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Teleparallel Newton–Cartan gravity

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Abstract

We discuss a teleparallel version of Newton–Cartan gravity. This theory arises as a formal large-speed-of-light limit of the teleparallel equivalent of general relativity (TEGR). Thus, it provides a geometric formulation of the Newtonian limit of TEGR, similar to standard Newton–Cartan gravity being the Newtonian limit of general relativity. We show how by a certain gauge-fixing the standard formulation of Newtonian gravity can be recovered.

Keywords: Newton–Cartan gravity, teleparallel gravity, Newtonian limit, Galilei geometry, teleparallel equivalent of general relativity

(Some figures may appear in colour only in the online journal)

1. Introduction

Any viable theory of gravity has to reproduce Newtonian gravity in the slow-velocity, weak-gravity Newtonian limit. For standard general relativity (GR), this Newtonian limit is traditionally exhibited by linearising the Einstein equations around a Minkowski spacetime background and imposing small velocities of matter, employing a suitable choice of coordinates (see, e.g., the textbook accounts [1, chapter 18], [2, section 5.1]).

However, there exists also a geometric, coordinate-free formulation of this limiting process: standard Newtonian gravity may be reformulated in geometric terms as so-called *Newton–Cartan gravity*, which features a Galilei-relativistic spacetime geometry and describes Newtonian gravity by a curved connection, similar to GR. Newton–Cartan gravity can be shown to arise as the Newtonian limit of GR. It was originally developed by Cartan in 1923 [3, 4] and independently by Friedrichs in 1926 (published in 1928) [5]. Important further contributions to Newton–Cartan gravity were made by Trautman in the 1960s [6, 7], by Künzle in the

1970s [8, 9], and by Ehlers in a 1981 article [10, 11] on the precise relation between GR and Newton–Cartan gravity. An elementary textbook account of Newton–Cartan gravity is given in a book on foundational issues of GR by Malament [12, chapter 4], to which we also refer for further historical references.

In this article, we answer the question what the corresponding geometric description of the Newtonian limit looks like for the teleparallel equivalent of general relativity (TEGR) [13]. Teleparallel gravity [14] is a class of gravitational theories in which the geometric framework for the description of gravity is modified compared to that of standard GR: in addition to the spacetime metric, one considers as basic ingredient a connection that is metric but, differently to the Levi-Civita connection, torsionful and flat. This framework allows for a dynamically equivalent formulation of GR, namely the above-mentioned TEGR, and modified teleparallel theories of gravity have recently gained a lot of interest in the context of classical modified gravity or effective descriptions of quantum gravity [14–17].

The geometry of Galilei manifolds, which underlies Newton–Cartan gravity, admits a 'gauge-theoretic' description in terms of the Bargmann group (i.e. the centrally extended inhomogeneous Galilei group), which was initially discovered in the 1980s in the context of massive matter coupling [18, 19]. Over the last decade, interest in Galilei geometries has significantly increased due to novel applications in condensed matter physics as well as in 'non-relativistic' string theory and related applications in quantum gravity [20–24]. In this context, the gauge-theoretic perspective has been further developed [25, 26]. It is this perspective on the geometry of Galilei manifolds in which a teleparallel formulation of Newton–Cartan gravity arises very naturally, which turns out to give the sought-for geometric description of the Newtonian limit of TEGR.

For the consideration of modifications of GR, there are some typical physical motivations, such as the problem of quantisation, observational issues in cosmology or astrophysics, or the prediction of singularities. Of those motivations, none applies to Newtonian gravity—for example, Newton–Cartan gravity coupled to a matter field can be rigorously quantised [27]. Therefore, one might argue that on physical grounds, the investigation of geometrically modified descriptions of Newtonian gravity just by itself does not seem necessary. Notwithstanding this argument, our investigation of a 'teleparallelised' version of Newton–Cartan gravity that gives the geometric description of the Newtonian limit of teleparallel gravity is still of fundamental importance on a both conceptual and theoretical level: the description of a physically interesting limit of an inherently geometric theory ought to be given in a geometric fashion.

A teleparallel formulation of Newton–Cartan gravity was also constructed by Read and Teh in [28]. Their approach is different from ours in two important aspects. First, in [28] the theory is constructed only in a restricted, 'gauge-fixed' situation; in the present paper, we develop instead a completely general teleparallel description of Newton–Cartan gravity, without introducing arbitrary assumptions on the connection or the frame. (In fact, the formalism of [28] may be shown to arise as a special case of ours.) Second, [28] does not analyse the Newtonian limit of TEGR, but instead shows that its teleparallel formulation of Newton–Cartan gravity can be obtained from higher-dimensional TEGR by null reduction, i.e. quotienting along a lightlike symmetry [19, 29].

The structure of this paper is as follows. First, we quickly introduce notation and conventions used throughout the article in section 1.1. In section 2, we introduce the geometric framework for our formulation of teleparallel Newton–Cartan gravity (starting with a review of the geometry of Galilei manifolds), and formulate the theory. Section 3 shows that this theory is the Newtonian $c \to \infty$ limit of TEGR; in section 4 we discuss how to recover the standard formulation of Newtonian gravity from teleparallel Newton–Cartan gravity. We conclude and discuss possible directions for future research in section 5.

1.1. Notation and conventions

1.1.1. Lie groups and algebras. For a Lie group denoted by a Latin capital letter (or by several letters), the corresponding Lie algebra will be denoted by the corresponding lowercase Fraktur letter, e.g. $\mathfrak{g} = \mathrm{Lie}(G)$.

The semidirect product of Lie groups H and N with respect to a homomorphism $\rho\colon H \to \mathsf{Aut}(N)$ will be denoted by

$$H \ltimes_\rho N. \tag{1.1}$$

Similarly, the semidirect sum of Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$ with respect to a homomorphism $\tilde{\rho}\colon \mathfrak{h} \to \mathsf{Der}(\mathfrak{n})$ will be denoted by

$$\mathfrak{h} \oplus_{\tilde{\rho}} \mathfrak{n}. \tag{1.2}$$

As a vector space, this is of course just the direct sum of $\mathfrak{h}$ and $\mathfrak{n}$, but we want to take account of the nontrivial Lie bracket structure in the notation. In the case of both semidirect products and sums, if the homomorphism is clear from context we will omit it from the notation.

1.1.2. The Galilei and Bargmann groups. When we speak of the Galilei group without further qualification, we will mean the orthochronous homogeneous Galilei group, which is a semidirect product

$$\mathsf{Gal} = \mathsf{O}(3) \ltimes \mathbb{R}^3 \tag{1.3}$$

whose parts are interpreted as spatial (improper) rotations and Galilei boosts, respectively, where the homomorphism for the semidirect product is the defining representation of $\mathsf{O}(3)$.[1]

The inhomogeneous (orthochronous) Galilei group is the semidirect product

$$\mathsf{IGal} = \mathsf{Gal} \ltimes \mathbb{R}^4 \tag{1.4}$$

of homogeneous transformations and spacetime translations, where the homomorphism $\mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^4)$ is given by

$$(R,k)(s,y) = (s, Ry + sk) \tag{1.5a}$$

for $(R,k) \in \mathsf{Gal}$ and $(s,y) \in \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$. We can thus view Gal as a subgroup of $\mathsf{GL}(4)$, with the element $(R,k) \in \mathsf{Gal}$ corresponding to the matrix

$$\begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix}. \tag{1.5b}$$

The Bargmann group, whose Lie algebra is the essentially unique one-dimensional central extension of the inhomogeneous Galilei algebra, is given as

$$\mathsf{Barg} = \mathsf{Gal} \ltimes (\mathbb{R}^4 \times \mathsf{U}(1)), \tag{1.6}$$

where the homomorphism $\mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^4 \times \mathsf{U}(1))$ is given by

$$(R,k)(s,y,\mathrm{e}^{\mathrm{i}\varphi}) = (s, Ry + sk, \mathrm{e}^{\mathrm{i}(\varphi + k\cdot Ry + \frac{1}{2}|k|^2 s)}). \tag{1.7}$$

[1] Note that the restriction to the orthochronous Galilei group is not essential at all; including time reversal would just introduce the necessity to define the notion of Galilei manifolds in section 2 in terms of a time metric instead of a clock one-form, which automatically introduces a choice of time orientation (and time-orientability).

Note that our sign conventions for the inhomogeneous Galilei group and the Bargmann group are different from those used in [26].

1.1.3. Forms on principal bundles. Forms living on the total space of a principal bundle will be denoted by boldface letters; their local representatives on the base manifold, i.e. pullbacks along local sections of the bundle, will be denoted by the corresponding non-boldface letters (the local section being understood from context). For example, if $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{g})$ is a connection form on a principal G-bundle $P \xrightarrow{\pi} M$, then for a local section $\sigma \in \Gamma(U, P)$ on $U \subset M$ the corresponding local connection form is $\omega = \sigma^* \boldsymbol{\omega} \in \Omega^1(U, \mathfrak{g})$.

Given a representation $\eta \colon G \to \mathsf{GL}(V)$, the space of η-tensorial k-forms on the total space P of a principal G-bundle will be denoted by $\Omega^k_\eta(P, V)$ or, if the representation is clear from context, by $\Omega^k_G(P, V)$.

1.1.4. Index notation. Unless otherwise specified, spacetime manifolds will be assumed as four-dimensional and denoted by M. The Einstein summation convention will be used throughout the article. Lowercase Greek letters will be used as coordinate indices on spacetime; for example, the coordinate component decomposition of a vector field reads $X = X^\mu \partial_\mu$.

Frame indices labelling a local frame of vector fields, i.e. a local section of the linear frame bundle $F(M)$ or a reduction thereof, will be uppercase Latin letters. These will also be used to denote the frame components of tensor fields; for example, the frame decomposition of a vector field reads $X = X^A \mathrm{e}_A$, where the local frame is (e_A) and the frame components of X are given by $X^A = \mathrm{e}^A(X) = \mathrm{e}^A_\mu X^\mu$ in terms of the dual frame (e^A). Put differently, denoting by

$$E = F(M) \times_{\mathsf{GL}(4)} \mathbb{R}^4 \tag{1.8}$$

the vector bundle associated to the linear frame bundle $F(M)$ via the fundamental representation of $\mathsf{GL}(4)$, this means that we freely use the canonical solder form of E to identify E with the tangent bundle TM, while always representing elements of E with respect to chosen local frames; and analogously for the corresponding tensor bundles.

If we reduce the structure group of the linear frame bundle to the Galilei group (for details see section 2.1), and therefore understand $\mathbb{R}^4$ as the space on which the Galilei group acts according to (1.5), we will often decompose frame indices according to

$$(A) = (t, a), \tag{1.9a}$$

using t as a 'temporal' index and lowercase Latin letters as 'spatial' indices running from 1 to 3. For example, a frame of local vector fields would then be decomposed as

$$(\mathrm{e}_A) = (\mathrm{e}_t, \mathrm{e}_a), \tag{1.9b}$$

or an element of $\mathbb{R}^4$ as

$$y = (y^A) = (y^t, y^a). \tag{1.9c}$$

If the relevant group is the *Lorentz* group instead, the temporal index will be denoted by 0 instead of t.

1.1.5. Linear connections. Given a linear connection ∇ on a manifold M, for its coordinate connection coefficients $\Gamma^\rho_{\mu\nu}$ the *first* lower index will be the 'form index'/differentiation index, i.e.

$$(\nabla_X Y)^\rho = X^\mu (\partial_\mu Y^\rho + \Gamma^\rho_{\mu\nu} Y^\nu). \tag{1.10}$$

In particular, the coordinate expression for the torsion is

$$T^{\rho}{}_{\mu\nu} = 2\Gamma^{\rho}_{[\mu\nu]} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}\,. \tag{1.11}$$

Similarly, if we have a 'teleparallel' connection ∇ and a 'non-teleparallel' connection $\widetilde{\nabla}$ (e.g. the Levi-Civita connection in standard Lorentzian GR), for the components of their difference tensor we will use the convention

$$K^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \widetilde{\Gamma}^{\rho}_{\mu\nu}\,. \tag{1.12}$$

Note that in some literature on teleparallel gravity, in particular in the review [14], the opposite convention is used for the lower indices of the connection coefficients, with the second one being the form index. Therefore, in [14] the index positioning on the contortion (1.12) is different to ours as well, even though the definition looks identical.

We use the same convention for the coordinate components of local connection forms with respect to local frames: the form index comes *before* the frame indices, i.e. for a local frame (e_A) the local connection form is given by

$$\nabla \mathrm{e}_B = \omega^A{}_B \otimes \mathrm{e}_A \text{ with } \omega^A_B = \omega^A_{\mu B}\,\mathrm{d}x^\mu. \tag{1.13}$$

Given a linear connection as a covariant derivative operator ∇ on TM, when extending it to higher-degree tensor bundles over M via a Leibniz rule, we will not always employ the identification of the tangent bundle TM with the associated vector bundle E from (1.8) via the canonical solder form. We will extend ∇ to tensor bundles of the form $(TM)^{\otimes r} \otimes (T^*M)^{\otimes s} \otimes E^{\otimes p} \otimes (E^*)^{\otimes q}$ in two different ways: one denoted (again) by ∇, acting *only* on tensor powers of TM or its dual, *not* acting on the E factors; and another one denoted by D, acting on all factors. In index notation, this means that covariant differentiation with ∇ 'acts only on spacetime indices', while covariant differentiation with D 'acts on both spacetime and frame indices'. For example, this means that for a tensor field

$$X = X^{\rho A}_{\sigma B}\,\partial_\rho \otimes \mathrm{d}x^\sigma \otimes \mathrm{e}_A \otimes \mathrm{e}^B \in \Gamma(TM \otimes T^*M \otimes E \otimes E^*) \tag{1.14a}$$

we have

$$\nabla_\mu X^{\rho A}_{\sigma B} = \partial_\mu X^{\rho A}_{\sigma B} + \Gamma^{\rho}_{\mu\nu} X^{\nu A}_{\sigma B} - \Gamma^{\nu}_{\mu\sigma} X^{\rho A}_{\nu B}\,, \tag{1.14b}$$

but

$$D_\mu X^{\rho A}_{\sigma B} = \partial_\mu X^{\rho A}_{\sigma B} + \Gamma^{\rho}_{\mu\nu} X^{\nu A}_{\sigma B} - \Gamma^{\nu}_{\mu\sigma} X^{\rho A}_{\nu B} + \omega^A_{\mu C} X^{\rho C}_{\sigma B} - \omega^C_{\mu B} X^{\rho A}_{\sigma C}\,. \tag{1.14c}$$

2. Bargmann structures and teleparallel Galilei connections

In this section we will introduce the geometric framework for our teleparallel version of Newton–Cartan gravity, and formulate the theory.

In section 2.1, we review the geometric framework of Galilei manifolds, which underlies usual Newton–Cartan gravity as well as its teleparallel variant. We present the classical description of the geometry as given by Künzle [9] and Ehlers [10, 11] and reviewed by Malament [12, chapter 4], but at some points provide a modernised perspective based on some aspects of the recent paper [26].

In section 2.2, we give a global, principal-bundle-based formulation of the 'Bargmann spacetime' framework by Geracie *et al* [26], which makes explicit in which sense the Bargmann group underlies the geometry of Galilei manifolds as a local symmetry group. Of course, this global description is in some sense implicit in the local description from [26];

however, an explicitly global formulation makes the invariant nature of the involved objects stand out more clearly.

Finally, in section 2.3 we employ the previously introduced framework to 'teleparallelise' Newton–Cartan gravity.

2.1. Galilei manifolds

A *Galilei manifold* is a four-dimensional manifold M with a nowhere vanishing *clock 1-form* $\tau \in \Omega^1(M)$ and a symmetric contravariant 2-tensor h of signature $(3,0,1)$—i.e. positive definite/negative definite/degenerate on subbundles of T^*M of rank 3/0/1 respectively—, called the *space metric*, whose degenerate direction is spanned by τ, i.e.

$$\tau_\mu h^{\mu\nu} = 0. \tag{2.1}$$

The kernel of τ at any point are the *spacelike* vectors at this point, on which h defines a positive definite scalar product. The integral of τ along any curve is interpreted as the time elapsed along the curve. In the following, we will mostly assume that $\mathrm{d}\tau = 0$, such that the time between two events is independent of the worldline between them chosen to measure it, i.e. we have an absolute notion of time. This also implies that the distribution of spacelike vectors is integrable, i.e. we have hypersurfaces of simultaneity/leaves of 'space', on each of which h defines a Riemannian metric. The assumption $\mathrm{d}\tau = 0$ of absolute time is made in standard Newton–Cartan gravity. Note however that by Frobenius' theorem the integrability of the distribution $\ker\tau$, i.e. the existence of an absolute notion of simultaneity, is equivalent to the weaker condition $\tau \wedge \mathrm{d}\tau = 0$. Newton–Cartan gravity in the more general context of a clock form satisfying only this integrability condition is referred to as *twistless torsional Newton–Cartan gravity* [22, 23]. In this article, we will however only be considering the teleparallel formulation of standard Newton–Cartan gravity, assuming absolute time.

A *Galilei frame* is a local frame $(\mathrm{e}_t = v, \mathrm{e}_a)$ of vector fields on M such that

$$\tau(v) = 1,\ h^{\mu\nu} = \delta^{ab}\mathrm{e}_a^\mu \mathrm{e}_b^\nu\ . \tag{2.2}$$

A choice of Galilei frame (or, indeed, of a temporal reference vector field v alone) gives us a projector onto space along v, namely

$$P^\mu_\nu = \delta^\mu_\nu - v^\mu \tau_\nu\ , \tag{2.3a}$$

and defines a v-dependent 'inverse' of h, whose components we denote by $h_{\mu\nu}$ (always understanding the implicit v-dependence), by

$$h_{\mu\nu} v^\nu = 0,\ h_{\mu\nu} h^{\nu\rho} = P^\rho_\mu\ . \tag{2.3b}$$

In terms of the vector fields of the frame and the covector fields of its dual frame, which we denote by $(\mathrm{e}^t = \tau, \mathrm{e}^a)$, these objects may be expressed as

$$P^\mu_\nu = \mathrm{e}_a^\mu \mathrm{e}^a_\nu\ ,\ h_{\mu\nu} = \delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu\ . \tag{2.4}$$

In the context of Galilei manifolds, we will use the common convention of 'raising indices' by contraction with the space metric $h^{\mu\nu}$, and of 'lowering' indices by contraction with $h_{\mu\nu}$ if a temporal reference field v is chosen. Note that due to the degeneracy of h these operations are not inverses of each other, but first raising and then lowering an index (or vice versa) corresponds to contracting with the spatial projector P^μ_ν. Similarly, if a local Galilei frame has been chosen, we will use δ^{ab} and δ_{ab} to raise and lower spatial frame indices.

As one easily checks, basis change matrices between Galilei frames (evaluated at a point) are precisely elements of the homogeneous Galilei group Gal, understood as a subgroup of

$\mathsf{GL}(4)$ according to (1.5). Thus, Galilei frames are local sections of a reduction of the structure group of the linear frame bundle $F(M)$ from $\mathsf{GL}(4)$ to Gal, which we denote by $G(M)$ and call the *Galilei frame bundle* of (M,τ,h). Given a four-dimensional manifold M, the specification of such a reduction is equivalent to the specification of τ and h making M into a Galilei manifold, and therefore we call such a reduction $G(M)$ a *Galilei structure* on M.

Any change of local Galilei frame, i.e. of local section of $G(M)$,[2] has the form

$$(v,\mathrm{e}_a)\to(v,\mathrm{e}_a)\cdot(R,k)^{-1} \tag{2.5a}$$

for a local Gal-valued function (R,k), where the dot denotes the right action of Gal on $G(M)$. Spelled out, this reads

$$(v,\mathrm{e}_a)\to\left(v-\mathrm{e}_b(R^{-1})^b{}_a k^a,\mathrm{e}_b(R^{-1})^b{}_a\right), \tag{2.5b}$$

and we call such a change of frame a *local Galilei transformation* by (R,k). This defines a left action on local Galilei frames by the group of local Gal-valued functions.

A *Galilei connection* on a Galilei manifold is a principal connection $\boldsymbol{\omega}$ on the Galilei frame bundle $G(M)$. The local connection form with respect to a Galilei frame (v,e_a), i.e. the pullback of the connection form $\boldsymbol{\omega}$ on the total space along the frame, is a local one-form ω on M with values in the homogeneous Galilei Lie algebra $\mathfrak{gal}=\mathfrak{so}(3)\oplus\mathbb{R}^3$. We decompose it as

$$\omega=(\omega^a{}_b,\varpi^a), \tag{2.6}$$

and following [26], we will sometimes call its $\mathfrak{so}(3)$-valued part $\omega^a{}_b$ the *spin connection* and its $\mathbb{R}^3$-valued part ϖ^a the *boost connection*[3].

Since $G(M)$ is a reduction of the structure group of $F(M)$ to Gal, a Galilei connection is equivalently given by a covariant derivative operator ∇ on the tangent bundle TM whose local connection form $\omega^A{}_B$ with respect to a Galilei frame, defined by

$$\nabla\mathrm{e}_B=\omega^A{}_B\otimes\mathrm{e}_A \tag{2.7}$$

and *a priori* taking values in $\mathfrak{gl}(4)$, takes values in $\mathfrak{gal}$, viewed as a subalgebra of $\mathfrak{gl}(4)$ according to (1.5). This means that $\omega^A{}_B$ satisfies

$$\omega^t{}_A=0, \tag{2.8a}$$

$$\delta_{ac}\omega^c{}_b=-\delta_{bc}\omega^c{}_a\,. \tag{2.8b}$$

The boost connection ϖ^a then arises as the 'spatio-temporal part' of $\omega^A{}_B$,

$$\varpi^a=\omega^a{}_t\,. \tag{2.9}$$

Equivalently to (2.8), a covariant derivative operator ∇ is a Galilei connection iff it is compatible with τ and h, i.e. satisfies

$$\nabla\tau=0,\ \nabla h=0. \tag{2.10}$$

Note that from compatibility with $\tau=\mathrm{e}^t$, it follows that for any tensor field with components $X^{\mu_1\cdots\mu_{r+1}}_{\nu_1\ldots\nu_s}$ satisfying $X^{\mu_1\cdots\mu_r t}_{\nu_1\ldots\nu_s}=0$ we have

$$D_\sigma X^{\mu_1\cdots\mu_r t}_{\nu_1\ldots\nu_s}=0. \tag{2.11}$$

[2] We avoid the ambiguous term 'gauge transformation' here, since it could mean either such a change of local section or a (global) fibre-preserving automorphism of the total space.

[3] Note that due to the semidirect product nature of Gal, the boost 'connection' does not define a connection by itself, other than the spin connection, which induces a connection on the bundle of orthonormal frames for the distribution $\ker\tau\subset TM$ of spacelike vectors together with the metric induced by h.

The *Newton–Coriolis form*[4] of a Galilei connection $\boldsymbol{\omega}$ with respect to a local Galilei frame (v, e_a) is the (local) two-form Ω on M with components

$$\Omega_{\mu\nu} := 2(\nabla_{[\mu} v^\kappa) h_{\nu]\kappa} \,. \tag{2.12a}$$

Due to $\nabla v = \nabla \mathrm{e}_t = \varpi^a \otimes \mathrm{e}_a$, we may also express Ω in terms of the boost connection,

$$\Omega_{\mu\nu} = 2\varpi_{[\mu}{}^a h_{\nu]\kappa} \mathrm{e}_a^\kappa = 2\varpi_{[\mu|a|} \mathrm{e}_{\nu]}^a \,, \tag{2.12b}$$

which may also be written as

$$\Omega = \varpi_a \wedge \mathrm{e}^a. \tag{2.12c}$$

Using the notation $\varpi_\mu{}^\nu := \varpi_\mu{}^a \mathrm{e}_a^\nu$, this may be expressed by the mnemonic that 'Ω is the antisymmetric part of the boost connection',

$$\Omega_{\mu\nu} = 2\varpi_{[\mu\nu]} \,. \tag{2.12d}$$

However, this involves some abuse of notation: since $\varpi_\mu{}^a$ are not the components of a tensor field on M, the components $\varpi_\mu{}^\nu$ do not have an invariant meaning.

A Galilei connection is determined by its torsion T and its Newton–Coriolis form Ω, with connection coefficients in coordinates given by

$$\Gamma^\sigma_{\mu\nu} = v^\sigma \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\sigma\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) + \frac{1}{2} T^\sigma{}_{\mu\nu} - T^\sigma_{(\mu\nu)} + \tau_{(\mu}\Omega_{\nu)}{}^\sigma. \tag{2.13}$$

The temporal component of the torsion of a Galilei connection is equal to $T^t{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu}$, i.e. if we assume absolute time, the temporal torsion vanishes. If conversely Ω is an arbitrary two-form and T any tensor field with the antisymmetry of a torsion that satisfies $T^t{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu}$, then (2.13) defines a Galilei connection with torsion T and Newton–Coriolis form Ω.

We note that, as for any principal connection on a reduction of the linear frame bundle, the torsion of a Galilei connection can be written in the following way: the tangent bundle TM is canonically isomorphic to the associated vector bundle $E = G(M) \times_{\mathsf{Gal}} \mathbb{R}^4$ via the canonical solder form $\theta \in \Omega^1(M, E)$ of E. Taking the exterior covariant derivative of the solder form with respect to the connection then yields the torsion $T \in \Omega^2(M, E) \cong \Omega^2(M, TM)$ ('Cartan's first structure equation'):

$$T = \mathrm{d}^{\boldsymbol{\omega}}\theta. \tag{2.14a}$$

In terms of a local Galilei frame, we have $\theta = \mathrm{e}^A \otimes \mathrm{e}_A$, i.e. the components of the canonical solder form with respect to the frame (viewed as a local frame for E) are given by the dual frame one-forms. Thus, expressed in components the first structure equation reads

$$T^A = (\mathrm{d}^{\boldsymbol{\omega}}\theta)^A = \mathrm{d}\mathrm{e}^A + \omega^A{}_B \wedge \mathrm{e}^B, \tag{2.14b}$$

or explicitly for Galilei connections and Galilei frames

$$(T^t, T^A) = (\mathrm{d}\tau, \mathrm{d}e^a + \omega^a{}_b \wedge \mathrm{e}^b + \varpi^a \wedge \tau). \tag{2.14c}$$

A Galilei connection on a Galilei manifold with absolute time ($\mathrm{d}\tau = 0$) is called *Newtonian* iff it is torsion-free and its curvature tensor satisfies symmetry in pairs,

$$R^\mu{}_\nu{}^\rho{}_\sigma = R^\rho{}_\sigma{}^\mu{}_\nu \,. \tag{2.15a}$$

[4] This name was introduced in [26], after previous authors had either given the form no name or called it just the 'Coriolis form'.

This is equivalent (which is not at all obvious!) to the Newton–Coriolis form (with respect to any frame) being closed,

$$\mathrm{d}\Omega = 0. \tag{2.15b}$$

At each point, the curvature tensor of a Newtonian connection has as many linearly independent components as that of the Levi-Civita connection of a (pseudo-)Riemannian metric on a manifold of the same dimension, i.e. 20 in the case of four spacetime dimensions.

We now consider how the objects introduced above transform under local Galilei transformations, i.e. under changes of Galilei frame parameterised by local Gal-valued functions (R,k) according to (2.5). To keep the formulae easier, we consider the transformation under local frame rotations and under local Galilei boosts separately. Spelled out, the purely rotational transformation of the frame with parameter R reads

$$(v, \mathrm{e}_a) \to \left(v, \mathrm{e}_b (R^{-1})^b{}_a\right), \tag{2.16a}$$

and those of the dual frame and the connection form are

$$(\tau, \mathrm{e}^a) \to (\tau, R^a{}_b \mathrm{e}^b), \tag{2.16b}$$

$$\omega^a{}_b \to R^a{}_c \omega^c{}_d (R^{-1})^d{}_b + R^a{}_c \mathrm{d}(R^{-1})^c{}_b \,, \tag{2.16c}$$

$$\varpi^a \to R^a{}_b \varpi^b, \tag{2.16d}$$

while for pure boosts with parameter k (which sometimes are called *Milne boosts* instead of 'local Galilei boosts'), we have

$$(v, \mathrm{e}_a) \to (v - \mathrm{e}_a k^a, \mathrm{e}_a)\,, \tag{2.17a}$$

$$(\tau, \mathrm{e}^a) \to (\tau, \mathrm{e}^a + k^a \tau), \tag{2.17b}$$

$$\omega^a{}_b \to \omega^a{}_b \,, \tag{2.17c}$$

$$\varpi^a \to \varpi^a - \mathrm{d}k^a - \omega^a{}_b k^b. \tag{2.17d}$$

2.2. Bargmann structures

Given a Galilei manifold (M, τ, h), we can extend its Galilei frame bundle $G(M)$ to a principal bundle with structure group the Bargmann group $\mathsf{Barg} = \mathsf{Gal} \ltimes (\mathbb{R}^4 \times \mathsf{U}(1))$ in the following way. We denote by

$$\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^4 \times \mathsf{U}(1)) \tag{2.18a}$$

the group homomorphism defining the Bargmann group, given in (1.7); by

$$\dot\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^4 \oplus \mathfrak{u}(1)) \tag{2.18b}$$

the induced representation of Gal by Lie algebra automorphisms, which explicitly takes the form

$$\dot\rho_{(R,k)}(y^t, y^a, \mathrm{i}\varphi) = \left(y^t, R^a{}_b y^b + y^t k^a, \mathrm{i}(\varphi + \tfrac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b)\right) \tag{2.18c}$$

for $(R,k) \in \mathsf{Gal}$ and $(y^A, \mathrm{i}\varphi) \in \mathbb{R}^4 \oplus \mathfrak{u}(1)$; and by

$$\tilde\rho\colon \mathsf{Gal} \to \mathsf{Diff}(\mathsf{Barg}) \tag{2.18d}$$

the left action of Gal on Barg by multiplication. The associated bundle

$$B(M) := G(M) \times_{\tilde\rho} \mathsf{Barg} \tag{2.19}$$

is then a principal Barg-bundle (with action given by right multiplication) that extends $G(M)$ along the natural inclusion map

$$\gamma\colon G(M) \to B(M), \gamma(p) = [p, \mathrm{e}_{\mathsf{Barg}}]. \tag{2.20}$$

(Here $\mathrm{e}_{\mathsf{Barg}}$ is the neutral element of Barg and we denoted the element of the Galilei frame bundle by p instead of our usual notation (e_A) in order to avoid confusion with the neutral element.)

The fundamental observation is now that connections $\hat{\boldsymbol{\omega}}$ on this bundle $B(M)$ are in one-to-one correspondence with pairs $(\boldsymbol{\omega}, \boldsymbol{\Theta})$ of connections $\boldsymbol{\omega}$ and $\dot{\rho}$-tensorial one-forms $\boldsymbol{\Theta} \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^4 \oplus \mathfrak{u}(1))$ on $G(M)$ via the pullback condition

$$\gamma^* \hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\Theta}). \tag{2.21a}$$

Furthermore, in this situation the pullback of the curvature form $\hat{\boldsymbol{R}}$ of $\hat{\boldsymbol{\omega}}$ is given by the curvature form $\boldsymbol{R}$ of $\boldsymbol{\omega}$ and the exterior covariant derivative of $\boldsymbol{\Theta}$ with respect to $\boldsymbol{\omega}$,

$$\gamma^* \hat{\boldsymbol{R}} = (\boldsymbol{R}, \mathrm{d}^{\boldsymbol{\omega}} \boldsymbol{\Theta}). \tag{2.21b}$$

This generalises the classical situation of connections on the affine frame bundle of a manifold (see, e.g., [30, section III.3]), and is generally true for 'semidirect extensions' of principal bundles as encountered here. Details and a general discussion of this extension construction may be found in appendix.

We further decompose

$$\boldsymbol{\Theta} = (\boldsymbol{\theta}, \mathrm{i}\boldsymbol{a}) \tag{2.22}$$

with $\boldsymbol{\theta} \in \Omega^1(G(M), \mathbb{R}^4)$ and $\boldsymbol{a} \in \Omega^1(G(M))$. Due to $\mathfrak{u}(1) \subset \mathbb{R}^4 \oplus \mathfrak{u}(1)$ being a $\dot{\rho}$-invariant subspace, we may view the $\mathbb{R}^4$-valued part $\boldsymbol{\theta}$ as transforming under the quotient representation $\mathsf{Gal} \to \mathsf{GL}((\mathbb{R}^4 \oplus \mathfrak{u}(1))/\mathfrak{u}(1)) \cong \mathsf{GL}(4)$, which is the usual representation (1.5) of Gal on $\mathbb{R}^4$. This means that $\boldsymbol{\theta}$ is by itself a tensorial form

$$\boldsymbol{\theta} \in \Omega^1_{\mathsf{Gal}}(G(M), \mathbb{R}^4), \tag{2.23}$$

which naturally corresponds to an associated-bundle-valued form

$$\theta \in \Omega^1(M, G(M) \times_{\mathsf{Gal}} \mathbb{R}^4) \tag{2.24}$$

on our Galilei manifold. If this is the canonical solder form of $G(M) \times_{\mathsf{Gal}} \mathbb{R}^4$, then we call $\boldsymbol{a} \in \Omega^1(G(M))$ a *Bargmann structure* on (M, τ, h).[5]

Summed up, a Bargmann structure on a Galilei manifold (M, τ, h) is a one-form $\boldsymbol{a} \in \Omega^1(G(M))$ on the Galilei frame bundle that together with the tensorial form $\boldsymbol{\theta} \in \Omega^1_{\mathsf{Gal}}(G(M), \mathbb{R}^4)$ corresponding to the canonical solder form combines into a $\dot{\rho}$-tensorial form $(\boldsymbol{\theta}, \mathrm{i}\boldsymbol{a}) \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^4 \oplus \mathfrak{u}(1))$, which in turn together with a Galilei connection $\boldsymbol{\omega} \in \Omega^1(G(M), \mathfrak{gal})$ would give a 'Bargmann connection' $\hat{\boldsymbol{\omega}}$ on $B(M)$. Note however that we consider the choice of Galilei connection $\boldsymbol{\omega}$ *not* to be part of the choice of Bargmann structure.

Given a Bargmann structure $\boldsymbol{a}$, the tensorial form $\boldsymbol{\Theta} = (\boldsymbol{\theta}, \mathrm{i}\boldsymbol{a}) \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^4 \oplus \mathfrak{u}(1))$ corresponds to an associated-bundle-valued form $\Theta \in \Omega^1(M, G(M) \times_{\dot{\rho}} (\mathbb{R}^4 \oplus \mathfrak{u}(1)))$. The local representative of this form with respect to a local Galilei frame $\sigma = (\mathrm{e}_A)$ defined on an open set $U \subset M$, i.e. the pullback

$$(\tau, \mathrm{e}^a, \mathrm{i}a) = \sigma^* \boldsymbol{\Theta} \in \Omega^1(U, \mathbb{R}^4 \oplus \mathfrak{u}(1)) \tag{2.25a}$$

[5] Note that in [19], the term 'Bargmann structure' was used differently, namely for the structure of a five-dimensional Lorentzian manifold with a null isometry from which a Galilei manifold may be obtained by null reduction; see also [29].

along the frame, using which we can locally express Θ as

$$\Theta = [\sigma, (\tau, \mathrm{e}^a, \mathrm{i}a)] \in \Omega^1\left(U, G(M) \times_{\dot{\rho}} (\mathbb{R}^4 \oplus \mathfrak{u}(1))\right), \tag{2.25b}$$

is what in [26] was called an *extended coframe*. (Note that since $\boldsymbol{\theta}$ corresponds to the canonical solder form, its components with respect to the local frame $\sigma = (\mathrm{e}_A) = (v, \mathrm{e}_a)$ are given by the dual frame (τ, e^a).)

Under local Galilei transformations, i.e. changes of local Galilei frame, the extended coframe transforms according to the representation $\dot{\rho}$. Using the explicit form (2.18c) of $\dot{\rho}$, we see that this means that while the dual frame (τ, e^a) transforms as in (2.16), (2.17), the one-form a locally representing the Bargmann structure is invariant under local rotations, and under local boosts with parameter k transforms as

$$a \to a + k_a \mathrm{e}^a + \frac{1}{2}|k|^2 \tau. \tag{2.26}$$

We now consider the exterior covariant derivative $\mathrm{d}^{\boldsymbol{\omega}}\Theta$ of the form Θ, locally represented by the extended coframe, with respect to a Galilei connection $\boldsymbol{\omega}$. According to (2.21b), it corresponds to the $\mathbb{R}^4 \oplus \mathfrak{u}(1)$-valued part of the curvature of the 'Bargmann connection' $\hat{\boldsymbol{\omega}}$ on $B(M)$ given by $\boldsymbol{\omega}$ and $\boldsymbol{\Theta}$. Since the $\mathbb{R}^4$-valued part of $\boldsymbol{\Theta}$ corresponds to the canonical solder form of $G(M) \times_{\mathsf{Gal}} \mathbb{R}^4$, according to Cartan's first structure equation (2.14) the $\mathbb{R}^4$-valued part of $\mathrm{d}^{\boldsymbol{\omega}}\boldsymbol{\Theta}$ corresponds to the torsion of $\boldsymbol{\omega}$. Following [26], we will call $\mathrm{d}^{\boldsymbol{\omega}}\Theta$ the connection's *extended torsion* with respect to the Bargmann structure, and denote its local components with respect to a local frame $\sigma = (\mathrm{e}_A)$ by

$$(T^A, \mathrm{i}f) := \sigma^*(\mathrm{d}^{\boldsymbol{\omega}}\boldsymbol{\Theta}). \tag{2.27}$$

Its $\mathfrak{u}(1)$-valued part f, which of course does not define an invariant geometric object on its own, we call the *mass torsion* following [26]. From the explicit form (2.18c) of $\dot{\rho}$, we obtain the induced Lie algebra representation $\dot{\rho}' \colon \mathfrak{gal} \to \mathsf{Der}(\mathbb{R}^4 \oplus \mathfrak{u}(1))$ as

$$\dot{\rho}'_{(X,k)}(y^A, \mathrm{i}\varphi) = \left(((X,k)y)^A, \mathrm{i}k_a y^a\right) \tag{2.28}$$

for $(X,k) \in \mathfrak{gal} = \mathfrak{so}(3) \oplus \mathbb{R}^3$ and $(y^A, \mathrm{i}\varphi) \in \mathbb{R}^4 \oplus \mathfrak{u}(1)$. Therefore, the local components of the extended torsion are[6]

$$\begin{aligned}(T^A, \mathrm{i}f) &= \sigma^*(\mathrm{d}^{\boldsymbol{\omega}}\boldsymbol{\Theta}) \\ &= \mathrm{d}(\mathrm{e}^A, \mathrm{i}a) + \dot{\rho}'_{(\omega,\varpi)} \wedge (\mathrm{e}^A, \mathrm{i}a) \\ &= (\mathrm{d}\mathrm{e}^A + \omega^A{}_B \wedge \mathrm{e}^B, \mathrm{i}(\mathrm{d}a + \varpi_a \wedge \mathrm{e}^a)),\end{aligned} \tag{2.30a}$$

and the mass torsion is given by

$$f = \mathrm{d}a + \varpi_a \wedge \mathrm{e}^a. \tag{2.30b}$$

In terms of the mass torsion f and the $\mathfrak{u}(1)$ component a of the extended coframe that locally represents the Bargmann structure, we can write the Newton–Coriolis form of the connection as

$$\Omega = \varpi_a \wedge \mathrm{e}^a = f - \mathrm{d}a, \tag{2.31}$$

[6] Under local Galilei transformations, the extended torsion of course again transforms according to $\dot{\rho}$, i.e. under local rotations the mass torsion is invariant, and under local Galilei boosts with parameter k it transforms as

$$f \to f + k_a T^A + \frac{1}{2}|k|^2 \mathrm{d}\tau. \tag{2.29}$$

However, we will not need this in our subsequent discussion.

i.e. given a choice of Bargmann structure, Ω is determined by (and determines) the mass torsion f. Since the freedom in the choice of a Galilei connection lies precisely in the torsion and the Newton–Coriolis form, we thus see that on a Galilei manifold with a Bargmann structure, *a Galilei connection is uniquely characterised by its extended torsion*. In particular, if we have absolute time (i.e. $\mathrm{d}\tau = 0$), which we will from now on assume for all Galilei manifolds unless otherwise stated, for each Bargmann structure there is a unique Galilei connection with vanishing extended torsion. Since its Newton–Coriolis form is closed (it is even exact!), this unique extended-torsion-free connection is Newtonian.

Note that we have a $\mathsf{U}(1)$ gauge freedom for Bargmann structures, corresponding to the $\mathsf{U}(1)$ direction in the Bargmann group: given a Bargmann structure $\boldsymbol{a}$ on (M,τ,h) and any $\mathsf{U}(1)$-valued function $\mathrm{e}^{\mathrm{i}\chi}$ on M, we may act with it (actively!) on $\boldsymbol{a}$ and obtain a *new* Bargmann structure[7]

$$\boldsymbol{a} \to \boldsymbol{a} + \pi^*(\mathrm{d}\chi), \tag{2.32a}$$

which on M is locally represented by

$$a \to a + \mathrm{d}\chi\,. \tag{2.32b}$$

Note that with respect to two Bargmann structures which are related to each other by such a $\mathsf{U}(1)$ gauge transformation, a Galilei connection has the same extended torsion, since $f = \mathrm{d}a + \Omega = \mathrm{d}(a + \mathrm{d}\chi) + \Omega$.

2.3. Teleparallel Galilei connections

In Lorentzian (or more generally pseudo-Riemannian) geometry, any metric connection is uniquely determined by its (arbitrarily specifiable) torsion. Therefore, the difference tensor between an arbitrary metric connection ∇ and the torsion-free Levi-Civita connection may be expressed purely in terms of the torsion of ∇. This allows for a reformulation of the Einstein equation, which is usually formulated in terms of the Levi-Civita connection, in terms of a flat torsionful connection ∇ and its torsion, giving rise to TEGR, the TEGR.

Differently to that, in the classical setting of Newton–Cartan gravity, such a reformulation is not possible, since Galilei connections on a Galilei manifold are not uniquely determined by their torsion: according to (2.13), we also need to specify the Newton–Coriolis form Ω. Since Ω depends on the choice of timelike Galilei frame vector field v, there is no naturally given unique connection $\widetilde{\nabla}$ the difference to which of a general connection ∇ we could use to reformulate Newton–Cartan gravity.

However, this is remedied by the introduction of a Bargmann structure. As we have seen above, on a Galilei manifold with absolute time with a chosen Bargmann structure, there is a unique Galilei connection with vanishing extended torsion. This allows us to reformulate Newton–Cartan gravity in a teleparallel way, which we will explain in the following.

Given a Galilei manifold (M,τ,h) with absolute time (i.e. $\mathrm{d}\tau = 0$) with a Bargmann structure $\boldsymbol{a}$, we define the *Newton–Cartan contortion* of a Galilei connection $\boldsymbol{\omega}$ to be the tensor field

[7] From the global point of view, this transformation arises as follows. From $\mathrm{e}^{\mathrm{i}\chi}$, we obtain a gauge transformation f_χ of the principal Barg-bundle $B(M) \xrightarrow{\hat{\pi}} M$, i.e. a fibre-preserving principal bundle automorphism $f_\chi : B(M) \to B(M)$, as $f_\chi(p) := p \cdot (\mathbb{1}, 0, 0, \mathrm{e}^{\mathrm{i}\chi\circ\hat{\pi}})$. Given a connection $\hat{\boldsymbol{\omega}}$ on $B(M)$, we may act on it with the gauge transformation f_χ, giving the new connection $f_\chi^*\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} + (0,0,0,\mathrm{i}\hat{\pi}^*(\mathrm{d}\chi))$. For the case of $\hat{\boldsymbol{\omega}}$ given by a Galilei connection $\boldsymbol{\omega}$ and a Bargmann structure $\boldsymbol{a}$, this gives rise to (2.32).

$$K^{\rho}{}_{\mu\nu} := \overset{(3)}{K}{}^{\rho}{}_{\mu\nu} + \tau_{(\mu} f_{\nu)}{}^{\rho} \tag{2.33a}$$

with

$$\overset{(3)}{K}{}^{\rho}{}_{\mu\nu} := \frac{1}{2} T^{\rho}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho}, \tag{2.33b}$$

where $(T^A, \mathrm{i}f)$ are the local components of the extended torsion of ω with respect to any local Galilei frame. Comparing this to the general form of the coordinate connection coefficients of a Galilei connection (2.13), we see that $K^{\rho}{}_{\mu\nu}$ is the difference tensor between our connection ω and the unique extended-torsion-free connection $\widetilde{\omega}$, i.e.

$$K^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \widetilde{\Gamma}^{\rho}_{\mu\nu} \, . \tag{2.34}$$

Note that this also implies that despite being defined in terms of objects depending on the choice of local Galilei frame in (2.33), the Newton–Cartan contortion is in fact independent of the choice of frame[8].

Using the notion of Newton–Cartan contortion, we can now formulate teleparallel Newton–Cartan gravity in terms of the following axioms:

2.3.1. Axioms for teleparallel Newton–Cartan gravity

(i) Spacetime is a Galilei manifold (M, τ, h) with absolute time, endowed with a Bargmann structure and a flat Galilei connection ω,

(ii) Ideal clocks measure time as defined by τ, and ideal rods measure spatial lengths as defined by the metric induced by h on spacelike vectors,

(iii) Free test particles move on timelike curves γ solving

$$(\nabla_{\dot\gamma}\dot\gamma)^{\rho} = K^{\rho}{}_{\mu\nu}\dot\gamma^{\mu}\dot\gamma^{\nu}, \tag{2.35}$$

(iv) The field equation

$$-D_{\sigma}K^{\sigma}{}_{AB} + D_{A}K^{\mu}{}_{\mu B} - K^{\mu}{}_{\sigma B}T^{\sigma}{}_{\mu A} + K^{\mu}{}_{\mu\sigma}K^{\sigma}{}_{AB} - K^{\mu}{}_{A\sigma}K^{\sigma}{}_{\mu B} = 4\pi G\rho\tau_A\tau_B \tag{2.36}$$

holds, where ρ is the mass density.

The theory defined by these axioms will be our subject of study in the rest of this article. Note that the left-hand side of the field equation (2.36) are just the components $\widetilde{R}_{AB}$ of the Ricci tensor of the extended-torsion-free connection $\widetilde{\omega}$, expressed in terms of the connection ω, its torsion and the difference tensor between the two connections (i.e. the Newton–Cartan contortion of ω). Therefore, the above theory really is an equivalent formulation of usual Newton–Cartan gravity, whose equation of motion and field equation read $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = 0$ and $\widetilde{R}_{\mu\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu}$, respectively.

Note that the formulation of teleparallel Newton–Cartan gravity given here is completely general, i.e. prior to any 'gauge-fixing' of either the connection or the frame. This distinguishes our formulation from that given in [28], where both the field equation and the test particle

[8] This may of course also be verified by direct calculation, using the transformation behaviour of the mass torsion and the local coframe (which enters the definition of $K^{\rho}{}_{\mu\nu}$ through lowering of indices with $h_{\mu\nu}$).

equation of motion were only constructed from those of usual Newton–Cartan gravity in a gauge-fixed situation.

3. Teleparallel Newton–Cartan gravity from TEGR

In this section, we will show that teleparallel Newton–Cartan gravity as introduced in the previous section arises as the formal $c \to \infty$ limit of TEGR, c being the speed of light, in direct analogy to standard Newton–Cartan gravity being the formal $c \to \infty$ limit of GR.

3.1. Formal expansions of Lorentzian geometry

In the following, we will describe how the formal $c \to \infty$ limit of a Lorentzian manifold gives rise to a Galilei manifold with a Bargmann structure, and how in this limit a Lorentzian metric connection becomes a Galilei connection. Most of the claims are not immediately obvious, but all of them may be verified by direct calculation.

In order to perform the formal $c \to \infty$ limit, we will expand all objects of Lorentzian geometry as formal power series in the parameter c^{-1}—or, more precisely, formal Laurent series, since we will need negative orders of c^{-1}. The 'formal $c \to \infty$ limit' of a quantity expanded as a power series will then be the term of order $\mathrm{O}(c^0)$, provided that there are no terms of negative order in c^{-1}.

Of course, analytically speaking, a 'Taylor expansion' in a dimensionful parameter like c does not make sense (even more so since c is a constant of nature); only for *dimensionless* parameters can a meaningful 'small-parameter approximation' be made. In physical realisations of the limit from Lorentzian to Galilei geometry, this means that the corresponding small parameter has to be chosen as, e.g., the ratio of some typical velocity of the system under consideration to the speed of light. A rigorous discussion of the Newtonian limit from standard GR to Newton–Cartan gravity in terms of an actual small parameter approaching zero is given by Ehlers in his article on 'frame theory' [10, 11]; a nice discussion on the relationship of formal Newtonian limits to actual physical slow-velocity approximations may be found in [31, section II].

In the following, however, we will ignore these issues and just expand in c^{-1} as a formal parameter, thus only considering how the Lorentzian theory may be viewed as a deformation of its formal 'Newtonian limit'. Our formal expansion in c^{-1} to implement the Newtonian limit is the same as that used in [32, 33] for the geometric description of post-Newtonian expansions of GR, specialised to the case $\mathrm{d}\tau = 0$ of absolute time, but at the same time generalised to allow for torsionful Lorentzian connections.

To notationally distinguish Lorentzian geometric objects from their Newton–Cartan counterparts, we will mostly denote the Lorentzian objects by an overset 'L'; for example, the torsion of a Lorentzian connection will be denoted by $\overset{\mathrm{L}}{T}$.

We start with a Lorentzian metric g on our spacetime manifold M, for which we have a local orthonormal frame/tetrad (E_A) with dual frame (E^A), such that the metric and inverse metric can be written as

$$g = \eta_{AB}\mathrm{E}^A \otimes \mathrm{E}^B, \quad g^{-1} = \eta^{AB}\mathrm{E}_A \otimes \mathrm{E}_B, \tag{3.1}$$

where η_{AB} denotes the components of the Minkowski metric in Lorentzian coordinates, i.e. $(\eta_{AB}) = \mathrm{diag}(-1,1,1,1)$. We assume that the frame and dual frame may be expanded as formal power series in c^{-1} as

$$\mathrm{E}^0_\mu = c\tau_\mu + c^{-1}a_\mu + \mathrm{O}(c^{-3}), \quad \mathrm{E}^a_\mu = \mathrm{e}^a_\mu + \mathrm{O}(c^{-2}), \tag{3.2a}$$

$$\mathrm{E}^\mu_0 = c^{-1}v^\mu + \mathrm{O}(c^{-3}), \qquad \mathrm{E}^\mu_a = \mathrm{e}^\mu_a + \mathrm{O}(c^{-2}) \tag{3.2b}$$

for some nowhere vanishing one-form τ. From these assumptions it follows that τ and $h := \delta^{ab}\mathrm{e}_a \otimes \mathrm{e}_b$ make M into a Galilei manifold. For this Galilei manifold that arises as the formal $c \to \infty$ limit of our Lorentzian manifold, (v, e_a) is a local Galilei frame with dual frame (τ, e^a). Let us stress that τ is to be viewed as an 'input' for the expansion; it is the object with respect to which we perform the formal Newtonian limit.

We now consider a local Lorentz boost $(\Lambda^A{}_B)$ parameterised by the $\mathbb{R}^3$-valued boost velocity function k as[9]

$$\Lambda^0{}_0 = 1 + c^{-2}\frac{|k|^2}{2} + \mathrm{O}(c^{-4}), \tag{3.3a}$$

$$\Lambda^a{}_0 = c^{-1}k^a + \mathrm{O}(c^{-3}) = \delta^{ab}\Lambda^0{}_b\,, \tag{3.3b}$$

$$\Lambda^a{}_b = \delta^a_b + c^{-2}\frac{k^a k_b}{2} + \mathrm{O}(c^{-4}). \tag{3.3c}$$

Transforming the Lorentzian frame (E_A) by Λ according to

$$(\mathrm{E}_A) \to (\tilde{\mathrm{E}}_A) = \left(\mathrm{E}_B(\Lambda^{-1})^B{}_A\right), \tag{3.4}$$

and expanding the new frame analogously to (3.2), we obtain a local Galilei boost of the Galilei frame (v, e_a) with boost velocity parameter k. Furthermore, under this change of frame, the local one-forms a that arise as the c^{-1} component of the timelike dual frame one-form E^0 transform according to (2.26), thereby defining a Bargmann structure $\boldsymbol{a}$ on (M, τ, h).

This means that *as the formal $c \to \infty$ limit of the Lorentzian manifold we started with, we obtain a Galilei manifold with a Bargmann structure*. We stress again that the only assumption that is needed for this result is an expansion of the Lorentzian tetrad and dual tetrad as in (3.2), with a nowhere vanishing τ.

On the Lorentzian manifold (M, g), we now also consider a metric connection $\overset{\mathrm{L}}{\boldsymbol{\omega}}$ which we assume to have a regular formal $c \to \infty$ limit, by which we mean that its coordinate components with respect to c-independent coordinates, or equivalently its local connection form with respect to the frame (v, e_a), have regular limits (i.e. no terms of negative order in the expansion in c^{-1}). This implies that its local connection form with respect to (E_A) expands as

$$\overset{\mathrm{L}}{\omega}{}^0{}_0 = 0, \tag{3.5a}$$

$$\overset{\mathrm{L}}{\omega}{}^a{}_0 = c^{-1}\varpi^a + \mathrm{O}(c^{-3}) = \delta^{ab}\overset{\mathrm{L}}{\omega}{}^0{}_b\,, \tag{3.5b}$$

$$\overset{\mathrm{L}}{\omega}{}^a{}_b = \omega^a{}_b + \mathrm{O}(c^{-2}) \tag{3.5c}$$

for local one-forms $\omega^a{}_b$, ϖ^a. Under local rotations and boosts of the frame, the $(\omega^a{}_b, \varpi^a)$ transform as the local connection form of a Galilei connection on (M, τ, h) would, thereby

[9] Note that this arises from expanding the standard form $\Lambda = \exp(\zeta^a K_a)$ of a boost, written in terms of the rapidity $\zeta^a = \operatorname{artanh}(|k|/c)\frac{k^a}{|k|}$ and the boost generators $(K_a)^A{}_B = \delta^A_0\eta_{aB} - \delta^A_a\eta_{0B}$.

in fact defining a Galilei connection $\boldsymbol{\omega}$. Cartan's first structure equation then implies that the torsion $\overset{\mathrm{L}}{T}$ of $\overset{\mathrm{L}}{\boldsymbol{\omega}}$ expands as

$$\begin{aligned}\overset{\mathrm{L}}{T}{}^0 &= \mathrm{dE}^0 + \overset{\mathrm{L}}{\omega}{}^0{}_a \wedge \mathrm{E}^a = c\mathrm{d}\tau + c^{-1}(\mathrm{d}a + \varpi_a \wedge \mathrm{e}^a) + \mathrm{O}(c^{-3}) \\ &= c\mathrm{d}\tau + c^{-1} f + \mathrm{O}(c^{-3}),\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\overset{\mathrm{L}}{T}{}^a &= \mathrm{dE}^a + \overset{\mathrm{L}}{\omega}{}^a{}_B \wedge \mathrm{E}^B = \mathrm{de}^a + \varpi^a \wedge \tau + \omega^a{}_b \wedge \mathrm{e}^b + \mathrm{O}(c^{-2}) \\ &= T^A + \mathrm{O}(c^{-2})\end{aligned} \tag{3.6b}$$

in terms of the extended torsion of $\boldsymbol{\omega}$ with respect to the Bargmann structure $\boldsymbol{a}$ obtained from the expansion of the frame. Assuming $\mathrm{d}\tau = 0$, we can further compute the expansion of the Lorentzian contortion as

$$\begin{aligned}\overset{\mathrm{L}}{K}{}^\rho{}_{\mu\nu} &= \frac{1}{2}\overset{\mathrm{L}}{T}{}^\rho{}_{\mu\nu} - \overset{\mathrm{L}}{T}{}_{(\mu\nu)}{}^\rho \\ &= \frac{1}{2}\underbrace{\overset{\mathrm{L}}{T}{}^\rho{}_{\mu\nu}}_{=T^\rho_{\mu\nu}+\mathrm{O}(c^{-2})} - \eta_{AB}\mathrm{E}^A_{(\mu}\overset{\mathrm{L}}{T}{}^B{}_{\nu)\sigma}\ \underbrace{\eta^{CD}\mathrm{E}^\sigma_C\mathrm{E}^\rho_D}_{=h^{\sigma\rho}+\mathrm{O}(c^{-2})} \\ &= \frac{1}{2}T^\rho_{\mu\nu} + \tau_{(\mu}f_{\nu)\sigma}h^{\sigma\rho} - h_{\kappa(\mu}T^\kappa{}_{\nu)\sigma}h^{\sigma\rho} + \mathrm{O}(c^{-2}) \\ &= K^\rho{}_{\mu\nu} + \mathrm{O}(c^{-2}).\end{aligned} \tag{3.7}$$

Finally, we want to comment on the transformation behaviour of the 'limiting' geometric objects on our Galilei manifold under (active) pushforward of the Lorentzian objects along diffeomorphisms. Transforming the Lorentzian geometric objects by c-independent diffeomorphisms, all limiting objects also transform simply by pushforward. Considering instead 'pushforward along an infinitesimal c-dependent diffeomorphism', i.e. the transformation

$$A \to A - c^{-2}\mathcal{L}_X A + \mathrm{O}(c^{-4}) \tag{3.8}$$

on natural Lorentzian geometric objects A for some vector field X, the only non-trivial transformation of the limiting Galilei-manifold objects arising from this is the transformation

$$a \to a - \mathcal{L}_X\tau = a - \mathrm{d}\tau(X,\cdot) - \mathrm{d}(\tau(X)) \tag{3.9}$$

of the local representative of the Bargmann structure. If the clock form satisfies our assumption of absolute time, $\mathrm{d}\tau = 0$, this amounts to a $\mathsf{U}(1)$ gauge transformation of the Bargmann structure (2.32). This means that under the assumption of absolute time, *we obtain all 'natural symmetries' of the framework of Galilei manifolds with Bargmann structure—diffeomorphisms, local Galilei transformations, and* $\mathsf{U}(1)$ *gauge transformations of the Bargmann structure—from the action of 'c-dependent' diffeomorphisms and local Lorentz transformations on Lorentzian objects, i.e. from the 'natural symmetries' of the Lorentzian setting*. For this to hold, the assumption of absolute time is crucial: otherwise, the limiting geometric objects would transform not under the Bargmann algebra, but under a certain Lie algebra expansion of the Poincaré algebra, and would instead define what has been termed a 'torsional Newton–Cartan type II' (TNC type II) geometry [32, 33].

3.2. Trace-reversing the field equation of TEGR

The field equation of TEGR is [14]

$$\mathrm{E}^{-1}\partial_\sigma(\mathrm{E}S_A{}^{\mu\sigma}) - \overset{\mathrm{L}}{T}{}^\sigma{}_{\nu A} S_\sigma{}^{\nu\mu} + \frac{\mathrm{E}^\mu_A}{4} S_\rho{}^{\sigma\nu}\overset{\mathrm{L}}{T}{}^\rho{}_{\sigma\nu} + \overset{\mathrm{L}}{\omega}{}_\nu{}^B{}_A S_B{}^{\nu\mu} = \frac{8\pi G}{c^4}\Theta_A{}^\mu\,, \tag{3.10}$$

where Θ denotes the energy–momentum tensor, $\mathrm{E} = \det(\mathrm{E}^A_\mu)$ is the determinant of the matrix of dual frame components, and the superpotential is given by

$$S_\rho{}^{\sigma\mu} = \overset{\mathrm{L}}{K}{}^\sigma{}_\rho{}^\mu - 2\delta_\rho^{[\sigma}\overset{\mathrm{L}}{T}{}^{|\nu|}{}_\nu{}^{\mu]}\,. \tag{3.11}$$

Note that we have rewritten the equations such as to conform to our notation and conventions (in particular regarding the index structure of the contortion), and inserted the expression of the torsion scalar in terms of the superpotential [14, equation (4.160)] into the field equation [14, equation (4.163)].

In order to consistently take the $c \to \infty$ Newtonian limit of the field equation, we have to consider it in trace-reversed form[10]. The trace of (3.10), obtained by contraction with E^A_μ, is

$$\mathrm{E}^{-1}\mathrm{E}^A_\mu\partial_\sigma(\mathrm{E}S_A{}^{\mu\sigma}) + \overset{\mathrm{L}}{\omega}{}_\nu{}^B{}_A S_B{}^{\nu A} = \frac{8\pi G}{c^4}\Theta. \tag{3.12}$$

Rewriting $\mathrm{E}^A_\mu\partial_\sigma(\mathrm{E}S_A{}^{\mu\sigma}) = \partial_\sigma(\mathrm{E}S_A{}^{A\sigma}) - \mathrm{E}S_A{}^{\mu\sigma}\partial_\sigma\mathrm{E}^A_\mu$, applying the identities $S_A{}^{A\sigma} = 2\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu$ (which directly follows from the definition of S) and $\partial_\sigma\mathrm{E}^A_\mu = \overset{\mathrm{L}}{\Gamma}{}^\kappa_{\sigma\mu}\mathrm{E}^A_\kappa - \overset{\mathrm{L}}{\omega}{}_\sigma{}^A{}_B\mathrm{E}^B_\mu$, and using the antisymmetry $S_A{}^{\mu\sigma} = S_A{}^{[\mu\sigma]}$, the trace equation takes the form

$$2\mathrm{E}^{-1}\partial_\sigma(\mathrm{E}\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu) + \frac{1}{2}S_\rho{}^{\sigma\nu}\overset{\mathrm{L}}{T}{}^\rho{}_{\sigma\nu} = \frac{8\pi G}{c^4}\Theta. \tag{3.13}$$

Now trace-reversing the field equation, i.e. considering (3.10)$-\frac{1}{2}\mathrm{E}^\mu_A$ (3.13), we obtain

$$\mathrm{E}^{-1}\partial_\sigma\left(\mathrm{E}S_A{}^{\mu\sigma}\right) - \mathrm{E}^{-1}\mathrm{E}^\mu_A\partial_\sigma\left(\mathrm{E}\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu\right) - \overset{\mathrm{L}}{T}{}^\sigma{}_{\nu A}S_\sigma{}^{\nu\mu} + \overset{\mathrm{L}}{\omega}{}_\nu{}^B{}_A S_B{}^{\nu\mu} = \frac{8\pi G}{c^4}\left(\Theta_A{}^\mu - \tfrac{1}{2}\Theta\mathrm{E}^\mu_A\right). \tag{3.14}$$

We will now further rewrite this equation, in order to express it purely in terms of the teleparallel connection, the torsion, and the contortion. Denoting the Lorentzian Levi-Civita connection by $\overset{\mathrm{L}}{\tilde{\nabla}}$, we have

$$\mathrm{E}^{-1}\partial_\sigma(\mathrm{E}\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu) = \overset{\mathrm{L}}{\tilde{\nabla}}_\sigma\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu = \overset{\mathrm{L}}{\nabla}_\sigma\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu - \underbrace{\overset{\mathrm{L}}{K}{}^\sigma{}_{\sigma\lambda}}_{=\overset{\mathrm{L}}{T}{}^\sigma{}_{\sigma\lambda}}\overset{\mathrm{L}}{T}{}^{\nu\lambda}{}_\nu\,, \tag{3.15}$$

implying

$$\mathrm{E}^{-1}\mathrm{E}^\mu_A\partial_\sigma(\mathrm{E}\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu) = \overset{\mathrm{L}}{\nabla}_\sigma(\mathrm{E}^\mu_A\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu) - \overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_\nu\underbrace{\overset{\mathrm{L}}{\nabla}_\sigma\mathrm{E}^\mu_A}_{=\overset{\mathrm{L}}{\omega}{}_\sigma{}^B{}_A\mathrm{E}^\mu_B} + \mathrm{E}^\mu_A\overset{\mathrm{L}}{T}{}^\sigma{}_{\lambda\sigma}\overset{\mathrm{L}}{T}{}^{\nu\lambda}{}_\nu\,, \tag{3.16}$$

[10] If we did not trace-reverse the equation, we would end up with a formal c^{-1} expansion not allowing us to easily extract meaningful information about the limit: inserting the expansions of the geometric objects as introduced above into the field equation, the only order at which the expanded equation would make a statement would be identically satisfied; the limit field equations proper would appear at the next order, which would no longer be contained in the expanded equation since the termination of the expansion of the geometric objects introduces unspecified terms into the equation.I.e. to obtain the leading-order equations for the expanded geometric objects from the original field equation, one would have to expand the objects to higher order than appear in the final equations. This is prevented by considering the trace-reversed equation instead.

and

$$
\begin{aligned}
\mathrm{E}^{-1}\partial_\sigma(\mathrm{E}S_A{}^{\mu\sigma}) &= \underbrace{\frac{\partial_\nu \mathrm{E}}{\mathrm{E}}}_{=\overset{\mathrm{L}}{\tilde{\Gamma}}{}^{\sigma}_{\sigma\nu}} S_A{}^{\mu\nu} + \partial_\sigma S_A{}^{\mu\sigma} \\
&= \overset{\mathrm{L}}{\tilde{\nabla}}_\sigma S_A{}^{\mu\sigma} - \overset{\mathrm{L}}{\tilde{\Gamma}}{}^{\mu}_{\sigma\nu} \underbrace{S_A{}^{\nu\sigma}}_{=S_A{}^{[\nu\sigma]}} \\
&= \overset{\mathrm{L}}{\tilde{\nabla}}_\sigma S_A{}^{\mu\sigma} \\
&= \overset{\mathrm{L}}{\nabla}_\sigma S_A{}^{\mu\sigma} - \overset{\mathrm{L}}{K}{}^{\mu}{}_{\sigma\nu} S_A{}^{\nu\sigma} - \overset{\mathrm{L}}{K}{}^{\sigma}{}_{\sigma\nu} S_A{}^{\mu\nu} \\
&= \overset{\mathrm{L}}{\nabla}_\sigma S_A{}^{\mu\sigma} - \tfrac{1}{2}\overset{\mathrm{L}}{T}{}^{\mu}{}_{\sigma\nu} S_A{}^{\nu\sigma} + \overset{\mathrm{L}}{T}{}^{\sigma}{}_{\nu\sigma} S_A{}^{\mu\nu},
\end{aligned}
\tag{3.17}
$$

where we used $\overset{\mathrm{L}}{K}{}^{\mu}{}_{[\sigma\nu]} = \frac{1}{2}\overset{\mathrm{L}}{T}{}^{\mu}{}_{\sigma\nu}$ and $\overset{\mathrm{L}}{K}{}^{\sigma}{}_{\sigma\nu} = -\overset{\mathrm{L}}{T}{}^{\sigma}{}_{\nu\sigma}$. Introducing the abbreviation

$$
\tilde{S}_A{}^{\mu\sigma} := S_A{}^{\mu\sigma} - \mathrm{E}^{\mu}_{A}\overset{\mathrm{L}}{T}{}^{\nu\sigma}{}_{\nu}\,, \tag{3.18}
$$

we can use (3.16) and (3.17) to rewrite the trace-reversed field equation (3.14) as

$$
\overset{\mathrm{L}}{\nabla}_\sigma \tilde{S}_A{}^{\mu\sigma} - \tfrac{1}{2}\overset{\mathrm{L}}{T}{}^{\mu}{}_{\sigma\nu} S_A{}^{\nu\sigma} + \overset{\mathrm{L}}{T}{}^{\sigma}{}_{\nu\sigma}\tilde{S}_A{}^{\mu\nu} - \overset{\mathrm{L}}{T}{}^{\sigma}{}_{\nu A} S_\sigma{}^{\nu\mu} - \overset{\mathrm{L}}{\omega}{}_{\sigma}{}^{B}{}_{A}\tilde{S}_B{}^{\mu\sigma} = \frac{8\pi G}{c^4}\left(\Theta_A{}^{\mu} - \tfrac{1}{2}\Theta\mathrm{E}^{\mu}_{A}\right). \tag{3.19}
$$

Using $\overset{\mathrm{L}}{\nabla}_\sigma \tilde{S}_A{}^{\mu\sigma} - \overset{\mathrm{L}}{\omega}{}_{\sigma}{}^{B}{}_{A}\tilde{S}_B{}^{\mu\sigma} = \overset{\mathrm{L}}{D}_\sigma \tilde{S}_A{}^{\mu\sigma}$, lowering the index μ, inserting the identity $\tilde{S}_{A\mu}{}^{\sigma} = -\overset{\mathrm{L}}{K}{}^{\sigma}{}_{\mu A} - \mathrm{E}^{\sigma}_{A}\overset{\mathrm{L}}{T}{}^{\nu}{}_{\mu\nu} = -\overset{\mathrm{L}}{K}{}^{\sigma}{}_{\mu A} + \mathrm{E}^{\sigma}_{A}\overset{\mathrm{L}}{K}{}^{\nu}{}_{\nu\mu}$ as well as the definition of S, and contracting with E^{μ}_{B}, this equation can be shown to be equivalent to

$$
-\overset{\mathrm{L}}{D}_\sigma \overset{\mathrm{L}}{K}{}^{\sigma}{}_{AB} + \overset{\mathrm{L}}{D}_A \overset{\mathrm{L}}{K}{}^{\mu}{}_{\mu B} - \overset{\mathrm{L}}{K}{}^{\mu}{}_{\sigma B}\overset{\mathrm{L}}{T}{}^{\sigma}{}_{\mu A} + \overset{\mathrm{L}}{K}{}^{\mu}{}_{\mu\sigma}\overset{\mathrm{L}}{K}{}^{\sigma}{}_{AB} - \overset{\mathrm{L}}{K}{}^{\mu}{}_{A\sigma}\overset{\mathrm{L}}{K}{}^{\sigma}{}_{\mu B} = \frac{8\pi G}{c^4}\left(\Theta_{AB} - \tfrac{1}{2}\Theta\eta_{AB}\right). \tag{3.20}
$$

Note that we could have arrived at this result more directly, or at least anticipated it beforehand: the left-hand side of (3.20) is simply the component $\overset{\mathrm{L}}{\tilde{R}}_{BA}$ of the Ricci tensor of the Lorentzian Levi-Civita connection expressed in terms of the teleparallel connection, its torsion, and the contortion, such that the equation is simply the trace-reversed Einstein equation. However, we wanted to keep the argumentation 'as teleparallel as possible' and therefore have presented the calculation leading to (3.20) with as little reference to the Levi-Civita connection as possible.

3.3. *Taking the limit*

We are now going to expand the field equation (3.20) in c^{-1}, and thus have to expand the trace-reversed energy–momentum tensor appearing on its right-hand side. First working in the Lorentzian setting without any c-expansion, we may decompose the energy–momentum tensor with respect to any unit future-directed timelike vector field ξ as

$$
\Theta_{\mu\nu} = \xi_\mu\xi_\nu w - c2\xi_{(\mu}\Pi_{\nu)} + \Sigma_{\mu\nu}\,, \tag{3.21}
$$

where w, Π and Σ are the energy density, the momentum density and the momentum flux density (stress tensor) with respect to ξ. Choosing $\xi^\mu = \mathrm{E}^\mu_0$ from our tetrad, which implies $\xi_\mu = -\mathrm{E}^0_\mu$, and expanding the energy density in this frame as

$$w = \rho c^2 + w^{(0)} + \mathrm{O}(c^{-2}) \tag{3.22}$$

with ρ being the (rest) mass density, while taking momentum density and momentum flux density to be of order c^0, we obtain the expansion

$$\Theta_{\mu\nu} = c^4 \rho \tau_\mu \tau_\nu + c^2 w^{(0)} \tau_\mu \tau_\nu + c^2 2\rho\tau_{(\mu} a_{\nu)} + c^2 2\tau_{(\mu}\Pi_{\nu)} + \mathrm{O}(c^0) \tag{3.23}$$

for the energy–momentum tensor, where we have used the expansion (3.2) of the tetrad[11]. Contracting this expansion again with that of the tetrad components (3.2), we obtain the trace $\Theta = -c^2\rho + \mathrm{O}(c^0)$. Thus, the trace-reversed energy–momentum tensor is

$$\begin{aligned} \Theta_{\mu\nu} - \tfrac{1}{2}\Theta \mathrm{E}^A_\mu \mathrm{E}^B_\nu \eta_{AB} &= \Theta_{\mu\nu} - \tfrac{1}{2}\Theta(-c^2\tau_\mu\tau_\nu + \mathrm{O}(c^0)) \\ &= \tfrac{1}{2}c^4\rho\tau_\mu\tau_\nu + \mathrm{O}(c^2), \end{aligned} \tag{3.24}$$

and the expansion of the right-hand side of the trace-reversed TEGR field equation (3.20) is $4\pi G\rho\tau_A\tau_B + \mathrm{O}(c^{-2})$. Thus, expanding the objects on the left-hand side as well, we see that in the $c\to\infty$ limit, the TEGR field equation goes over to the field equation (2.36) of teleparallel Newton–Cartan gravity.

Regarding curvature, one easily checks that the components of the curvature form of any Lorentzian connection expanded as in (3.5) take the form

$$\overset{\mathrm{L}}{R}{}^a{}_0 = c^{-1}(\mathrm{d}\varpi^a + \omega^a_b \wedge \varpi_b) + \mathrm{O}(c^{-3}) = c^{-1}R^a{}_t + \mathrm{O}(c^{-3}), \tag{3.25a}$$

$$\overset{\mathrm{L}}{R}{}^a{}_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b + \mathrm{O}(c^{-2}) = R^a{}_b + \mathrm{O}(c^{-2}), \tag{3.25b}$$

$$\overset{\mathrm{L}}{R}{}^0{}_0 = \mathrm{O}(c^{-4}), \tag{3.25c}$$

$$\overset{\mathrm{L}}{R}{}^0{}_a = c^{-1}\delta_{ab}R^b{}_t + \mathrm{O}(c^{-3}) \tag{3.25d}$$

in terms of the curvature form of the limiting Galilei connection. Therefore, if $\overset{\mathrm{L}}{\omega}$ is flat, then also the limiting Galilei connection ω will be flat.

Finally, it is clear that the TEGR test particle equation of motion

$$(\nabla_{\dot\gamma}\dot\gamma)^\rho = \overset{\mathrm{L}}{K}{}^\rho{}_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \tag{3.26}$$

goes over to its teleparallel Newton–Cartan equivalent (2.35) in the $c\to\infty$ limit. Thus, we have shown that the formal $c\to\infty$ limit of TEGR with respect to a closed clock one-form τ is teleparallel Newton–Cartan gravity as introduced in section 2.

4. Recovering Newtonian gravity

As for usual Newton–Cartan gravity, the standard formulation of Newtonian gravity can be recovered from teleparallel Newton–Cartan gravity, as we will show now.

[11] Note that the mass density ρ is invariant under changes of frame.

4.1. The field equation

The *tt*-component of the teleparallel Newton–Cartan field equation (2.36), i.e. the equation for $A = B = t$, can be rewritten as

$$-D_\sigma K^\sigma{}_{tt} - D_t T^\mu{}_{t\mu} - K^a{}_{bt} T^b{}_{at} + T^a{}_{ab} K^b{}_{tt} - K^a{}_{tb} K^b{}_{at} = 4\pi G\rho, \tag{4.1}$$

where we used that $T^t = 0$ and therefore $K^t{}_{\mu\nu} = 0$. In the following, we are going to show how to recover from this the standard formulation of the Newtonian field equation.

As a first step, we 'gauge-fix' the connection $\boldsymbol{\omega}$ to vanishing purely spatial torsion, i.e. we assume

$$T^a{}_{bc} = 0. \tag{4.2}$$

Here we put the term 'gauge-fix' in quotation marks since we do *not* use it to refer to the fixing of any gauge redundancy in the proper sense (e.g. in a Hamiltonian analysis). Instead, we only use it to mean that we add (4.2) as an additional assumption on the connection $\boldsymbol{\omega}$, which is compatible with the field equations and does not introduce any further restrictions. That the choice of vanishing purely spatial torsion is always possible follows from the purely spatial components ($A = a$, $B = b$) of the field equation (2.36): this part of the equation reads $\widetilde{R}_{ab} = 0$ in terms of the Ricci tensor of the extended-torsion-free connection $\widetilde{\boldsymbol{\omega}}$. This means that the spatial leaves are Ricci-flat as Riemannian manifolds (with the spatial metric induced by h); since they are three-dimensional, this implies that they are flat. Therefore, by choosing the spatial connection $\omega^a{}_b$ to be the Levi-Civita connection $\widetilde{\omega}^a{}_b$ of the spatial leaves, we may indeed satisfy (4.2) as well as flatness of $\boldsymbol{\omega}$.

Let us stress here again that this 'gauge-fixing' assumption of vanishing purely spatial torsion is, differently to the situation considered in [28], *not* part of the formulation of the theory, but only added *afterwards* for the recovery of standard Newtonian gravity. Note also that it is more general than the torsion constraint from [28], where the *total* spatial torsion $T^a{}_{\mu\nu}$ is assumed to vanish (i.e. including its mixed spatio-temporal components $T^a{}_{tb}$).[12]

Now we are going to rewrite (4.1). From

$$K^\sigma{}_{(AB)} = \overset{(3)}{K}{}^\sigma{}_{(AB)} + \tau_{(A} f_{B)}{}^\sigma = -T_{(AB)}{}^\sigma + \tau_{(A} f_{B)}{}^\sigma \tag{4.3}$$

we obtain

$$\begin{aligned} D_\sigma K^\sigma{}_{tt} &= -D_\sigma T_{tt}{}^\sigma + D_\sigma f_t{}^\sigma \\ &= \varpi_\sigma{}^a T_{at}{}^\sigma + D_a f_t{}^a \\ &= -\varpi_b{}^a T_a{}^b{}_t - D_a f^a{}_t\,, \end{aligned} \tag{4.4}$$

where at the second equality sign we used $T_{t\mu\nu} = 0$ for the first and (2.11) for the second term. Furthermore, from

$$D_A T^\mu{}_{B\mu} = \partial_A T^\mu{}_{B\mu} - \omega_A{}^C{}_B T^\mu{}_{C\mu} \tag{4.5}$$

we obtain

$$D_t T^\mu{}_{t\mu} = \partial_t T^\mu{}_{t\mu} - \varpi_t{}^c T^\mu{}_{c\mu} = -\partial_t T^a{}_{at}\,, \tag{4.6}$$

[12] In [28], this torsion constraint is not even emphasised as an assumption, but instead it is stated that 'for a flat connection, the Bianchi identities imply that [the spatial torsion vanishes]'. This clearly is not true at all; one can easily give examples of flat metric torsionful connections in, for example, the two-dimensional Euclidean plane.

where we used our 'gauge assumption' $T^a{}_{bc} = 0$. A direct calculation further shows that

$$-K^a{}_{bt}T^b{}_{at} - K^a{}_{tb}K^b{}_{at} = -T^{ab}{}_t T_{(ab)t} + \frac{1}{4}f^{ab}f_{ab}\,. \tag{4.7}$$

Using (4.4), (4.6), (4.7) and (again) vanishing of the purely spatial torsion, we can rewrite the tt-component (4.1) of the field equation as

$$D_a f^a{}_t + \varpi_b{}^a T_a{}^b{}_t + \partial_t T^a{}_{at} - T^{ab}{}_t T_{(ab)t} + \frac{1}{4}f^{ab}f_{ab} = 4\pi G\rho. \tag{4.8}$$

From now on, we are going to assume the 'absolute rotation' condition introduced by Trautman [6, 7]. This is an additional curvature condition that has to be assumed in usual Newton–Cartan gravity in order to recover Newtonian gravity in the usual sense, taking the form

$$\widetilde{R}^{\mu\nu}{}_{\rho\sigma} = 0. \tag{4.9}$$

Without assuming this, Newton–Cartan gravity actually describes a slightly generalised version of Newtonian gravity, in which the vorticity of rigid timelike flows need not be spatially constant. If in a Newton–Cartan spacetime there is one rigid timelike vector field with spatially variable vorticity, then actually *all* rigid timelike vector fields have spatially non-constant vorticity, and in this sense in such a spacetime no absolute notion of rotation exists. In the 'recovered' Newtonian equations this leads to non-eliminable Coriolis force terms. If one however *assumes* the absolute rotation condition (4.9), rigid non-rotating frames exist, and in those one recovers Newtonian gravity proper[13]. Note that (4.9) also implies $\widetilde{R}^a{}_t = 0$, i.e. the at-component of the field equation (2.36).

Assuming absolute rotation, we now choose the timelike frame field $v = \mathrm{e}_t$ to be rigid and non-rotating. In usual Newton–Cartan gravity, i.e. in terms of the extended-torsion-free connection, this means that the frame field satisfies $\mathcal{L}_v h = 0$ or equivalently $\widetilde{\nabla}^{(\mu}v^{\nu)} = 0$ (rigidness), as well as $\widetilde{\nabla}^{[\mu}v^{\nu]} = 0$ (it is non-rotating). Expressed in terms of our 'teleparallel' Newton–Cartan connection ∇, this means[14]

$$\varpi_{(ab)} = T_{(ab)t}\,, \tag{4.10a}$$

$$\varpi_{[ab]} = \frac{1}{2}f_{ab}\,. \tag{4.10b}$$

The second of these equations is equivalent to $(\mathrm{d}a)_{ab} = 0$, i.e. $0 = (\mathrm{d}a)|_\Sigma = \mathrm{d}(a|_\Sigma)$ on any spatial leaf Σ (where a is the frame representative of the Bargmann structure). Thus, we locally have $a|_\Sigma = \mathrm{d}u$ for some function u on Σ; choosing u smoothly between leaves, we thus have $a = \mathrm{d}u + \phi\tau$ for functions u, ϕ. In particular, we have

$$\mathrm{d}a = \mathrm{d}\phi \wedge \tau. \tag{4.11}$$

Obviously, this conversely implies $(\mathrm{d}a)_{ab} = 0$, i.e. (4.10b). The (locally defined, frame-dependent) function ϕ defined by (4.11) will play the rôle of the Newtonian potential, and is defined up to addition of a time-dependent spatially constant function[15].

[13] The interpretation of the additional curvature condition (4.9) in terms of rotation was, as far as I (the author) know, not given by Trautman, at least not explicitly. To my knowledge, it first appears in explicit form in Ehlers' 1981 article on frame theory [10, 11].

[14] This may be verified by direct computation, using the rigidness and non-rotation conditions in terms of $\widetilde{\nabla}$ and the explicit form of the Newton–Cartan contortion (2.33).

[15] Note that this is precisely the same as for usual Newton–Cartan gravity: there, the Newtonian potential in the reconstruction of Newtonian gravity in a rigid non-rotating frame is defined by $\tilde{\Omega} = -\mathrm{d}\phi \wedge \tau$.

From now on working in a rigid non-rotating frame in the sense introduced above, we have $f = \mathrm{d}a + \Omega = \mathrm{d}\phi \wedge \tau + \Omega$, in particular implying

$$f_{at} = \partial_a \phi + 2\varpi_{[at]} = \partial_a \phi + \cancel{\varpi_{at}} - \varpi_{ta} \,. \tag{4.12}$$

Thus, we can rewrite

$$\begin{aligned} D_a f^a{}_t &= \partial_a f^a{}_t + \omega_a{}^a{}_b f^b{}_t - \underbrace{\varpi_a{}^b f^a{}_b}_{\overset{(4.10b)}{=} 2\varpi^{[ab]}\varpi_{[ab]}} \\ &= \underbrace{\partial_a \partial^a \phi + \omega_a{}^a{}_b \partial^b \phi}_{= D_a D^a \phi} - \partial_a \varpi_t{}^a - \omega_a{}^a{}_b \varpi_t{}^b - 2\varpi^{[ab]}\varpi_{[ab]} \,. \end{aligned} \tag{4.13}$$

For any one-form α, we have the general identity

$$(\mathrm{d}\alpha)_{\mu\nu} = 2\partial_{[\mu}\alpha_{\nu]} = 2\nabla_{[\mu}\alpha_{\nu]} + T^\rho{}_{\mu\nu}\alpha_\rho \,, \tag{4.14}$$

giving in particular

$$\begin{aligned} (\mathrm{d}\alpha)_{at} &= 2D_{[a}\alpha_{t]} + T^B{}_{at}\alpha_B \\ &= 2\partial_{[a}\alpha_{t]} - \varpi_a{}^b \alpha_b + \omega_t{}^b{}_a \alpha_b + T^b_{at}\alpha_b \,. \end{aligned} \tag{4.15}$$

Applied to the one-form ϖ^a, we obtain

$$(\mathrm{d}\varpi^a)_{at} = 2\partial_{[a}\varpi_{t]}{}^a - \underbrace{\varpi_a{}^b \varpi_b{}^a}_{= \varpi^{ab}\varpi_{ba}} + \omega_t{}^b{}_a \varpi_b{}^a + T^b{}_{at}\varpi_b{}^a \,. \tag{4.16}$$

Flatness of the connection implies $0 = \mathrm{d}\varpi^a + \omega^a{}_b \wedge \varpi^b$, which combined with the previous equation leads to

$$\begin{aligned} 0 &= (\mathrm{d}\varpi^a)_{at} + (\omega^a{}_b \wedge \varpi^b)_{at} \\ &= \partial_a \varpi_t{}^a - \underbrace{\partial_t \varpi_a{}^a}_{\overset{(4.10a)}{=} T^a{}_{at}} - \varpi^{ab}\varpi_{ba} + \cancel{\omega_t{}^b{}_a \varpi_b{}^a} + T^b{}_{at}\varpi_b{}^a + \omega_a{}^a{}_b \varpi_t{}^b - \cancel{\omega_t{}^a{}_b \varpi_a{}^b} . \end{aligned} \tag{4.17}$$

Combining (4.13) and (4.17), several terms cancel out and we arrive at

$$D_a f^a{}_t + \partial_t T^a{}_{at} = D_a D^a \phi \underbrace{- 2\varpi^{[ab]}\varpi_{[ab]} - \varpi^{ab}\varpi_{ba}}_{= -\varpi^{ab}\varpi_{ab}} + T^b{}_{at}\varpi_b{}^a . \tag{4.18}$$

Inserting this, we can rewrite the tt-component (4.8) of the field equation as

$$D_a D^a \phi - \varpi^{ab}\varpi_{ab} + \varpi_{(ba)} 2T^{(ab)}{}_t - T^{ab}{}_t T_{(ab)t} + \frac{1}{4} f^{ab} f_{ab} = 4\pi G\rho, \tag{4.19}$$

which using the conditions (4.10) on the frame reduces to

$$D_a D^a \phi = 4\pi G\rho, \tag{4.20}$$

the Newtonian field equation. Note that due to our choice of vanishing purely spatial torsion, the induced spatial connection is the Levi-Civita connection of the induced spatial metric, such that the expression $D_a D^a \phi$ is the spatial metric Laplace operator acting on ϕ.

4.2. The test particle equation of motion

Of course, to warrant the claim that we can recover Newtonian gravity, we not only have to show that we have a field ϕ satisfying the Newtonian field equation, but also that test particles couple to it in the correct way. We will now show how the test particle equation of motion may be reduced to its usual Newtonian counterpart.

We consider the equation of motion (2.35) for a unit timelike vector field ξ, i.e.

$$\xi^\sigma \nabla_\sigma \xi^\rho = K^\rho{}_{\mu\nu} \xi^\mu \xi^\nu, \quad \tau_\mu \xi^\mu = 1. \tag{4.21}$$

Inserting the definition of K, this equation is equivalent to

$$\xi^\sigma D_\sigma \xi^a = (-T_{bt}{}^a + f_b{}^a)\xi^b + f_t{}^a. \tag{4.22}$$

Now in a rigid, non-rotating frame (4.10) with ϕ defined by (4.11), we have

$$-T_{bt}{}^a = T_b{}^a{}_t \overset{(4.10a)}{=} \varpi^a{}_b + \varpi_b{}^a - T^a{}_{bt}\,. \tag{4.23}$$

Using this, (4.10*b*) and (4.12), we can rewrite the equation of motion as follows:

$$\begin{aligned}\xi^\sigma D_\sigma \xi^a &= 2\xi^b \varpi_b{}^a + \varpi_t{}^a - T^a{}_{bt}\xi^b - \partial^a \phi \\ &= \xi^\sigma \varpi_\sigma{}^a + (\varpi_b{}^a - T^a{}_{bt})\xi^b - \partial^a \phi.\end{aligned} \tag{4.24}$$

We now specialise to a frame such that the spacelike frame fields commute with the timelike one, i.e. $[\mathrm{e}_a, v] = 0$. Such a frame may always be constructed by starting with orthonormal vector fields e_a on one spatial leaf and then extending those along the flow of v to vector fields on spacetime. The above commutation requirement is equivalent to $T(\mathrm{e}_b, v) = \nabla_{\mathrm{e}_b} v - \nabla_v \mathrm{e}_b$, or in components to

$$\varpi_b{}^a - T^a{}_{bt} = \omega_t{}^a{}_b \tag{4.25}$$

(where we used $T^t = 0$). Thus, in such a frame the equation of motion (4.24) takes the form

$$\xi^\sigma D_\sigma \xi^a = \xi^\sigma \varpi_\sigma{}^a + \omega_t{}^a{}_b \xi^b - \partial^a \phi. \tag{4.26}$$

Explicitly expressing the left-hand side in terms of components, we obtain

$$\xi^\sigma \partial_\sigma \xi^a + \omega_c{}^a{}_b \xi^c \xi^b = -\partial^a \phi, \tag{4.27}$$

which for a flow line γ of ξ becomes[16]

$$\ddot\gamma^a(t) + \omega_c{}^a{}_b(\gamma(t))\,\dot\gamma^c(t)\,\dot\gamma^b(t) = -\partial^a \phi(\gamma(t)). \tag{4.28}$$

This is the Newtonian equation of motion for a test particle, expressed in a non-Cartesian orthonormal spatial frame.

5. Conclusion

In this paper, we have shown how the local formulation of the geometry of Galilei spacetimes in terms of the Bargmann group gives rise to a natural notion of teleparallel Galilei connections, allowing for a teleparallel formulation of Newton–Cartan gravity that generalises the special case from [28]. We have shown that this theory is the natural $c \to \infty$ limit of TEGR (with respect to a closed clock form/leading order timelike dual frame field), and explained how to

[16] Note that due to $\tau_\mu \xi^\mu = 1$, the parameter of γ is Newtonian time t as defined by τ.

recover standard Newtonian gravity from it. Thus, teleparallel Newton–Cartan gravity provides a geometric description of how TEGR gives rise to Newtonian gravity as its Newtonian limit, in the same way as usual Newton–Cartan gravity does this for GR.

The work presented here lends itself to several interesting generalisations. First, one may wonder about a generalisation to the situation of so-called *torsional Newton–Cartan gravity* (TNC gravity) [22, 23], i.e. to a non-closed clock form $d\tau \neq 0$, allowing for non-absolute time. Note, however, that in order to obtain such a theory as a (formal) limit of TEGR (with respect to a non-closed τ), one probably would have to take as the 'gauged' algebra that locally describes the geometry not the Bargmann algebra, but a specific Lie algebra expansion of the Poincaré algebra. This means that one would have to consider what has been termed a 'torsional Newton–Cartan type II' (TNC type II) geometry instead of a TNC type I geometry [32, 33].

A second direction for further investigations concerns the formulation of teleparallel Newton–Cartan gravity in terms of an action principle, instead of the sole consideration of the equations of motion as done here. Since in particular *modified* teleparallel theories of gravity are commonly formulated in terms of an action principle, this would enable the investigation of the geometric Newtonian limit of modified teleparallel theories of gravity. Note that similar to the previous point, for developing an action principle one probably needs to consider TNC type II geometry, since standard Newton–Cartan gravity and TNC type I gravity cannot be given a variational formulation [32].

One may also seek modifications of the present work describing the Newtonian limit of theories of gravity based on further different geometries. For example, one could consider so-called *symmetric teleparallel* gravity theories, in which the 'Lorentzian' connection is flat and torsion-free but has non-metricity, or analyse even more general metric-affine theories in which the connection can have some combination of curvature, torsion and non-metricity.

Finally, the formulation of teleparallel Newton–Cartan gravity allows for the investigation of the post-Newtonian expansion of (modified) teleparallel gravity theories in a geometric, coordinate-free way, complementing the coordinate approach from [34–36]. Such a geometric description of the post-Newtonian expansion of standard GR, starting with and going beyond usual Newton–Cartan gravity, has been developed in [31, 37] and widely extended in [32, 33] (although the latter references take a somewhat more 'field-theoretic'/gauge-theoretic perspective on the topic). The development of such a geometric description of the post-Newtonian expansion of modified (teleparallel) theories of gravity we view as the most interesting direction for further research, since, to quote the introduction, questions regarding an inherently geometric theory ought to be answered in a geometric fashion[17].

Data availability statement

No new data were created or analysed in this study.

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[17] Of course, for practical computations in post-Newtonian theory the standard coordinate approach has some clear advantages. Nevertheless, from a conceptual point of view a geometric understanding is of fundamental importance, and it may also offer insights for concrete calculations, e.g. regarding questions of coordinate (in-)dependence of observed phenomena.

Appendix. Semidirect extension of principal bundles

Here we will explain the 'semidirect extension' construction for principal bundles and connections on them that is used in the main text in the definition of Bargmann structures in section 2.2.

Let H, N be Lie groups and $\rho\colon H \to \mathsf{Aut}(N)$ a smooth homomorphism. By $\dot{\rho}\colon H \to \mathsf{Aut}(\mathfrak{n})$, we denote the induced representation of H on $\mathfrak{n}$ by Lie algebra automorphisms, defined by

$$\dot{\rho}_h := \mathrm{D}(\rho_h)|_{\mathrm{e}_N} \in \mathsf{Aut}(\mathfrak{n}). \tag{A.1}$$

Let $P \xrightarrow{\pi} M$ be a principal H-bundle. We can extend P to a principal $H \ltimes N$-bundle $Q \xrightarrow{\hat{\pi}} M$ as follows: denoting by $\tilde{\rho}\colon H \to \mathsf{Diff}(H \ltimes N)$ the natural left action of H on $H \ltimes N$ by multiplication, i.e.

$$\tilde{\rho}_{h_2}(h_1, n) := (h_2, \mathrm{e}_N)(h_1, n) = (h_2 h_1, \rho_{h_2}(n)), \tag{A.2a}$$

we define Q as the associated bundle

$$Q := P \times_{\tilde{\rho}} (H \ltimes N). \tag{A.2b}$$

The natural right action of $H \ltimes N$ on itself induces a free right action on Q which is transitive on the fibres and compatible with the local trivialisations, thus making Q into a principal bundle as desired. We also obtain natural bundle homomorphisms $Q \underset{\gamma}{\overset{\beta}{\rightleftarrows}} P$ satisfying $\beta \circ \gamma = \mathrm{id}_P$, namely

$$\gamma(p) = [p, (\mathrm{e}_H, \mathrm{e}_N)], \quad \beta([p, (h, n)]) = ph\,. \tag{A.3}$$

By construction, with respect to local trivialisations $P \xrightarrow{\gamma} Q$ looks like the inclusion $H \hookrightarrow H \ltimes N$, such that it really exhibits Q as an extension of P.

Theorem A.1. *Let $\hat{\boldsymbol{\omega}} \in \Omega^1(Q, \mathfrak{h} \oplus \mathfrak{n})$ be a connection on Q. We decompose its pullback along γ as $\gamma^* \hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\theta})$ with $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{h})$ and $\boldsymbol{\theta} \in \Omega^1(P, \mathfrak{n})$. Then $\boldsymbol{\omega}$ is a connection and $\boldsymbol{\theta}$ is a $\dot{\rho}$-tensorial form on P.*

Conversely, given a connection $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{h})$ and a $\boldsymbol{\theta} \in \Omega^1_{\dot{\rho}}(P, \mathfrak{n})$, there is a unique connection $\hat{\boldsymbol{\omega}} \in \Omega^1(Q, \mathfrak{h} \oplus \mathfrak{n})$ such that $\gamma^* \hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\theta})$.

The curvature form $\hat{\boldsymbol{R}} \in \Omega^2(Q, \mathfrak{h} \oplus \mathfrak{n})$ of $\hat{\boldsymbol{\omega}}$ satisfies

$$\gamma^* \hat{\boldsymbol{R}} = \left(\boldsymbol{R}, \mathrm{d}^{\boldsymbol{\omega}} \boldsymbol{\theta} + \tfrac{1}{2}[\boldsymbol{\theta} \wedge \boldsymbol{\theta}]\right), \tag{A.4}$$

where $\boldsymbol{R} \in \Omega^2(p, \mathfrak{h})$ is the curvature form of $\boldsymbol{\omega}$.

Idea of proof. Using the explicit form of the adjoint representation of a semidirect product group, the Ad-equivariance of $\hat{\boldsymbol{\omega}}$, and the H-equivariance of γ, one can check the properties needed of $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ by direct computation.

For the converse, $\hat{\boldsymbol{\omega}}$ is determined on the image of γ in the 'N directions' of TQ, which are generated by fundamental vector fields, by being a connection, and in the remaining directions by its pullback $\gamma^* \hat{\boldsymbol{\omega}}$. Ad-equivariance by N then uniquely determines $\hat{\boldsymbol{\omega}}$ on the whole of Q. A computation which is quite direct, only somewhat tedious to write down, then shows that the $\hat{\boldsymbol{\omega}}$ thus defined really is a connection.

The expression for the pullback of the curvature form follows directly by pulling back the structure equation $\hat{\boldsymbol{R}} = \mathrm{d}\hat{\boldsymbol{\omega}} + \frac{1}{2}[\hat{\boldsymbol{\omega}} \wedge \hat{\boldsymbol{\omega}}]$. □

As noted in the main text, this construction generalises the classical situation of connections on the affine frame bundle of a manifold (see, e.g., [30, section III.3]). Note also that in the case $\dim M = \dim N$, our construction may be seen as a special case of Cartan geometry [38]—for the application in the main text however, namely the consideration of Barg bundles over Galilei manifolds, this is not satisfied.

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