On selected problems in multivariate analysis

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Abstract

Selected problems in the field of multivariate statistical analysis are treated. Thereby, one focus is on the paired sample case. Among other things, statistical testing problems of marginal homogeneity are under consideration. In detail, properties of Hotelling's T^2 test in a special parametric situation are obtained. Moreover, the nonparametric problem of marginal homogeneity is discussed on the basis of possibly incomplete data. In the bivariate data case, properties of the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic on the basis of partly not identically distributed data are investigated. Similar testing problems are treated within the scope of the application of a result for the empirical process of the concomitants for partly categorial data. Furthermore, testing changes in the modeled solvency capital requirement of an insurance company by means of a paired sample from an internal risk model is discussed. Beyond the paired sample case, a new asymptotic relative efficiency concept based on the expected volumes of multidimensional confidence regions is introduced. Besides, a new approach for the treatment of the multi-sample goodness-of-fit problem is presented. Finally, a consistent test for the treatment of the goodness-of-fit problem is developed for the background of huge or infinite dimensional data.

Zusammenfassung

Es werden ausgewählte Probleme aus dem Bereich der multivariaten statistischen Analyse behandelt. Ein Schwerpunkt stellt dabei die statistische Analyse von gepaarten Stichproben dar. Unter anderem werden hierbei statistische Testprobleme der marginalen Homogenität betrachtet. Konkret werden in einer speziellen parametrischen Situation Eingenschaften des Hotellingschen T^2-Tests erhalten. Darüber hinaus wird die Behandlung des nichtparametrischen Problems der marginalen Homogenität auf der Basis von möglicherweise unvollständigen Daten diskutiert. Im Fall von bivariaten Daten werden Eigenschaften der Hoeffding-Blum-Kiefer-Rosenblatt Teststatistik für das Testproblem der Unabhängigkeit auf Basis von teilweise nicht identisch verteilten Daten untersucht. Ähnliche Testprobleme werden im Rahmen der Anwendung eines Resultats für den empirischen Prozess der Konkomitanten von teilweise kategoriellen Daten behandelt. Weiter wird das Testen von Veränderungen in der modellierten Solvenzkapitalanforderung eines Versicherungsunternehmens anhand einer gepaarten Stichprobe aus einem internen Risikomodell diskutiert. Über den gepaarten Stichprobenfall hinaus wird ein neues Konzept für eine asymptotische relative Effizienz basierend auf den erwarteten Volumina von mehrdimensionalen Konfidenzbereichen vorgestellt. Außerdem wird ein neuer Ansatz zur Behandlung des Anpassungsproblems bei multiplen Stichproben präsentiert. Schließlich wird vor dem Hintergrund von hoch- bzw. unendlichdimensionalen Daten ein konsistenter Test zur Behandlung des Anpassungsproblems entwickelt.

Publications or accepted manuscripts submitted for the cumulative habilitation thesis

Baringhaus, L., Gaigall, D. (2017a). On Hotelling's T^2 test in a special paired sample case. *Communications in Statistics - Theory and Methods* 48, 1–11.

Ditzhaus, M., Gaigall, D. (2018). A consistent goodness-of-fit test for huge dimensional and functional data. *Journal of Nonparametric Statistics* 30, 834–859.

Gaigall, D. (2019). On a new approach to the multi-sample goodness-of-fit problem. *Communications in Statistics - Simulation and Computation*, 1–19.

Baringhaus, L., Gaigall, D. (2019). On an asymptotic relative efficiency concept based on expected volumes of confidence regions. *Statistics* 53, 1396–1436.

Gaigall, D. (2020a). Testing marginal homogeneity of a continuous bivariate distribution with possibly incomplete paired data. *Metrika* 83, 437–465.

Gaigall, D. (2020b). Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic on partly not identically distributed data. *Communications in Statistics - Theory and Methods*, 1–23.

Gaigall, D., Gerstenberg, J., Trinh, T.T.H. (2021). Empirical process of concomitants for partly categorial data and applications in statistics. *Accepted for publication in Bernoulli Journal.*

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Further Publications

Baringhaus, L., Gaigall, D. (2015). On an independence test approach to the goodness-of-fit problem. *Journal of Multivariate Analysis* 140, 193–208.

Baringhaus, L., Gaigall, D. (2017b). Hotelling's T^2 tests in paired and independent survey samples - an efficiency comparison. *Journal of Multivariate Analysis* 144, 177–198.

Baringhaus, L., Gaigall, D. (2018). Efficiency comparison of the Wilcoxon tests in paired and independent survey samples. *Metrika* 81, 891–930.

Baringhaus, L., Gaigall, D, Thiele, J.P. (2018). Statistical inference for L^2-distances to uniformity. *Computational Statistics & Data Analysis* 33, 1863–1896.

Gaigall, D. (2020c). Rothman-Woodroofe symmetry test statistic revisited. *Computational Statistics & Data Analysis* 142, 1–12.

Preprint

Ditzhaus, M., Gaigall, D. Testing marginal homogeneity in Hilbert spaces with applications to stock market returns. *Submitted to TEST (in Revision).*

Published thesis

Gaigall, D. (2016). Vergleich von statistischen Tests im verbundenen und unabhängigen Stichprobenfall. *Dissertation. Gottfried Wilhelm Leibniz Universität Hannover.*

Introduction

In multivariate statistical analysis the paired sample case is of particular interest. Classical examples for a paired sample are values for the systolic and diastolic blood pressure of patients, see Kossmann (1946), or the values of the head length and head breadth of the sons in different families, see Frets (1921). Other examples from more modern research are concentrations of potent broadly neutralizing antibodies, which can prevent a HIV infection at genital surfaces in males, from both inner and outer foreskin from male persons, see Fong et al. (2018), or bivariate data obtained by antidepressant clinical trials for the degree of depression, measured in terms of the Hamilton 17–item rating scale, before and after an intake of a drug or a placebo, see Detke et al. (2004) and Goldstein et al. (2004).

On Hotelling's T^2 test in a special paired sample case

In Baringhaus and Gaigall (2017a), we consider the paired sample case that $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are random $d \in \mathbb{N}$ column vectors with the property that the random $2d$ column vectors

$$\begin{bmatrix} X_j \\ \hdashline Y_j \end{bmatrix},\ j = 1, \dots, n,$$

are independent and identically distributed, where $n \in \mathbb{N}$ is a given sample size. One statistical problem of interest is the testing problem of marginal homogeneity, i.e., verifying whether or not the first and second marginal distributions coincide. We focus on the $2d$ dimensional multivariate normal model with complete block symmetric covariance matrix, see Perlman (1987), i.e., the underlying distribution is the multivariate normal distribution

$$N_{2d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} \Sigma & \Delta \\ \Delta & \Sigma \end{bmatrix}\right),$$

where the mean vectors $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^d$ of X_j and Y_j, the common covariance matrix Σ of X_j and Y_j as well as the cross covariance matrix Δ of X_j and Y_j are unknown, Δ is symmetric, and the covariance matrix

$$\Xi = \begin{bmatrix} \Sigma & \Delta \\ \Delta & \Sigma \end{bmatrix}$$

is positive definite. Here, the testing problem of marginal homogeneity reduces to the testing problem

$$H : a = b,\ K : a \neq b,$$

where H stands for the null hypothesis that the marginal distributions coincide and K represents the alternative that this is not the case. As a problem of a multivariate normal model with special symmetric covariance structure it belongs to the family of statistical problems called "amenable problems of classical multivariate analysis" (Perlman (1987)), i.e., problems marked in the way that explicit expressions for maximum likelihood estimators and likelihood ratio statistics, for example, are available. In fact, Arnold (1973) and also Olkin (1974) especially deals with the family of normal distributions with so-called complete block symmetric covariance matrix. The problem of testing that the mean vector is also complete block symmetric can be transformed to the product of a trivial problem and a certain multivariate analysis of variance testing problem. The likelihood ratio test of the latter is identified to be the likelihood ratio test for the hypothesis that the mean vector is complete block symmetric. For a specific dimension under consideration, the $2d$ dimensional multivariate normal model with complete block symmetric covariance matrix and the testing problem of marginal homogeneity obtain. Then, the likelihood ratio test is Hotelling's T^2 test based on the differences $W_j = X_j - Y_j$, $j = 1, \dots, n$, that is the test at significance level $\alpha \in (0, 1)$ witch rejects the null hypothesis if and only if

$$T_n^2 > \frac{d(n-1)}{n-d} F_{d,n-d;1-\alpha},$$

where $F_{d,n-d;1-\alpha}$ denotes the $1-\alpha$ quantile of the $F_{d,n-d}$ distribution, and

$$T_n^2 = (n-1)\overline{W}' S_{WW}^{-1} \overline{W},\ S_{WW} = \frac{1}{n}\sum_{j=1}^{n}(W_j - \overline{W})(W_j - \overline{W})'.$$

In fact, Hotelling's T^2 test based on the differences is also the uniformly most powerful invariant test (Arnold (1973), p.690). We give an elementary straightforward proof of this result, which is of special interest in practice. Without using distributional results on matrix transformed random normal vectors and Wishart matrices, and avoiding the general technical apparatus developed and described by Arnold, the proof essentially takes advantage of the fact that explicit expressions of the inverse and the positive definite square root of $2d \times 2d$ covariance matrices with complete block symmetry are available; indeed, the inverse and the positive definite square root are seen to be of complete block symmetry as well. The testing problem is of interest even if the assumption of complete block symmetry of the covariance matrix cannot be accepted. Then, the likelihood ratio test for the testing problem is not Hotelling's T^2 test based on the differences $X_j - Y_j$, an explicit expression of the likelihood ratio statistic does not exist, optimality results are not available. Thus, the gain ensued from the assumption of complete block symmetry of the covariance matrix is obvious. Needless to say that a check of this assumption is indispensable. The likelihood ratio test for testing the hypothesis of complete block symmetry is easily derived. As asserted by Perlman (1987), the likelihood ratio statistic is the ratio of the determinants of the maximum likelihood estimators of the covariance matrix in the hypothesis case and the general case.

Testing marginal homogeneity of a continuous bivariate distribution with possibly incomplete paired data

Inference on the basis of a paired sample is a classical statistical problem. For an overview over problems arising with bivariate data and well-known or rather new testing procedures, we refer to Gaigall (2020a). In Gaigall (2020a), we consider the full nonparametric testing problem of homogeneity of the marginal distributions on the basis of a possibly incomplete paired sample from a continuous bivariate distribution. To the best of the author's knowledge, this testing problem had not yet been studied rigorously in this general form, not even for a complete paired sample. We discuss a plausible testing criterion which reaches the significance level and is consistent as the sample size tends to infinity. A sample of size $n \in \mathbb{N}$ of independent and identically distributed bivariate random vectors

$$\begin{bmatrix} X_{1,1} \\ X_{2,1} \end{bmatrix}, \dots, \begin{bmatrix} X_{1,n} \\ X_{2,n} \end{bmatrix}$$

is considered. Denoting by F_1 and F_2 the first and second marginal distribution of the (unknown) underlying bivariate distribution $\boldsymbol{F}$, the testing problem

$$\mathcal{H} : F_1 = F_2,\ \mathcal{K} : F_1 \neq F_2$$

is treated. Clearly, tests for detecting a difference in the means of the first and second components of the bivariate random vectors are not suitable. Moreover, tests for verifying symmetry about zero of the differences of the first and the second components are not applicable in this general situation, and the same result applies to tests for verifying exchangeability of the first and the second components. For a simple example which illustrates this fact we refer to Gaigall (2020a). We consider the complicating case that some components in the paired sample are missing, where $n_1 \in \{1, \dots, n\}$ and $n_2 \in \{1, \dots, n\}$ are the numbers of data remaining in the first and second components. It is assumed that $\boldsymbol{F}$ is absolutely continuous with density $\boldsymbol{f}$, where $\boldsymbol{R} = \{(x, y) \in \mathbb{R}^2; \boldsymbol{f}(x, y) > 0\}$ is open as well as convex and $\boldsymbol{f}$ is continuous and bounded on $\boldsymbol{R}$. Furthermore, it is supposed that the data are missing completely at random such that

$$\lim_{n\to\infty} \frac{n_1}{n} = \rho_1 \in (0, 1],\ \lim_{n\to\infty} \frac{n_2}{n} = \rho_2 \in (0, 1].$$

Denoting by F_{1,n_1} and F_{2,n_2} the empirical distribution functions of the data remaining in the first and second components, the application of the two-sample Cramér-von-Mises distance, see Anderson (1962), yields the test statistic

$$T_{n_1,n_2} = \frac{n_1 n_2}{n_1 + n_2} \int \left(F_{1,n_1}(x) - F_{2,n_2}(x)\right)^2 \mathrm{d}\overline{F}_{n_1,n_2}(x),$$

where

$$\overline{F}_{n_1,n_2}(x) = \frac{n_1}{n_1 + n_2} F_{1,n_1}(x) + \frac{n_2}{n_1 + n_2} F_{2,n_2}(x),\ x \in \mathbb{R}.$$

Because of the dependencies in the data, T_{n_1,n_2} is not distribution free under $\mathcal{H}$ but the distribution depends on $\boldsymbol{F}$. Application of empirical processes theory, in particular under usage of results in Dudley (1984), Gänßler & Ziegler (1994), van der Vaart & Wellner (1996), and Ziegler (1997), yields the following result for the limit distribution of the test statistic under the null hypothesis.

Theorem 1. *Under $\mathcal{H}$, it holds*

$$T_{n_1,n_2} \xrightarrow{d} T \text{ as } n \to \infty,$$

where

$$T = \sum_{i=1}^{\infty} \lambda_i Z_i^2,$$

$Z_1, Z_2, \ldots$ is a sequence of independent standard normal distributed random variables, $\lambda_1, \lambda_2, \ldots$ are the non-negative eigenvalues of a certain integral operator, and $\lambda_i > 0$ for at least one $i \in \mathbb{N}$. In particular, the distribution function of T is continuous on $\mathbb{R}$ and strictly increasing on $[0, \infty)$.

For a proof we refer to Gaigall (2020a). To obtain critical values, we suggest a resampling procedure. By rearranging the incomplete observations we obtain three independent samples

$$\begin{bmatrix} X_{1,k_{1,1}} \\ X_{2,k_{1,1}} \end{bmatrix}, \ldots, \begin{bmatrix} X_{1,k_{1,m_1}} \\ X_{2,k_{1,m_1}} \end{bmatrix},\ X_{1,k_{2,1}}, \ldots, X_{1,k_{2,m_2}},\ X_{2,k_{3,1}}, \ldots, X_{2,k_{3,m_3}}$$

of sizes $m_1, m_2, m_3 \in \{0, \ldots, n\}$ of independent and identically distributed random variables, respectively, where $(k_{1,1}, \ldots, k_{1,m_1}, k_{2,1}, \ldots, k_{2,m_2}, k_{3,1}, \ldots, k_{3,m_3})$ is a suitable permutation of $(1, \ldots, n)$. Given that $\alpha \in (0, 1)$ is the significance level, sampling with replacement from the three samples and the application of a specific statistic yields a critical value $\widehat{C}_{m_1,m_2,m_3,1-\alpha}$. For details we refer to Gaigall (2020a). Finally, we obtain the following properties of the test.

Theorem 2.

a) Under $\mathcal{H} : F_1 = F_2$, the test asymptotically reaches the significance level

$$\lim_{n\to\infty} P(T_{n_1,n_2} > \widehat{C}_{m_1,m_2,m_3,1-\alpha}) = \alpha.$$

b) Under $\mathcal{K} : F_1 \neq F_2$, the test is consistent

$$\lim_{n\to\infty} P(T_{n_1,n_2} > \widehat{C}_{m_1,m_2,m_3,1-\alpha}) = 1.$$

c) Under suitable local alternatives $\mathcal{K}_n$, the test is asymptotically unbiased

$$\lim_{n\to\infty} P(T_{n_1,n_2} > \widehat{C}_{m_1,m_2,m_3,1-\alpha}) = \beta \geqslant \alpha.$$

For details and proofs we refer to Gaigall (2020a).

Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic on partly not identically distributed data

Another problem of interest in the bivariate data case is the question of independence. The paper Gaigall (2020b) deals with the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic, that is an established test statistic for the treatment of the nonparametric testing problem of independence. If the data are independent and identically distributed, standard regularity assumptions ensure that the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic is distribution-free under the null hypothesis of independence and converges in distribution to a real-valued random variable as the sample size tends to infinity, see Hoeffding (1948) and Blum, Kiefer, and Rosenblatt (1961). Recent studies for the classical Rothman-Woodroofe symmetry test demonstrate that the application of classical statistical procedures to not identically distributed data is possible in specific cases, see Gaigall (2020c). In fact, a bulk of works treat statistical problems on the basis of independent but not identically distributed data; for a list of literature with a focus on statistical tests, we refer to Gaigall (2020c). We apply the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic to partly not identically distributed data. Given that $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ let

$$(X_{i,j}, Y_{i,j}),\ j = 1, \ldots, n_i,\ i = 1, \ldots, k,$$

be $n = n_1 + \cdots + n_k$ independent bivariate random vectors with values in $\mathbb{R} \times \mathbb{R}$. Using F as a generic notation for the distribution (function), we assume

$$F_{(X_{i,j}, Y_{i,j})} = F_{(X,Y_i)}, \ j = 1, \ldots, n_i, \ i = 1, \ldots, k,$$

where in general

$$F_{Y_i} \neq F_{Y_j}, \ i, j = 1, \ldots, k, \ i \neq j.$$

Here, (X, Y_i), $i = 1, \ldots, k$, are bivariate random vectors with values in $\mathbb{R} \times \mathbb{R}$. We assume that the distribution functions $F_{(X,Y_i)}$, $i = 1, \ldots, k$, are uniformly continuous. The Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic on the partly not identically distributed data is

$$\mathrm{HBKR}_n = n \int \left(\widehat{F}_{(X,Y),n}(x, y) - \widehat{F}_{X,n}(x) \widehat{F}_{Y,n}(y) \right)^2 \mathrm{d}\widehat{F}_{(X,Y),n}(x, y),$$

with the empirical distribution functions

$$\widehat{F}_{(X,Y),n}(x, y) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \mathbb{I}(X_{i,j} \leqslant x, Y_{i,j} \leqslant y), \ (x, y) \in \mathbb{R}^2,$$

and

$$\widehat{F}_{X,n}(x) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \mathbb{I}(X_{i,j} \leqslant x), \ x \in \mathbb{R},$$

as well as

$$\widehat{F}_{Y,n}(y) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \mathbb{I}(Y_{i,j} \leqslant y), \ y \in \mathbb{R},$$

where $\mathbb{I}$ is the indicator function. Consider the null hypothesis of independence

$$\mathcal{H} : F_{(X,Y_i)} = F_X \otimes F_{Y_i}, \ i = 1, \ldots, k.$$

We investigate asymptotic properties of the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic. In this regard, we point out that n tends to infinity if some of the $n_1, \ldots, n_k$ tend to infinity or (possibly simultaneously) k tends to infinity. Throughout the paper, we suppose that $n \to \infty$ such that $n_i/n \to \rho_i$ for all i, where $\rho_i \in [0, 1]$ and $\sum_i \rho_i = 1$. Although the mentioned result for the distribution-freeness of the classical Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic in the finite sample case does not apply in this situation, we obtain that the statistic converges to the same distribution-free random variable as the classical criterion if the null hypothesis of independence is true. Defining the stochastic process $U_n = (U_n(x, y), (x, y) \in \overline{\mathbb{R}}^2)$ by

$$U_n(x, y) = \sqrt{n}\big(\widehat{F}_{(X,Y),n}(x, y) - \widehat{F}_{X,n}(x) \widehat{F}_{Y,n}(y)\big), \ (x, y) \in \overline{\mathbb{R}}^2,$$

the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic can be rewritten as

$$\mathrm{HBKR}_n = \int U_n(x, y)^2 \mathrm{d}\widehat{F}_{(X,Y),n}(x, y).$$

We will deduce the limiting null distribution of the test statistic from the convergence in distribution of the stochastic process U_n under the null hypothesis of independence and from the convergence of the distribution function $F_{(X,Y),n}$. The latter one is separately treated in Gaigall (2020b). We speak about convergence in distribution of stochastic processes in the sense of van der Vaart & Wellner (1996). Firstly, let us go to the stochastic process U_n. We obtain the following convergence in distribution.

Proposition 1. *Let the null hypothesis of independence $\mathcal{H}$ be true. Then, we have the convergence in distribution*

$$U_n \rightsquigarrow U,$$

where $U = (U(x, y), (x, y) \in \overline{\mathbb{R}}^2)$ is a centered Gaussian process with a.s. uniformly d_2-continuous sample paths and covariance function

$$u(x, y, r, s) = \big(F_X(x \wedge r) - F_X(x)F_X(r)\big)\big(F_Y(y \wedge s) - F_Y(y)F_Y(s)\big), \ (x, y), (r, s) \in \overline{\mathbb{R}}^2,$$

and Y denotes a real-valued random variable with distribution function

$$F_Y(y) = \sum_i \rho_i F_{Y_i}(y), \ y \in \overline{\mathbb{R}}.$$

For a proof we refer to Gaigall (2020b). We note the surprising observation that the stochastic process U has the structure of the well-known Brownian pillow, also called Wiener pillow, completely tucked Brownian sheet, or tied-down Kiefer process, see p. 137 in Piterbarg (1996), p. 368 in van der Vaart & Wellner (1996), p. 320 in Csörgő and Horváth (1997), or Koning and Protasov (2003), arising with the classical independence empirical process based on independent and identically distributed bivariate random vectors. Finally, we obtain the limiting distribution of the Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic under the null hypothesis of independence.

Theorem 3. *Let the null hypothesis of independence $\mathcal{H}$ be true. Then we have the convergence in distribution*

$$\mathrm{HBKR}_n \rightsquigarrow \mathrm{HBKR},$$

where the real-valued random variable HBKR *is given by*

$$\mathrm{HBKR} = \int\int U(x,r)^2 \mathrm{d}F_X(x)\mathrm{d}F_Y(y).$$

In particular, HBKR *is distribution-free, has a continuous and strictly increasing distribution function, and the characteristic function*

$$\varphi_{\mathrm{HBKR}}(t) = \prod_{j,\ell=1}^{\infty}\left(1 - \frac{2it}{\pi^4 j^2 \ell^2}\right)^{-\frac{1}{2}},\ t \in \mathbb{R}.$$

For a proof we refer to Gaigall (2020b). We make the interesting observation, that the characteristic function φ_{HBKR} is the well-known characteristic function of the limiting null distribution of the classical Hoeffding-Blum-Kiefer-Rosenblatt independence criterion if the approach based on independent and identically distributed bivariate random vectors with continuous marginal distributions, see Hoeffding (1948) and Blum, Kiefer, and Rosenblatt (1961). As an application, we consider a nonparametric random effects meta-regression model, which is very popular in meta-analysis. Here, the statistical problem of interest is testing goodness-of-fit for the regression function. The Hoeffding-Blum-Kiefer-Rosenblatt independence test statistic is applied to pairs of inputs and residual, which are partly not identically distributed. We use a quantile of the limit distribution of the classical criterion and obtain a test which reaches the significance level exactly as the number of data tends to infinity and is also consistent.

Empirical process of concomitants for partly categorial data and applications in statistics

Similar testing problems are treated within the scope of the application of a result for the empirical process of the concomitants for partly categorial data in Gaigall, Gerstenberg, and Trinh (2021). Given that $n \in \mathbb{N}$ is the sample size, let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent and identically distributed. We suppose that the random variable X_1 is categorial with values in a set Σ of cardinality $m \in \mathbb{N}$, $\Sigma = \{1, \ldots, m\}$, say, and that the probability that X_1 takes the value $1, \ldots, m$ is different from zero, respectively. In addition, we assume that the random variable Y takes values in $\mathbb{R}$ with continuous distribution function. It is supposed that the joint distribution of (X_1, Y_1) and the marginal distributions of X_1 and Y_1 are unknown. Concomitants (also called induced order statistic) appear when we sort the values of X-attributes according to real-valued Y-attributes. There is a large number of works dealing with concomitants. For a list of literature we refer to Gaigall, Gerstenberg, and Trinh (2021). Let us assume that $Y_1, \ldots, Y_n$ are pair-wise distinct without loss of generality. Let $R_{1:n}, \ldots, R_{n:n}$ be the ranks of $Y_1, \ldots, Y_n$,

$$R_{j:n} = \sum_{k=1}^{n} 1(Y_k \leqslant Y_j),\ j = 1, \ldots, n,$$

and denote by $R_{1:n}^{-1}, \ldots, R_{n:n}^{-1}$ the inverse ranks of $Y_1, \ldots, Y_n$ such that

$$Y_{R_{1:n}^{-1}} < \cdots < Y_{R_{n:n}^{-1}}.$$

Then, the random variables

$$X_{[j:n]} = X_{R_{j:n}^{-1}},\ j = 1, \ldots, n,$$

are called the concomitants, also called induced order statistic. Suppose we have only the concomitants $X_{[1:n]}, \dots, X_{[n:n]}$ as our observations. We define the empirical distribution function of $Y_1, \dots, Y_n$ by

$$\hat{F}_n(t) = \frac{1}{n}\sum_{j=1}^{n} 1(Y_j \leqslant t),\ t \in \mathbb{R},$$

and set

$$\hat{N}_n(i,l) = \sum_{j=1}^{l} 1(X_{[j:n]} = i),\ i \in \Sigma,\ l = 1, \dots, n.$$

The unknown joint distribution of X_1 and $F(Y_1)$ is determined by

$$\rho(i,t) = P(X_1 = i, F(Y_1) \leqslant t),\ (i,t) \in \Sigma \times [0,1].$$

The latter term can be consistently estimated only on the basis of the concomitants $X_{[1:n]}, \dots, X_{[n:n]}$ by

$$\begin{aligned}\frac{1}{n}\hat{N}_n(i, \lfloor nt \rfloor) &= \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} 1\Big(X_{[j:n]} = i\Big) = \frac{1}{n}\sum_{j=1}^{n} 1\Big(X_{[j:n]} = i, \frac{j}{n} \leqslant t\Big) \\ &= \frac{1}{n}\sum_{j=1}^{n} 1\Big(X_{R^{-1}_{j:n}} = i, \frac{j}{n} \leqslant t\Big) = \frac{1}{n}\sum_{j=1}^{n} 1\Big(X_j = i, \frac{R_{j:n}}{n} \leqslant t\Big) \\ &= \frac{1}{n}\sum_{j=1}^{n} 1(X_j = i, \hat{F}_n(Y_j) \leqslant t),\ (i,t) \in \Sigma \times [0,1].\end{aligned}$$

This estimator is strong uniformly consistent in $i \in \Sigma,\ t \in [0,1]$ meaning that

$$\sup_{i \in \Sigma, t \in [0,1]} \Big| \frac{\hat{N}_n(i, \lfloor nt \rfloor)}{n} - \rho(i,t) \Big| \longrightarrow 0 \text{ almost surely as } n \to \infty.$$

We define

$$G_n(i,t) = \sqrt{n}\Big(\frac{\hat{N}_n(i, \lfloor nt \rfloor)}{n} - \rho(i,t)\Big),\ (i,t) \in \Sigma \times [0,1],$$

and introduce the empirical process of the concomitants $G_n = (G_n(i,t))_{(i,t) \in \Sigma \times [0,1]}$ as the subject of our investigation. A functional Central Limit Theorem for the concomitants is under consideration in Theorem 24.3.1 in Davydov and Egorov (2001) in a very general setting. In particular, the concrete structure of the related limit processes are not transparent there. We obtain a functional central limit theorem for the empirical process of the concomitants G_n, where the argumentation in our proof is rather straightforward and use classical empirical process theory such as in Ziegler (1997) and knowledge about the well-known Bahadur-Kiefer process, see Bahadur (1966). We obtain the concrete structure of the limit process and we deal with a fairly general setting of triangular arrays of random variables and by that extend the result of Davydov and Egorov (2001). For the detailed formulation of the theorem and the related proof, we refer to Gaigall, Gerstenberg, and Trinh (2021). We consider the testing problem of independence

$$\begin{aligned}\mathcal{H}:\ &\forall\ (i,t) \in \Sigma \times \mathbb{R} : P(X_1 = i, Y_1 \leqslant t) = P(X_1 = i)P(Y_1 \leqslant t), \\ \mathcal{K}:\ &\exists\ (i,t) \in \Sigma \times \mathbb{R} : P(X_1 = i, Y_1 \leqslant t) \neq P(X_1 = i)P(Y_1 \leqslant t),\end{aligned}$$

as an example for the application of our main result, noticing that another application in the context of a two-sample homogeneity problem is given in Gaigall, Gerstenberg, and Trinh (2021). For literature about tests of independence for partly continuous and partly categorial data, see Gaigall, Gerstenberg, and Trinh (2021) for details. Defining

$$p_i = P(X_1 = i),\ i \in \Sigma,$$

it is clear that the testing problem is equivalent to

$$\mathcal{H}:\ \forall\ (i,t) \in \Sigma \times [0,1] : \rho(i,t) = tp_i,\ \mathcal{K}:\ \exists\ (i,t) \in \Sigma \times [0,1] : \rho(i,t) \neq tp_i.$$

Similar as above, we can estimate $p_1, \ldots, p_m$ consistently on the basis of the concomitants $X_{[1:n]}, \ldots, X_{[n:n]}$ by

$$\frac{1}{n}\hat{N}_n(i,n) = \frac{1}{n}\sum_{j=1}^{n} 1(X_{[j:n]} = i), \ i \in \Sigma.$$

It is easily seen that

$$\sup_{i\in\Sigma, t\in[0,1]} \left| \frac{\hat{N}_n(i,\lfloor nt \rfloor) - t\hat{N}_n(i,n)}{\hat{N}_n(i,n)} \right| \longrightarrow 0 \text{ almost surely as } n \to \infty$$

holds under the null hypothesis $\mathcal{H}$ as well. These deliberations motivate to use the test statistic

$$T_n = \int_0^1 \sum_{i=1}^{m} \hat{N}_n(i,n) \left(\frac{\hat{N}_n(i,\lfloor nt \rfloor) - t\hat{N}_n(i,n)}{\hat{N}_n(i,n)} \right)^2 dt.$$

Large values of T_n should be significant. In fact, by splitting the integration in parts of length $1/n$ we obtain the following simple expression of the test statistic

$$T_n = \frac{1}{6n} - \frac{1}{2} - \frac{n}{3} + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\hat{N}_n(i,j)^2}{n\hat{N}_n(i,n)},$$

useful for calculation purposes in practice. A simple consequence of our results is that under the null hypothesis of independence $\mathcal{H}$ it holds the following convergence in distribution

$$T_n \xrightarrow{d} T = \sum_{k=1}^{\infty} \frac{W_k}{k^2\pi^2},$$

where $W_k, k \in \mathbb{N}$, is a sequence of independent χ^2-distributed random variables with $m-1$ degrees of freedom. In particular, T is distribution free. For that reason, the related asymptotic test at significance level $\alpha \in (0,1)$, that is the test which rejects the null hypothesis $\mathcal{H}$ if and only if $T_n > c$, where c is the $(1-\alpha)$-quantile of T, is suitable for the treatment of the testing problem of independence. Under the null hypothesis $\mathcal{H}$, the test reaches the significance level exactly in the limit. Under any fixed alternative $\mathcal{K}$, the test is consistent, i.e.,

$$P(T_n > c) \to 1.$$

Based on the full observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ it is possible to translate our testing problem of independence to the multi-sample testing problem of homogeneity in Kiefer (1959) by grouping the Y-observations with respect to the X-observations. Among others, Kiefer (1959) treat a Cramér-von-Mises type test statistic. It is not obvious at first sight that the Cramér-von-Mises type test statistic introduced by Kiefer (1959) is measurable with respect to the concomitants. Local alternatives are of interest, e.g., for efficiency deliberations, in particular for the Pitman-efficiency, see Puri and Sen (1971), or the Volume-efficiency, see Baringhaus and Gaigall (2019). Under suitable local alternatives, see Gaigall, Gerstenberg, and Trinh (2021) for details and proofs, we obtain

$$T_n \xrightarrow{d} T = \sum_{k=1}^{\infty}\sum_{i=2}^{m} \left[\frac{Z_{k,i}}{k\pi} + c_{k,i} \right]^2,$$

where $Z_{k,i}$, $k \in \mathbb{N}$, $i = 2, \ldots, m$, are independent standard normal distributed random variables and $c_{k,i} \in \mathbb{R}$, $k \in \mathbb{N}$, $i = 2, \ldots, m$, are constants (depending on the local alternative), where the explicit expression is given in Gaigall, Gerstenberg, and Trinh (2021), and the test is asymptotically unbiased,

$$P(T_n > c) \to \beta \in [\alpha, 1].$$

Test for changes in the modeled solvency capital requirement in an internal risk model

Paired samples also occur in the context of the validation of an internal risk model of an insurance company in Gaigall (2021). Requests on the internal risk model are the modeling of a forecast distribution of the own funds of the insurance company and a related modeled solvency capital requirement based on the Value-at-Risk at level 99.5% of the modeled forecast distribution of the own funds, where the time horizon is one year. We consider two different model runs. The source of the internal risk model in model run $k = 1, 2$ is an input $(X_1^{(k)}, \ldots, X_d^{(k)})$, where $X_1^{(k)}, \ldots, X_d^{(k)}$ are $d \in \mathbb{N}$ risk factors, given by real-valued random variables. The joint distributions of these risk factors model the framework conditions for the insurance company with a forecast horizon of one year and are assumed to be known. The modeled forecast of the own funds of the insurance company in model run $k = 1, 2$ is obtained by the application of a company specific deterministic and measurable function $r^{(k)} : \mathbb{R}^d \to \mathbb{R}$, which models the current asset and liability portfolio of the corporate. For the reasons explained in Gaigall (2021), we deal with this function as unknown. The forecast of the own funds in model run $k = 1, 2$ is modeled by the real-valued random variable

$$Y^{(k)} = r^{(k)}(X_1^{(k)}, \ldots, X_d^{(k)}).$$

Reminding that the Value-at-Risk at level $(1 - \gamma) \in (0, 1)$ of a real-valued random variable Y is defined as

$$\mathrm{VaR}_{1-\gamma}(Y) = \inf\{x \in \mathbb{R}; P(Y \leqslant x) \geqslant 1 - \gamma\},$$

we suppose that the distributions $F^{(k)}$ of $Y^{(k)}$, $k = 1, 2$, are absolutely continuous, where the related densities $f^{(k)}$ satisfy $f^{(k)}(\mathrm{VaR}_\gamma(Y^{(k)})) > 0$. We define the modeled solvency capital requirement at level γ in model run $k = 1, 2$ as the Value-at-Risk at level $1 - \gamma$ of the difference of the (known) current own funds of the corporate $y^{(k)} \in \mathbb{R}$ and the modeled forecast of the own funds $Y^{(k)}$ in one year, that is

$$\mathrm{SCR}^{(k)} = \mathrm{VaR}_{1-\gamma}(y^{(k)} - Y^{(k)}) = y^{(k)} - \mathrm{VaR}_\gamma(Y^{(k)}).$$

For $\gamma = 0.5\%$ we obtain the modeled solvency capital requirement of the insurance company. We suppose that the modeled solvency capital requirement at level γ in model run $k = 1, 2$ does not vanish, and that the modeled solvency capital requirements at level γ for model runs 1 and 2 do not coincide. We are interested in the relative change of the modeled solvency capital requirement from model run 1 to model run 2, defined by

$$\Delta = \left| \frac{\mathrm{SCR}^{(2)} - \mathrm{SCR}^{(1)}}{\mathrm{SCR}^{(1)}} \right|.$$

We aim to check whether there is a significant change in the modeled solvency capital requirement in the sense that the relative change of the modeled solvency capital requirement from model run 1 to model run 2 exceeds a given level $\delta \in (0, \infty)$, that is the testing problem

$$H_0 : \Delta \geqslant \delta \text{ versus } H_1 : \Delta < \delta.$$

In practice, a Monte-Carlo procedure is customary to generate a model output. For validation purposes, an appropriate implementation of the Monte-Carlo simulation is suggested to obtain corresponding results for the empirical own funds for different model runs. As a result, we obtain the empirical own funds in model run $k = 1, 2$ by

$$Y_j^{(k)} = r^{(k)}(X_{1,j}^{(k)}, \ldots, X_{d,j}^{(k)}),$$

$j = 1, \ldots, n$, and we have that

$$(Y_1^{(1)}, Y_1^{(2)}), \ldots, (Y_n^{(1)}, Y_n^{(2)})$$

is a paired sample of independent bivariate random vectors, each with the same bivariate distribution function F, which will be used as data for the treatment of the testing problem. Because $r^{(k)}$ is unknown, $k = 1, 2$, the underlying bivariate distribution F is unknown as well. To circumvent technical problems, we suppose that F is uniformly continuous. We obtain the empirical solvency capital requirement at level γ of the own funds in model run $k = 1, 2$ by

$$\mathrm{SCR}_n^{(k)} = y^{(k)} - Y_{\lceil \gamma n \rceil : n}^{(k)},$$

where $Y^{(k)}_{\lceil \gamma n\rceil:n}$ is the sample quantile at level γ of the $Y^{(k)}_1, \dots, Y^{(k)}_n$. The relative change of the empirical solvency capital requirement from model run 1 to model run 2 is defined by

$$\Delta_n = \left| \frac{\mathrm{SCR}^{(2)}_n - \mathrm{SCR}^{(1)}_n}{\mathrm{SCR}^{(1)}_n} \right|.$$

We suggest the test statistic

$$T_n = \sqrt{n}(\Delta_n - \delta)$$

for the implementation of the test. In mathematical terms, we treat the testing problem whether or not the absolute value of the relative deviation between two quantiles exceeds a given level on the basis of a paired sample as data. There is a lot of literature about inference for two quantiles, where usually the two samples case is considered and two independent samples from the underlying distributions are used as data. For a list of references, we refer to Gaigall (2021). Our inference based on a paired sample as data. Because the paired sample case can be seen as a generalization of the two samples case if the sample sizes for both samples coincide, our approach can be regarded as an extension in this sense. We investigate asymptotic properties of the suggested test statistic, where we apply empirical process theory in Dudley (1984), van der Vaart & Wellner (1996), and Ziegler (1997), combined with the concept of Hadamard differentiability and the functional Delta method in van der Vaart (1998).

Theorem 4. *a) In the interior of the null hypothesis $\Delta > \delta$, the divergence of the test statistic holds almost surely*

$$T_n \longrightarrow +\infty \ \text{as}\ n \to \infty.$$

b) On the boundary of the null hypothesis $\Delta = \delta$, the convergence in distribution of the test statistic holds

$$T_n \xrightarrow{d} N \ \text{as}\ n \to \infty,$$

where N is a real-valued random variable with a centered normal distribution and variance

$$\begin{aligned}\sigma^2 =& \gamma(1-\gamma)\frac{(\mathrm{SCR}^{(1)})^2 f^{(1)}(\mathrm{VaR}_\gamma(Y^{(1)}))^2 + (\mathrm{SCR}^{(2)})^2 f^{(2)}(\mathrm{VaR}_\gamma(Y^{(2)}))^2}{f^{(1)}(\mathrm{VaR}_\gamma(Y^{(1)}))^2 f^{(2)}(\mathrm{VaR}_\gamma(Y^{(2)}))^2 (\mathrm{SCR}^{(1)})^4} \\ &+ 2(F(\mathrm{VaR}_\gamma(Y^{(1)}), \mathrm{VaR}_\gamma(Y^{(2)})) - \gamma^2)\frac{\mathrm{SCR}^{(1)}\mathrm{SCR}^{(2)}}{f^{(1)}(\mathrm{VaR}_\gamma(Y^{(1)}))f^{(2)}(\mathrm{VaR}_\gamma(Y^{(2)}))(\mathrm{SCR}^{(1)})^4}.\end{aligned}$$

c) Under the alternative hypothesis $\Delta < \delta$, the divergence of the test statistic holds almost surely

$$T_n \longrightarrow -\infty \ \text{as}\ n \to \infty.$$

For a proof we refer to Gaigall (2021). The results motivate a bootstrap procedure for the approximation of the standard deviation of the test statistic and finally for the determination of critical values. Using that the theory is available in a general setting of triangular arrays of random variables and a Glivenko-Cantelli result in Gänßler & Ziegler (1994), we obtain that the bootstrap procedure is suitable.

Dealing with given testing problems, statisticians usually try to pick up or develop statistical tests being most efficient in a certain sense. Thereby, as in finite sample cases such tests often do not exist, the main focus is on efficiency concepts, that enable the comparison of competing procedures by means of its specific asymptotic properties. The most familiar concepts in this respect are the concepts of Pitman, Bahadur and Hodges-Lehmann. For a systematic overview of these and other approaches we refer to the books Serfling (1980) and Nikitin (1985). In typical cases, the (local) asymptotic efficiencies of tests compared are seen to be ratios of (local) slopes of functions related to its power functions in different specific ways. Taking into account the correspondence principle, comparison of statistical tests can be also done via its associated confidence regions.

On an asymptotic relative efficiency concept based on expected volumes of confidence regions

Beyond the bivariate case, we introduce a new asymptotic relative efficiency concept based on the expected volumes of multidimensional confidence regions in Baringhaus and Gaigall (2019). The expected volume $\mathrm{E}_\vartheta(V_n)$ is a quality criterion for a confidence region B_n, see, e.g., Baringhaus and Gaigall (2017b) and Baringhaus and Gaigall (2018). Here the symbol ϑ can be seen as a generic notation for the parameter of the underlying distribution; we refer to Baringhaus and Gaigall (2019) for details and a precise formulation of the model under consideration. It is

$$N_{\gamma,\vartheta} = \inf\{n \in \mathbb{N}; \mathrm{E}_\vartheta(V_m) \leqslant \gamma \ \forall m \geqslant n\}$$

the smallest sample size from that $\mathrm{E}_\vartheta(V_n)$ does not exceed a given value $\gamma \in (0,1)$. We define the asymptotic relative volume efficiency of B' with respect to B as

$$\mathrm{arve}_\vartheta(B', B) = \lim_{\gamma \downarrow 0} \frac{N_{\gamma,\vartheta}}{N'_{\gamma,\vartheta}}.$$

if the limit exists. The crucial condition to obtain the asymptotic relative volume efficiency in applications is

$$\lim_{n\to\infty} n^r \mathrm{E}_\vartheta(V_n) = c_\vartheta \in (0,\infty)$$

for some $r \in (0,\infty)$. If this condition is satisfied for B and B' with the same r, we obtain the asymptotic relative volume efficiency by

$$\mathrm{arve}_\vartheta(B', B) = \left(\frac{c_\vartheta}{c'_\vartheta}\right)^{1/r}.$$

For details, see Baringhaus and Gaigall (2019). Given a dimension $d \in \mathbb{N}$, let us consider the case of a multivariate normal distribution $\mathcal{N}_d(\mu, \Sigma)$, $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ symmetric positive definite. On the basis of a sample from this distribution, a confidence ellipsoid for μ is obtained by

$$B_n = \left\{\eta \in \mathbb{R}^d; n(\overline{X}_n - \eta)^\top S_n^{-1}(\overline{X}_n - \eta) < \frac{nd}{n-d} F_{d,n-d;1-\alpha}\right\},$$

where $\overline{X}_n$ is the sample mean, S_n is the sample covariance matrix, and $F_{d,n-d;1-\alpha}$ denotes the quantile of order $(1-\alpha) \in (0,1)$ of the F distribution with d and $n-d$ degrees of freedom. In this situation, the expected volume of B_n satisfies the condition

$$\lim_{n\to\infty} n^{d/2}\mathrm{E}_\vartheta(V_n) = \frac{(\chi^2_{d;1-\alpha})^{d/2}\pi^{d/2}}{\Gamma(d/2+1)} \det \Sigma^{1/2}.$$

This follows from the more general results in Baringhaus and Gaigall (2019). As an example, we consider the common distributional model for 2×2 contingency tables is the 4-dimensional multinomial distribution family

$$\{\mathfrak{M}_4(1; p_{11}, p_{12}, p_{21}, p_{22}); p_{ij} \in (0,1), 1 \leqslant i,j \leqslant 2, p_{11}+p_{12}+p_{21}+p_{22} = 1\}.$$

We are interested in confidence regions for the parameter vector $\left(\frac{p_{11}}{p_{11}+p_{21}}, \frac{p_{22}}{p_{12}+p_{22}}\right)^\top$ that is of special interest in various studies in applied sciences; in fact, the fractions $\frac{p_{11}}{p_{11}+p_{21}}$ and $\frac{p_{22}}{p_{12}+p_{22}}$ are known there as the positive predictive value and the negative predictive value. We use the different alternative parameterisation

$$\{\mathfrak{M}_4(1; \xi_1\xi_3, (1-\xi_2)(1-\xi_3), (1-\xi_1)\xi_3, \xi_2(1-\xi_3)); \xi_i \in (0,1), i = 1,2,3\}$$

of the distributional model obtained by putting

$$\xi = (\xi_1, \xi_2, \xi_3)^\top = \left(\frac{p_{11}}{p_{11}+p_{21}}, \frac{p_{22}}{p_{12}+p_{22}}, p_{11}+p_{21}\right)^\top \in (0,1)^3.$$

Given $\xi = (\xi_1, \xi_2, \xi_3)^\top \in (0,1)^3$, let

$$X_j = (X_{j,11}, X_{j,12}, X_{j,21}, X_{j,22})^\top \sim \mathfrak{M}_4(1; \xi_1\xi_3, (1-\xi_2)(1-\xi_3), (1-\xi_1)\xi_3, \xi_2(1-\xi_3)).$$

The confidence regions to be derived for the subvector $(\xi_1, \xi_2)^\top$ are based on the sufficient statistic $N_n = (N_{n,11}, N_{n,12}, N_{n,21}, N_{n,22})^\top = \sum_{j=1}^{n} X_j$ that for given ξ has the multinomial distribution $\mathfrak{M}_4\left(n; \xi_1\xi_3, (1-\xi_2)(1-\xi_3), (1-\xi_1)\xi_3, \xi_2(1-\xi_3)\right)$. Setting

$$\widehat{\xi}_{n,1} = \frac{N_{n,11}}{N_{n,11} + N_{n,21}}, \quad \widehat{\xi}_{n,2} = \frac{N_{n,22}}{N_{n,12} + N_{n,22}}, \quad \widehat{\xi}_{n,3} = \frac{1}{n}\left(N_{n,11} + N_{n,21}\right),$$

It is easily seen that observations of $(\widehat{\xi}_{n,1}, \widehat{\xi}_{n,2}, \widehat{\xi}_{n,3})^\top$ in $(0,1)^3$ are maximum likelihood estimates of $\xi = (\xi_1, \xi_2, \xi_3)^\top$. For other observations, a redefinition of the estimators is appropriate, see Baringhaus and Gaigall (2019) for details. Motivated by the correspondence principle for confidence regions and statistical tests, we consider the confidence regions

$$\begin{aligned} B_n^{\mathrm{P}} &= \left\{\eta \in (0,1)^2; T_{\eta,n}^{\mathrm{P}} < \chi^2_{2;1-\alpha}\right\}, \\ B_n^{\mathrm{W}} &= \left\{\eta \in (0,1)^2; T_{\eta,n}^{\mathrm{W}} < \chi^2_{2;1-\alpha}\right\}, \\ B_n^{\mathrm{LR}} &= \left\{\eta \in (0,1)^2; T_{\eta,n}^{\mathrm{LR}} < \chi^2_{2;1-\alpha}\right\}, \end{aligned}$$

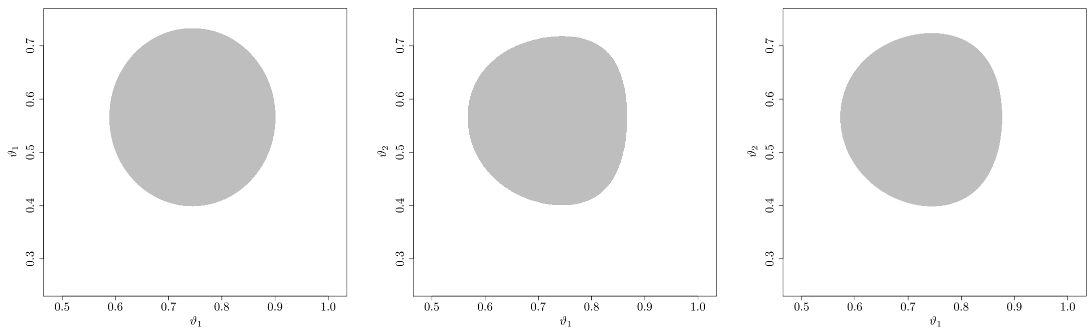
where

$$\begin{aligned} T_{\eta,n}^{\mathrm{P}} &= n\left((\widehat{\xi}_{n,1} - \eta_1)^2 \frac{\widehat{\xi}_{n,3}}{\widehat{\xi}_{n,1}(1-\widehat{\xi}_{n,1})} + (\widehat{\xi}_{n,2} - \eta_2)^2 \frac{1-\widehat{\xi}_{n,3}}{\widehat{\xi}_{n,2}(1-\widehat{\xi}_{n,2})}\right), \\ T_{\eta,n}^{\mathrm{W}} &= n\left((\widehat{\xi}_{n,1} - \eta_1)^2 \frac{\widehat{\xi}_{n,3}}{\eta_1(1-\eta_1)} + (\widehat{\xi}_{n,2} - \eta_2)^2 \frac{1-\widehat{\xi}_{n,3}}{\eta_2(1-\eta_2)}\right), \\ T_{\eta,n}^{\mathrm{LR}} &= 2\left(N_{n,11}\log\frac{\widehat{\xi}_{n,1}}{\eta_1} + N_{n,12}\log\frac{1-\widehat{\xi}_{n,2}}{1-\eta_2} + N_{n,21}\log\frac{1-\widehat{\xi}_{n,1}}{1-\eta_1} + N_{n,22}\log\frac{\widehat{\xi}_{n,2}}{\eta_2}\right), \end{aligned}$$

that are a plug-in type test statistic, a Wald type test statistic, and a likelihood ratio test statistic for testing the hypothesis $(\xi_1, \xi_2) = (\eta_1, \eta_2)$. Applying the asymptotic relative volume efficiency concept and the more general results in Baringhaus and Gaigall (2019), we obtain that the asymptotic relative volume efficiency of each of these three statistics with respect to each one of its competitors is equal to 1, i.e.,

$$\mathrm{arve}_\vartheta(B_n^{\mathrm{P}}, B_n^{\mathrm{W}}) = \mathrm{arve}_\vartheta(B_n^{\mathrm{P}}, B_n^{\mathrm{LR}}) = \mathrm{arve}_\vartheta(B_n^{\mathrm{W}}, B_n^{\mathrm{LR}}) = 1.$$

For a rough impression on the finite sample performance we refer to the following Figure. There, for given $\alpha = 0.05$ are shown the observed associated confidence regions for the pair (ξ_1, ξ_2) of the positive predicted value $\xi_1 = \frac{p_{11}}{p_{11}+p_{21}}$ and the negative predicted value $\xi_2 = \frac{p_{22}}{p_{12}+p_{22}}$ based on the value $(35, 23, 12, 30)$ of the sufficient statistic N_n obtained by generating by Monte Carlo simulation a single sample of the multinomial distribution $\mathfrak{M}_4(n; p_{11}, p_{12}, p_{21}, p_{22})$ where $n = 100$ and $p_{11} = 3/8$, $p_{12} = 1/4$, $p_{21} = 1/8$, $p_{22} = 1/4$, equivalently, $p_{11}/(p_{11}+p_{21}) = 3/4$, $p_{22}/(p_{12}+p_{22}) = 1/2$, $p_{11}+p_{21} = 1/2$. The observed region B_n^{P} is an ellipse with half axes $a = 0.1556841$ and $b = 0.1666392$.



Confidence regions B_n^{P} (left), B_n^{W} (middle), and B_n^{LR} (right)

A lot of well-known statistical problems are related to statistical inference for multiple samples. Some popular and very basic data examples, originally intended for ANOVA, are online available and provided by the Cengage College (2018). The examples include data for the depths for significant archaeological discoveries at different excavation sites at an archeological area in New Mexico. Another data set represents the extension growth after four years for different types of root-stock used in an apple orchard grafting experiment. Moreover, the result of a study is presented, where the researchers fed mice different doses of red dye number 40 and recorded the time of death in weeks for female mice. A further data set gives business startup costs for different types of businesses, namely pizza startups, baker or donut startups, shoe stores, gift shops, and pet stores. Finally, data for the weight of professional football players of the Dallas Cowboys, the Green Bay Packers, the Denver Broncos, the Miami Dolphins, and the San Francisco Forty Niners are provided.

On a new approach to the multi-sample goodness-of-fit problem

A new approach for the treatment of the multi-sample goodness-of-fit problem is presented in Gaigall (2019). Suppose we have $k \in \mathbb{N}$ samples $X_{1,1}, \ldots, X_{1,n_1}, \ldots, X_{k,1}, \ldots, X_{k,n_k}$ with different sample sizes $n_1, \ldots, n_k$ and unknown underlying distribution functions $F_1, \ldots, F_k$ as observations plus k families of distribution functions $\{G_1(\cdot, \vartheta); \vartheta \in \Theta\}, \ldots, \{G_k(\cdot, \vartheta); \vartheta \in \Theta\}$, each indexed by elements ϑ from the same parameter set Θ, we consider the new goodness-of-fit problem whether or not $(F_1, \ldots, F_k)$ belongs to the parametric family $\{(G_1(\cdot, \vartheta), \ldots, G_k(\cdot, \vartheta)); \vartheta \in \Theta\}$. We study this new multi-sample goodness-of-fit problem and new test statistics as a generalization and unification of the one-sample goodness-of-fit problem in Stute, Gonzáles-Manteiga and Presedo-Quindimil (1993) and the multi-sample goodness-of-fit problem in Kiefer (1959) (null hypothesis H_2). The new test statistics are presented and a parametric bootstrap procedure for the approximation of the unknown null distributions is discussed. Under regularity assumptions, it is proved that the approximation works asymptotically, and the limiting distributions of the test statistics in the null hypothesis case are determined. If the null hypothesis

$$\mathcal{H} : (F_1, \ldots, F_k) \in \big\{\big(G_1(\cdot, \vartheta), \ldots, G_k(\cdot, \vartheta)\big); \vartheta \in \Theta\big\}$$

is true, the statistician does not know the true underlying parameter $\vartheta \in \Theta$. An estimation on the basis of the pool of all $N = n_1 + \cdots + n_k$ observations is necessary. For this purpose, let

$$\widehat{\vartheta}_n = v_n(X_{1,1}, \ldots, X_{1,n_1}, \ldots, X_{k,1}, \ldots, X_{k,n_k})$$

be an estimator of ϑ. For $i = 1, \ldots, k$, we define the empirical distribution function based on the sample from the i-th population by

$$\widehat{F}_{i,n_i}(x) = \frac{1}{n_i} \sum_{j=1}^{n_i} I(X_{i,j} \leqslant x), \; x \in \mathbb{R},$$

where $I(\cdot)$ denotes the indicator function. We suggest the test statistic

$$T_n = \sum_{i=1}^{k} \frac{n_i}{N} \sup_{x \in \mathbb{R}} |\widehat{F}_{i,n_i}(x) - G_i(x, \widehat{\vartheta}_n)|,$$

or the test statistic

$$S_n = \sum_{i=1}^{k} \frac{n_i}{N} \int \big(\widehat{F}_{i,n_i}(x) - G_i(x, \widehat{\vartheta}_n)\big)^2 G_i(\mathrm{d}x, \widehat{\vartheta}_n),$$

for verifying the null hypothesis $\mathcal{H}$. For details and necessary measurability assumptions we refer to Gaigall (2019). In general, the distributions of the test statistics in the null hypothesis case depend on the underlying parameter $\vartheta \in \Theta$, likewise asymptotically. Because the true parameter is unknown in applications, we suggest a parametric bootstrap procedure for the approximation of the distributions of the test statistics under the null hypothesis. Under regularity assumptions, it follows from the deliberations in Gaigall (2019) that this approximation works asymptotically. Moreover, the limiting null distributions of the test statistics are determined there. In particular, the special case $G_1 = \cdots = G_k$ of the testing problem is of interest. Here, the null hypothesis is given by

$$\mathcal{H} : F_1 = \cdots = F_k \in \{G_1(\cdot, \vartheta); \vartheta \in \Theta\}.$$

This special case is in itself already a generalization of the one-sample goodness-of-fit problem in Stute, Gonzáles-Manteiga and Presedo-Quindimil (1993) and of the multi-sample goodness-of-fit problem in

Kiefer (1959) (null hypothesis H_2). As examples, we consider two parametric families of distributions for the null hypothesis, namely the family of inverse Gaussian distributions, denoted by $\mathrm{IG}(\mu, \lambda)$, $\mu \in (0, \infty)$, $\lambda \in (0, \infty)$, and the family of Rayleigh distributions, denoted by $\mathrm{R}(\sigma)$, $\sigma \in (0, \infty)$. Denoting by Φ the distribution function of the standard normal distribution, the null hypothesis of inverse Gaussian distribution is given by

$$G_i\big(x, (\mu, \lambda)'\big) = \Phi\bigg(\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\mu} - 1\Big)\bigg) + \exp\Big(\frac{2\lambda}{\mu}\Big)\Phi\bigg(-\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\mu} + 1\Big)\bigg),\ x > 0,\ (\mu, \lambda)' \in \Theta,\ i = 1, \ldots, k,$$

where $\Theta = (0, \infty)^2$. Maximum-likelihood estimator $(\widehat{\mu}_n, \widehat{\lambda}_n)'$ is given by

$$\widehat{\mu}_n = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{i,j}, \quad \widehat{\lambda}_n = \frac{1}{\frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{1}{X_{i,j}} - \frac{1}{\widehat{\mu}_n}\right)}.$$

The null hypothesis of Rayleigh distribution is given by

$$G_i(x, \sigma^2) = 1 - \exp\Big(-\frac{x^2}{2\sigma^2}\Big),\ x > 0,\ \sigma^2 \in \Theta,\ i = 1, \ldots, k,$$

where $\Theta = (0, \infty)$. Maximum-likelihood estimator $\widehat{\sigma}_n^2$ is now

$$\widehat{\sigma}_n^2 = \frac{1}{2N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{i,j}^2.$$

Huge dimensional data analysis and functional data analysis are two topics which are discussed frequently in current statistic literature. Although both fields deal with the same problem of large dimensional data they were developed mainly independently of each other. Both fields would benefit from more interaction. One possibility to achieve this is working on a very general space including both fields, like a separable Hilbert space. Applications are given, e.g., for stock market returns, see Ditzhaus and Gaigall (2021).

A consistent goodness-of-fit test for huge dimensional and functional data

In Ditzhaus and Gaigall (2018), we consider a goodness of fit problem in a separable Hilbert space. We develop a non-parametric goodness-of-fit test which can be applied in functional data analysis for infinite dimensional spaces, for data of huge but finite dimension or even for the more common low dimensional data case. For real-valued data our approach coincides with the well-known Cramér-von-Mises test, which was discussed by Anderson (1962) and, in the multivariate case, by Rosenblatt (1952). Our test's basic idea is projecting the data from the Hilbert space to real numbers and then applying an appropriate Cramér-von-Mises test. Using projections is not new at all, for example, Cuesta-Albertos et al. (2006) and Cuesta-Albertos et al. (2007) used a finite number of random projections in order to obtain real-valued data and applied a Kolmogorov-Smirnov goodness-of-fit test to these data. In contrast to these papers, we will not work with random projections but with all projections from an appropriate set. The advantage is that the result of the test only depends on the data and not on some additional randomness. Let H be a separable Hilbert space with countable orthonormal basis $\{e_i; i \in I\}$, where I is an index set. Given that $(\Omega, \mathcal{A}, , P)$ is a probability space, we denote by $Y : \Omega \longrightarrow H$ some $(\mathcal{A}, \mathfrak{B}(H))$-measurable random variable with respect to the σ-algebra $\mathcal{A}$ and the Borel σ-field $\mathfrak{B}(H)$ on H, where P^Y is assumed to be known, and by $X : \Omega \longrightarrow H$ some $(\mathcal{A}, \mathfrak{B}(H))$-measurable random variable,where P^X is assumed to be unknown. On the basis of $X_1, X_2, \ldots$, given by independent and identically distributed copies of X, we study the null hypothesis

$$\mathcal{H} : P^X = P^Y.$$

Let $\pi : H \to \mathbb{R}$ be a linear and continuous map, $F(\pi, \cdot)$ be the distribution function of $\pi(Y)$, and $\widehat{F}_n(\pi, \cdot)$ be the empirical distribution function of $\pi(X_1), \ldots, \pi(X_n)$. It is

$$\mathcal{T}_n(\pi) = n\int \big(\widehat{F}_n(\pi, t) - F(\pi, t)\big)^2 F(\pi, \mathrm{d}t)$$

the classical one-sample Cramér-von-Mises type test statistic on the basis of the distribution function $F(\pi,\cdot)$ and the real-valued random variables $\pi(X_1),\ldots,\pi(X_n)$. We consider the test statistic

$$T_n = \int \mathcal{T}_n(\pi)\mathcal{P}(\mathrm{d}\pi) = n\int\int \left(\widehat{F}_n(\pi,t) - F(\pi,t)\right)^2 F(\pi,\mathrm{d}t)\mathcal{P}(\mathrm{d}\pi)$$

for a probability measure $\mathcal{P}$ on $(H,\mathfrak{B}(H))$ chosen by the statistician. We have

$$\forall x \in H : x = \sum_{i\in I}\langle x, e_i\rangle e_i,$$

and the characteristic function $\varphi_Y : H \to \mathbb{C}$,

$$\varphi_Y(x) = \mathrm{E}\left(e^{i\langle x,Y\rangle}\right),\ x \in H,$$

determines the distribution of Y uniquely. It holds that

$$\forall x \in H\backslash\{0\} : \varphi_Y(x) = \varphi_{\langle Y,x/||x||\rangle}(||x||) = \varphi_{\langle Y,-x/||x||\rangle}(-||x||).$$

Focusing on

$$h = \left\{\sum_{j=1}^{k} m_j e_{i_j}; k\in I, (i_1,\ldots,i_k)\in I_<^k, (m_1,\ldots,m_k)\in S_+^{k-1}\right\}$$

$$\text{with } S_+^{k-1} = \left\{(m_1,\ldots,m_k)\in\mathbb{R}^k; m_1 \geqslant 0, \sum_{j=1}^{k} m_j^2 = 1\right\}$$

$$\text{and } I_<^k = \{(i_1,\ldots,i_k)\in I^k; i_1 < i_2 < \cdots < i_k\} \text{ for } k\in I,$$

we have $h \in \mathfrak{B}(H)$, and it is sufficient to consider $\mathcal{P}$ on $\mathfrak{B}(H)$ with

$$\mathcal{P}(h) = 1$$

to obtain a consistent testing procedure. In the case of $H = \mathbb{R}$ the property $\mathcal{P}(h) = 1$ leads to

$$h = \{1\} \text{ and } \mathcal{P} = \delta_1$$

and so

$$T_n = n\int\left(\frac{1}{n}\sum_{j=1}^{n}\mathrm{I}(X_j \leqslant t) - P(Y\leqslant t)\right)^2 P(Y\leqslant \mathrm{d}t),$$

i.e. our approach is a direct generalization of the classical one-sample Cramér-von-Mises test. Although the results obtained in Ditzhaus and Gaigall (2018) covers more general cases we limit ourselves to the consideration of specific $\mathcal{P}$ in what follows, determined by the following procedure for the generation of a realization from $\mathcal{P}$. At first, the statistician chooses two distributions ξ and η on $(I,\mathfrak{P}(I))$ with support I. Then the following steps are conducted.

1. Generate a realization k of the distribution ξ.
2. Independently of Step 1, generate $(i_1,\ldots,i_k)\in I_<^k$ by k-times sampling from I according to law η without replacement and by arranging the sample according to size.
3. Independently of Steps 1 and 2, generate a realization $(m_1,\ldots,m_k)$ of the uniform distribution u_k on $(S_+^{k-1},\mathfrak{B}^k(S_+^{k-1}))$.
4. Set $\pi = \sum_{j=1}^{k} m_j e_{i_j}$.

The convergence in distribution of the test statistic under the null hypothesis is proved and the test's consistency is concluded. A general approach enables the treatment of incomplete data. Properties under local alternatives are also discussed. With the help of the theory of U-statistics, see Koroljuk & Borovskich (1994), we obtain for local alternatives of the form

$$\frac{\mathrm{d}P^X}{\mathrm{d}P^Y} = 1 + \frac{\psi}{\sqrt{n}},$$

where ψ is in $\mathcal{L}^2(H,\mathfrak{B}(H),P^Y)$, the following result for the limit distribution of the test statistic under local alternatives.

Theorem 5. *Suppose the distributions of* $\langle X, e_i \rangle$ *and* $\langle Y, e_i \rangle$ *are continuous for all* $i \in I$*. Then we have*

$$T_n \xrightarrow{\mathcal{D}} \sum_{k=1}^{\infty} \lambda_k (\tau_k + \mathrm{E}(\varphi_k(Y)\psi(Y)))^2 \text{ as } n \to \infty,$$

where $(\tau_i)_{i \in \mathbb{N}}$ *is a sequence of independent standard normal distributed random variables and* $(\lambda_i)_{i \in \mathbb{N}}$ *is a sequence on non-negative numbers with* $\lambda_i > 0$ *for at least one* $i \in \mathbb{N}$.

For a proof we refer to Ditzhaus and Gaigall (2018). Applications are given for data of huge but finite dimension and for functional data in infinite dimensional spaces. Application to huge dimensional data takes place for the Hilbert space $H = \mathbb{R}^d$, where $d \in \mathbb{N}$. Here the random variables X and Y are random vectors

$$X = (X(1), \ldots, X(d)) \text{ and } Y = (Y(1), \ldots, Y(d)).$$

With $\mu = 1_{d \times 1}$ and $\Sigma = \frac{d}{d+1}(I_d - \frac{1}{d+1} 1_{d \times d})$, where $1_{d \times s}$ denotes the $d \times s$ matrix of ones and I_d represents the $d \times d$ identity matrix, we test the null hypothesis

$$P^Y = \ell_d(\mu, \Sigma)$$

versus contamination alternatives of the form

$$P^X = (1-a)\ell_d(\mu, \Sigma) + a\mathcal{N}_d(5\mu, \Sigma) \text{ or } P^X = (1-a)\ell_d(\mu, \Sigma) + at_d(\mu, 2\Sigma),$$

i.e., we consider the d-dimensional Laplace distribution $\ell_d(\mu, \Sigma)$, the d-dimensional normal distribution $\mathcal{N}_d(\mu, \Sigma)$ and the d-dimensional t distribution $t_d(\mu, \Sigma)$ with one degree of freedom, where all these distributions are parametrized by a location parameter $\mu \in \mathbb{R}^d$ and a symmetric positive definite covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$. Application to functional data takes place for the Hilbert space $H = \mathcal{L}^2([0,1], \mathfrak{B}_{[0,1]}, \lambda_{[0,1]})$ with orthonormal basis given by the normalized Legendre polynomials. Here the random variables X and Y are stochastic processes

$$X = \big(X(t); t \in [0,1]\big) \text{ and } Y = \big(Y(t); t \in [0,1]\big).$$

We test the null hypothesis

$$Y(t) = B(t),\ t \in [0,1],$$

for a standard Brownian bridge $B = (B(t); t \in [0,1])$ versus alternatives of the form

$$X(t) = aB(t) + bt(t-1),\ t \in [0,1].$$

References

Anderson, T.W. (1962). On the Distribution of the Two-Sample Cramer-von Mises Criterion. *The Annals of Mathematical Statistics* 33, 1148–1159.

Arnold, S.F. (1973). Application of the theory of products of problems to certain patterned covariance matrices. *Ann. Statist.* 1, 682–699.

Bahadur, R. (1966). A Note on Quantiles in Large Samples. *The Annals of Mathematical Statistics* 37: 577–580.

Blum, J. R., Kiefer, J., Rosenblatt, M. (1961). Distribution Free Tests of Independence Based on the Sample Distribution Function. *Annals of Mathematical Statistics* 32, 485–498.

Cengage College. (2018). `http://college.cengage.com/mathematics/brase/understandable_statistics/7e/students/datasets/owan/frames/frame.html`

Csörgő, M., Horváth, L. (1997). *Limit Theorems in Change-Point Analysis* . Wiley, NewYork.

Cuesta-Albertos, J. A., Fraiman, R. & Ransford, T. (2006). Random projections and goodness-of-fit tests in infinite-dimensional spaces. *Bull. Braz. Math. Soc. (N.S.)* **37**, no. 4, 477–501.

Cuesta-Albertos, J. A., del Barrio, E., Fraiman, R. & Matrán, C. (2007). The random projection method in goodness of fit for functional data. *Computational Statistics & Data Analysis* **51**, no. 10, 4814–4831.

Davydov, Y., Egorov, V. (2001). Functional CLT and LIL for induced order statistics. *Asymptotic Methods in Probability and Statistics with Applications (St. Petersburg, 1998). Stat. Ind. Technol.* 333–349. Boston, MA: Birkhäuser.

Detke M.J., Wiltse, C.G., Mallinckrodt, C.H., McNamara, R.K., Demitrack, M.A., Bitter I. (2004). Duloxetine in the acute and long-term treatment of major depressive disorder: a placebo- and paroxetine-controlled trial. *European Neuropsychopharmacology* 14, 457–470.

Dudley, R. M. (1984). *A course on empirical processes.* Lecture Notes in Mathematics 1097, 1–142. Springer, New York.

Fong, Y., Huang, Y., Lemos, M. P., Mcelrath, M. J. (2018). Rank-based two-sample tests for paired data with missing values. *Biostatistics* 19, 281–294.

Frets, G.P. (1921). Heredity of head form in man. *Genetica* 3, 193.

Gänßler, P., Ziegler, K. (1994). A Uniform Law of Large Numbers for Set-Indexed Processes with Applications to Empirical and Partial-Sum Processes. *Probability in Banach Spaces* 9, 385–400.

Goldstein, D,J,, Lu, Y., Detke, M.J., Wiltse, C., Mallinckrodt, C., Demitrack, M.A. (2004). Duloxetine in the treatment of depression: a double-blind placebo-controlled comparison with paroxetine. *Journal of Clinical Psychopharmacology* 24, 389–99.

Hoeffding, W. (1948). A nonparametric test of independence. *Annals of Mathematical Statistics* 19, 546–557.

Kiefer, J. (1959). K-Sample Analogues of the Kolmogorov-Smirnov and Cramér-V. Mises Tests. *The Annals of Mathematical Statistics* 30: 420–447.

Koning, A.J., Protasov, V. (2003). Tail behaviour of Gaussian processes with applications to the Brownian pillow. *Journal of Multivariate Analysis* **87**, 370–397.

Koroljuk, V.S. & Borovskich, Y. V. (1994). *Theory of U-Statistics.* Kluwer Academic Publishers Group, Dordrecht.

Kossmann, C.E. (1946). Relative importance of certain variables in the clinical determination of blood pressure. *American Journal of Medicine* 1, 464–467.

Nikitin Y. (1995). Asymptotic efficiency of nonparametric tests. Cambridge: Cambridge University Press.

Olkin, I. (1974). Inference for a normal population when the parameters exhibit some structure. In: *Reliability and biometry: statistical analysis of lifelength.* Proc. Conf., Florida State Univ., Tallahassee, 1973, 759–773. SIAM, Philadelphia.

Perlman, M.D. (1987). A Review of Multivariate Analysis: Comment: Group Symmetry Covariance Models. *Statist. Sci.* 2, 421–425.

Piterbarg, V.I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* American Mathematical Society, Providence, RI.

Puri, M. L., Sen, P. K. (1971). *Nonparametric Methods in Multivariate Analysis.* Wiley.

Rosenblatt, M. (1952). Limit theorems associated with variants of the von Mises statistic. *Ann. Math. Statist.* **23**, 617-623.

Serfling R. (1980). Approximation theorems of mathematical statistics. New York: Wiley.

Stute, W., Gonzáles-Manteiga, W. and Presedo-Quindimil, M. (1993). Bootstrap based goodness-of-fit-tests. *Metrika* 40: 243–256. doi:10.1007/BF02613687.

van der Vaart, A., Wellner, J.A. (1996). *Weak convergence and empirical processes.* Springer, New York.

van der Vaart. (1998). *Asyptotic Statistics.* Cambridge.

Ziegler, K. (1997). Functional Central Limit Theorems for Triangular Arrays of Function-Indexed Processes under Uniformly Integrable Entropy Conditions. *Journal of Multivariate Analysis* 62, 233–272.