moore, 1966; Alefeld and Herzberger, 2012; Zadeh, 1965; Zadeh, 1978), can serve as appropriate mathematical models for imprecise variables, as noted in the literature (e.g. Möller and Beer, 2008; Beer et al., 2013). In general, set-theoretic models used to describe the imprecise parameters in probability models, in conjunction with the probability models themselves, serve to fully account for the inherent randomness of phenomena and the lack of knowledge (e.g. Beer et al., 2013). Thus, the concept of imprecise probability is introduced to deal with the mix of probability model and imprecise parameters, wherein the approaches include Bayesian methods (e.g. Bi et al., 2019), random sets approaches (e.g. Tonon and Bernardini, 1998), sets of probability measures (e.g. Fetz and Oberguggenberger, 2004), evidence theory-based methods (e.g. Shafer, 2016) and fuzzy stochastic methods (e.g. Beer et al., 2011). The interested reader is referred to Beer et al., 2013; Helton and Oberkampf, 2004; Fellin et al., 2005; Augustin et al., 2014; Bradley, 2019; Faes et al., 2021.

The set-theoretic models include intervals (e.g. Moore, 1966; Alefeld and Herzberger, 2012), Bayesian sets (e.g. Zimmermann, 2011), rough sets (e.g. Pawlak, 1991), clouds and convex models (e.g. Neumaier, 2004). Specifically, interval imprecise probability theory describes a hybrid of uncertainties where the parameter value of random tools, such as power density spectrum or probability density function, falls within a range of upper and lower bounds without any other additional information. In this setting, the interval probability assumption is necessary for the probability model with insufficient information (e.g. Sun et al., 2018). In interval imprecise probability analysis, the interval serves as the input and then is translated to the interval output without any subjective assumption. Interval imprecise probability is considered in this thesis.

1.7.2 Uncertainty propagation: current imprecise probability-based solution framework

Uncertainty propagation for the combination of aleatory and epistemic uncertainty is usually achieved in a double loop, where the inner loop deals with the aleatory uncertainty and the outer loop deals with the epistemic uncertainty (e.g. Moens and Vandepitte, 2004). The solution of the double loop defines an optimization problem that can be solved using the interval Monte Carlo method (e.g. Zhang et al., 2010) or the interval quasi-Monte Carlo method (e.g. Zhang et al., 2013). However, fully accurate modeling of epistemic uncertainty requires a sufficient number of values to be sampled in the interval, and the failure probability of structures must be calculated for each sample, which is computationally expensive. Furthermore, despite the complexity and high cost of the failure probability computation, probability estimation in the inner loop can lead to non-smooth behavior of the objective function, making the computation of the bounds of the reliability problem intractable (Faes et al., 2021).

In this setting, some decoupling methods, such as the importance sampling-based method (e.g. Wei et al., 2019a,b) and the advanced line sampling method (e.g. de Angelis et al., 2015), have been developed to efficiently handle the double loop problem. Another computationally efficient technique is the surrogate modeling method, which uses supervised learning to train a surrogate model. These approaches include models such as the polynomial chaos expansion model (e.g. Liu et al., 2020), the Kriging model (e.g. Lelièvre et al., 2018; Ling et al., 2019; Schöbi and Sudret, 2017), the high dimensional model representation (e.g. Wei et al., 2019a), and the interval predictor model (e.g. Crespo et al., 2016). These are "black box" techniques, and the accuracy of the failure probability bounds is ensured by the convexity assumption of the training and the rigorous framework of scenario optimization; the interested reader is referred to the review paper (Faes et al., 2021).

In particular, research on stochastic processes with imprecise probabilities has only recently been initiated. Gao et al., 2018 proposed a framework for assessing the reliability of structures with imprecise random and interval fields. In Faes and Moens, 2019, an imprecise random field analysis was presented, including an interval of correlation length of the auto-correlation function. Faes et al., 2020 proposed an operator norm based framework to bound the first excursion probability of linear systems under interval stochastic loads. Faes et al., 2022 developed a distribution-free p-boxes method for assessing the system reliability under imprecise non-Gaussian stochastic process. Alternatively, the response and reliability assessment of systems subject to stochastic processes with missing data are investigated in Comerford et al., 2017; Zhang et al., 2017; Pasparakis et al., 2022d.

1.7.3 Aims and objectives

As discussed, the field of stochastic dynamics faces the challenge of dealing with imprecise probability problems. However, current techniques for imprecise probability quantification are computationally demanding. Additionally, the research on imprecise probability stochastic processes is just recently initiated with less corresponding research. Thus, the need for a solution to the first excursion problem of dynamic structures arises when dealing with imprecise probability stochastic processes. Recently, Faes et al., 2020 proposed a technique to address this challenge. Nevertheless, it is important to note that the current operator norm-based methods are only suitable for linear systems and may not be suitable for various engineering applications. In this context, this thesis aims to bound the first excursion probability of nonlinear dynamical systems subject to imprecise stochastic loading.

1.8 Contributions

This thesis focuses on analyzing and assessing nonlinear stochastic structural dynamical systems. It is still a challenge to efficiently determine the response statistics and estimate the failure probability of nonlinear stochastic dynamical systems, especially for those with singular parameter matrices and constraints, as well as those endowed with fractional derivative elements. This thesis covers three key topics in stochastic dynamics. The first topic deals with the determination of response statistics of nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations. The second topic relates to efficiently determining the peak response of nonlinear structural systems with fractional derivative elements subject to excitations compatible with a given seismic design spectrum. The third one focuses on the bounds on the first-excursion probability of nonlinear dynamical systems under imprecise Gaussian loads.

1.8.1 Stochastic response determination of nonlinear systems with singular parameter matrices and constraints

Firstly, a method is proposed for determining the response of nonlinear systems with singular matrices subject to deterministic and stationary stochastic excitations simultaneously. This is driven by the presence of singular matrices in nonlinear system modeling due to various factors such as considering a redundant modeling of the system's governing equations or incorporating additional constraint equations. In addition, it is common for nonlinear structural and mechanical systems to operate under combined deterministic and stochastic excitations simultaneously. The proposed methodology relies on the combination of the statistical linearization and the harmonic balance methods, and also uses elements of the generalized inverse matrix theory to determine the response statistics of nonlinear systems with singular matrices. Specifically, the harmonic balance method is extended to handle the deterministic component of the response of nonlinear systems with singular matrices. The stochastic component response is achieved by using the statistical linearization method with averaging treatment. Subsequently, an iteration scheme is developed to break the loop of the two coupled deterministic and stochastic systems. The methodology and results can be found in Ni et al., 2021; Mitseas et al., 2021.

Then, driven by the need to model in a more accurate way for the excitations in nature, the problem of determining non-stationary response statistics of structural systems arises. To address this issue, a methodology for determining the response statistics of nonlinear systems with singular matrices subject to combined deterministic and non-stationary stochastic excitations is considered. Specifically, a matrix differential equation is formulated by combining a generalized statistical linearization methodology and a state space formulation to analyze the nonlinear subsystem with singular matrices under non-stationary stochastic excitation. The methodology is applied to cases of combined deterministic and modulated white noise excitations, as well as to cases of combined deterministic and modulated filter noise excitation. The results are presented in Ni et al., 2023a.

The applications of the two proposed techniques have been demonstrated by considering the MDOF nonlinear structural system with singular matrices due to the redundant coordinates modeling. This is driven by the flexibility and cost-effectivity of system modeling with additional DOFs, especially for the complex systems with many DOFs. In addition, the cases of a vibration energy harvester device subject to considered excitation are presented. In this context, the electrical system in the energy harvester device is considered as a constraint equation for the system. The relevant results for nonlinear energy harvesting device under the case of combined deterministic and stationary loads, and the case of combined deterministic and non-stationary loads have been published in Ni et al., 2022a and in Ni et al., 2023a, respectively.

1.8.2 Code-compliant stochastic structural aseismic analysis

An approximate method is developed to efficiently estimate the peak response of nonlinear structural systems with fractional derivative elements subject to seismic excitations compatible with a given design spectrum, without undertaking the nonlinear time history analysis. The seismic excitations in the method are represented by an evolutionary power spectrum in a stochastic sense, which is compatible with the given design spectrum. This avoids the computational demand and bias of selecting and scaling the ground motions. Further, due to the non-stationary characteristics of excitations, the time-variant equivalent stiffness and damping elements are obtained by utilizing the combination of statistical linearization and stochastic averaging method, which is more effective compared to the time-invariant linearization method in the stationary case. This leads to more accurate estimates in ensuing analysis. Then, the peak response displacement is approximated by using the global minimum and the global maximum of the time-variant stiffness and damping elements, in conjunction with the given design spectrum, which makes it accessible to engineers. In addition, for more accurate modeling of viscoelastic material in vibration control, the nonlinear system endowed with fractional derivative elements in the governing equations of motion is considered. Thus, the framework takes into account the comprehensive and thorough understanding of controlled structural performance and aligns with the seismic design spectrum specified in current codes. These results have been published in Kougioumtzoglou et al., 2022.

Further, a stochastic incremental dynamics analysis method is proposed for nonlinear systems with fractional derivative elements. The proposed method facilitates engineers in practice due to the code-compatible stochastic seismic modeling. The probability density function of response displacement is generated efficiently and comprehensively, based on the stochastic averaging and statistical linearization methods. Especially, a stochastic incremental dynamical analysis surface is obtained, resulting in more reliable response statistics in comparison with the traditional incremental dynamical analysis curves. In addition, a significant property refers to the derivation of response evolutionary power spectrum function for spectral seismic accelerations. These results have been accepted in Ni et al., 2023b.

1.8.3 Imprecise probability of stochastic dynamical structures

An operator norm-based statistical linearization technique is proposed for bounding the first excursion probability of nonlinear systems under imprecise stochastic loading. This typically involves a nested double-loop problem, where the propagation of aleatory uncertainty for stochastic loading must be performed for each realization of the epistemic parameters. This task is generally intractable, especially for nonlinear dynamical systems. These challenges are overcome with a generalized operator norm framework based on the statistical linearization methodology. The proposed scheme succeeds in breaking the double loop and determining the values of the epistemic uncertain parameters that produce bounds on the probability of failure a priori. It can significantly reduce the computational burden and provide a reliable estimate of the probability of failure. These results have been published in Ni et al., 2022b.

1.9 Organization of the thesis

This thesis comprises five chapters, followed by the list of published articles. It includes five articles except for Chapter 1 and Chapter 8. Each article corresponds to the pertinent acticals. Specifically, this thesis is organized as follows.

Chapter 1 serves as an introduction to the thesis and outlines the motivation and objectives of the current research effort. Furthermore, the contributions and organization of thesis are also briefly presented.

Chapter 2 focuses on the response statistics determination of nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations.

Chapter 3 proposes a technique for determining the response statistics of nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations of non-stationary kind.

In Chapter 4 an application pertaining to the response determination of the vibration energy harvesting devices subject to combined deterministic and stochastic excitations is showcased.

In Chapter 5, a novel approximate framework is proposed for estimating the peak response of nonlinear systems with fractional derivative elements subject to stochastic seismic excitations, compatible with a given design spectrum. In this chapter, the author's contributions focus on the methodology, software, writing - original draft, visualization.

In Chapter 6, a novel stochastic incremental dynamical analysis method is proposed for nonlinear systems with fractional derivative elements under code-compliant seismic excitation.

Chapter 7 presents a generalized operator norm framework to bound the first excursion probability of nonlinear systems under imprecise stochastic excitations, this is attained by resorting to the statistical linearization method.

In Chapter 8 the concluding remarks as well as some potential future research directions are presented.

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Chapter 2

Research article 1: Response Determination of Nonlinear Systems with Singular Matrices Subject to Combined Stochastic and Deterministic Excitations

Response Determination of Nonlinear Systems with Singular Matrices Subject to Combined Stochastic and Deterministic Excitations

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Abstract: A new technique is proposed for determining the response of multi-degree-of-freedom nonlinear systems with singular parameter matrices subject to combined stochastic and deterministic excitations. Singular matrices in the governing equations of motion potentially account for the presence of constraint equations in the system. Further, they also appear when a redundant coordinates modeling is adopted to derive the equations of motion of complex multi-body systems. In this regard, considering that the system is subject to both stochastic and deterministic excitations, its response also has two components, namely a deterministic and a stochastic one. Therefore, employing first the harmonic balance method to treat the deterministic component leads to an overdetermined system of equations, to be solved for computing the associated coefficients. Then, the generalized statistical linearization method for deriving the stochastic response of nonlinear systems with singular matrices, in conjunction with an averaging treatment, are utilized to determine the stochastic component of the response. The validity of the proposed technique is demonstrated by pertinent numerical examples.

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2.1 Introduction

Utilizing the minimum number of independent generalized coordinates constitutes the commonly followed practice for modeling the equations of motion of multi-degree-of-freedom (MDOF) dynamical systems (e.g., Roberts and Spanos, 2003; Li and Chen, 2009). Clearly, this is due to the symmetric and positive definite system parameter matrices appearing in the governing equations of motion. These facilitate the development of efficient stochastic response determination techniques, such as these based on the Wiener path integral (e.g., Petromichelakis and Kougioumtzoglou, 2020), but also on recently developed efficacious sparse representations of the stochastic system response based on compressive sampling concepts and tools (e.g., Kougioumtzoglou et al., 2020). However, also taking into account the effort involved in the modeling procedure, it can be argued that modeling based on the minimum number of coordinates can be a rather daunting task. This especially applies for classes of complex multi-body systems and/or systems subject to constraints (e.g., Udwadia and Kalaba, 1992; Udwadia and Kalaba, 2001). In particular, depending on the number of bodies which constitute the system under consideration, on the topology and nature of their connections (e.g., linear, nonlinear, hysteretic), as well as on the presence of constraint equations, utilizing the minimum number of coordinates/degrees-of-freedom (DOFs) can even become impractical. Moreover, it can be argued that following the standard minimum number of DOFs-based formulation of the equations of motion in multi-body system modeling (instead of adopting a redundant DOFs one), apart from providing the modeler with limited flexibility, it also relates to solution frameworks of increased computational cost; see, indicatively, Udwadia and Phohomsiri, 2006; Featherstone, 1984; Schutte and Udwadia, 2011; Pappalardo and Guida, 2018b; Pappalardo and Guida, 2018a; Udwadia and Wanichanon, 2013; Pirrotta et al., 2019 for a more detailed discussion.

Further, it is worth noting that the degree of simplicity and the amount of effort required for deriving the equations of motion are critical for assessing the performance of an applied solution framework.

In this regard, an alternative approach has been developed for bypassing some of the previous limitations, where the formulation of the governing equations of motion relies on adopting additional dependent coordinates/DOFs (e.g., Udwadia and Kalaba, 2001; Udwadia and Phohomsiri, 2006; Schutte and Udwadia, 2011). However, due to the dependence among the utilized DOFs, singular matrices appear in the system equations of motion, rendering all standard system analyses inapplicable. Therefore, it is necessary to develop new tools and techniques for studying the behavior and assessing the reliability of engineering systems with singular parameter matrices in the governing equations of motion. The first steps towards this direction have been recently made by resorting to the theory of generalized matrix inverses. In particular, the Moore-Penrose (M-P) matrix inverses theory has been invoked to extend standard time- and frequency-domain approaches of random vibration theory to account for linear and nonlinear systems with singular matrices (Fragkoulis et al., 2016a; Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017; Pasparakis et al., 2021; Pirrotta et al., 2021); see also Refs. Fragkoulis et al., 2015; Pantelous and Pirrotta, 2017; Pirrotta et al., 2019 for additional applications based on an M-P matrix inverses framework.

The machinery of the M-P matrix inverses-based solution framework is further enhanced in this paper by introducing a technique for determining the response of MDOF nonlinear systems with singular parameter matrices subject to combined stochastic and deterministic excitations. This is a rather substantial extension with applications, for instance, in the response determination of slender structures (e.g., wind turbines, submission towers, etc.), which are often subject to stochastic wind loading as well as deterministic loading due to vortex-shedding (Davenport, 1995; Tessari et al., 2017). In such cases, depending on the complexity of the system under consideration, adopting the herein proposed multi-body system modeling approach potentially facilitates the derivation of its dynamics, and subsequently, of the system response determination. Further, the proposed approach can be used in vibration energy harvesting applications. Specifically, it can be used in applications related to contemporary vibration energy harvesters (VEHS) designed to operate in tandem with larger structures, such as bridges vibrating due to wind loads and harmonic loads caused by vehicles (Cai and Harne, 2020). In particular, when the problem of combined VEHs is considered for maximizing the energy production (e.g., Lee et al., 2019), a redundant DOFs modeling can be employed to facilitate the derivation of the system dynamics.

The herein proposed technique can be construed as a generalization of a recently developed framework for deriving the response of MDOF nonlinear systems subject to combined stochastic and deterministic excitations (Spanos et al., 2019) to account for systems with singular parameter matrices. In this regard, the harmonic balance method (e.g., Mickens, 2010; Krack and Gross, 2019) and the recently derived statistical linearization methodology for systems with singular matrices (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017) are invoked to determine the response of systems exhibiting singular matrices, and subject to combined stochastic and deterministic excitation. Specifically, considering the form of the excitation, first, it is assumed that the corresponding system response is composed of a deterministic and a stochastic part. Next, the harmonic balance method is employed to treat the deterministic response. However, in contrast to the standard implementation of the method (i.e., Spanos et al., 2019), an overdetermined system of equations (e.g., Lindfield and Penny, 2018) is constructed, to be solved for computing the harmonic coefficients of the method. Therefore, a novel M-P matrix inverses-based theoretical framework is introduced to solve the system, and thus, to determine the associated harmonic coefficients (e.g., Ben-Israel and Greville, 2003; Campbell and Meyer, 2009). Then, the generalized statistical linearization methodology for systems with singular matrices in conjunction with an averaging treatment are employed for treating the stochastic component of the response. It is noted that the combination of the two methods (i.e., of the harmonic balance and the statistical linearization) leads to a coupled system of algebraic equations, which is solved iteratively and both the stochastic and the deterministic response components are derived. Two numerical examples are used to demonstrate the validity of the proposed technique. Specifically, systems with mass, damping as well as stiffness nonlinearities of several magnitudes are considered. The obtained results are compared and found in complete agreement with corresponding results derived by applying the standard approach in Spanos et al., 2019.

2.2 Mathematical formulation

2.2.1 Nonlinear multi-degree-of-freedom Systems with Singular Parameter Matrices

The matrix form of the equations of motion of an l-DOF nonlinear system, where $\mathbf{x}$ denotes an l-dimensional dependent coordinates vector is given by

$$\mathbf{M_x}\ddot{\mathbf{x}} + \mathbf{C_x}\dot{\mathbf{x}} + \mathbf{K_x}\mathbf{x} + \mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{Q_x}(t), \tag{2.1}$$

where $\mathbf{M_x}$, $\mathbf{C_x}$ and $\mathbf{K_x}$ correspond to the $l \times l$ mass, damping and stiffness matrices of the system. Further, $\mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ denotes the l-dimensional vector of the system nonlinearities, which depends on the displacement $\mathbf{x}$ and its first and second derivatives. Finally, $\mathbf{Q_x}(t)$ represents a zero-mean Gaussian stochastic excitation. Next, it is considered that the system of Eq. (2.1) is subject to additional constraints of the form (Schutte and Udwadia, 2011; Fragkoulis et al., 2016a)

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{2.2}$$

which, for simplicity, are expressed as $\mathbf{A}\ddot{\mathbf{x}} + \mathbf{E}\dot{\mathbf{x}} + \mathbf{L}\mathbf{x} = \mathbf{F}$, with $\mathbf{A}, \mathbf{E}, \mathbf{L}$ and $\mathbf{F}$ denoting, respectively, $m \times l$ matrices and an l-dimensional vector. Then, Eq. (2.1) is recast into

$$\bar{\mathbf{M}}_\mathbf{x}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_\mathbf{x}\dot{\mathbf{x}} + \bar{\mathbf{K}}_\mathbf{x}\mathbf{x} + \bar{\mathbf{\Phi}}_\mathbf{x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{Q}}_\mathbf{x}(t). \tag{2.3}$$

In Eq. (2.3), $\bar{\mathbf{M}}_\mathbf{x}$, $\bar{\mathbf{C}}_\mathbf{x}$ and $\bar{\mathbf{K}}_\mathbf{x}$ denote the augmented $(l+m)\times l$ mass, damping and stiffness matrices of the system, which are given by (Fragkoulis et al., 2016a)

$$\bar{\mathbf{M}}_\mathbf{x} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{M}_\mathbf{x} \\ \mathbf{A} \end{bmatrix}, \quad \bar{\mathbf{C}}_\mathbf{x} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{C}_\mathbf{x} \\ \mathbf{E} \end{bmatrix}, \quad \bar{\mathbf{K}}_\mathbf{x} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{K}_\mathbf{x} \\ \mathbf{L} \end{bmatrix}, \tag{2.4}$$

whereas

$$\bar{\mathbf{\Phi}}_\mathbf{x} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{\Phi}_\mathbf{x} \\ \mathbf{0} \end{bmatrix}, \quad \bar{\mathbf{Q}}_\mathbf{x}(t) = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{Q}_\mathbf{x}(t) \\ \mathbf{F} \end{bmatrix} \tag{2.5}$$

are the augmented $(l+m)$-dimensional vectors of the system nonlinearities and stochastic excitation, respectively. Finally, $\mathbf{I}_l$ corresponds to the $l \times l$ identity matrix, and "+" denotes the M-P matrix inverse operation (see Appendix I). A detailed derivation of Eqs. (2.3)-(2.5) can be found in ibid.

2.2.2 Generalized Statistical Linearization Methodology for multi-degree-of-freedom Systems with Singular Parameter Matrices

The statistical linearization methodology for solving approximately and efficiently nonlinear stochastic differential equations (e.g., Roberts and Spanos, 2003; Socha, 2007), has been recently extended and generalized to determine the response statistics of nonlinear dynamical systems with singular parameter matrices (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017). A concise presentation of the generalized method is included in this section for completeness. The major objective of the methodology lies in replacing the originally given nonlinear system with an equivalent linear one. This becomes feasible by minimizing, in some sense, the error that is formed by the difference between the two systems. The rationale behind this approach stems from that there are readily available closed form analytical expressions in time and frequency domains for the response characterization of linear systems, which are used to approximate the response of the original nonlinear system. The method is widely utilized in diverse engineering applications due to its versatility in addressing a wide range of nonlinear behaviors, and also due to that it leads to closed-form expressions for determining the parameter matrices of the equivalent linear system (e.g., Spanos and Evangelatos, 2010; Spanos and Kougioumtzoglou, 2012; Fragkoulis et al., 2019; Mitseas and Beer, 2019; Pasparakis et al., 2021). The interested reader is directed to Fragkoulis et al., 2016b and Kougioumtzoglou et al., 2017 for a detailed presentation of the method.

For the application of the generalized statistical linearization methodology, first, an equivalent linear system to the nonlinear system defined in Eq. (2.3) is considered as

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e)\ddot{\mathbf{x}} + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e)\dot{\mathbf{x}} + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e)\mathbf{x} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{2.6}$$

where $\bar{\mathbf{M}}_e, \bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ denote the augmented equivalent linear mass, damping and stiffness $(l+m)\times l$ matrices. Then, the error

$$\boldsymbol{\varepsilon} = \bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) - \bar{\mathbf{M}}_e\ddot{\mathbf{x}} - \bar{\mathbf{C}}_e\dot{\mathbf{x}} - \bar{\mathbf{K}}_e\mathbf{x} \tag{2.7}$$

is defined as the difference between the nonlinear and the equivalent linear systems, and is minimized in the mean square sense. Further, by adopting the standard Gaussian response assumption (Roberts and Spanos, 2003) a linear set of equations is derived, whose solution leads to the determination of the elements of the equivalent linear matrices. Thus, denoting by $\mathbf{m}_{i*}^{e\mathrm{T}}, \mathbf{c}_{i*}^{e\mathrm{T}}$ and $\mathbf{k}_{i*}^{e\mathrm{T}}$ the i-th row of $\bar{\mathbf{M}}_e, \bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$, and utilizing the M-P matrix inverses theory yields (Fragkoulis et al., 2016b)

$$\begin{bmatrix} \mathbf{k}_{i*}^{e\mathrm{T}} \\ \mathbf{c}_{i*}^{e\mathrm{T}} \\ \mathbf{m}_{i*}^{e\mathrm{T}} \end{bmatrix} = \mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]\mathbb{E}\begin{bmatrix} \frac{\partial\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial\mathbf{x}} \\ \frac{\partial\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial\dot{\mathbf{x}}} \\ \frac{\partial\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial\ddot{\mathbf{x}}} \end{bmatrix} + \mathbf{g}(\mathbf{y}), \tag{2.8}$$

for $i = 1, 2, \dots, l+m$, where $\hat{\mathbf{x}}$ is the $3l$-dimensional vector $\hat{\mathbf{x}}^{\mathrm{T}} = \begin{bmatrix} \mathbf{x}_s & \dot{\mathbf{x}}_s & \ddot{\mathbf{x}}_s \end{bmatrix}$, $\mathbb{E}[\cdot]$ denotes the expectation operator and "T" represents the matrix transpose operation. Further, $\mathbf{g}(\mathbf{y})$ is an arbitrary $3l$-dimensional vector (see also Appendix I), which leads to a family of solutions for the determination of the equivalent linear elements. Nevertheless, based on the adoption of the mean square error minimization criterion, it has been proved in Fragkoulis et al., 2016b that the solution obtained by setting the arbitrary term equal to zero is at least as good, as any other solution that corresponds to a non-zero value for the arbitrary term.

Next, a frequency domain treatment is applied to derive the response statistics of the equivalent system in Eq. (2.6). This is attained by resorting to the standard input-output relationship of random vibration theory, which connects the power spectrum of the system response to the corresponding excitation spectra. Specifically, the recently derived generalized input-output relationship for systems with singular parameter matrices is employed (Kougioumtzoglou et al., 2017)

$$\mathbf{S}_{\mathbf{x}}(\omega) = \boldsymbol{\alpha}_{\mathbf{x}}(\omega)\mathbf{S}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(\omega)\boldsymbol{\alpha}_{\mathbf{x}}^{\mathrm{T}*}(\omega), \tag{2.9}$$

where $\mathbf{S}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(\omega)$ and $\mathbf{S}_{\mathbf{x}}(\omega)$ denote, respectively, the excitation and response power spectrum matrices, and $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$ represents the frequency response function (FRF) matrix of the system. Further, "$*$" corresponds to the conjugate matrix operation. The FRF matrix is given by (Kougioumtzoglou et al., 2017)

$$\boldsymbol{\alpha}_{\mathbf{x}}(\omega) = \left(-\omega^2(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_{\mathbf{e}}) + i\omega(\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{e}}) + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{e}})\right)^{+}. \tag{2.10}$$

Finally, for the determination of the second order response statistics, Eq. (2.9) is used in conjunction with

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \int_{-\infty}^{\infty} \mathbf{S}_{\mathbf{x}}(\omega)d\omega. \tag{2.11}$$

2.2.3 Combined Harmonic Balance and Statistical Linearization Methods for MDOF Systems with Singular Parameter Matrices

In this section a new approach is proposed for determining the response of nonlinear systems with singular matrices subject to stochastic and deterministic excitations. It consists of a combination of the harmonic balance method, which is used for deriving the periodic solution of nonlinear differential equations (Krack and Gross, 2019; Mickens, 2010; Chatterjee, 2003) and the generalized statistical linearization methodology (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017). The proposed approach can be construed as a generalization of the methodology developed in Spanos et al., 2019 to account for systems with singular matrices; see also Kong and Spanos, 2021 for an extension to nonlinear systems with hysteretic behavior. Further applications of systems subject to combined stochastic and deterministic excitations are found, indicatively, in Anh and Hieu, 2012; Haiwu et al., 2001; Chen and Zhu, 2011; Megerle et al., 2013; Spanos and Malara, 2020.

Generalized harmonic balance solution framework

Following closely the formulation of Eq. (2.3), the equations of motion for an l-DOF nonlinear system subject to constraint equations of the form in Eq. (2.2), as well as to combined deterministic and stochastic excitations, are given by

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} + \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{f}}_{d,\mathbf{x}}(t) + \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{2.12}$$

where $\bar{\mathbf{M}}_{\mathbf{x}}, \bar{\mathbf{C}}_{\mathbf{x}}, \bar{\mathbf{K}}_{\mathbf{x}}$ are defined in Eq. (2.4) and $\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ is given by Eq. (2.5). Further, the deterministic component of the excitation is given by the $(l + m)$-dimensional vector

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{f}_{d,\mathbf{x}}(t) \\ \mathbf{0}_{m\times 1} \end{bmatrix}, \tag{2.13}$$

whereas the stochastic component $\bar{\mathbf{Q}}_{\mathbf{x}}(t)$ is also given by Eq. (2.5).

Then, considering the combined excitation of the augmented system in Eq. (2.12), it is assumed that the system response is written as

$$\mathbf{x}(t) = \mathbf{x}_s(t) + \mathbf{x}_d(t), \tag{2.14}$$

where $\mathbf{x}_s(t)$ and $\mathbf{x}_d(t)$ denote its stochastic and deterministic components, which account for the corresponding components of the excitation. Next, assuming for simplicity that the stochastic excitation is modeled as a zero-mean Gaussian process, substituting Eq. (2.14) into the augmented equations of motion in Eq. (2.12) and ensemble averaging leads to

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_d + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_d + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_d + \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)] = \bar{\mathbf{f}}_{d,\mathbf{x}}(t). \tag{2.15}$$

Clearly, Eq. (2.15) consists of a deterministic and an additional stochastic component, which are treated separately in the ensuing analysis. Specifically, first, an extended harmonic balance methodology in conjunction with M-P matrix inverses-based theoretical concepts are applied to the deterministic component in Eq. (2.15). Then, the application of the generalized statistical linearization methodology to treat the stochastic component of the system follows.

Next, directing attention to treating the deterministic component of the response, it is assumed that the system nonlinearities are of the polynomial kind. Note that, apart from simplicity, since it facilitates the derivation of closed form solutions for determining the equivalent linear system, this assumption is directly related to the application of the harmonic balance method (Mickens, 1984). Moreover, it is commonly adopted in nonlinear engineering system modeling (Roberts and Spanos, 2003). Further, $\bar{\mathbf{f}}_{d,\mathbf{x}}(t)$ in Eq. (2.13) is modeled as a monochromatic function of period $T = \frac{2\pi}{\omega_d}$, i.e.,

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \bar{\mathbf{f}}_{d_1,\mathbf{x}} \cos(\omega_d t) + \bar{\mathbf{f}}_{d_2,\mathbf{x}} \sin(\omega_d t), \tag{2.16}$$

where $\bar{\mathbf{f}}_{d_1,\mathbf{x}}$ and $\bar{\mathbf{f}}_{d_2,\mathbf{x}}$ are the constant coefficient $(l+m)$-dimensional vectors for the new coordinates system in the phase plane (Krack and Gross, 2019; Hayashi, 2014). In this regard, the deterministic response is written as

$$\mathbf{x}_d(t) = \mathbf{x}_{d_1} \cos(\omega_d t) + \mathbf{x}_{d_2} \sin(\omega_d t), \tag{2.17}$$

where $\mathbf{x}_{d_1}, \mathbf{x}_{d_2}$ are constant l-dimensional vectors. Substituting Eqs. (2.16) and (2.17) into Eq.

(2.15) yields

$$
\begin{aligned}
- &\quad \omega_d^2 \bar{\mathbf{M}}_{\mathbf{x}} (\mathbf{x}_{d_1} \cos(\omega_d t) + \mathbf{x}_{d_2} \sin(\omega_d t)) + \omega_d \bar{\mathbf{C}}_{\mathbf{x}} (-\mathbf{x}_{d_1} \sin(\omega_d t) + \mathbf{x}_{d_2} \cos(\omega_d t)) \\
+ &\quad \bar{\mathbf{K}}_{\mathbf{x}} (\mathbf{x}_{d_1} \cos(\omega_d t) + \mathbf{x}_{d_2} \sin(\omega_d t)) \\
+ &\quad \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}} (\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)] = \bar{\mathbf{f}}_{d_1,\mathbf{x}} \cos(\omega_d t) + \bar{\mathbf{f}}_{d_2,\mathbf{x}} \sin(\omega_d t).
\end{aligned} \tag{2.18}
$$

Then, applying the harmonic balance method, Eq. (2.18) leads to a set of $2(l+m)$ equations with $2l$ unknowns. Specifically, these are given by

$$
\begin{aligned}
- &\quad \omega_d^2 \sum_{j=1}^{l} \left(\bar{\mathbf{M}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_1}(j) \right) + \omega_d \sum_{j=1}^{l} \left(\bar{\mathbf{C}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_2}(j) \right) + \sum_{j=1}^{l} \left(\bar{\mathbf{K}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_1}(j) \right) \\
+ &\quad \frac{2}{T} \int_0^T \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}} (\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)](i) \ \cos(\omega_d t) dt = \bar{\mathbf{f}}_{d_1}(i)
\end{aligned} \tag{2.19}
$$

and

$$
\begin{aligned}
- &\quad \omega_d^2 \sum_{j=1}^{l} \left(\bar{\mathbf{M}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_2}(j) \right) - \omega_d \sum_{j=1}^{l} \left(\bar{\mathbf{C}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_1}(j) \right) + \sum_{j=1}^{l} \left(\bar{\mathbf{K}}_{\mathbf{x}}(i,j) \mathbf{x}_{d_2}(j) \right) \\
+ &\quad \frac{2}{T} \int_0^T \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}} (\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)](i) \ \sin(\omega_d t) dt = \bar{\mathbf{f}}_{d_2}(i),
\end{aligned} \tag{2.20}
$$

for $i = 1, 2, \ldots, l+m$, where the indexes (i,j) and $(j), (i)$ denote, respectively, the elements in position (i,j), and in positions j and i of the corresponding $(l+m) \times l$ matrices and l-dimensional vectors.

For the solution of the algebraic system defined by Eqs. (2.19) and (2.20), and thus, for the computation of the deterministic response component, Eqs. (2.19) and (2.20) are equivalently written in the form

$$
\mathbf{P}\mathbf{u} = \mathbf{v}, \tag{2.21}
$$

where

$$\mathbf{P} = \begin{bmatrix} \bar{\mathbf{K}}_{\mathbf{x}} - \omega_d^2 \bar{\mathbf{M}}_{\mathbf{x}} & \omega_d \bar{\mathbf{C}}_{\mathbf{x}} \\ -\omega_d \bar{\mathbf{C}}_{\mathbf{x}} & \bar{\mathbf{K}}_{\mathbf{x}} - \omega_d^2 \bar{\mathbf{M}}_{\mathbf{x}} \end{bmatrix} \tag{2.22}$$

is a $2(l+m) \times 2l$ matrix whose components are given by Eq. (2.4). Further, the $2l$-dimensional and $2(l+m)$-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\mathbf{u} = \begin{bmatrix} \mathbf{x}_{d_1} \\ \mathbf{x}_{d_2} \end{bmatrix} \tag{2.23}$$

and

$$\mathbf{v} = \begin{bmatrix} \bar{\mathbf{f}}_{d_1} - \frac{2}{T} \int_0^T \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}] \cos(\omega_d t) dt \\ \bar{\mathbf{f}}_{d_2} - \frac{2}{T} \int_0^T \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}] \sin(\omega_d t) dt \end{bmatrix}, \tag{2.24}$$

respectively. Clearly, Eqs. (2.19) and (2.20) or, equivalently, Eqs. (2.21)-(2.24) define an overdetermined system of equations, whose solution is derived by resorting to the generalized matrix inverses theory (Campbell and Meyer, 2009; Ben-Israel and Greville, 2003). In particular, by utilizing the concept of the M-P matrix inverses, the general solution to Eq. (2.21) is given by

$$\mathbf{u} = \mathbf{P}^{+}\mathbf{v} + (\mathbf{I} - \mathbf{P}^{+}\mathbf{P})\mathbf{y}, \tag{2.25}$$

where $\mathbf{y}$ denotes an arbitrary $2l$-dimensional vector (see also Appendix I). It is readily seen that due to the arbitrary vector $\mathbf{y}$, Eq. (2.25) corresponds to a family of solutions for obtaining the deterministic component of the response, instead of a uniquely defined solution.

However, depending on the rank of the matrix $\mathbf{P}$ in Eq. (2.22), the selection of a uniquely defined solution is feasible. In particular, if $\mathbf{P}$ has full column rank (Meyer, 2000), the M-P inverse matrix $\mathbf{P}^{+}$ is written in closed-form as (Lindfield and Penny, 2018; Campbell and Meyer, 2009)

$$\mathbf{P}^{+} = (\mathbf{P}^{*}\mathbf{P})^{-1}\mathbf{P}^{*}. \tag{2.26}$$

Thus, substituting Eq. (2.26) into Eq. (2.25), and taking into account that the M-P inverse of any matrix is uniquely defined (Campbell and Meyer, 2009), Eq. (2.25) attains a unique solution

$$\mathbf{u} = \mathbf{P}^{+}\mathbf{v}. \tag{2.27}$$

In passing, it is worth noting that the augmented matrix $\bar{\mathbf{M}}_{\mathbf{x}}$ in the diagonal entries of matrix $\mathbf{P}$ in Eq. (2.22) ensures that the columns of the latter are independent of each other or, equivalently, that $\mathbf{P}$ has full column rank. Therefore, Eq. (2.27) constitutes the uniquely defined solution of the system in Eq. (2.21) or, equivalently, in Eqs. (2.19) and (2.20) for determining $\mathbf{x}_{d_1}$ and $\mathbf{x}_{d_2}$. Subsequently, this leads to the derivation of the deterministic response component.

Generalized statistical linearization and averaging treatments

In this section, the recently proposed generalized statistical linearization methodology for systems with singular parameter matrices (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017) is applied to treat the stochastic component $\mathbf{x}_s(t)$ of the system response.

In this regard, forming the difference between the systems in Eqs. (2.12) and (2.15) yields

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_s + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_s + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_s + \tilde{\boldsymbol{\Phi}}_{\mathbf{x}} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{2.28}$$

where

$$\tilde{\boldsymbol{\Phi}}_{\mathbf{x}} = \bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d) - \mathbb{E}[\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)] \tag{2.29}$$

is the zero-mean vector of the system nonlinearities, and $\mathbf{x}_s$ is the stochastic component of the response. Next, following closely the formulation of Eq. (2.6), the linear equivalent system to Eq. (2.28) becomes

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e)\ddot{\mathbf{x}}_s + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e)\dot{\mathbf{x}}_s + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e)\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t). \tag{2.30}$$

Then, the error function which is defined as the difference between Eqs. (2.28) and (2.30) is formed, and minimized by adopting the mean square minimization criterion (Fragkoulis et al., 2016b). Further, considering that the arbitrary vector $\mathbf{g}(\mathbf{y})$ in Eq. (2.8) is the null vector, the elements of the $(l+m)\times l$ matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ are readily determined by

$$\begin{bmatrix} \mathbf{k}_{i*}^{e\mathrm{T}} \\ \mathbf{c}_{i*}^{e\mathrm{T}} \\ \mathbf{m}_{i*}^{e\mathrm{T}} \end{bmatrix} = \mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]\mathbb{E}\begin{bmatrix} \frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\mathbf{x}} \\ \frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\dot{\mathbf{x}}} \\ \frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\ddot{\mathbf{x}}} \end{bmatrix}, \tag{2.31}$$

where $\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)$, $i = 1,2,\ldots,l+m$, denotes the i-th component of the nonlinear vector in Eq. (2.29).

Clearly, the nonlinear vector $\tilde{\mathbf{\Phi}}_{\mathbf{x}}$ in Eq. (2.29) not only depends on the stochastic response component $\mathbf{x}_s(t)$ (and its first and second order derivatives) but also on the deterministic (harmonic) component of the system response, i.e., $\mathbf{x}_d(t)$, and its first and second order derivatives. Thus, the elements $m_{ij}^e, c_{ij}^e, k_{ij}^e$, for $i = 1,2,\ldots,l+m$ and $j = 1,2,\ldots,m$, obtained in Eq. (2.31) are also time dependent. Nevertheless, by relying on the harmonic balance method, the slowly varying over a period T of oscillation components of matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ are approximated by their average over T (Spanos et al., 2019; Hayashi, 2014), i.e.,

$$\bar{\mathbf{M}}_e^{av} = \frac{1}{T}\int_0^T \bar{\mathbf{M}}_e dt, \quad \bar{\mathbf{C}}_e^{av} = \frac{1}{T}\int_0^T \bar{\mathbf{C}}_e dt, \quad \bar{\mathbf{K}}_e^{av} = \frac{1}{T}\int_0^T \bar{\mathbf{K}}_e dt. \tag{2.32}$$

The matrices of Eq. (2.32) serve, in essence, as the closed form solutions which are used to approximate the equivalent mass, damping and stiffness matrices of the linear system in Eq. (2.30), which becomes

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e^{av})\ddot{\mathbf{x}}_s + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e^{av})\dot{\mathbf{x}}_s + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e^{av})\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t). \tag{2.33}$$

Subsequently, a frequency domain approach is invoked to derive the response statistics of the equivalent system in Eq. (2.33). In this regard, taking into account Eqs. (2.32) and (2.33), the FRF matrix is derived by Eq. (2.10), i.e.,

$$\boldsymbol{\alpha}_{\mathbf{x}}(\omega) = \left(-\omega^2(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e^{av}) + i\omega(\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e^{av}) + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e^{av})\right)^+, \tag{2.34}$$

and thus, the response power spectrum $\mathbf{S}_{\mathbf{x}_s}(\omega)$ is found by Eq. (2.9). Finally, the second order response statistics of the equivalent system in Eq. (2.33), are computed by Eq. (2.11), i.e.,

$$\mathbb{E}[\mathbf{x}_s^2(i)] = \int_{-\infty}^{\infty} S_{\mathbf{x}_s(i)\mathbf{x}_s(i)}(\omega)d\omega, \quad \mathbb{E}[\dot{\mathbf{x}}_s^2(i)] = \int_{-\infty}^{\infty} \omega^2 S_{\mathbf{x}_s(i)\mathbf{x}_s(i)}(\omega)d\omega, \tag{2.35}$$

for $i = 1, 2, \ldots, l$. Note, in passing, that the integrals in Eq. (2.35) are calculated numerically in the ensuing analysis. However, closed-form solutions for calculating random vibration integrals are also available (Roberts and Spanos, 2003).

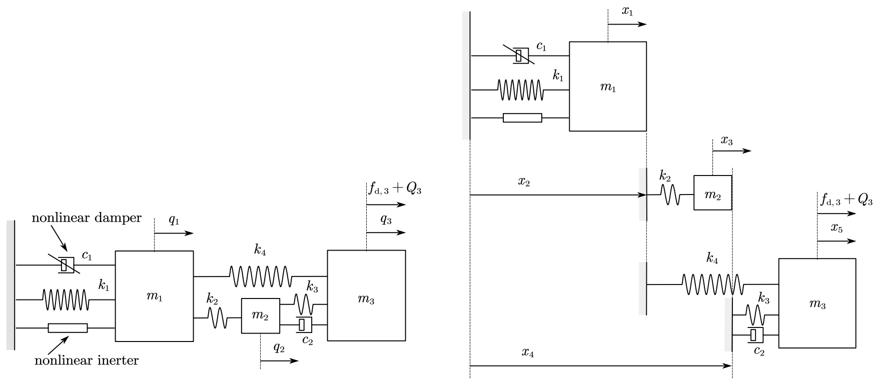
Clearly, Eq. (2.35) in conjunction with the generalized input-output relationship in Eq. (2.9), as well as Eq. (2.27), constitute a coupled nonlinear system of equations to be solved for determining the system response. The following simple iterative procedure is used to solve the coupled nonlinear system: $i.$ The scheme is initialized by setting the nonlinear vector $\tilde{\mathbf{\Phi}}_{\mathbf{x}}$ in the governing equations of motion equal to the null vector. Then, the deterministic response $\mathbf{x}_d$ is obtained. $ii.$ Employing Eq. (2.9), as well as Eq. (2.35), the variance of the stochastic response $\mathbf{x}_s$ is derived. $iii.$ Using step $(ii.)$, Eq. (2.27) yields the deterministic response $\mathbf{x}_d$. Then, the (updated) values of matrices $\bar{\mathbf{M}}_e^{av}$, $\bar{\mathbf{C}}_e^{av}$ and $\bar{\mathbf{K}}_e^{av}$ are calculated. $iv.$ Steps $(ii.)$ and $(iii.)$ are repeated until satisfactory accuracy for the response variance is attained.

2.3 Numerical examples

In this section, two numerical examples are used to validate the herein proposed approach and assess its versatility. The obtained results are compared with corresponding results which are derived by following the standard solution framework in Spanos et al., 2019.

2.3.1 3-DOF Nonlinear System with Singular Matrices

The 3-DOF nonlinear system in Fig. 1(a) is considered, where mass m_1 is connected to the foundation by a linear spring of stiffness k_1, a nonlinear inerter (e.g., Smith, 2002; Marian and Giaralis, 2014) and a nonlinear damper. The damping force is given by $c_1\dot{q}_1(1+\varepsilon_2\dot{q}_1^2)$ and the force due to the nonlinear inerter is given by $m_1\ddot{q}_1(1+\varepsilon_1\dot{q}_1^2)$, where q_i ($i=1,2,3$) denotes the displacement of the i-th mass, and ε_1 and ε_2 denote the magnitude of the nonlinearity for each case. Further, mass m_1 is connected to masses m_2 and m_3 by linear springs of stiffness k_2 and k_4, respectively. Finally, mass m_2 is connected to mass m_3 by a linear spring of stiffness k_3 and a linear damper of damping c_2. A force $Q_3(t)$, which is modeled as a Gaussian white noise stochastic process with constant spectral density S_0, and a deterministic force given by $f_{d_2,3}\sin(\omega_d t)$ are applied on mass m_3.



(a) (b)

Figure 2.1: (a) A 3-DOF nonlinear system subject to stochastic and deterministic excitations. (b) The nonlinear system of Fig. 2.1(a) modeled by employing redundant coordinates.

Next, the standard solution framework in Spanos et al., 2019 is applied for deriving the system response variance. In this regard, the parameter values $m_1=m_3=2$, $m_2=1$, $c_1=c_2=0.1$, $k_1=k_2=k_3=k_4=1$, in conjunction with the parameter values $\varepsilon_1=\varepsilon_2=1$ as well as $S_0=10^{-3}$ for $0<\omega<2\pi$, and $f_{d_2,3}=0.4, \omega_d=\pi$, are considered. The standard approach leads to

$$\sigma_{q_1}^2 = 0.0478, \quad \sigma_{\dot{q}_1}^2 = 0.0103, \quad \sigma_{\ddot{q}_1}^2 = 0.0061, \tag{2.36}$$

$$\sigma_{q_2}^2 = 0.0051, \quad \sigma_{\dot{q}_2}^2 = 0.0029, \quad \sigma_{\ddot{q}_2}^2 = 0.0052, \tag{2.37}$$

$$\sigma_{q_3}^2 = 0.0033, \quad \sigma_{\dot{q}_3}^2 = 0.0082, \quad \sigma_{\ddot{q}_3}^2 = 0.0438. \tag{2.38}$$

Then, considering the redundant coordinates vector $\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}$, the 3-DOF system in Fig. 1(a) is decomposed into its constituent parts as shown in Fig. 1(b). Further, taking into account the constraint equations connecting the subsystems in Fig. 1(b), matrix $\mathbf{A}$ in Eq. (2.2) becomes

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \tag{2.39}$$

whereas $\mathbf{E} = \mathbf{L} = \mathbf{0}_{2\times5}$ and $\mathbf{F} = \mathbf{0}_{2\times1}$. Thus, Eq. (2.12) is formed, where

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.4m_1 & 0.2m_2 & 0.2m_2 & 0.2m_3 & 0.2m_3 \\ 0.4m_1 & 0.2m_2 & 0.2m_2 & 0.2m_3 & 0.2m_3 \\ -0.2m_1 & 0.4m_2 & 0.4m_2 & 0.4m_3 & 0.4m_3 \\ 0.2m_1 & 0.6m_2 & 0.6m_2 & 0.6m_3 & 0.6m_3 \\ 0 & 0 & 0 & m_3 & m_3 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.4c_1 & 0 & 0 & 0 & 0 \\ 0.4c_1 & 0 & 0 & 0 & 0 \\ -0.2c_1 & 0 & 0 & 0 & 0 \\ 0.2c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.40}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.4k_1 & 0.2k_4 & -0.2k_2 & -0.2k_4 & -0.2k_4 \\ 0.4k_1 & 0.2k_4 & -0.2k_2 & -0.2k_4 & -0.2k_4 \\ -0.2k_1 & -0.6k_4 & 0.6k_2 & 0.6k_4 & 0.6k_4 \\ 0.2k_1 & -0.4k_4 & 0.4k_2 & 0.4k_4 & 0.4k_4 \\ 0 & -k_4 & 0 & k_4 & k_3+k_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.41}$$

and the nonlinear vector in Eq. (2.5) becomes

$$\bar{\boldsymbol{\Phi}}_{\mathbf{x}}^{\mathrm{T}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = (\varepsilon_1 m_1 \dot{x}_1^2 \ddot{x}_1 + \varepsilon_2 c_1 \dot{x}_1^3) \begin{bmatrix} 0.4 & 0.4 & -0.2 & 0.2 & 0 & 0 & 0 \end{bmatrix}. \tag{2.42}$$

Also, Eqs. (2.5) and (2.13) yield, respectively,

$$\begin{aligned} \bar{\mathbf{Q}}_{\mathbf{x}}^{\mathrm{T}} &= Q_3(t) \begin{bmatrix} 0.2 & 0.2 & 0.4 & 0.6 & 1 & 0 & 0 \end{bmatrix}, \\ \bar{\mathbf{f}}_{d,\mathbf{x}}^{\mathrm{T}} &= f_{d_2,3} \sin(\omega_d t) \begin{bmatrix} 0.2 & 0.2 & 0.4 & 0.6 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.43}$$

Next, the herein generalized harmonic balance method for systems with singular matrices is applied to the system defined by the singular parameter matrices in Eqs. (2.40) and (2.41). Thus, taking into account the decomposition of the system response into a stochastic and a deterministic component, i.e., $\mathbf{x}_s^{\mathrm{T}} = \begin{bmatrix} x_{s,1} & x_{s,2} & x_{s,3} & x_{s,4} & x_{s,5} \end{bmatrix}$ and $\mathbf{x}_d^{\mathrm{T}} = \begin{bmatrix} x_{d,1} & x_{d,2} & x_{d,3} & x_{d,4} & x_{d,5} \end{bmatrix}$, Eq. (2.42) yields

$$\begin{aligned} \mathbb{E}[\bar{\boldsymbol{\Phi}}_{\mathbf{x}}]^{\mathrm{T}} = &\Big(\varepsilon_1 m_1 (\dot{x}_{d,1}^2 \ddot{x}_{d,1} + \sigma_{\dot{x}_{s,1}}^2 \ddot{x}_{d,1}) + \varepsilon_2 c_1 (\dot{x}_{d,1}^3 + 3\dot{x}_{d,1}\sigma_{\dot{x}_{s,1}}^2)\Big) \\ &\times \begin{bmatrix} 0.4 & 0.4 & -0.2 & 0.2 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.44}$$

Further, since the 14×10 matrix $\mathbf{P}$ in Eq. (2.22) has full rank, i.e., $rank(\mathbf{P}) = 10$, Eq. (2.27) is used instead of Eq. (2.25) to derive a unique solution for the periodic response vector (see also Eqs. (2.19) and (2.20)). Finally, applying the generalized statistical linearization method, in conjunction with the averaging treatment, Eqs. (2.32) yields

$$\bar{\mathbf{C}}_e^{av} = 0.6\varepsilon_2 c_1 \sigma^2_{\dot{x}_{s,1}} \begin{bmatrix} 2H(6,6) & 2H(7,6) & 2H(8,6) & 2H(9,6) & 2H(10,6) \\ 2H(6,6) & 2H(7,6) & 2H(8,6) & 2H(9,6) & 2H(10,6) \\ -H(6,6) & -H(7,6) & -H(8,6) & -H(9,6) & -H(10,6) \\ H(6,6) & H(7,6) & H(8,6) & H(9,6) & H(10,6) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \varepsilon_2 c_1 \omega_d^2 (x^2_{d_1,1} + x^2_{d_2,1}) \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ -0.3 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.45}$$

and

$$\bar{\mathbf{M}}_e^{av} = 0.2\varepsilon_1 m_1 \sigma^2_{\dot{x}_{s,1}} \begin{bmatrix} 2H(11,11) & 2H(12,11) & 2H(13,11) & 2H(14,11) & 2H(15,11) \\ 2H(11,11) & 2H(12,11) & 2H(13,11) & 2H(14,11) & 2H(15,11) \\ -H(11,11) & -H(12,11) & -H(13,11) & -H(14,11) & -H(15,11) \\ H(11,11) & H(12,11) & H(13,11) & H(14,11) & H(15,11) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \varepsilon_1 m_1 \omega_d^2 (x^2_{d_1,1} + x^2_{d_2,1}) \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.46}$$

The terms $H(i,j), i,j = 1,2,\ldots,15$, in Eqs. (2.45) and (2.46) denote the (i,j) element of the 15×15 matrix $\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ in Eq. (2.31) (see also Fragkoulis et al., 2016b).

Then, the coupled set of algebraic equations formed by Eq. (2.35), Eq. (2.9) and Eq. (2.27) is solved for determining the stochastic and deterministic components of the response. This is attained by employing the iterative scheme included in section "Generalized statistical linearization and averaging treatments". In this regard, considering the initial values $\bar{\mathbf{M}}_e^{av} = \bar{\mathbf{C}}_e^{av} = \mathbf{0}$ and $x_{d_1} = x_{d_2} = 0$, the stochastic component is derived based on the criterion $\left|\frac{\bar{\mathbf{M}}^{av}_{e,j+1} - \bar{\mathbf{M}}^{av}_{e,j}}{\bar{\mathbf{M}}^{av}_{e,j}}\right| < 10^{-5}$ and $\left|\frac{\bar{\mathbf{C}}^{av}_{e,j+1} - \bar{\mathbf{C}}^{av}_{e,j}}{\bar{\mathbf{C}}^{av}_{e,j}}\right| < 10^{-5}$, whereas a similar criterion is used to obtain the deterministic components x_{d_1}, x_{d_2}. The iterative scheme stops after 5 iterations, when satisfactory accuracy for the response velocity variance $\sigma^2_{\dot{x}_{s,1}}$ is attained (see Eqs. (2.45) and (2.46)).

Finally, substituting Eq. (2.17) into Eq. (2.14), and successively ensemble and temporal averaging to treat, respectively, the stochastic and deterministic components of the response, yields

$$\left\langle \mathbb{E}[x_i^2] \right\rangle = \sigma_{x_{s,1}}^2 + \frac{x_{d_1,i}^2 + x_{d_2,i}^2}{2}, \quad \left\langle \mathbb{E}[\dot{x}_i^2] \right\rangle = \sigma_{\dot{x}_{s,1}}^2 + \frac{\omega_d^2 (x_{d_1,i}^2 + x_{d_2,i}^2)}{2} \tag{2.47}$$

and

$$\left\langle \mathbb{E}[\ddot{x}_i^2] \right\rangle = \sigma_{\ddot{x}_{s,1}}^2 + \frac{\omega_d^4}{2} (x_{d_1,i}^2 + x_{d_2,i}^2), \tag{2.48}$$

for $i = 1, 2, \ldots, 5$, where $\langle \cdot \rangle$ denotes the temporal averaging operation. Eqs. (2.47) and (2.48), in conjunction with the results of the iterative scheme above yield

$$\sigma_{x_1}^2 = 0.0478, \quad \sigma_{\dot{x}_1}^2 = 0.0103, \quad \sigma_{\ddot{x}_1}^2 = 0.0061, \tag{2.49}$$

$$\sigma_{x_3}^2 = 0.0051, \quad \sigma_{\dot{x}_3}^2 = 0.0029, \quad \sigma_{\ddot{x}_3}^2 = 0.0052, \tag{2.50}$$

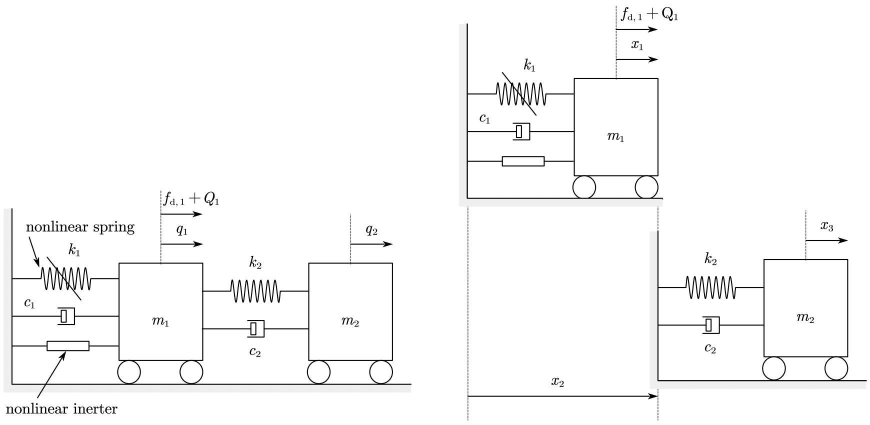
$$\sigma_{x_5}^2 = 0.0033, \quad \sigma_{\dot{x}_5}^2 = 0.0082, \quad \sigma_{\ddot{x}_5}^2 = 0.0438. \tag{2.51}$$

Comparing Eqs. (2.49)-(2.51) with Eqs. (2.36)-(2.38), it is readily seen that the herein proposed framework is in total agreement with the standard approach in Spanos et al., 2019.

2.3.2 2-DOF Nonlinear Structural System with Singular Parameter Matrices

In this example, the application of the herein proposed framework to a wider magnitude range of system nonlinearities is demonstrated. In this regard, the 2-DOF system of rigid masses m_1 and m_2 in Fig. 2(a) is considered. Mass m_1 is connected to the foundation by a nonlinear inerter and a nonlinear spring, whose forces are $m_1 \ddot{q}_1 (1 + \varepsilon_1 \dot{q}_1^2)$ and $k_1 q_1 (1 + \varepsilon_2 q_1^2)$, respectively, where q_i $(i = 1, 2)$ denotes the displacement of the i-th mass, and $\varepsilon_1, \varepsilon_2$ the magnitude of the nonlinearities. Further, mass m_1 is connected to mass m_2 by a linear spring of stiffness k_2 and a linear damper of damping c_2. The system is excited by combined stochastic and deterministic forces applied on mass m_1. In particular, $Q_1(t)$ is modeled as a Gaussian white noise stochastic process with constant spectral density S_0 and the deterministic force has the form $f_{d_2,1} \sin(\omega_d t)$.

Further, considering the parameter values $m_1 = m_2 = 1$, $c_1 = c_2 = 0.2$, $k_1 = k_2 = 1$,



(a) (b)

Figure 2.2: (a) A 2-DOF nonlinear system subject to stochastic and deterministic excitations. (b) The nonlinear system of Fig. 2.2(a) modeled by employing an additional redundant coordinate.

$S_0 = 10^{-2}$ $(0 < \omega < 2\pi)$ and $f_{d_2,1} = 0.4$, $\omega_d = \pi$, the system response variance is determined by applying the standard approach in Spanos et al., 2019. In addition, the magnitude ε of nonlinearities, where $\varepsilon_1 = \varepsilon_2 = \varepsilon$, is taking values in the interval $[0, 5]$. The results are depicted by the solid line in Fig. 3.

Next, considering the redundant coordinates vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$, the 2-DOF system of Fig. 2(a) is decomposed into its partial subsystems, as shown in Fig. 2(b). In this regard, Eq. (2.2) is formed, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \tag{2.52}$$

$\mathbf{E} = \mathbf{L} = \mathbf{0}_{1\times3}$ and the vector $\mathbf{F}$ degenerates to $\mathbf{F} = 0$. Thus, the parameter matrices in Eq. (2.12) are given by

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5m_1 & 0.5m_2 & 0.5m_2 \\ 0.5m_1 & 0.5m_2 & 0.5m_2 \\ 0 & m_2 & m_2 \\ 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.5c_1 & 0 & 0 \\ 0.5c_1 & 0 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5k_1 & 0 & 0 \\ 0.5k_1 & 0 & 0 \\ 0 & 0 & k_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.53}$$

whereas Eqs. (2.5) and (2.13), respectively, yield

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}^{\mathrm{T}}(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}}) = (m_1\varepsilon_1\dot{x}_1^2\ddot{x}_1 + k_1\varepsilon_2 x_1^3)\begin{bmatrix}0.5 & 0.5 & 0 & 0\end{bmatrix}, \quad \bar{\mathbf{Q}}_{\mathbf{x}}^{\mathrm{T}} = Q_1(t)\begin{bmatrix}0.5 & 0.5 & 0 & 0\end{bmatrix} \tag{2.54}$$

and

$$\bar{\mathbf{f}}_{d,\mathbf{x}}^{\mathrm{T}} = f_{d_2,1}\sin(\omega_d t)\begin{bmatrix}0.5 & 0.5 & 0 & 0\end{bmatrix}. \tag{2.55}$$

For the application of the harmonic balance method, the system response is decomposed into its stochastic $\mathbf{x}_s^{\mathrm{T}} = \begin{bmatrix}x_{s,1} & x_{s,2} & x_{s,3}\end{bmatrix}$ and deterministic $\mathbf{x}_d^{\mathrm{T}} = \begin{bmatrix}x_{d,1} & x_{d,2} & x_{d,3}\end{bmatrix}$ components, and thus, substituting Eq. (2.17) into Eq. (2.54) and ensemble averaging yields

$$\mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}]^{\mathrm{T}} = \left(m_1\varepsilon_1(\dot{x}_{d,1}^2\ddot{x}_{d,1} + \sigma_{\dot{x}_{s,1}}^2\ddot{x}_{d,1}) + k_1\varepsilon_2(x_{d,1}^3 + 3x_{d,1}\sigma_{x_{s,1}}^2)\right)\begin{bmatrix}0.5 & 0.5 & 0 & 0\end{bmatrix}. \tag{2.56}$$

Then, the overdetermined system of equations defined by Eq. (2.21) (or, equivalently, by Eqs. (2.19) and (2.20)) is solved. To this end, it is noted that the 8×6 matrix $\mathbf{P}$ in Eq. (2.22) has full rank. Hence, Eq. (2.27) leads to a uniquely defined periodic response component. Subsequently, the generalized statistical linearization method is used in conjunction with the averaging treatment to treat the stochastic component of the response. In this regard, Eq. (2.32) implies

$$\bar{\mathbf{K}}_e^{av} = 1.5k_1\varepsilon_2\sigma_{x_{s,1}}^2\begin{bmatrix}H(1,1) & H(2,1) & H(3,1)\\ H(1,1) & H(2,1) & H(3,1)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + 3k_1\varepsilon_2\begin{bmatrix}\frac{(x_{d_1,1}^2+x_{d_2,1}^2)}{2} & 0 & 0\\ \frac{(x_{d_1,1}^2+x_{d_2,1}^2)}{2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{2.57}$$

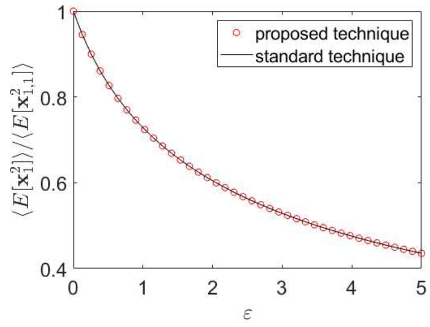
and

$$\bar{\mathbf{M}}_e^{av} = 0.5m_1\varepsilon_1\sigma^2_{\dot{x}_{s,1}}\begin{bmatrix} H(7,7) & H(8,7) & H(9,7) \\ H(7,7) & H(8,7) & H(9,7) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + m_1\varepsilon_1\begin{bmatrix} \frac{\omega_d^2(x^2_{d_1,1}+x^2_{d_2,1})}{2} & 0 & 0 \\ \frac{\omega_d^2(x^2_{d_1,1}+x^2_{d_2,1})}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.58)$$

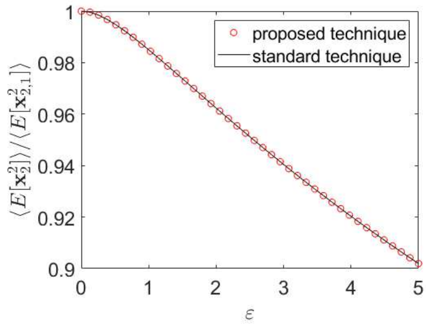
where $H(i,j), i,j = 1,2,\ldots,9$, denote the (i,j) element of matrix $\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ in Eq. (2.31).

Then, the iterative scheme in section "Generalized statistical linearization and averaging treatments" is employed to solve the coupled set of algebraic equations formed by Eqs. (2.35), Eq. (2.9) and Eq. (2.27), and thus, to derive the variance of the stochastic response. Considering the dependence between the stochastic and deterministic components (see Eqs. (2.56)-(2.58)), the scheme is initialized by using $\bar{\mathbf{M}}_e^{av} = \mathbf{0}, \bar{\mathbf{K}}_e^{av} = \mathbf{0}$ and $x_{d_1} = x_{d_2} = 0$. Then, the stochastic and deterministic components are derived based on the criterion $\left|\frac{\bar{\mathbf{M}}_{e,j+1}^{av} - \bar{\mathbf{M}}_{e,j}^{av}}{\bar{\mathbf{M}}_{e,j}^{av}}\right| < 10^{-5}$ and $\left|\frac{\bar{\mathbf{K}}_{e,j+1}^{av} - \bar{\mathbf{K}}_{e,j}^{av}}{\bar{\mathbf{K}}_{e,j}^{av}}\right| < 10^{-5}$, as well as a similar criterion for x_{d_1}, x_{d_2}. The iterative scheme continues until reaching satisfactory accuracy for the response displacement and velocity variance $\sigma^2_{x_{s,1}}$ and $\sigma^2_{\dot{x}_{s,1}}$.

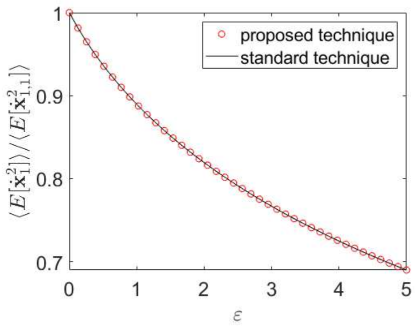
Finally, the system response variance is determined by utilizing Eqs. (2.47) and (2.48). The obtained results for different values of $\varepsilon_1 = \varepsilon_2 = \varepsilon \in [0,5]$ are represented by dots in Fig. 3. They are in complete agreement with the corresponding results obtained by applying the standard approach in Spanos et al., 2019 (solid line). Thus, the herein developed combination of the M-P matrix inverses-based statistical linearization and harmonic balance scheme constitutes a generalization of the formulation in ibid. to account for systems with singular parameter matrices. Note, in passing, that a normalization with respect to the analytical results for the linear case, i.e., $\varepsilon_1 = \varepsilon_2 = 0$, is considered for both solution frameworks to show the considerable nonlinearity effect on the system response.



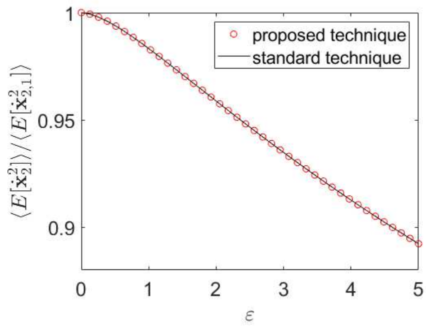
(a)



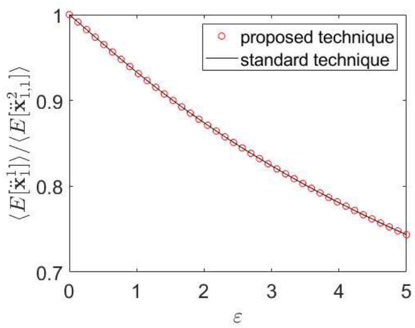
(b)



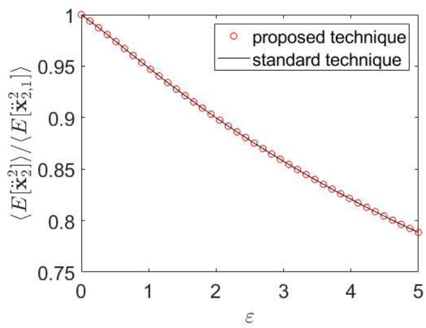
(c)



(d)



(e)



(f)

Figure 2.3: Normalized response variance of the nonlinear structural system in Figs. 2.2(a)-2.2(b) vs. nonlinearity magnitude. Comparison between the standard and the proposed techniques. (a) 1st DOF response displacement variance; (b) 2nd DOF response displacement variance; (c) 1st DOF response velocity variance; (d) 2nd DOF response velocity variance; (e) 1st DOF response acceleration variance; (f) 2nd DOF response acceleration variance

2.4 Conclusions

In this paper, a novel technique has been developed for bounding the responses and probability of failure of nonlinear structural models subjected to imprecisely defined stochastic Gaussian loads. The proposed technique can be construed as a generalization of a recently developed operator norm-based method to account for nonlinear dynamical systems. This is attained by resorting to the statistical linearization methodology for defining a linear system equivalent to the nonlinear system under consideration. In this regard, the double loop that is typically associated with estimating the bounds on the probability of failure of nonlinear dynamical systems is effectively decoupled, and the associated computational cost is reduced by several orders of magnitude. Thus, it can be argued that integrating statistical linearization into the operator norm framework allows for bounding the probability of failure of nonlinear systems with acceptable accuracy and at greatly reduced numerical cost.

The validity and numerical efficiency of the proposed technique have been demonstrated by considering two nonlinear structural systems. However, since the linearization scheme has been performed in a mean square error minimization sense, the representation of the nonlinear system is less accurate in the tails of the distribution. This aspect renders the proposed approach mostly suitable for estimating the bounds of moderate to large failure probabilities. Nevertheless, future work is directed toward developing an enhanced operator norm-based linearization scheme capable of estimating bounds on smaller failure probabilities. This can be achieved, in principle, by combining the application of the statistical linearization methodology with a stochastic averaging treatment.

Further, the proposed framework can be integrated with more advanced simulation methods, such as importance sampling or subset simulation. Another path for future work consists of extending the range of application of the proposed framework to more general models for stochastic loading (other than Gaussian). Finally, the evaluation of the proposed approach for more complex and numerically demanding structural models involving multiple types of nonlinearities constitutes an additional subject for future research.

Data Availability Statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

2.5 Acknowledgments

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2.6 APPENDIX I. Elements of the theory of Moore-Penrose matrix inverses

In this appendix, a concise presentation of the fundamental results of the Moore-Penrose (M-P) generalized matrix inverses theory is presented for completeness. The interested reader is directed to Campbell and Meyer, 2009 and Ben-Israel and Greville, 2003 for a detailed presentation.

The mathematical problem that gave rise to the generalized matrix inverses theory is related to the solution of the algebraic system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{2.59}$$

In the general case, $\mathbf{A}$ in Eq. (2.59) denotes a rectangular $m \times n$ matrix and $\mathbf{x}, \mathbf{b}$ correspond, respectively, to n- and m-dimensional vectors. However, it is noted that the ensuing results also hold for the case of square, but singular matrix $\mathbf{A}$. Taking into account that the general solution to the problem in Eq. (2.59) is not possible due to the nature of matrix $\mathbf{A}$, and also considering that such problems are often encountered in theoretical as well as in applied science, the concept of a "partial inverse" of matrix $\mathbf{A}$ was introduced (Campbell and Meyer, 2009).

Definition 1. Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, there is a uniquely defined matrix $\mathbf{A}^{+} \in \mathbb{C}^{n \times m}$ such that:
(i) $\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}$, (ii) $\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}$, (iii) $(\mathbf{A}\mathbf{A}^{+})^{*} = \mathbf{A}\mathbf{A}^{+}$, (iv) $(\mathbf{A}^{+}\mathbf{A})^{*} = \mathbf{A}^{+}\mathbf{A}$.

Matrix $\mathbf{A}^{+}$ in Definition 1 is the so-called M-P inverse of $\mathbf{A}$. In general, when $\mathbf{A}$ is invertible, $\mathbf{A}^{+}$ coincides with the regular inverse $\mathbf{A}^{-1}$. Considering the solution of the algebraic system in Eq. (2.59), the M-P inverse holds an exceptional place among the family of generalized inverses, since it leads to the family of solutions

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + (\mathbf{I}_n - \mathbf{A}^{+}\mathbf{A})\mathbf{y}, \tag{2.60}$$

where $\mathbf{I}_n$ is the identity $n \times n$ matrix and $\mathbf{y}$ accounts for an arbitrary n-dimensional vector (ibid.; Ben-Israel and Greville, 2003).

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Chapter 3

Research article 2: Non-stationary response of nonlinear systems with singular parameter matrices subject to combined deterministic and stochastic excitation

Non-stationary response of nonlinear systems with singular parameter matrices subject to combined deterministic and stochastic excitation

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Abstract: A new technique is proposed for determining the response of multi-degree-of-freedom nonlinear systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. Singular matrices in the governing equations of motion potentially account for the presence of constraints equations in the system. Further, they also appear when a redundant coordinates modeling is adopted to derive the equations of motion of complex multi-body systems. In this regard, the system response is decomposed into a deterministic and a stochastic component corresponding to the two components of the excitation. Then, two sets of differential equations are formulated and solved simultaneously to compute the system response. The first set pertains to the deterministic response component, whereas the second one pertains to the stochastic component of the response. The latter is derived by utilizing the generalized statistical linearization method for systems with singular matrices, while a formula for determining the time-dependent equivalent elements of the generalized statistical lineariza-

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tion methodology is also derived. The efficiency of the proposed technique is demonstrated by pertinent numerical examples. Specifically, a vibration energy harvesting device subject to combined deterministic and modulated white noise excitation and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations are considered. **Keywords:** Stochastic Dynamics; Combined Excitation; Moore-Penrose Matrix Inverse; Statistical Linearization; Energy Harvester

3.1 Introduction

Assessing the reliability of nonlinear multi-degree-of-freedom (MDOF) systems subject to combined deterministic and stochastic loading constitutes a persistent challenge in random vibration, which finds a plethora of applications in several engineering fields. Indicatively, these span from vibration energy harvesting (e.g., Ando et al., 2017; Dai and Harne, 2018; Huang et al., 2022) to the problem of turbine blades vibration under turbulent flow (e.g., Namachchivaya, 1991; Zhu and Wu, 2004), or nonlinear vibration of beams and plates (e.g., Spanos and Malara, 2020), and vibration of gear systems (e.g., Zhang and Spanos, 2020b).

In this context, considerable research effort has been put over the last decades into developing methodologies and techniques aiming at determining the response of nonlinear MDOF systems subject to combined deterministic and stochastic excitation. This has been done by utilizing and combining standard deterministic and stochastic analysis tools such as, indicatively, the harmonic balance and statistical linearization or Gaussian closure methods (e.g., Zhu and Guo, 2015; Zhang and Spanos, 2020a; Spanos et al., 2019; Zhang and Spanos, 2020b; Kong and Spanos, 2021; Kong et al., 2022c), the harmonic balance and stochastic averaging methods (e.g., Haiwu et al., 2001), and the equivalent linearization and deterministic or stochastic averaging methods (e.g., Anh and Hieu, 2012; Anh et al., 2014). Further, the need for more accurate media behavior modeling dictated by recent advances in theoretical and applied mechanics (e.g., Di Paola et al., 2013) has propelled the use of fractional calculus which, in turn, resulted to the development of pertinent frameworks (e.g., Spanos and Malara, 2020; Kong et al., 2022b). Yet, most of the approaches available in the literature to-date treat systems whose stochastic excitation component is modeled as a stationary stochastic process. However, a more accurate modeling of the applied stochastic excitation component necessitates considering the non-stationary characteristics corresponding to excitations often met in nature, such as wave, wind and earthquake loads. This has recently led to the extension of relevant tools, and approaches accounting for non-stationary stochastic excitations (e.g., Kong et al., 2022a) and non-stationary excitations described by evolutionary power spectrum forms (e.g., Han et al., 2022) have been proposed.

An additional aspect of the response determination problem for MDOF systems subject to combined deterministic and stochastic excitation relates to the complexity of the system under consideration. In this regard, a technique accounting for singular parameter matrices and constraints in the equations governing the dynamics of the MDOF system has been recently developed in Ni et al., 2021. Examples of such systems are often met in engineering applications including, indicatively, systems with massless joints (e.g., Pirrotta et al., 2019, 2021), oscillators modeled via additional auxiliary state equations (e.g., Petromichelakis et al., 2020), energy harvesting devices and specific classes of non-viscously damped systems (e.g., Adhikari, 2013). Hence, utilizing tools from the theory of generalized matrix inverses (e.g., Campbell and Meyer, 2009) has led to the extension of known input-output (excitation-response) expressions in random vibration theory, and subsequently, to the development of various frameworks for determining the response of MDOF linear and nonlinear systems (e.g., Fragkoulis et al., 2016a,b; Kougioumtzoglou et al., 2017; Antoniou et al., 2017; Karageorgos et al., 2021), conducting joint time-frequency analysis of the system response (e.g., Pasparakis et al., 2021, 2022), or solving random eigenvalue problems for systems with singular random parameter matrices (Fragkoulis et al., 2022).

In this paper, the technique developed in Ni et al., 2021 is extended to MDOF nonlinear systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. This is done by formulating and solving simultaneously two sets of differential equations, corresponding to the deterministic and the stochastic components of the response, respectively. An additional contribution relates to the generalization of the expression derived in Fragkoulis et al., 2016b to determine the time-dependent equivalent elements of the generalized statistical linearization methodology for systems with singular parameter matrices. Three numerical examples are considered to assess the reliability of the proposed technique. These include a vibration energy harvesting device subject to combined deterministic and modulated white noise excitation and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations. The obtained results are compared with pertinent Monte Carlo simulation (MCS) data as well as with corresponding results obtained by the approach proposed in Kong et al., 2022a.

3.2 Mathematical formulation

3.2.1 Nonlinear MDOF systems with singular parameter matrices

The governing equations of motion of an l-DOF nonlinear system subject to combined deterministic and non-stationary stochastic excitation are given by

$$\mathbf{M_x}\ddot{\mathbf{x}} + \mathbf{C_x}\dot{\mathbf{x}} + \mathbf{K_x}\mathbf{x} + \mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{f}_{d,\mathbf{x}}(t) + \mathbf{Q_x}(t), \tag{3.1}$$

where $\mathbf{x}$ denotes the (possibly dependent) l-dimensional response displacement vector and $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ are the response velocity and acceleration l-dimensional vectors, respectively. Further, $\mathbf{M_x}$, $\mathbf{C_x}$ and $\mathbf{K_x}$ correspond to the $l \times l$ mass, damping and stiffness matrices of the system, whereas $\mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ denotes the l-dimensional nonlinear vector of the system. Lastly, $\mathbf{f}_{d,\mathbf{x}}(t)$ and $\mathbf{Q_x}(t)$ are the l-dimensional vectors of the deterministic and the zero-mean non-stationary stochastic excitation, respectively. It is noted that considering a zero-mean excitation is rather for simplicity and not restrictive for the ensuing analysis, which can be generalized also to the case of a nonzero-mean process (e.g., Roberts and Spanos, 2003).

Considering, next, that the l-DOF system of Eq. (3.1) is subject to additional constraints (Udwadia and Phohomsiri, 2006; Fragkoulis et al., 2016a)

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{3.2}$$

Eq. (3.1) is recast into

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} + \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{f}}_{d,\mathbf{x}}(t) + \bar{\mathbf{Q}}_{\mathbf{x}}(t). \tag{3.3}$$

In Eq. (3.3), $\bar{\mathbf{M}}_{\mathbf{x}}$, $\bar{\mathbf{C}}_{\mathbf{x}}$ and $\bar{\mathbf{K}}_{\mathbf{x}}$ denote the augmented $(l + m) \times l$ mass, damping and stiffness matrices of the system given by (Karageorgos et al., 2021; Pasparakis et al., 2022)

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{J}\mathbf{M}_{\mathbf{x}} \\ \mathbf{A} \end{bmatrix}, \; \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{J}\mathbf{C}_{\mathbf{x}} \\ \mathbf{E} \end{bmatrix}, \; \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{J}\mathbf{K}_{\mathbf{x}} \\ \mathbf{L} \end{bmatrix}, \tag{3.4}$$

the augmented $(l + m)$-dimensional nonlinearity vector has the form

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{J}\mathbf{\Phi}_{\mathbf{x}} \\ \mathbf{0}_{m\times 1} \end{bmatrix}, \tag{3.5}$$

whereas the augmented $(l + m)$-dimensional vectors of the applied deterministic and stochastic excitations are given by

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \begin{bmatrix} \mathbf{J}\mathbf{f}_{d,\mathbf{x}}(t) \\ \mathbf{0}_{m\times 1} \end{bmatrix} \tag{3.6}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{J}\mathbf{Q}_{\mathbf{x}}(t) \\ \mathbf{F} \end{bmatrix}, \tag{3.7}$$

respectively. For the derivation of the system parameter matrices in Eq. (3.4), as well as the excitation vectors in Eq. (3.7), the system constraints Eq. (3.2) is written, for simplicity, in the form $\mathbf{A}\ddot{\mathbf{x}} + \mathbf{E}\dot{\mathbf{x}} + \mathbf{L}\mathbf{x} = \mathbf{F}$, with $\mathbf{A}$, $\mathbf{E}$, $\mathbf{L}$ denoting $m \times l$ matrices and $\mathbf{F}$ denoting an

m-dimensional vector (e.g., Fragkoulis et al., 2016a). Further, $\mathbf{J}$ represents an $l \times l$ matrix connecting the constraints Eq. (3.2) with the system governing equations of motion Eq. (3.1) (e.g., Antoniou et al., 2017; Pasparakis et al., 2022). A detailed derivation of Eqs. (3.3-3.7) can be found in Fragkoulis et al., 2016a; Kougioumtzoglou et al., 2017; Ni et al., 2021.

3.2.2 System response determination

In this section, a semi-analytical technique is proposed for determining the response of MDOF systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. This is attained by decomposing the nonlinear system into two subsystems, i.e., one subject to the non-stationary stochastic excitation and one subject to the deterministic excitation. The former is simplified by resorting to the generalized statistical linearization method for systems with singular parameter matrices (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017), followed by a state variable treatment. This involves the formulation of a time-dependent matrix differential equation, whose solution yields the standard deviation of the stochastic component of the response. Further, a set of deterministic differential equations corresponding to the subsystem subject to the deterministic excitation, and thus, governing the deterministic response component, is derived and solved simultaneously with the matrix differential equation above. This can be done by resorting to any standard numerical scheme, such as the Runge–Kutta method.

Generalized statistical linearization based framework

Consider the augmented system in Eq. (3.3) which is subject to combined deterministic and non-stationary stochastic excitation. The system response is decomposed into two components, namely the stochastic and the deterministic one, accounting, respectively, for the corresponding components of the excitation. That is

$$\mathbf{x}(t) = \mathbf{x}_s(t) + \mathbf{x}_d(t), \tag{3.8}$$

where the l-dimensional vector $\mathbf{x}_s(t)$ denotes the zero-mean stochastic component of the response and the l-dimensional vector $\mathbf{x}_d(t)$ represents the deterministic response component. Then, taking into account that the stochastic displacement component is modeled as a zero-mean process, substituting Eq. (3.8) into Eq. (3.3) and ensemble averaging yields

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_d + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_d + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_d + \mathrm{E}\left[\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)\right] = \bar{\mathbf{f}}_{d,\mathbf{x}}(t), \tag{3.9}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. Eq. (3.9) constitutes a subsystem of deterministic differential equations to be solved for computing the deterministic response of the system. Then, subtracting Eq. (3.9) from Eq. (3.3) yields a subsystem of equations subject to non-stationary stochastic excitation, namely

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_s + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_s + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_s + \tilde{\bar{\boldsymbol{\Phi}}}_{\mathbf{x}} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{3.10}$$

where

$$\tilde{\bar{\boldsymbol{\Phi}}}_{\mathbf{x}} = \bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d) - \mathrm{E}\left[\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)\right]. \tag{3.11}$$

Clearly, the nonlinear terms in Eqs. (3.9) and (3.10) consist of a deterministic and a stochastic response components, which are intertwined. Therefore, Eqs. (3.9) and (3.10) constitute a coupled set of differential equations to be solved for determining the system response.

In this regard, the generalized statistical linearization method is applied and a linear system equivalent to the subsystem of Eq. (3.10) is defined as

$$\left(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e(t)\right)\ddot{\mathbf{x}}_s + \left(\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e(t)\right)\dot{\mathbf{x}}_s + \left(\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e(t)\right)\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{3.12}$$

where $\bar{\mathbf{M}}_e(t)$, $\bar{\mathbf{C}}_e(t)$ and $\bar{\mathbf{K}}_e(t)$ are the time-varying $(l+m) \times l$ mass, damping and stiffness matrices of the equivalent linear system. Then, the error function is defined as the difference between the nonlinear system in Eq. (3.10) and the equivalent linear system in Eq. (3.12), and it is minimized by adopting a mean square minimization criterion in conjunction with the Gaussian response assumption (Roberts and Spanos, 2003).

Clearly, one of the advantages of the standard statistical linearization method relates to its capacity to provide closed-form expressions for determining the equivalent linear elements of Eq. (3.12). In this context, consider that $\mathbf{m}_{i*}^{e\mathrm{T}}(t)$, $\mathbf{c}_{i*}^{e\mathrm{T}}(t)$ and $\mathbf{k}_{i*}^{e\mathrm{T}}(t)$ for $i = 1, 2, \ldots, l+m$ denote the i-th row of the $(l+m) \times l$ time-varying matrices $\bar{\mathbf{M}}_e(t)$, $\bar{\mathbf{C}}_e(t)$ and $\bar{\mathbf{K}}_e(t)$, respectively, and that $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_s & \dot{\mathbf{x}}_s & \ddot{\mathbf{x}}_s \end{bmatrix}^{\mathrm{T}}$ is a $3l$-dimensional vector with "T" denoting the matrix transpose operation. A key aspect in determining $\mathbf{m}_{i*}^{e\mathrm{T}}(t)$, $\mathbf{c}_{i*}^{e\mathrm{T}}(t)$ and $\mathbf{k}_{i*}^{e\mathrm{T}}(t)$ is that the covariance matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ is invertible (Roberts and Spanos, 2003). However, due to the possibly dependent coordinates utilized to model the system governing equations of motion in Eq. (3.1), $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ is singular. Nevertheless, generalized expressions for the equivalent elements have been proposed in Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017 for the case where the system is subject to stationary stochastic excitation, as well as in Ni et al., 2021 for MDOF systems subject to combined deterministic and stationary stochastic excitation.

In this regard, the equivalent linear elements for systems with singular parameter matrices and subject to deterministic and non-stationary stochastic excitations are given by

$$\begin{bmatrix} \mathbf{k}_{i*}^{e\mathrm{T}}(t) \\ \mathbf{c}_{i*}^{e\mathrm{T}}(t) \\ \mathbf{m}_{i*}^{e\mathrm{T}}(t) \end{bmatrix} = \mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]\mathrm{E}\begin{bmatrix} \frac{\partial \tilde{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial \mathbf{x}} \\ \frac{\partial \tilde{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial \dot{\mathbf{x}}} \\ \frac{\partial \tilde{\boldsymbol{\Phi}}_{\mathbf{x}}(i)}{\partial \ddot{\mathbf{x}}} \end{bmatrix}, \tag{3.13}$$

for $i = 1, 2, \ldots, l+m$, where "+" denotes the Moore-Penrose generalized inverse matrix operation (e.g., Campbell and Meyer, 2009). In passing, note that an arbitrary term should also be included in Eq. (3.13) due to utilizing the generalized inverse matrix theory for its derivation. Thus, Eq. (3.13) corresponds, in essence, to a family of solutions for the equivalent linear elements rather than a unique expression. Nevertheless, it has been proved in Fragkoulis et al., 2016b that the solution derived by setting the arbitrary term equal to zero is at least as good as any other solution corresponding to a nonzero value for the arbitrary term. Therefore, Eq. (3.13) constitutes the counterpart of the expression in Fragkoulis et al., 2016b used for determining the time-dependent equivalent elements of the generalized statistical linearization methodology for systems with singular parameter matrices. The interested reader is directed to Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017 for details on the derivation of Eq. (3.13); corresponding expressions accounting for joint time-frequency response analysis of nonlinear systems with singular matrices are found in Pasparakis et al., 2021.

Finally, it is noted that the response covariance matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ as well as its Moore-Penrose generalized matrix inverse $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}$ are required for computing the equivalent linear elements in Eq. (3.13), and subsequently, for determining the system response. In addition, it is readily seen that the equivalent linear elements are time-dependent, and thus, in contrast to the stationary case Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017, a set of differential equations is derived and solved in the ensuing analysis. This is attained by utilizing a state variable formulation, which leads to a matrix differential equation governing the time-variant covariance matrix of the system response.

State variable analysis for MDOF systems with singular parameter matrices

In this section, the state variable formulation developed in Fragkoulis et al., 2016a for MDOF systems with singular parameter matrices is further extended to treat the linear system with time-dependent equivalent elements in Eq. (3.12). Ultimately, this leads to a time-varying matrix differential equation to be solved for determining the standard deviation of the non-stationary response component.

In this regard, suppose for simplicity that $\bar{\mathbf{M}}_{\mathbf{x},t} = \bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_{e}(t)$, $\bar{\mathbf{C}}_{\mathbf{x},t} = \bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_{e}(t)$ and $\bar{\mathbf{K}}_{\mathbf{x},t} = \bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_{e}(t)$. Then, taking into account the properties of the generalized matrix inverse theory (e.g., Campbell and Meyer, 2009), Eq. (3.12) yields

$$\ddot{\mathbf{x}} = \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\left(-\bar{\mathbf{C}}_{\mathbf{x},t}\dot{\mathbf{x}} - \bar{\mathbf{K}}_{\mathbf{x},t}\mathbf{x} + \bar{\mathbf{Q}}_{\mathbf{x}}(t)\right) + \left(\mathbf{I} - \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{M}}_{\mathbf{x},t}\right)\mathbf{y}, \tag{3.14}$$

where $\mathbf{y}$ is an arbitrary l-dimensional vector. Clearly, the presence of $\mathbf{y}$ in Eq. (3.14) defines a family of equations for the response acceleration. Nevertheless, it is noted that for the special case when the $(l+m) \times l$ matrix $\bar{\mathbf{M}}_{\mathbf{x},t}$ has full rank, i.e., $\mathrm{rank}(\bar{\mathbf{M}}_{\mathbf{x},t}) = l$, its Moore-Penrose generalized matrix inverse simplifies to $\bar{\mathbf{M}}^{+}_{\mathbf{x},t} = \left(\bar{\mathbf{M}}^{*}_{\mathbf{x},t}\bar{\mathbf{M}}_{\mathbf{x},t}\right)^{-1}\bar{\mathbf{M}}^{*}_{\mathbf{x},t}$, where "$*$" denotes the conjugate transpose matrix operation. Substituting the latter into Eq. (3.14), the arbitrary part becomes zero and Eq. (3.14) is recast into the state space form

$$\dot{\mathbf{p}} = \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{p} + \mathbf{q}_{\mathbf{x}}, \tag{3.15}$$

where

$$\bar{\mathbf{G}}_{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{I}_{l\times l} \\ -\bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{K}}_{\mathbf{x},t} & -\bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{C}}_{\mathbf{x},t} \end{bmatrix} \tag{3.16}$$

is a $2l \times 2l$ matrix with time-dependent elements and

$$\mathbf{p} = \begin{bmatrix} \mathbf{x}_s \\ \dot{\mathbf{x}}_s \end{bmatrix}, \; \mathbf{q}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{Q}}_{\mathbf{x}}(t) \end{bmatrix} \tag{3.17}$$

are $2l$-dimensional vectors. The interested reader is also referred to Pirrotta et al., 2021, where a generalized state variable formulation for MDOF systems with fractional derivative terms and singular parameter matrices is introduced.

Next, for an initially at rest system it is assumed that the time-dependent system response vector $\mathbf{p}$ in Eq. (3.17) is a zero-mean stochastic process. Then, defining the $2l \times 2l$ matrix of the system response variance $\mathbf{V} = \mathrm{E}[\mathbf{p}\mathbf{p}^{\mathrm{T}}]$ and resorting to the standard theory of linear systems (e.g., Chen, 1998), the general solution of the state space equation Eq. (3.15) is derived. It takes the form

$$\dot{\mathbf{V}}(t) = \mathbf{V}\bar{\mathbf{G}}^{\mathrm{T}}_{\mathbf{x}}(t) + \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{V} + \int_0^t \exp\left(\bar{\mathbf{G}}_{\mathbf{x}}(t-\tau)\right)\left(\mathbf{w}(t,\tau) + \mathbf{w}^{\mathrm{T}}(t,\tau)\right)\mathrm{d}\tau, \tag{3.18}$$

where

$$\mathbf{w}(t,\tau) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{0}_{l\times l} \\ \mathbf{0}_{l\times l} & \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau)(\bar{\mathbf{M}}^{+}_{\mathbf{x},t})^{\mathrm{T}} \end{bmatrix} \tag{3.19}$$

is the $2l \times 2l$ covariance matrix of the system excitation with $\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \mathrm{E}[\bar{\mathbf{Q}}_{\mathbf{x}}\bar{\mathbf{Q}}_{\mathbf{x}}^{\mathrm{T}}]$ denoting the $(l+m)\times(l+m)$ covariance matrix of $\bar{\mathbf{Q}}_{\mathbf{x}}$.

Solution of the proposed matrix differential equation subject to modulated white noise

In this section, the zero-mean non-stationary excitation in Eq. (3.1) is modeled as the product of a stationary excitation with a modulated time-function. That is

$$\mathbf{Q}_{\mathbf{x}}(t) = \mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}(t), \tag{3.20}$$

where $\mathbf{a}(t)$ is a deterministic $l\times n$ matrix of modulating functions and $\mathbf{Q}_{\mathbf{x},s}(t)$ is an n-dimensional stationary stochastic process. Therefore, the $(l+m)\times(l+m)$ covariance matrix of the excitation in Eq. (3.19) takes the form

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \begin{bmatrix} \mathbf{J}\mathbf{a}(t)\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau)\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{J}\mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\mathbf{Q}_{\mathbf{x},s}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{F}\mathbf{F}^{\mathrm{T}} \end{bmatrix}, \tag{3.21}$$

where $\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau) = \mathrm{E}[\mathbf{Q}_{\mathbf{x},s}\mathbf{Q}_{\mathbf{x},s}^{\mathrm{T}}]$. Eq. (3.21) is further simplified if the stationary excitation $\mathbf{Q}_{\mathbf{x},s}(t)$ in Eq. (3.20) is modeled as a Gaussian white noise process with $\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau) = \delta(t-\tau)\mathbf{S}$, where $\mathbf{S}$ is a real, symmetric and non-negative $n\times n$ matrix of constants, and $\delta(\cdot)$ denotes the Dirac delta function. Thus, taking into account Eq. (3.21), the matrix differential equation in Eq. (3.18) becomes

$$\dot{\mathbf{V}}(t) = \mathbf{V}\bar{\mathbf{G}}_{\mathbf{x}}^{\mathrm{T}}(t) + \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{V} + \boldsymbol{\Theta}(t), \tag{3.22}$$

where

$$\boldsymbol{\Theta}(t) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{0}_{l\times l} \\ \mathbf{0}_{l\times l} & \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau)\left(\bar{\mathbf{M}}^{+}_{\mathbf{x},t}\right)^{\mathrm{T}} \end{bmatrix} \tag{3.23}$$

is a $2l \times 2l$ matrix with

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \begin{bmatrix} \mathbf{J}\mathbf{a}(t)\mathbf{S}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{J}\mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\mathbf{Q}^{\mathrm{T}}_{\mathbf{x},s}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{F}\mathbf{F}^{\mathrm{T}} \end{bmatrix}. \tag{3.24}$$

Clearly, in the special case where the system excitation is a stationary process, Eq. (3.22) degenerates to the standard Lyapunov matrix differential equation governing the covariance matrix of the system response (e.g., Roberts and Spanos, 2003; Fragkoulis et al., 2016a).

The matrix differential equation Eq. (3.22) in conjunction with the generalized equivalent linear elements derived by Eq. (3.13) constitute a coupled set of equations to be solved for determining the response of the subsystem subject to the non-stationary excitation. The deterministic component of the response is derived by considering Eq. (3.9), i.e., the subsystem subject to the deterministic excitation. Overall, the differential equations corresponding to the deterministic and stochastic response components are solved simultaneously by resorting to any standard numerical algorithm, such as the Runge-Kutta method.

Solution of the proposed matrix differential equation subject to modulated colored noise

In this section, the non-stationary non-white system excitation is modeled by considering additional auxiliary linear filter equations. In general, linear and nonlinear filters are widely used to model non-white excitation processes in engineering dynamics in various cases, such as the Kanai-Tajimi excitation, or even to provide sufficiently accurate approximations in cases where the excitation power spectrum cannot be represented in the time domain as the response of a filter (e.g., Spanos, 1986; Chai et al., 2015; Psaros et al., 2018).

In this regard, each one of the nonzero elements of the stationary excitation vector $\mathbf{Q}_{\mathbf{x},s}(t)$ in Eq. (3.20) are considered as the output of a linear r-order filter equation whose input is a

Gaussian white noise process. Specifically, the filter equations are

$$v_{r-1}u^{(r-1)} + v_{r-2}u^{(r-2)} + \cdots + v_0 u^{(0)} = Q_s(t) \tag{3.25}$$

and

$$u^{(r)} + \lambda_{r-1}u^{(r-1)} + \cdots + \lambda_0 u^{(0)} = w(t), \tag{3.26}$$

where λ_i and v_i ($i = 0, 1, \dots, r-1$) denote the filter coefficients, $w(t)$ is a white noise process with constant power spectrum density S_0, and the superscript "(j)" denotes the j-th order derivative ($j = 0, 1, \dots, r$).

Next, assuming that $\mathbf{v} = \begin{bmatrix} v_0 & v_1 & \cdots & v_{r-1} \end{bmatrix}^{\mathrm{T}}$ is the vector of the filter constants and that $\mathbf{u} = \begin{bmatrix} u^{(0)} & u^{(1)} & \cdots & u^{(r-1)} \end{bmatrix}^{\mathrm{T}}$ represents the pre-filter output, combining Eqs. (3.7), (3.20) and (3.25) yields

$$\bar{\mathbf{D}}_{\mathbf{x}}\mathbf{u} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{3.27}$$

where

$$\bar{\mathbf{D}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times r} \\ \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{P}}a(t)\mathbf{v}^{\mathrm{T}} \end{bmatrix} \tag{3.28}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{P}}a(t)Q_s(t) \end{bmatrix}. \tag{3.29}$$

The $(l+m)$-dimensional vector $\bar{\mathbf{P}}$ in Eqs. (3.28-3.29) corresponds to the nonzero elements of the excitation $\bar{\mathbf{Q}}_{\mathbf{x}}$ in Eq. (3.7). Therefore, Eq. (3.7) is equivalently written as

$$\bar{\mathbf{Q}}_{\mathbf{x}}(t) = a(t)Q_s(t)\bar{\mathbf{P}} \tag{3.30}$$

with $a(t)$ denoting a time-modulating function and

$$\bar{\mathbf{P}} = \begin{bmatrix} \mathbf{J}\mathbf{I}_{\bar{\mathbf{P}}} \\ \left(a(t)Q_s(t)\right)^{-1}\mathbf{F} \end{bmatrix}. \tag{3.31}$$

For instance, assuming for simplicity that $\mathbf{Q_x}(t)$ in Eq. (3.20) contains only a single zero-mean process in its first entry yields $\mathbf{I}_{\bar{\mathbf{P}}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$, and $\mathbf{J}$, $\mathbf{F}$ correspond to the $l \times l$ matrix and m-dimensional vector of Eq. (3.7), respectively. Further, Eq. (3.26) is written in the standard state variable form

$$\dot{\mathbf{u}} = \mathbf{\Lambda}\mathbf{u} + \mathbf{w}_s, \tag{3.32}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ -\lambda_0 & -\lambda_1 & \cdots & -\lambda_{r-1} \end{bmatrix} \tag{3.33}$$

denotes an $r \times r$ matrix and $\mathbf{w}_s = \begin{bmatrix} 0 & 0 & \cdots & w(t) \end{bmatrix}^{\mathrm{T}}$ is an r-dimensional vector. Overall, the governing equations of the system under consideration are derived by combining the equations of the original system defined in Eq. (3.15) and the filter equations Eqs. (3.25-3.26). Specifically, considering the new variable $\mathbf{z} = \begin{bmatrix} \mathbf{p}^{\mathrm{T}} & \mathbf{u}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, the augmented state space system is written as

$$\dot{\mathbf{z}} = \bar{\mathbf{N}}\mathbf{z} + \mathbf{W}, \tag{3.34}$$

where

$$\bar{\mathbf{N}} = \begin{bmatrix} \bar{\mathbf{G}}_{\mathbf{x}} & \bar{\mathbf{D}}_{\mathbf{x}} \\ \mathbf{0}_{r\times 2l} & \mathbf{\Lambda} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{0}_{2l\times 1} \\ \mathbf{w}_s \end{bmatrix}. \tag{3.35}$$

Finally, denoting by $\mathbf{V} = \mathrm{E}[\mathbf{z}\mathbf{z}^{\mathrm{T}}]$ the response covariance matrix, the matrix differential equation corresponding to Eq. (3.22) for the case where the system excitation is modeled as modulated colored noise takes the form

$$\dot{\mathbf{V}} = \mathbf{V}\bar{\mathbf{N}}^{\mathrm{T}}(t) + \bar{\mathbf{N}}(t)\mathbf{V} + \mathbf{W}_s, \tag{3.36}$$

where $\mathbf{W}_s = \mathrm{diag}(0, 0, \ldots, 2\pi S_0)$ is a $(2l + r) \times (2l + r)$ diagonal matrix.

The matrix differential equation Eq. (3.36) is considered in conjunction with Eq. (3.13) to determine the stochastic response component. Moreover, similar to the formulation in section 3.2.2, the deterministic component of the response is computed by considering the subsystem subject to the deterministic excitation in Eq. (3.9). Finally, the Runge-Kutta method is used to solve simultaneously the set of differential equations governing the stochastic and the deterministic response.

Mechanization of the proposed technique

The mechanization of the proposed technique is concisely described by the following steps:

1. Consider Eq. (3.8) to decompose the system response into deterministic and stochastic parts. Then, form the subsystems of deterministic and stochastic differential equations defined by Eqs. (3.9) and (3.10), respectively.

2. Apply the generalized statistical linearization methodology in section 3.2.2 to derive the equivalent linear system in Eq. (3.12) corresponding to the nonlinear stochastic differential equation Eq. (3.10). This is done by utilizing Eq. (3.13) for determining the time-varying equivalent linear elements.

3. Apply the state variable analysis for systems with singular parameter matrices in section 3.2.2. First, construct matrix $\bar{\mathbf{G}}_{\mathbf{x}}$ in Eq. (3.16). Then,

   Case 1: Nonlinear system subject to modulated white noise.

   determine matrix $\mathbf{\Theta}$ in Eq. (3.23), and thus, formulate the matrix differential equation Eq. (3.22).

   Case 2: Nonlinear system subject to modulated colored noise.

determine matrices $\bar{\mathbf{D}}_{\mathbf{x}}$ and $\mathbf{\Lambda}$ in Eqs. (3.28) and (3.33), respectively, and thus, construct matrix $\bar{\mathbf{N}}$ in Eq. (3.35). Next, form the matrix differential equation Eq. (3.36).

4. Finally, solve simultaneously the matrix differential equation derived in step 3, i.e., Eq. (3.22) for the white noise excitation, or Eq. (3.36) for the colored noise excitation, in conjunction with the deterministic differential equation Eq. (3.9) derived in step 1. This can be done by resorting to any standard numerical algorithm, such as the Runge-Kutta method.

3.3 Numerical examples

In this section, three numerical examples are used to demonstrate the validity of the proposed technique and assess its reliability. The first one pertains to a nonlinear piezoelectric energy harvesting device subject to combined deterministic and modulated white noise excitation. The technique is applied to determine the response displacement and induced voltage of the device, while a comparison with pertinent MCS data (500 realizations) is used to demonstrate the accuracy of the obtained results. The second example refers to a 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated white noise excitation, whereas in the third example the same system is considered subject to combined deterministic and modulated colored noise excitation. In both cases the results obtained by the proposed technique are compared with corresponding results obtained by the standard approach in Kong et al., 2022a.

3.3.1 Nonlinear energy harvesting device subject to combined deterministic and modulated white noise excitation

In this example, the proposed technique is used for determining the response of a typical nonlinear piezoelectric energy harvesting device. Such devices consist of a mechanical part, which is usually a cantilever beam moving as a result of applied excitation and a corresponding piezoelectric part, which is used to transform the mechanical energy into electric current or voltage. They often operate in tandem with large scale infrastructure such as bridges and high-rise buildings (e.g., Roccia et al., 2020), which, in turn, are potentially subject to combined deterministic and non-stationary stochastic excitation (e.g., Quaranta et al., 2018).

The coupled electro-mechanical equations governing the dynamics of the system subject to combined deterministic and non-stationary excitation are given by

$$\ddot{q} + 2\zeta\dot{q} + \frac{dU(q)}{dq} + \kappa^2 y = f_d(t) + Q(t) \tag{3.37}$$

$$\dot{y} + \alpha y - \dot{q} = 0 \tag{3.38}$$

where q, $\dot{q}$ and $\ddot{q}$ denote the response displacement, velocity and acceleration of the mechanical part, and y is the induced voltage of a capacitive harvester (e.g., Daqaq et al., 2014; Petromichelakis et al., 2018; Karageorgos et al., 2021). ζ denotes the damping coefficient of the mechanical part, κ is a coupling coefficient, α is a constant and $U(q)$ represents the potential function. The nonlinear function of the system is given by

$$\frac{dU(q)}{dq} = q + \lambda q^2 + \delta q^3, \tag{3.39}$$

where λ and δ are coefficients classifying a typical harvesting device into distinctive classes; the interested reader is directed to He and Daqaq, 2016; Petromichelakis et al., 2018 for a detailed discussion. Further, assume that the deterministic component of the excitation is given by $f_d = f_{d,1}\sin(\omega_d t)$. The non-stationary stochastic excitation component is modeled as a modulated white noise stochastic process $Q(t) = a(t)Q_s$, where $a(t) = A\exp(-\mu t)$ with $A, \mu > 0$ is a time-modulating function and $Q_s(t)$ is a Gaussian white noise process with $\mathrm{E}\left[Q_s(t)Q_s(t+\tau)\right] = 2\pi S_0\delta(\tau)$.

Next, the proposed technique is used to treat the system of Eqs. (3.37-3.39). In this regard, considering the coordinates vector $\mathbf{x}(t) = \begin{bmatrix} q(t) & y(t) \end{bmatrix}^{\mathrm{T}}$, Eqs. (3.37-3.39) are written in the form of Eq. (3.1), where

$$\mathbf{M_x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; \mathbf{C_x} = \begin{bmatrix} 2\zeta & 0 \\ -1 & 1 \end{bmatrix}, \; \mathbf{K_x} = \begin{bmatrix} 1 & \kappa^2 \\ 0 & \alpha \end{bmatrix}, \tag{3.40}$$

$$\mathbf{\Phi_x} = \begin{bmatrix} \lambda q^2 + \delta q^3 \\ 0 \end{bmatrix} \tag{3.41}$$

and

$$\mathbf{f}_{d,\mathbf{x}} = \begin{bmatrix} f_d(t) \\ 0 \end{bmatrix}, \; \mathbf{Q_x} = \begin{bmatrix} Q(t) \\ 0 \end{bmatrix}. \tag{3.42}$$

Clearly, the matrix $\mathbf{M_x}$ in Eq. (3.40) is singular, and thus, a direct treatment of the system of Eqs. (3.37-3.39) is not possible. However, in the ensuing analysis a solution is derived in a direct manner by resorting to the generalized matrix inverse theory. Specifically, considering that Eq. (3.38) constitutes the constraints equation of the harvesting device (e.g., Petromichelakis et al., 2018; Karageorgos et al., 2021; Pasparakis et al., 2022) and differentiating it once with respect to time, Eq. (3.2) is formulated, where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \; \mathbf{E} = \begin{bmatrix} 0 & \alpha \end{bmatrix}, \; \mathbf{L} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{3.43}$$

and

$$F = 0. \tag{3.44}$$

Further, the $l \times l$ matrix $\mathbf{J}$ in Eqs. (3.4) and (3.5) interconnecting the constraints to the system governing equations takes the form

$$\mathbf{J} = \mathbf{I}_l - \mathbf{A}^+\mathbf{A}, \tag{3.45}$$

where $\mathbf{I}_l$ denotes the $l \times l$ identity matrix. The interested reader is directed to indicative Refs. Fragkoulis et al., 2016a; Pirrotta et al., 2021; Karageorgos et al., 2021 for more details. Therefore, the system of Eqs. (3.37-3.39) is equivalently written in the form of Eq. (3.3), where

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ -1 & 1 \end{bmatrix}, \; \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} -0.5\alpha & 0.5 \\ -0.5\alpha & 0.5 \\ 0 & \alpha \end{bmatrix}, \; \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5\kappa^2+\alpha \\ 0.5 & 0.5\kappa^2+\alpha \\ 0 & 0 \end{bmatrix}, \tag{3.46}$$

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}) = \left(\lambda q^2 + \delta q^3\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{3.47}$$

and

$$\bar{\mathbf{f}}_{d,\mathbf{x}} = f_{d,1}\sin\left(\omega_d t\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \; \bar{\mathbf{Q}}_{\mathbf{x}} = Q(t) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{3.48}$$

Next, considering that the system response consists of a stochastic and a deterministic component, namely $\mathbf{x}_s = \begin{bmatrix} q_s & y_s \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{x}_d = \begin{bmatrix} q_d & y_d \end{bmatrix}^{\mathrm{T}}$, ensemble averaging the nonlinear vector in Eq. (3.47) yields

$$\mathrm{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}] = \left(\lambda\sigma_{q_s}^2 + \lambda q_d^2 + 3\delta\sigma_{q_s}^2 q_d + \delta q_d^3\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{3.49}$$

Then, applying the generalized statistical linearization method with $\hat{\mathbf{x}} = \mathbf{x}_s$ the equivalent linear matrix $\bar{\mathbf{K}}_e$ is determined by Eq. (3.13) in the form

$$\bar{\mathbf{K}}_e = \left(\lambda q_d + 1.5\delta\left(\sigma_{q_s}^2 + q_{d_2}^2\right)\right) \begin{bmatrix} R(1,1) & R(2,1) \\ R(1,1) & R(2,1) \\ 0 & 0 \end{bmatrix}, \tag{3.50}$$

where $R(i,j)$, $i,j = 1,2$, denotes the (i,j) element of the matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$. For the numerical evaluation, the following set of parameter values are used for the system $\zeta = 0.1$,

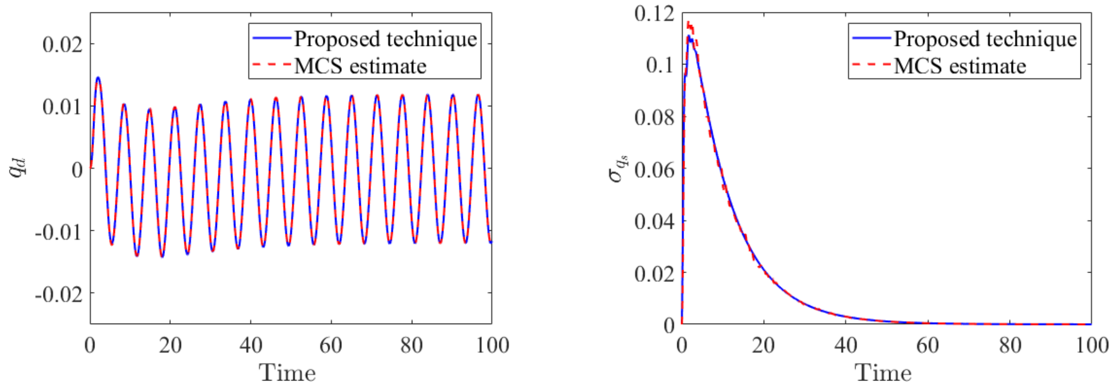
$\kappa = 3.25$, $\alpha = 0.8$, $\delta = 0.2$, $\lambda = 2\sqrt{\delta} \approx 0.89$, and $f_{d_1} = 0.1$, $\omega_d = 1$, $A = 1$, $\mu = 0.1$ and $S_0 = 0.2/\pi$ for the excitation. In this regard, the matrix differential equation Eq. (3.22) is formed, where

$$\mathbf{w}_{\bar{\mathbf{Q}}_\mathbf{x}} = \exp\left(-0.2t\right) \begin{bmatrix} 0.0159 & 0.0159 & 0 \\ 0.0159 & 0.0159 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.51}$$

and

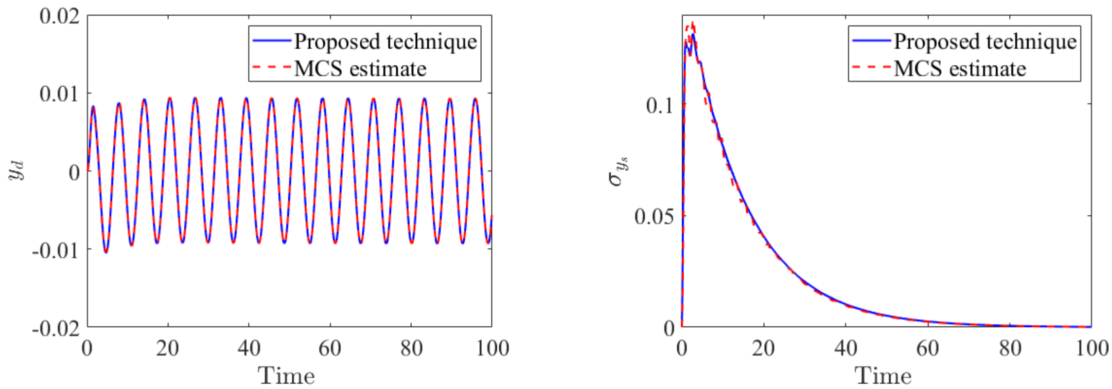
$$\boldsymbol{\Theta}(t) = \exp\left(-0.2t\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0637 & 0.0637 \\ 0 & 0 & 0.0637 & 0.0637 \end{bmatrix}. \tag{3.52}$$

Finally, the deterministic response component and the standard deviation of the stochastic response component for both the mechanical and the piezoelectric parts of the device are determined by considering the coupled set of Eqs. (3.9) and (3.22). Specifically, 10 differential equations governing the stochastic response of the system are derived by Eq. (3.22), whereas 4 additional differential equations governing the deterministic response are derived by Eq. (3.9). These are solved simultaneously by the Runge-Kutta method. The solid lines in Figs. 3.1(a) and 3.1(b) show the obtained results corresponding to the mechanical part of the device, namely the deterministic response displacement and the standard deviation of the stochastic response displacement, respectively. Further, the solid lines in Figs. 3.2(a) and 3.2(b) correspond to the piezoelectric part of the device. Fig. 3.2(a) shows the deterministic component of the induced voltage y, whereas Fig. 3.2(b) shows the standard deviation of the stochastic component of y. The obtained results are compared and found in good agreement with MCS data (500 realizations) generated by the spectral representation method Liang et al., 2007, with a signal duration $T_0 = 100$ s and a cut-off frequency equal to 2π rad/s.



(a) (b)

Figure 3.1: Response of the mechanical part of the nonlinear energy harvesting device described by Eqs. (3.37-3.39) subject to combined deterministic and modulated white noise excitation: (a) deterministic response displacement; (b) standard deviation of the stochastic response displacement. MCS data (500 realizations) are included for comparison.

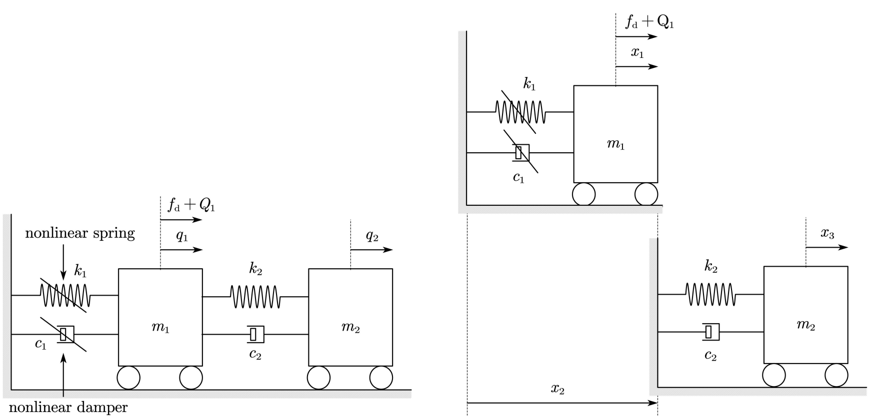


(a) (b)

Figure 3.2: Response of the piezoelectric part of the nonlinear energy harvesting device described by Eqs. (3.37-3.39) subject to combined deterministic and modulated white noise excitation: (a) deterministic component of the induced voltage; (b) standard deviation of the stochastic component of the induced voltage. MCS data (500 realizations) are included for comparison.

3.3.2 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated white noise excitation

The 2-DOF nonlinear structural system in Fig. 3.3(a) is considered, where mass m_1 is connected to the foundation with a nonlinear spring with stiffness coefficient k_1 and a nonlinear damper with damping coefficient c_1. The corresponding forces are $k_1 q_1(1 + \varepsilon_1 q_1^2)$ and $c_1 \dot{q}_1(1 + \varepsilon_2 \dot{q}_1^2)$, respectively, where ε_1 and ε_2 are positive constants, and q_1 denotes the response displacement of mass m_1. Further, mass m_2 is connected to mass m_1 via a linear spring and a linear damper with stiffness and damping coefficients k_2 and c_2, respectively. q_2 denotes the response displacement of mass m_2. The system is subject to a combined deterministic and non-stationary stochastic excitation, which is applied on mass m_1. The deterministic excitation component is $f_d = f_{d,1} \sin(\omega_d t)$. The stochastic excitation is modeled as a modulated white noise $Q_1(t) = a(t) Q_s(t)$, where $a(t) = A \exp(-\mu t)$ is a time-modulating function with $t \geq 0$ and $A, \mu > 0$, and $Q_s(t)$ is a Gaussian white noise process with $\mathrm{E}\left[Q_s(t) Q_s(t+\tau)\right] = 2\pi S_0 \delta(\tau)$.



(a) (b)

Figure 3.3: (a) A 2-DOF nonlinear structural system subject to combined deterministic and non-stationary stochastic excitation. (b) The nonlinear system of Fig. 3.3(a) modeled by employing an additional redundant coordinate.

The system governing equations of motion are derived by considering the (generalized) coordinates vector $\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\mathrm{T}}$. The mass, damping and stiffness matrices of the system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} c_1 & -c_2 \\ 0 & c_2 \end{bmatrix}, \ \mathbf{K} = \begin{bmatrix} k_1 & -k_2 \\ 0 & k_2 \end{bmatrix}. \tag{3.53}$$

Further, the system nonlinearity is written as

$$\boldsymbol{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \begin{bmatrix} \varepsilon_1 k_1 q_1^3 + \varepsilon_2 c_1 \dot{q}_1^3 \\ 0 \end{bmatrix}, \tag{3.54}$$

whereas the deterministic and non-stationary excitation vectors are

$$\mathbf{f}_d = \begin{bmatrix} f_{d,1} \sin(\omega_d t) \\ 0 \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}. \tag{3.55}$$

For the numerical evaluation, the system parameters take the values $m_1 = m_2 = 1$, $c_1 = c_2 = 0.2$, $k_1 = k_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 0.1$, $S_0 = \frac{0.2}{\pi}$, $A = 1$, $\mu = 0.1$ and the excitation parameter values are $f_{d,1} = 1$, $\omega_d = 1$. The deterministic response component and the standard deviation of the stochastic response component of the nonlinear system are derived by applying the standard technique proposed in Kong et al., 2022a. The obtained results for the response displacement and the response velocity for each of the system DOFs are shown by dashed line in Figs. 3.4 and 3.5, respectively.

Next, the system governing equations of motion are derived by adopting a redundant coordinates modeling. The nonlinear system in Fig. 3.3(a) is decomposed into its constituent parts as seen in Fig. 3.3(b), and considering the coordinates vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$, the equation of motion Eq. (3.1) is formed. Further, differentiating twice with respect to time the constraints equation connecting the two subsystems in Figs. 3.3(a) and 3.3(b), i.e., $x_2 = x_1 + d$, where d denotes the length of mass m_1, Eq. (3.2) is formed, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \tag{3.56}$$

$\mathbf{E} = \mathbf{L} = \mathbf{0}_{1\times 3}$ and $F = 0$. In addition, matrix $\mathbf{J}$ in Eq. (3.45) becomes

$$\mathbf{J} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.57}$$

In this regard, the parameter matrices in Eq. (3.4) are given by

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.58}$$

whereas the nonlinearity of Eq. (3.5) becomes

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \left(k_1 \varepsilon_1 x_1^3 + c_1 \varepsilon_2 \dot{x}_1^3\right) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{3.59}$$

Lastly, the deterministic and non-stationary stochastic excitation components are given by

$$\bar{\mathbf{f}}_{d,\mathbf{x}} = f_{d,1} \sin(\omega_d t) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{3.60}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}} = Q_1(t) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{3.61}$$

respectively.

Then, for the application of the proposed technique the system response is decomposed into a deterministic component $\mathbf{x}_d = \begin{bmatrix} x_{d,1} & x_{d,2} & x_{d,3} \end{bmatrix}^{\mathrm{T}}$ and a corresponding stochastic component

$\mathbf{x}_s = \begin{bmatrix} x_{s,1} & x_{s,2} & x_{s,3} \end{bmatrix}^{\mathrm{T}}$. Next, ensemble averaging the nonlinear function in Eq. (3.59), i.e.,

$$\mathrm{E}\left[\bar{\mathbf{\Phi}}_{\mathbf{x}}\right] = \begin{bmatrix} 0.5k_1\varepsilon_1\left(x_{d,1}^3 + 3x_{d,1}\sigma_{x_{s,1}}^2\right) + 0.5c_1\varepsilon_2\left(\dot{x}_{d,1}^3 + 3\dot{x}_{d,1}\sigma_{\dot{x}_{s,1}}^2\right) \\ 0.5k_1\varepsilon_1\left(x_{d,1}^3 + 3x_{d,1}\sigma_{x_{s,1}}^2\right) + 0.5c_1\varepsilon_2\left(\dot{x}_{d,1}^3 + 3\dot{x}_{d,1}\sigma_{\dot{x}_{s,1}}^2\right) \\ 0 \\ 0 \end{bmatrix}, \tag{3.62}$$

Eq. (3.9) is formed for the subsystem subject to deterministic excitation, while the generalized statistical linearization method is applied to treat the subsystem subject to non-stationary excitation. Thus, considering the 6-dimensional vector $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_s & \dot{\mathbf{x}}_s \end{bmatrix}^{\mathrm{T}}$, the equivalent linear elements in Eq. (3.13) are given by

$$\bar{\mathbf{K}}_e = 1.5k_1\varepsilon_1\left(x_{d,1}^2 + \sigma_{x_{s,1}}^2\right) \begin{bmatrix} R(1,1) & R(2,1) & R(3,1) \\ R(1,1) & R(2,1) & R(3,1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.63}$$

and

$$\bar{\mathbf{C}}_e = 1.5c_1\varepsilon_2\left(\dot{x}_{d,1}^2 + \sigma_{\dot{x}_{s,1}}^2\right) \begin{bmatrix} R(4,4) & R(5,4) & R(6,4) \\ R(4,4) & R(5,4) & R(6,4) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.64}$$

where $R(i,j), i,j = 1,2,\ldots,6$ denotes the (i,j) element of matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ of Eq. (3.13). Then, following the presentation in section 3.2.2, the matrix differential equation Eq. (3.22) is formed, where

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}} = \frac{\exp(-0.2t)}{20\pi} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.65}$$

and

$$\mathbf{\Theta}(t) = \frac{\exp\left(-0.2t\right)}{5\pi} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}. \tag{3.66}$$

This leads to 21 differential equations pertaining to the determination of the stochastic response component, which are solved simultaneously with 6 additional differential equations derived by Eq. (3.9). The set of all differential equations is solved by the Runge-Kutta method. The obtained results for the deterministic and stochastic components of the response displacement and velocity for both DOFs of the system are shown by solid line in Figs. 3.4 and 3.5, respectively. Clearly, these are in total agreement with the corresponding results (dashed line) obtained by the standard method proposed in Kong et al., 2022a.

3.3.3 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated colored noise excitation

In this section, the system shown in Figs. 3.3(a) and 3.3(b) is subject to combined deterministic and non-stationary stochastic excitation, with the latter modeled as modulated colored noise. Similar to the case in section 3.3.2, the deterministic excitation component is given by $f_{d,1}\sin(\omega_d t)$. The stochastic component is modeled as $Q_1(t) = a(t)Q_s$, where $a(t) = A\exp(-\mu t)$ is a time-modulating function with $t \geq 0$ and $A, \mu > 0$, and Q_s is a non-white stochastic excitation process with a Kanai-Tajimi power spectrum

$$S(\omega) = \frac{1 + 4\xi_g \frac{\omega^2}{\omega_g^2}}{\left(1 - \frac{\omega^2}{\omega_g^2}\right)^2 + 4\xi_g \frac{\omega^2}{\omega_g^2}} S_0. \tag{3.67}$$

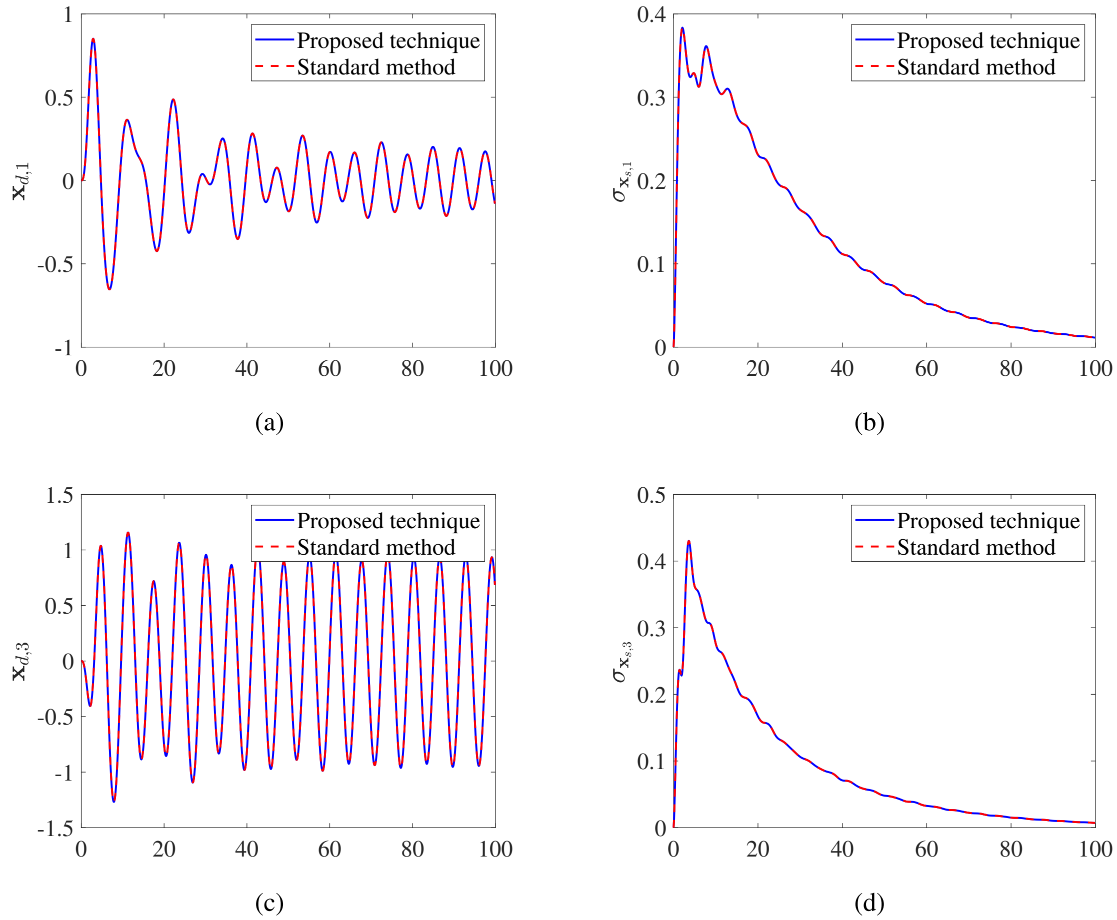


Figure 3.4: Response of the 2-DOF nonlinear structural system in Figs. 3.3(a) and 3.3(b) subject to combined deterministic and modulated white noise excitation: (a) deterministic response displacement of the 1st DOF; (b) standard deviation of the stochastic response displacement of the 1st DOF; (c) deterministic response displacement of the 3rd DOF; and (d) standard deviation of the stochastic response displacement of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in Kong et al., 2022a (dashed line).

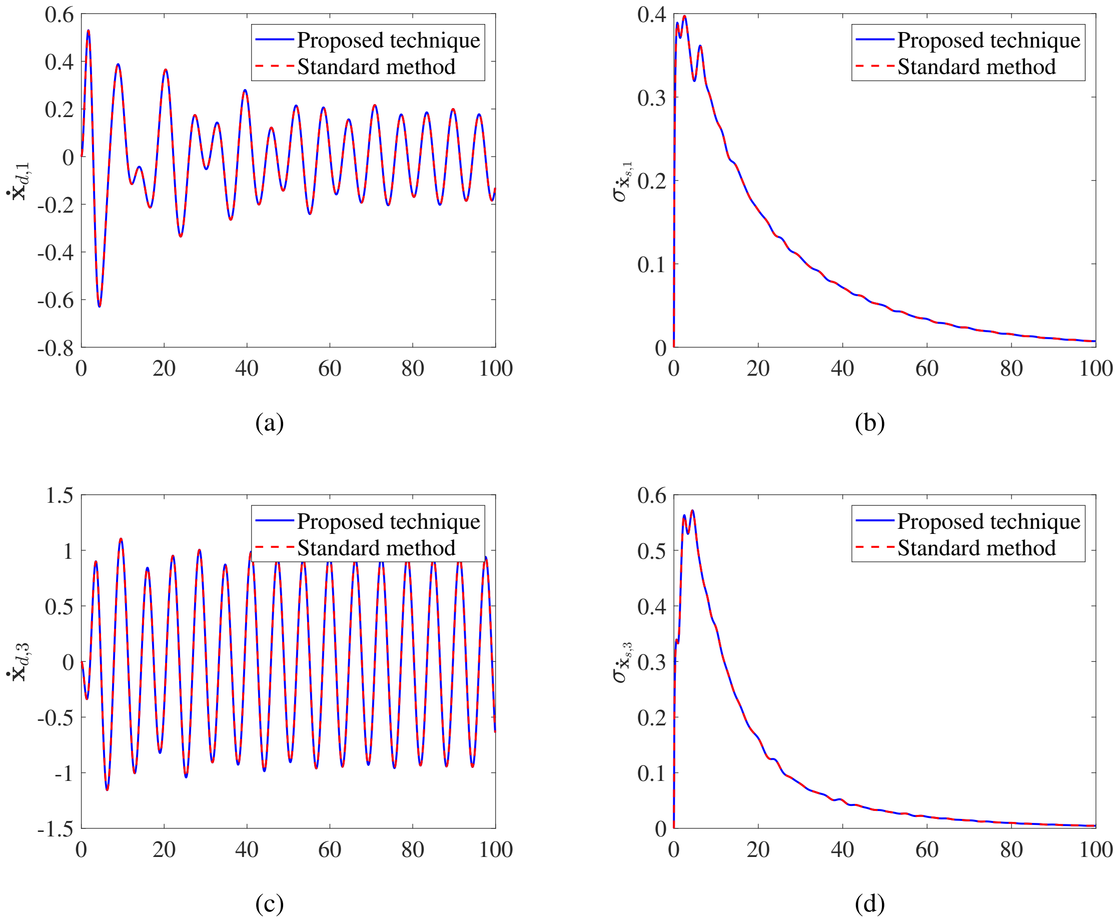


Figure 3.5: Response of the 2-DOF nonlinear structural system in Figs. 3.3(a) and 3.3(b) subject to combined deterministic and modulated white noise excitation: (a) deterministic response velocity of the 1st DOF; (b) standard deviation of the stochastic response velocity of the 1st DOF; (c) deterministic response velocity of the 3rd DOF; and (d) standard deviation of the stochastic response velocity of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in Kong et al., 2022a (dashed line).

The values of the parameters in the power spectrum of Eq. (3.67) are $\xi_g = 0.5$, $\omega_g = 1$ and $S_0 = 0.2\pi$.

Next, the technique proposed in section 3.2.2 is applied for determining the response of the system. In this regard, Eqs. (3.25) and (3.26) reduce to a second order linear filter with coefficients λ_0, λ_1, v_0 and v_1, given by

$$v_1 u^{(1)} + v_0 u^{(0)} = Q_s(t) \tag{3.68}$$

and

$$u^{(2)} + \lambda_1 u^{(1)} + \lambda_0 u^{(0)} = w(t), \tag{3.69}$$

where $v_0 = \omega_g^2, v_1 = 2\zeta_g\omega_g, \lambda_0 = \omega_g^2, \lambda_1 = 2\zeta_g\omega_g$, and $w(t)$ is a white noise process. Further, since the excitation is applied only on the first DOF of the system (see Fig. 3.3(b)), $\mathbf{I}_{\bar{\mathbf{P}}}$ in Eq. (3.31) is equal to $\mathbf{I}_{\bar{\mathbf{P}}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and using Eq. (3.45), Eq. (3.31) yields $\bar{\mathbf{P}} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. This leads to the computation of the 6×2 matrix $\bar{\mathbf{D}}_{\mathbf{x}}$ in Eq. (3.28). In addition, Eqs. (3.63-3.64) are used for computing the 6×6 matrix $\bar{\mathbf{G}}_{\mathbf{x}}(t)$ in Eq. (3.16), whereas $\boldsymbol{\Lambda}$ is readily found by Eq. (3.33).

Finally, taking into account Eq. (3.35), the matrix differential equation Eq. (3.36) is formed and solved simultaneously with the deterministic sub-equations derived by Eq. (3.9). Overall, Eq. (3.36) yields 36 differential equations governing the stochastic response component, whereas Eq. (3.9) yields 6 additional differential equations governing the deterministic component of the response. The set of differential equations is solved by the Runge–Kutta method. The results obtained for the deterministic and the stochastic components of the response displacement and velocity for both DOFs of the system are shown by solid line in Figs. 3.6 and 3.7, respectively. To demonstrate the validity of the proposed technique, corresponding results obtained by the standard method in Kong et al., 2022a are also included in Figs. 3.6 and 3.7 for comparison. The latter are depicted by dashed line and practically coincide with the results obtained by the proposed technique.

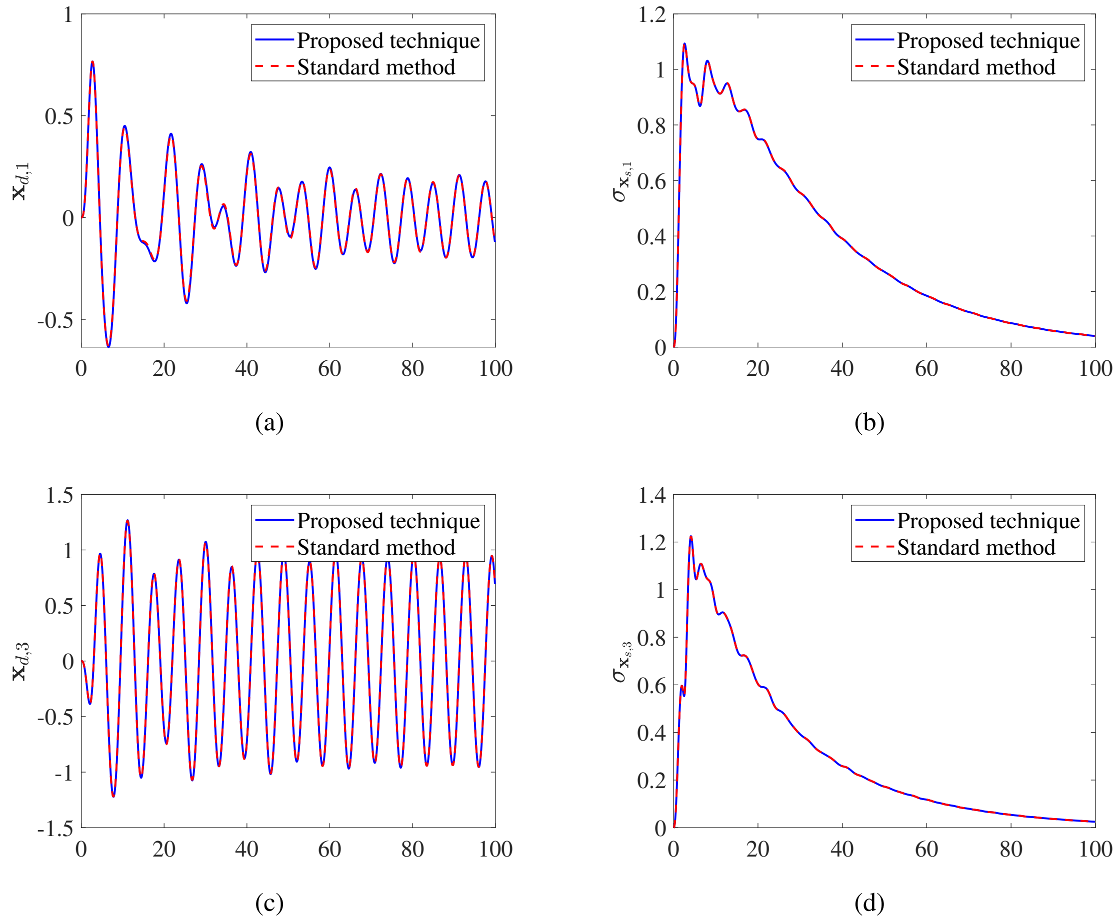


Figure 3.6: Response of the 2-DOF nonlinear structural system in Figs. 3.3(a) and 3.3(b) subject to combined deterministic and modulated colored noise excitation: (a) deterministic response displacement of the 1st DOF; (b) standard deviation of stochastic response displacement of the 1st DOF; (c) deterministic response displacement of the 3rd DOF; and (d) standard deviation of the stochastic response displacement of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in Kong et al., 2022a (dashed line).

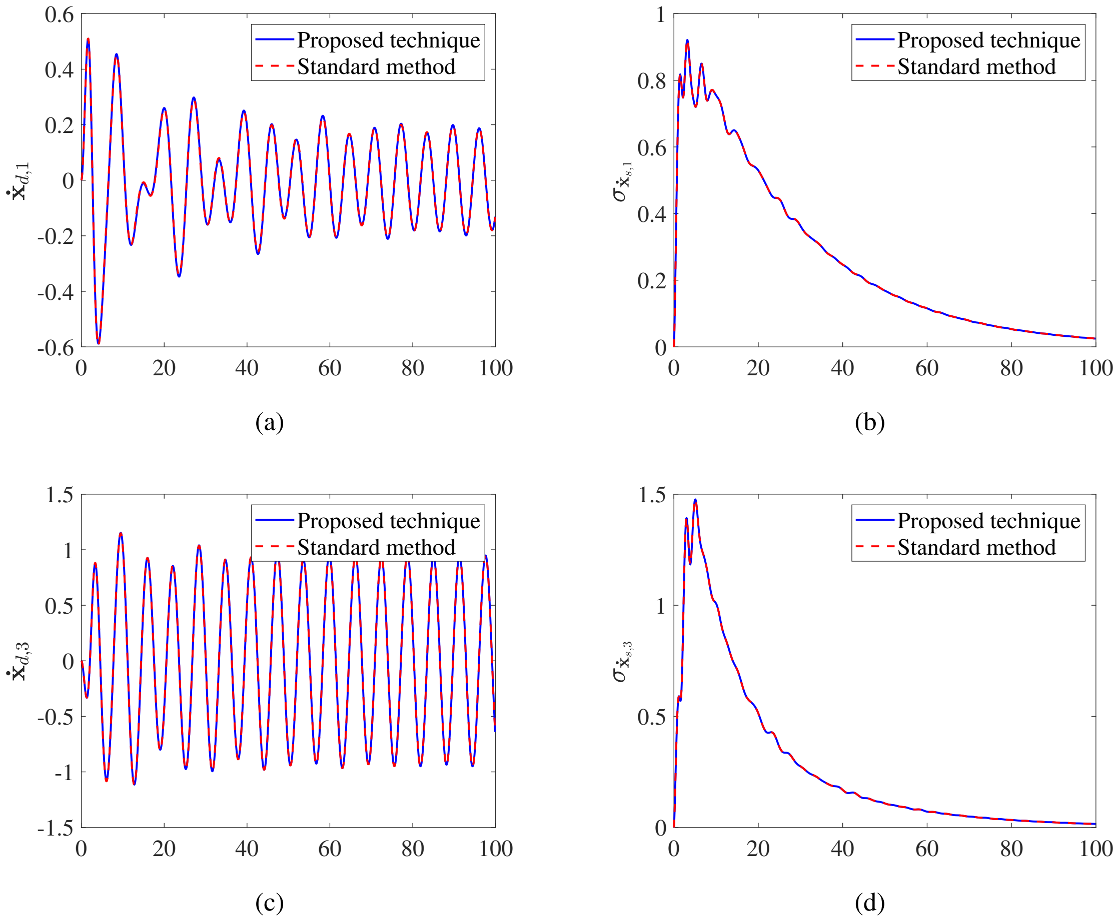


Figure 3.7: Response of the 2-DOF nonlinear structural system in Figs. 3.3(a) and 3.3(b) subject to combined deterministic and modulated colored noise excitation: (a) deterministic response velocity of the 1st DOF; (b) standard deviation of stochastic response velocity of the 1st DOF; (c) deterministic response velocity of the 3rd DOF; and (d) standard deviation of stochastic response velocity of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in Kong et al., 2022a (dashed line).

3.4 Concluding remarks

In this paper, a new technique has been proposed for determining the response of MDOF systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitations. The appearance of singular matrices in the equations of motion pertain to additional constraints equations in the system, or to a redundant coordinates modeling of its governing dynamics. Further, the stochastic excitation component is modeled as a non-stationary process driven by the need to develop response analysis frameworks accounting for the non-stationary characteristics of excitations such as wave, wind and earthquake loads.

In this regard, the MDOF nonlinear system has been decomposed into two subsystems based on the applied excitation, and a coupled set of equations has been derived and solved to determine the system response. First, a subsystem of deterministic equations governing the response of the system subject to deterministic excitation has been derived. Next, the generalized statistical linearization method has been utilized to treat the nonlinear subsystem subject to non-stationary stochastic excitation. This has been done in conjunction with a state space formulation, which resulted a matrix differential equation governing the stochastic response. The latter has been solved simultaneously with the deterministic equation above by applying a standard Runge-Kutta numerical scheme. In addition, a closed form expression for determining the time-dependent equivalent elements of the generalized statistical linearization methodology (Fragkoulis et al., 2016b) has been derived. Overall, the proposed technique can be construed as an extension of the approach in Ni et al., 2021 to systems subject to combined deterministic and non-stationary stochastic excitation. It has been assessed by considering three numerical examples including a vibration energy harvesting device subject to combined deterministic and modulated white noise excitation, and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations. The reliability of the obtained results has been demonstrated by comparisons to MCS data and corresponding results obtained by the approach proposed in Kong et al., 2022a.

CRediT authorship contribution statement

Peihua Ni: Methodology, Software, Writing - original draft, Visualization. **Vasileios C. Fragkoulis:** Conceptualization, Methodology, Writing - review & editing, Supervision, Project administration, Funding acquisition. **Fan Kong:** Conceptualization, Methodology, Writing - review & editing, Funding acquisition. **Ioannis P. Mitseas:** Conceptualization, Methodology, Writing - review & editing, Supervision, Funding acquisition. **Michael Beer:** Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Chapter 4

Research article 3: Response determination of a nonlinear energy harvesting device under combined stochastic and deterministic loads

Response determination of a nonlinear energy harvesting device under combined stochastic and deterministic loads

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Abstract: An approximate analytical technique for determining the response statistics of a nonlinear piezoelectric energy harvesting device is proposed. This is attained by resorting to a recently developed method for determining the response of multi-degree-of-freedom dynamical systems with singular matrices subject to combined deterministic and stochastic loads. Such systems are often met in engineering applications, for instance, as a result of modeling the governing equations of motion of complex multi-body systems by utilizing dependent coordinates. In this regard, the governing equations of the harvesting system dynamics are treated separately. Specifically, the harmonic balance method is used for treating the deterministic component of the response, while the corresponding stochastic response component is treated by combining the stochastic averaging and the statistical linearization methodologies. A numerical example is used to demonstrate the validity of the proposed technique. The obtained results are verified by using pertinent MCS data.

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4.1 Introduction

In general, formulating the system governing equations of motion of engineering systems relies on the use of the minimum number of (generalized) coordinates Roberts and Spanos, 2003. This, in turn, results system parameter matrices with some appealing properties, such as positive definiteness and symmetry. However, for several classes of complex engineering systems and/or systems subject to constraint equations, it is often more efficient to derive the governing equations based on a dependent coordinates modeling, i.e., by considering additional degrees-of-freedom (DOF) (e.g., Udwadia and Kalaba, 2001; Udwadia and Phohomsiri, 2006; Schutte and Udwadia, 2011). As a result the aforementioned appealing properties of the system parameter matrices do not apply anymore, since the latter are singular. Subsequently, this aspect necessitates the development of pertinent methodologies for conducting response analyses of such systems.

In this regard, considering the problem of multi-DOF linear and nonlinear systems with singular matrices, as well as with constraint equations, has led to the development of pertinent solution frameworks for determining the stochastic response of such system in time and frequency domains, as well as for conducting a joint time-frequency response analysis; see indicatively, Fragkoulis et al., 2016a,b; Kougioumtzoglou et al., 2017; Antoniou et al., 2017; Fragkoulis et al., 2015; Pantelous and Pirrotta, 2017; Pirrotta et al., 2019; Pasparakis et al., 2021; Pirrotta et al., 2021; Karageorgos et al., 2021; Fragkoulis et al., 2022. This has been attained by resorting to the theory of generalized matrix inverses (Ben-Israel and Greville, 2003), and particularly, by considering the concept of the Moore-Penrose inverse of a matrix.

In this paper, a recently proposed generalized matrix inverses-based framework for deriving the response of MDOF nonlinear systems with singular matrices subject to combined periodic and stochastic excitations (Ni et al., 2021) is used to compute in a direct way the stochastic response of a nonlinear piezoelectric energy harvesting device (**Petromichelakis2021**; Petromichelakis et al., 2018; Karageorgos et al., 2021). This is attained by considering the harmonic balance method for treating the periodic component of the response (Mickens, 2010; Spanos et al., 2019; Kong and Spanos, 2021; Kong et al., 2022) in conjunction with the statistical linearization methodology for systems with singular matrices for treating the corresponding stochastic response component (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017). The obtained results are compared with pertinent Monte Carlo simulation data.

4.2 Mathematical formulation

4.2.1 Governing equations of motion

The governing equations of motion of an l-DOF nonlinear system subjected to combined stochastic $\mathbf{Q}_{\mathbf{x}}(t)$ and deterministic $\mathbf{f}_{d,\mathbf{x}}(t)$ excitations have the form (Fragkoulis et al. 2016b; Spanos et al. 2019)

$$\mathbf{M}_{\mathbf{x}}\ddot{\mathbf{x}} + \mathbf{C}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{K}_{\mathbf{x}}\mathbf{x} + \mathbf{\Phi}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{f}_{d,\mathbf{x}}(t) + \mathbf{Q}_{\mathbf{x}}(t). \tag{4.1}$$

In Eq. (4.1), $\mathbf{x}$ is an l dependent coordinates vector, $\mathbf{M}_{\mathbf{x}}, \mathbf{C}_{\mathbf{x}}$ and $\mathbf{K}_{\mathbf{x}}$ denote the $l \times l$ system parameter matrices, whereas $\mathbf{\Phi}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ corresponds to the l vector of the system nonlinearity. Next, the system of Eq. (4.1) is subject to additional constraint equations, which are written for simplicity in the form (Schutte and Udwadia, 2011)

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{E}\dot{\mathbf{x}} + \mathbf{L}\mathbf{x} = \mathbf{F}, \tag{4.2}$$

where $\mathbf{A}, \mathbf{E}, \mathbf{L}$ are $m \times l$ matrices and $\mathbf{F}$ is an l vector. In this regard, Eq. (4.1) is equivalently written as (Kougioumtzoglou et al., 2017)

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} + \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{f}}_{d,\mathbf{x}}(t) + \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{4.3}$$

where

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{M}_{\mathbf{x}} \\ \mathbf{A} \end{bmatrix}, \tag{4.4}$$

$$\bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{C}_{\mathbf{x}} \\ \mathbf{E} \end{bmatrix} \tag{4.5}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{K}_{\mathbf{x}} \\ \mathbf{L} \end{bmatrix}, \tag{4.6}$$

are the $(l+m) \times l$ parameter matrices of the system, whereas

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{\Phi}_{\mathbf{x}} \\ \mathbf{0} \end{bmatrix} \tag{4.7}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}}(t) = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{Q}_{\mathbf{x}}(t) \\ \mathbf{F} \end{bmatrix}, \tag{4.8}$$

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{f}_{d,\mathbf{x}}(t) \\ \mathbf{0} \end{bmatrix}, \tag{4.9}$$

are, respectively, the $(l+m)$ vectors of the system nonlinearities, as well as the stochastic and deterministic excitations. Also, $\mathbf{I}_l$ denotes the $l \times l$ identity matrix and "+" is used for the Moore-Penrose (M-P) matrix inverse operation. A detailed derivation of Eqs. (4.3-4.9) is found in Kougioumtzoglou et al., 2017.

4.2.2 Determination of the system response

Considering that $\bar{\mathbf{Q}}_{\mathbf{x}}(t)$ and $\bar{\mathbf{f}}_{d,\mathbf{x}}(t)$ in Eq. (4.3) correspond to the stochastic and deterministic excitations of the system, where the former is modeled as a zero-mean Gaussian process and the latter is modeled as a monochromatic function of period $T = \frac{2\pi}{\omega_d}$; i.e.,

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \bar{\mathbf{f}}_{d_1,\mathbf{x}} \cos(\omega_d t) + \bar{\mathbf{f}}_{d_2,\mathbf{x}} \sin(\omega_d t), \tag{4.10}$$

where $\bar{\mathbf{f}}_{d_1,\mathbf{x}}$ and $\bar{\mathbf{f}}_{d_2,\mathbf{x}}$ are constants. It is assumed that the system response has also a stochastic and a periodic component. These are denoted by $\mathbf{x}_s(t)$ and $\mathbf{x}_d(t)$, respectively. Therefore, ensemble averaging Eq. (4.3), an expression consisting of a periodic and a stochastic component arises. This is given by

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_d + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_d + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_d + \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)] = \bar{\mathbf{f}}_{d,\mathbf{x}}(t), \tag{4.11}$$

which is used next for deriving the system response. To this end, a framework is proposed which is based on the combination of the harmonic balance method (for treating the deterministic component), and the statistical linearization methodology for systems with singular matrices (for treating the stochastic component).

Application of the harmonic balance and statistical linearization treatments

First, considering the system in Eq. (4.3), the harmonic balance method is applied for determining the periodic component of the response. It is assumed for simplicity that the nonlinear vector $\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)$ in Eq. (4.3) contains polynomial nonlinear functions. This assumption facilitates the derivation of closed form solutions for the system response (Spanos et al., 2019), as well as simplifies the application of the harmonic balance method (Mickens, 2010).

In this regard, the deterministic response becomes

$$\mathbf{x}_d(t) = \mathbf{x}_{d_1} \cos(\omega_d t) + \mathbf{x}_{d_2} \sin(\omega_d t), \tag{4.12}$$

where $\mathbf{x}_{d_1}, \mathbf{x}_{d_2}$ are constant l vectors. Next, applying the harmonic balance method yields

$$\mathbf{P}\mathbf{u} = \mathbf{v}. \tag{4.13}$$

In Eq. (4.13), $\mathbf{P}$ is a $2(l+m) \times 2l$ matrix whose elements are functions of ω_d and the augmented parameter matrices defined in Eqs. (4.4)-(4.6). Further, $\mathbf{v}$ is a $2(l+m)$ vector containing the deterministic excitation, as well as the ensemble average of the stochastic excitation, whereas the $2l$ vector

$$\mathbf{u} = \begin{bmatrix} \mathbf{x}_{d_1} \\ \mathbf{x}_{d_2} \end{bmatrix} \tag{4.14}$$

contains the deterministic response of the system.

Then, employing the M-P inverse of the matrix $\mathbf{P}$ (Ben-Israel and Greville, 2003), the solution to the overdetermined system of equations defined in Eq. (4.14) is given by

$$\mathbf{u} = \mathbf{P}^{+}\mathbf{v} + (\mathbf{I} - \mathbf{P}^{+}\mathbf{P})\mathbf{y}. \tag{4.15}$$

In Eq. (4.15), $\mathbf{y}$ is an arbitrary $2l$ vector, and thus, this expression corresponds to a family of possible solutions for the deterministic response component of the system. However, a unique solution is attained when $\mathbf{P}$ has full column rank. Specifically, it such case the M-P inverse matrix of $\mathbf{P}$ is given by $\mathbf{P}^{+} = (\mathbf{P}^{*}\mathbf{P})^{-1}\mathbf{P}^{*}$, and substituting the latter into Eq. (4.15), a simplified expression is derived.

Next, the stochastic response component is treated by resorting to the statistical linearization methodology for systems with singular matrices (Fragkoulis et al., 2016b; Kougioumtzoglou et al., 2017); see also Mitseas et al., 2016, 2018; Fragkoulis et al., 2019; Mitseas and Beer, 2019; Pasparakis et al., 2021; Mitseas and Beer, 2021; Ni et al., 2022 for indicative application frameworks of the method.

In this regard, considering Eqs. (4.3) and (4.11) leads to

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_s + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_s + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_s + \tilde{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s, \mathbf{x}_d) = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{4.16}$$

where

$$\tilde{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s, \mathbf{x}_d) = \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d) - \mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)] \tag{4.17}$$

is the zero-mean nonlinear vector of the system, to be replaced by equivalent linear elements. Specifically, applying the statistical linearization yields the equivalent linear system

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e)\ddot{\mathbf{x}}_s + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e)\dot{\mathbf{x}}_s + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e)\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{4.18}$$

where $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ denote the unknown equivalent linear $(l+m)\times l$ matrices of the system, which are used to account for neglecting from Eq. (4.16) the nonlinear vector. It is noted that closed form expressions for the equivalent linear matrices are found in Fragkoulis et al., 2016b and Kougioumtzoglou et al., 2017. Further, it is noted that since the nonlinear vector in Eq. (4.17) is written in terms of both the stochastic and deterministic response components, this will also hold for the equivalent linear elements. However, considering that the elements of the equivalent matrices are slowly varying over a period T of oscillation, they are approximated by their average over T Spanos et al., 2019. Therefore, Eq. (4.18) becomes

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e^a)\ddot{\mathbf{x}}_s + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e^a)\dot{\mathbf{x}}_s + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e^a)\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t). \tag{4.19}$$

Eq. (4.19) corresponds to the equivalent linear system, whose solution is derived by following either a time-domain treatment, where the system response is derived by solving a Lyapunov equation Fragkoulis et al., 2016a. Alternatively, applying a frequency-domain treatment, the system response is determined by Roberts and Spanos, 2003

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \int_{-\infty}^{\infty} \mathbf{S}_{\mathbf{x}}(\omega) d\omega, \tag{4.20}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator and $\mathbf{S}_{\mathbf{x}}(\omega)$ is the response power spectrum. The latter is determined by resorting to the input-output expression

$$\mathbf{S}_{\mathbf{x}}(\omega) = \boldsymbol{\alpha}_{\mathbf{x}}(\omega)\mathbf{S}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(\omega)\boldsymbol{\alpha}_{\mathbf{x}}^{\mathrm{T}*}(\omega), \tag{4.21}$$

where $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$ is the frequency response matrix and $\mathbf{S}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(\omega)$ the power spectrum of the excitation; see Kougioumtzoglou et al., 2017 for a detailed presentation.

4.3 Numerical examples

In this section, the M-P generalized inverse matrix-based framework is used to compute the response of a piezoelectric energy harvesting device. An indicative piezoelectric energy harvester, consists of a mechanical system, such as a cantilever beam moving as a result of applied excitation, and a corresponding piezoelectric system, which is used for transforming the mechanical energy into electric current. Such devices are used in several applications, mostly for powering adjoining low power level devices. Specifically, they often operate in tandem with large scale infrastructure, such as bridges and high-rise buildings (Roccia et al., 2020), which are potentially subjected to combined deterministic and stochastic excitations.

The equations governing the dynamics of the system are given by (Daqaq et al., 2014; Petromichelakis et al., 2018; Karageorgos et al., 2021)

$$\ddot{q} + 2\zeta\dot{q} + \frac{dU(q)}{dq} + \kappa^2 y = w(t) + f_d(t), \tag{4.22}$$

$$\dot{y} + \alpha y - \dot{q} = 0. \tag{4.23}$$

In the coupled system of Eqs. (4.22) and (4.23), q denotes the response displacement of the mechanical part and y is either the induced voltage or the induced current. Further, ζ is the damping coefficient of the mechanical system, κ denotes a coupling coefficient, α is a constant and $U(q)$ denotes the potential function (He and Daqaq, 2016). The system is subjected to the stochastic excitation $w(t)$, which is modeled as a Gaussian white noise stochastic process with constant spectral density S_0, and also to the deterministic component, which is given by $f_d = f_{d_1} \cos \omega_d t + f_{d_2} \sin \omega_d t$. It is assumed that the nonlinear function of the system has the form (Petromichelakis et al., 2018)

$$\frac{dU(q)}{dq} = q + \lambda q^2 + \delta q^3, \tag{4.24}$$

where λ and δ denote parameters which control the intensity of the nonlinearity. The following set of parameter values are used: $\alpha = 0.8$, $S_0 = 0.05$, $\delta = 0.1$, $\kappa = 3.25$, $\omega_d = \pi$, $f_{d_1} = 0$ and $f_{d_2} = 0.1$.

Setting

$$\mathbf{x}(t) = \begin{bmatrix} q(t) \\ y(t) \end{bmatrix} \tag{4.25}$$

and also considering Eq. (4.24), the system of Eqs. (4.22) and (4.23) is written in the form of Eq. (4.1), where the parameter matrices are given by

$$\mathbf{M_x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{C_x} = \begin{bmatrix} 2\zeta & 0 \\ -1 & 1 \end{bmatrix} \tag{4.26}$$

and

$$\mathbf{K_x} = \begin{bmatrix} 1 & \kappa^2 \\ 0 & \alpha \end{bmatrix}, \tag{4.27}$$

whereas the deterministic and stochastic excitation vectors become, respectively,

$$\mathbf{f}_{d,\mathbf{x}} = \begin{bmatrix} f_d(t) \\ 0 \end{bmatrix} \tag{4.28}$$

and

$$\mathbf{Q}_{\mathbf{x}} = \begin{bmatrix} w(t) \\ 0 \end{bmatrix}. \tag{4.29}$$

Clearly, the matrix $\mathbf{M}_{\mathbf{x}}$ in Eq. (4.26) is singular, which hinders the direct treatment of the system. However, considering that Eq. (4.23) denotes the constraint equation of the harvester (see also Petromichelakis et al., 2018) facilitates the ensuing analysis. Specifically, differentiating Eq. (4.23) once with respect to time, Eq. (4.2) is formulated, where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} 0 & \alpha \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{4.30}$$

and

$$F = 0. \tag{4.31}$$

In this regard, the system of Eqs. (4.22) and (4.23) is equivalently written in the form of Eq. (4.3), where

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ -1 & 0 \end{bmatrix}, \ \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} -0.5\alpha & 0.5 \\ -0.5\alpha & 0.5 \\ 0 & \alpha \end{bmatrix} \tag{4.32}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5k^2 + \alpha \\ 0.5 & 0.5k^2 + \alpha \\ 0 & 0 \end{bmatrix}. \tag{4.33}$$

Further, Eq. (4.7) becomes

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}) = (\lambda q^2 + \delta q^3) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \tag{4.34}$$

and Eqs. (4.8) and (4.9) yield, respectively,

$$\bar{\mathbf{Q}}_{\mathbf{x}} = w(t) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{4.35}$$

and

$$\bar{\mathbf{f}}_{d,\mathbf{x}} = f_{d_2} \sin(\omega_d t) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{4.36}$$

Next, considering that the voltage process $y(t)$ has zero mean (Grigoriu, 2021), the herein generalized harmonic balance method for systems with singular matrices is employed. Considering further that the system response in Eq. (4.25) has a stochastic and a deterministic component, i.e.,

$$\mathbf{x}_s(t) = \begin{bmatrix} q_s(t) \\ y_s(t) \end{bmatrix}, \ \mathbf{x}_d(t) = \begin{bmatrix} q_d(t) \\ y_d(t) \end{bmatrix} \tag{4.37}$$

and ensemble averaging Eq. (4.34), leads to

$$\mathbb{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}] = \left(\lambda \sigma_{q_s}^2 + \lambda q_d^2 + 3\delta \sigma_{q_s}^2 q_d + \delta q_d^3\right) \times \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{4.38}$$

Then, the 6×4 matrix $\mathbf{P}$ in Eq. (4.13) is formed and since it has full rank, a unique solution for the periodic response vector is found by solving Eq. (4.15). Further, applying the generalized statistical linearization method, the equivalent matrices $\bar{\mathbf{M}}_e^a, \bar{\mathbf{C}}_e^a$ and $\bar{\mathbf{K}}_e^a$ are derived and the equivalent linear system in Eq. (4.19) is formed. Indicatively, the matrix $\bar{\mathbf{K}}_e^a$ is given by

$$\bar{\mathbf{K}}_e^a = 1.5\delta\sigma_{q_s}^2 \begin{bmatrix} H(1,1) & H(2,1) \\ H(1,1) & H(2,1) \\ 0 & 0 \end{bmatrix} + 1.5\delta \begin{bmatrix} \frac{q_{d_1}^2+q_{d_2}^2}{2} & 0 \\ \frac{q_{d_1}^2+q_{d_2}^2}{2} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.39}$$

In Eq. (4.39), $H(i,j)$, $i,j = 1,2$, denote the (i,j) element of the matrix $\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$, where $\hat{\mathbf{x}}^{\mathrm{T}} = \begin{bmatrix} \mathbf{x} & \dot{\mathbf{x}} \end{bmatrix}$ and $\mathbf{x}$ is defined in Eq. (4.25); see Fragkoulis et al., 2016b for a detailed discussion.

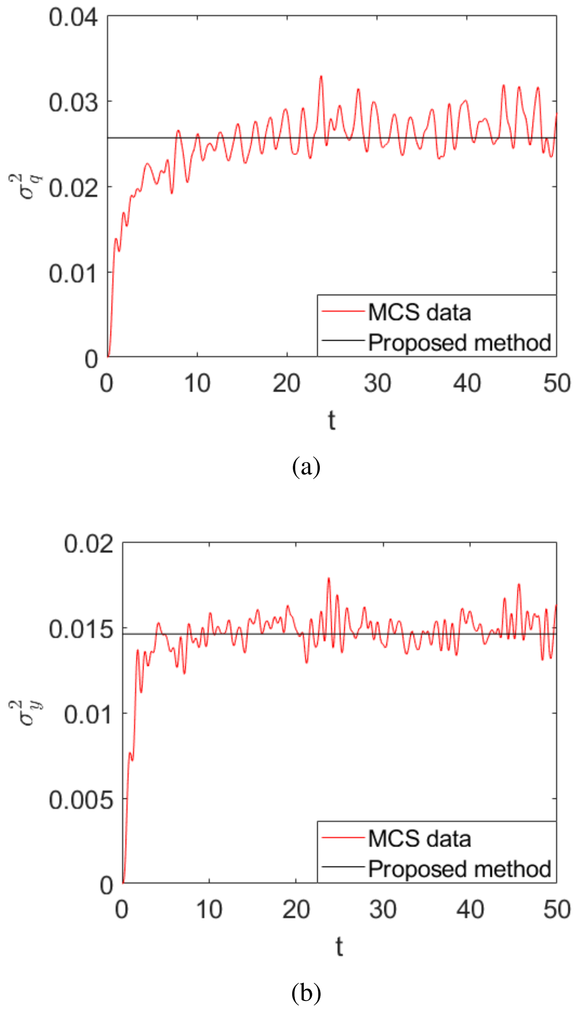

Figure 4.1: Figure 1. Response variance of the energy harvesting system of Eqs. Eqs. (4.22) and (4.23) subjected to combined stochastic and deterministic excitations ($S_0 = 0.05$, $f_{d_2} = 0.4$, $\omega_d = \pi$). Analytical solution vis-à-vis MCS estimate (500 realizations): (a) response displacement variance; (b) response voltage variance.

Finally, the variance of the stochastic response is computed by solving the coupled set of Eqs. (4.15), (4.20) and (4.21). In addition, considering Eqs. (4.12), (4.25) and (4.37), and successively ensemble and temporal averaging to treat, respectively, the stochastic and deterministic components of the response, yields

$$\left\langle \mathbb{E}[x_i^2] \right\rangle = \sigma_{x_{s,1}}^2 + \frac{\omega_d (x_{d_1,i}^2 + x_{d_2,i}^2)}{2}, \tag{4.40}$$

$i = 1, 2$, where $\langle \cdot \rangle$ denotes the temporal averaging operation.

The response displacement variance and the variance of the response voltage of the nonlinear harvester of Eqs. (4.22) and (4.23) subjected to combined stochastic and deterministic excitations are shown, respectively, in Figs. 4.1(a) and 4.1(b). The validity of the results obtained by the proposed method is verified by also considering pertinent MCS data. Specifically, 500 realizations are generated by the spectral representation method (Shinozuka and Deodatis, 1991) for duration $T_0 = 50\ s$ and cut-off frequency equal to 2π. Then, the system response variance is derived by utilizing a standard 4^{th} order Runge-Kutta numerical integration scheme to solve the governing equations of the system.

4.4 Concluding remarks

In this paper, the problem of determining the response statistics of a nonlinear piezoelectric energy harvesting device subjected to combined stochastic and deterministic excitation has been considered. The system response has been computed in a direct way by utilizing a recently developed method for determining the response of multi-degree-of-freedom nonlinear systems with singular parameter matrices (Ni et al., 2021). The method relies on the combination of the generalized statistical linearization treatment for systems with singular matrices and the harmonic balance method. Specifically, since the system excitation consists of a periodic and a stochastic component, the system response has been decomposed into two corresponding components. Then, the statistical linearization and harmonic balance methods have been utilized to treat, respectively, the former and latter. The validity of the obtained results has been verified by considering pertinent MCS data.

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Chapter 5

Research article 4: An approximate stochastic dynamics approach for design spectrum based response analysis of nonlinear structural systems with fractional derivative elements

An approximate stochastic dynamics approach for design spectrum based response analysis of nonlinear structural systems with fractional derivative elements

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Abstract: A novel approximate approach is developed for determining, in a computationally efficient manner, the peak response of nonlinear structural systems with fractional derivative elements subject to a given seismic design spectrum. Specifically, first, an excitation evolutionary power spectrum is derived that is compatible with the design spectrum in a stochastic sense. Next, relying on a combination of statistical linearization and stochastic averaging yields an equivalent linear system (ELS) with time-variant stiffness and damping elements. Further, the values of the ELS elements at the most critical time instant, i.e., the time instant associated with the highest degree of nonlinear/inelastic response behavior exhibited by the structural system, are used in conjunction with the design spectrum for determining approximately the nonlin-

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ear system peak response displacement. The herein developed approach can be construed as an extension of earlier efforts in the literature to account for fractional derivative terms in the governing equations of motion. Furthermore, the approach exhibits the significant novelty of exploiting the localized time-dependent information provided by the derived time-variant ELS elements. Indeed, the values of the ELS stiffness and damping elements at the most critical time instant capture the system dynamics better than an alternative standard time-invariant statistical linearization treatment. This leads to enhanced accuracy when determining nonlinear system peak response estimates. An illustrative numerical example is considered for assessing the performance of the approximate approach. This pertains to a bilinear hysteretic structural system with fractional derivative elements subject to a Eurocode 8 elastic design spectrum. Comparisons with pertinent Monte Carlo simulation data are included as well, demonstrating a high degree of accuracy.

Keywords: Earthquake engineering; Design spectrum; Stochastic dynamics; Fractional derivative; Nonlinear system; Statistical linearization

5.1 Introduction

Contemporary seismic codes favor design spectrum based response analyses of building structures. In this regard, the input seismic action is defined through elastic design spectra that provide the peak response of linear single-degree-of-freedom (SDOF) oscillators as a function of their natural period T and damping ratio ζ (e.g., Chopra, 2001). These are developed, typically, for a nominal damping ratio $\zeta = 0.05$ and are complemented with damping adjustment factors in case a different damping ratio needs to be considered (e.g., Lin et al., 2005). Nevertheless, seismic codes and regulatory agencies allow ordinary structures to exhibit nonlinear/inelastic response behaviors towards achieving cost-effective designs (e.g., CEN, 2004). In this setting, the problem of estimating the peak nonlinear/inelastic system response subject to a given elastic design spectrum arises naturally in code-compliant structural design applications, and remains a persistent research challenge in the field of earthquake engineering.

Irrespective of the nonlinearity type, the above problem can be addressed by performing nonlinear system response time-history analyses in a Monte Carlo simulation (MCS) context. Specifically, the nonlinear system is subjected to an ensemble of ground motion records, whose average design spectrum matches approximately the target one provided by the codes (e.g., Katsanos et al., 2010). Such design spectrum compatible excitation records can comprise artificial accelerograms and/or judiciously selected records from relevant databanks (e.g., Giaralis and Spanos, 2009; Cacciola, 2010; Araújo et al., 2016). In many cases, these records need to be further scaled and modified to achieve the desired compatibility with the given design spectrum. Note, however, that the operation of scaling accelerograms has raised significant concerns in the literature from a theoretical perspective (e.g., Grigoriu, 2011). Further, to reduce the variability of the peak response data obtained based on the minimum number of accelerograms allowed by the seismic codes (e.g., Beyer and Bommer, 2007), a relatively large number of excitation records are required for the system response analysis. In this manner, numerical integration of the nonlinear equations of motion needs to be performed in MCS fashion; thus, rendering the approach computationally demanding.

Obviously, the aforementioned computational cost becomes higher with increasing complexity of the mathematical model representing the nonlinear/hysteretic system under consideration. In this regard, the need for more accurate modeling of viscoelastic material behavior has led recently to the utilization of advanced mathematical tools such as fractional calculus (e.g., Makris, 1997; Sabatier et al., 2007; Rossikhin and Shitikova, 2010; Di Paola et al., 2013). Indeed, models based on fractional derivatives have exhibited a high degree of accuracy compared with experimental viscoelastic response data obtained via creep and relaxation tests. Notably, in contrast to traditional models that utilize combinations of Maxwell and/or Kelvin elements and depend on several parameters, the fractional derivative model requires the identification of two parameters only for capturing both relaxation and creep tests (e.g., Di Paola et al., 2011). Remarkably, structural engineering has benefited significantly from exploiting fractional calculus concepts. In fact, several research efforts pertaining to seismic isolation and vibration control applications have demonstrated the capability of fractional derivatives to model successfully the response behavior of viscoelastic dampers, e.g., Koh and Kelly, 1990; Makris and Constantinou, 1991; Lee and Tsai, 1994; Shen and Soong, 1995; Rüdinger, 2006. The interested reader is also directed to Petromichelakis et al., 2021 for a recent paper referring to fractional derivative modeling of the capacitance term in the governing equations of a broad class of nonlinear electromechanical energy harvesters. It is noted that solving numerically the corresponding fractional differential equation of motion can be a highly demanding task computationally. This is due to the need for treating numerically the convolution integral associated with the fractional derivative operator in conjunction with complex nonlinearities and hysteresis. In this context, various solution schemes have been developed for determining the response of deterministically and/or stochastically excited nonlinear oscillators with fractional derivative elements (e.g., Koh and Kelly, 1990; Spanos and Evangelatos, 2010; Di Matteo et al., 2014; Fragkoulis et al., 2019; Pirrotta et al., 2021; Kong et al., 2022a,b).

It is readily seen that there is merit in developing alternative, more efficient, approaches for treating the problem of estimating the peak response of a nonlinear/hysteretic system with fractional derivative elements subject to a given elastic design spectrum. In this regard, a rather popular class of approaches relates to deriving an equivalent linear system (ELS) based on various deterministic or stochastic linearization criteria (e.g., Iwan, 1980; Iwan and Gates, 1979a; Jennings, 1968; Iwan and Gates, 1979b; Hadjian, 1982; Koliopulos et al., 1994; Giaralis and Spanos, 2010; Mitseas et al., 2018; Mitseas and Beer, 2019). Further, the ELS is characterized by effective stiffness and damping elements that can be used in conjunction with the elastic design spectrum for estimating approximately the peak response of the original system.

In this paper, an approximate stochastic dynamics approach is developed for determining the peak response displacement of nonlinear structural systems with fractional derivative elements subject to a given seismic design spectrum. This is done in a computationally efficient manner without resorting to numerical integration of the governing equations of motion. Specifically, first, an approximate scheme by Cacciola Cacciola, 2010 is employed for deriving an excitation evolutionary power spectrum (EPS) compatible in a stochastic sense with the design spectrum. Note that the choice of utilizing the above scheme is not restrictive, and other alternative approaches for deriving design spectrum compatible power spectra can be adopted. Further, a solution treatment based on a combination of statistical linearization and stochastic averaging is employed that yields an equivalent linear system (ELS) with time-variant stiffness and damping elements. Without loss of generality, systems with softening response behaviors reflecting structural degradation are considered in the ensuing analysis. In this regard, the time instant corresponding to the global minimum and the global maximum of the time-variant stiffness and damping elements, respectively, is treated as the most critical time instant associated with the highest degree of nonlinear/inelastic response behavior exhibited by the structural system. In passing, it is remarked that Dos Santos et al. dos Santos et al., 2016 relied on a somewhat similar concept to develop an efficient stochastic incremental dynamic analysis methodology for circumventing computationally expensive nonlinear system response analyses in a MCS context. Next, the stiffness and damping values at this critical time instant are used in conjunction with the design spectrum for determining approximately the nonlinear system peak response displacement. Note that the peak response estimate is evaluated in an iterative manner till convergence, which ensures that the damping ratio of the imposed design spectrum matches the damping ratio of the ELS.

Compared to earlier relevant efforts in the literature (e.g., Giaralis and Spanos, 2010; Mitseas et al., 2018), the herein developed approach can be construed as an extension to treat structural systems with fractional derivative elements. Furthermore, its significant novel aspect of providing localized time-dependent information via the derived time-variant ELS elements leads to an enhanced accuracy degree when determining nonlinear system peak response estimates. Indeed, it is shown that the values of the ELS stiffness and damping elements at the most critical time instant capture the system dynamics better than an alternative standard statistical linearization solution treatment yielding time-invariant (stationary) ELS stiffness and damping elements (e.g., Giaralis and Spanos, 2010; Mitseas et al., 2018). An illustrative numerical example is considered pertaining to a bilinear hysteretic structural system with fractional derivative elements subject to a Eurocode 8 elastic design spectrum. Comparisons with relevant MCS data are included as well for assessing the accuracy of the approximate approach.

5.2 Mathematical formulation

5.2.1 Auxiliary concepts: Equivalent linear system time-dependent damping and stiffness elements, and stochastic averaging solution treatment

The governing equation of motion describing the dynamics of a stochastically excited nonlinear SDOF system with fractional derivative terms takes the form

$$m\ddot{x}(t) + cD_{0,t}^{\alpha}x(t) + g(t, x, \dot{x}) = ma_g(t), \tag{5.1}$$

where x represents the response displacement process, a dot over a variable denotes differentiation with respect to time t, m is the mass and c is a damping coefficient. Further, $g(t, x, \dot{x})$ is an arbitrary nonlinear function that can account also for hysteretic response behaviors, and $D_{0,t}^{\alpha}x(t)$ denotes the Caputo fractional derivative of order α defined as

$$D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}}d\tau, \quad 0 < \alpha < 1, \tag{5.2}$$

where $\Gamma(\cdot)$ represents the Gamma function. Equivalently, Eq. (5.1) can be cast in the form

$$\ddot{x}(t) + \beta D_{0,t}^{\alpha}x(t) + g_0(t, x, \dot{x}) = a_g(t), \tag{5.3}$$

where $\beta = c/m$ and $g_0 = g/m$. Furthermore, $a_g(t)$ is a non-stationary stochastic excitation process with an EPS $S_{a_g}(\omega, t)$ that is compatible with a prescribed design spectrum $S(\omega, \zeta)$, where ω denotes the frequency in rad/s and ζ is the damping ratio.

Note that various approaches have been developed in the literature over the past few decades for deriving stochastic process power spectra that are compatible in a statistical sense with design spectra provided by seismic building codes; see Pfaffinger, 1983; Spanos and Loli, 1985; Christian, 1989; Park, 1995; Gupta and Trifunac, 1998; Cacciola, 2010; Giaralis and Spanos, 2009; Shields, 2015; Brewick et al., 2018 for some indicative references. Without loss of generality, the approach proposed in Cacciola, 2010 is employed in the ensuing analysis for generating $S_{a_g}(\omega, t)$ based on a given $S(\omega, \zeta)$. The salient aspects of the approach are included in section 5.6 for completeness.

Next, the fundamental ingredients of a recently developed approximate analytical technique for determining the stochastic response of oscillators governed by Eq. (5.3) are delineated. The interested reader is also directed to Fragkoulis et al., 2019 for more details. Specifically, relying on a combination of statistical linearization and stochastic averaging (see Roberts and Spanos, 1986; Roberts and Spanos, 2003 for a broad perspective), the technique in Fragkoulis et al., 2019 yields the non-stationary response amplitude PDF of nonlinear/hysteretic oscillators endowed with fractional derivative elements.

More specifically, considering relatively light damping, the system response exhibits a pseudo-harmonic behavior described by the equations

$$x(t) = A(t)\cos(\omega(A)t + \psi(t)) \tag{5.4}$$

and

$$\dot{x}(t) = -\omega(A)A(t)\sin(\omega(A)t + \psi(t)), \tag{5.5}$$

where the response amplitude $A(t)$ and phase $\psi(t)$ are considered to be slowly-varying quantities with respect to time, and thus, approximately constant over one cycle of oscillation. Next, manipulating Eqs. (5.4) and (5.5) yields

$$A^2(t) = x^2(t) + \left(\frac{\dot{x}(t)}{\omega(A)}\right)^2. \tag{5.6}$$

Further, Eq. (5.3) is recast, equivalently, in the form

$$\ddot{x}(t) + \beta_0 \dot{x}(t) + h(t, x, D^{\alpha}_{0,t}x, \dot{x}) = a_g(t), \tag{5.7}$$

where

$$h(t, x, D^{\alpha}_{0,t}x, \dot{x}) = \beta D^{\alpha}_{0,t}x + g_0(t, x, \dot{x}) - \beta_0 \dot{x}, \tag{5.8}$$

with $\beta_0 = 2\zeta_0\omega_0$ representing a damping coefficient, and ω_0 and ζ_0 denoting, respectively, the natural frequency and damping ratio of the corresponding linear oscillator (i.e., $h(t, x, D^{\alpha}_{0,t}x, \dot{x}) = \omega_0^2 x(t)$). Furthermore, an ELS is defined as

$$\ddot{x}(t) + (\beta_0 + \beta(A))\,\dot{x}(t) + \omega^2(A)x(t) = a_g(t). \tag{5.9}$$

In the following, applying a mean square error minimization between Eqs. (5.7) and (5.9), and approximating the involved fractional derivatives according to Spanos et al., 2016; Li et al., 2015; Di Matteo et al., 2018, yields the ELS amplitude-dependent damping and stiffness coefficients in the form Fragkoulis et al., 2019

$$\beta(A) = \frac{\omega_0^2}{A\omega(A)} S(A) + \frac{\beta}{\omega^{1-\alpha}(A)} \sin\left(\frac{\alpha\pi}{2}\right) - \beta_0 \tag{5.10}$$

and

$$\omega^2(A) = \frac{\omega_0^2}{A} F(A) + \beta\omega^{\alpha}(A) \cos\left(\frac{\alpha\pi}{2}\right), \tag{5.11}$$

where

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} g_0(A\cos\varphi, -A\omega(A)\sin\varphi)\sin\varphi \mathrm{d}\varphi, \tag{5.12}$$

$$F(A) = \frac{1}{\pi}\int_0^{2\pi} g_0(A\cos\varphi, -A\omega(A)\sin\varphi)\cos\varphi \mathrm{d}\varphi, \tag{5.13}$$

and $\varphi(t) = \omega(A)t + \psi(t)$.

Note that the ELS elements $\omega(A)$ and $\beta(A)$ depend on the response non-stationary amplitude

A to account for the nonlinearities and the fractional derivative terms of the original system. Thus, $\omega(A)$ and $\beta(A)$ can be construed as non-stationary stochastic processes, whose time-varying mean values are given by applying the expectation operator on Eqs. (5.10) and (5.11). This yields

$$\beta_{eq}(t) = \int_0^\infty \beta(A)p(A,t)\mathrm{d}A \tag{5.14}$$

and

$$\omega_{eq}^2(t) = \int_0^\infty \omega^2(A)p(A,t)\mathrm{d}A, \tag{5.15}$$

respectively. Further, Eqs. (5.14-5.15) can be associated with an alternative to Eq. (5.9) ELS of the form

$$\ddot{x}(t) + (\beta_0 + \beta_{eq}(t))\,\dot{x}(t) + \omega_{eq}^2(t)x(t) = a_g(t). \tag{5.16}$$

In passing, note that the potent concept of a time-dependent ELS natural frequency, such as the one defined in Eq. (5.15), has been of considerable importance in the field of structural dynamics. Indicative applications include damage detection (e.g., Spanos et al., 2007) and identification of moving resonance phenomena (e.g., Beck and Papadimitriou, 1993; Tubaldi and Kougioumtzoglou, 2015). The latter can occur, for example, during a seismic event when the decrease of the fundamental system frequency due to yielding tends to track the decrease of the predominant frequency of the ground motion. As a result, nonlinear systems can exhibit significant response amplifications.

Next, it is readily seen that the evaluation of the ELS time-dependent damping $\beta_{eq}(t)$ and stiffness $\omega_{eq}^2(t)$ elements via Eqs. (5.14-5.15) requires knowledge of the non-stationary response amplitude PDF $p(A,t)$. In this regard, the stationary response amplitude PDF corresponding to a linear oscillator with fractional derivative terms and subjected to Gaussian white noise was obtained in closed-form in Spanos et al., 2018 based on stochastic averaging. Motivated by this analytical solution, a generalized form of this PDF was considered in Fragkoulis et al., 2019 for modeling the non-stationary response amplitude PDF of the nonlinear oscillator governed by

Eq. (5.3), or equivalently, by Eq. (5.7). This takes the form

$$p(A,t) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) A}{\omega_0^{1-\alpha} c(t)} \exp\left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2c(t)}\right), \tag{5.17}$$

where $c(t)$ is a time-dependent coefficient to be determined. Further, based on a stochastic averaging solution treatment of Eq. (5.16), it was shown in Fragkoulis et al., 2019 that substituting Eq. (5.17) into the associated Fokker-Planck partial differential equation governing the evolution in time of the response amplitude PDF, i.e.,

$$\begin{aligned}\frac{\partial p(A,t)}{\partial t} &= -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}(\beta_0 + \beta_{eq}(t))A + \frac{\pi S_{a_g}(\omega_{eq}(t),t)}{2\omega_{eq}^2(t)A}\right)p(A,t)\right\} \\ &+ \frac{1}{4}\frac{\partial}{\partial A}\left\{\frac{\pi S_{a_g}(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}\frac{\partial p(A,t)}{\partial A} + \frac{\partial}{\partial A}\left(\frac{\pi S_{a_g}(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}p(A,t)\right)\right\}\end{aligned} \tag{5.18}$$

and manipulating, leads to

$$\dot{c}(t) = -(\beta_0 + \beta_{eq}(c(t)))c(t) + \left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}}\right)\frac{\pi S_{a_g}(\omega_{eq}(c(t)),t)}{\omega_{eq}^2(c(t))}. \tag{5.19}$$

Eq. (5.19) constitutes a deterministic first-order nonlinear ordinary differential equation. This can be solved readily by any standard numerical integration scheme, such as the Runge–Kutta, for determining the time-dependent coefficient $c(t)$. Furthermore, $c(t)$ can be used for evaluating the ELS time-dependent damping and stiffness elements by employing Eqs. (5.14) and (5.15). Note that the ELS elements are expressed in Eq. (5.19) as $\beta_{eq}(t) = \beta_{eq}(c(t))$ and $\omega_{eq}(t) = \omega_{eq}(c(t))$ to highlight the explicit dependence of $\beta_{eq}(t)$ and $\omega_{eq}(t)$ on the time-varying coefficient $c(t)$ via Eqs. (5.14-5.15).

It is remarked that various approaches have been proposed in the literature for nonlinear system peak response estimation based on deriving a stationary power spectrum compatible with the provided design spectrum (e.g., Giaralis and Spanos, 2010; Mitseas et al., 2018). In other words, $a_g(t)$ in Eq. (5.3) is modeled as a stationary process with a power spectrum $S_{a_g}(\omega)$ compatible with the design spectrum $S(\omega,\zeta)$. In this case, it can be readily seen that employing the technique developed in Fragkoulis et al., 2019 for a stationary excitation process, i.e., $S_{a_g}(\omega, t) = S_{a_g}(\omega)$, leads to a time-invariant stationary PDF for the response amplitude. In fact, Eq. (5.17) becomes

$$p(A) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) A}{\omega_0^{1-\alpha} c(t_\infty)} \exp\left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2c(t_\infty)}\right), \tag{5.20}$$

where $c(t_\infty)$ is the stationary constant value of $c(t)$ as $t \to \infty$, and thus, Eqs. (5.14) and (5.15) degenerate to

$$\beta_{eq} = \int_0^\infty \beta(A) p(A) \mathrm{d}A \tag{5.21}$$

and

$$\omega_{eq}^2 = \int_0^\infty \omega(A) p(A) \mathrm{d}A, \tag{5.22}$$

respectively. Obviously, Eqs. (5.21-5.22) represent the stationary mean values of the damping and stiffness elements corresponding to the time-invariant, in this case, ELS of Eq. (5.16).

5.2.2 A novel approximate approach for nonlinear system peak response estimation exhibiting enhanced accuracy and accounting for fractional derivative modeling

In this section, a novel approximate approach is developed for determining, in a computationally efficient manner, the peak response of nonlinear structural systems with fractional derivative elements subject to a design spectrum $S(\omega, \zeta)$ provided by seismic building codes. The approach can be construed as an extension of the work in Mitseas et al., 2018 to account for systems with fractional derivative terms. Further, compared to the scheme proposed in ibid., the herein developed approach exhibits an enhanced accuracy degree in determining nonlinear system peak response estimates. This is primarily due to the novel aspect of exploiting the localized time-dependent information provided by the derived ELS elements of Eqs. (5.14-5.15). In this regard, the proposed approach is capable of identifying the critical time instant where the nonlinear/inelastic behavior of the system response is most prevalent. Clearly, the values of the ELS elements of Eqs. (5.14-5.15) corresponding to this specific time instant capture the localized system dynamics better than the time-invariant (stationary) ELS elements derived in Mitseas et al., 2018 based on a standard statistical linearization treatment. In fact, it is shown that more accurate estimates are obtained for the nonlinear system peak response by employing Eqs. (5.14-5.15) compared to their stationary counterparts in Eqs. (5.21-5.22).

Specifically, the mechanization of the proposed approach comprises the following steps:

i) Derivation of an excitation EPS $S_{a_g}(\omega, t)$ compatible with the provided elastic design spectrum $S(\omega, \zeta)$; see Cacciola, 2010 and section 5.6 for more details.

ii) Stochastic averaging/linearization solution treatment of the nonlinear/hysteretic oscillator with fractional derivative terms, and determination of ELS time-dependent stiffness $\omega_{eq}(t)$ and damping $\beta_{eq}(t)$ elements via Eqs. (5.14-5.15).

iii) Identification of the most critical time instant t_{cr} corresponding to the global maximum and global minimum of $\beta_{eq}(t)$ and $\omega_{eq}(t)$, respectively. This time instant is treated as being associated with the highest degree of nonlinear/inelastic response behavior exhibited by the oscillator.

iv) Evaluation of $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr}) = \frac{\beta_0+\beta_{eq}(t_{cr})}{2\omega_{eq}(t_{cr})}$. If $\frac{|\zeta-\zeta_{eq}(t_{cr})|}{\zeta} < \varepsilon$, then go to step v), otherwise set $\zeta = \zeta_{eq}(t_{cr})$ and repeat steps i)-iv) until convergence. This iterative scheme ensures that the damping ratio of the imposed design spectrum matches the damping ratio of the ELS; see also Mitseas et al., 2018 for more details.

v) Peak response estimation by employing the updated design spectrum $S(\omega, \zeta_{eq}(t_{cr}))$ and considering the ELS natural frequency value $\omega_{eq}(t_{cr})$.

The mechanization of the proposed approach is depicted graphically in Fig. 5.1, where it is also compared with the original approach in ibid. The novel aspects of the herein developed approach are highlighted in bold red. Clearly, not only the approach in ibid. is extended to treat systems with fractional derivative terms, but it also exploits localized time-dependent information for evaluating the ELS elements $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr})$. This leads to enhanced accuracy when determining peak response estimates.

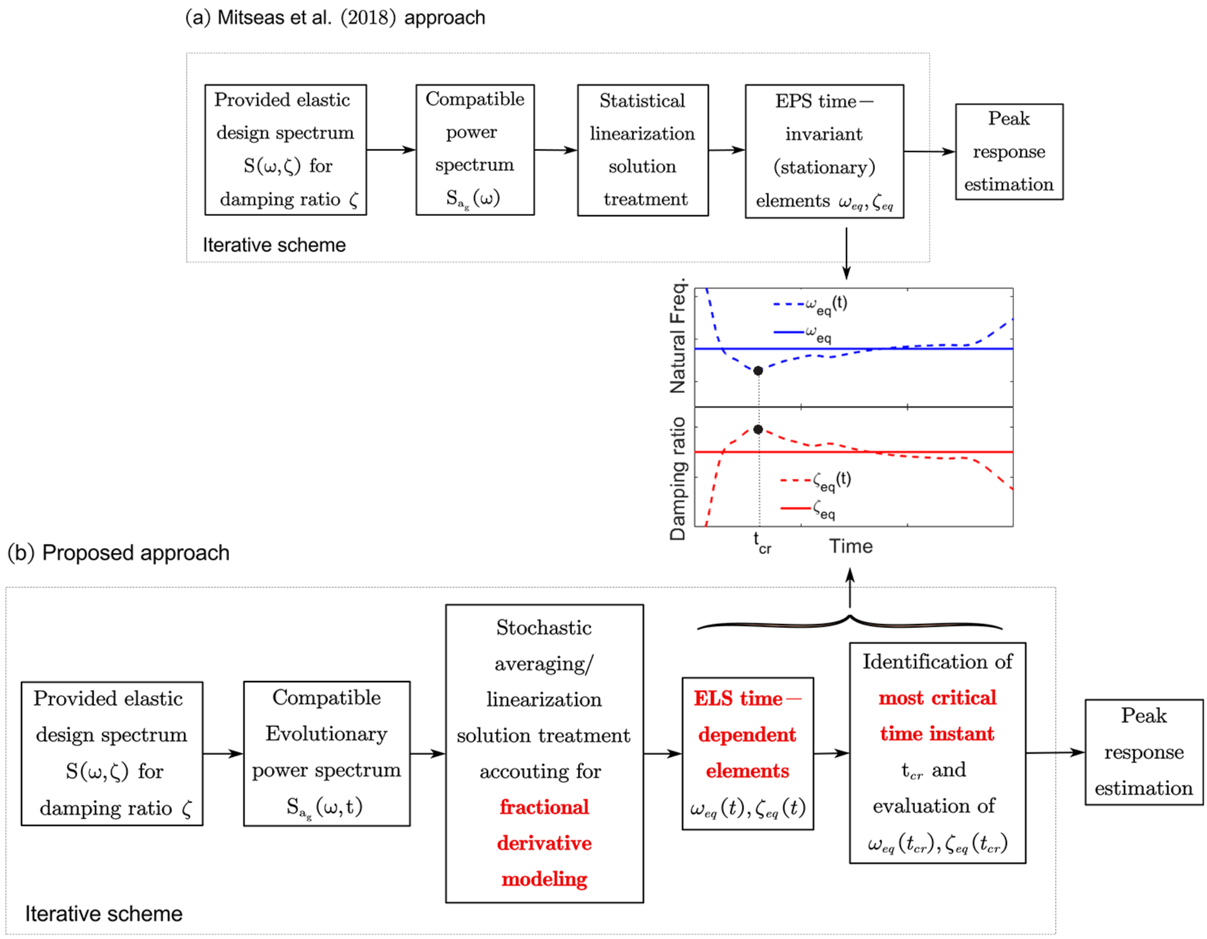

Figure 5.1: Nonlinear system peak response estimation: (a) Approach in Mitseas et al., 2018, (b) Proposed approach exhibiting enhanced accuracy and accounting for fractional derivative modeling.

5.3 Illustrative application

In this section, a bilinear hysteretic structural system with fractional derivative terms subject to a Eurocode 8 elastic design spectrum is considered as an illustrative numerical example for demonstrating the reliability of the developed approach. The achieved accuracy of the predicted peak displacements is quantified by comparison with pertinent results derived from nonlinear response history analyses for an ensemble of time-histories compatible with the considered Eurocode 8 design spectrum. 5.7 includes details on the definition of the imposed Eurocode 8 elastic design spectrum.

5.3.1 Governing equations of a bilinear hysteretic structural system with fractional derivative elements, and ELS time-dependent elements

The governing equation of the bilinear hysteretic oscillator, which has been widely utilized in earthquake engineering applications (e.g., Caughey, 1960; Roberts and Spanos, 2003), takes the form of Eq. (5.3) with

$$g_0(t, x, \dot{x}) = \gamma\omega_0^2 x + (1-\gamma)\omega_0^2 x_y z \tag{5.23}$$

and

$$x_y \dot{z} = \dot{x}\{1 - H(\dot{x})H(z-1) - H(-\dot{x})H(-z-1)\}. \tag{5.24}$$

In Eqs. (5.23-5.24), $H(\cdot)$ denotes the Heaviside step function, γ is the post- to pre-yield stiffness ratio, z is the hysteretic force corresponding to the elasto-plastic characteristic, and x_y is the critical value of the displacement at which the yield occurs.

Next, taking into account Eq. (5.23), Eqs. (5.10-5.11) become (e.g., Fragkoulis et al., 2019)

$$\beta(A) = \frac{(1-\gamma)\omega_0^2 S(A)}{A\omega(A)} + \frac{\beta}{\omega^{1-\alpha}(A)} \sin\left(\frac{\alpha\pi}{2}\right) - \beta_0 \tag{5.25}$$

and

$$\omega^2(A) = \omega_0^2 \left[\gamma + \frac{(1-\gamma)F(A)}{A} + \beta\omega^\alpha(A)\cos\left(\frac{\alpha\pi}{2}\right)\right], \tag{5.26}$$

respectively, where

$$S(A) = \begin{cases} \frac{4x_y}{\pi}\left(1 - \frac{x_y}{A}\right), & A > x_y \\ 0, & A \le x_y \end{cases} \tag{5.27}$$

$$F(A) = \begin{cases} \frac{A}{\pi}\left[\Lambda - \frac{1}{2}\sin(2\Lambda)\right], & A > x_y \\ A, & A \le x_y \end{cases} \tag{5.28}$$

and

$$\cos(\Lambda) = 1 - \frac{2x_y}{A}. \tag{5.29}$$

Further, substituting Eqs. (5.25) and (5.26) into Eqs. (5.14) and (5.15), respectively, yields

$$\begin{aligned}\beta_{eq}(t) =& -\beta_0 + \frac{\beta \sin^2(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}c(t)} \times \int_0^\infty \frac{A}{\omega^{1-\alpha}(A)} \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) \mathrm{d}A \\ &+ \frac{4x_y\omega_0^2(1-\gamma)\sin(\frac{\alpha\pi}{2})}{\pi\omega_0^{1-\alpha}c(t)} \times \int_1^\infty \frac{1-\frac{x_y}{A}}{\omega(A)} \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) \mathrm{d}A\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}\omega_{eq}^2(t) =& \omega_0^2 - (1-\gamma)\omega_0^2 \left\{ \exp\left(-\frac{{x_y}^2 \sin(\frac{\alpha\pi}{2})}{2c(t)\omega_0^{1-\alpha}}\right) - \frac{\sin(\frac{\alpha\pi}{2})}{\pi\omega_0^{1-\alpha}c(t)} \right. \\ &\left. \times \int_1^\infty (\Lambda - \frac{1}{2}\sin(2\Lambda))A \times \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) \mathrm{d}A \right\} \\ &+ \frac{\beta\sin(\frac{\alpha\pi}{2})\cos(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}c(t)} \times \int_0^\infty \omega^\alpha(A) A \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) \mathrm{d}A.\end{aligned} \tag{5.31}$$

5.3.2 Peak inelastic response determination and comparisons with Monte Carlo simulation data

First, following the approach by Cacciola Cacciola, 2010 described succinctly in section 5.6, the excitation EPS $S_{a_g}(\omega, t)$ compatible with the Eurocode 8 design spectrum $S(\omega, \zeta = 0.05)$ (see 5.7) is determined. Specifically, Fig. 5.2(a) shows the $a_g^R(t)$ component of Eq. (5.32) referring to a recorded time history at El Centro site of the Imperial Valley earthquake of May 18, 1940. Fig. 5.2(b) shows a joint time-frequency analysis of the recorded time history at El Centro based on the short-time Thompson's multiple window spectrum estimation scheme proposed in Conte and Peng, 1997. It is readily seen that not only the intensity, but also the frequency content of the time history changes with time. In Fig. 5.2(c), the power spectrum $G^S(\omega)$ is plotted corresponding to the stationary component $a_g^S(t)$ of Eq. (5.32). This is determined based on the iterative scheme of Eq. (5.43). Further, Fig. 5.2(d) shows the calculated excitation EPS $S_{a_g}(\omega, t)$ compatible with the Eurocode 8 design spectrum. This is used in the ensuing analysis as the input EPS for evaluating the ELS time-dependent elements via Eqs. (5.30-5.31). Furthermore, to compare the herein developed approach shown graphically in Fig. 5.1(b) with the approach in Mitseas et al., 2018 shown in Fig. 5.1(a), a design spectrum compatible stationary power spectrum is also determined. Specifically, setting $\alpha = 0$ and $\varphi(t) = 1$ in Eq. (5.32) yields $S_{a_g}(\omega, t) = S_{a_g}(\omega) = G^S(\omega)$, which is computed based on Eq. (5.43) and plotted in Fig. 5.3. Clearly, the $G^S(\omega)$ in Fig. 5.3 corresponds to a larger variance than the $G^S(\omega)$ in Fig. 5.2(c). This is anticipated since the additional component of $a_g^R(t)$ in Eq. (5.32) is omitted in this case, and thus, the intensity of $a_g^S(t)$ needs to increase to counteract the absence of $a_g^R(t)$.

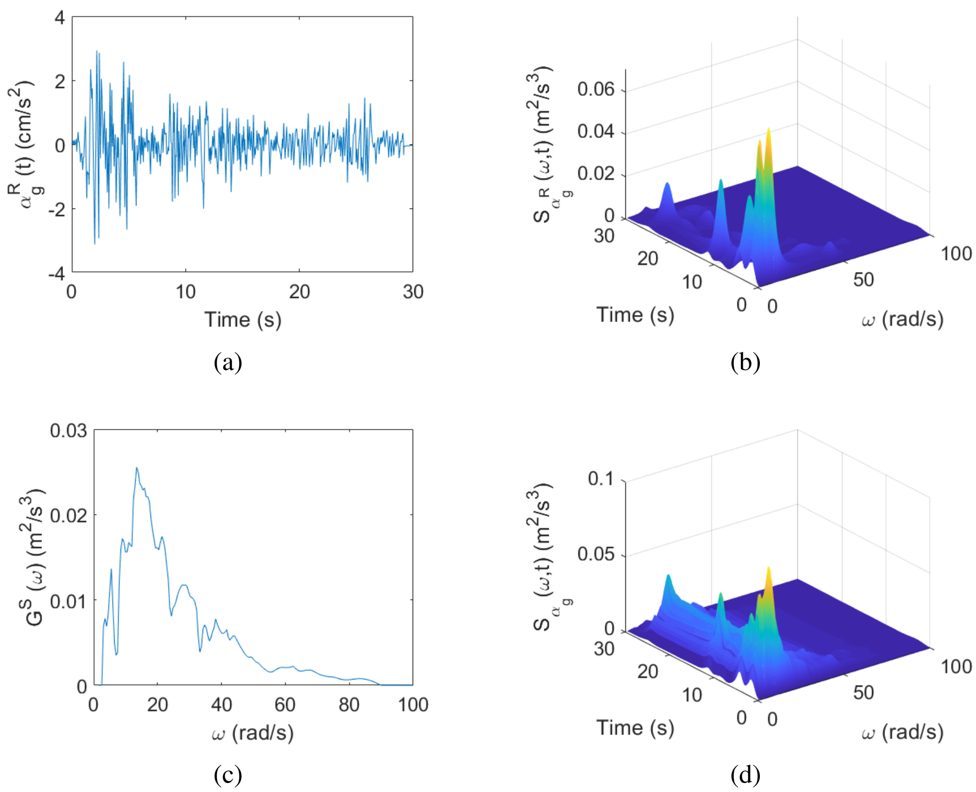

Figure 5.2: (a) The recorded time history at El Centro site; (b) EPS estimate of the recorded time history at El Centro site; (c) Calculated power spectrum $G^S(\omega)$ corresponding to the stationary process $a_g^S(t)$; (d) Excitation EPS $S_{a_g}(\omega, t)$ compatible with a Eurocode 8 type B design spectrum $S(\omega, \zeta = 0.05)$.

Further, the parameter values $\omega_0 = 5.48\ rad/sec$, $\beta_0 = 0.7$, $\gamma = 0.4$, $x_y = 7\ cm$ and $\alpha = 0.5$

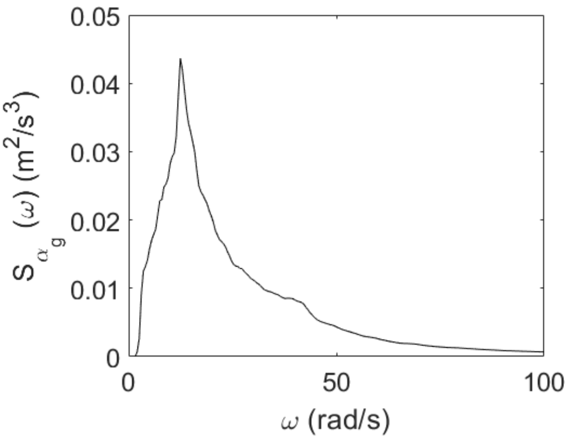

Figure 5.3: Calculated stationary power spectrum $S_{a_g}(\omega, t) = S_{a_g}(\omega) = G^S(\omega)$ compatible with a Eurocode 8 type B design spectrum $S(\omega, \zeta = 0.05)$.

are used in conjunction with the bilinear hysteretic oscillator with fractional derivative elements described by Eqs. (5.23-5.24). The herein developed approach for nonlinear system peak response estimation is applied next. Specifically, considering the excitation EPS in Fig. 5.2(d) and utilizing Eqs. (5.30-5.31), Eq. (5.19) is solved numerically for $c(t)$. This is substituted into Eqs. (5.30-5.31) and the ELS time-variant stiffness $\omega_{eq}(t)$ and damping $\zeta_{eq}(t)$ elements are evaluated. Also, the most critical time instant t_{cr} is identified corresponding to the global minimum and global maximum of $\omega_{eq}(t)$ and $\zeta_{eq}(t)$, respectively.

Next, to ensure that the input design spectrum $S(\omega, \zeta)$ and the ELS of Eq. (5.16) share the same value of damping ratio ζ, an iterative scheme till convergence is applied, where the design spectrum is updated at each step by setting $S(\omega, \zeta = \zeta_{eq}(t_{cr}))$; see also Mitseas et al., 2018 for more details. In this regard, Fig. 5.4 shows the computed time-variant elements $\omega_{eq}(t)$ and $\zeta_{eq}(t)$ corresponding to the 4-th iteration when convergence has been reached. These are compared with time-invariant (stationary) elements ω_{eq} and ζ_{eq} obtained by Eqs. (5.21-5.22). It is readily seen that $\omega_{eq}(t)$ and $\zeta_{eq}(t)$ are capable of capturing time-localized dynamics of the nonlinear system response. In fact, the value of the ELS natural frequency at the most critical time instant t_{cr}, i.e., $\omega_{eq}(t_{cr})$, is considerably smaller than the time-invariant value ω_{eq}. In other words, $\omega_{eq}(t_{cr})$ reflects a higher degree of nonlinear/inelastic response behavior than ω_{eq}. In a similar manner, and in agreement with the above argument, $\zeta_{eq}(t_{cr})$ is larger than the time-invariant value ζ_{eq}; thus, reflecting a higher degree of system nonlinearity. Further, the calculated ELS elements $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr})$ are plotted in Fig. 5.4 corresponding to successive iterations of the scheme, and compared with stationary ω_{eq} and ζ_{eq} estimates. It is seen that convergence has been achieved practically after 4 iterations.

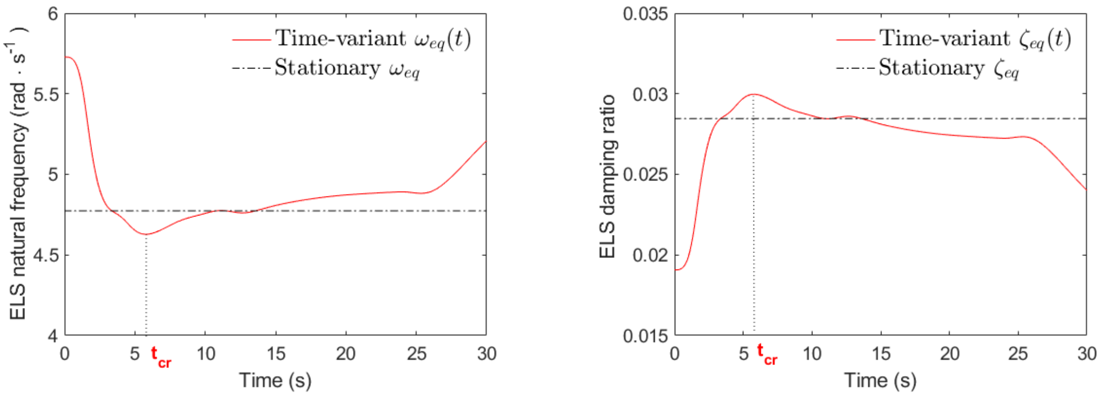

Figure 5.4: ELS time-variant elements and most critical time instant t_{cr} based on Eqs. (5.14-5.15) and corresponding to the 4-th iteration when convergence of the scheme has been reached: (a) natural frequency $\omega_{eq}(t)$, and (b) damping ratio $\zeta_{eq}(t)$. Comparisons with stationary estimates based on Eqs. (5.21-5.22).

Following convergence of the scheme, i.e., $|\zeta - \zeta_{eq}(t_{cr})|/\zeta < \varepsilon$, the obtained ELS elements $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr})$ for $k = 4$ in Fig. 5.5 are used to estimate the nonlinear system peak response in conjunction with the Eurocode 8 elastic design spectrum. This procedure is shown schematically in Fig. 5.6 where the Eurocode 8 design spectrum is plotted against the natural period $T = 2\pi/\omega$, in terms of spectral acceleration $S(\omega, \zeta_{eq}(t_{cr}))$, left vertical axis, and in terms of spectral displacement $S(\omega, \zeta_{eq}(t_{cr}))/\omega^2$, right vertical axis. Next, the peak inelastic displacement is read on the right vertical axis using the pair $(T_{eq}(t_{cr}) = 2\pi/\omega_{eq}(t_{cr}), \zeta_{eq}(t_{cr}))$ indicated on the figure.

Further, to assess the accuracy of the herein developed approach for nonlinear system peak response estimation, comparisons with pertinent MCS data are included as well. In this regard, an ensemble of 1000 acceleration time-histories are generated compatible with the Eurocode 8

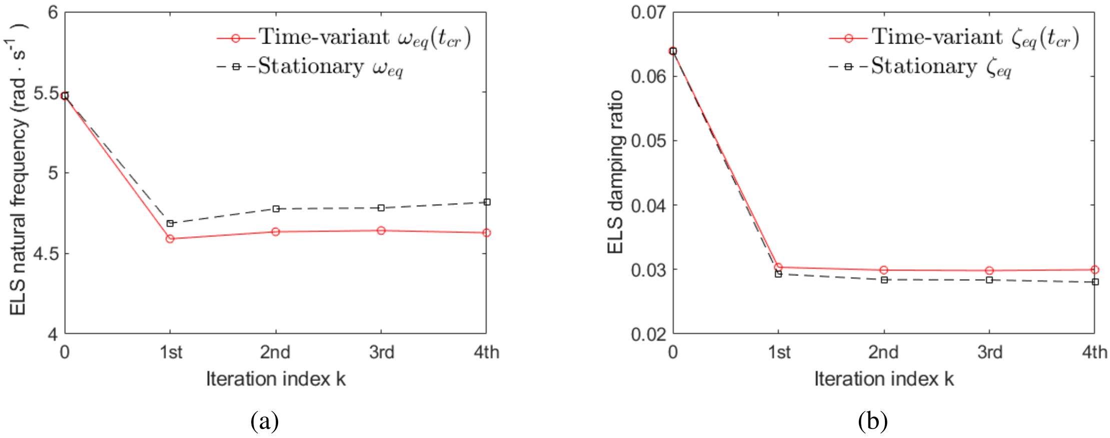

Figure 5.5: ELS time-variant elements based on Eqs. (5.14-5.15) corresponding to successive iterations and evaluated at the most critical time instant t_{cr}: (a) natural frequency $\omega_{eq}(t_{cr})$, and (b) damping ratio $\zeta_{eq}(t_{cr})$. Comparisons with stationary estimates based on Eqs. (5.21-5.22).

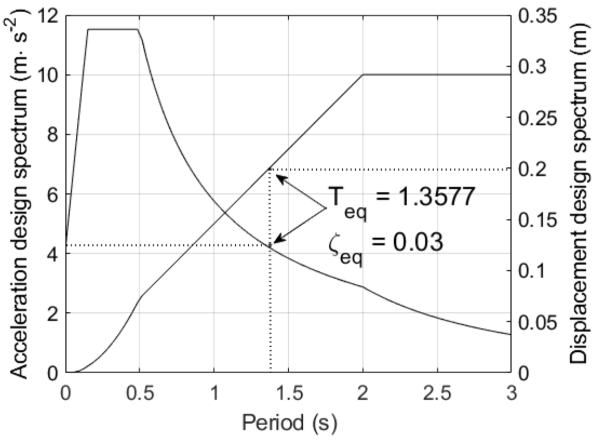

Figure 5.6: Nonlinear system peak response displacement determination using the ELS elements $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr})$ for $k = 4$ in Fig. 5.5 in conjunction with the design spectrum of 5.7.

design spectrum $S(\omega, \zeta = 0.05)$ based on Eq. (5.44) of section 5.6. Furthermore, the governing Eq. (5.1) is numerically integrated for the above ensemble by resorting to an L1-algorithm (e.g., Koh and Kelly, 1990), and the mean peak response estimate is obtained based on statistical analysis of the response time-histories. In passing, note that the nonlinearity degree exhibited by the oscillator is significant as shown by the MCS-based average ductility demand estimate. This is calculated as $x_{max}/x_y = 0.1965/0.07 = 2.8$, indicating that the oscillator enters well into the inelastic range.

Table 5.1 compares the MCS-based estimate with peak displacements obtained by using both the time-variant elements $(\omega_{eq}(t_{cr}),\ \zeta_{eq}(t_{cr}))$ and the stationary elements $(\omega_{eq},\ \zeta_{eq})$, reported in Fig. (5.5) for $k = 4$, in conjunction with the Eurocode 8 design spectrum as illustrated in Fig. (5.6). Also, results corresponding to various values of the fractional derivative order α and of the nonlinearity parameter γ are included in Table 5.1 as well. In all cases, it is seen that the peak response obtained by the proposed approach not only agrees well with the MCS-based estimate, but it also consistently exhibits a higher accuracy degree compared with the results obtained by a stationary treatment of the ELS elements.

Table 5.1: Peak response displacement of bilinear/hysteretic oscillator with fractional derivative elements using the ELS elements $\omega_{eq}(t_{cr})$ and $\zeta_{eq}(t_{cr})$ for various values of the fractional derivative order α and of the nonlinearity parameter γ. Comparisons with stationary estimates based on Eqs. (5.21-5.22), and with MCS data.

Peak displacement estimates					
(α, γ)	MCS	Time-invariant (stationary) elements ω_{eq}, ζ_{eq}	error (based on MCS)	Time-variant elements $\omega_{eq}(t_{cr}), \zeta_{eq}(t_{cr})$	error (based on MCS)
$(0.5, 0.4)$	0.1965	0.1939	1.3%	0.1977	0.6%
$(0.75, 0.4)$	0.1623	0.1587	2.2%	0.1618	0.3%
$(0.5, 0.2)$	0.1941	0.1852	4.6%	0.1901	2.1%
$(0.75, 0.2)$	0.1675	0.1585	5.4%	0.1630	2.7%

5.4 Concluding remarks

In this paper, an approximate approach has been developed for determining the peak response displacement of nonlinear structural systems with fractional derivative elements subject to a given seismic design spectrum. This has been done in a computationally efficient manner without resorting to numerical integration of the governing equations of motion. Specifically, first, an approximate scheme has been utilized for deriving an excitation EPS compatible in a stochastic sense with the design spectrum. Further, employing a solution treatment based on a combination of statistical linearization and stochastic averaging has yielded an ELS with time-variant stiffness and damping elements. Without loss of generality, systems with softening response behaviors reflecting structural degradation have been considered. In this setting, it has been shown that the global minimum and the global maximum of the time-variant stiffness and damping elements, respectively, correspond to the time instant associated with the highest degree of nonlinear/inelastic response behavior exhibited by the oscillator. In this regard, the stiffness and damping values at this critical time instant have been used in conjunction with the design spectrum for determining approximately the nonlinear oscillator peak response displacement. Compared to earlier relevant efforts in the literature (e.g., Giaralis and Spanos, 2010; Mitseas et al., 2018), the herein developed approach can be construed as an extension to treat systems with fractional derivative elements. Furthermore, its significant novel aspect of providing localized time-dependent information via the derived time-variant ELS elements leads to an enhanced accuracy degree when determining nonlinear system peak response estimates. Indeed, it has been shown that the values of the ELS stiffness and damping elements at the most critical time instant capture the system dynamics better than an alternative standard statistical linearization solution treatment yielding time-invariant (stationary) ELS stiffness and damping elements. An illustrative numerical example has been considered for assessing the performance of the approximate approach, pertaining to a bilinear hysteretic oscillator with fractional derivative elements subject to a Eurocode 8 elastic design spectrum. Comparisons with relevant Monte Carlo simulation data have demonstrated a high degree of accuracy.

5.5 Acknowledgment

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5.6 Appendix: Derivation of design spectrum compatible excitation evolutionary power spectrum

Following Cacciola, 2010, the non-stationary excitation stochastic process $a_g(t)$ comprises a fully non-stationary component $a_g^R(t)$ modeled by a recorded earthquake time-history, and a time-modulated stationary zero-mean Gaussian process $a_g^S(t)$, i.e.,

$$a_g(t) = \alpha a_g^R(t) + \varphi(t) a_g^S(t). \tag{5.32}$$

In Eq. (5.32), both the scaling factor α and the power spectrum $G^S(\omega)$ of the stationary process $a_g^S(t)$ are unknowns to be determined, and the time-modulating function $\varphi(t)$ is given as

$$\varphi(t) = \begin{cases} \left(\frac{t}{t_1}\right)^2, & t < t_1 \\ 1, & t_1 \leq t \leq t_2 \\ \exp\left[-\beta(t - t_2)\right], & t > t_2 \end{cases} \tag{5.33}$$

where $t_2 = t_1 + T_s$, with T_s representing the time window during which stationarity is assumed. Next, an approximate relationship can be derived for the corresponding design spectra; that is,

$$S(\omega, \zeta) = \sqrt{\alpha^2 S^R(\omega, \zeta)^2 + S^S(\omega, \zeta)^2}, \tag{5.34}$$

where $S^R(\omega, \zeta)$ and $S^S(\omega, \zeta)$ are the design spectra referring to the response a linear oscillator subject to $a_g^R(t)$ and $a_g^S(t)$, respectively. Taking into account Eq. (5.34), the value of α lies in the range $(0, 1]$ and is estimated as

$$\alpha = \min \left\{ \frac{S(\omega, \zeta)}{S^R(\omega, \zeta)} \right\}. \tag{5.35}$$

Next, attention is directed to determining $G^S(\omega)$. This is done by relying on an approximate solution treatment of the first-passage time problem according to Vanmarcke, 1976, and to an iterative scheme proposed by Cacciola et al., 2004. Specifically, consider the n-th order stationary response spectral moment of a linear SDOF oscillator

$$\lambda_n = \int_0^\infty \omega^n \frac{1}{(\omega_0^2 - \omega^2)^2 + (2\zeta_0\omega_0\omega)^2} G^S(\omega) \mathrm{d}\omega, \tag{5.36}$$

where ω_0 and ζ_0 are the natural frequency and the damping ratio of the oscillator. Further, assuming a sufficiently long duration of $T_s \geq 15\ s$, $G^S(\omega)$ can be related to $S^S(\omega_0, \zeta_0)$ in a statistical manner via the concept of the "peak factor" η. That is,

$$S^S(\omega_0, \zeta_0) = \omega_0^2 \eta \sqrt{\lambda_0(\omega_0, \zeta)}, \tag{5.37}$$

where the peak factor can be estimated by the semi-empirical expression Vanmarcke, 1976

$$\eta = \sqrt{2\ln(2\mu) \left[1 - \exp\left(-\delta\sqrt{\pi \ln(2\mu)}\right)\right]}. \tag{5.38}$$

In Eq. (5.38), the mean zero crossing rate μ and the spread factor δ are defined as

$$\mu = \frac{T_s}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} (-\ln p)^{-1}, \tag{5.39}$$

and

$$\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}, \tag{5.40}$$

respectively. Herein, the probability p in Eq. (5.39) is set equal to 0.5, so that $S^S(\omega_0, \zeta_0)$ in Eq. (5.37) is interpreted as the "median" pseudo-acceleration design spectrum. That is, half of the displacement response spectral ordinates of an ensemble of stationary samples of duration T_s compatible with the power spectrum $G^S(\omega)$ lie below $S^S(\omega_0, \zeta_0)/\omega_0^2$; see also Giaralis and Spanos, 2010 and Mitseas et al., 2018 for more details. Next, relying on the approximate expression (Vanmarcke, 1976)

$$\lambda_0 = \frac{G^S(\omega_0)}{\omega_0^3}\left(\frac{\pi}{4\zeta_0} - 1\right) + \frac{1}{\omega_0^4}\int_0^{\omega_i} G^S(\omega)\mathrm{d}\omega, \tag{5.41}$$

substituting Eq. (5.41) into Eq. (5.37), and manipulating, yields

$$S^S(\omega_0, \zeta_0) = \eta^2\omega_0 G^S(\omega_0)\left(\frac{4 - \pi\zeta_0}{4\zeta_0}\right) + \eta^2\int_0^{\omega_i} G^S(\omega)\mathrm{d}\omega. \tag{5.42}$$

Applying a discretization of the frequency domain into a uniform grid of M frequency points $\omega_i = \omega_b^l + (i - 0.5)\Delta\omega$, $i = 1, 2, \ldots, M$ within the range (ω_b^l, ω_b^u), and manipulating Eq. (5.42), yields Cacciola et al., 2004

$$G^S(\omega_i) = \begin{cases} \frac{4\zeta}{\omega_i\pi - 4\zeta_0\omega_{i-1}}\left(\frac{(S^S(\omega_0,\zeta_0))^2}{\eta^2} - \Delta\omega\sum_{q=1}^{i-1} G^S(\omega_q)\right), & \omega_b^l < \omega_i < \omega_b^u \\ 0, & \omega_i \le \omega_b^l \end{cases} \tag{5.43}$$

Eq. (5.43) can be recursively applied for $i = 1, 2, \ldots, M$ to evaluate the ordinates of the power spectrum $G^S(\omega)$ at the M frequency points ω_i lying $\delta\omega$ apart in the range (ω_b^l, ω_b^u).

Further, following determination of $G^S(\omega)$, a k-th non-stationary acceleration time-history can be generated based on spectral representation theory (e.g., Shinozuka and Deodatis, 1991; Liang et al., 2007). That is,

$$a_g^{(k)}(t) = \alpha a_g^R(t) + \varphi(t)\sum_{i=1}^{N_a}\sqrt{4G^S(i\Delta\omega)\Delta\omega}\cos\left(i\Delta\omega t + \varphi_i^{(k)}\right), \tag{5.44}$$

where $\varphi_i^{(k)}$ are independent random phases uniformly distributed in the interval $[0, 2\pi)$, and N_a

is the number of harmonics to be considered in the summation. Clearly, the non-separable EPS $S_{a_g}(\omega, t)$ can be estimated by various joint time-frequency analysis techniques based on statistical analysis of an ensemble of realizations generated by Eq. (5.44) (e.g., Qian, 2002; Spanos and Failla, 2004; Kougioumtzoglou et al., 2012, 2020). Furthermore, iterative improvement of the $G^S(\omega)$ may be required for satisfying code provisions. To this aim, the iterative scheme

$$G^{S(j)}(\omega) = G^{S(j-1)}(\omega) \left(\frac{S(\omega, \zeta)^2}{\hat{S}^{(j-1)}(\omega, \zeta)^2} \right), \tag{5.45}$$

can be applied, where $\hat{S}^{(j-1)}(\omega, \zeta)^2$ is the mean design spectrum of the ground acceleration $a_g(t)$ at the $(j-1)$-th iteration; see also Cacciola, 2010 and references therein for more details.

Note that the approach presented succinctly in this Appendix degenerates to the scheme in Cacciola et al., 2004 by setting $\alpha = 0$ and $\varphi(t) = 1$ in Eq. (5.32). In this regard, the design spectrum compatible power spectrum becomes $S_{a_g}(\omega, t) = G^S(\omega)$ corresponding to the stationary process $a_g^S(t)$.

5.7 Appendix: Eurocode 8 design spectrum

The Eurocode 8 design spectrum for peak ground acceleration $0.36g$ ($g = 981\ cm/s^2$) and ground type B used in the numerical example of this paper is defined as CEN, 2004

$$S(T, \zeta) = 0.432g \times \begin{cases} 1 + \frac{T}{0.15}(2.5\delta - 1), & 0 \le T \le 0.15 \\ 2.5\delta, & 0.15 \le T \le 0.5 \\ \frac{1.25\delta}{T}, & 0.5 \le T \le 2 \\ \frac{2.5\delta}{T^2}, & 2 \le T \le 4 \end{cases} \tag{5.46}$$

where

$$\delta = \sqrt{\frac{10}{5 + \zeta}} \ge 0.55, \tag{5.47}$$

$T = 2\pi/\omega$ is the natural period and ζ is the damping ratio.

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Chapter 6

Research article 5: Stochastic Incremental Dynamics Methodology for Nonlinear Structural Systems Endowed with Fractional Derivative Terms Subjected to Code-compliant Seismic Excitation

Stochastic Incremental Dynamics Methodology for Nonlinear Structural Systems Endowed with Fractional Derivative Terms Subjected to Code-compliant Seismic Excitation

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Abstract: A novel stochastic incremental dynamics analysis methodology is developed for nonlinear structural systems with fractional derivative elements exposed to seismic excitation consistently aligned with contemporary aseismic codes provisions (e.g. Eurocode 8). Rendering to the concept of non-stationary stochastic processes, the vector of the imposed seismic excitations is characterized by evolutionary power spectra compatible in a stochastic sense with elastic response acceleration spectra of specified modal damping ratio and scaled ground acceleration. The proposed stochastic dynamics technique relies on a combination of the stochastic averaging and statistical linearization methods, which permits the determination of the response displacement probability density function in an efficient and comprehensive manner. The commonly encountered in the literature incremental dynamics analysis curves have been replaced by a stochastic incremental dynamics analysis surface providing with reliable higher order statistics of the system response. A significant attribute of the method pertains to the derivation of an associated response evolutionary power spectrum as a function of spectral acceleration.

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The method retains the coveted attribute of a particularly low associated computational cost. A structural system comprising the nonlinear model endowed with fractional derivative terms subject to a Eurocode 8 elastic design spectrum serves as a numerical example for demonstrating the reliability of the proposed methodology, whose accuracy is demonstrated by comparisons with pertinent Monte Carlo simulation data.

6.1 Introduction

In the engineering discipline of earthquake resistant structures nonlinearities arise naturally in various forms. In this setting, there is a well-detected need for rigorous and consistent representation of the system model by considering thoroughly the underlying mechanisms which determine the overall system behavior. Constant needs for enhanced modeling purposes dictate a reasonable transition to more advanced mathematical tools such as fractional calculus. Notably, structural engineering has significantly benefited from the utilization of fractional calculus concepts. The emerged number of research efforts pertaining to seismic isolation, vibration control and energy harvesting applications reveals the capabilities of fractional calculus to offer upgraded system modeling services in numerous cases of structural engineering interest (Makris and Constantinou, 1991; Rüdinger, 2006; Koh and Kelly, 1990; Kougioumtzoglou et al., 2022; Di Paola et al., 2013; Rossikhin and Shitikova, 2010). Further, an appropriate stochastic representation of seismic excitation in conjunction with nonlinear system modeling and in alignment with aseismic codes provisions secures a solid basis for formulating a realistic structural analysis procedure (Mitseas and Beer, 2021).

Reliable numerical estimations related to the performance of structural systems necessitate a proper quantitative treatment of uncertainties. The emerging concept of performance-based earthquake engineering (PBEE) advocates the assessment of the structural system performance in a comprehensive and rigorous manner by properly accounting for the presence of uncertainties (Mitseas et al., 2016; Mitseas and Beer, 2020). Specifically, basic notions pertaining to PBEE comprise the definition of excitation related variables, known as intensity measures (IMs) (e.g., spectral acceleration, peak ground acceleration, etc.), and of system response related variables known as engineering demand parameters (EDPs) (e.g., peak story drift, inter-story drift ratio, etc.). Moreover, the information provided via the functional relationship between the IMs and the EDPs in conjunction with judicially defined damage-state rules (DSs), is utilized for quantifying a decision variable (DV) (e.g. financial loss).

In the earthquake engineering field, one of the customarily employed methodologies for estimating the functional relationship between the IMs and the EDPs is the incremental dynamic analysis (IDA) (Vamvatsikos and Cornell, 2002). IDA aims at assessing the structural performance of systems subject to a suite of ground motion records, each scaled to several levels of seismic intensity; thus, conducting a nonlinear response time-history analysis (RHA) for each and every scaled record. It is noteworthy that each IDA curve is related to a specific ground motion record whereas each point of the curve corresponds to a specific ground motion intensity level and respective structural system response magnitude. Clearly, the determination of the above-mentioned functional relationship is associated with a significant computational cost. Further, IDA provides with simple statistics of the selected EDP such as the standard deviation and the mean whereas potential higher order statistics requirements under a fully probabilistic framework could render the whole process computationally prohibitive. Notably, some recent research efforts have been made in the area harnessing the potential of advanced random vibration concepts (dos Santos et al., 2016; Mitseas and Beer, 2021).

The developed stochastic incremental dynamics analysis methodology pertains to nonlinear structural systems with fractional derivative elements exposed to seismic excitation consistently determined with contemporary aseismic codes provisions (e.g. Eurocode 8). Specifically, the imposed scaled seismic excitation is characterized by a series of evolutionary power spectra (EPS) compatible in a stochastic sense with an elastic response acceleration spectrum of specified modal damping ratio and scaled ground acceleration (Cacciola, 2010). At the core of the proposed technique lies a combination of the stochastic averaging and statistical linearization methodologies (Roberts and Spanos, 2003; Fragkoulis et al., 2019), which permits the determination of the response displacement probability density function (PDF) in an efficient manner. The generated stochastic incremental dynamics analysis surface provides with reliable higher order statistics of the system response. In addition, a significant attribute of the proposed method is the derivation of the associated response EPS as a function of spectral acceleration. Notably, the method keeps the associated computational cost at a minimum level. An illustrative numerical example pertaining to a bilinear hysteretic structural system with fractional derivative elements subject to a Eurocode 8 elastic design spectrum serves as a numerical example for demonstrating the reliability of the proposed methodology, while comparisons with relevant Monte Carlo simulation (MCS) data are included as well for assessing its accuracy.

6.2 Mathematical formulation

6.2.1 Equivalent linear system determination

The governing equation of motion of a nonlinear single-degree-of-freedom (SDOF) system endowed with fractional derivative elements subject to a non-stationary excitation is given by

$$\ddot{x}(t) + \beta D^{\alpha}_{0,t} x(t) + g(t, x, \dot{x}) = a_g(t), \tag{6.1}$$

where x is the response displacement and a dot over a process denotes differentiation with respect to time. $g(t, x, \dot{x})$ is an arbitrary nonlinear/hysteretic function and $D^{\alpha}_{0,t}(\cdot)$ represents the Caputo fractional derivative of fractional order α ($0 < \alpha < 1$)

$$D^{\alpha}_{0,t} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{6.2}$$

where $\Gamma(\cdot)$ is the Gamma function and β is a damping coefficient. Lastly, $a_g(t)$ is a stochastic seismic excitation process whose evolutionary power spectrum (EPS) $S_{a_g}(\omega, t)$ is compatible with a prescribed design spectrum $S(\omega, \zeta_0)$ (Cacciola, 2010).

Next, a recently developed approximate analytical technique (Fragkoulis et al., 2019), which relies on a combination of statistical linearization and stochastic averaging methods, is applied to determine the non-stationary response amplitude PDF of the oscillator in Eq. (6.1). Considering that the oscillator is lightly damped, its response follows a pseudo-harmonic behavior, given by

$$x(t) = A(t)\cos(\omega(A)t + \psi(t)), \tag{6.3}$$

$$\dot{x}(t) = -\omega(A)A(t)\sin(\omega(A)t + \psi(t)), \tag{6.4}$$

where $\psi(t)$ and $A(t) = A$ denote the response phase and amplitude, respectively. The latter vary slowly with respect to time, and thus, can be regarded as constant over one cycle of oscillation (Roberts and Spanos, 1986). Taking into account Eqs. (6.3)-(6.4), a decoupling of the corresponding differential equations is attained in the form

$$A^2(t) = x^2(t) + \left(\frac{\dot{x}(t)}{\omega(A)}\right)^2, \tag{6.5}$$

$$\psi(t) = -\omega(A)t - \arctan\left(\frac{\dot{x}(t)}{x(t)\omega(A)}\right). \tag{6.6}$$

Then, applying a statistical linearization scheme, Eq. (6.1) is recast into

$$\ddot{x}(t) + (\beta_0 + \beta(A))\,\dot{x}(t) + \omega^2(A)x(t) = a_g(t), \tag{6.7}$$

where $\beta_0 = 2\zeta_0\omega_0$ with ω_0 and ζ_0 denoting the natural frequency and damping ratio of the corresponding linear oscillator. Further, defining an error function as the difference between Eqs. (6.1) and (6.7) and minimizing it in a mean square sense, leads to the equivalent linear amplitude-dependent elements

$$\beta(A) = -\beta_0 + \frac{S(A)}{A\omega(A)} + \frac{\beta \sin\left(\frac{\alpha\pi}{2}\right)}{\omega^{1-\alpha}(A)}, \tag{6.8}$$

$$\omega^2(A) = \frac{F(A)}{A} + \beta\omega^{\alpha}(A)\cos\left(\frac{\alpha\pi}{2}\right), \tag{6.9}$$

where

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} g_0(A,\varphi)\sin\varphi d\varphi, \tag{6.10}$$

$$F(A) = \frac{1}{\pi}\int_0^{2\pi} g_0(A,\varphi)\cos\varphi d\varphi, \tag{6.11}$$

with $g_0(A,\varphi) = g\left(A\cos\varphi, -A\omega(A)\sin\varphi\right)$ and $\varphi(t) = \omega(A)t + \psi(t)$. Since Eqs. (6.8) and (6.9) are amplitude-dependent, $\beta(A)$ and $\omega(A)$ are also non-stationary processes. Hence, taking expectations on Eqs. (6.8) and (6.9) yields the time-varying mean values (Kougioumtzoglou and Spanos, 2009)

$$\beta_{eq}(t) = \int_0^{\infty} \beta(A)p(A,t)\mathrm{d}A, \tag{6.12}$$

$$\omega_{eq}^2(t) = \int_0^{\infty} \omega^2(A)p(A,t)\mathrm{d}A, \tag{6.13}$$

where $p(A,t)$ denotes the non-stationary response amplitude PDF. In passing, note that Eqs. (6.12) and (6.13) correspond to the equivalent linear system

$$\ddot{x}(t) + \left(\beta_0 + \beta_{eq}(t)\right)\dot{x}(t) + \omega^2(t)x(t) = a_g(t). \tag{6.14}$$

Clearly, $p(A,t)$ is required for evaluating the time-varying equivalent elements in Eqs. (6.12)-(6.13), which is given by (Fragkoulis et al., 2019)

$$p(A,t) = \frac{\sin\left(\frac{\alpha\pi}{2}\right)A}{\omega_0^{1-\alpha}c(t)}\exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right). \tag{6.15}$$

In Eq. (6.15), $c(t)$ denotes an unknown time-dependent coefficient, which is determined by resorting to a stochastic averaging treatment of Eq. (6.14). In this regard, first, a first-order stochastic differential equation for $A(t)$ is derived. Then, substituting Eq. (6.15) into the corresponding Fokker-Planck partial differential equation governing the evolution in time of $p(A,t)$, i.e.,

$$\begin{aligned}\frac{\partial p(A,t)}{\partial t} &= -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}(\beta_0+\beta_{eq}(t))A+\frac{\pi S_{a_g}(\omega_{eq}(t),t)}{2\omega_{eq}^2(t)A}\right)p(A,t)\right\} \\ &+\frac{1}{4}\frac{\partial}{\partial A}\left\{\frac{\pi S_{a_g}(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}\frac{\partial p(A,t)}{\partial A}+\frac{\partial}{\partial A}\left(\frac{\pi S_{a_g}(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}p(A,t)\right)\right\},\end{aligned} \tag{6.16}$$

and manipulating yields

$$\dot{c}(t) = -\left(\beta_0+\beta_{eq}\left(c(t)\right)\right)c(t)+\left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}}\right)\frac{\pi S_{a_g}(\omega_{eq}(c(t)),t)}{\omega_{eq}^2(c(t))}. \tag{6.17}$$

Eq. (6.17) constitutes a deterministic first-order nonlinear ordinary differential equation, which can be readily solved by the Runge-Kutta numerical integration scheme; the interested reader is directed to Fragkoulis et al., 2019; Kougioumtzoglou et al., 2022; Fragkoulis and Kougioumtzoglou, 2023 for more details on the derivation of Eqs. (6.7)-(6.17).

Lastly, considering Eqs. (6.8)-(6.9) and following Kougioumtzoglou, 2013, the amplitude-dependent response EPS is determined by

$$S_{xx}(\omega,t) = \int_0^\infty \frac{S_{a_g}(\omega,t)p(A,t)dA}{(\omega^2(A)-\omega^2)^2+(\omega\beta(A))^2}. \tag{6.18}$$

6.2.2 Code-compliant stochastic incremental dynamics analysis methodology

Numerous systems of real engineering interest can be modeled adequately as SDOF systems (Roberts and Spanos, 2003). Consider a quiescent nonlinear SDOF system base-excited by a response spectrum compatible acceleration stochastic process $a_g(t)$ whose dynamic behavior is governed by Eq. (6.1). Following Cacciola, 2010, the non-stationary acceleration process $a_g(t)$

is characterized in the frequency domain by an associated EPS $S_{a_g}(\omega, t)$, compatibly defined with Eurocode 8 provisions. An incremental mechanization analogous to that used in normal IDA is adopted herein, where a_g^0 stands for the scaled image of the excitation magnitude leading to the introduction of the definition of $S_{a_g}(\omega, t; a_g^0)$. In the present study, the selected EDP is that of the response displacement amplitude A at the most critical time instant t_{in}, which stands for the time instant when the parameter $c(t)$ found in Eq. (6.17) reaches its maximum value. In this regard, the response amplitude PDF at t_{in} with respect to specific level of the scaled excitation a_g^0 is given by

$$p(A, t_{in}; a_g^0) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) A}{\omega_0^{1-\alpha} c(t_{in})} \exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2c(t_{in})}\right). \tag{6.19}$$

The generated $p(A, t_{in}; a_g^0)$ for each and every scaled level of the excitation a_g^0 leads to the efficient determination of the stochastic IDA response amplitude PDF surface, comprising valuable higher order statistics under a fully probabilistic consideration. Manipulating Eqs. (6.18)-(6.19) yields the response power spectrum with respect to a specified level of excitation a_g^0 and time instant,

$$S_{xx}(\omega, a_g^0) = \int_0^\infty \frac{S_{a_g}(\omega, t_{in}; a_g^0)}{(\omega^2(A) - \omega^2)^2 + (\omega\beta(A))^2} \times p(A, t_{in}; a_g^0)\, dA, \tag{6.20}$$

where $S_{xx}(\omega, a_g^0)$ is the system response EPS at the time instant when the response variance reaches its maximum value for a given ground acceleration a_g^0.

The above quoted relation leads to the efficient determination of the coveted response power spectrum which pertains evolutionary characteristics as a function of spectral acceleration. The mechanization of the proposed methodology is provided in the following steps:

1. Derive the excitation EPS $S_{a_g}(\omega, t; a_g^0)$ in a stochastically compatible manner with an assigned elastic response acceleration spectrum of specified modal damping ratio and scaled ground acceleration a_g^0; see Cacciola, 2010 for more details.
2. Following the proposed stochastic averaging and linearization method shown in sec-

tion 6.2.1, determine the maximum value $c_{max}(t_{in})$ and the corresponding time instant t_{in}. Practically, this is achieved by employing Eqs. (6.12)-(6.13) and Eq. (6.17).

3. For a specific level of excitation a_g^0, estimate the response EDP PDF and the response EPS at t_{in} by Eqs. (6.19) and (6.20), respectively.

4. Repeat steps 1-3 for all scaled images of the excitation a_g^0 to determine the stochastic IDA response amplitude PDF surface and the response EPS as function of the spectral acceleration.

6.3 Numerical application

Employing the bilinear hysteretic force-deformation law is a common practice to capture the behavior of structural members and structures under seismic excitation (Mitseas and Beer, 2019; Giaralis and Spanos, 2010). Therefore, in this section, a bilinear hysteretic oscillator with fractional derivative elements subject to a Eurocode 8 elastic pseudo-acceleration response spectrum is utilized to demonstrate the reliability of the proposed stochastic IDA framework. The obtained results are compared and found in good agreement with corresponding results derived from nonlinear RHA in a MCS-based context.

6.3.1 Bilinear hysteretic SDOF system with fractional derivative elements

The equation of motion of a nonlinear bilinear hysteretic SDOF system with fractional derivative elements is considered. The restoring force of the system is given by

$$g(t, x(t), \dot{x}(t)) = \gamma\omega_0^2 x(t) + (1-\gamma)\omega_0^2 x_y z(t), \tag{6.21}$$

$$x_y \dot{z}(t) = \dot{x}\{1 - \Phi(\dot{x}(t))\Phi(z(t)-1) - \Phi(-\dot{x}(t))\Phi(-z(t)-1)\}, \tag{6.22}$$

where $\Phi(\cdot)$ denotes the Heaviside step function. Further, $z(t)$ is an auxiliary state representing the hysteretic component, γ denotes the post- to pre-yield stiffness ratio and x_y is the yielding displacement.

Next, taking into account Eq. (6.21), Eqs. (6.10) and (6.11) become

$$S(A) = \begin{cases} \frac{4x_y\omega_0^2}{\pi}\left(1 - \frac{x_y}{A}\right), & A > x_y \\ 0, & A \leq x_y \end{cases} \tag{6.23}$$

$$F(A) = \begin{cases} \frac{A\omega_0^2}{\pi}\left(\Lambda - \frac{1}{2}\sin(2\Lambda)\right), & A > x_y \\ A\omega_0^2, & A \leq x_y \end{cases} \tag{6.24}$$

with $\cos(\Lambda) = 1 - \frac{2x_y}{A}$. Thus, Eq. (6.12) yield

$$\begin{aligned} \beta_{eq}(c(t)) = -\beta_0 + \frac{\beta \sin^2(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}c(t)} \times \int_0^\infty \frac{A}{\omega^{1-\alpha}(A)} \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) dA \\ + \frac{4x_y\omega_0^2(1-\gamma)\sin(\frac{\alpha\pi}{2})}{\pi\omega_0^{1-\alpha}c(t)} \times \int_{x_y}^\infty \frac{1-\frac{x_y}{A}}{\omega(A)} \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) dA, \end{aligned} \tag{6.25}$$

whereas Eq. (6.13) leads to

$$\begin{aligned} \omega_{eq}^2(c(t)) = \omega_0^2 - (1-\gamma)\omega_0^2 \times \left\{\exp\left(-\frac{{x_y}^2\sin(\frac{\alpha\pi}{2})}{2c(t)\omega_0^{1-\alpha}}\right) - \frac{\sin(\frac{\alpha\pi}{2})}{\pi\omega_0^{1-\alpha}c(t)}\right. \\ \left.\times \int_{x_y}^\infty \frac{2\Lambda - \sin(2\Lambda)}{2A^{-1}} \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) dA\right\} + \frac{\beta\sin(\frac{\alpha\pi}{2})\cos(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}c(t)} \\ \times \int_0^\infty \omega^\alpha(A) A \exp\left(-\frac{\sin(\frac{\alpha\pi}{2})}{\omega_0^{1-\alpha}}\frac{A^2}{2c(t)}\right) dA. \end{aligned} \tag{6.26}$$

6.3.2 Response statistics stochastic IDA surfaces determination

The elastic pseudo-acceleration design spectrum $S(\omega, \zeta_0 = 0.05)$ for soil type B according to Eurocode 8 is selected as the reference input spectrum. In addition, the recorded time history at the El Centro site corresponding to the SOOE (N-S) component of the Imperial Valley earthquake of May 18, 1940, is used to model the excitation's non-stationary attributes. The scaled images of the excitation are determined as $a_g^0 = 0.35g \times [0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6]$ where g stands for the acceleration of gravity.

The following parameters pertaining to the bilinear SDOF system under consideration have been employed: $m = 1$, $\omega_0 = 8$, $\zeta_0 = 0.05$, $a = 0.5$, $\gamma = 0.2$ and $x_y = 0.02\ m$. It can be readily seen that following the mechanization presented in section 6.2.2, the stochastic IDA response EDP PDF surface and the response EPS stochastic IDA surface can be efficiently determined at a particularly low computational cost. Note, in passing, that the corresponding time instants t_{in} differ with respect to the scaled image of the ground acceleration a_g^0 following the criterion of the maximum value $c_{max}(t_{in})$.

Fig. 6.1 shows the response EDP PDF surface of the bilinear SDOF system with fractional derivative order $\alpha = 0.5$. Note that the red solid line depicts the modes of the EDP. To assess the accuracy of the developed approach, the response EDP PDF surface from MCS data is plotted as well in Fig. 6.2. In this regard, utilizing the spectral representation method of Liang et al., 2007, an ensemble of $10,000$ acceleration time histories is generated, compatible with the reference design spectrum corresponding every time to a specified scaled image of the excitation a_g^0. Subsequently, the governing equation of motion Eq. (6.1) subject to the above ensemble of accelerograms is numerically solved by resorting to an L1-algorithm (Koh and Kelly, 1990). Considering the approximations involved in the proposed approach, it can be clearly stated that the results obtained by the proposed methodology are in good agreement with the MCS-based estimates.

The response EPS stochastic IDA surface is shown in Fig. 6.3. It is noted that exceeding an intensity threshold signals a gradual transition from elastic into the plastic region. The noted break, which is expressed with a transition to lower values of frequency, is indicative of the system stiffness degradation. It is noteworthy that the proposed method provides with an insight into the underlying dynamic character of the system; this significant operation cannot be determined following typical nonlinear RHA.

6.4 Conclusions

In this paper, a novel stochastic incremental dynamics analysis methodology has been developed for nonlinear systems with fractional derivative elements subject to a seismic excitation

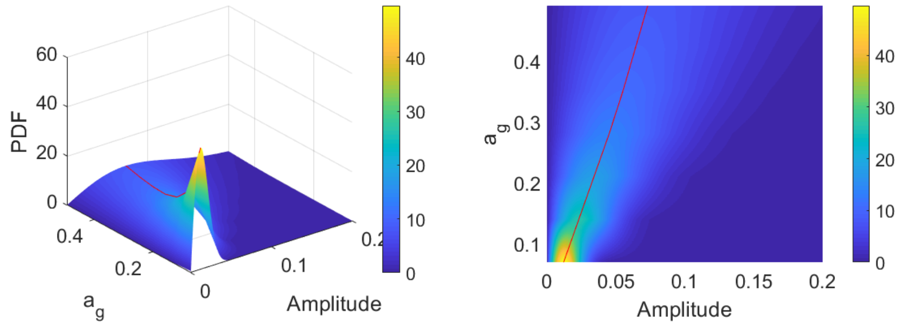

Figure 6.1: Response EDP PDF surface of a bilinear hysteretic oscillator with fractional derivative elements by the proposed method: (a) 3D view; (b) planar view.

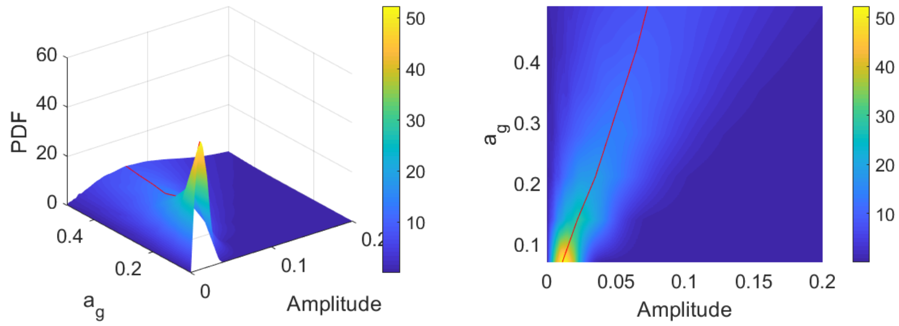

Figure 6.2: Response EDP PDF surface of a bilinear hysteretic oscillator with fractional derivative elements by MCS method $(10,000$ realizations): (a) 3D view; (b) planar view.

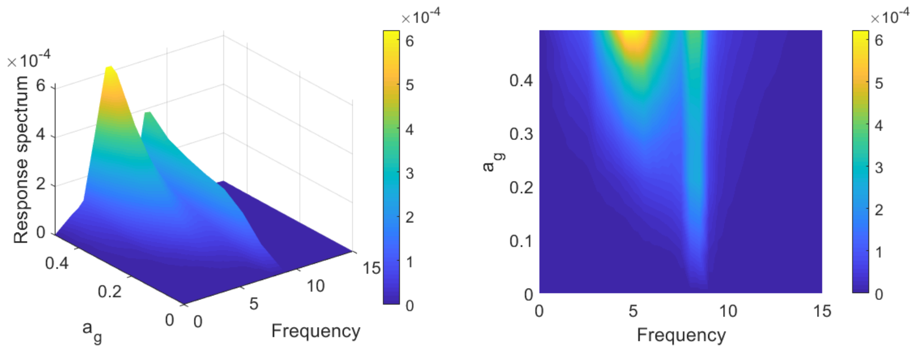

Figure 6.3: Response EPS stochastic IDA surface of a bilinear hysteretic oscillator with fractional elements: (a) 3D view; (b) planar view.

vector consistently aligned with contemporary aseismic codes provisions. In this regard, an incremental mechanization analogous to the one used in normal incremental dynamic analysis is adopted to ensure the necessary compatibility for pertinent applications in structural engineering field. Specifically, rendering to the concept of non-stationary stochastic processes, the vector of the imposed seismic excitations is characterized by EPS stochastically compatible with elastic response acceleration spectra of specified modal damping ratio and scaled ground acceleration. Harnessing the potential of a combination of the stochastic averaging and statistical linearization methods, the response displacement PDFs are determined in an efficient and comprehensive manner. The proposed methodology provides with reliable higher order statistics of the selected EDP rather than simple estimates only of the mean and standard deviation, which is currently the norm in the IDA relevant literature. Further, a particularly interesting attribute of the proposed methodology is the derivation of the associated response EPS as a function of spectral acceleration. This coveted element has a twofold meaning; it performs structural behavior monitoring considering intensity, whereas it provides with an insight into the underlying dynamic character of the system. The efficient identification of the latter cannot be determined following nonlinear RHA. Lastly, the associated low computational cost renders the proposed methodology particularly useful for related performance-based engineering applications. A structural system comprising the bilinear model endowed with fractional derivative elements serves as a numerical example for demonstrating the reliability of the proposed methodology, whereas comparisons with relevant MCS data demonstrate the accuracy of the proposed code-compliant stochastic IDA technique.

6.5 Acknowledgment

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Chapter 7

Research article 6: Operator Norm-based Statistical Linearization to Bound the First Excursion Probability of Nonlinear Structures Subjected to Imprecise Stochastic Loading

Operator Norm-based Statistical Linearization to Bound the First Excursion Probability of Nonlinear Structures Subjected to Imprecise Stochastic Loading

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Abstract: This paper presents a highly efficient approach for bounding the responses and probability of failure of nonlinear models subjected to imprecisely defined stochastic Gaussian loads. Typically, such computations involve solving a nested double loop problem, where the propagation of the aleatory uncertainty has to be performed for each realization of the epistemic parameters. Apart from near-trivial cases, such computation is generally intractable without resorting to surrogate modeling schemes, especially in the context of performing nonlinear dynamical simulations. The recently introduced operator norm framework allows for breaking this double loop by determining those values of the epistemic uncertain parameters that produce bounds on the probability of failure a priori. However, the method is in its current form only applicable to linear models due to the adopted assumptions in the derivation of the involved operator norms. In this paper, the operator norm framework is extended and generalized

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by resorting to the statistical linearization methodology, to account for nonlinear systems. Two case studies are included to demonstrate the validity and efficiency of the proposed approach.

Keywords: Uncertainty quantification; Imprecise probabilities; Operator norm theorem; Statistical linearization

7.1 Introduction

Uncertainties about the true properties of, and loads acting on, structural systems are commonly encountered in the context of all fields of engineering, including structural dynamics. For instance, natural phenomena such as earthquakes or wind loads are especially hard to model exactly, since the corresponding dynamical loads acting on the system often cannot be described in a crisp way due to the sheer complexity of the underlying phenomena. Further, when designing structures with natural or highly engineered materials, such uncertainties may arise as well. To treat these issues effectively, stochastic processes (Shinozuka and Sato 1967, Vanmarcke and Grigoriu 1983) have been introduced as a rigorous framework to account for the aleatory uncertainties and corresponding correlations in space and time of uncertain loads and properties. This is obtained by resorting to the well-documented framework of probability theory, which is highly suited to treat aleatory uncertainties.

However, the definition of such stochastic processes may require prohibitive amounts of informative data to fully characterize the probabilistic descriptors, including the auto-correlation function. In a practical engineering context, such information may not always be available due to scarcity, incompleteness or even conflicted nature of typically available data sources. As a potential remedy, one can model the additional (epistemic) uncertainty by means of subjective probability density functions, which might be a valid approach in case sufficient reasons are present to validate the considered assumptions. However, in general, this includes unwarranted subjectivity in the analysis, which might give a wrong sense of reliability to the model. Alternatively, set theoretical approaches, such as intervals (Faes and Moens 2019) or fuzzy numbers (Beer 2004), can be used to include the epistemic uncertainty. By imposing such set-theoretical descriptors on top of probabilistic models for the uncertainty, a full set of probabilistic models that is consistent with the lack of knowledge is considered, which allows for an objective judgement on the bounds of the system reliability. In this context, utilizing the concept of imprecise probabilities (Beer et al. 2013) provides the analyst with a concrete theoretical framework to define and compute (with such hybrid forms) the uncertainties. In structural dynamics, for instance, given a set of stochastic processes that are consistent with the epistemic uncertainty, an imprecise probabilities-based solution treatment leads to bounds on the first excursion probability. The latter not only allows to assess the sensitivity of the model reliability to the existing epistemic uncertainty, but also yields an estimate of the lower bound of the reliability.

In engineering practice, however, the effective application of such methods is typically hindered by the corresponding computational cost. In essence, the propagation of the epistemic and aleatory uncertainty has to be performed such that their effects on the reliability are kept separated (Moens and Vandepitte 2004). This gives rise to double loop approaches, where the outer loop takes care of epistemic uncertainty while the inner loop deals with aleatory uncertainty. Many efficient methods have been introduced in recent years to alleviate this computational cost; see, indicatively, Faes et al., 2021a for a recent review paper. Examples of such approaches are based on Extended Monte Carlo simulation (Wei et al. 2019), surrogate modeling schemes (Schöbi and Sudret 2017), Bayesian probabilistic propagation (Wei et al. 2021) or Line Sampling (de Angelis et al. 2015). A recent development in this context is based on operator norm theory to decouple the double loop into a deterministic optimization, followed by a single reliability analysis per bound on the reliability (Faes et al. 2020; 2021), which is capable of reducing the corresponding computational cost by several orders of magnitude. However, the methods based on operator norm theory are limited to linear systems subject to Gaussian loading, which renders their application to realistic engineering models impossible.

In this regard, directing attention to extending the operator norm framework to nonlinear dynamical systems subject to imprecise Gaussian loading, a new technique is developed herein for computing moderate to large failure probabilities. This is attained by resorting to the statistical linearization methodology (Roberts and Spanos 2003, Socha 2007), which is used for defining an equivalent linear system of equations to account for the nonlinear system under consideration. Then, an operator norm theory-based solution treatment (Faes et al. 2021) is employed to obtain the bounds on the probability of failure. Two pertinent numerical examples demonstrate the validity and efficiency of the proposed methodology.

7.2 Bounds on the reliability of nonlinear dynamical systems

7.2.1 Nonlinear stochastic dynamics

A nonlinear dynamical system subjected to a stochastic load $p(t, \boldsymbol{\xi})$ is represented using the Finite Element representation of the dynamical equation, by the following set of ordinary differential equations:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) + \boldsymbol{\Phi}\left(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)\right) = \boldsymbol{\rho} p(t, \boldsymbol{\xi}), \tag{7.1}$$

where $\mathbf{M} \in \mathbb{R}^{n_d \times n_d}$, $\mathbf{C} \in \mathbb{R}^{n_d \times n_d}$ and $\mathbf{K} \in \mathbb{R}^{n_d \times n_d}$ represent, respectively, the mass, damping and stiffness matrices of the system, and n_d denotes the degrees of freedom in the model. Further, $\boldsymbol{\xi}$ represents a realization of a random variable vector, whereas the vector $\boldsymbol{\rho} \in \mathbb{R}^{n_d \times 1}$ links the stochastic load $p(t, \boldsymbol{\xi})$ to the appropriate degrees of freedom in the structure. The vectors $\mathbf{q} \in \mathbb{R}^{n_d}$, $\dot{\mathbf{q}} \in \mathbb{R}^{n_d}$ and $\ddot{\mathbf{q}} \in \mathbb{R}^{n_d}$ represent, respectively, the nodal displacements, velocities and accelerations, where a dot over a variable denotes differentiation with respect to time $t \in \mathbb{R}$. Finally, $\boldsymbol{\Phi}\left(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)\right) \in \mathbb{R}^{n_d}$ represents the nonlinear restoring force, which depends on the nodal displacement, velocity and acceleration vectors.

In Eq. (7.1), $p(t, \boldsymbol{\xi})$ represents the load to which the system is subjected, which in the context of stochastic dynamical systems is usually modeled as a stochastic process. If $p(t, \boldsymbol{\xi})$ is a stationary zero-mean Gaussian process, it can be characterized using its power spectral density function $S_{PP}(\omega)$, where $\omega \in \mathbb{R}$ denotes the circular frequency. The Wiener-Khintchine theorem allows for the calculation of the autocorrelation function corresponding to $S_{PP}(\omega)$, and vice versa. This is attained by utilizing the Fourier transforms:

$$S_{PP}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{PP}(\tau)e^{-\mathrm{i}\omega\tau}d\tau, \;\; R_{PP}(\tau) = \int_{-\infty}^{+\infty} S_{PP}(\omega)e^{\mathrm{i}\omega\tau}d\omega, \tag{7.2}$$

where $R_{PP}(\tau)$ denotes the autocorrelation function with time lag $\tau \in \mathbb{R}$ and 'i' is the imaginary unit. Sample paths of this stochastic process can be generated, for example, by applying the Karhunen-Loève (KL) expansion (e.g., Schenk and Schuëller 2005, Stefanou 2009). In this regard, assume that the loading is applied for time T, where $t_k = (k-1)\Delta t,\; k = 1, 2, \ldots, n_T$, corresponds to time discretization with step Δt and n_T denotes the number of discrete time steps. Then, the associated discrete correlation matrix $\mathbf{R_{PP}} \in \mathbb{R}^{n_T \times n_T}$ becomes:

$$\mathbf{R_{PP}} = \begin{bmatrix} R_{PP}(0) & R_{PP}(t_1 - t_2) & \ldots & R_{PP}(t_1 - t_{n_T}) \\ R_{PP}(t_2 - t_1) & R_{PP}(0) & \ldots & R_{PP}(t_2 - t_{n_T}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{PP}(t_{n_T} - t_1) & R_{PP}(t_{n_T} - t_2) & \ldots & R_{PP}(0) \end{bmatrix}. \tag{7.3}$$

Note that the framework described above can be also extended to account for non-stationary Gaussian processes, see e.g. Li and Chen, 2009a. Utilizing the matrix-vector form of the KL expansion, i.e.:

$$\mathbf{p}(\boldsymbol{\xi}) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\xi}, \tag{7.4}$$

sample paths compatible with the stochastic ground acceleration are generated. In Eq. (7.4), $\mathbf{p}$ denotes an n_T-dimensional vector containing the sample of the loading; $\boldsymbol{\xi}$ is a realization of the random variable vector Ξ, which follows an n_{KL}-dimensional standard Gaussian distribution, where n_{KL} corresponds to the number of terms retained in the KL expansion; $\boldsymbol{\Psi} \in \mathbb{R}^{n_T \times n_{KL}}$ is a matrix whose columns contain the eigenvectors associated with the largest n_{KL} eigenvalues of the discrete covariance matrix $\mathbf{R_{PP}}$; and $\boldsymbol{\Lambda} \in \mathbb{R}^{n_{KL} \times n_{KL}}$ denotes a diagonal matrix whose elements contain the largest n_{KL} eigenvalues of $\mathbf{R_{PP}}$. A criterion for selecting the number of terms to be retained in the KL expansion is to find the minimum value of n_{KL}, such that $\sum_{p=1}^{n_{KL}} \lambda_p \geq p_v \sum_{p=1}^{n_T} \lambda_p$, where p_v denotes the fraction of the total variance of the underlying stochastic process that is retained by the approximate representation, and λ_p is the p-th eigenvalue of $\mathbf{R_{PP}}$ (Lee and Verleysen 2007). For a recent overview of numerical methods to solve the associated Fredholm integral eigenvalue problem in a continuous case, the reader is directed to Betz et al., 2014. Alternatively, the sample paths can also be generated using frequency domain methods, such as described in Chen and Li, 2013.

In a structural engineering context, one is usually interested in finding the reliability of the structure, which is related to its performance by means of Eq. (7.1). Practically, the structural reliability can be quantified by its complement, i.e., the failure probability P_f. In this context, failure is encoded in the performance function $g(\boldsymbol{\xi})$, i.e., $g(\boldsymbol{\xi}) \leq 0$ indicates that the realization of values $\boldsymbol{\xi}$ leads to a structural failure. The probability of failure is calculated by solving the integral equation:

$$P_f = \int_{\boldsymbol{\xi} \in \mathbb{R}^{n_{KL}}} I_F(\boldsymbol{\xi}) f_{\boldsymbol{\Xi}}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{7.5}$$

where $f_{\boldsymbol{\Xi}}(\cdot)$ is a standard n_{KL}-dimensional Gaussian probability density function and $I_F(\cdot)$ is the indicator function, whose value is equal to one in case $g(\boldsymbol{\xi}) \leq 0$ and zero otherwise. Note, in passing, that the exact formulation of $g(\boldsymbol{\xi})$ is highly case dependent. For instance, when considering the first-passage problem, which is a classical problem in stochastic dynamics (e.g., Spanos and Kougioumtzoglou 2014, Spanos et al. 2016), $g(\boldsymbol{\xi})$ is given by:

$$g(\boldsymbol{\xi}) = 1 - \max_{i=1,\ldots,n_\eta} \left(\max_{k=1,\ldots,n_T} \left(\frac{|\eta_i(t_k, \boldsymbol{\xi})|}{b_i} \right) \right). \tag{7.6}$$

where $\eta_i(t_k, \boldsymbol{\xi}),\ i = 1, 2, \ldots, n_\eta$, indicates the i-th response of the system at time instant t_k (e.g., q_i or one of its time derivatives), $|\cdot|$ denotes the absolute value and b_i is a predefined threshold value above which a structural failure occurs (e.g., a maximally allowed displacement).

The integral in Eq. (7.5) usually comprises a high number of dimensions, as n_{KL} may be in the order of hundreds or thousands for realistic stochastic processes. Furthermore, $g(\boldsymbol{\xi})$, and hence, $I_F(\boldsymbol{\xi})$ is only known point-wise for realizations $\boldsymbol{\xi}$ of $\boldsymbol{\Xi}$. Therefore, such an integral cannot be solved analytically. In general, simulation methods should be applied to evaluate P_f (Schuëller and Pradlwarter 2007). However, using simulation methods to calculate the probability of failure of a non-linear dynamical system can become quite challenging (Pradlwarter et al. 2007). For instance, the definition of appropriate importance sampling density functions to be used within the context of Importance Sampling might not always be trivial in this case (Au 2009). Moreover, it is highlighted that the nonlinear restoring force $\boldsymbol{\Phi}(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t))$ in Eq. (7.1) hinders the determination of $\eta_i(t_k), i = 1, 2, \ldots, n_\eta, k = 1, 2, \ldots, n_T$, since its presence necessitates the employment of pertinent numerical algorithms (Chopra 1995). In particular, combining simulation algorithms with these nonlinear solvers potentially leads to solution frameworks of prohibitively high computational cost.

7.2.2 Imprecise stochastic dynamical analysis

The characterization of the stochastic process $p(t, \boldsymbol{\xi})$ in Eq. (7.1) in terms of its power spectral density, or autocorrelation function, usually relies on a prescribed model. This, in turn, depends on a number of parameters, which are grouped in a vector $\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}$. In this case, the parameters that determine the covariance matrix $\mathbf{R_{PP}}(\tau|\boldsymbol{\theta})$ reflect some specific characteristics of the process, such as dominant frequencies, amplitude, etc. When selecting the appropriate value of these quantities, the analyst may be faced with considerable uncertainty, such as lack of knowledge, vague or ambiguous information, etc., which leads to epistemic uncertainty concerning the correct parameter value. Therefore, instead of selecting a crisp value, it is often preferred to explicitly account for this epistemic uncertainty by resorting to non-traditional models for uncertainty quantification (Beer et al. 2013).

In this regard, it is herein assumed that the epistemic uncertainty in the definition of $\boldsymbol{\theta}$ can be bounded by an interval, i.e., $\boldsymbol{\theta} \in \boldsymbol{\theta}^I = [\underline{\boldsymbol{\theta}}, \overline{\boldsymbol{\theta}}]$, where $\underline{\boldsymbol{\theta}}$ and $\overline{\boldsymbol{\theta}}$ denote, respectively, the lower and upper bound between which the *true* parameter value is believed to lie. Techniques to infer these bounds based on limited data have been reported; see, indicatively, Imholz et al., 2020.

Taking these uncertainties explicitly into account, Eq. (7.1) becomes:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) + \boldsymbol{\Phi}\left(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)\right) = \boldsymbol{\rho} p(t, \boldsymbol{\xi}, \boldsymbol{\theta}^I). \tag{7.7}$$

Close inspection of Eq. (7.7) reveals that both interval and random variables are present. The fact that the input parameters of the stochastic loading model are described by means of intervals has important implications on the evaluation of the structural reliability of the model under consideration. In particular, both loading and the structural system responses become interval stochastic processes (Faes and Moens 2019). This, in turn, leads to an interval valued performance function, which causes the failure probability to become interval valued as well. Therefore, instead of calculating a single probability of failure associated with the structure (using Eq. (7.5)), given the epistemic uncertainty represented by $\boldsymbol{\theta}^I$, one has to estimate the bounds on P_f. These bounds are calculated by solving the optimization problems:

$$\underline{P}_f = \min_{\boldsymbol{\theta} \in \boldsymbol{\theta}^I} \left(P_f(\boldsymbol{\theta})\right) = \min_{\boldsymbol{\theta} \in \boldsymbol{\theta}^I} \left(\int_{\boldsymbol{\xi} \in \mathbb{R}^{nKL}} I_F\left(\boldsymbol{\xi}, \boldsymbol{\theta}\right) f_{\boldsymbol{\Xi}}\left(\boldsymbol{\xi}\right) d\boldsymbol{\xi} \right), \tag{7.8}$$

$$\overline{P}_f = \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}^I} \left(P_f(\boldsymbol{\theta})\right) = \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}^I} \left(\int_{\boldsymbol{\xi} \in \mathbb{R}^{nKL}} I_F\left(\boldsymbol{\xi}, \boldsymbol{\theta}\right) f_{\boldsymbol{\Xi}}\left(\boldsymbol{\xi}\right) d\boldsymbol{\xi} \right). \tag{7.9}$$

In general, the solution of the optimization problems defined in Eqs. (7.8) and (7.9) is extremely demanding from a computational perspective. Specifically, as pointed out earlier, the solution of the reliability problem for nonlinear dynamical systems is rather cumbersome. In addition, solving the corresponding optimization problems is not straightforward, since this constitutes a double loop problem, where the inner loop comprises probability calculation, while the outer loop explores the possible values of the parameters $\boldsymbol{\theta}$. Hence, besides considering near-trivial simulation models, such computation is generally intractable without resorting to surrogate modelling strategies.

7.3 Operator norm theory as a tool to decouple the double loop

A highly efficient operator norm theory-based approach to decouple the double loop associated with the solution of Eqs. (7.8) and (7.9) has already been developed by some of the authors of the present paper (Faes et al. 2021; 2020). In this section, a concise presentation of the results in Faes et al., 2021b, 2020 is provided for completeness. Then, directing attention to computing the bounds on the probability of failure of the nonlinear system given by Eq. (7.7), a novel methodology is proposed, which is based on the combination of the statistical linearization method (Roberts and Spanos 2003) with the theoretical framework described above.

7.3.1 Linear problems

The operator norm method introduced in Faes et al., 2021b, 2020, specifically focuses on models whose relation between the response $\boldsymbol{\eta}$ and the uncertain inputs $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ is given by:

$$\boldsymbol{\eta}(\boldsymbol{\theta},\boldsymbol{\xi}) = \mathbf{A}\mathbf{B}(\boldsymbol{\theta})\boldsymbol{\xi}. \tag{7.10}$$

In Eq. (7.10), $\mathbf{A} : \mathbb{R}^{n_t} \mapsto \mathbb{R}^{n_\eta}$ denotes a continuous linear map that represents the translation of the model input to the responses of interest, whereas $\mathbf{B} : \mathbb{R}^{n_{KL}} \mapsto \mathbb{R}^{n_t}$ is a linear map that transforms the random vector $\boldsymbol{\xi}$ to the sample paths of the stochastic process which serves as model input. For instance, using the KL series expansion, $\mathbf{B}$ is given in its discrete form as:

$$\mathbf{B} = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{1/2}, \tag{7.11}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ are the matrices which contain, respectively, the eigenvectors and eigenvalues of the matrix $\mathbf{R}_{\mathbf{PP}}$ (see also section "Bounds on the reliability of nonlinear dynamical systems"). Note that eq. (7.10) allows modeling the dynamic response of linear structural systems comprising classical or non-proportional damping subject to dynamic loading. Details about the numerical formulation of eq. (7.10) can be found in, e.g., Chopra, 1995; Jensen and Valdebenito, 2007.

Considering the linear map defined in Eq. (7.10) and also defining $\mathbf{D}(\boldsymbol{\theta}) = \mathbf{A}\mathbf{B}(\boldsymbol{\theta})$ for simplicity, it can be shown that the inequality:

$$\|\mathbf{D}(\boldsymbol{\theta})\boldsymbol{\xi}\|_{p_1} \leq |c|\|\boldsymbol{\xi}\|_{p_2}, \tag{7.12}$$

with $\|\cdot\|_p$ denoting a certain L_p norm, always holds. In essence, this equation states that the length of the uncertain model input $\boldsymbol{\xi}$, quantified via a prescribed L_{p_i} norm, can be amplified at most by a factor c towards the model responses $\boldsymbol{\eta}$ when applying the linear mapping defined by $\mathbf{D}(\boldsymbol{\theta})$. A measure for *how much* a certain deterministic linear map $\mathbf{D}(\boldsymbol{\theta})$ increases the length of the uncertain model input $\mathbf{v}$ in the maximum case, is given by the operator norm $\|\mathbf{D}(\boldsymbol{\theta})\|_{p_1,p_2}$, which is defined in a deterministic sense (i.e., for one realization of the uncertain parameters) as:

$$\|\mathbf{D}(\boldsymbol{\theta})\|_{p_1,p_2} = \inf\left\{c \geq 0 : ||\mathbf{D}(\boldsymbol{\theta})\mathbf{v}||_{p_1} \leq |c| \cdot \|\mathbf{v}\|_{p_2}, \forall \mathbf{v} \in \mathbb{R}^{n_v}\right\}, \tag{7.13}$$

or, equivalently:

$$\|\mathbf{D}(\boldsymbol{\theta})\|_{p_1,p_2} = \sup\left\{\frac{\|\mathbf{D}(\boldsymbol{\theta})\mathbf{v}\|_{p_1}}{\|\mathbf{v}\|_{p_2}} : \mathbf{v} \in \mathbb{R}^{n_v} \text{ with } \mathbf{v} \neq 0\right\}. \tag{7.14}$$

Clearly, the calculation of a specific value $\|\mathbf{D}(\boldsymbol{\theta})\|_{p_1,p_2}$ depends on the choice of p_1 and p_2. The interested reader is directed to Faes et al., 2021b, 2020 for an analytical presentation of the method and for guidance on the optimal selection of p_1 and p_2; and to Faes and Valdebenito, 2020; Faes and Valdebenito, 2021 for a practical application of the framework in the context of reliability-based design optimization.

In case of calculating first excursion probabilities, taking into account Eq. (7.6), experience shows that selecting $p_1 \to \infty$ and $p_2 = 2$ provides a good correlation between the operator norm $\|\mathbf{D}(\boldsymbol{\theta})\|_{\infty,2}$ and the failure probability P_f. This happens since the operator norm $\|\mathbf{D}(\boldsymbol{\theta})\|_{\infty,2}$ describes the amount of 'energy' amplification in the random signal towards the 'extremes' of the responses $\boldsymbol{\eta}_i$, and hence, its corresponding effect on P_f. Thus, it is readily seen that finding those values of the epistemic uncertain parameters $\boldsymbol{\theta}$ that minimize and maximize, respectively, $\|\mathbf{D}(\boldsymbol{\theta})\|_{\infty,2}$ will provide a good approximation of the realizations that minimize and maximize P_f. Hence, the double loop that is presented in Eqs. (7.8) and (7.9) can be efficiently decoupled, first, by determining $\boldsymbol{\theta}^U$ via:

$$\boldsymbol{\theta}^U = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\operatorname{argmax}}\|\mathbf{D}(\boldsymbol{\theta})\|_{\infty,2} \tag{7.15}$$

to find the parameters that yield $\overline{P}_f$, and then, by determining $\boldsymbol{\theta}^L$ via:

$$\boldsymbol{\theta}^L = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\operatorname{argmin}}\|\mathbf{D}(\boldsymbol{\theta})\|_{\infty,2} \tag{7.16}$$

to find the parameters that yield $\underline{P}_f$. Next, the bounds on P_f, i.e., $\underline{P}_f$ and $\overline{P}_f$, are obtained by solving Eq. (7.5) twice, corresponding to $\boldsymbol{\theta}^U$ and $\boldsymbol{\theta}^L$. It is noted that any pertinent optimization solver can be employed to solve Eqs. (7.15) and (7.16). Further, it is readily seen that recasting the problem in the form given by Eq. (7.10) is critical for the application of the method. In essence, this means that the underlying model must be linear, and that the aleatory uncertainty can only be present in the load description (Faes et al. 2021). This feature of the method hinders its direct application to nonlinear systems defined by Eq. (7.7). Nevertheless, this limitation is addressed in the following by resorting to the statistical linearization method, i.e., by defining an equivalent linear system for the nonlinear system of Eq. (7.7).

7.3.2 Statistical linearization methodology

In this section, a concise presentation of the statistical linearization methodology is provided for completeness. The main objective of the method is to replace the originally considered nonlinear system with an equivalent linear one and minimize (in some sense) the difference between the two systems. Clearly, the readily available solution frameworks for treating the equivalent linear system are used to estimate the stochastic response of its nonlinear counterpart. In general, several variations of the method have been used to solve approximately and efficiently nonlinear stochastic differential equations associated with engineering applications; see, indicatively, Fragkoulis et al., 2016a; Kougioumtzoglou et al., 2017; Fragkoulis et al., 2019; Spanos and Malara, 2020; Pasparakis et al., 2021; Ni et al., 2021 and references therein. Its extensive utilization in stochastic dynamics is associated with its capacity to treat a wide range of nonlinear behaviors in a straightforward manner.

The statistical linearization method is invoked herein to obtain an equivalent linear system that is compatible with the operator norm framework. For the application of the method, the nonlinear system in Eq. (7.1) is replaced by an equivalent linear system of the form:

$$(\mathbf{M} + \mathbf{M}_e)\,\ddot{\mathbf{q}}(t) + (\mathbf{C} + \mathbf{C}_e)\,\dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{K}_e)\,\mathbf{q}(t) = \boldsymbol{\rho} p(t, \boldsymbol{\xi}). \tag{7.17}$$

In Eq. (7.17), $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{K}_e$ denote, respectively, the mass, damping and stiffness $n_d \times n_d$ matrices of the equivalent linear system that account for neglecting the nonlinearity from Eq. (7.1) . Next, the error $\boldsymbol{\varepsilon} \in \mathbb{R}^{n_d}$ is defined as the difference between Eqs. (7.1) and (7.17), i.e.:

$$\boldsymbol{\varepsilon} = \boldsymbol{\Phi}\left(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)\right) - \mathbf{M}_e\ddot{\mathbf{q}}(t) - \mathbf{C}_e\dot{\mathbf{q}}(t) - \mathbf{K}_e\mathbf{q}(t), \tag{7.18}$$

and its mean square is minimized. Note that although several criteria are available for minimizing ε (e.g., Socha 2007, Elishakoff and Andriamasy 2012), adopting a mean square error minimization in conjunction with the Gaussian assumption for the system response probability density functions (Roberts and Spanos 2003) facilitates the determination of the equivalent linear system in Eq. (7.17). Specifically, the elements of matrices $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{K}_e$ are given in closed form by:

$$m_{ij}^e = \mathbb{E}\left[\frac{\partial \boldsymbol{\Phi}_i}{\partial \ddot{q}_j}\right], \quad c_{ij}^e = \mathbb{E}\left[\frac{\partial \boldsymbol{\Phi}_i}{\partial \dot{q}_j}\right], \quad k_{ij}^e = \mathbb{E}\left[\frac{\partial \boldsymbol{\Phi}_i}{\partial q_j}\right], \tag{7.19}$$

where $\mathbb{E}[\cdot]$ is the expectation operator and the indices $i, j = 1, 2, \ldots, n_d$ denote the corresponding element of the $n_d \times n_d$ matrices and n_d-dimensional vectors.

Next, note that the equivalent linear system response variance is also required to compute the elements of the equivalent matrices given by Eq. (7.19). This is attained by employing either a time- or a frequency-domain solution framework (Roberts and Spanos 2003, Fragkoulis et al. 2016, Kougioumtzoglou et al. 2017). For instance, following the latter, the system response variance is determined by resorting to the input-output relationship of random vibration theory:

$$\mathbf{S}_{\mathbf{qq}}(\omega) = \boldsymbol{\alpha}(\omega)\mathbf{S}_{\mathbf{PP}}(\omega)\boldsymbol{\alpha}^{\mathrm{T}*}(\omega), \tag{7.20}$$

where $\mathbf{S}_{\mathbf{qq}}(\omega)$ and $\mathbf{S}_{\mathbf{PP}}(\omega)$ denote, respectively, the response and excitation power spectrum, and '$\mathrm{T}*$' corresponds to the conjugate transpose matrix operator. Further, $\boldsymbol{\alpha}(\omega)$ is the frequency response function matrix of the equivalent system in Eq. (7.17), i.e.:

$$\boldsymbol{\alpha}(\omega) = \left[-\omega^2(\mathbf{M}+\mathbf{M}_e) + \mathrm{i}\omega(\mathbf{C}+\mathbf{C}_e) + (\mathbf{K}+\mathbf{K}_e)\right]^{-1}, \tag{7.21}$$

Thus, taking into account Eqs. (7.20) and (7.21), the system response variance is determined by:

$$\mathbb{E}\left[q_i^2(t)\right] = \int_{-\infty}^{\infty} S_{q_iq_i}(\omega)\mathrm{d}\omega,\ \mathbb{E}\left[\dot{q}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 S_{q_iq_i}(\omega)\mathrm{d}\omega,\ \mathbb{E}\left[\ddot{q}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^4 S_{q_iq_i}(\omega)\mathrm{d}\omega, \tag{7.22}$$

where $S_{q_iq_i}(\omega)$, $i = 1, 2, \ldots, n_d$, are the diagonal elements of the system response spectrum $\mathbf{S}_{\mathbf{qq}}(\omega)$. Clearly, Eq. (7.19) and Eq. (7.22) define a coupled set of nonlinear equations to be solved for determining $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{K}_e$. For its solution, the following iterative scheme is used. First, the equivalent parameter matrices in Eq. (7.17) are set equal to null matrices. Then, initial values for the response variance are computed by Eq. (7.22). Next, the latter are used in conjunction with Eq. (7.19) to update the values for $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{K}_e$. The last two steps are repeated until convergence.

Finally, it is noted that since the linearization is performed in a mean-square error minimization sense, the approximation of the true system is generally less accurate in the tails of the distribution. Hence, the accuracy of the method tends to decrease when considering smaller failure probabilities. That is, using the equivalent linear system does not generally provide sufficiently accurate estimates for smaller failure probabilities. In this regard, in the proposed approach the equivalent linear system is only used for identifying the epistemic parameter values that yield the extrema of P_f. After these values have been identified, they are used to obtain the corresponding lower and upper bounds of P_f for the original nonlinear system by means of direct Monte Carlo simulation. Nonetheless, as it is shown in the numerical examples section, the proposed framework provides practical advantages in the sense that the failure probability bounds can be computed with significantly greater numerical efficiency.

7.3.3 Solution of the equivalent linear system

Clearly, Eq. (7.17) represents a linear structural system subject to stochastic Gaussian loading. However, it is noted that, depending on the form of nonlinearity $\mathbf{\Phi}\left(\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)\right)$ in Eq. (7.7), the parameter matrices of the equivalent system in Eq. (7.17) are no longer necessarily symmetric. Nevertheless, this poses no difficulty in applying the proposed methodology. In general, new approaches have been recently developed for treating linear and nonlinear multi-degree-of-freedom systems which lack mathematically appealing properties, such as symmetry and positive definiteness; see, indicatively, Fragkoulis et al., 2016b,a. Further, note that matrix $\mathbf{C} + \mathbf{C}_e$ represents a 'full' damping matrix. Therefore, commonly applied solution schemes based on convolution, as described in Chopra, 1995 cannot be applied directly.

In this regard, Eq. (7.17) is recast into a state-space form (Chopra 1995; Jensen and Valdebenito 2007):

$$\mathbf{M}^{*}\dot{\mathbf{q}}^{*}(t) + \mathbf{K}^{*}\mathbf{q}^{*}(t) = \mathbf{P}^{*}(t, \boldsymbol{\xi}), \tag{7.23}$$

where $\mathbf{M}^* \in \mathbb{R}^{2n_d \times 2n_d}$, $\mathbf{K}^* \in \mathbb{R}^{2n_d \times 2n_d}$ and $\mathbf{P}^* \in \mathbb{R}^{2n_d \times 1}$ are block matrices given by:

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{0} & \mathbf{M}+\mathbf{M}_e \\ \mathbf{M}+\mathbf{M}_e & \mathbf{C}+\mathbf{C}_e \end{bmatrix}, \quad \mathbf{K}^* = \begin{bmatrix} -(\mathbf{M}+\mathbf{M}_e) & \mathbf{0} \\ \mathbf{0} & \mathbf{K}+\mathbf{K}_e \end{bmatrix}, \quad \mathbf{P}^* = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\rho} p(t,\boldsymbol{\xi}) \end{bmatrix}, \tag{7.24}$$

and $\mathbf{q}^*(t)$ denotes the $2n_d$-dimensional vector

$$\mathbf{q}^*(t) = \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}. \tag{7.25}$$

The impulse response function $h_i(t)$ corresponding to the system in Eq. (7.23) is defined as:

$$h_i(t) = \sum_{r=1}^{2n_d} \frac{\boldsymbol{\beta}_i^T \boldsymbol{\Phi}_r \boldsymbol{\Upsilon}_{pr}^T \boldsymbol{\rho}}{(2\lambda_r T_r + S_r)} e^{\lambda_r t}, \tag{7.26}$$

where $i = 1, 2, \ldots, n_r$ denotes the number of responses, and $\boldsymbol{\beta}_i$ is a constant vector such that a response of interest η_i is generated as $\eta_i = \boldsymbol{\beta}_i^T \boldsymbol{q}$. Variables T_r and S_r are the modal energies given by:

$$T_r = \boldsymbol{\Upsilon}_{pr}^T (\mathbf{M}+\mathbf{M}_e) \boldsymbol{\Phi}_{pr}, \quad S_r = \boldsymbol{\Upsilon}_{pr}^T (\mathbf{C}+\mathbf{C}_e) \boldsymbol{\Phi}_{pr}, \tag{7.27}$$

where $\boldsymbol{\Upsilon}_{pr}$ and $\boldsymbol{\Phi}_{pr}$ are, respectively, the position parts (i.e., the last n_d components) of the right and left eigenvectors, associated with the right and left eigenproblems of Eq. (7.23); λ_r contains the corresponding eigenvalues.

The dynamic responses $\boldsymbol{\eta}_i, i = 1, 2, \ldots, n_\eta$, that solve Eq. (7.17) are calculated by applying the convolution integral between the corresponding unit impulse response functions $h_i(t)$, $i = 1, 2, \ldots, n_\eta$, and the stochastic loading $p(t, \boldsymbol{\xi})$, i.e.:

$$\eta_i(t, \boldsymbol{\xi}) = \int_0^t h_i(t-\tau)\, p(t, \boldsymbol{\xi}) d\tau, \; i = 1, 2, \ldots, n_\eta. \tag{7.28}$$

In view of the excitation model introduced in Eq. (7.4), evaluating Eq. (7.28) at time t_k yields:

$$\eta_i(t_k, \boldsymbol{\xi}) = \sum_{l_1=1}^{k} \Delta t \epsilon_{l_1} h_i(t_k - t_{l_1}) \left(\sum_{l_2=1}^{n_{KL}} \psi_{l_1,l_2} \sqrt{\lambda_{l_2}} \xi_{l_2} \right) = \boldsymbol{\gamma}_{i,k} \boldsymbol{\xi}, \tag{7.29}$$

for $i = 1, 2, \ldots, n_\eta,\ k = 1, 2, \ldots, n_T$, where ψ_{l_1,l_2} is the (l_1, l_2)-th element of matrix $\boldsymbol{\Psi}$; $\boldsymbol{\gamma}_{i,k}$ is a n_{KL}-dimensional vector such that:

$$\boldsymbol{\gamma}_{i,k} = \begin{bmatrix} \sum_{l_1=1}^{k} \Delta t \epsilon_{l_1} h_i(t_k - t_{l_1}) \psi_{l_1,1} \sqrt{\lambda_1} & \ldots & \sum_{l_1=1}^{k} \Delta t \epsilon_{l_1} h_i(t_k - t_{l_1}) \psi_{l_1,n_{KL}} \sqrt{\lambda_{n_{KL}}} \end{bmatrix} \tag{7.30}$$

and ϵ_{l_1} is a coefficient depending on the numerical integration scheme used in the evaluation of the convolution integral. When the trapezoidal integration rule is chosen (Gautschi 2012), $\epsilon_{l_1} = 1/2$, if $l_1 = 1$ or $l_1 = k$; otherwise, $\epsilon_{l_1} = 1$. As such, $\boldsymbol{\eta}_i$ is calculated as a linear transformation that maps the standard normal random vector $\boldsymbol{\xi}$ to the responses $\boldsymbol{\eta}_i$ for each time instant:

$$\boldsymbol{\eta}_i(\boldsymbol{\xi}) = \boldsymbol{\Gamma}_i(\boldsymbol{\theta}) \boldsymbol{\xi}, \tag{7.31}$$

where:

$$\boldsymbol{\eta}_i(\boldsymbol{\xi}) = \begin{bmatrix} \eta_i(t_1, \boldsymbol{\xi}) \\ \eta_i(t_2, \boldsymbol{\xi}) \\ \vdots \\ \eta_i(t_{n_T}, \boldsymbol{\xi}) \end{bmatrix}, \ \boldsymbol{\Gamma}_i(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\gamma}_{i,1}(\boldsymbol{\theta}) \\ \boldsymbol{\gamma}_{i,2}(\boldsymbol{\theta}) \\ \vdots \\ \boldsymbol{\gamma}_{i,n_T}(\boldsymbol{\theta}) \end{bmatrix}, \tag{7.32}$$

in which $\boldsymbol{\Gamma}_i(\boldsymbol{\theta})$ is a $n_T \times n_{KL}$ matrix that represents a linear map from the standard normal random vector $\boldsymbol{\xi}$ to the i-th response of interest. Note that $\boldsymbol{\Gamma}_i(\boldsymbol{\theta})$ depends directly on the epistemic uncertain parameters $\boldsymbol{\theta}$ through the eigenvalues and eigenvectors of the KL series expansion.

7.3.4 Bounds on the first excursion probability

As explained in section "Linear problems", the operator norm theorem can be used to bound the probability of failure of linear models under epistemic uncertainty in the definition of the load.

To extend the method towards treating nonlinear dynamical simulation models, a framework based on the combination of the operator norm-based treatment and the statistical linearization methodology is proposed. Hereto, the linearized system of Eq. (7.31) is considered. Specifically, the epistemic uncertain parameters of the imprecisely defined stochastic load that bound P_f are defined as:

$$\boldsymbol{\theta}^U = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\text{argmax}} \max_{i=1,2,\ldots,n_\eta} \|\boldsymbol{\Gamma}_i(\boldsymbol{\theta})\|_{\infty,2} \tag{7.33}$$

and:

$$\boldsymbol{\theta}^L = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\text{argmin}} \max_{i=1,2,\ldots,n_\eta} \|\boldsymbol{\Gamma}_i(\boldsymbol{\theta})\|_{\infty,2}, \tag{7.34}$$

with $\boldsymbol{\Gamma}_i$ as defined in Eq. (7.32). These parameter realizations are used for finding the parameters that yield $\overline{P}_f$ and $\underline{P}_f$, respectively. Note that the explicit dependence of $\boldsymbol{\Gamma}_i$ on $\boldsymbol{\theta}$ is highlighted in these equations. The parameters $\boldsymbol{\theta}$ influence $\boldsymbol{\Gamma}_i$ through the eigenfunctions and corresponding eigenvalues of the KL expansion shown in Eq. (7.4) and the interaction with the structural nonlinearities. Based on the derivations in Tropp, 2004, Eqs. (7.33) and (7.34) are recast into:

$$\boldsymbol{\theta}^U = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\text{argmax}} \max_{i=1,2,\ldots,n_\eta} \max_{j=1,2,\ldots,n_T} \|\boldsymbol{\Gamma}_i^{j:}(\boldsymbol{\theta})\|_2 \tag{7.35}$$

and:

$$\boldsymbol{\theta}^L = \underset{\boldsymbol{\theta}\in\boldsymbol{\theta}^I}{\text{argmin}} \max_{i=1,2,\ldots,n_\eta} \max_{j=1,2,\ldots,n_T} \|\boldsymbol{\Gamma}_i^{j:}(\boldsymbol{\theta})\|_2, \tag{7.36}$$

respectively, where the superscript 'j :' denotes the j-th row of matrix $\boldsymbol{\Gamma}_i$ and $\|\cdot\|_2$ denotes the regular L_2 vector norm.

To summarize, the proposed procedure can be described as follows:

1. Represent the nonlinear model including the epistemic uncertainty by using Eq. (7.7).
2. Solve the optimization problems in Eqs. (7.35) and (7.36) to identify $\boldsymbol{\theta}^U$ and $\boldsymbol{\theta}^L$, by using any appropriate algorithm. Then, compute matrix $\boldsymbol{\Gamma}(\boldsymbol{\theta})$ for a given realization $\boldsymbol{\theta}$. This is done in two steps. First, applying the statistical linearization method, solve iteratively Eqs. (7.19)-(7.22). Secondly, taking into account Eqs. (7.24)-(7.32), perform

modal analysis over the equivalent linear system to derive matrix $\boldsymbol{\Gamma}(\boldsymbol{\theta})$.

3. Once $\boldsymbol{\theta}^U$ and $\boldsymbol{\theta}^L$ are identified, perform reliability analysis using the full nonlinear model in order to determine the upper and lower bounds of the failure probability.

7.4 Numerical examples

7.4.1 Case study 1: two-degrees-of-freedom nonlinear system

In this case study, the two-degrees-of-freedom (DOF) system in Fig. 7.1 is considered. The system consists of masses m_1 and m_2, which are connected to each other by a linear damper of damping coefficient c_2 and a linear spring of stiffness coefficient k_2. Further, mass m_1 connects to the foundation by a linear damper of damping coefficient c_1 and a nonlinear spring of stiffness coefficient k_1.

Figure 7.1: A two-degrees-of-freedom nonlinear system under stochastic excitation.

Next, considering the coordinates vector $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$ and following the standard Newtonian approach to derive the system governing equations of motion (Roberts and Spanos 2003), Eq. (7.1) is formulated. The system parameter matrices are given by:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \ \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \tag{7.37}$$

whereas:

$$\boldsymbol{\rho} p(t, \boldsymbol{\xi}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} p(t, \boldsymbol{\xi}) \tag{7.38}$$

denotes the stochastic excitation. Further, the nonlinear restoring force of the system is given by:

$$\boldsymbol{\Phi}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) = \begin{bmatrix} k_1 \nu q_1^3 \\ 0 \end{bmatrix}, \tag{7.39}$$

where ν corresponds to the intensity of the nonlinearity. Finally, the load $p(t, \boldsymbol{\xi})$ acting on the system is modeled as a zero-mean Gaussian stochastic process, described by the Clough-Penzien spectrum (Li and Chen 2009):

$$S_{PP}(\omega) = \frac{\omega^4 \left(\omega_g^4 + (2\zeta_g\omega_g\omega)^2\right) S_0}{\left((\omega_g^2 - \omega^2)^2 + (2\zeta_g\omega_g\omega)^2\right)\left((\omega_f^2 - \omega^2)^2 + (2\zeta_f\omega_f\omega)^2\right)}. \tag{7.40}$$

The following parameter values are considered for the system in Fig. 7.1, $m_1 = m_2 = 1$ [kg], $c_1 = c_2 = 0.2$ [N· s/m], $k_1 = k_2 = 1$ [N/m], whereas the intensity of the nonlinearity is $\nu = 1$ and the nominal parameters of the excitation spectrum are $[\omega_g, \omega_f, \zeta_g, \zeta_f, S_0] = [4\pi, 0.4\pi, 0.7, 0.7, 3 \times 10^{-4}]$. Failure of the system is considered as the first passage of any of the displacements of the masses over a threshold value of $b = 0.040$ [m]. Further, it is considered that the analyst is unsure about the exact values of the stochastic load acting on the system. Specifically, the definition of the parameters of the Clough-Penzien spectrum is subject to epistemic uncertainty. The intervals that are applied for bounding this epistemic uncertainty are shown in Table 7.1.

Next, the herein proposed operator norm theory-based statistical linearization framework is employed for computing the bounds on the probability of failure. In this regard, first, the governing equation of motion with parameter matrices and nonlinear vector given by Eqs. (7.37) and Eq. (7.39), respectively, is replaced by an equivalent linear system of the form of Eq. (7.17). Then, considering the error function in Eq. (7.18) and adopting a mean square minimization of the error, Eq. (7.19) leads to the equivalent parameter matrices:

$$\mathbf{M}_e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}_e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K}_e = \begin{bmatrix} 3k_1\nu\sigma_{q_1}^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.41}$$

Regarding the numerical implementation, considering as stopping criterion $\left|\frac{\mathbf{K}_e^{i+1} - \mathbf{K}_e^i}{\mathbf{K}_e^i}\right| < 10^{-5}$, where the index '$i$' denotes the i-th iteration and the initial value $\mathbf{K}_e^0$ is set equal to zero, the iterative scheme described in the section "Statistical linearization methodology" converges after three iterations. Thus, the nonlinear system shown in Fig. 7.1 is approximated by the equivalent linear system whose governing equations of motion are given by Eq. (7.17).

Next, the augmented state-space system in Eq. (7.23) is formulated and taking into account Eqs. (7.26)-(7.31), the linear map $\mathbf{\Gamma}_i(\boldsymbol{\theta})$ is calculated. Then, following the presentation in the section "Bounds on the first excursion probability", and considering the derived equivalent linear matrices, the operator norm that corresponds to any given realization of the epistemically uncertain Gaussian process load is computed. In addition, the optimization over the operator norm can be performed using the Matlab built-in patternsearch optimization tool. Finally, two optimization problems have to be solved; the first one for determining $\boldsymbol{\theta}^U$ (see Eq. (7.35)) and the second one for determining $\boldsymbol{\theta}^L$ (see Eq. (7.36)), which require approximately 100 iterations to converge.

So far, the operator norm-based statistical linearization framework is used for determining the bounds on P_f. Next, the validity of the obtained results is verified by using a brute-force implementation of the double-loop problem. Hereto, the Newmark solver is considered in conjunction with Monte Carlo simulation (MCS) as the 'inner loop' in Eqs. (7.8) and (7.9) for computing P_f for each realization of the epistemic uncertainty. It is noted that a total of 1000 samples are considered for estimating the failure probability at each realization of the epistemic parameters. A patternsearch optimization algorithm (Kolda et al. 2003) is used to solve the optimization problem in the 'outer loop'. This result serves as the benchmark for the bounds on P_f against which the result of the proposed operator norm-based statistical linearization framework is compared.

Results and discussion

The functional relationship between the operator norm $\|\mathbf{\Gamma}\|_{2,\infty}$, as computed over the linearized system, and P_f, as computed using MCS combined with the Newmark solver, is shown in Fig. 7.2. The black dots in this figure are obtained by drawing 1000 uniformly distributed samples in between the bounds of $\boldsymbol{\theta}^I$. First, it is noted that the relation between the operator norm $\|\mathbf{\Gamma}_i(\boldsymbol{\theta})\|_{\infty,2}$ and P_f is not bijective. In addition, there is a clear trend between these two quantities, where higher operator norm values correspond to higher probability of failure values and vice-versa. This illustrates the validity of the proposed approach in the sense that minimizing (or maximizing) the operator norm also yields a minimum (or maximum) of the failure probability. Further, Table 7.2 shows the parameters that yield an extremum in P_f by optimizing directly over P_f (indicated DL), as well as over the operator norm (indicated ON). These parameters are grouped in the rows indicated with $\boldsymbol{\theta}$. Furthermore, the corresponding optima are reported, as well as the number of required function calls (n^0). It is important to stress that to obtain a value for the operator norm, only the linear map $\boldsymbol{\Gamma}$ (see Eq. (7.31)) needs to be assembled and the corresponding operator norm needs to be calculated. On the other hand, the calculation of one value of P_f requires the full solution of Eq. (7.5).

Finally, in order to evaluate the performance of the proposed approach for different threshold levels, Fig. 7.3 presents the failure probability bounds obtained by the proposed method (denoted ON) and the reference bounds obtained by a direct double loop implementation (denoted DL) for different values of b. First, note that the failure probability values tend to decrease for higher threshold levels, as expected. In addition, it is seen that the lower bounds for the failure probability obtained by the proposed approach agree very well with the reference values for smaller threshold levels, i.e., $b \leq 0.040$ m. On the other hand, the deviations between the operator norm-based estimates for the lower bounds and the corresponding reference values tend to increase for larger values of b, which are associated with smaller failure probabilities. For instance, the proposed scheme overestimates the lower failure probability bound in 30% for the case $b = 0.050$ m. This illustrates that the proposed statistical linearization-based method is more suitable for problems involving moderate to large failure probabilities, as already pointed out. In this regard, the integration of the ON-based framework with alternative linearization techniques can (potentially) improve the performance of the proposed scheme for smaller failure probabilities.

Figure 7.2: Comparison of the operator norm, computed on the linearized system with the probability of failure as computed by Monte Carlo simulation in combination with Newmark method.

Figure 7.3: Failure probability bounds for different threshold levels obtained by the proposed method (ON) and a double loop implementation (DL).

7.4.2 Case study 2: six degrees-of-freedom structure

In this example, a 6-DOF system of rigid masses m_i $(i = 1, 2, \cdots, 6)$ connected to each other by nonlinear dampers as shown in Fig. 7.4 is considered. In this regard, considering the coordinates vector $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}$, the matrix form of the system governing equations of motion is formulated (see Eq. (7.1)), whose parameter matrices are given by:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ m_2 & m_2 & 0 & 0 & 0 & 0 \\ m_3 & m_3 & m_3 & 0 & 0 & 0 \\ m_4 & m_4 & m_4 & m_4 & 0 & 0 \\ m_5 & m_5 & m_5 & m_5 & m_5 & 0 \\ m_6 & m_6 & m_6 & m_6 & m_6 & m_6 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 & -c_2 & 0 & 0 & 0 & 0 \\ 0 & c_2 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & c_3 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & c_4 & -c_5 & 0 \\ 0 & 0 & 0 & 0 & c_5 & -c_6 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{bmatrix} \tag{7.42}$$

and:

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_2 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_4 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_5 & 0 \\ 0 & 0 & 0 & 0 & k_5 & -k_6 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}. \tag{7.43}$$

Further, it is assumed that the system is subjected to ground acceleration, which is modeled as a stochastic process, whose corresponding power spectrum is given by:

$$\mathbf{S}(\omega) = \begin{bmatrix} S_1(\omega) & 0 & 0 & 0 & 0 & 0 \\ 0 & S_2(\omega) & 0 & 0 & 0 & 0 \\ 0 & 0 & S_3(\omega) & 0 & 0 & 0 \\ 0 & 0 & 0 & S_4(\omega) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_5(\omega) & 0 \\ 0 & 0 & 0 & 0 & 0 & S_6(\omega) \end{bmatrix}, \tag{7.44}$$

where $S_i(\omega)$, $i = 1, 2, \ldots, 6$, is modeled as a Clough-Penzien spectrum (see Eq. (7.40)) with the epistemic uncertainty on the parameters ω_g, ω_f, ζ_g and ζ_f characterized by the intervals given in Table 7.1, whereas the epistemic uncertainty on parameter S_0 is characterized by the interval $[0.8, 1.2] \times 0.05$. In addition, the nonlinear function $\mathbf{\Phi}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q})$ takes the form:

$$\mathbf{\Phi}^{\mathrm{T}}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) = \\ \begin{bmatrix} c_1\nu\dot{q}_1^3 - c_2\nu\dot{q}_2^3 & c_2\nu\dot{q}_2^3 - c_3\nu\dot{q}_3^3 & c_3\nu\dot{q}_3^3 - c_4\nu\dot{q}_4^3 & c_4\nu\dot{q}_4^3 - c_5\nu\dot{q}_5^3 & c_5\nu\dot{q}_5^3 - c_6\nu\dot{q}_6^3 & c_6\nu\dot{q}_6^3 \end{bmatrix}, \tag{7.45}$$

with ν describing the intensity of the nonlinearity in Eq. (7.45). The system parameter values are $m_1 = m_2 \cdots = m_6 = 1$, $c_1 = c_2 \cdots = c_6 = 0.2$, $k_1 = k_2 \cdots = k_6 = 1$ and $\nu = 3$. In addition, failure is defined in this case as the first passage of any interstory drift beyond the maximum allowable threshold $b = 0.6$ m.

Figure 7.4: A six-degrees-of-freedom nonlinear system under stochastic excitation.

Then, the herein proposed operator norm theory-based statistical linearization framework is applied. In this regard, the equivalent linear mass and stiffness 6×6 matrices take the form:

$$\mathbf{M_e} = \mathbf{K_e} = \mathbf{0}, \tag{7.46}$$

whereas the equivalent linear damping 6×6 matrix becomes:

$$\mathbf{C_e} = \begin{bmatrix} 3c_1\nu\sigma^2_{\dot{q}1} & -3c_2\nu\sigma^2_{\dot{q}2} & 0 & 0 & 0 & 0 \\ 0 & 3c_2\nu\sigma^2_{\dot{q}2} & -3c_3\nu\sigma^2_{\dot{q}3} & 0 & 0 & 0 \\ 0 & 0 & 3c_3\nu\sigma^2_{\dot{q}3} & -3c_4\nu\sigma^2_{\dot{q}4} & 0 & 0 \\ 0 & 0 & 0 & 3c_4\nu\sigma^2_{\dot{q}4} & -3c_5\nu\sigma^2_{\dot{q}5} & 0 \\ 0 & 0 & 0 & 0 & 3c_5\nu\sigma^2_{\dot{q}5} & -3c_6\nu\sigma^2_{\dot{q}6} \\ 0 & 0 & 0 & 0 & 0 & 3c_6\nu\sigma^2_{\dot{q}6} \end{bmatrix}. \tag{7.47}$$

The elements of the equivalent matrix in Eq. (7.47) are determined by utilizing the iterative scheme described in the section 'Statistical linearization methodology'. Specifically, using $\left|\frac{\mathbf{C}_e^{i+1} - \mathbf{C}_e^i}{\mathbf{C}_e^i}\right| < 10^{-5}$ as stopping criterion, where 'i' denotes the i-th iteration of the scheme, and also considering the initial value $\mathbf{C}_e^0 = \mathbf{0}$, the scheme converges after five iterations. Thus, the nonlinear system shown in Fig. 7.4 is approximated by the equivalent linear system whose governing equations of motion are given by Eq. (7.17).

Next, the augmented state-space system in Eq. (7.23) is formulated and taking into account Eqs. (7.26)-(7.31), the linear map $\mathbf{\Gamma}_i(\boldsymbol{\theta})$ is calculated. Subsequently, following the presentation in the section "Bounds on the first excursion probability", and considering the derived equivalent linear matrices, the operator norm that corresponds to a certain realization of the epistemically uncertain Gaussian process load is computed. In addition, the optimization over the operator norm is performed using the Matlab built-in patternsearch optimization tool. Finally, two optimization problems have to be solved; the first one for determining $\boldsymbol{\theta}^U$ (see Eq. (7.35)) and the second one for determining $\boldsymbol{\theta}^L$ (see Eq. (7.36)), which require approximately 200 iterations to converge.

Results and discussion

The results of the herein proposed framework are shown in Table 7.3, which shows the parameters that yield an extremum in P_f by either optimizing directly over P_f (indicated DL) or over the operator norm (indicated ON). These parameters are grouped in the rows indicated with $\boldsymbol{\theta}$. Clearly, the proposed method is capable of adequately approximating the true bounds on P_f. The results are compared to a brute-force double loop implementation using Newmark method to solve the nonlinear ODE, MCS to calculate P_f, and patternsearch in Matlab to optimize over the epistemic parameter space. It is highlighted that the results obtained by following the proposed approach are in reasonable agreement with the corresponding results obtained by following a classic double loop approach. The small discrepancy between the results is expected and is due to adopting an approximate linearization scheme to enable the application of the operator norm framework. Nonetheless, it can be argued that these bounds are highly reasonable given the immense reduction in computational cost that is required to calculate them. For instance, considering the upper bound on P_f, the required number of deterministic model solutions can be reduced from 292000 to just 626, with 1000 additional samples for computing the associated failure probability.

7.5 Conclusions

In this paper, a novel technique has been developed for bounding the responses and probability of failure of nonlinear structural models subjected to imprecisely defined stochastic Gaussian loads. The proposed technique can be construed as a generalization of a recently developed operator norm-based method to account for nonlinear dynamical systems. This is attained by resorting to the statistical linearization methodology for defining a linear system equivalent to the nonlinear system under consideration. In this regard, the double loop that is typically associated with estimating the bounds on the probability of failure of nonlinear dynamical systems is effectively decoupled and the associated computational cost is reduced by several orders of magnitude. Thus, it can be argued that integrating statistical linearization into the operator norm framework allows for bounding the probability of failure of nonlinear systems with acceptable accuracy and at greatly reduced numerical cost. The validity and numerical efficiency of the proposed technique has been demonstrated by considering two nonlinear structural systems. It

Table 7.1: Tested values for $\boldsymbol{\theta}^I$.

ω_g^I	ω_f^I	ζ_g^I	ζ_f^I	S_0^I
$[0.8, 1.2] \times 4\pi$	$[0.8, 1.2] \times 0.4\pi$	$[0.8, 1.2] \times 0.7$	$[0.8, 1.2] \times 0.7$	$[0.8, 1.2] \times 3 \times 10^{-4}$

Table 7.2: Results of the optimization problems. Case study 1.

	parameter	$\underline{P}_f$ (DL)	$\underline{P}_f$ (ON)	$\overline{P}_f$ (DL)	$\overline{P}_f$ (ON)
	S_0^*	$2.409 \cdot 10^{-04}$	$2.409 \cdot 10^{-04}$	$3.534 \cdot 10^{-04}$	$3.591 \cdot 10^{-04}$
	ω_g^*	11.782	15.080	11.195	10.056
$\boldsymbol{\theta}$	ω_f^*	1.507	1.508	1.007	1.005
	ζ_g^*	0.700	0.840	0.575	0.840
	ζ_f^*	0.825	0.840	0.575	0.560
	P_f	0.084	0.088	0.977	0.974
Output	ON	0.0072	0.0069	0.0354	0.0375
	n^0	354000	$520 + 1000$	28900	$595 + 1000$

Table 7.3: Results of the optimization problems. Case study 2.

	parameter	$\underline{P}_f$ (DL)	$\underline{P}_f$ (ON)	$\overline{P}_f$ (DL)	$\overline{P}_f$ (ON)
	S_0^*	0.040	0.040	0.060	0.060
	ω_g^*	12.557	12.684	14.570	10.053
$\boldsymbol{\theta}$	ω_f^*	1.507	1.508	1.007	1.005
	ζ_g^*	0.809	0.840	0.700	0.560
	ζ_f^*	0.827	0.840	0.567	0.560
	P_f	0.097	0.123	0.859	0.855
Output	ON	0.081	0.079	0.307	0.319
	n^0	281000	$1804 + 1000$	292000	$626 + 1000$

is noted, however, that since the linearization scheme has been performed in a mean-square error minimization sense, the representation of the nonlinear system is less accurate in the tails of the distribution. This aspect renders the proposed approach mostly suitable for estimating the bounds of moderate to large failure probabilities. Nevertheless, future work is directed towards developing an enhanced operator norm-based linearization scheme capable of estimating bounds on smaller failure probabilities. This can be achieved, in principle, by combining the application of the statistical linearization methodology with a stochastic averaging treatment. Further, the proposed framework can be integrated with more advanced simulation methods, such as importance sampling or subset simulation. Another path for future work consists of extending the range of application of the proposed framework to more general models for stochastic loading (other than Gaussian). Finally, the evaluation of the proposed approach for more complex and numerically demanding structural models involving multiple types of nonlinearities constitutes an additional subject for future research.

Data availability statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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Chapter 8
Concluding remarks

In this chapter, the main conclusions are presented and discussed along with the pertinent remarks of the present thesis. Potential future research is also outlined.

This thesis aims to address several key challenges of response and reliability analysis for nonlinear dynamical systems in random vibration. Structural and mechanical systems commonly exhibit nonlinear behavior and operate in uncertain environments during their lifetime due to various factors, leading to the need of developing methods for nonlinear stochastic dynamical analysis. However, Efficient determining response statistics and estimating reliability estimates for nonlinear systems remain challenging, especially those with singular matrices or endowed with fractional derivative elements. Various applications of these complex nonlinear systems are met in engineering.

In this context, this thesis has made efforts to address the challenges of three main topics, including determining the response statistics of nonlinear systems with singular matrices, estimating the peak response of nonlinear systems with fractional derivative elements subject to stochastic code-compatible excitations, and bounding the first excursion probability of nonlinear systems under imprecise stochastic loading. The motivation and objectives of the thesis have been demonstrated in chapter 1.

In chapter 2 and 3, two techniques have been developed for determining the response statistics of nonlinear systems with singular matrices under simultaneous deterministic and stochastic loading. The first technique pertains to the case of combined deterministic and stochastic excitation of the stationary kind, while the other focuses on the case of combined deterministic and stochastic excitation of the non-stationary kind. These techniques are motivated by the presence of singular matrices in the governing equation of motion, resulting from redundant coordinates modeling for complex multi-body systems or due to system modeling using additional constraint equations. Further, the stochastic excitation component is modeled as a non-stationary process accounting for the inherent non-stationary characteristics of natural phenomena, such as wave, wind and earthquake.

These methods involve decoupling the system response into two components, namely a stochastic and a deterministic part, to account for both stochastic and deterministic excitations. For the case of combined deterministic and stationary stochastic excitation, a generalized harmonic balance method is employed to determine the deterministic response component, resulting in an over-determined system of equations to determine the coefficients. The use of generalized matrix inverse theory overcomes this challenge. Subsequently, the stochastic response component is derived by using the generalized statistical linearization methodology for systems with singular matrices, in conjunction with the averaging treatment.

For the case of combined deterministic and non-stationary stochastic excitation, two subsystems are first generalized for governing the deterministic and non-stationary stochastic response, separately. Next, the generalized statistical linearization method is developed to determine the time-dependent equivalent elements of the equivalent linear systems of the stochastic sub-systems with singular matrices. Then, the equivalent linear systems associated with the state space method formulate a matrix differential equation, which is solved together with the deterministic subsystems by numerical methods. The proposed techniques are demonstrated on nonlinear MDOF systems with redundant coordinates modeling. This is driven by the flexibility and cost-effectivity of system modeling with additional DOFs, especially for the complex systems with many DOFs. Finally, the proposed techniques are applied for determining the response of vibration energy harvester devices, which further illustrates the accuracy and efficiency of the techniques in the problem of systems modeling with additional constraint equations. The application of the energy harvester devices can be found in chapter 4.

The proposed techniques for nonlinear systems with singular matrices in this thesis are applied to the Duffing model for nonlinear systems, without this being restrictive for their application. Therefore, future research could focus on more complex and diverse nonlinear systems, such as bilinear or Bouc-Wen hysterestic nonlinear systems. In addition, the proposed techniques can also be extended to assess the reliability of the nonlinear systems with singular matrices, examining, for instance, the first passage probability determination problem.

Chapter 5 presents an approximate approach for estimating the peak response of nonlinear structural systems with fractional derivative elements subject to seismic excitations compatible with a given design spectrum. Specifically, the seismic excitation is modeled by an evolutionary power spectrum compatible with the design spectrum in a stochastic sense. Then, the time-dependent equivalent stiffness and damping elements are obtained by utilizing the combination of statistical linearization and stochastic averaging. Subsequently, the global minimum and maximum of the time-variant stiffness and damping elements are selected to estimate the peak response displacement in conjunction with the given design spectrum.

The proposed approach avoids the need for undertaking the nonlinear time history analysis. The use of evolutionary power spectrum compatible with design spectrum to represent the seismic excitation not only captures the non-stationary characteristics of earthquakes but also facilitates engineers in practical structural design. Further, the peak response is estimated by the global minimum and maximum of the time-variant stiffness and damping elements leading to a more accurate assessment than the time-invariant elements obtained by the linearization methods in the stationary sense. Moreover, nonlinear system modeling endowed with fractional derivative elements is considered in the framework, making its application for the viscoelastic material in vibration control. Thus, the framework takes into account the comprehensive and thorough understanding of controlled structural performance and aligns with the seismic design spectrum specified in current codes.

In Chapter 6, a stochastic incremental dynamical analysis method has been proposed for nonlinear systems with fractional derivative elements under code-compliant stochastic seismic excitations. Specifically, the proposed method generates an incremental dynamical analysis surface, which ensures a more reliable assessment of systems compared to the traditional incremental dynamical analysis curves. In addition, a significant novelty of the proposed method refers to the response evolutionary power spectrum function, which is aligned with the spectrum for different ground acceleration levels.

Further, potential future work can be found in the first-passage probability determination for nonlinear systems with fractional derivative elements subject to stochastic excitation compatible with a design spectrum. Future research work in performance-based earthquakes engineering can be also in the study of fragility analysis problems. In addition, these proposed code-compitable frameworks can also be extended to more complex systems.

In chapter 7 an operator norm-based statistical linearization technique is proposed to bound the responses and probability of failure of nonlinear structural models under imprecisely stochastic loading. Specifically, the statistical linearization method is utilized to define the equivalent linear systems to the original nonlinear systems. Then, the operator norm-based statistical linearization technique is proposed to decouple the double loop and bound the probability of failure of nonlinear dynamical systems. The proposed technique is much more efficient with significantly lower computational cost by several orders of magnitude compared to Monte Carlo simulation method. Two nonlinear structural systems are provided to demonstrate the validity and efficiency of the proposed technique.

However, the statistical linearization methodology may result in less accurate response statistics, especially in determining the tails of the response distribution, due to the adopted Gaussian response assumption. This renders its application mostly suitable for bounding large response failure probabilities. In this context, future research may focus on an enhanced operator norm-based linearization scheme that can be used for bounding smaller failure probabilities. This may be achieved by resorting to a combination of the statistical linearization and stochastic averaging methods. Further, the proposed framework can be enhanced by using more advanced simulation methods, such as importance sampling or subset simulation. In addition, due to the imprecise Gaussian loading assumption in the thesis, the proposed technique can also be extended for more general models for stochastic loading, such as non-Gaussian processes.

In summary, the first three developments pertains to response statistics determination of nonlinear systems with singular matrices subject to combined deterministic and stochastic excitations. The fourth and fifth proposed methods are developed for the peak response determination and incremental dynamic analysis of nonlinear systems with fractional derivative elements subject to code-compliant seismic stochastic excitations. And the last development deals with the first excursion probability of nonlinear systems subject to imprecise stochastic loading. These developments find applications in the nonlinear stochastic structural dynamical analysis of engineering systems. However, there are still numerous challenges that need to be addressed, particularly in the realm of stochastic dynamics for nonlinear systems. Further potential research in this area is warranted.

List of publications

In preparation

- Ni P., Jerez D. J., Fragkoulis V. C., Mitseas, I. P, Faes M. G. R., Valdebenito M. A., Beer M., Operator norm-based linearization framework to bound failure probability of nonlinear systems with fractional derivative terms subject to imprecise stationary (Gaussian) loads.
- Ni P., Mitseas, I. P., Fragkoulis V. C., Beer M., Stochastic incremental dynamics methodology for nonlinear structural systems endowed with fractional derivative terms subjected to code-compliant seismic excitation.

Peer-Reviewed International Journals

- Ni, P., Fragkoulis, V. C., Kong, F., Mitseas, I. P., Beer, M., 2023. Non-stationary response of nonlinear systems with singular parameter matrices subject to combined deterministic and stochastic excitation. Mechanical Systems and Signal Processing, 188, 110009.
- Kougioumtzoglou, I. A., Ni, P., Mitseas, I. P., Fragkoulis, V. C., Beer, M., 2022. An approximate stochastic dynamics approach for design spectrum based response analysis of nonlinear structural systems with fractional derivative elements. International Journal of Non-Linear Mechanics, 146, 104178.
- Ni P., Jerez D. J., Fragkoulis V. C., Faes M. G. R., Valdebenito M. A., Beer M., 2022. Operator Norm-based Statistical Linearization to Bound the First Excursion Probability of Nonlinear Structures Subjected to Imprecise Stochastic Loading, ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering. DOI: 10.1061/AJRUA6.0001217.
- Ni P., Fragkoulis V. C., Kong F., Mitseas I. P., Beer M., 2021. Response Determina-

tion of Nonlinear Systems with Singular Matrices Subject to Combined Stochastic and Deterministic Excitations, ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering, 7(4), 04021049: 1-11.

Peer-Reviewed Conference Proceedings (Papers)

- Ni P., Mitseas I.P., Fragkoulis V.C., Beer M., Stochastic incremental dynamics methodology for nonlinear structural systems endowed with fractional derivative terms subjected to code-compliant seismic excitation, In: Proceedings of the 14th International Conference on Application of Statistics and Probability in Civil Engineering (ICASP 14), Trinity college Dublin, Ireland, 9 - 13 July, 2023. (Accepted)
- Ni P., Jerez D.J., Fragkoulis V.C., Mitseas I. P., Faes M. G. R., Valdebenito M. A., Beer M., Probability of failure of nonlinear oscillators with fractional derivative elements subject to imprecise Gaussian loads, In: Proceedings of the XII International Conference on Structural (EURODYN 2023), Delft, The Netherlands, 02 - 15 July, 2023. (Accepted)
- Ni P., Fragkoulis V. C., Kong F., Mitseas I. P., Beer M., Response of an MDOF nonlinear system with constraints under combined deterministic and non-stationary stochastic excitation. In: Proceedings of the 8th International Symposium on Reliability Engineering and Risk Management (ISRERM), Hannover, Germany, 4-7 September 2022, Leibniz University Hannover, Hannover, Germany.
- Ni P., Fragkoulis V. C., Kong F., Mitseas I. P., Beer M. Response determination of a nonlinear energy harvesting device under combined stochastic and deterministic loads. In: Proceedings of the 13th International Conference On Structural Safety And Reliability (ICOSSAR 2022), 20-24 June, 2022, Tongji University, Shanghai, China.
- Mitseas I. P., Ni P., Fragkoulis V. C., Kong F., Beer M., Fragiadakis M. Stochastic nonlinear response of structural systems endowed with singular matrices subject to combined periodic and stochastic excitations. In: Proceedings of the 8th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering (COMPDYN 2021), Athens, Greece, 27-30 June, 2021.