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Critical mass phenomena in higher dimensional quasilinear Keller–Segel systems with indirect signal production

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In this paper, we deal with quasilinear Keller–Segel systems with indirect signal production,

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + w, & x \in \Omega,\ t > 0, \\ w_t + w = u, & x \in \Omega,\ t > 0, \end{cases}$$

complemented with homogeneous Neumann boundary conditions and suitable initial conditions, where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded smooth domain, $m \geq 1$ and

$$\mu(t) := \fint_\Omega w(\cdot, t) \qquad \text{for } t > 0.$$

We show that in the case $m \geq 2 - \frac{2}{n}$, there exists $M_c > 0$ such that if either $m > 2 - \frac{2}{n}$ or $\int_\Omega u_0 < M_c$, then the solution exists globally and remains bounded, and that in the case $m \leq 2 - \frac{2}{n}$, if either $m < 2 - \frac{2}{n}$ or $M > 2^{\frac{n}{2}} n^{n-1} \omega_n$, then there exist radially symmetric initial data such that $\int_\Omega u_0 = M$ and the solution blows up in finite or infinite time, where the blow-up time is infinite if $m = 2 - \frac{2}{n}$. In particular, if $m = 2 - \frac{2}{n}$, there is a critical mass phenomenon in the sense that

$$\inf \left\{ M > 0 : \exists u_0 \text{ with } \int_\Omega u_0 = M \text{ such that the corresponding solution blows up in infinite time} \right\}$$

is a finite positive number.

KEYWORDS
chemotaxis, indirect signal production, infinite-time blow-up

MSC CLASSIFICATION
35B33, 35B44, 35A01, 35K55, 92C17

1 | INTRODUCTION

1.1 | Critical mass in the two-dimensional Keller–Segel system with direct signal production

Chemotaxis, which is the motion of cells oriented toward higher concentrations of a chemical substance, is an important cause for aggregation in different biological contexts, for example, the formation of bacterial colonies or tumor invasion [1]. For its mathematical description, the Keller–Segel model is often used, which, in a simplified parabolic–elliptic form, reads

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - v + u. \end{cases} \tag{1.1}$$

Here, u and v denote the density of cells and the concentration of a signal substance, respectively, and the chemical signal is directly produced by the cells. This system and its variants have been studied extensively (see, e.g., the surveys [2–4]).

For the present article, it is of particular interest that (1.1) features a critical-mass phenomenon in that in two-dimensional domains, radially symmetric initial data u_0 with mass $M = \int_\Omega u_0 < 8\pi$ lead to global and bounded solutions, whereas for any larger mass, some initial data with this mass can be found, which evolve into solutions blowing up in finite time, [5]. In higher-dimensional settings, blow-up solutions can be found for any prescribed positive initial mass, [5]. In the radially symmetric case, the same has been observed for other parabolic–elliptic and fully parabolic versions of (1.1), see [6–10].

1.2 | Indirect signal production

A more recent line of investigations is concerned with systems where the signal is produced indirectly, for example, as in

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + w, & x \in \Omega,\ t > 0, \\ w_t + w = u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Here, $\Omega := B_1(0) \subset \mathbb{R}^n$ $(n \geq 2)$ is a ball and

$$\mu(t) := \fint_\Omega w \ \text{ for } \ t > 0;$$

ν is the outward normal vector to $\partial\Omega$; $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ are nonnegative. This is the simplified version of a chemotaxis model proposed by Strohm et al. [11], which describes the spread and aggregative behavior of the mountain pine beetle (MPB). Here, u, w represent the densities of flying MPB and of nesting MPB, and v denotes the concentration of the beetle pheromone. For the above model, Tao and Winkler [12] established boundedness and infinite-time blow-up in the two-dimensional case; more precisely, in the radial setting, the solution remains bounded when $\int_\Omega u_0 < 8\pi$, and there exist initial data such that $\int_\Omega u_0 > 8\pi$ and the solution blows up in infinite time. For $\Omega = \mathbb{R}^2$, a corresponding result has recently been obtained in [13]. Based on a Lyapunov functional, Laurençot [14] uncovered the same phenomenon in the fully parabolic and in the nonradial setting, where the critical mass decreases to 4π. Also in a related model concerned with a population split into a static, signal-producing and a motile, chemotactically active group, a dichotomy between initial masses leading to global boundedness of all solutions or unboundedness, respectively, has been observed [15]. From these results, we understand that a critical mass phenomenon happens also for the two-dimensional Keller–Segel system with indirect signal production given above and that the difference from systems with direct signal production is that solutions always exist globally in time.

In summary, also for indirect signal production, there is still some critical mass phenomenon in 2D, but now it discriminates between boundedness and unboundedness of global solutions.

However, up to now, it is not known whether such results are satisfied in the higher dimensional cases. Thus the question, whether a critical mass phenomenon happens in the higher dimensional Keller–Segel system with indirect signal production, naturally arises.

As to be shown below, the answer is no. In fact, we always obtain unbounded solutions in the higher dimensional cases.

Proposition 1.1. *Let $\Omega := B_1(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball. For each $M > 0$, there exist initial data (u_0, w_0) with $\int_\Omega u_0 = M$ such that the corresponding solution of (1.2) is unbounded.*

We remark that this is in contrast to the indirect taxis system considered in [16, 17], where also the third component diffuses and where the mass $(8\pi)^2$ has been observed to be critical in dimension four.

1.3 | Critical mass phenomenon in higher dimensional *quasilinear* Keller–Segel system with indirect signal production

Quasilinear diffusion is occasionally relevant in applications of chemotaxis systems, for example, when concerned with the motion of cells (like tumor invasion, cf. [18]). In the present indirect setting, the inclusion of porous medium type diffusion leads to the following system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + w, & x \in \Omega,\ t > 0, \\ w_t + w = u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded smooth domain and $\mu(t) := \fint_\Omega w$ for $t > 0$; $m \geq 1$; ν is the outward normal vector to $\partial\Omega$; $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ are nonnegative.

For a related direct production system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

it is known that the size of m determines whether solutions remain bounded or blow up; large diffusion exponents m counteract explosions; indeed, when $m > 2 - \frac{2}{n}$, boundedness of solutions was obtained in [19, 20]; on the other hand, when $m < 2 - \frac{2}{n}$, initial data with corresponding solutions blowing up in finite time were constructed in [21, 22]. Also, for the system such that the diffusion term is replaced with the degenerate diffusion Δu^m, similar results were proved (boundedness in [23, 24] and finite-time blow-up in [25]). Moreover, in the critical case $m = 2 - \frac{2}{n}$, a critical mass phenomenon is observed in the degenerate system posed on $\Omega = \mathbb{R}^n$; in this case, there exists $M_c > 0$ such that if $\int_\Omega u_0 < M_c$, then solutions are global and bounded [26, 27] whereas for all $M > M_c$, one can find initial data with $\int_\Omega u_0 = M_c$ leading to finite-time blow-up (see [26] for the parabolic–elliptic and [28] for the fully parabolic three- and four-dimensional setting). Regarding bounded domains, corresponding boundedness results have also been obtained [24, 29].

From these results, the questions naturally arise whether behavior of solutions is determined by conditions on m also in (1.3) and whether a critical mass phenomenon possibly happens for some value of m. The purpose of this paper is to give a positive answer to the aforementioned question in the system (1.3).

1.4 | Main results

As preparation—and in order to specify what type of solutions and possible blow-up we are dealing with—we introduce the following proposition on local existence of solutions to (1.3).

Proposition 1.2 (Local existence)**.** *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded smooth domain and let $m \geq 1$. Assume that $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ are nonnegative. Then there exist $T_{\max} \in (0, \infty]$ and uniquely determined nonnegative functions*

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ &v \in C^{2,0}(\overline{\Omega} \times [0, T_{\max})), \\ &w \in C^{0,1}(\overline{\Omega} \times [0, T_{\max})) := \{w \in C^0(\overline{\Omega} \times [0, T_{\max})) | w_t \in C^0(\overline{\Omega} \times [0, T_{\max}))\}, \end{aligned}$$

which solve (1.3) classically in $\overline{\Omega} \times [0, T_{\max})$ *and which are such that*

$$\text{if } T_{\max} < \infty, \text{ then } \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \text{ as } t \nearrow T_{\max}. \tag{1.4}$$

Moreover, if u_0 *and* w_0 *are radially symmetric, then so are* $u(\cdot, t), v(\cdot, t)$ *and* $w(\cdot, t)$ *for any* $t \in (0, T_{\max})$.

The first theorem is concerned with global existence and boundedness of solutions to (1.3). Similarly as in [19, 20, 23, 24, 27], solutions remain bounded when $m > 2 - \frac{2}{n}$, and the same result holds under a smallness condition for the size of initial data u_0 when $m = 2 - \frac{2}{n}$. Moreover, we note that global existence is ensured regardless of the size of initial data when $m \geq 2 - \frac{2}{n}$.

Theorem 1.3 (Global existence and boundedness). *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ *be a bounded smooth domain and let* $m \geq 2 - \frac{2}{n}$. *Then there exists* $M_c > 0$ *satisfying the following property: For all nonnegative initial data* $u_0 \in C^0(\overline{\Omega})$ *and* $w_0 \in C^1(\overline{\Omega})$, *the corresponding solution* (u, v, w) *of (1.3) exists globally in time. Moreover, if* u_0 *and* m *satisfy either*

$$m > 2 - \frac{2}{n} \text{ or } \int_\Omega u_0 < M_c,$$

then the solution (u, v, w) *of (1.3) is bounded in* $\Omega \times (0, \infty)$ *in the sense that there is* $C > 0$ *such that*

$$u(x, t) \leq C, \ v(x, t) \leq C \text{ and } w(x, t) \leq C \tag{1.5}$$

for all $x \in \Omega$ *and* $t \in (0, \infty)$.

The next theorem gives unboundedness of solutions to (1.3).

Theorem 1.4 (Unboundedness). *Let* $\Omega := B_1 := B_1(0) \subset \mathbb{R}^n$ $(n \geq 3)$ *be a ball and let* $m \in \left[1, 2 - \frac{2}{n}\right]$. *If either*

$$\left(m \in \left[1, 2 - \frac{2}{n}\right) \text{ and } M > 0\right) \text{ or } \left(m = 2 - \frac{2}{n} \text{ and } M > 2^{\frac{n}{2}} n^{n-1} \omega_n\right),$$

where $\omega_n := |\partial B_1(0)|$, *then there exists* $R \in (0, 1)$ *such that for each* $\eta > 0$ *one can find constants* $\Gamma_u > 0$, $\gamma > 0$, $\Gamma_w > 0$, $\alpha > 0$ *and* $C > 0$ *with the property that for all nonnegative radially symmetric initial data* $u_0 \in C^0(\overline{\Omega})$ *and* $w_0 \in C^1(\overline{\Omega})$ *satisfying*

$$\begin{aligned} &\int_\Omega u_0 = M, \\ &\fint_{B_r} u_0 \geq \Gamma_u \quad \text{for all } r \in (0, R), \end{aligned} \tag{1.6}$$

$$\fint_{B_1 \setminus B_r} u_0 \leq \gamma \qquad \text{for all } r \in (R, 1), \tag{1.7}$$

$$\fint_{B_r} w_0 \geq \fint_{B_1} w_0 + \Gamma_w \qquad \text{for all } r \in (0, R), \tag{1.8}$$

$$\fint_{B_1 \setminus B_r} w_0 \leq \fint_{B_1} w_0 - \eta \qquad \text{for all } r \in (R, 1), \tag{1.9}$$

the solution (u, v, w) of (1.3) blows up in finite or infinite time in the sense that $\left\{\|u(\cdot,t)\|_{L^\infty(\Omega)} | t \in (0, T_{\max})\right\}$ is unbounded. More precisely, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \geq Ce^{at}$ holds for all $t \in (0, T_{\max})$.

From this theorem, we can immediately prove Proposition 1.1.

Proof of Proposition 1.1. Let $m = 1$ and $M > 0$. Then, from Theorem 1.4, we can find initial data u_0, w_0 with $\int_\Omega u_0 = M$ such that the corresponding solution is unbounded.

In Theorem 1.4, we see that blow-up occurs for the system (1.3) in the case $m < 2 - \frac{2}{n}$ as in [21, 22, 25, 28]. However, it is not clear whether the blow-up time is finite or infinite. In contrast, in the case $m = 2 - \frac{2}{n}$, we can obtain initial data leading to infinite-time blow-up by a combination of Theorems 1.3 and 1.4.

Corollary 1.5. *Let $\Omega := B_1(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball and let $m = 2 - \frac{2}{n}$. Then for all*

$$M > 2^{\frac{n}{2}} n^{n-1} \omega_n,$$

there exist nonnegative radially symmetric initial data (u_0, w_0) such that $\int_\Omega u_0 = M$ and the solution blows up in infinite time.

Remark 1.6. From Theorem 1.3 and Corollary 1.5, we know that the critical value is $m = 2 - \frac{2}{n}$ in the higher dimensional cases, where a critical mass phenomenon occurs. In contrast to the Keller–Segel system with direct signal production (see, e.g., [22]), now the blow-up time is always infinite if $m = 2 - \frac{2}{n}$.

1.5 | Main ideas and plan of the paper

In Section 3, we will prove Theorem 1.3. Both global existence and boundedness of solutions are obtained by showing that the functional

$$\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}$$

(which we consider for sufficiently large p) is a subsolution to a certain ODE, whose solution is global and, under certain conditions, also bounded. A crucial step for the corresponding estimates is the observation that the third equation in (1.3) regularizes in time (but not in space), which is inter alia manifested by the identity $\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{p+1} + \int_\Omega w^{p+1} = \int_\Omega u w^p$. If $m \geq 2 - \frac{2}{n}$, the Gagliardo–Nirenberg inequality allows us then to favorably bound the worrisome terms by, essentially, the dissipative terms; if either $m > 2 - \frac{2}{n}$ or $\int_\Omega u_0$ is sufficiently small, this can be done (see Lemmata 3.3 and 3.4) in such a way that we can infer not only global existence but also boundedness of the solutions. This proof of global existence is based on [12, Section 3].

Section 4 is devoted to proving Theorem 1.4. To that end, we consider the function $U(\xi, t) := \int_0^{\xi^{\frac{1}{n}}} r^{n-1} u(r,t)\mathrm{d}r$ for $\xi \in [0, 1]$ and $t \in [0, T_{\max})$, which is introduced in [30]. As in [12], our aim is to construct an unbounded subsolution $\underline{U}$. Here, we take the ansatz $\underline{U}(\xi, t) := \frac{a(t)\xi}{b(t)+\xi}$ for $\xi \in [0, \xi_0]$ and $t \in [0, \infty)$ with certain a, b and a small $\xi_0 \in (0, 1)$, so that the derivative of $\underline{U}$ is unbounded at the origin. One of the new challenges compared to [12] is to deal with the term

$$2n^2\left(\frac{na(t)b(t)}{(b(t)+\xi)^2} + 1\right)^{m-1} \frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi},$$

which appears when applying the parabolic operator $\mathcal{P}$ defined in (4.2) to $\underline{U}$. Evidently, this term is simplified if one sets $m = 1$. However, if $m \in \left[1, 2 - \frac{2}{n}\right]$, this term can still be controlled by $\xi_0^{2-\frac{2}{n}-m}$ and next absorbed by a negative term depending on $\int_\Omega u_0$; if either $m \in \left[1, 2 - \frac{2}{n}\right)$ or $\int_\Omega u_0$ is sufficiently large, this approach works, and thus, we can indeed show that $\underline{U}$ and, hence, U is unbounded.

2 | PRELIMINARIES

In this section, let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with a smooth boundary and $m \geq 1$. To begin with, we consider the local existence result of Proposition 1.2.

Proof of Proposition 1.2. This proposition is proved by a standard fixed point argument as in [31].

We next collect a basic property of solutions to (1.3).

Lemma 2.1. *Let $m \geq 1$ and let $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ be nonnegative. Then the solution (u, v, w) of (1.3) given by Proposition 1.2 with maximal existence time $T_{\max}$ satisfies that*

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \tag{2.1}$$

and

$$\mu(t) = \fint_\Omega w(\cdot, t) \geq 0 \tag{2.2}$$

for all $t \in (0, T_{\max})$.

Proof. While (2.1) follows immediately upon integrating the first equation in (1.3), (2.2) is obtained by nonnegativity of w. □

3 | GLOBAL EXISTENCE AND BOUNDEDNESS

Throughout this section, we fix a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), parameters $m \geq 1$ and $M > 0$ as well as nonnegative initial data $u_0 \in C^0(\overline{\Omega})$, $w_0 \in C^1(\overline{\Omega})$ with $M = \int_\Omega u_0$. Moreover, we denote the solution of (1.3) provided by Proposition 1.2 by (u, v, w) and its maximal existence time by $T_{\max}$.

In order to prove global existence and boundedness, we will establish L^p-estimates for u and L^{p+1}-estimates for w. To this end, we begin by preparing an estimate for $\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}$.

Lemma 3.1. *For all $p > 1$ and $k > 0$, the solution (u, v, w) of (1.3) satisfies*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} + \frac{4(p-1)}{(p+m-1)^2}\int_\Omega |\nabla u^{\frac{p+m-1}{2}}|^2 + \int_\Omega w^{p+1} \\
&\leq 2k\int_\Omega u^{p+1} + (k^{-p} + k^{-\frac{1}{p}})\int_\Omega w^{p+1} \qquad \textit{in } (0, T_{\max}).
\end{aligned} \tag{3.1}$$

Proof. For henceforth fixed $p > 1$, we multiply the first equation in (1.3) by u^{p-1}, integrate over Ω, and use integration by parts to obtain that

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p &= \int_\Omega u^{p-1}\nabla \cdot ((u+1)^{m-1}\nabla u) - \int_\Omega u^{p-1}\nabla \cdot (u\nabla v) \\
&= -(p-1)\int_\Omega u^{p-2}(u+1)^{m-1}|\nabla u|^2 + (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v \\
&\leq -\frac{4(p-1)}{(p+m-1)^2}\int_\Omega |\nabla u^{\frac{p+m-1}{2}}|^2 - \frac{p-1}{p}\int_\Omega u^p \Delta v \qquad \text{in } (0, T_{\max}).
\end{aligned}$$

Here, from the second equation in (1.3) and (2.2), we have $\Delta v = \mu(t) - w \geq -w$, so that it follows that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \leq -\frac{4(p-1)}{(p+m-1)^2}\int_\Omega |\nabla u^{\frac{p+m-1}{2}}|^2 + \frac{p-1}{p}\int_\Omega u^p w \qquad \text{in } (0, T_{\max}). \tag{3.2}$$

Next, multiplying the third equation in (1.3) by w^p and integrating over Ω, we can observe that

$$\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^{p+1} + \int_\Omega w^{p+1} = \int_\Omega uw^p,$$

which together with (3.2) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} + \frac{4(p-1)}{(p+m-1)^2}\int_\Omega |\nabla u^{\frac{p+m-1}{2}}|^2 + \int_\Omega w^{p+1} \le \frac{p-1}{p}\int_\Omega u^p w + \int_\Omega u w^p$$

in $(0, T_{\max})$. Finally, we apply Young's inequality to two terms on the right-hand side of this inequality and thereby derive (3.1). □

To estimate the first term on the right-hand side of (3.1), we next state the following consequence of the Gagliardo–Nirenberg inequality.

Lemma 3.2. *For all* $p > \max\left\{1, \frac{n}{2}\left(2-\frac{2}{n}-m\right)\right\}$, *there exists* $C_0 := C_0(p) > 0$ *such that for arbitrary* $\varphi \in C^1(\bar{\Omega})$,

$$\int_\Omega \varphi^{p+1} \le C_0 \|\nabla \varphi^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1)}{p+m-1}\theta} \|\varphi\|_{L^1(\Omega)}^{(p+1)(1-\theta)} + C_0\|\varphi\|_{L^1(\Omega)}^{p+1}, \tag{3.3}$$

where $\theta := \theta(p) := \dfrac{\frac{p+m-1}{2} - \frac{p+m-1}{2(p+1)}}{\frac{p+m-1}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$.

Proof. Applying the Gagliardo–Nirenberg inequality (e.g., in the variant of [18, Lemma 2.3], which allows for integrability exponents below 1), with some $C > 0$, we can estimate

$$\begin{aligned}\int_\Omega \varphi^{p+1} = \|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+1)}{p+m-1}}(\Omega)}^{\frac{2(p+1)}{p+m-1}} &\le C\|\nabla\varphi^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1)}{p+m-1}\theta}\|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+1)}{p+m-1}(1-\theta)} + C\|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+1)}{p+m-1}}\\ &= C\|\nabla\varphi^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1)}{p+m-1}\theta}\|\varphi\|_{L^1(\Omega)}^{(p+1)(1-\theta)} + C\|\varphi\|_{L^1(\Omega)}^{p+1}\end{aligned}$$

for all $\varphi \in C^1(\bar{\Omega})$, which concludes the proof. □

We next prove L^p-estimates for u and L^{p+1}-estimates for w, provided $m \ge 2 - \frac{2}{n}$.

Lemma 3.3. *Assume that* $p > 1$.

(i) If $m = 2 - \frac{2}{n}$ *and*

$$M = \int_\Omega u_0 < M_c(p) := \left[\frac{1}{4\cdot 2^p C_0(p)}\cdot\frac{4(p-1)}{(p+m-1)^2}\right]^{\frac{1}{(1-\theta(p))(p+1)}} \tag{3.4}$$

with $C_0(p)$ *and* $\theta(p)$ *from Lemma 3.2, then there exists* $C(p) > 0$ *such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(p) \text{ and } \|w(\cdot,t)\|_{L^{p+1}(\Omega)} \le C(p) \qquad \text{for all } t \in (0, T_{\max}). \tag{3.5}$$

(ii) In the case $m > 2 - \frac{2}{n}$, *there exists* $C(p) > 0$ *such that (3.5) holds.*

Proof. We fix $k := 2\cdot 2^p > 2^p$, so that $k^{-p} + k^{-\frac{1}{p}} < 1$. Let us first consider the case (i). The condition $m = 2-\frac{2}{n}$ ensures that θ defined in Lemma 3.2 satisfies $\frac{2(p+1)}{p+m-1}\theta = 2$. Thus, invoking Lemma 3.2 and (2.1), we obtain $c_1 := C_0(p) > 0$ such that with $M = \int_\Omega u_0$

$$\int_\Omega u^{p+1} \le c_1 M^{(p+1)(1-\theta)}\|\nabla u^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^2 + c_1 M^{p+1} \qquad \text{in } (0, T_{\max}). \tag{3.6}$$

If $M < M_c(p)$, we can pick $\eta \in (0,1)$ so small that

$$M \le \left[\frac{1}{2(k+\eta)c_1}\cdot\frac{4(p-1)}{(p+m-1)^2}\right]^{\frac{1}{(1-\theta)(p+1)}}. \tag{3.7}$$

By virtue of (3.6), (3.7), and Young's inequality, in $(0, T_{\max})$, we have

$$\begin{aligned}2k\int_\Omega u^{p+1} &= (2k+\eta)\int_\Omega u^{p+1} - \eta\int_\Omega u^{p+1}\\ &\le (2k+\eta)c_1M^{(p+1)(1-\theta)}\|\nabla u^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + (2k+\eta)c_1M^{p+1} - \eta\int_\Omega u^{p+1}\\ &\le \frac{4(p-1)}{(p+m-1)^2}\|\nabla u^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + c_2 - c_3\int_\Omega u^p,\end{aligned}$$

where $c_2 := (2k+\eta)c_1M^{p+1} + \frac{\eta}{p}|\Omega|$ and $c_3 := \frac{p+1}{p}\eta$. We infer from (3.1) and the above inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} + c_3\int_\Omega u^p + (1-(k^{-p}+k^{-\frac{1}{p}}))\int_\Omega w^{p+1} \le c_2, \tag{3.8}$$

in $(0, T_{\max})$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} + c_4\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} \le c_2 \qquad \text{in } (0, T_{\max}),$$

where $c_4 := \min\{c_3p, (1-(k^{-p}+k^{-\frac{1}{p}}))(p+1)\} > 0$. From this differential inequality, we arrive at (3.5).

Next, we deal with the case (ii). The condition $m > 2 - \frac{2}{n}$ implies $\frac{(p+1)}{p+m-1}\theta < 1$, where θ is again as in Lemma 3.2. Therefore, applying Young's inequality to the first term on the right-hand side of (3.3), we see that there exists $c_5 > 0$ such that

$$\int_\Omega u^{p+1} \le \frac{1}{2k+1}\cdot\frac{4(p-1)}{(p+m-1)^2}\|\nabla u^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + c_5 \qquad \text{in } (0, T_{\max})$$

and inserting this in (3.1), we again achieve (3.8) (albeit with different constants) and conclude (3.5) exactly as in case (i). □

As to the case (i) in Lemma 3.3, the L^p-estimate independent of time for u was shown by imposing a smallness condition on the size of initial data. On the other hand, in the absence of such a smallness condition, we can derive a time-dependent L^p-estimate for u, which will allow us to establish global existence regardless of the size of initial data in the case $m = 2-\frac{2}{n}$.

Lemma 3.4. *If $m \ge 2 - \frac{2}{n}$, then for all $p > 1$ and $T \in (0, T_{\max}] \cap (0, \infty)$, there exists $C(p,T) > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(p,T) \text{ and } \|w(\cdot,t)\|_{L^{p+1}(\Omega)} \le C(p,T)$$

for all $t \in (0, T)$.

Proof. The case of $m > 2 - \frac{2}{n}$ is already covered by Lemma 3.3; we therefore assume $m = 2 - \frac{2}{n}$ and note that this condition warrants that $\frac{2(p+1)}{p+m-1}\theta = 2$ with θ as in Lemma 3.2. Thanks to Lemma 3.2 and Hölder's inequality, we can find $c_1 > 0$ such that

$$\begin{aligned}\int_\Omega u^{p+1} &\le c_1M^{(p+1)(1-\theta)}\|\nabla u^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + c_1M\left(\int_\Omega u\right)^p\\ &\le c_1M^{(p+1)(1-\theta)}\|\nabla u^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + c_1M|\Omega|^{p-1}\int_\Omega u^p \qquad \text{in } (0, T_{\max}),\end{aligned}$$

which together with (3.1) entails that for any $k > 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} + \frac{4(p-1)}{(p+m-1)^2}\int_\Omega|\nabla u^{\frac{p+m-1}{2}}|^2 + \int_\Omega w^{p+1}\\ &\le 2kc_1M^{(p+1)(1-\theta)}\int_\Omega|\nabla u^{\frac{p+m-1}{2}}|^2 + 2kc_1M|\Omega|^{p-1}\int_\Omega u^p + (k^{-p}+k^{-\frac{1}{p}})\int_\Omega w^{p+1} \qquad \text{in } (0, T_{\max}).\end{aligned}$$

Here, choosing $k = \frac{1}{2c_1 M^{(p+1)(1-\theta)}} \cdot \frac{4(p-1)}{(p+m-1)^2}$, we can observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} \le c_2\left\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\right\} \qquad \text{in } (0, T_{\max})$$

with $c_2 := \max\{2kc_1 M|\Omega|^{p-1}p, (k^{-p} + k^{-\frac{1}{p}})(p+1)\}$. Therefore, for any $T \in (0, T_{\max}] \cap (0, \infty)$ we obtain $c_3(T) > 0$ such that $\|u(\cdot, t)\|_{L^p(\Omega)} \le c_3(T)$ and $\|w(\cdot, t)\|_{L^{p+1}(\Omega)} \le c_3(T)$ for all $t \in (0, T)$. □

Now, we prove global existence and boundedness of solutions to (1.3) by applying Lemmata 3.3 and 3.4.

Proof of Theorem 1.3. We first show that $T_{\max} = \infty$. Let $p > n + 2$. Assuming $T_{\max} < \infty$, we can apply Lemma 3.4 to $T = T_{\max} < \infty$ to obtain $c_1 > 0$ satisfying $\|u(\cdot, t)\|_{L^p(\Omega)} \le c_1$ and $\|w(\cdot, t)\|_{L^{p+1}(\Omega)} \le c_1$ for all $t \in (0, T_{\max})$. Therefore, according to elliptic regularity theory (cf. [32, Theorem I.19.1]), there exists $c_2 > 0$ such that $\|v(\cdot, t)\|_{W^{2,p+1}(\Omega)} \le c_2$ for all $t \in (0, T_{\max})$, and then the Sobolev embedding theorem tells us that $\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le c_3$ for all $t \in (0, T_{\max})$ with some $c_3 > 0$. Hence, applying the Moser-type iteration of [20, Lemma A.1], we can obtain $c_4 > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_4$$

for all $t \in (0, T_{\max})$. However, this contradicts (1.4), meaning that $T_{\max}$ cannot be finite.

In the case $m > 2 - \frac{2}{n}$ or if $M < M_c := M_c(n+3)$ (with $M_c(n+3)$ as in (3.4), which crucially does not depend on M or the solution (u, v, w)) in the case of $m = 2 - \frac{2}{n}$, by means of Lemma 3.3, elliptic regularity theory and the Sobolev embedding theorem, we can similarly verify that $\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le c_5$ for all $t \in (0, \infty)$ with some $c_5 > 0$, where c_5 is independent of time. Thus, by a Moser-type iteration, we see that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_6$ for all $t \in (0, \infty)$ with some $c_6 > 0$, and so (1.5) holds.

4 | UNBOUNDEDNESS

In the following, we let $\Omega := B_1(0) \subset \mathbb{R}^n$ ($n \ge 3$), $m \in [1, 2 - \frac{2}{n}]$ and $M > 0$. For simplicity, we also fix nonnegative radially symmetric initial data $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ as well as the solution (u, v, w) of (1.3) given by Proposition 1.2 and denote its maximal existence time by $T_{\max}$. However, we emphasize that all constants below only depend on Ω, m and M, not explicitly on the initial data or the solution. Moreover, the uniqueness statement in Proposition 1.2 implies that (u, v, w) is radially symmetric and henceforth we write $u(|x|, t)$ instead of $u(x, t)$ etc.

In order to prove Theorem 1.4, referring to [12], we define the function U as

$$U(\xi, t) := \int_0^{\xi^{\frac{1}{n}}} r^{n-1} u(r, t)\mathrm{d}r \quad \text{for } \xi \in [0, 1] \text{ and } t \in [0, T_{\max}), \tag{4.1}$$

which belongs to $C^{1,0}([0, 1] \times [0, T_{\max})) \cap C^{2,1}([0, 1] \times (0, T_{\max}))$ and introduce the parabolic operator $\mathcal{P}$ as

$$\begin{aligned}\mathcal{P}\widetilde{U}(\xi, t) := {} & \widetilde{U}_t(\xi, t) - n^2\xi^{2-\frac{2}{n}}(n\widetilde{U}_\xi(\xi, t) + 1)^{m-1}\widetilde{U}_{\xi\xi}(\xi, t) - n\left\{\int_0^t \mathrm{e}^{-(t-s)}\left(\widetilde{U}(\xi, s) - \frac{M}{\omega_n}\xi\right)\mathrm{d}s\right\}\widetilde{U}_\xi(\xi, t) \\ & - n(W_0(\xi) - K_0\xi)\mathrm{e}^{-t}\widetilde{U}_\xi(\xi, t)\end{aligned} \tag{4.2}$$

for $\xi \in (0, 1)$, $t \in (0, T)$ and $\widetilde{U} \in C^1((0, 1) \times (0, T)) \cap C^0((0, T); W^{2,\infty}((0, 1)))$, $T > 0$, with

$$W_0(\xi) := \int_0^{\xi^{\frac{1}{n}}} r^{n-1} w_0(r)\mathrm{d}r \quad \text{for } \xi \in [0, 1] \text{ and } K_0 := W_0(1), \tag{4.3}$$

where $\omega_n = |\partial B_1(0)| = n|B_1(0)|$. Now, we first collect properties of U.

Lemma 4.1. *The function U satisfies that*

$$U(0,t)=0 \text{ and } U(1,t)=\frac{M}{\omega_n} \tag{4.4}$$

for all $t\in[0,T_{\max})$ as well as

$$U_\xi(\xi,t)=\frac{1}{n}u(\xi^{\frac{1}{n}},t)\geq 0 \tag{4.5}$$

for all $\xi\in(0,1)$ and $t\in(0,T_{\max})$. Moreover,

$$\mathcal{P}U(\xi,t)=0 \tag{4.6}$$

for all $\xi\in(0,1)$ and $t\in(0,T_{\max})$.

Proof. We immediately see that (4.4) holds from the definition of U and (2.1) and that (4.5) is obtained by a direct computation and nonnegativity of u. Also, transforming the system (1.3) exactly as in [12, Lemma 4.1] (there only for 2D and linear diffusion), we arrive at (4.6). □

As a preparation to the proof of Theorem 1.4, let us prove a comparison principle. In stating the result, we consider the functions A,B,D such that for arbitrary $T>0$,

$$A\in C^0((0,1)\times[0,T)\times[0,\infty)),\ B\in C^0((0,1)\times[0,T)),\ D\in C^0([0,1]\times[0,T]\times[0,T]) \tag{4.7}$$

and the operator $\mathcal{Q}$ such that

$$\mathcal{Q}\widetilde{U}(\xi,t):=\widetilde{U}_t(\xi,t)-A(\xi,t,\widetilde{U}_\xi)\widetilde{U}_{\xi\xi}(\xi,t)-\left\{B(\xi,t)+\int_0^t D(\xi,t,s)\widetilde{U}(\xi,s)\mathrm{d}s\right\}\widetilde{U}_\xi(\xi,t) \tag{4.8}$$

for $\xi\in(0,1)$, $t\in[0,T)$ and sufficiently regular $\widetilde{U}:(0,1)\times(0,T)\to\mathbb{R}$. We note that the operator $\mathcal{Q}$ slightly differs from the definition in [12, (4.9)], where A depends on ξ and t only. In the case $\mathcal{Q}=\mathcal{P}$, we will deal with

$$\begin{aligned}
A(\xi,t,\widetilde{U}_\xi)&=n^2\xi^{2-\frac{2}{n}}(n\widetilde{U}_\xi+1)^{m-1},\\
B(\xi,t)&=-n\int_0^t \mathrm{e}^{-(t-s)}\frac{M}{\omega_n}\xi\mathrm{d}s+n(W_0(\xi)-K_0\xi)\mathrm{e}^{-t} \text{ and}\\
D(\xi,t,s)&=n\mathrm{e}^{-(t-s)}.
\end{aligned}$$

Lemma 4.2. *Let $t_1\geq 0$ and $T>t_1$. Suppose that A, B and D satisfy (4.7),*

$$A\geq 0 \text{ in } (0,1)\times(t_1,T)\times[0,\infty) \text{ and } D\geq 0 \text{ in } [0,1]\times[0,T]\times[0,T].$$

Moreover, assume that

$$\underline{U},\overline{U}\in C^0([0,1]\times[0,T])\cap C^1((0,1)\times(t_1,T))\cap C^0((t_1,T);W^{2,\infty}((0,1)))$$

are nonnegative and such that

$$0\leq \underline{U}_\xi(\xi,t)\leq L \text{ for all } \xi\in(0,1) \text{ and } t\in(t_1,T)$$

with some $L>0$ and such that

$$\mathcal{Q}\underline{U}(\xi,t)\leq\mathcal{Q}\overline{U}(\xi,t) \text{ for a.e. } \xi\in(0,1) \text{ and all } t\in(t_1,T),$$

where $\mathcal{Q}$ is as in (4.8). If

$$\underline{U}(\xi,t)\leq\overline{U}(\xi,t) \text{ for all } \xi\in[0,1] \text{ and } t\in[0,t_1] \tag{4.9}$$

and

$$\underline{U}(0,t) \le \overline{U}(0,t) \text{ for all } t \in [t_1, T], \ \underline{U}(1,t) \le \overline{U}(1,t) \quad \text{for all } t \in [t_1, T], \tag{4.10}$$

then

$$\underline{U}(\xi,t) \le \overline{U}(\xi,t) \text{ for all } \xi \in [0,1] \text{ and } t \in [0,T].$$

Proof. This can be shown as in the proof of [12, Lemma 4.2], with the only difference being that in our setting A also depends on $\widetilde{U}_\xi$. The main idea is to show that for arbitrary $\varepsilon > 0$ and certain $\beta > 0$, the function

$$d(\xi,t) := \underline{U}(\xi,t) - \overline{U}(\xi,t) - \varepsilon e^{\beta t} \text{ for } \xi \in [0,1] \text{ and } t \in [0,T)$$

is negative. If this were not the case, then (4.9) and (4.10) would ensure that there exists $(\xi_\star, t_\star) \in (0,1) \times (t_1, T)$ such that $d(\xi_\star, t_\star) = 0$. As we could further assume $t_\star$ to be minimal, $d(\cdot, t_\star)$ would attain a maximum at $\xi_\star$; hence, $d_\xi(\xi_\star, t_\star) = 0$. Since then, $A(\xi_\star, t_\star, \underline{U}_\xi(\xi_\star, t_\star)) = A(\xi_\star, t_\star, \overline{U}_\xi(\xi_\star, t_\star))$, and due to the required regularity of A, $\underline{U}$, and $\overline{U}$, the fact that A depends on $\widetilde{U}_\xi$ turns out to cause no additional challenges, and we can arrive at a contradiction just as in the proof of [12, Lemma 4.2]. □

To prove blow-up, we shall construct a subsolution with unbounded space derivative to (4.6). Referring to [12, (6.1)], we put the function $\underline{U}$ as

$$\underline{U}(\xi,t) := \begin{cases} \frac{a(t)\xi}{b(t)+\xi} & \text{if } \xi \in [0,\xi_0] \text{ and } t \in [0,\infty), \\ \frac{a(t)b(t)\xi + a(t)\xi_0^2}{(b(t)+\xi_0)^2} & \text{if } \xi \in (\xi_0, 1] \text{ and } t \in [0,\infty), \end{cases} \tag{4.11}$$

where $\xi_0 \in (0,1)$ and the functions a and b are defined as

$$a(t) := \frac{M}{\omega_n} \cdot \frac{(b(t)+\xi_0)^2}{b(t)+\xi_0^2} \text{ and } b(t) := b_0 e^{-\alpha t} \qquad \text{for } t \in [0,\infty) \tag{4.12}$$

with some $b_0 > 0$ and $\alpha > 0$. Then the function $\underline{U}$ satisfies (cf. [12, Lemma 6.1])

$$\underline{U} \in C^1([0,1] \times [0,\infty)) \cap C^0([0,\infty); W^{2,\infty}((0,1))) \cap C^0([0,\infty); C^2_{\text{loc}}([0,1] \setminus \{\xi_0\})).$$

Moreover, by computing $\mathcal{P}\underline{U}$, we have the following lemma.

Lemma 4.3. *Let $\alpha, b_0 > 0$, $\xi_0 \in (0,1)$ and a, b be defined as in (4.12). Then the function $\underline{U}$ defined in (4.11) satisfies that*

$$\begin{aligned} \frac{(b(t)+\xi)^2}{a(t)b(t)\xi} \mathcal{P}\underline{U}(\xi,t) = {} & \frac{a'(t)(b(t)+\xi)}{a(t)b(t)} - \frac{b'(t)}{b(t)} + 2n^2 \left(\frac{na(t)b(t)}{(b(t)+\xi)^2} + 1 \right)^{m-1} \frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi} \\ & - n \int_0^t e^{-(t-s)} \left(\frac{a(s)}{b(s)+\xi} - \frac{M}{\omega_n} \right) ds - n \left(\frac{W_0(\xi)}{\xi} - K_0 \right) e^{-t} \end{aligned} \tag{4.13}$$

for all $\xi \in (0, \xi_0)$ and $t \in (0,\infty)$ and

$$\begin{aligned} \frac{(b(t)+\xi_0)^2}{a(t)b(t)} \mathcal{P}\underline{U}(\xi,t) = {} & \frac{a'(t)\xi}{a(t)} + \frac{b'(t)\xi}{b(t)} + \frac{a'(t)\xi_0^2}{a(t)b(t)} - 2\frac{b'(t)\xi + \frac{b'(t)}{b(t)}\xi_0^2}{b(t)+\xi_0} \\ & - n \int_0^t e^{-(t-s)} \left(\frac{a(s)b(s)\xi + a(s)\xi_0^2}{(b(s)+\xi_0)^2} - \frac{M}{\omega_n}\xi \right) ds \\ & - n(W_0(\xi) - K_0\xi)e^{-t} \end{aligned} \tag{4.14}$$

for all $\xi \in (\xi_0, 1)$ and $t \in (0,\infty)$.

Proof. This results from straightforward computations; for details, see [12, Lemma 6.1], where only the third term in the right-hand side of (4.13) differs slightly from the one of [12, (6.2)]. □

Our goal is to make sure that $\mathcal{P}\underline{U} \le 0$ in $(0,1)\times(0,\infty)$. Since the function $\underline{U}$ fulfills (4.14) in $(\xi_0,1)\times(0,\infty)$, which is similar to [12, (6.3)], we can immediately obtain the following lemma.

Lemma 4.4. *Assume that $\xi_0 \in (0,1)$ and $\eta_0 > 0$ satisfy*

$$\frac{W_0(\xi) - K_0\xi}{1-\xi} \ge \eta_0 \tag{4.15}$$

for all $\xi \in (\xi_0,1)$. Then for all $\alpha_\star > 0$, there exists $\alpha \in (0,\alpha_\star)$ such that for any choice of $b_0 \in (0,\xi_0^2)$, the function $\underline{U}$ in (4.11) satisfies

$$\mathcal{P}\underline{U}(\xi,t) \le 0 \tag{4.16}$$

for all $\xi \in (\xi_0,1)$ and $t \in (0,\infty)$.

Proof. The main difference of the operator $\mathcal{P}$ introduced in (4.2) compared to the one considered in [12] is the factor in front of $\widetilde{U}_{\xi\xi}$, which of course is inconsequential in regions where $\underline{U}_{\xi\xi}$ vanishes. Accordingly, this lemma can be shown as in [12, Lemma 6.2 and Lemma 6.3], with a slightly different choice of η_0 depending on n. □

Before estimating the right-hand side of (4.13), we recall an estimate for its first term.

Lemma 4.5. *Let $\alpha, b_0 > 0$, $\xi_0 \in (0,1)$ and a, b be defined as in (4.12). The function a satisfies*

$$\frac{a'(t)(b(t)+\xi)}{a(t)b(t)} \le \frac{\alpha}{\xi_0}$$

for all $\xi \in (0,\xi_0)$ and $t \in (0,\infty)$.

Proof. This is derived from the definitions of a and b in (4.12); for the computation, see [12, Lemma 6.4]. □

Now, we prove the estimate $\mathcal{P}\underline{U} \le 0$ in $(0,\xi_0)\times(0,\infty)$ with some $\xi_0 \in (0,1)$. We first consider this estimate for suitably large times. When $m = 2 - \frac{2}{n}$, a largeness condition for M is needed, whereas when $m \in \left[1, 2-\frac{2}{n}\right)$, the estimate $\mathcal{P}\underline{U} \le 0$ is satisfied for any $M > 0$.

Lemma 4.6. *Assume that*

$$W_0(\xi) - K_0\xi \ge 0$$

for all $\xi \in (0,1)$.

(i) If $m = 2 - \frac{2}{n}$ and

$$M > 2^{\frac{n}{2}} n^{n-1}\omega_n,$$

or

(ii) if $m \in \left[1, 2-\frac{2}{n}\right)$ and $M > 0$,

then there exist $\xi_0 \in (0,1)$ and $\alpha_\star > 0$ with the property that for all $\alpha \in (0,\alpha_\star)$, one can find $b_0 \in (0,\xi_0^2)$ and $t_0 \in (0,\infty)$ such that the function $\underline{U}$ in (4.11) satisfies

$$\mathcal{P}\underline{U}(\xi,t) \le 0$$

for all $\xi \in (0,\xi_0)$ and $t \in [t_0,\infty)$.

Proof. In part, this proof is similar to [12, Lemma 6.5]. However, we choose to still give a full proof for two reasons: First, this is the point where the conditions on m and M play a crucial role. Second, unlike in same of the proofs above, here, we need to introduce new ideas for dealing with the nonlinear diffusion present in (1.3) (and hence in (4.2)) for $m > 1$.

We assume that m and M are as required by (i) or (ii) and take $\varepsilon \in (0,1)$ and $\xi_0 \in (0,1)$ satisfying

$$\xi_0 \le \frac{\varepsilon}{2} \tag{4.17}$$

and, in the case of (ii), also

$$\xi_0^{2-\frac{2}{n}-m} < \frac{(1-\varepsilon)^3 nM}{(1+\varepsilon)\omega_n} \cdot \frac{1}{2n^2}\left((\varepsilon+1)^2\frac{nM}{\omega_n} + \frac{\varepsilon}{2}\right)^{-(m-1)}, \tag{4.18}$$

so that, in both cases,

$$c_1 := \frac{(1-\varepsilon)^3 nM}{(1+\varepsilon)\omega_n} - 2n^2\left((\varepsilon+1)^2\frac{nM}{\omega_n} + \frac{\varepsilon}{2}\right)^{m-1}\xi_0^{2-\frac{2}{n}-m} > 0,$$

either by (4.16) and sufficiently small choice of ε or by (4.18). We let

$$\alpha_\star = \min\left\{\frac{\log\frac{1}{1-\varepsilon}}{\log\frac{1}{\varepsilon}}, \frac{c_1}{4}\right\}. \tag{4.19}$$

Moreover, given $\alpha \in (0, \alpha_\star)$, we pick $b_0 > 0$ and $t_0 > 0$ fulfilling that

$$b_0 \le \varepsilon\xi_0^2 \le \xi_0^2 \le \xi_0 \tag{4.20}$$

and

$$t_0 \ge \frac{1}{\alpha}\log\frac{1}{1-\varepsilon}. \tag{4.21}$$

The condition (4.15) and the identity (4.13) assert that

$$\begin{aligned} \frac{(b(t)+\xi)^2}{a(t)b(t)\xi}\mathcal{P}\underline{U}(\xi,t) \le{}& \frac{a'(t)(b(t)+\xi)}{a(t)b(t)} - \frac{b'(t)}{b(t)} + 2n^2\left(\frac{na(t)b(t)}{(b(t)+\xi)^2}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi} \\ & - n\int_0^t \mathrm{e}^{-(t-s)}\left(\frac{a(s)}{b(s)+\xi} - \frac{M}{\omega_n}\right)\mathrm{d}s \end{aligned} \tag{4.22}$$

for all $\xi \in (0, \xi_0)$ and $t \in (0, \infty)$. It follows from Lemma 4.5 and the definition of b in (4.12) that

$$J_1(t) := \frac{a'(t)(b(t)+\xi)}{a(t)b(t)} - \frac{b'(t)}{b(t)} \le \frac{\alpha}{\xi_0} + \alpha \le \frac{2}{\xi_0}\alpha \tag{4.23}$$

and from the definition of a in (4.12) and (4.17) (and (4.20)) that

$$\varepsilon\frac{a(t)}{b(t)+\xi} = \frac{M}{\omega_n}\cdot\frac{(b(t)+\xi_0)^2\varepsilon}{(b(t)+\xi)(b(t)+\xi_0^2)} \ge \frac{M}{\omega_n}\cdot\frac{(b(t)+\xi_0)\varepsilon}{b(t)+\xi_0^2} \ge \frac{M}{\omega_n}\cdot\frac{\xi_0\cdot 2\xi_0}{\xi_0^2+\xi_0^2} = \frac{M}{\omega_n} \tag{4.24}$$

as well as from (4.12) and (4.20) that

$$a(t) = \frac{M}{\omega_n}\cdot\frac{(b(t)+\xi_0)^2}{b(t)+\xi_0^2} \ge \frac{M}{\omega_n}\cdot\frac{\xi_0^2}{b_0+\xi_0^2} \ge \frac{M}{(1+\varepsilon)\omega_n} \tag{4.25}$$

for all $\xi \in (0, \xi_0)$ and $t \in (0, \infty)$ (see also [12, p. 3671]). Furthermore, as in [12, pp. 3671–3672], the conditions (4.19) and (4.21) ensure that

$$\int_0^t \mathrm{e}^{-(t-s)}\frac{1}{b(s)+\xi}\mathrm{d}s \ge \frac{(1-\varepsilon)^2}{b(t)+\xi} \tag{4.26}$$

for all $\xi \in (0, \xi_0)$ and $t \in [t_0, \infty)$. Therefore, in light of (4.24)–(4.26), we obtain

$$\begin{aligned}
-n\int_0^t \mathrm{e}^{-(t-s)}\left(\frac{a(s)}{b(s)+\xi}-\frac{M}{\omega_n}\right)\mathrm{d}s &\le -(1-\varepsilon)n\int_0^t \mathrm{e}^{-(t-s)}\frac{a(s)}{b(s)+\xi}\mathrm{d}s\\
&\le -\frac{(1-\varepsilon)nM}{(1+\varepsilon)\omega_n}\int_0^t \mathrm{e}^{-(t-s)}\frac{1}{b(s)+\xi}\mathrm{d}s\\
&\le -\frac{(1-\varepsilon)^3 nM}{(1+\varepsilon)\omega_n}\cdot\frac{1}{b(t)+\xi}
\end{aligned}\tag{4.27}$$

for all $\xi \in (0, \xi_0)$ and $t \in [t_0, \infty)$. Also, since (4.12), (4.20) and the relation $\xi_0 < 1$ yield

$$a(t) \le \frac{M}{\omega_n}\cdot\frac{(\varepsilon\xi_0^2+\xi_0)^2}{\xi_0^2} \le (\varepsilon+1)^2\cdot\frac{M}{\omega_n}$$

for all $t \in [0, \infty)$, we infer from this inequality, the fact that $\frac{b(t)}{b(t)+\xi} \le 1$ and (4.17) that

$$\begin{aligned}
2n^2\left(\frac{na(t)b(t)}{(b(t)+\xi)^2}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi} &\le 2n^2\left((\varepsilon+1)^2\cdot\frac{nM}{\omega_n}\cdot\frac{1}{b(t)+\xi}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi}\\
&\le 2n^2\left((\varepsilon+1)^2\cdot\frac{nM}{\omega_n}\cdot\frac{1}{\xi}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi}\\
&= 2n^2\left((\varepsilon+1)^2\cdot\frac{nM}{\omega_n}+\xi\right)^{m-1}\frac{\xi^{2-\frac{2}{n}-m}}{b(t)+\xi}\\
&\le 2n^2\left((\varepsilon+1)^2\cdot\frac{nM}{\omega_n}+\frac{\varepsilon}{2}\right)^{m-1}\frac{\xi_0^{2-\frac{2}{n}-m}}{b(t)+\xi}
\end{aligned}$$

for all $\xi \in (0, \xi_0)$ and $t \in [0, \infty)$, which together with (4.27) implies that

$$\begin{aligned}
J_2(t) &:= 2n^2\left(\frac{na(t)b(t)}{(b(t)+\xi)^2}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi}-n\int_0^t \mathrm{e}^{-(t-s)}\left(\frac{a(s)}{b(s)+\xi}-\frac{M}{\omega_n}\right)\mathrm{d}s\\
&\le \left[2n^2\left((\varepsilon+1)^2\cdot\frac{nM}{\omega_n}+\frac{\varepsilon}{2}\right)^{m-1}\xi_0^{2-\frac{2}{n}-m}-\frac{(1-\varepsilon)^3 nM}{(1+\varepsilon)\omega_n}\right]\cdot\frac{1}{b(t)+\xi} = -\frac{c_1}{b(t)+\xi}
\end{aligned}\tag{4.28}$$

for all $\xi \in (0, \xi_0)$ and $t \in [t_0, \infty)$. Noting from (4.20) that $b(t)+\xi \le \varepsilon\xi_0^2+\xi_0 \le 2\xi_0$, we have from (4.22), (4.23), and (4.28) that

$$\frac{(b(t)+\xi)^2}{a(t)b(t)\xi}\mathcal{P}\underline{U}(\xi,t) \le \frac{2}{\xi_0}\alpha-\frac{c_1}{b(t)+\xi} \le \frac{2}{\xi_0}\alpha-\frac{c_1}{2\xi_0} = \frac{2}{\xi_0}\left(\alpha-\frac{c_1}{4}\right) \le 0$$

for all $\xi \in (0, \xi_0)$ and $t \in [t_0, \infty)$. □

Next, we derive the estimate $\mathcal{P}\underline{U} \le 0$ near the origin.

Lemma 4.7. *Let $\alpha, b_0 > 0$ and $\xi_0 \in (0, 1)$. Then, for all $t_0 \in (0, \infty)$, there exists $\Gamma_0 > 0$ such that whenever W_0 and K_0 satisfy*

$$\frac{W_0(\xi)}{\xi} - K_0 \ge \Gamma_0$$

for all $\xi \in (0, \xi_0)$, the function $\underline{U}$ in (4.11) satisfies

$$\mathcal{P}\underline{U}(\xi, t) \le 0$$

for all $\xi \in (0, \xi_0)$ and $t \in (0, t_0)$.

Proof. In estimating the terms on the right-hand side of (4.13), we follow [12, Lemma 6.6]. In the following, we mainly point out the differences. Since we have from (4.12) that

$$\frac{b(t)}{(b(t)+\xi)^2} \le \frac{1}{b(t)} = \frac{e^{\alpha t}}{b_0} \text{ and } a(t) \le \frac{M}{\omega_n}\cdot\frac{(b_0+1)^2}{b(t)} = \frac{M}{\omega_n}\cdot\frac{(b_0+1)^2}{b_0}e^{\alpha t}, \tag{4.29}$$

it follows that

$$\frac{na(t)b(t)}{(b(t)+\xi)^2} \le \frac{nM}{\omega_n}\cdot\frac{(b_0+1)^2}{b_0^2}e^{2\alpha t_0} =: c_1$$

for all $\xi \in (0,\xi_0)$ and $t \in (0,t_0)$. Thus, in light of this and the first identity in (4.29), we can estimate

$$\begin{aligned} 2n^2\left(\frac{na(t)b(t)}{(b(t)+\xi)^2}+1\right)^{m-1}\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi} &\le 2n^2(c_1+1)^{m-1}\cdot\frac{\xi^{1-\frac{2}{n}}}{b(t)+\xi} \\ &\le 2n^2(c_1+1)^{m-1}\cdot\frac{\xi_0^{1-\frac{2}{n}}}{b_0}e^{\alpha t_0} =: c_2 \end{aligned} \tag{4.30}$$

for all $\xi \in (0,\xi_0)$ and $t \in (0,t_0)$. Therefore, by Lemma 4.5 and (4.12) and (4.30),

$$\frac{(b(t)+\xi)^2}{a(t)b(t)\xi}\mathcal{P}\underline{U}(\xi,t) \le \frac{\alpha}{\xi_0}+\alpha+c_2-n\int_0^t e^{-(t-s)}\left(\frac{a(s)}{b(s)+\xi}-\frac{M}{\omega_n}\right)ds - n\left(\frac{W_0(\xi)}{\xi}-K_0\right)e^{-t}.$$

As in [12, Lemma 6.6], $-n\int_0^t e^{-(t-s)}\left(\frac{a(s)}{b(s)+\xi}-\frac{M}{\omega_n}\right)ds \le 0$; therefore, $\mathcal{P}\underline{U} \le 0$ for all $\xi \in (0,\xi_0)$ and $t \in (0,t_0)$ if we set

$$\Gamma_0 := \frac{1}{n}\left(\left(\frac{1}{\xi_0}+1\right)\alpha + c_2\right)e^{t_0}.$$

□

Finally, we construct initial data to show Theorem 1.4.

Lemma 4.8. *Let $\alpha, b_0, \eta, \Gamma_0 > 0$, $\xi_0 \in (0,1)$ and a, b be defined as in (4.12). Set $R := \xi_0^{\frac{1}{n}}$, $\eta_0 := \frac{\eta}{n}$ as well as*

$$\Gamma_u := \frac{na(0)}{b(0)} = \frac{nM}{\omega_n}\cdot\frac{(b_0+\xi_0)^2}{b_0+\xi_0^2}\cdot\frac{1}{b_0},\ \gamma := \frac{nM}{\omega_n}\cdot\frac{b_0}{b_0+\xi_0^2} \text{ and } \Gamma_w := n\Gamma_0. \tag{4.31}$$

If u_0 and w_0 satisfy (1.6)–(1.9), then

$$\frac{W_0(\xi)-K_0\xi}{1-\xi} \ge \eta_0 \text{ for all } \xi \in (\xi_0,1) \text{ and } \frac{W_0(\xi)}{\xi}-K_0 \ge \Gamma_0 \text{ for all } \xi \in (0,\xi_0), \tag{4.32}$$

and moreover, the function $\underline{U}$ defined in (4.11) satisfies that

$$\underline{U}(\xi,0) \le U(\xi,0) \text{ for all } \xi \in (0,1). \tag{4.33}$$

Proof. In the same way as in [12, p. 3674], from (4.3) and (1.9), we can make sure that $W_0(\xi) - K_0\xi = (1-\xi)\left(K_0 - \frac{1}{n}⨍_{B_1\setminus B_{\xi^{\frac{1}{n}}}} w_0\right) \ge (1-\xi)\frac{\eta}{n} = (1-\xi)\eta_0$ for all $\xi \in (\xi_0,1)$, that is, the first inequality in (4.32) holds. Moreover, condition (1.8) implies that $\frac{W_0(\xi)}{\xi} - K_0 \ge \frac{1}{n}\Gamma_w = \Gamma_0$ for all $\xi \in (0,\xi_0)$, see also [12, (6.43)]. Finally, as in [12, p. 3675], combining the definitions (4.11) and (4.12) with the condition (1.6), we find that $\underline{U}(\xi,0) \le \frac{1}{n}\Gamma_u\xi \le U(\xi,0)$ for all $\xi \in (0,\xi_0)$ and from (4.11), (4.12) and (1.7) and the definition of γ that $\underline{U}(\xi,0) \le \frac{M}{\omega_n} - \frac{M}{\omega_n}\cdot\frac{b_0}{b_0+\xi_0^2}(1-\xi) \le U(\xi,0)$ for all $\xi \in (\xi_0,1)$. □

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. This theorem can be shown similarly to [12, Theorem 1.3]. Thus, we only give a sketch of the proof. First, we may assume $T_{\max} = \infty$ as the case $T_{\max} < \infty$ is already covered by Proposition 1.2. We then take $\xi_0 \in (0,1)$, $\alpha_\star > 0$, $b_0 \in (0, \xi_0^2)$ and $t_0 \in (0,\infty)$ given by Lemma 4.6, and next set $R := \xi_0^{\frac{1}{n}}$. Also, we put $\eta_0 := \frac{\eta}{n}$ and pick $\alpha \in (0, \alpha_\star)$ provided in Lemma 4.4. Moreover, we define Γ_u, γ and Γ_w as in (4.31). Our goal is to prove that

$$u(0,t) = nU_\xi(0,t) \ge n\underline{U}_\xi(0,t) \text{ for all } t \in (0,\infty). \tag{4.34}$$

As $\underline{U}_\xi(0,t)$ grows at least exponentially because we have from the definition of b in (4.12) and the inequality $b(t) \le b_0 < \xi_0^2$ that

$$\underline{U}_\xi(0,t) = \lim_{\xi \searrow 0} \frac{\underline{U}(\xi,t)}{\xi} = \frac{M}{\omega_n} \cdot \frac{(b(t)+\xi_0)^2}{b(t)+\xi_0^2} \cdot \frac{1}{b(t)} \ge \frac{M}{\omega_n} \cdot \frac{\xi_0^2}{2\xi_0^2} \cdot \frac{1}{b_0} e^{\alpha t} = \frac{M}{2\omega_n b_0} e^{\alpha t}$$

for all $t \in (0,\infty)$, (4.34) implies that also $u(0,\cdot)$ grows at least exponentially. In order to obtain (4.34), we show that

$$U(\xi,t) \ge \underline{U}(\xi,t) \text{ for all } \xi \in [0,1] \text{ and } t \in [0,\infty). \tag{4.95}$$

Thanks to Lemma 4.8, we find that (4.32) holds. Therefore, it follows from Lemmata 4.4,4.6, and 4.7 that

$$\mathcal{P}\underline{U}(\xi,t) \le 0 \text{ for all } \xi \in (0,1) \setminus \{\xi_0\} \text{ and } t \in (0,\infty). \tag{4.36}$$

Also, we can immediately observe from (4.1) and (4.11) that $\underline{U}(0,t) = U(0,t) = 0$ and $\underline{U}(1,t) = U(1,t) = \frac{M}{\omega_n}$ for all $t \in (0,\infty)$. In light of these identities and (4.33) as well as (4.36), we apply the comparison principle in Lemma 4.2 with $t_1 := 0$, $\overline{U} := U$ and $\mathcal{Q} := \mathcal{P}$ to derive (4.35). Thus, we can see that

$$u(0,t) = nU_\xi(0,t) = \lim_{\xi \searrow 0} \frac{nU(\xi,t)}{\xi} \ge \lim_{\xi \searrow 0} \frac{n\underline{U}(\xi,t)}{\xi} = n\underline{U}_\xi(0,t) \text{ for all } t \in (0,\infty),$$

which shows (4.34) and hence concludes the proof.

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AUTHOR CONTRIBUTIONS

Mario Fuest: Conceptualization; investigation; writing—original draft; writing—review & editing. **Johannes Lankeit**: Conceptualization; investigation; writing—original draft; writing—review & editing. **Yuya Tanaka**: Investigation; conceptualization; writing—original draft; writing—review & editing.

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REFERENCES

1. T. Hillen and K. J. Painter, *A user's guide to PDE models for chemotaxis*, J. Math. Biol. **58** (2009), no. 1-2, 183–217. DOI 10.1007/s00285-008-0201-3
2. N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, *Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. **25** (2015), no. 9, 1663–1763. DOI 10.1142/S021820251550044X
3. D. Horstmann, *From 1970 until present: the Keller–Segel model in chemotaxis and its consequences I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), no. 3, 103–165.
4. J. Lankeit and M. Winkler, *Facing low regularity in chemotaxis systems*, Jahresber. Dtsch. Math.-Ver. **122** (2020), no. 1, 35–64. DOI 10.1365/s13291-019-00210-z

5. T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. **5** (1995), no. 2, 581–601.
6. M. A. Herrero and J. J. L. Velázquez, *A blow-up mechanism for a chemotaxis model*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24** (1997), no. 4, 633–683.
7. N. Mizoguchi and M. Winkler, *Blow-up in the two-dimensional parabolic Keller–Segel system*. Preprint.
8. T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997), no. 3, 411–433.
9. T. Senba, *Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domains*, Funkcial. Ekvac. **48** (2005), no. 2, 247–271. DOI 10.1619/fesi.48.247
10. M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system*, J. Math. Pures Appl. **100** (2013), no. 5, 748–767. DOI 10.1016/j.matpur.2013.01.020
11. S. Strohm, R. C. Tyson, and J. A. Powell, *Pattern formation in a model for mountain pine beetle dispersal: linking model predictions to data*, Bull. Math. Biol. **75** (2013), no. 10, 1778–1797. DOI 10.1007/s11538-013-9868-8
12. Y. Tao and M. Winkler, *Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production*, J. Eur. Math. Soc. **19** (2017), no. 12, 3641–3678. DOI 10.4171/JEMS/749
13. Z. Xiang and L. Yang, *Critical mass for the Cauchy problem of a chemotaxis model with indirect signal production mechanism*, 2023. Preprint.
14. Ph. Laurençot, *Global bounded and unbounded solutions to a chemotaxis system with indirect signal production*, Discrete Contin. Dyn. Syst. B **24** (2019), no. 12, 6419–6444. DOI 10.3934/dcdsb.2019145
15. Ph. Laurençot and Ch. Stinner, *Mass threshold for infinite-time blowup in a chemotaxis model with split population*, SIAM J. Math. Anal. **53** (2021), no. 3, 3385–3419. DOI 10.1137/20M1371968
16. K. Fujie and T. Senba, *Application of an Adams type inequality to a two-chemical substances chemotaxis system*, J. Differ. Equ. **263** (2017), no. 1, 88–148. DOI 10.1016/j.jde.2017.02.031
17. K. Fujie and T. Senba, *Blowup of solutions to a two-chemical substances chemotaxis system in the critical dimension*, J. Differ. Equ. **266** (2019), no. 2-3, 942–976. DOI 10.1016/j.jde.2018.07.068
18. Y. Li and J. Lankeit, *Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion*, Nonlinearity **29** (2016), no. 5, 1564–1595. DOI 10.1088/0951-7715/29/5/1564
19. S. Ishida, K. Seki, and T. Yokota, *Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains*, J. Differ. Equ. **256** (2014), no. 8, 2993–3010. DOI 10.1016/j.jde.2014.01.028
20. Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity*, J. Differ. Equ. **252** (2012), no. 1, 692–715. DOI 10.1016/j.jde.2011.08.019
21. T. Cieślak and Ch. Stinner, *Finite-time blowup global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions*, J. Differ. Equ. **252** (2012), no. 10, 5832–5851. DOI 10.1016/j.jde.2012.01.045
22. T. Cieślak and Ch. Stinner, *New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models*, J. Differ. Equ. **258** (2015), no. 6, 2080–2113. DOI 10.1016/j.jde.2014.12.004
23. S. Ishida and T. Yokota, *Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, J. Differ. Equ. **252** (2012), no. 2, 1421–1440. DOI 10.1016/j.jde.2011.02.012
24. Y. Mimura, *The variational formulation of the fully parabolic Keller–Segel system with degenerate diffusion*, J. Differ. Equ. **263** (2017), no. 2, 1477–1521. DOI 10.1016/j.jde.2017.03.020
25. T. Hashira, S. Ishida, and T. Yokota, *Finite-time blow-up for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, J. Differ. Equ. **264** (2018), no. 10, 6459–6485. DOI 10.1016/j.jde.2018.01.038
26. A. Blanchet, J. A. Carrillo, and Ph. Laurençot, *Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differ. Equ. **35** (2009), no. 2, 133–168. DOI 10.1007/s00526-008-0200-7
27. A. Blanchet and Ph. Laurençot, *The parabolic–parabolic Keller–Segel system with critical diffusion as a gradient flow in* $\mathbb{R}^d$, $d \geq 3$, Comm. Partial Differ. Equ. **38** (2013), no. 4, 658–686. DOI 10.1080/03605302.2012.757705
28. Ph. Laurençot and N. Mizoguchi, *Finite time blowup for the parabolic–parabolic Keller–Segel system with critical diffusion*, Ann. Inst. H. Poincaré, C Anal. Non Linéaire **34** (2017), no. 1, 197–220. DOI 10.1016/j.anihpc.2015.11.002
29. E. Nasreddine, *Global existence of solutions to a parabolic-elliptic chemotaxis system with critical degenerate diffusion*, J. Math. Anal. Appl. **417** (2014), no. 1, 144–163. DOI 10.1016/j.jmaa.2014.02.069
30. W. Jäger and S Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), no. 2, 819–824. DOI 10.2307/2153966
31. Y. Tao and M. Winkler, *A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source*, SIAM J. Math. Anal. **43** (2011), no. 2, 685–704. DOI 10.1137/100802943
32. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, Inc., New York-Montreal Que.-London, 1969.

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