Sometimes Mathematics is Different

Studies on Mathematical Practices in Electrical Engineering

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Referent:	Prof. Dr. Reinhard Hochmuth
Koreferent:	Prof. Dr. Frode Rønning
Koreferent:	Prof. Dr. Andres Wernet
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Abstract

The six studies collected in this thesis deal with praxeological analyses of mathematical practices in a Signal Transmission module of an electrical engineering study programme. The research framework is the Anthropological Theory of the Didactic (ATD). Our analyses concern the introduction of the Dirac delta impulse in textbooks on signals and systems, the lecturer's sample solution to an exercise on the envelope demodulator, and the lecturer's sample solution and the students' solutions to an exercise on amplitude modulation. Three research foci are developed in the collected studies. The first focus is *The subject-specific reconstruction of mathematical practices* (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021). We analyse two exercises and the corresponding sample solutions given by the lecturer and students' solutions. Based on methodological developments we were able to reconstruct aspects of two different mathematical discourses as well as how they interact within the analysed practices. We also propose a graphical representation of the results of our analyses.

The second focus is *The epistemological and philosophical relationship between mathematics and electrical engineering* (Hochmuth & Peters, 2020, 2022). Our aim is to gain a better understanding of the mathematical practices involved in the introduction of the Dirac delta impulse in a textbook on signals and systems. We realised that certain mathematical steps could be better understood from an engineering point of view if historical-philosophical studies of the relationship between physics and mathematics were also taken into account. Here, we are not referring to the philosophical positions of students or lecturers. We are referring to philosophical studies that focus on the societal aspects that play a role in the historical concrete formation of practices. The case of the Dirac impulse is well suited to illustrate the fruitfulness of such studies for subject-specific analyses.

Finally, the third focus is *Revisiting the relationship between mathematics and electrical engineering*, on how ideas for the development of teaching can be derived from the other two perspectives (Peters, 2022; Peters & Hochmuth, 2022). Looking back at our previous studies, we show how our research findings allow for a conceptualisation of the relationship between mathematics and engineering that differs from the standard application and modelling approaches. We also focus on the phenomenon of disconnectedness of the mathematical practices in mathematics service courses and engineering courses.Based on our analyses of engineering mathematical practices, we develop the idea of modifying exercises from a mathematics service course. The key feature of this approach is that it is an alternative to the usual approaches that propose the use of engineering application examples or the use of modelling tasks.

Keywords: Anthropological Theory of the Didactic, Mathematical Practices, Engineering Education

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Part I.

Introduction and Background

Prologue: Humpty Dumpty

In many publications (e.g. Bosch, Chevallard, García, & Monaghan, 2019; Chevallard, 2020; Chevallard, Farràs, Bosch, Florensa, Gascón, Nicolás, & Ruiz-Munzón, 2022) and whenever I have heard him speak about ATD, Yves Chevallard used the famous quote from Lewis Carroll to emphasise the importance of ATD-specific meanings of terms, sometimes in contrast to what they mean elsewhere:

> "I don't know what you mean by 'glory,'" Alice said.
> Humpty Dumpty smiled contemptuously. "Of course you don't– till I tell you. I meant 'there's a nice knock-down argument for you!'"
> "But 'glory' doesn't mean 'a nice knock-down argument,'" Alice objected.
> "When *I* use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean – neither more nor less."
> "The question is," said Alice, "whether you can make words mean so many different things."
> "The question is," said Humpty Dumpty, "which is to be master – that's all." (Carroll and Gardner, 1960, p. 268/9)

I was intrigued by this from the beginning, because on the one hand it is an odd quote to point out the relevance of theory-specific meanings of notions. Do we need Humpty Dumpty to remind us that concepts are theory-specific?[1] And doesn't the seemingly absurd exchange between Alice and Humpty Dumpty also harbour the danger of the misunderstanding that absurd attributions of meaning are to be given power here by means of an argument of authority? But on the other hand, it also seemed to resonate somehow with our work on mathematical practices in engineering education. Of course, the why and how of this resonance was not clear to me from the start. It has merely evolved from a hunch, in parallel with our research.

In the context of ATD, the *Humpty Dumpty principle* states that ATD words, even if they have an everyday meaning or a meaning in other theoretical frameworks, primarily mean what they mean according to ATD. Chevallard (2020) writes, addressing friends and colleagues: "In the case in point, we are collectively the 'master'. A praxeology [praxéologie, praxeología] is exactly what the ATD says it is – neither more nor less." (p. 16). I would like to take a more detailed look at the passage from Lewis Carroll with the help of Hancher (1981). On her way through the world behind the looking glass, Alice meets Humpty Dumpty, who is sitting on a wall and whom she already knows from the nursery rhyme[2]. They start to discuss birthdays and un-birthdays and after explaining to Alice why it is better to celebrate un-birthdays, Humpty Dumpty closes with "There's glory for you!". The conversation then proceeds as quoted. Hancher elaborates on the linguistic interpretation of this passage and how it relates to different models of verbal meaning. I will not go into details here but use his observation that

[1] This questioning is of course based on my background in German academia. The ATD community includes a variety of people from diverse backgrounds, not all from academia. So I may not be the main addressee here.

[2] Humpty Dumpty sat on a wall, / Humpty Dumpty had a great fall. / All the king's horses and all the king's men / Couldn't put Humpty together again.

> [t]he usual reading of the passage shades off in two different directions. At one extreme Humpty Dumpty stands guilty of a secret arbitrariness in his use of words. A monster of private language, he deserves the fall that is in store for him. But in the other direction it has been noticed that Humpty Dumpty resembles his creator, C. L. Dodgson, who as a professional logician approved of the practice that has since been named "stipulative definition." "Any writer may mean exactly what he pleases by a phrase so long as he explains it *beforehand*," Dodgson once wrote in a letter, apropos of his forthcoming book on logic.[3] (Hancher, 1981, p. 49, emphasis added by Hancher, footnote added by J.P.).

The practice of definition is what the *Humpty Dumpty principle* of the ATD refers to. The other direction of interpretation, which sees Humpty Dumpty as a "monster of private language", is what makes this quote seem so odd. But if we take a closer look, once again following Hancher, at Humpty Dumpty's conversation with Alice, we can see that his use of words is not secretly arbitrary. Humpty Dumpty is eager to explain the assigned meanings to Alice and he later helps her to understand the meanings of the words in the poem "Jabberwocky". The only odd thing about Humpty Dumpty in the quoted passage (and the discussion with Alice that precedes it) is that he defines the words after he has used them, rather than "beforehand", as Dodgson wrote.

The reason why the quoted discussion between Alice and Humpty Dumpty is interesting even beyond the *Humpty Dumpty principle* of the ATD lies in the following final quote from Hancher:

> But Humpty Dumpty is not really to blame; for everything, including temporal sequence, is reversed on his side of the looking-glass. He is no more accountable for using words in a novel way before he stipulatively defines them than the White Queen is for crying out before she pricks her finger with a pin, or than Alice is for distributing the plum-cake before she cuts it. To clear Humpty Dumpty of the charge that he practices a perverse private language, it is only necessary to take into account, as Alice and most readers do not, the peculiar order of the world in which he speaks. (p. 50)

This piece of literature illustrates the importance of taking into account the "world" in which people live and act, if we want to understand their practices. Who is the master of assigning meanings to words, acknowledgement to practices? And this is why Chevallard's quote resonated so well with our work on mathematical practices of engineers from the very beginning. When we started analysing passages from a text on signal theory, we found that the argumentations was unclear at certain steps. We had the impression that the unclear steps occurred at places that were significant both for an understanding what it means that an engineering practice is pragmatic and for a better understanding of the relationship between mathematics and engineering sciences. Additionally, in discussions with other researchers, we noticed a deficit-oriented view on engineering practices. Sometimes people judged engineering mathematics according to the norms, values, and rules of their own academic mathematics background. At the risk of exaggerating the illustrative example, let me use it again to present the stance of this work:

[3] In this book Dodgson writes in a chapter about The Existential Import of Propositions: "They [writers and editors of Logical text books, J.P.] speak of the Copula of a Proposition 'with bated breath,' almost as if it were a living, conscious Entity, capable of declaring for itself what it chose to mean, and that we, poor human creatures, had nothing to do but to ascertain *what* was its sovereign will and pleasure, and submit to it. In opposition to this view, I maintain that any writer of a book is fully authorised in attaching any meaning he likes to any word or phrase he intends to use. [...] I meekly accept his ruling, however injudicious I may think it." (Bartley, 1986, p. 323)

If we, as researchers in mathematics education who mainly have a background in academic mathematics, look at engineers' mathematical practices like Alice wonders about Humpty Dumpty's way of talking. And if we misjudge engineers like "most readers" misjudge Humpty Dumpty. Then we not only restrict ourself in our endeavour to understand engineers' practices, but we may also fail to design learning environments that aim to help engineering students learn mathematics. The general research interest underlying this thesis is to understand mathematical practices of electrical engineers at university, while avoiding a deficit-oriented perspective from the point of view of academic mathematics. I will present research foci and research questions that concretise this general research interest in chapter 1. Since it is important to understand the "peculiar world" in which electrical engineers act, we have included in our studies, at least the part of the technical context that is relevant to our research. In chapter 2, I will both summarise the basic context relevant ideas and present the data underlying our studies. The Anthropological Theory of the Didactic (ATD) has already made an appearance in this introduction, although only in the form of the *Humpty Dumpty principle*. I have also touched on other principles of ATD on my way to the general research interest and stance of this work without naming them. I will do so in chapter 3, where I will also present the aspects of ATD that are relevant to the studies. A literature review and the embedding of the work in the international research field is done in chapter 4, while the discussion of the research questions is done in chapter 5. The published studies on mathematical practices in electrical engineering education are presented in Part II of this thesis:

Study I: Hochmuth, R., & Peters, J. (2020). About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *22*(4), 454–471. https://doi.org/10.23925/1983-3156.2020v22i4p454-471 (page 120)

Study II: Peters, J., & Hochmuth, R. (2021). Praxeologische Analysen mathematischer Praktiken in der Signaltheorie. In R. Biehler, A. Eichler, R. Hochmuth, S. Rach, & N. Schaper (Eds.), *Lehrinnovationen in der Hochschulmathematik: praxisrelevant – didaktisch fundiert – forschungsbasiert* (pp. 109–139). Springer Spektrum. https://doi.org/10.1007/978-3-662-62854-6_6 (page 59)

Study III: Hochmuth, R., & Peters, J. (2021). On the Analysis of Mathematical Practices in Signal Theory Courses. *International Journal of Research in Undergraduate Mathematics Education*, *7*(2), 235–260. https://doi.org/10.1007/s40753-021-00138-9 (page 91)

Study IV: Hochmuth, R., & Peters, J. (2022). About two epistemological related aspects in mathematical practices of empirical sciences. In Y. Chevallard, B. B. Farràs, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & N. Ruiz-Munzón (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 327–342). Birkhäuser Basel. https://doi.org/10.1007/978-3-030-76791-4_26 (page 139)

Study V: Peters, J., & Hochmuth, R. (2022). Sometimes mathematics is different in electrical engineering. *Hiroshima Journal of Mathematics Education*, (15), 115–127. https://doi.org/10.24529/hjme.1510 (page 157)

Study VI: Peters, J. (2022). Modifying Exercises in Mathematics Service Courses for Engineers Based on Subject-Specific Analyses of Engineering Mathematical Practices. In R. Biehler, G. Gueduet, M. Liebendörfer, C. Rasmussen, & C. Winsløw (Eds.), *Practice-Oriented Research in Tertiary Mathematics Education: New Directions.* (pp. 581–601). Springer. https://doi.org/10.1007/978-3-031-14175-1_28 (page 171)

1. Overview of the studies and research questions

The six studies in this dissertation are based on my collaboration with Reinhard Hochmuth during my time as a research associate in his research group at Leibniz Universität Hannover. Our work builds on the completed KoM@ING project[1], which also provided the data for the studies. Stefan Schreiber was also involved in the KoM@Ing project. Study I in particular builds on the results of his work with Reinhard Hochmuth (e.g. Hochmuth & Schreiber, 2015, 2016). The studies in this thesis are numbered chronologically, in the order in which we worked on the topics: Study I has a stronger focus on ATD than the Kom@ING studies. Here we already talk about discourses and how they "mix" but the notion of discourse as well as the reconstructed mathematical practices are still rather superficial (from today's perspective, of course). Another focus in Study I is to refer to historical-philosophical studies in order to clarify certain vague passages in a textbook. We also embedded this reference in the theoretical framework of ATD. In Studies II and III we went from textbook analysis to the analyses of lecturer sample solutions to exercises and students' solutions. Here we deepened the subject-specific reconstruction of mathematical practices. In particular, we have advanced two important methodological developments: institutional discourses and methodological steps to relate the results of institutional analyses to individual student work. In Study IV we took up and expanded the historical-philosophical perspective from Study I. We also took the developments from Studies II and III into account. Studies V and VI elaborate in two directions what we learned from previous work.

In addition to a chronological order, a thematic order is also suitable. This is particularly useful for structuring the main research foci and associated research questions in this chapter. In presenting the foci and research questions, I also refer to the studies in which these questions were elaborated. The research foci now arrange the studies differently, this time thematically. In this order, with the research foci as structuring elements, the studies are also presented in Part II.

For this PhD project, we focused on material from the module Signal Transmission of the electrical engineering study programme at the University of Kassel (see chapter 2 for details). The general research interest that was also already stated in the Prologue can be formulated as the following research question:

RQ: How can we understand the mathematical practices in electrical engineering courses at University while avoiding a deficit-oriented perspective from the point of view of academic mathematics?

This general research question has developed into three research foci and four research questions. The first research focus is concerned with the specific mathematical content: *The subject-specific reconstruction of mathematical practices*. A basic observation that was fundamental to our work was that mathematical practices of higher semester electrical engineering courses could be roughly classified into practices from mathematics service courses, practices from basic electrical engineering courses, and new mathematical

[1] See https://www.kompetenzen-im-hochschulsektor.de/koming/, Teilprojekt A

practices (e.g. concerning the Dirac impulse). This observation itself is evident when looking at study guidelines, module descriptions or textbooks. However, it was not clear how those practices from the different course contexts interacted, how they differed and what possible connections looked like. This leads to the first research question:

RQ1: How can mathematical practices in the module Signal Transmission be reconstructed, and what connections, differences and interactions can be identified with regard to mathematical practices from mathematics service courses, from basic electrical engineering courses, and newly introduced mathematical practices?

This question is addressed in Study II on page 59 of this thesis, were we analyse two exercises and corresponding lecturer sample solutions from the course Signals and Systems. On the basis of methodological developments, we were able to reconstruct aspects of two different mathematical discourses as well as their interactions within the analysed practices. We also propose a graphical representation of our analyses results. As our results refer to institutional mathematical practices, the question arises:

RQ2: How can analysis results of institutional mathematical practices be related to individual students' actions?

This question is addressed in Study III on page 91 of this thesis. Here we explicitly draw attention to the difference between institutional practices and individual actions. With a focus on exercises we have also developed methodological steps to apply institutional analysis results to individual students' solutions.

The second research focus is: *The epistemological and philosophical relationship between mathematics and electrical engineering*. We explore this in Study I on page 120 and Study IV on page 139 of this thesis:

RQ3: How can epistemological and philosophical studies contribute to analyses of mathematical practices in electrical engineering courses and how can this lead to an alternative conceptualisation of the relationship between mathematics and engineering?

In Study I we aim to gain a better understanding of the mathematical practices involved in the introduction of the Dirac delta impulse in a textbook on signals and systems. Therefore this study also addresses the first research focus[2]. We realised that certain mathematical steps could be better understood from an engineering point of view, if historic-philosophical studies on the relationship between physics and mathematics were also taken into account. However, we are not concerned here with the philosophical positions of students or teachers. We are referring to philosophical studies that focus on the societal aspects that play a role in the historical concrete formation of practices. The case of the Dirac impulse is very well suited to illustrate the fruitfulness of such studies for subject-specific analyses. We take this up in Study IV and deepen our considerations. In this study, we also have the results of Study II and Study III at our disposal and can thus relate the considerations of societal aspects of the historical formation of practices to the subject-specific analyses. In particular, we show how our considerations apply at a more general level to all empirical sciences. For example, we also consider psychology.

The last two studies constitute the third research focus: *Revisiting the relationship between mathematics and electrical engineering*. Here we explore the question:

[2] To a certain extent, all studies contain aspects of all three research foci. Subject-specific analyses, epistemological and philosophical considerations about the relationship between electrical engineering and mathematics, and a conceptualisation of this relationship with a view to teaching design are not independent of each other. See chapter 5 for more details on this.

RQ4: How can institutional analyses of mathematical practices and an alternative conceptualisation of the relationship between mathematics and engineering contribute to teaching design and lecturer support?

In Study V on page 157 of this thesis we look back at our previous studies and show how our research results allow for a conceptualisation of the relationship between mathematics and engineering which differs from the standard application- and modelling approaches. Overall, we developed an understanding that sometimes mathematics is different *in*[3] electrical engineering. Correspondingly, the question of how mathematics can become different *for* electrical engineering comes to mind, where mathematics *for* engineering refers to mathematics service courses. Considering the *in* and *for* and their relationship also relates to an understanding of the relationship between mathematics and engineering. In Study VI on page 171 of this thesis we focus on the phenomenon of disconnectedness of the mathematical practices in mathematics service courses and engineering courses. We develop the idea of modifying exercises from a mathematics service course on the basis of our analyses of engineering mathematical practices. This small scale approach on teaching design for mathematics service courses illustrates how considerations of the in/for relationship and correspondingly the relationship between mathematics and engineering can be productive for teaching development. This approach is characterised by the fact that it presents an alternative to the usual approaches that propose to use application examples from engineering or deal with modelling tasks.

[3] The notions of mathematics *in* and *for* engineering originate from the title of the *Special Issue: Mathematics in/for Engineering Education* of the International Journal of Research in Undergraduate Mathematics Education (Pepin, Bieler, & Gueudet, 2021).

2. Institutional and subject-specific context

This chapter describes the institutional- and subject-specific context of the studies I–VI, as well as the data used for the analyses. Our analyses focus on mathematical practices in the module Signal Transmission in the 4th semester of the Bachelor study programme of electrical engineering at the University of Kassel. This module is called Signalübertragung in Figure 2.1.

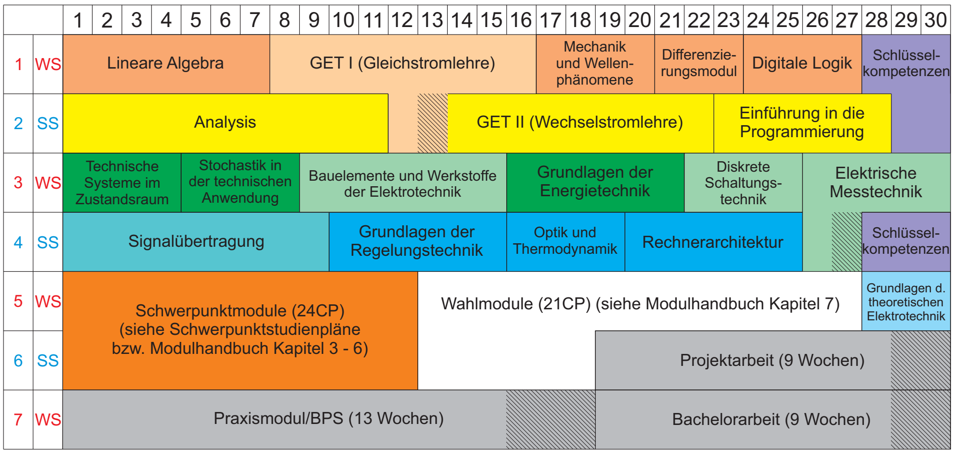

Figure 2.1.: General study plan for the Bachelor of Electrical Engineering at the University of Kassel[1]

Our analyses concern

- the introduction of the Dirac delta impulse by Fettweis (1996) in Study I. This is taken up and related to the introduction of the Dirac delta impulse by Frey and Bossert (2009) in Study IV;
- the lecturer sample solution to an exercise about the envelop demodulator in Study II;
- the lecturer sample solution to an exercise about amplitude modulation in Study II, Study III. Study V and Study VI draw on these results;
- students' solutions to an exercise on amplitude modulation in Study III.

Our analyses have shown that the differentiation of two mathematical discourses, one related to the electrical engineering courses and one related to the mathematics service courses, is very fruitful[2]. The

[1]The study plan is the one that is valid in 2011–2013. Actual study plans can be found at https://www.uni-kassel.de/eecs/studium/bachelor/elektrotechnik. By February 2023, there have been no relevant differences.
[2]See chapter 3 for our ATD related understanding of the notion discourse and chapter 5 for a more detailed discussion of the methodological aspects with respect to our results.

subject-specific content concerning the notions of signal, the Dirac delta impulse and amplitude modulation and -demodulation is important as background for our analyses and for the characterisation of the mathematical discourse related to electrical engineering. In particular, our work with the amplitude modulation exercise has shown that complex numbers is a very suitable topic to illustrate the two different mathematical discourses more concretely. A summary of the most important aspects of the introduction of the Dirac delta impulse is given in section 2.1. A summary of the most important aspects of amplitude modulation and complex numbers in electrical engineering and in the mathematics service course as well as references to the studies in which those aspects are introduced more thoroughly can be found in section 2.2.

The distribution of courses within the study programme can be seen in Figure 2.1. Here the modules per semester are shown together with the appointed Credit Points. The mathematics service courses are Analysis in the 1st semester and Lineare Algebra in the 2nd semester. At the University of Kassel complex numbers are treated in Lineare Algebra (see Strampp, 2012, Chapter 3). The relevant basic electrical engineering module is GET II (Wechselstromlehre) in the 2nd semester[3]. Thus, the complex numbers are first dealt with in the mathematics service course before they are introduced in GET II in the following semester. The data we used for our analysis was originally collected within the Kom@ING project. Of this data, we selected that which was relevant for our research questions. The data is summarised in Table 2.1 and consists of lecture notes: LN, exercises (with lecturer sample solutions): E(S); literature/textbooks: Lit; a description of the boundary conditions by the lecturer: BC; module descriptions: MD; student's lecture notes: Stud LN; and students' exercise solutions: Stud E.

Module	Semester	LN	E(S)	Lit	BC	MD	Stud LN	Stud E
Lineare Algebra	Winter 2011/12	-	x	x[a]	x	x	x	x
Analysis	Summer 2012	-	x	x[b]	x	x	x	x
GET II	Summer 2012	x	x(S)	x[c]	x	x	-	-
Signalübertragung	Summer 2013	x	x(S)	x[d]	x	x	x	(x)

[a] Strampp (2012) [b] Strampp (2015) [c] Albach (2011) [d] Fettweis (1996) and Frey and Bossert (2009)

Table 2.1.: Data for the analyses in studies I–VI.

For the modules Analysis and Lineare Algebra, all the student's lecture notes and the student's exercise solutions are by the same person, as are the student's lecture notes for Signal Transmission. The solutions to the exercises in the module Signal Transmission come from a group of students[4]. To indicate the difference in the origin of the students' related data, the latter is bracketed. There are no lecture notes for the mathematics service course modules Analysis and Lineare Algebra because the lecturer used the textbooks of which he is also the author of. The module Signal Transmission consists of the two lectures Signals and Systems and Digital Communication. Both lectures were offered together as a unit in the summer semester 2013. Lecture notes and exercise sessions covered both lectures without differentiating between them. A total of five exercise sheets were integrated into the lectures, which were handed out and addressed at suitable times in the lectures. Analyses of two exercises, about the envelope demodulator and

[3] The University of Kassel also allows for a start in the summer semester. In this case the mathematics service course starts with Analysis. Lineare Algebra is in the second semester and Signalübertragung and GET II in the third semester.

[4] The number varies depending on the task. We explicitly consider those solutions in Study III on page 91.

about amplitude modulation, with lecturer sample solutions are central for our studies. Both exercises are from the second exercise sheet and represent two subtasks of problem 4. The first subtask is about the envelope demodulator. The second subtask deals with amplitude modulation and is divided into three items, of which the third is the main focus of our analysis. The relevant part of the exercise sheet with lecturer sample solutions is shown in the appendix of Study II on pages 83–87 of this thesis. An English version of the exercise and lecturer sample solution on amplitude modulation can be found in Study VI on pages 187–188.

2.1. The introduction of the Dirac delta impulse in Signal Transmission

An important topic in the Signal Transmission module is the concept of signal. Aspects of the notion of signal gathered in our studies are on the one hand relevant to our epistemological considerations, see Study I and Study IV. On the other hand, they also contribute to the characterisation of the ET-discourse which is central to our subject-specific praxeological analyses in Study II and Study III. Since the term is important for both epistemological studies and subject-specific analyses, we can also use it to discuss the relationships between them. We do so in Study IV.

Concerning our analysis of the introduction of the Dirac delta impulse by Fettweis (1996) in Study I, the distinction between two types of signals, real signals and idealised signals is of central importance. This distinction is based on Fettweis' preference for a physical understanding of the concepts of signal theory over a rigorous mathematical approach and his explicit reflection of this. He presents the relation between mathematics and physics as a dilemma between mathematical precision and the understanding of physical reasoning. He states that

> with increasing refinement of the underlying mathematical relationships, understanding the physical justification of the chosen approach becomes increasingly difficult. So what is gained in mathematical rigour on the one hand is lost again on the other when it comes to insight into the actual applicability to physical actualities. (p. iii, translated from German by J.P.)

By stating a dilemma, Fettweis upholds the importance of mathematical rigour but prefers physical understanding. This leads Fettweis to explicitly address the deviation from mathematical rigour by discussing a different kind of argument to justify mathematical practices that can in principle be fully justified with mathematics but the necessary apparatus for this would reduce the focus on the physical understanding. Before I go more into detail, I want to note that this is not specific for Fettweis (1996). Similar aspects are also found in the work of Frey and Bossert (2009), who are, considering Fettweis' dilemma, more on the side of mathematical rigour, and, in the case of the Dirac delta impulse, in the work of Dirac (1981) himself.

Fettweis' explicit handling of the situation makes his textbook interesting for our analyses (see Fettweis, 1996, p. 4ff): He first introduces real- and idealised signals. Real signals occur in communication transmission and are irregular and highly diverse. They are of finite duration, continuous and sufficiently differentiable. Due to their irregularity and high diversity, real signals are not suitable for numerical- and analytical calculations and cannot be used as measurement signals. Therefore, idealised signals are introduced that necessarily violate some of the properties of real signals. Fettweis gives the unit step function $u(t)$ as an example and states:

> A real signal, which we want to approximate idealised by a unit step function, would thus correspond to a function $f_n(t)$, which increases very rapidly but steadily from 0 to 1 in the vicinity of $t = 0$. If difficulties arise when using the unit step function, we would therefore have to replace $u(t)$ by $f_n(t)$ and, after carrying out the intended analysis, let $f_n(t)$ approximate the ideal course more and more. (Fettweis, 1996, p. 9, translated from German by J.P.)

Fettweis' deviation from mathematical rigour plays explicitly a role in the introduction of the Dirac delta impulse, an important tool for the description and characterisation of signals and systems. Briefly, in the introduction of the Dirac pulse, specific sequences of functions are considered. In the course of the process, limit and integration are then interchanged at one step, although, from the perspective of the mathematics service course, the mathematical prerequisites for this are not fulfilled (p. 14). Frey and Bossert (2009) make a similar step, except that they interchange limit and differentiation (p. 109). This situation, viewed from different perspectives, allows different conclusions to be drawn in each case. From the perspective of the mathematics service course this step is wrong. A mathematical reference model[5], which could have been built on the basis of an academic mathematics perspective, would probably find a deficit here. Both authors refer also to distribution theory for a mathematically precise introduction of the Dirac delta impulse. If those steps are read from the perspective of distribution theory, they are justified. The justifications and explanations are just not given, a well accepted method in engineering and also physics[6]. This is a typical technique for holding up mathematical rigour without explicitly performing it. But as Bueno (2005) states: "The suggestion that the delta function is ultimately a distribution misses a significant point about Dirac's strategy; namely, the pragmatic role that the function plays in describing—in an elegant and simple way—the relevant relations." (p. 471). This pragmatic role of the Delta impulse, the third perspective, is also at the core of Fettweis' introduction and very important for electrical engineering practices in Signal Transmission. Our general interest in avoiding a deficit oriented perspectives and explicitly looking out for engineering specific justifications allows us to bring this pragmatic role and its importance into the foreground. In our praxeological analysis in Study I, we also refer to epistemological considerations concerning the relationship of mathematics and physics (Wahsner & Borzeszkowski, 1992) to flesh this out more. In Study IV, besides others, we bring this together with the two different mathematical discourses that are important in our subject-specific reconstructions.

2.2. Amplitude modulation and complex numbers

In addition to the description and characterisation of different transmission channels (systems), the question of the realisation of signal transmission via a specific channel then also plays a major role. An important criterion here is the possible multiple utilisation of the transmission channel: Several signals are to be transmitted simultaneously without crosstalk occurring between signals at the receiver. A

[5] See chapter 3 and for a critical discussion of the epistemological reference model in relation to our approach concerning institutional discourses, see chapter 5.

[6] This is also mentioned by Fettweis: "Eine solche Vorgehensweise ist in der Tat sowohl aus physikalischer als auch aus mathematischer Sicht geboten und entspricht der auch in anderen physikalischen Bereichen üblichen Vorgehensweise." (p. 6)

simple procedure that can be carried out with little technical effort is analogue amplitude modulation and demodulation. Therefore, amplitude modulation is an important topic in the Signal Transmission module as it allows to discuss general aspects of modulation and demodulation, which are also important in modern, much more complex devices, in a more technically accessible context. As the signal-theoretical background is vital for our analyses, it is presented in most of our studies. The most detailed version concerning amplitude modulation can be found in Study VI. The envelope demodulator is covered in Study II.

The basic principle of amplitude modulation is that the amplitude of a high-frequency carrier signal $\cos(\omega t)$ is varied in relation to that of the low-frequency message signal $\cos(\Omega t)$. The AM signal can then be represented as $x(t) = A[1 + m\cos(\Omega t)]\cos(\omega t)$. The message signal usually is something much more complex, e.g. music from a radio broadcast. The frequency of the carrier signal is the one people tune to on the radio when they select a station[7]. Tuning to a radio station means configuring a filter that only lets through carrier signals with the chosen frequency. Every message that is modulated onto the carrier signal then can be received by reconstructing it via a demodulation process.

An important aspect of our analyses lies within the graphical representation of amplitude modulation. This is also the topic of the exercise on amplitude modulation whose sample solution we have analysed (exercise and sample solution are given in the appendix in Study VI on pages 187–188 of this thesis). There are two important graphical representations of signals in electrical engineering: the waveform representation, which is the shape of the graph when the signal is interpreted as a function, and the phasor representation, which is the representation of the signal when it is interpreted as a complex number, as an arrow in the Argand diagram. Both types are shown in Study VI on page 177 of this thesis in Figure 28.1, which shows the waveform representation of amplitude modulation, and in Figure 28.2, which shows the relationship between the phasor- and waveform representation of a sinusoidal signal. In circuits in which only signals of one frequency occur, the periodicity of the signal, or the rotation of the phasor, can be neglected. In amplitude modulation, where signals with two different frequencies occur, this is no longer possible. The AM applet by Liu and Kernetzky (2018) provides an excellent dynamic visualisation of amplitude modulation both, in phasor- and waveform representation: https://www.lntwww.de/lnt_applets/physAnSignal_en/index.html.

The exercise on amplitude modulation asks for developing the phasor representation from the algebraic expression of an amplitude modulated signal. The property of complex numbers to describe periodic signals as a rotating phasor or as a combination of several rotating phasors is at the core of the characterisation of the mathematical discourse related to electrical engineering. In Study II we analysed mathematical practices in lecturer sample solutions to exercises on the envelope demodulator and amplitude modulation. On the basis of these analyses, especially the one concerning the amplitude modulation exercise, we have reconstructed the two mathematical discourses and, in doing so, also sharpened discourse as a methodological notion of ATD (see chapter 5 for more details). In Study III we look at students' solutions to the amplitude modulation task and analyse their practices using our institutional analysis as a reference. In Study IV we relate our findings on the two mathematical discourses to epistemological and societal considerations on the relationship between mathematics and empirical sciences like physics or electrical engineering. The question of how the relationship between mathematics and electrical engineering can be understood as something other than application or modelling is explored in Study V, and in Study VI I show how our analyses can lead to new ideas for teaching design.

[7] Usually people use FM broadcasting because frequency modulation (FM) has some advantages above amplitude modulation.

3. The Anthropological Theory of the Didactic[1]

In the prologue, I have already illustrated with a famous passage about Humpty Dumpty and Alice how important it is to take their peculiar world into account when trying to understand people's actions. In particular, our interest in the mathematical practices in an engineering study programme has made it necessary to find a theoretical framework in which this can be considered. The Anthropological Theory of the Didactic (ATD), which has already made its appearance in the Prologue, is, in my view, particularly well suited for this purpose.

The Anthropological Theory of the Didactic stands in a French research tradition[2] and has its origins in the Theory of Didactic Situations, which was mainly developed in the 1970s and 1980s (Brousseau, 2002), and in the Theory of Didactic Transpositions (Chevallard, 1985, 1989). Comprehensive introductions in ATD can be found, for example, in Bosch and Gascón (2014) and Chevallard (1992, 2019).

ATD allows to look at the world of engineers in such a way that the mathematical practices of engineers can be modelled in a way that takes into account their subject-specific peculiarities. A fundamental principle of the ATD that enables this is the *emancipatory principle* that consists in

> avoiding taking for granted the elements of the social world we are studying, a world we know very well because of our experience as students and citizens and sometimes also as teachers, educators or parents. [...] The strategy proposed by the ATD consists in setting forth a wide set of basic notions used to model—or conceptually reconstruct—the didactic world in a "fresh" perspective, to avoid being contaminated by the visions of the persons and institutions that are part of this world. (Bosch, Chevallard, García, and Monaghan, 2019, p. xii)

The "basic notions" from the ATD that are central to the studies are the institutional standpoint of ATD, according to which human practices, such as doing mathematics, are always located in *institutions* and institutional conditions determine which actions and justifications are considered adequate; the modelling of knowledge or practices as *praxeologies* that exist depending on the respective given institutional conditions; and the model of *(didactic) transposition*, which allows us to examine and describe the development, change and dissemination of knowledge across different institutions. In the following I will explain these terms in more detail. In doing so, I will refer back to chapter 2 at appropriate points to show how the theoretical models are concretely fleshed out in our context. I will also highlight important aspects and refer to the respective studies in which they are relevant.

[1] The presentation of the ATD in this chapter is based on elaborations presented in varying degrees of detail in our studies, mainly in Study II and Study III and in my unpublished Master's thesis (Peters, 2021).

[2] An impression of the French research tradition can be gained from Laborde (2016), who takes a look at German subject matter didactics (Stoffdidaktik) from a French perspective.

3.1. The institutional standpoint of ATD

ATD is a research programme for the study of human practices from an institutional perspective. The notion of institution is addressed on different levels. Chevallard (1992) introduces it by way of example:

> It is now time to say a few words about this central character, *institutions*. Once again, an institution can be just about anything. In reality, because of the current meaning of the word, some of you may be surprised to see on which objects I may be led to stick this label. A school is an institution, as is a class; however, there is also the institution of "practical work", the institution of "lectures", the institution of "family". Daily life is an institution (in a given social setting), and so is the state of being in love (in a given culture), etc. (p. 144)

Later on in his 1992 publication Chevallard refers to the social anthropologist Douglas (1986) who elaborates the idea that all knowledge depends on (social) institutions and, conversely, all institutions are based on shared knowledge. She focuses on the question of how systems of knowledge, or "the collective foundation of knowledge" (p. 45), are created. The central idea in her work is that the emergence and entrenchment of knowledge is not only an individual cognitive process, but a genuinely social process. Conversely, the process of institutionalisation is not only a social-political but also an intellectual process (p. 45). Knowledge and institutions are not independent aspects that merely relate to each other, but cannot be understood separately. Mary Douglas elaborates on the process by which institutions come into being: "Minimally, an institution is only a convention." (p. 45) However, certain conditions are required for communities to form stable institutions on the basis of conventions. In particular, the legitimisation of conventions plays a major role.

It is important to emphasise that already from Chevallard's exemplary introduction of the term, but also from Douglas' elaborations, that by institution is not merely understood the bureaucratic or social organisations that are called institutions in everyday life (such as school or university). Also "surprising" things like family, or being in love in Western society are understood as institutions. Elsewhere, Chevallard explicitly includes (social) entities and structures that fulfil a certain formative function (Chevallard, 2019, p. 72). Institution understood as a stable social organisation is also taken up by Castela (2015), whose work was an important basis for our studies:

> I define an institution I as a stable social organisation that offers a framework in which some different groups of people carry out different groups of activities. These activities are subjected to a set of constraints, - rules, norms, rituals - which specifies the institutional expectations towards the individuals intending to act within the institution I. An individual has to satisfy these expectations, at least, to a certain extent depending on the institution. Hence, using the ATD vocabulary, the individual (s/he) is subjected to the institution's expectations and becomes an institutional subject (from Latin *sub-jectus*: literally thrown under). [...] Institutions tend to constrain their subjects but conversely they provide the resources (material and cultural) necessary for activities to take place. Epistemologically, the existence of institutions is an absolute precondition for the development of human culture. They foster collective processes for facing and solving human problems; and they favour the dissemination of inventions/innovations, even when they do not create specific schools for that. (S. 7)

Legitimation, Douglas (1986) writes about legitimisation of conventions, is noted by Castela (2015) in the form of "rules norms and rituals" and the subjection of individuals to the institution's expectations. Legitimisation with a view to institutional norms is also taken up by Castela (2020). However, this subjection is understood to be productive and dialectical[3].

In addition to an exemplary introduction and a reference to sociological notions of institution, institution is also explained through its relation to persons, institutional positions and objects of knowledge. Chevallard (1992) writes, if a person is aware of a certain object of knowledge or if an object of knowledge exists within an institution, it is said that the object O exists for the person X or the institution I. The *personal relation of X to O* is noted by $R(X, O)$[4] and the *institutional relation of I to O* is noted by $R_I(O)$ (cf. p. 142). Within an institution, there are different positions that people can hold. For example, in a teaching institution there is usually the position of student and the position of teacher. The institutional relations to one object of knowledge differ for different institutional positions. To indicate this one writes $R_I(p, O)$, where p stands for the institutional position. Chevallard then writes

> A person X becomes a good subject of I relative to the institutional object O when his personal relation $R(X, O)$ is judged to be consistent with the institutional relation $R_I(O)$. This person may also prove to be a bad subject, [...] and may, in the end, be expelled from I. Here is where a development relating to intra-institutional evaluation comes into play, relating to the mechanisms according to which I is led to pronounce, through some of its agents, a verdict of conformity (or non-conformity) of $R(X, O)$ to $R_I(O)$. [...] In particular, the institutional relation [...] is nobody's personal relation, [...]: conformity is not identity. (p. 146/7)

The aspect of legitimation is also central when looking at the relationship between persons, institutions, and objects of knowledge. It emerges in the form of the "good- and bad subject", the "intra-institutional evaluation" and the "non-/conformity" of the individual with the institutional relation. The aspect, that personal relations are not identical to institutional relations is central to our work in Study III and gives rise to the question of how the work of an individual can be analysed based on analyses of institutional relations (see RQ2).

Based on what I have written here so far about the ATD, we can now look back at the institutional and subject specific context in chapter 2. The term institutional in the chapter title refers merely to the organisation of the study programme of electrical engineers and the distribution of the courses that are relevant for our studies. But now we can start to use ATD as a lens. The module Signal Transmission is the central institution for our studies. As a module with accompanying lecture and exercise sessions it is part of the teaching institution University. It was set as a concrete topic, i.e. it is part of the thematic classification[5] of electrical engineering and is considered relevant for the teaching of future engineers.

[3] According to Bosch, Chevallard, García, and Monaghan (2019) in ATD dialectic is understood praxeologically (see section 3.2): "Any *praxeology* that enables one to overcome two opposed types of *constraints* by turning them into a new kind of *conditions* that supersede them. In this context, one , therefore, speaks of supersession (French *dépassement*, German *Aufhebung*, Spanish superación)" (S. xxii).

[4] This formal notation of the relationship of institutions, persons and practices may seem to be unusual and abstract, it highlights the relations and abstracts from concrete qualities. This has the advantage that complex relationships can be written down in a concise way. An interesting example where this is productively used is the work by Winsløw (2017), who introduces a compact ATD version of Klein's "double discontinuity".

[5] With regard to the classification and legitimation of conventions and their significance for processes of institutionalisation, I would like to refer again to Douglas (1986).

We use the abbreviation SST[6] to index this institution. From the organisation of the study programme the importance of mathematics service courses for the programme is apparent. In our analyses, we have also noted the presence of accompanying institutional influences from the outset. In Study I we identified "higher mathematical ideas" in the justifications of practices in the introduction of the Dirac delta impulse in the textbook by Fettweis (1996), see also Figure 7.1. So we took practices associated to the mathematics service courses into account, indicating those institutional influences with HM. In our analyses of the exercise and sample solution on amplitude modulation complex numbers became an important mathematical topic. A comparison of the way complex numbers are introduced in the mathematics course Lineare Algebra (considered as an institution, denoted by HM) and how they are used in GET II (considered as an institution, denoted by ET) show how two different institutions can have different relations to an object of knowledge (see also section 28.2.2.1 in Study VI).

Coming back to the introduction of ATD, we can now have a more detailed look on the relation between persons and institutions, in particular on its role in ATD studies. Bosch and Gascón (2014) write that

> [i]n order to answer the question of why people do what they do, what makes it possible for them to do what they do, etc., ATD postulates that what explains the behavior of people are not only their personal idiosyncrasies but also the existence (or availability) of institutional constructions that each person adapts, adopts, and develops either individually or collectively. An ATD analysis therefore starts by approaching *institutional praxeologies* and then referring individual behavior to them, talking in terms of the "praxeological equipment" of a given person. Observable behavior obviously consists of a mixture of personal and institutional ingredients. This dialectic between the personal and the institutional makes it possible to explain both the regularities of our behavior and its personal "footprint". People evolve as they enter different institutions and, at the same time, these individual participations enable institutions to appear, run, and change. (S. 69)

Institutional praxeologies, see section 3.2, thus function in ATD studies as reference points for the analysis of individual actions (or their products). The key point here is that individual actions are not to be understood exclusively as "personal idiosyncrasies" but that aspects of the institution always play a constitutive role in them. What Bosch and Gascón calls "adapts, adopts, and develops" is described by Chevallard as establishing conformity between the individual and the institutional relation to the object (of knowledge or action). Here, and this is an important point, conformity is not to be understood as identity. This point is central to our Study III.

The institutional point of view is important in the ATD research programme in two respects. On the one hand, any form of knowledge, understood as practices, is located in institutions and is subject to institutional conditions. It follows that mathematics education research must take these conditions into account. The Humpty Dumpty principle resonated so well with our work because, in order to make

[6] The indices we use to refer to different institutions come from German standard abbreviations: SST stands for Signal- und Systemtheorie, i.e. signals and system theory; HM stands for Höhere Mathematik, i.e. higher mathematics, a collecting term for mathematics service courses; and ET stands for Elektrotechnik, i.e. electrical engineering, or GET in the study plan in Figure 2.1 which stands for fundamentals of electrical engineering. A note on the translation of "Höhere Mathematik" as higher mathematics: in German, "Höhere Mathematik" denotes mathematics courses that cover mathematics beyond school mathematics. Higher mathematics is a verbal translation and does not imply that this kind of mathematics is somehow better or has a higher status or standpoint, e.g. in relation to mathematics of engineering courses.

sense, Humpty Dumpty has to be taken seriously. And that means taking into account the conditions of his world. The practices of engineering students cannot be understood sufficiently well if they are understood exclusively as individual actions. In this context, it is important to emphasise once again that institutional conditions are not only *external* conditions, but are constitutive within practices. On the other hand, the institutional point of view also concerns researchers themselves. They too are subject to institutional conditions and the "researchers' institutional subjections affect the way of conceiving and understanding reality" (Bosch, 2015, p. 53). The analysis of institutional conditions is thus also an essential component of mathematics education research[7].

> Mathematics is a body of knowledge … produced through specific research activities; both, knowledge and activities, are acknowledged as mathematics by international mathematics institutions. Mathematics is at the same time a body of knowledge, a field of activities and an institution. This looks very much like a closed world. When someone of this world, that is, a mathematician, begins to investigate on mathematics education, especially but not only in vocational education, he needs tools to distance himself with the "*alma mater*". (p. 18)

The most widely used tool in ATD for analysing institutional conditions, which also includes reflection on one's own standpoint, is the reference epistemological model (REM). Lucas, Fonseca, Gascón, and Schneider (2019) consider the REM as a phenomenotechnique (in the sense of Bachelard[8]), with which didactic phenomena can be produced and thus studied in the research process, depending on the concrete choice of reference model. They write:

> REMs facilitate a detachment from the DEMs of the considered educational institutions and help give visibility to phenomena that could remain unnoticed and unexplained. In particular, in the light of a REM, it can become clear that the official raison d'être assigned by the DEM to a certain domain of school mathematics presents limitations, contradictions or incompleteness. (p. 78)

Here, in the difference between REM and DEM (dominant epistemological model), didactic phenomena become visible and thus investigable.

An alternative to this is the explicit reconstruction of institutional conditions, such as done by Castela (2015), Castela and Romo-Vázquez (2011), and Romo-Vázquez (2009). Their work is a vantage point for our own development. Furthermore, I discuss the epistemological reference model in relation to our own approaches in chapter 5.

3.2. The modelling of practices as praxeologies

In ATD, knowledge is understood as human practices that include not only practical aspects of "know-how" but also knowledge in the sense of "know-why". This is subsumed under term praxeology, a double word composed of the two components praxis and logos. The *anthropological principle* states that in the ATD,

[7] Bosch (2015) calls this, following Elias (1956), "detachment principle" (p. 52).

[8] "Phenomenotechnics understands the phenomena to be observed as co-generated by tools that are themselves a materialisation of the theory under whose premise and for whose purpose they were constructed." (Diaz-Bone, 2008, p. 51, translated from German by J.P.)

> didactic phenomena are considered as inherent to any group of human beings, as part of humanity. Being human beings means co-creating and disseminating knowledge, and also failing to do so. [...] [A]ny human activity can be described as a praxeology or as an amalgamation of praxeologies of different "sizes". (Bosch, Chevallard, García, and Monaghan, 2019, p. xiii).

A praxeology is a basic epistemological model of knowledge and associated practices in the form of the two inseparable and interrelated blocks of praxis P, denoting the practical parts of practices and logos L, denoting the part that explains, makes intelligible, or justifies praxis "in whatever style of 'reasoning' such as an explanation or justification may be cast. *Praxis* thus entails *logos* which in turn backs up *praxis*" (Chevallard, 2006, p. 23). This model was used in Study I. In our other studies we needed the more differentiated 4T-model: The praxis block P consists of types of tasks T and relevant solution techniques τ. The logos block L consists of a two-level reasoning discourse. At the first level, technology θ describes, justifies, explains, etc. the technique. At the second level, theory θ organises, supports and explains technology. Overall, a praxeology can be represented as a 4T-model: $[T, \tau, \theta, \Theta]$.

The set of tasks which can be solved with a certain technique within an institution is called the *scope of the technique*. An important aspect of technology is the *raison d'être* of a body of knowledge. This is the reason why it exists in an institution, what it is good for, and why it is studied. Practices can be modelled on different levels of granularity. On the one hand, point praxeologies, containing a single type of task, can be integrated into local praxeologies, containing a set of types of tasks with a common technological discourse. All point- and local praxeologies that share a common theory are called regional praxeologies. Integrated praxeologies are sometimes called mathematical organisations (MOs). On the other hand, praxeologies can be differentiated further: Techniques that are complex can be interpreted as a (sub)task that are then part of (sub) praxeologies. In our studies we use the latter to flesh out engineering praxeologies in high details, see specifically Study II and Study III. In the quotation from Chevallard (2006) above, the part "in whatever style of 'reasoning' such as an explanation or justification may be cast" refers to another principle of ATD: The principle of *institutional relativity* states that

> a given object can be part of a type of tasks, a technique, a technological or a theoretical element depending on the praxeology we are considering and the way this praxeology exists and evolves in a given institutional setting. Not only does a praxeology change when moving from one institution to another [...], the very elements of a praxeology can have different functions depending on the type of praxeology that is considered. (Bosch, Chevallard, García, and Monaghan, 2019, p. xv).

So, institutional conditions constitute the technological-theoretical discourse, the available techniques and the relevant types of tasks. To incorporate the references to the institutions that are relevant in our studies, we introduced an extended praxeological model in Study II:

$$\left[T, \begin{matrix} \tau_{\mathrm{HM}}, \theta_{\mathrm{HM}} \\ \tau_{\mathrm{ET}}, \theta_{\mathrm{ET}} \end{matrix}, \Theta\right]_{SST}$$

The index SST means that we are looking at practices within the institution Signal Transmission. Those practices, modelled as praxeologies, possibly contain techniques or technologies originating from the institutions HM and ET respectively. From an ATD point of view, the different institutional conceptualisations of complex numbers give rise to different praxeologies. The characterisations of the two contexts, complex numbers in the mathematics service course and in electrical engineering, can be understood as descriptions of institutional aspects that shape the logos block and thus, due to the dialectic of praxis and logos, also the practical part of praxeologies. In our analyses we understand the characterisations of the two contexts as characterisations of two different *mathematical discourses* and associate praxeologies or praxeological elements to the mathematical *ET-discourse* or the mathematical *HM-discourse*. Within ATD, the term discourse is used in the etymological sense, e.g. in expressions such as "reasoning discourse" for logos aspects (see Bosch & Gascón, 2014, p. 68). In our work, we understand discourse methodologically, and more deeply linked to ATD concepts such as the institutional dependence of knowledge and logos aspects like the raison d'être. I will take this up again and discuss it more deeply in chapter 5. There I will also discuss the graphical representation of our analysis results that is based on the extended praxeological model.

The institutional influences on practices can have diverse origins. ATD proposes to model those by a hierarchy of mutually constraining and conditioning levels of codetermination, as shown in Figure 3.1.

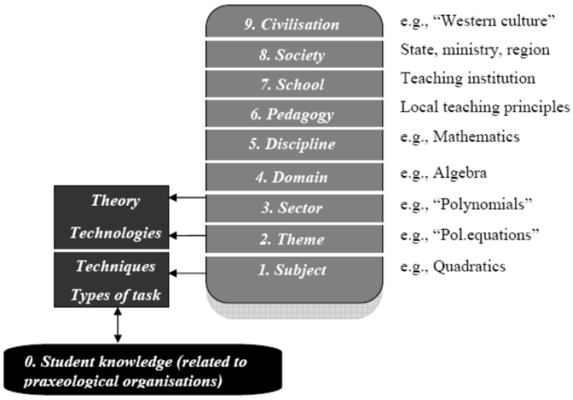

Figure 3.1.: Levels of didactic codetermination, mathematical praxeologies and the knowledge of students related to the mathematics taught. (Artigue & Winsløw, 2010, p. 6)

The levels of codetermination model institutional or societal influences (constraints, possibilities, etc.) on practices. With the levels of codetermination, ATD proposes an order of these influences on practices that are particularly relevant to phenomena of teaching and learning. The different levels now provide categories that help to identify institutional and societal influences and to name and structure them accordingly. Among other things, this means identifying influences as institutional rather than personalising them as individual characteristics.

In our works we consider the impacts of developments also located on higher levels on the constitution of practices. In Study II and Study III we refer to historical studies on electrical engineering to highlight differences in, and describe the two different institutional mathematical discourses. The analyses of practices and their institutionalisation can also be informed by the results of sociological or philosophical studies. In Study I and Study IV we refer to historical-philosophical studies to reconstruct societal aspects, located at the higher levels of codetermination, and their reflection in concrete practices.

Artigue and Winsløw (2010) also show the relationship between institutional praxeologies and the "student knowledge" to the levels of codetermination. The double arrow in the figure highlights that the students' praxeologies are not identical to the institutional praxeologies but also not independent idiosyncrasies. This is an important aspect in Study III where we analyse students' solutions to the exercises.

3.3. The (didactic) transpositions of knowledge

Praxeologies are a model to capture mathematical practices in their institutional constitution. With the model of (didactic) transposition, ATD also offers the possibility to examine and describe aspects of the production, development, change and dissemination of knowledge between institutions, and thus also their relationships with each other. The Figure 3.2 represents a diagram of the basic model of didactic

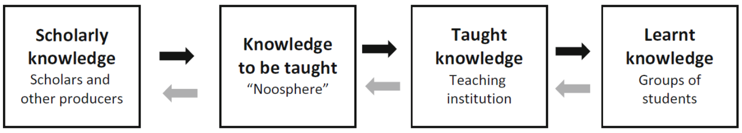

Figure 3.2.: The process of didactic transposition (see Bosch & Gascón, 2014, p. 70)

transposition. This model is related to processes of organisation and transformation of knowledge with a view to teaching institutions. Originating from studies about school mathematics the didactic transposition is a tool to make institutional conditions on the knowledge to be taught in the mathematics classroom accessible for analyses. Chevallard (1989) assumes here that "[s]chool mathematics, for example, has essentially evolved from mathematicians' mathematics. More generally, taught bodies of knowledge have been derived from corresponding scholarly bodies of knowledge." (p. 7). In general, scholarly knowledge is produced e.g. in universities or research institutes. The knowledge to be taught is determined by official curricula. Politicians, scientists, educators and other members of the noosphere[9] are involved in this process. This ultimately becomes taught knowledge, which is taught in teaching institutions and in turn results from the curricular documents via a didactic transposition process. Here, one can then additionally differentiate the learned knowledge. In summary, the didactic transposition refers to the transformation of disciplinary or "scholarly" knowledge performed in order to make it teachable (and learnable). From a critical point of view, this model of the transformation of scholarly knowledge into knowledge to be taught is quite narrowly conceived. Bergsten, Jablonka, and Klisinska (2010) and Sträßer (1992), among others, formulate some critical remarks on didactical transposition theory. A particularly relevant point for our studies is that from this conception, where scholarly knowledge is transformed into knowledge to be taught, the institutional dependence of knowledge may not be sufficiently taken into account. The researcher's own institutional location, usually in university mathematics, might be ignored and used as a (natural?) reference model. Even if this is explicitly taken into account and a REM is used to "detach" the researcher from its own institutional involvement, a REM based on university mathematics seems to be the natural point of reference based on this model

[9] In the ATD, noosphere refers to "the sphere of those who 'think' (noos) about teaching-, its relationship to 'scholarly knowledge' which usually legitimates its introduction in educational institutions, and the specific form it takes when arriving in the classroom [...]." (Bosch and Gascón, 2014, p. 71). The noosphere comprises all agents involved in the process of (didactic) transposition from scholarly knowledge to knowledge to be taught. This comprehensive term also expresses the fact that the agents involved in this transposition process and the associated historical and institutional conditions are not always easy to identify.

of didactical transformation. I will resume a critical reflection on the REM in chapter 5. Nevertheless, the model of (didactical) transposition draws attention to the possible variety of institutional references and corresponding actors and can be used fruitfully. Castela (2015), for example, refers in particular to external transpositive effects concerning the production and legitimation of mathematical knowledge, which are relevant in the transition from scientific mathematics to vocational domains.

Based on this basic concept, there are different uses within ATD, depending on the research question. The transition from scholarly mathematical knowledge to knowledge to be taught is also referred to as external didactic transposition, the transition to taught knowledge as internal didactic transposition, see Figure 3.3. The institutional dependence of knowledge, e.g. of mathematics, raises the question

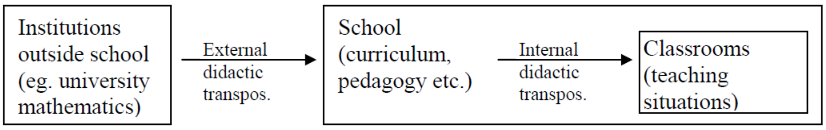

Figure 3.3.: Internal- and external didactic transposition (Winsløw, 2011)

of which mathematics should be taught in other areas, such as in study programmes like electrical engineering. Schmidt and Winsløw (2021) pose this question in the context of mathematics service courses for engineering students. They note that while the selection of mathematical content takes place according to the needs of the engineering study programme, the actual teaching is carried out according to generic standards and methods for teaching mathematics. The specific institutional conditions of engineering thus enter into the external didactic transposition, but not into the internal one. They call this the parallel model of didactic transposition, see Figure 3.4. The figure implies that mathematics in

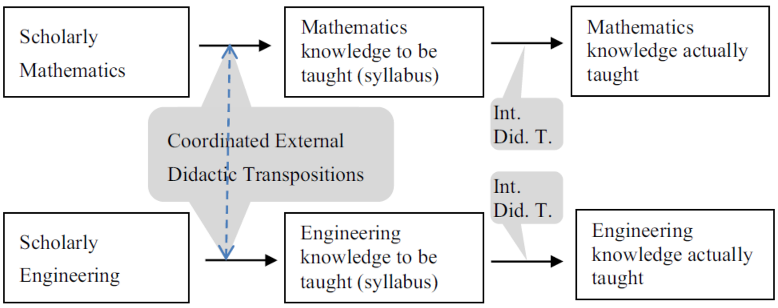

Figure 3.4.: The parallel model for didactic transposition in engineering education (Schmidt & Winsløw, 2021, p. 266)

engineering study programmes is essentially taught in the mathematics service courses. Since Schmidt and Winsløw focus specifically on these courses, its adequate that they use this model. In our own studies, however, we point out that mathematical practices, especially in higher semester engineering modules such as Signal Transmission, can be described as a mixture of practices of mathematics from service courses, mathematics as developed and used in basic electrical engineering courses, and specific signal theory content. Moreover, our analyses show that mathematical practices in these courses cannot be understood solely as the application of mathematics taught in service courses. Schmidt and Winsløw especially highlight an important issue that also becomes relevant in Study V and Study VI: While the internal didactical transpositions are developed by different teachers (i.e. persons occupying the institutional teaching positions) the student position is occupied by the same person in each institutional setting (i.e. engineering courses and mathematics service courses). Therefore, the detached internal didactical transpositions potentially causes a disconnection of mathematics taught in mathematics service courses and mathematics in engineering courses. This situation of detached mathematics motivates questions on how the relationship of mathematics and engineering can be understood (Study V) to enable teaching developments (Study VI) that potentially help to promote connections.

I would now like to conclude this section by reiterating my description of the work of Artaud (2020), given in Study VI on page 180 of this thesis. Artaud identifies two different transposition processes through which mathematical knowledge, in subjects such as electrical engineering, is created[10]. (1) Either the mathematical knowledge required in electrical engineering institutions is already elaborated and developed in other institutions, for example academic mathematical research institutes. This knowledge then enters the electrical engineering institution via didactic transposition processes, so to speak externally, and serves the mathematical education of future electrical engineers. Here, one can localise the HM-discourse and the idea of mathematics for engineering. Through this *exogenous* didactic transposition process, however, the academic mathematical knowledge is changed and adapted especially to the needs of electrical engineering institutions for the education of future engineers, but maintains the orientation towards academic mathematics. Schmidt and Winsløw (2021) also note this and it is to this aspect that their parallel model for didactic transposition refers. (2) Or, the relevant mathematical knowledge has been developed in the course of a historical process by actors specialising in electrical engineering. In this case, the mathematical knowledge entered the electrical engineering institution a long time ago via an institutional transposition process to be put to use. The investigation by Bissell (2004) of the introduction of complex quantities in electrical engineering, driven by Steinmetz (1893) among others, that allows to manipulate graphical and pictorial representations instead of complicated mathematical expressions and also led to systems thinking and black box analysis Bissell (2004, p. 309), give a glimpse on such an institutional transposition process. In the course of time this knowledge was used in electrical engineering and was didactically transformed in order to be taught. This didactic transposition process is *endogenous*. Here the ET-discourse and the idea of mathematics *in* engineering can be situated. We show in our studies that mathematical practices differ depending on the transposition process and that these differences are relevant for didactic questions.

[10] Artaud's studies come from the field of economics, but she makes statements on a general level.

4. Research on engineering mathematical practices

This literature review is based on the literature discussed in the studies of this thesis and aims to situate them within the field of research. The research programme of this thesis, the ATD, has already been introduced in chapter 3. Important references that focus on specific aspects of ATD can be found there. I would like to highlight once again the work of Castela (2015), Castela and Romo-Vázquez (2011), and Romo-Vázquez (2009), which is an important starting point for our own work. In particular, they refer to external transpositive effects regarding the production and legitimation of mathematical knowledge that are relevant when moving from academic mathematics to vocational domains. In doing so, they differentiate the model of the didactic transposition process with regard to different institutional influences and extend the praxeological 4T-model: they distinguish between a theoretical and a practical component of the technology. This distinction serves to reconstruct the institutional dependence of technological discourses of different courses with regard to higher levels of codetermination. Castela and Romo-Vázquez focus on tracing the impact of external didactic transformation and different modalities of institutional validations. Our studies extend and adapt this idea of a further differentiation of praxeological blocks. However, our focus is on aspects of internal didactic transposition.

Considering ATD as a research programme, Bosch, Chevallard, García, and Monaghan (2019), Bosch, Gascón, Ruiz Olarria, Artaud, Bronner, Chevallard, Cirade, Ladage, and Larguier (2011), and Chevallard, Farràs, Bosch, Florensa, Gascón, Nicolás, and Ruiz-Munzón (2022) give an insight into typical ATD studies. Overall, one can identify four broad areas in which ATD work can be classified: (1) ATD as a research programme in mathematics (or other fields) education, (2) teacher education in mathematics and questions around professionalisation, (3) studies on teaching design, essentially in the paradigm of "Questioning the World"[1] and (4) didactic studies in (university) mathematics and mathematics-related university courses or other contexts (such as museum or vocational training). The Studies I-VI in this thesis belong to the fourth area. Before focussing more specifically on didactic studies in university mathematics courses in electrical engineering, I would like to mention the work by Gascón and Nicolás (2017, 2019). Referring to Weber's 1917 distinction between normative and scientific statements, the authors specified the ATD's position regarding the responsibilities and goals of didactics as a science. They formulate clear criticism of the lack of separation of value judgement and research results:

> However, some expressions appearing in works by people using the ATD – for instance "to turn the study of probability into something *meaningful*", "*true* raison d'être of the elementary algebra", "*undesirable* consequences of the disintegration of school mathematics" – can be disturbing and misleading, as they assume value judgements which, in turn, induce normative prescriptions or proscriptions concerning teaching. Actually, these judgements

[1] This "paradigm" (Chevallard, 2015) is comparable to approaches of inquiry-based learning, see also the introductory remarks of Florensa Ferrando (2018).

> and norms, although frequent in ATD work, cannot be presented as research results, not even as consequences of them. (Gascón and Nicolás, 2019, p. 10)

I agree with the criticism of value-laden assessments in research studies. This applies in particular to my criticism of the deficit-oriented view on engineering practices. However, I disagree with the rejection of formulating teaching proposals based on research findings. Research focus 3 is specifically concerned with developing ideas for teaching design and lecturer support based on our findings. I think the reason for the disagreement here might be, among other things, the status given to REM as a fundamental methodology that is also adopted by Gascón and Nicolás (2019). As explained in more detail in chapter 5, REM as a methodology aims in principle to identify differences, which in turn can be interpreted as deficits. Didactic design then aims to overcome these deficits. Value-laden teaching designs that are justified by research results thus could follow from the value-laden interpretation of the REM methodology. Our methodological approach differs from this. Our teaching design and support proposals are generally not aimed at making anything better (in the above sense). We show which possibilities for teaching design and lecturer support can be found on the basis of our studies. What end these design ideas then serve is independent of this. Here we refer, among other things, to aims stated in the literature, for example promoting connections between mathematics and engineering (see below). The work in research focus 2 also serves as an explication of the basis of my general stance. Here I agree with Gascón and Nicolás (2019): “To raise those principles may help to not only illuminate the link between research and teaching proposals but also increase the degree of self-awareness of the theory” (p. 10). Considering Weber’s ideal type concept, as far as I know, there has been no discussion in the ATD so far.

The study of engineering mathematical practices is an important topic in research on university mathematics education (Winsløw, Gueudet, Hochmuth, & Nardi, 2018). It is an integral part of current publications (Biehler, Liebendörfer, Gueudet, Rasmussen, & Winsløw, 2022) and a regular feature of conferences such as the INDRUM conferences (Trigueros, Barquero, Hochmuth, & Peters, 2023). It can be divided into the two strands mathematics *for* engineering, referring to mathematics service courses and mathematics *in* engineering, referring to mathematics as an integrated part of engineering courses (Pepin, Bieler, & Gueudet, 2021). Our studies can mainly be associated with the research focus mathematics *in* engineering. Romo-Vázquez and Artigue (2022) give a state of the art overview of mathematics education research for engineers with a specific focus on ATD. They begin with a historical overview, focusing on the challenges in the field, and continue with examples of recent research and development, especially of ATD. Regarding the role of mathematics in engineering, in addition to the work of Castela and Romo-Vázquez already mentioned, the work of González-Martín (2021, 2022) should also be mentioned. The author addresses the issue of coherence of practice in relation to aspects of the integral concept and its use in calculus and mechanics and electricity and magnetism courses. These essentially curricular differences have also been observed and investigated by Dammann (2016). More specifically, González-Martín (2021) conducts praxeological analyses of the introduction of integrals in the textbooks of the engineering courses and contrasts this analyses with lecturers interviews. Although Gonzales focuses more on curricular differences, he finds a similar “entanglement of elements from mathematics and engineering (both in the technique and in the technology)” (p. 229). Frode Rønning also conducted praxeological analyses in signal theory Rønning (2021) and electrical engineering Rønning (2022). Rønning (2021) bases his praxeological analysis in the context of Fourier analysis on textbooks in (basic and advanced) mathematics and in signal theory and aims at answering the question “What characterises the praxeologies connected to Fourier series in mathematics and in signal theory?”

(p. 191). He also complements his data by a lecturer's video and gives a historic background on Fourier series. Similar to González-Martín, Rønning (2021) also examines the differences in basic mathematics praxeologies, advanced mathematics praxeologies, and engineering praxeologies. Identifying differences is an important result and certainly gives cause to think about making appropriate adjustments and to promote the development of teaching to address this. Contrary to these studies, our research interest is directed towards a more refined analysis of mathematical practices in engineering and how different mathematical discourses interact within these practices. Rønning (2022) presents a project that aims at redesigning a mathematics service courses for engineering programmes. One aim of the project is to adapt the curriculum of mathematical topics to the needs of engineers (see also Schmidt and Winsløw's 2021 parallel model for didactic transposition, Figure 3.4). In addition, the electrical engineering lecturer and the mathematics lecturer cooperated closely in the project. One aim of this cooperation was to introduce examples from electrical engineering into the mathematics course. Using the ATD, his aim is to

> study the discourses that develop to see how the praxeologies in mathematics and engineering influence and interact with each other. I will inquire into the challenges and opportunities that arise at the interface between mathematics and electronics, as seen from the viewpoint of the teachers in the two subjects. (p. 604)

Unlike in our studies, Rønning does not start from the institutional dependence of knowledge but assumes that the didactic systems differ because they refer to separate courses. Looking at the didactic system in the electronics course, he now examines how mathematical and electronic praxeologies interact. In addition to his ATD analysis he also interviewed the two lecturers. A very interesting aspect from the short interview he presents is that both lecturers talk positively about the cooperation while emphasising the importance of "division of labour" (p. 614) and therefore reflecting the institutional separation of the mathematics service course and engineering courses. Both, González-Martín and Rønning include analyses of didactic praxeologies or -systems in their studies, and thus also study processes that are not considered in Studies I-VI of this thesis.

The studies in research focus 3 also refer to the strand mathematics *for* engineering, especially mathematics service courses. Alpers (2020) and Hochmuth (2020) present overviews of the research field in relation to this topic. Our studies can also be understood as aiming for an integration of the two perspectives mathematics in/for engineering, especially our teaching design proposal in study VI. The lack of connections between mathematics courses and engineering is an important research topic. A literature review with regard to this topic was made in Study VI on pages 171ff of this thesis. Modelling and application examples are the standard proposals to facilitate connections. In Study V, we reflect on the standard approaches to application and modelling to make the epistemology of this relationship accessible for discussion. Combined with epistemological reflections from Study IV we argue that a naive application of mathematics (see "applicationism" by Barquero, Bosch, and Gascón (2013)) or modelling cycles (Blum & Leiss, 2007) are insufficient for our purposes. Kortemeyer (2019) and Kortemeyer and Biehler (2022) proposes a modification of the modelling cycle that forms the basis of the Studi-Expert solution they developed in the context of basic electrical engineering courses. They use it to analyse mathematical exercises from basic electrical engineering courses. For the more complex exercises in signal theory in our studies, this does not seem sufficient to us. Here, too, the relationships between the mathematical practices and orientations that we have addressed cannot be adequately represented. An approach to modelling developed within the ATD is study and research paths (SRP). This approach belongs to the

"Questioning the World" (Chevallard, 2015) paradigm and generally represents an important research area within ATD (Winsløw, Matheron, & Mercier, 2013). With reference to engineering, the work by Bartolomé, Florensa, Bosch, and Gascón (2019), Bosch, Florensa, Markulin, and Ruiz-Munzon (2022), and Florensa Ferrando (2018) should be mentioned here. Study and research paths is a design-based teaching development framework, therefore the mentioned studies have a different focus than the studies collected in this thesis. Subject-specific analyses can inform the design of study and research paths. Especially from our research perspective, the focus on a crucial question that is the start of the study and research path and guides its processes is relevant here. This question is not only a question from a specific context, but also a question that has the potential to challenge the content specific institutional rationales. Although her work is thematically outside the scope of this review, I would also like to mention the study by Jessen (2014)[2]. She investigates design and implementation of a bidisciplinary study and research path combining mathematics and biology at the school level. She notes:

> The SRP enables most students to make the two disciplines interact. As already said it is crucial to choose a strong generating question that engages the students to develop the intended praxeologies, and the quality of this choice could secure the possibility of actual interdisciplinary work. This means that a thorough a priori analysis must be the starting point of all bidisciplinary SRP designs since the interaction between disciplines is clearly not obvious or automatic. (p. 216)

In my opinion, Studies I-VI collected in this thesis show how such a thorough a priori analysis could be done. Additionally, Jessen discusses interesting questions regarding the cooperation of the mathematics- and biology teacher:

> Is it sufficient that two teachers (representing each discipline) formulate the design? or is it necessary for the teachers to do an analysis of the didactic transposition [...] of the interplay of the involved scientific disciplines in order to identify interdisciplinary praxeologies combining the school disciplines? (p. 2017)

Given the institutional constraints and possibilities, which are also reflected in the "division of labour" from Rønning's interview, these questions are highly relevant to the collaboration between mathematics and engineering lecturers. In a study on the views of engineers and mathematicians on the concept of continuity, Alpers (2018) shows that there is a separation reflected in the different views of mathematicians teaching service courses and lecturers of the engineering faculty.

[2] This work is part of her dissertation thesis (Jessen, 2017).

5. Discussion of the research questions

Before discussing the research questions in this chapter, I will first summarise the overall research contribution of the studies collected in this thesis. At the methodological level, the studies show three important developments: (1) The notion of discourse has been developed as a phenomenotechnique. This will be discussed together with RQ1. (2) Methodological steps were elaborated to relate institutional analyses to individual student works. This is at the essence of RQ2. (3) Results of historical-philosophical studies were considered in order to clarify concrete subject-specific practices via higher levels of codetermination. I will take this up when discussing RQ3.

Another contribution of the studies is the concrete demonstration of how each of these methodological developments can be fruitfully used for praxeological analysis. Related to this is our proposal for the graphical representation of the results of praxeological analyses. This proposal aims in particular to make the partially intertwined different mathematical discourses explicit. These developments are now leading to a new way of looking at the relationship between engineering and mathematics, especially mathematics service courses. And last but not least, conclusions for teaching support and teaching design can be drawn from all these developments, as I will elaborate in the discussion of RQ4.

Since Study I–Study V were jointly written by me and Reinhard Hochmuth and Study VI is based on our collaboration, it is appropriate that I specify Reinhard Hochmuths and my own contribution. Considering the three methodological developments, the elaboration of the notion of discourse from a mere synonym of technology to a methodological tool for our praxeological analyses was mainly my responsibility. The question how to relate institutional analyses to individual student works, especially working out the steps based on Schwemmer (1975) and Weber (1904) was Reinhard Hochmuths work. The same applies to working on the question how influences from higher levels of codetermination on concrete practices can be reconstructed. References to the work of Borzeszkowski and Wahsner (2012) and Wahsner and Borzeszkowski (1992) were mainly elaborated by Reinhard Hochmuth and the section "About Mathematical Practices in Psychology" in Study IV was solely written by him. The concrete execution of the methodological developments in praxeological analyses was my responsibility. This includes the concrete praxeological analyses and the subject-specific elaboration of the two mathematical discourses, taking into account references to studies by Steinmetz (1893) and Bissell (2004) and Bissell and Dillon (2000, 2012). The same holds for the proposal of the graphical representation of our analyses results. The development of a new view of the relationship between engineering and mathematics has taken place across all studies. Here I cannot assign any main responsibility. But I have elaborated the references to modelling and application in Study V. The same holds for the elaboration of the example for teaching development based on our studies in Study VI. The idea for this example was a mutual development, the detailed formulation was done by me. In addition, we have also developed ideas for teaching support in the respective studies. Since these were developed in joint discussions, it is difficult to assign responsibilities here.

5.1. Focus 1: The subject-specific reconstruction of mathematical practices

Two research questions are grouped under this focus, which I will discuss below. First, I will address RQ1, for which Study II is central. In addition, the methodological contribution (1), the notion of discourse and its concrete practical implementation for praxeological analysis will be presented. Since Study II was worked out at the beginning of the project and the methodological concept of discourse in particular developed over the whole period, I will also address here developments from later studies. The same holds for descriptions of the two mathematical discourses. The best formulated description can be found in Study VI. The central contribution for RQ2 is Study III. In this part, I also discuss the methodological contribution (2), steps to relate institutional analyses to individual student works.

RQ1: How can mathematical practices in the module Signal Transmission be reconstructed, and what connections, differences and interactions can be identified with regard to mathematical practices from mathematics service courses, from basic electrical engineering courses, and newly introduced mathematical practices?

A general observations relating to this question is the occurrence of mathematics in the electrical engineering study programme in different institutionally situated courses. However, as mentioned in chapter 1, it is not clear how those practices from the different courses interact in the module Signal Transmission, how they differ and what possible connections look like. Similarly, the problem of the disconnection between mathematics as it occurs in engineering and mathematics as it is taught in service courses[1], which is an important topic in university mathematics education research (Hochmuth, 2020; Winsløw, Gueudet, Hochmuth, & Nardi, 2018). From our own analysis experience and from discussions with others, the phenomenon of a potentially deficit-oriented view of the mathematical practices of engineering students and teachers, but also of the mathematical practices in engineering textbooks, has also become apparent. Apart from the associated (research) ethical problem that such a deficit-oriented position more or less clearly denies engineers their own and engineering-specific justification of such practices, this deficit-oriented view also obscures insights into the engineering-specificity of these practices and potentially promotes problematic epistemological views. We have not systematically explored the problem of research ethics or epistemological views of students and lecturers further. At the institutional level, this touches on the question of the epistemological conception of the relationship between electrical engineering and mathematics (see also "applicationism as the dominant epistemology at university" in the works of Barquero, Bosch, & Gascón, 2013). This is discussed further under research focus 2 with RQ3. But the aim to understand the engineering specificity of mathematical practices in Signal Transmission courses became one of the main driving factors for our studies. At this point, research foci 1 and 2 meet: We showed that the consideration of philosophical-historical studies is suitable for clarifying this engineering specificity to some extent with regard to the historical-societal development located at higher levels of codetermination. This is the core of research focus 2, to which I will return below. In the following, the first research focus aims at a detailed subject-specific praxeological analysis. Thus, while focus 2 elucidates influences from higher levels of codetermination, in focus 1 we carry out a detailed differentiation with regard to different institutional influences at the level of the praxeological model.

[1] This aspect is explored further under RQ4.

The answer to RQ1 touches on the three levels of theory, methodology and method: The extension of the praxeological model reflects our methodological developments at the theoretical level. The starting point for this development were methodical problems that we encountered in the analysis of the lecturer sample solutions[2]: A central methodical step of a praxeological analysis is the assignment of praxeological elements (i.e. type of task, techniques, technologies, and theory) to segments of a text. In our analyses we mainly focus on techniques and technologies . The texts we analysed are lecturer sample solutions or textbooks of Signal Transmission, and the segments are text passages in textbooks or solution steps in the lecturer sample solutions like the transformation of an algebraic expression. One transformation in the solution to the amplitude modulation exercise is particularly interesting and suitable to illustrate our methodological approach: The transformation from line (2) to line (3), see e.g. the appendix of Study VI on pages 187–188. In the following I will focus on this step. For the full analysis see Study II and Study III[3]. Highlights of the analysis are also given in Study IV, Study V, and Study VI each to illustrate aspects to address the different questions of the studies.

The techniques that we could assign to this transformation from line (2) to line (3) are techniques taught in the mathematics service course. This step is only possible because we as researchers can recognise and explain these techniques in a way that makes it possible to relate them to the mathematics service course. This requires two things: firstly, an analysis of the content of the mathematics service course to be sure that the corresponding techniques[4] can be related to it. We have done this based on the corresponding course materials, see chapter 2. And secondly, the recognition and explanation of the mathematical techniques themselves, based, among other things, on the institutional situatedness of the researcher. Researchers in mathematics education are often representatives (understood as institutional position within ATD) of the institution academic mathematics[5]. The mathematics education researcher as a representative of academic mathematics, who takes the exogenous didactic transposition (Artaud, 2020) from academic mathematics to the mathematics service course of the electrical engineering study programme (cf. Figure 3.4) into account is then able to recognise and explain mathematical steps in a solution and assign techniques from the mathematics service course. Here I would like to make a brief comment on the aspect of explaining. With reference to the theory of rational explanation by Schwemmer (1975), "an action is explained precisely when the 'calculations' (Dray) that led to it are reconstructed, i.e. when they can be interpreted as means to an end pursued by the agent." (p. 45, translated from German by J.P.). Since we are looking at a lecturer sample solution, that is, institutionally accepted adequate techniques and technologies, the means and ends are the institutional means and ends. The "agent" is not seen as an individual but a representative of institutionally accepted knowledge. In research focus 2, where we analysed students' solutions, this is no longer the case. Therefore, in Study III, we introduced methodological steps to deal with the situation.

Interestingly, this transformation makes a "clearly" structured mathematical expression more "complicated". The overall task of this exercise is to draw a phasor diagram and, again from a mathematics service

[2] In Study V, we presented the extended praxeological model by exemplifying part of the analysis and introducing our extensions at the point in the analysis where the extension became necessary. Introducing aspects of the research framework integrated with an illustrative excerpt of analysis is well suited to describe this methodological development triggered by methodical difficulties (cf. pages 161ff of this thesis)

[3] The analyses in those two studies differ a bit as they address different questions. We slightly reworked the analysis for Study III to focus our research interest in this study.

[4] In this example, only techniques are considered for now. But in the context of the overall analysis, technologies of the mathematics service course also play a role, of course.

[5] Scholarly mathematics in the model of the didactic transposition.

course perspective and how drawing phasor diagrams was introduced there, it seemed to be much easier to draw this diagram from the less complicated algebraic expression[6]. So we couldn't assign technologies that justify or explain this step from the perspective of the mathematics service course. To be able to do so we had to look for the electrical engineering reasoning (means and ends) behind this transformation. Analyses of the course material from Signal Transmission served as a basis here. In principle, this transformation allows to draw the phasor diagram in a way to graphically represent important aspects of amplitude modulation. Relevant for this step is knowledge about how the two different frequencies are assigned to message- and carrier signal, how the relationship between those two signals is defined by the modulation process, and how this is reflected in the multiplication and the additions in the mathematical expression (see also the description of Amplitude Modulation in section 2.2). Complex numbers and how they are used in electrical engineering became a central topic, so we took textbooks from basic electrical engineering courses (especially GET II, cf. chapter 2), historical studies by Bissell (2004) and Bissell and Dillon (2000, 2012), and original historical documents (Steinmetz, 1893) into account. Thus, we could describe the institutional relation of electrical engineering to complex numbers and show how this relation differs from the institutional relation of the mathematics service course to complex numbers (see section 28.2.2.1 in Study VI, on pages 176ff of this thesis for a detailed description.).

From this institutional electrical engineering perspective, we could assign technologies to this step in the sample solution because we have reconstructed mathematical ends and means specific to electrical engineering based on endogenous didactical transpositions (Artaud, 2020). In order to distinguish between the different institutional references, we have given the techniques and technologies corresponding indices: HM referring to the mathematics service course and ET referring to electrical engineering courses (see footnote 6 on page 20). In order to be able to describe these different institutional conceptions of mathematical practices well in our work, we have introduced the notions of institutional mathematical ET-discourse and institutional mathematical HM-discourse.

I understand discourse as a methodological tool that bundles certain aspects of models that already exist in ATD[7]. The starting point is the informal notion of discourse within ATD: Here, the term discourse is already used in the etymological sense, e.g. in expressions such as "reasoning discourse", according to Bosch and Gascón (2014, p. 68). Chevallard (2019) also refers to the etymological origin in "logos":

> To understand a given technique τ is to understand why τ requires to perform the task t_1 (using the technique τ_1), then the task t_2 (adopting the technique τ_2), and so on – we suppose here that performing τ on t equates to performing the succession of tasks $t_1, t_2, \ldots, t_n$. To understand (why) τ amounts to understanding why performing $t_1, t_2, \ldots, t_n$ add up to performing t. To this need responds in the position p of I a "discourse" on τ that purports to explain τ, that is an account of τ that claims to "make it clear." It is such a discourse, that can vary from institution to institution, and even from position to position within a given institution, that is called a technology of the technique τ and is denoted by the Greek letter θ (theta). (p. 87)

Thus, an institutional mathematical discourse refers to mathematical technologies that are justified in that specific institution. Since technologies and techniques are dialectically related, we can also

[6] In Study III we look at students' solutions to this exercise and, in fact, there is one student who tries exactly this and fails to produce an adequate diagram: See Figure 9 on page 105 of this thesis.

[7] The reflections on the concept of discourse also go back to my unpublished master's thesis (Peters, 2021).

assign techniques according to the specific institutional discourse. In order to justify the assignment of a technique or technology to a specific institutional discourse, it is necessary to describe the relevant institutional knowledge and to characterise its institutional specificity. This is based on the ATD principle of institutional relativity. Modelling tools of ATD that we used to flesh out this institutional specificity are: The scope of techniques, different raisons d'être, processes of exogenous- and endogenous transpositions (Artaud, 2020), and research on external- and internal didactic transposition processes. We also used historical and philosophical studies (Bissell, 2004; Bissell & Dillon, 2000, 2012; Borzeszkowski & Wahsner, 2012; Wahsner & Borzeszkowski, 1992) and historical documents (e.g. Dirac, 1981; Steinmetz, 1893), which are particularly useful for, besides other, identifying raisons d'être.

This approach is based on the institutional situatedness of the researcher, self-reflection and the explicit analysis of institutional conditions that are probably initially alien (e.g. electrical engineering ends and means of mathematical actions that are results of endogenous and internal (didactical) transpositions) to the researcher. In some of its aspects this opposes the standard approach of ATD to construct a reference epistemological model (see also REM on page 21). The REM is meant to help facilitating a detachment from the social reality under investigation. Bosch (2015) underlines the importance of the researcher emancipating herself from the prevailing societal reality:

> I will call it the "detachment" principle, after the work of the German sociologist Norbert Elias (1987). Because researchers in didactics deal with a reality that takes place in social institutions, and because they often participate at these institutions (as researchers, teachers, students, or in several positions at the same time), we need to protect ourselves– to emancipate–from the institutional points of view about this reality, that is, from the common-sense models used to understand it. (p. 52)

Here, emancipation is understood as detachment, the researcher's position is "outside" the studied system. The reconstruction of institutional discourses is based to some extent on the involvement of the researcher in institutional conditions in order to be able to identify ends and means that are not part of her own institutional situatedness. In order to provide some justification for this approach, I would like to refer here to Schlaudt (2014)[8], who also deals with everyday realism in his book "Was ist empirische Wahrheit" (What is empirical truth). He states that we can only be (naive) realists in everyday life because we locate ourselves in an epistemic distance to this world from the outset (p. 44). He quotes Holzkamp (1973), who puts it this way: "The world-relation of conscious knowledge implies a distance of the knower from the object of knowledge, through which the human being can reflectively grasp himself in his relation to the world and to other people[9]" (p. 157). And if everyday knowledge is possible on the basis of this reflected epistemic distance, research is even more so.

However, it is unclear and would be interesting to be discussed what similarities the reconstruction of institutional discourses and the construction of REMs share. For example, the construction of a REM seems to require reflection on one's own institutional situatedness too. Otherwise, academic mathematics would be naively set as a naturalised reference. I have already referred to this point in the introduction of didactic transposition on page 25. From the perspective of the REM methodology, the

[8] This reference to Schlaudt and everyday realism at this point is somewhat artificial and also not necessary in the course of the text so far. I will refer to Schlaudt's pragmatic theory of truth in the discussion of RQ3 and briefly justify why it is useful for our studies and also compatible with ATD.

[9] Das Weltverhältnis des gewußten Wissens impliziert eine Distanz des Erkennenden zum Erkenntnisgegenstand, durch welche der Mensch sich selbst in seiner Beziehung zur Welt und zu anderen Menschen reflektierend erfassen kann

mathematical ET-discourse could possibly be interpreted as the dominant epistemological model (DEM) of the electrical engineering institution. According to Lucas, Fonseca, Gascón, and Schneider (2019),

> [t]he DEM of a domain of the mathematical activity assigns an *official raison d'être* to it–that is, a set of possible questions whose answer requires, in an essential way, the use of the knowledge components of that domain and, consequently, gives meaning to the school study of the domain. Both the DEM and the official raison d'être (of this domain in I) are usually quit transparent to and unquestioned by the subjects of I (p.78).

Both methodological approaches can be seen as phenomenotechniques in the sense of Bachelard. But both produce different didactical phenomena. The REM is usually meant to serve as a reference for a comparison with the institutional knowledge to be taught or actually taught (the DEM). The identified differences are probably understood in the form of deficits (incompletenesses, limitations and contradictions, see Lucas et al. (2019, p. 78)). This is then usually referred to in the design of teaching environments or tasks that are intended to overcome these deficits. The focus on these kinds of phenomena and didactic questions is certainly justified, but the possible tendency towards a deficit-oriented view of the DEM should be reflected.

Our methodological approach enables us to ask a different kind of questions and to focus on another kind of phenomena. In our ATD analyses in the context of Signal Transmission, we are not concerned with identifying the "incomplete, limited, or contradictory" nature of the mathematics found in signal theory in order to make a proposal for better teaching of signal theory or related mathematics service courses that overcomes those. Our question, and it is in this context that the need for a methodological notion of discourse has arisen, is that of an understanding of and the relationship of institutionally different subject-specific mathematical practices. This, of course, also enables the development of learning environments and tasks, see focus 3.

By analysing two lecturer sample solutions in Study II, we showed that an extended praxeological model is suitable for reconstructing institutional mathematical practices that show references to practices of mathematics service courses (HM) and to practices of other electrical engineering courses (ET), see also page 22:

$$\left[T, \begin{matrix} \tau_{\mathrm{HM}}, \theta_{\mathrm{HM}} \\ \tau_{\mathrm{ET}}, \theta_{\mathrm{ET}} \end{matrix}, \Theta\right]_{SST}$$

Two mathematical institutional discourses could be characterised: a mathematical HM-discourse and a mathematical ET-discourse. In terms of the two tasks and their solutions, the characterisations of the two mathematical discourses address identifiable differences and are sufficiently precise for this purpose. In our analyses, these two discourses are intermingled in many ways[10]. We referred to these as types: Type a) HM technique as a moment of an ET-discourse $[T, \tau_{HM}, \theta_{ET}, \Theta]$, Type b) ET technique as a moment of an ET-discourse $[T, \tau_{ET}, \theta_{ET}, \Theta]$, and Type c) HM technique as a moment of an HM-discourse $[T, \tau_{HM}, \theta_{HM}, \Theta]$.

Based on the extended praxeological model, we have developed a graphical presentation of our analysis results. In addition to the textual presentation of analysis results, tables are also common in ATD studies to present praxeological analysis results. However, in the case of very complex analyses, a table can be

[10] This detailed differentiation does not seem to me to be achievable with the REM/DEM methodology.

confusing and, due to the linear presentation, does not clarify the interrelationships of the praxeological elements (e.g. techniques differentiated into subtasks) as well. I used the CmapTools programme[11] to

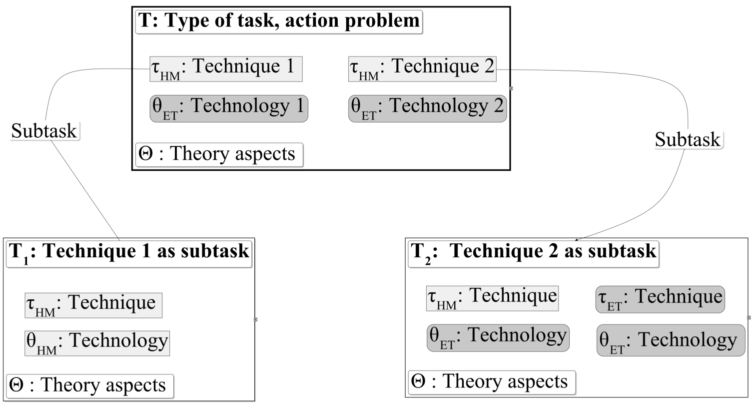

Figure 5.1.: Schema for the graphical representation of a potential praxeological analysis

draw the diagrams of the analyses results. For a better graphical representation, the extended praxeological model is tilted on its side, see the example in Figure 5.1: tasks are on top, all techniques necessary to solve the task are in the second line, below each technique is the corresponding technology, and theory aspects at the bottom. Performing a complex technique can become a task in itself. In these cases we have also assigned techniques, technologies and theoretical aspects. To indicate the two different mathematical discourses, I have used different colours (or shades of grey), indices and slightly different geometric shapes. The diagrams usually start at the top or in the dark framed rectangle. The different solution steps are shown one after the other by technique-technology pairs from left to right. Sometimes subtasks branch off, usually drawn below the main task (sometimes I deviate from this to make good use of space). The subdivision of the text to be analysed into sequences, for example into solution steps, and the differentiation of subtasks are each analytical steps. These may vary depending on the research interest, see for example the slightly different analyses of the amplitude modulation task in Study II and Study III. As such, they are of course open to discussion. An interpretation of the result of the analysis can now, on the one hand, concern the individual detailed praxeologies, but on the other hand, think of these praxeologies as integrated again. Diagrams of praxeological analyses in the studies can be found on the following pages:

- Task about the envelope demodulator: in Study II on page 76.
- Task about amplitude modulation: in Study II on page 78 and in Study III on page 98.

We also used the diagram of the analysis of the amplitude modulation task in Study III as a scheme of reference to analyse the students' solutions. We produced similar diagrams for the students' solutions and marked the found differences, see pages 103ff. I will elaborate on this in the following discussion of the second research question:

[11] https://cmap.ihmc.us/

RQ2: How can analysis results of institutional mathematical practices be related to individual students' actions?

ATD focuses on institutional knowledge and provides models to research this in a variety of ways. In discussing the first research question, I have shown how in exercise solutions references can be made to different institutions relevant to the engineering study programme. We have modelled these references with two different institutional mathematical discourses that are intertwined in these praxeologies modelling the exercise solutions. Solving these exercises is also about integrating the respective institutional mathematical discourses and transforming them into significant mathematical practices for Signal Transmission (SST praxeologies). Thus, the question arises whether the steps at which the different mathematical discourses each occur and interact with each other could be possible breaking points in the students' solution processes. The knowledge of individuals is not central to ATD. The notion of the personal relation to an object of knowledge can be used to distinguish individual actions from institutional praxeologies. The most important aspect here is that the personal relation to an object of knowledge should become conform to the institutional one, but this conformity should not be mistaken for identity[12]. We could consider the sample solutions written by the lecturers as institutionally accepted knowledge because of their institutional position. This changes when we look at the students' solutions, because a student's institutional position implies that the knowledge expressed must be validated against the institutionally recognised knowledge. In educational institutions, the validation is done, among other things, through the correction of exercises and through exams. In our studies we have not focused on these, or other aspects of validation. What is important for us is the different status of the texts to be analysed (i.e. the students' solutions) and the importance of taking this into account methodologically. To explain students' actions, we have to reconstruct means and ends pursued by them (cf. Schwemmer (1975)) and cannot simply identify them with the institutional ones. However, although individuals do not need to reproduce the specific institutional praxeologies (in the sense of identity), they provide points of reference for their actual practices (conformity). Therefore, institutional praxeological analyses results are important points of references for analyses of individual students' works.

In Study III we refer to Weber's 1904 concept of ideal type and

> interpret the two mathematical discourses as ideal types and the underlying characterisations of the discourses as a result of "mental enhancements" regarding aspects of mathematical practices within specified institutional contexts. Furthermore, we use the two ideal typical mathematical discourses as heuristics for our subject-specific analyses of the exercises and the student solutions as well as to formulate hypotheses. Thus, following Weber's formulation, we assume that the ideal typical discourses are to a certain extent effective and represent a means of expressing something real in sample solutions and students works on exercises. Thereby the relevance of this something in individual productions cannot be substantiated by the ideal typical discourses themselves, but has to be shown in the individual productions in a concrete and subject-specific way. (p. 245)

In this study, we also make two methodological remarks that are also important beyond this specific study (p. 245): 1. From a subject-specific point of view, the emergence has to be shown in the specific context of exercises and related praxeologies. For example, it could be shown empirically that a subject-specific

[12] See also the quote by Chevallard (1992) on page 19.

context cannot be reconstructed as a particular case of the application of mathematical discourses. 2. With regard to the question of whether a student experiences the identified aspects as such, corresponding claims are accessible to empirical criticism. It might thus be possible to prove empirically that the connections formulated by means of mathematical discourses do not contribute to the understanding of students' thoughts about their actions.

Regarding the subjective experiences of the students in 2., we would have to consider different data material, for example interviews, and other research frameworks. In our studies we refer to the works of Holzkamp (1985, 1995) who enables, among other things, the reconstruction of subject-related patterns of reasoning. The analyses we have presented in our studies would then, in terms of research logic, be prerequisites for such reconstructions of subjective reasoning patterns (see also Hochmuth (2018) and Hochmuth and Schreiber (2015)). In Study III we focussed on 1., the subject-specific aspect. We proposed four steps, based on Schwemmer's 1976 theory of rational explanation, to guide analyses of students' works:

1. At the institutional level, praxeologies are to be identified and connected to ideal typical discourses.

2. Hypotheses in the specific context of signal theory or the exercises are to be formulated.

3. Concrete material (exercises, sample solutions and students' solutions) is to be validated with regard to corresponding observations.

4. If these are available, concrete material can be explained on this basis.

For step 1 we reworked the praxeological analysis of the lecturer sample solution to the amplitude modulation task. We also presented our analysis result graphically and used this graphical representation as a scheme for presenting the analyses of the students' solutions: Praxeological aspects that we could not identify in the student solutions, because the data did not provide this, were crossed out in the corresponding diagram. Aspects that could be identified in the student solution and correspond to the aspects in the lecturer sample solution were displayed without specific marking. Aspects that were present but different were circled with dashes. Step 3 required in particular to justify and argue the correspondence between observations in the students' solutions and aspects of the lecturer's sample solution, as well as the difference of aspects in the students' solutions to the lecturer's sample solution with reference the two mathematical discourses. See pages 103ff of this thesis for the full analysis.

Unlike the first research question, I consider the REM approach appropriate here. We used the results of the institutional praxeological analysis as a reference for the analysis of students' solutions. The identified differences and similarities become important when the question of conformity with institutional praxeologies becomes relevant. As a reference point for this conformity, we have used the analysis result of RQ1, where we have particularly taken into account the engineering specificity of mathematical practices.

5.2. Focus 2: The epistemological and philosophical relationship between mathematics and electrical engineering

In focus 1, especially under RQ1, we investigated the engineering specificity of mathematical practices and were able to work out detailed references to two different institutional mathematical discourses. I will now shift the perspective from a detailed subject-specific reconstruction to the influences of higher levels of codetermination on the engineering specificity of mathematical practices. The reconstruction of influences from higher levels of codetermination (cf. Figure 3.1) poses a certain problem in the ATD: The reference epistemological model (REM) the standard approach in ATD, is an important and valuable methodology, we also used it when analysing students' solutions. But it has certain limitations. One limitation is that this approach metaphorically positions the researcher "outside" the system she studies to enable her to identify differences with respect to a reference model that serves as a phenomenotechnique. If we are interested in influences, conditions and constraints from higher levels of codetermination this detachment becomes more difficult or even impossible. Bosch (2019), speaking about changes at different levels of codetermination necessary for changing the organisation of study processes, notes that "[t]hose at the level of civilisations are possibly the most hidden ones, since they correspond to beliefs or assumptions that are difficult to identify, unless we move to another civilisation, through the space or the time" (p. 4051).

In our work we most notably refer to studies by Borzeszkowski and Wahsner (2012) and Wahsner and Borzeszkowski (1992) who raise several epistemological issues which must be resolved in each mathematics based field that intends to describe aspects of "nature". How those aspects are resolved in each field has then influences on the concrete formation of practices within the historical process of the development of the field. This can also be interpreted as being part of the endogenous transposition process (Artaud, 2020). This is the basis of the third research question and addressed in Study I and Study IV:

RQ3: How can epistemological and philosophical studies contribute to analyses of mathematical practices in electrical engineering courses and how can this lead to an alternative conceptualisation of the relationship between mathematics and engineering?

The short answer to this question is that we understand mathematical practices in electrical engineering as results of historical societal processes, which also go back to these respective engineering specific resolutions of epistemological issues. Therefore, referring to epistemological and philosophical studies can specifically enrich the logos-block in praxeological analyses and contribute to a wider understanding of actual justifications of practices. In Study I we illustrated this with an analysis of the introduction of the Dirac delta impulse in the textbook by Fettweis (1996). Summarising our findings, the technological-theoretical discourse is a mixture of higher mathematics ideas, engineering reasoning[13] and a principle concerning the interplay between real- and idealised signals reflecting the connection of mathematics and physics in general. The praxeological analysis can be found on pages 130ff of this thesis, an unpublished graphical representation of this analysis is given in Figure 7.1. In section 2.1 I noted that the interplay between real signals, which are in principle measurable and observable, and idealised signals, which are

[13] In Study I, we distinguished between higher mathematical ideas and engineering reasoning. In particular, we had not yet understood engineering reasoning as mathematical ET-discourse.

neither observable nor measurable, is important for the justification of a pragmatic introduction of the Dirac delta impulse. In Study IV we summarise

> But the central point in our argumentation is not that distribution theory is not used here as the basis for justification, which would correspond to a rather deficit-oriented view. Rather, our point is that the electrotechnical mathematics discourse, in its reference to empirical objects, not only allows for a justification of the step, but also establishes a reference of symbols and argumentation to empirical objects and contexts. A purely distribution-theoretical argumentation could not make this possible. (p. 336)

In Study IV we refer to the pragmatic theory of truth by Schlaudt (2014) and historic-philosophical studies by Wahsner and Borzeszkowski (1992) to reflect on the relationship of mathematics and empirical sciences, especially physics. Since physics and electrical engineering have important properties in common with regard to the following considerations, I consider the reference to physics appropriate at this point. According to Schlaudt (2014), empirical truth consists in the mastery of objective means to achieve subjective[14] (respectively intersubjective) ends (p. 160, translated from German by J.P.). A statement is thus considered true, or valid accepted knowledge, if it offers a successful action procedure. Schlaudt gives the example of a weight measurement. The statement "*y* weighs 5 kg" does not mean here that the numerical property 5 kg is assigned to the object *y* but provides information about the behaviour of *y*, i.e. about what effect the object *y* has, had and will have on this scale under standard conditions (calibrated, undisturbed scale). (p. 158/9). Schlaudt's philosophy of truth is thus well compatible with ATD[15]. In Study IV we write

> Mathematics abstracts from behaviour and focuses on the pure quantity as well as presupposes the existence of objects in an axiomatic system of relations. But empirical sciences cannot "forget" these constituents. Instead, they are inherently reflected in the practices by constituting, incorporating and framing specific mathematical practices. ... [In empirical sciences, J.P.], truth (valid knowledge) must always establish a reference beyond theory. Empirical sciences cannot be divided into an empirical (non-mathematical) part that regulates the relationship to reality and a mathematical part that is free of this relationship to reality. (p. 330/1)

Thus, on an epistemological level, it is at least made clear that the relationship between electrical engineering and mathematics cannot be understood as a naive application of mathematics (applicationism, cf. the work by Barquero, Bosch, and Gascón (2013)) or as separate worlds (modelling cycles, e.g. by Blum and Leiss (2007)).

From the ATD perspective, I base the understanding of this relationship on the acknowledgement of the mathematical practices of engineers as institutional mathematical practices in their own right and

[14] The relationship between objectivity and subjectivity in Schlaudt's theory of truth cannot be presented here. I would like to quote only one short sentence that may be suitable to counteract misunderstandings that this theory of truth is subjectivist: Truth depends "not on the skill of the individual, but on what is socially possible" (p. 159, translated from German by J.P.).

[15] Job and Schneider (2014) also refer to a pragmatic perspective, although differently than Schlaudt. They "envision the development of calculus as an epistemological transition between two types of praxeologies, pragmatic and deductive, a praxeology being an anthropological and epistemological model of knowledge" (p. 635). This also fits well with the presented perspectives I discussed at the introduction of the Dirac delta impulse, where limit and integration are interchanged at one step, see section 2.1.

with engineering-specific conceptualisations of mathematical knowledge. In our analyses (see Focus 1), we showed that the engineering-specificity of the mathematical practices in the Signal Transmission courses could be reconstructed as a mixture of mathematical practices related to academic mathematics (the HM discourse) and mathematical practices related to mathematics developed within engineering institutions (the ET discourse).

5.3. Focus 3: Revisiting the relationship between mathematics and electrical engineering

In research foci 1 and 2, the aim was to model mathematical practices in a subject-specific way and to relate their justifications also to societal influences of higher levels of codetermination. We have developed an understanding of the relationship between mathematics and electrical engineering that enables us to work out the engineering specificity of mathematical practices. I have interpreted our methodological developments as phenomenotechnique in the sense of Bachelard, see also footnote 8 on page 21. With the elaboration of new phenomena, one can now also ask corresponding didactic questions. In this third focus, I take this new understanding of the relationship, and the reconstructed two mathematical discourses as a starting point for exploring the possibilities for teaching design and lecturer support:

RQ4: How can institutional analyses of mathematical practices and an alternative conceptualisation of the relationship between mathematics and engineering contribute to teaching design and lecturer support?

In Study V, we made initial reflections on application and modelling, which are commonly used in studies of mathematics education in engineering to capture the relationship between mathematics and engineering, and to serve for the improvement of learning and for teaching design (Alpers, 2020). Our aim in this study is to make the epistemology of this relationship accessible for discussion. In summary, the standard approaches of modelling and application have the following problem: "Application and modelling both entail a conceptualisation of electrical engineering knowledge as consisting of inner-mathematically justified mathematical practices and extra-mathematical engineering knowledge" (Peters and Hochmuth, 2022, p. 123). In this understanding, the mathematical practices are learned in mathematics service courses and later applied in the engineering context. And the extra-mathematical engineering knowledge provides the application context or the source of the modelling problem. This characterisation is certainly an exaggeration, but I think it highlights the problem of the separation of "mathematics" from "the rest of the world", or the problem of the idea of an application of mathematical concepts that remain unchanged. In Study V we also refer to understandings of modelling (e.g. Bissell & Dillon, 2000) and application (e.g. Rønning, 2022; Schmidt & Winsløw, 2021) that differ from this problematic one. We also present an ATD based approach to modelling that takes an epistemological and institutional point of view and can be connected to the ATD notion of study and research paths (e.g. Bartolomé, Florensa, Bosch, & Gascón, 2019; Chevallard, 2006; Florensa Ferrando, 2018). Study and research paths focus on a crucial initial question that guides the learning process and have the potential to question the the content specific institutional rationales.

The considerations in Study V make general epistemological backgrounds explicit and thus accessible for discussions in the context of teaching design and lecturer support. In Study I and Study IV, we also worked out what it means for electrical engineering mathematical practices to be pragmatic. Pragmatic does not necessarily mean that the mathematical practices cannot be represented as adequately as would be necessary from a academic mathematics point of view, but due to the time constraints and conditions of engineering study programmes, one simply has to deviate from this. Pragmatic, as we have worked it out, means that the mathematical practices of electrical engineering have been developed as historic-specific manifestation in societal institutionalised teaching contexts with a view to solve epistemological problems specific to engineering. In their pragmatics, they fulfil subject-specific purposes. These subject-specific purposes can then be addressed explicitly in the teaching. Both, mathematics and electrical engineering (or physics) have a relevant part in solving these epistemological issues. This represents something to be learned, and references to historical-philosophical work can help to highlight these aspects, for example as part of logos blocks of praxeologies, and thus make them accessible for teaching. This could, for example, also enrich the search for rich and fruitful initial questions for SRPs. These aspects are obscured if the relationship between electrical engineering and mathematics is only understood in terms of (naive) application or modelling.

In Study VI, I take our findings on the two mathematical discourses as a starting point and show, with the formulation of a concrete example, how tasks from the mathematics service course can be modified to potentially promote connections between mathematics and electrical engineering. The core idea of this approach is to bring aspects of the mathematical ET-discourse into the mathematical HM-discourse without adopting the engineering context. In the worked example I show this with the topic complex numbers. Here the two mathematical discourses differ, for example, in the raisons d'être for complex numbers. In the mathematics service course, complex numbers allow for generalisation, they are useful to solve equations, and they are formal objects of calculation. Phasor representations serve to illustrate calculation rules, and the Euler equation $e^{i\varphi} = \cos(\varphi) + \sin(\varphi)i$ is introduced as a convenient abbreviation (pointwise). In electrical engineering complex numbers are, among other things, central to describe periodic signals. This aspect is not present in the HM-discourse, but can be introduced by using complex numbers to describe closed curves. In Study VI, an existing task on ellipses from the service course, which was originally only intended to train students to transform an unusual expression into a known formula, is modified to enable students to explore this new, engineering related, aspects of complex numbers. This modified exercise contains, in particular, rotational aspects of complex numbers that have not been covered in the mathematics course so far. An important aspect that is also central to this exercise modification, but cannot be elaborated on here, is that this new exercise is still purely an inner-mathematical exercise without any electrical engineering context.

In addition to the exercise modification proposal in Study VI, we also made suggestions in Study II and Study III on how institutional analyses could be fruitfully used in teaching support and exercise design: The analysis of the two exercises in Study II has shown that the two mathematical discourses can be reconstructed in lecturer sample solutions. In particular, the graphical representation of the analysis results makes their intertwining explicit. This can reveal discourse switches and associated passages in the solution process that may represent structural obstacles. These can be addressed explicitly in class. If students have difficulties with these structural hurdles in their solution process, this can be explicitly addressed. Difficulties with techniques, for example, can also be considered with the justifying and explaining technologies and located in the respective discourse. In Study III we worked out methodological steps that enabled us to use the institutional analysis results to explain students solutions. Our use of the graphical representation of our analysis results as a schema for the analysis of student solutions could, in a simplified form, also be used by lecturers to give individual feedback.

Epilogue: Sometimes mathematics is different - Some perspectives

At the beginning (p. 7), I formulated the general research interest: *How can we understand the mathematical practices in electrical engineering courses at University while avoiding a deficit-oriented perspective from the point of view of academic mathematics?* This has now been discussed with regard to the three research foci. Our aim to avoid a deficit-oriented perspective from the point of view of academic mathematics allowed us to shine a light on the pragmatic role of mathematical practices in engineering and the intermingling of the two mathematical discourses. Our understanding of pragmatic does not mean that every practice that deviates from academic mathematical rigour is simply justified as pragmatic. Not every vague step in a textbook is clear from the perspective of the ET-discourse, not every "wrong" transformation from the perspective of the HM-discourse can be explained from the ET-discourse. The means must be institutionally recognised, as must the ends. The recognising institution here is not necessarily, naturally, and unquestioningly academic mathematics. This enables us to ask questions like: are identified deviations or vague steps justified mathematical practices in electrical engineering? If so, how do these differ from mathematical practices taught in the mathematics service courses? Could these differences be made fruitful for teaching? But we can also ask the same questions to practices taught in the mathematics service courses. Are they actually also meaningful practices in terms of electrical engineering means and ends? Praxeological analyses of higher level electrical engineering courses can also identify HM-techniques and logos aspects that are particularly relevant. This does not mean that logos aspects that do not appear should be dropped from the mathematics service courses. The HM-discourse is characterised by a rather close relationship to academic mathematics and this should be maintained[1]. The justificatory and argumentative power of mathematical proofs, rigour and other logos aspects is still important. But the question can now be, which aspects of praxeologies one wants to keep, even if they no longer occur later, and what didactic purpose this serves. And which ones should be dropped in favour for other things.

To continue the idea from Study VI, one would now need to identify engineering raisons d'etre and other ET-discourse aspects that could be integrated into mathematics service courses in a compatible way, analogous to the example in Study VI. After the very focused analyses in the studies collected in this thesis, this would now have to be pursued more broadly across all topics from the mathematics service courses.

[1] Of course, this position can also be questioned. I want to present a short argument why maintaining this orientation towards academic mathematics might be desirable: The first one is connected to Fettweis' dilemma. Referring to mathematical rigour had an important function and, besides other, enabled him to deviate from it when introducing the Dirac delta impulse. References to distribution theory for a mathematically sound introduction require students to trust the mathematical rigour without checking it themselves. Experiences with mathematical rigour and the power of proof in other, more accessible topics can help to build and foster this trust. A similar argument, but focussing the societal need for trust, is given by Bissell and Dillon (2000): "One reason arises from seeing the operation of communities of professional practice in terms of trust (Porter, 1996). Engineers, scientists, doctors – professional groupings – need to establish a tradition of trust within wider society – if only because technical matters cannot be resolved elsewhere." (p. 10).

Since mathematics service courses usually have students from different engineering study programmes, potential engineering discourses from other engineering disciplines would also have to be considered. Studies from the engineering mathematics education community can serve as fruitful reference points for interesting topics and logos aspects. I think the methodological approaches developed in our studies could also be fruitful for the design of study and research paths or projects as presented by Rønning (2022).

Small-scale exercise redesign approaches as presented in Study VI can become alternatives to more complex course design approaches that are always in danger of coming into conflict with rigid institutional conditions[2], see also the considerations by Barquero, Bosch, and Gascón (2013). A distinctive feature of the approach presented in Study VI is "that identified important aspects from engineering mathematical practices are brought into the mathematics service course without also introducing the engineering context" (p. 583). It would be interesting to study if this this approach allows to promote meaningful connections between engineering and mathematics service courses, while maintaining the division of labour mentioned by Rønning (2022). The idea of division of labour is strongly related to the institutional relativity of knowledge and, on the organisational level also to the institutional separation of mathematics service courses and engineering courses. Therefore, helping to maintain division of labour has the potential to consolidate existing institutional structures and can thus also prevent changes. On the other hand, institutional conditions of this magnitude are difficult, if not impossible, to change. Therefore, I see this more as an extension of action possibilities under existing conditions. Subject science (Schraube & Osterkamp, 2013), which is well compatible with ATD (Hochmuth, 2018; Hochmuth & Schreiber, 2015), provides a very well elaborated framework to explore such questions about lecturers' action possibilities in addition to subject-specific analyses.

In addition to research projects, our work also offers possibilities for the development of professional development for lecturers. We have already worked in this direction: Ruge and Peters (2021) developed an understanding of professional development that is based on subject science, which, among other things, adopts a view of professional development that goes beyond the derivation of practical and applicable tools from research. The approaches presented in this thesis do not provide directly applicable tools either. Instead, they show how there is potential for lecturer development in the process of analysing the respective institutional mathematical discourses and reflecting on the institutional situatedness of mathematical practices. As part of the European project PLATINUM [3], which aims to develop inquiry-based mathematics education (IBME), we have developed a professional development workshop for lecturers (I. Gómez-Chacón, Hochmuth, Rogovchenko, & Brouwer, 2021). A tool in the form of a table was designed to facilitate reflections on IBME aspects of exercises with a view to redesign. Among the characteristics of IBME we suggested for reflection were "enabling discourses on techniques" and "enabling interdisciplinary knowledge linking" (p. 131). Together with our colleagues from Spain, who developed a similar workshop at the Complutense University in Madrid as part of the PLATINUM project, we cooperatively developed a follow-up workshop that was held in Madrid (I. M. Gómez-Chacón, Hochmuth, & Peters, 2022). The methodological developments presented in this thesis could serve to develop this further.

[2] See also the discussion in Study VI.

[3] https://platinum.uia.no/

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Part II.

Published Studies

6. The subject-specific reconstruction of mathematical practices

Study II: Praxeologische Analysen mathematischer Praktiken in der Signaltheorie

Study II has first been published as Peters, J., & Hochmuth, R. (2021). Praxeologische Analysen mathematischer Praktiken in der Signaltheorie. In R. Biehler, A. Eichler, R. Hochmuth, S. Rach, & N. Schaper (Eds.), *Lehrinnovationen in der Hochschulmathematik: praxisrelevant – didaktisch fundiert – forschungsbasiert* (pp. 109–139). Springer Spektrum. https://doi.org/10.1007/978-3-662-62854-6_6.

Study III: On the Analysis of Mathematical Practices in Signal Theory Courses

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Remarks on the figures

The illustrations in this publication are inconveniently placed. Figure 1 on page 240 is mentioned at the bottom of page 238 and would be best placed at the top of page 239. Figure 3 on page 241 belongs to the section "Step 1: Institutional Analysis" beginning on page 247. Figure 4 on page 242 also belongs to this section. Figures 5 to 11, which are all placed in the section "Step 1: Institutional Analysis", are results of the analyses of the students solutions and therefore belong to "Steps 2 to 4: Analyses of Students' Solutions". Figure 12, which is on page 253, is part of the "Discussion" on page 257 and Figure 13 is part of the original exercise sheet and belongs in the "Appendix" on page 258.

6 Praxeologische Analysen mathematischer Praktiken in der Signaltheorie

Jana Peters und Reinhard Hochmuth

Zusammenfassung

Im Fokus dieses Beitrags steht die Analyse mathematischer Praktiken, wie sie in der Signaltheorie eines Elektrotechnik-Studiengangs gelehrt werden. Den theoretischen Rahmen der Analyse bildet die Anthropologische Theorie der Didaktik (ATD). Im Sinne dieser werden die mathematischen Praktiken der Signaltheorie als institutionalisierte Verknüpfungen von Praktiken der Höheren Mathematik für Ingenieure, der Mathematik, wie sie in elektrotechnischen Grundvorlesungen entwickelt und verwendet wird, und spezifischen signaltheoretischen Inhalten verstanden. Dabei unterscheiden wir zwei Mathematikdiskurse, einen Höhere-Mathematik- und einen elektrotechnischen Mathematik-Diskurs. Auf der Basis eines entsprechend erweiterten praxeologischen 4T-Modells rekonstruieren wir im Folgenden exemplarisch an zwei signaltheoretischen Aufgaben die jeweiligen Diskursaspekte sowie deren Verknüpfungen und stellen diese Ergebnisse grafisch dar. Die beiden Beispiele zeigen, dass das erweiterte praxeologische Modell geeignet ist, um aufgabenbezogen potenzielle, mit der Verknüpfung der analytisch unterschiedenen Diskurse verbundene Hürden bei studentischen Aufgabenbearbeitungen zu identifizieren und fachbezogene Anregungen für die Lehrpraxis zu generieren.

J. Peters (✉) · R. Hochmuth
Leibniz Universität Hannover, Hannover, Deutschland
E-Mail: peters@idmp.uni-hannover.de

R. Hochmuth
E-Mail: hochmuth@idmp.uni-hannover.de

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6.1 Einleitung

Die Mathematik in Lehrveranstaltungen zur Signaltheorie ist unter anderem dadurch gekennzeichnet, dass sie Praktiken der Höheren Mathematik für Ingenieure, der Mathematik, wie sie in elektrotechnischen Grundvorlesungen entwickelt und verwendet wird, und spezifisch signaltheoretische Inhalte verknüpft. Diese Feststellung als solche bedarf zu ihrer Begründung keiner eigenen Forschung, da sie sich aus Prüfungsordnungen und Modulbeschreibungen sowie der darin vorgenommenen zeitlichen und inhaltlichen Verortung von Lehrveranstaltungen bzw. deren Inhalten ergibt. Aber auch ein Blick in einschlägige Literatur zur Signaltheorie (etwa Fettweis 1996; Frey und Bossert 2009) offenbart auf den ersten Blick die genannten Bezüge. Schließlich scheint es auch einen gewissen Konsens darüber zu geben, die Bezüge und Verknüpfungen als „pragmatisch" zu betrachten und in den Zusammenhang von Praxisbezogenheit (etwa im Sinne von Anforderungen aus elektrotechnischen Anwendungen) zu stellen (vgl. z. B. Rach et al. 2014). Wie die Verknüpfungen allerdings im Detail aussehen, welche Überlegungen, Vorstellungen, Praktiken jeweils einfach übernommen, neu eingeführt, ggf. modifiziert usw. werden, erschließt sich nicht unmittelbar.

In dieser Arbeit werfen wir nun einen genaueren Blick auf die Verknüpfungen der verschiedenen mathematikbezogenen Praktiken. Dabei geht es uns insbesondere darum, den die mathematischen Praktiken in der Signaltheorie rechtfertigenden Diskurs zu rekonstruieren, wobei ein eher defizitorientierter Blick aus Sicht der Universitätsmathematik vermieden werden soll. Dieser bestünde etwa darin, einerseits anzumerken, dass gewisse Techniken oder Aussagen der Universitätsmathematik (etwa aus der Fourier-Analysis oder der Distributionentheorie) in elektrotechnischen Signaltheorie-Lehrveranstaltungen nicht den universitätsmathematischen Normen genügend verwendet werden, und andererseits nicht im Detail nach fachbezogenen Gründen und Rechtfertigungen für die spezifischen Abweichungen sowie nach deren (ggf. ingenieurwissenschaftlichem) Mehrwert zu fragen. Wir gehen diesbezüglich zum einen davon aus, dass dem spezifischen Diskurs in der Signaltheorie die Lösung gewisser Aufgaben zukommt, und zum anderen, dass die häufig in Lehre und Literatur nicht explizit gemachte Verknüpfung von verschiedenen fachlichen Orientierungen gehorchenden Praktiken den Studierenden beim Einstieg in die Signaltheorie potenziell Probleme bereitet.

Unseres Erachtens stellen ein adäquater Umgang mit den verschiedenen fachlich-institutionellen Orientierungen und deren spezifische signaltheoretische Integration ein wichtiges Lernziel signaltheoretischer Lehrveranstaltungen dar. Die gegebenenfalls auftretenden Probleme von Studierenden sind also auch als darauf bezogene Lerngelegenheiten zu verstehen. Probleme treten insbesondere beim Bearbeiten von Übungsaufgaben auf, wenn etwa nicht klar ist, welche Argumente gerade eben zulässig sind oder nicht, aber auch bei der Einführung neuer Begriffe oder Objekte, wie etwa dem Dirac-Impuls. Hochmuth und Peters (2020) fokussierten auf diesen zweiten Problembereich und rekonstruierten zentrale Aspekte der Verknüpfung eines elektrotechnischen und eines HM-bezogenen Mathematik-Diskurses im Kontext der Einführung des

Dirac-Impulses in der Signaltheorie. Deren Rekonstruktion erforderte es, insbesondere auch epistemologisch-philosophische Vorstellungen bezüglich des Verhältnisses von Mathematik und Ingenieurwissenschaften einzubeziehen. Damit konnte das, was allgemein als „pragmatische" ingenieurwissenschaftliche Verwendung von Mathematik umschrieben wird, an diesem Fall praxeologisch charakterisiert werden. Unsere Analysen wiesen ebenfalls darauf hin, dass ein Verständnis darüber, was es bedeutet, dass eine mathematische Praxis in der Elektrotechnik „pragmatisch" ist, über die Auffassung, dass Mathematik in der Elektrotechnik lediglich angewendet wird[1], hinausgehen muss. Die reine Anwendungsinterpretation schien uns prinzipiell mit einer mehr oder weniger expliziten defizitorientierten Sicht auf mathematische Praxen in der Elektrotechnik verbunden. Stattdessen konnten wir epistemologische Probleme aufzeigen, die die Mathematik nicht lösen kann, sondern erst deren geeigneter Einbau in den „pragmatischen" Signaltheorie-Diskurs.

In dieser Arbeit fokussieren wir auf den ersten Problembereich, also potenzielle Probleme beim Bearbeiten von Übungsaufgaben. Wir werden anhand der Analyse zweier Beispielaufgaben aus der Signaltheorie zeigen, dass ein spezifisch erweitertes praxeologisches 4T-Modell der Anthropologischen Theorie der Didaktik (ATD) geeignet ist, um aufgabenspezifische Verknüpfungen eines elektrotechnischen und eines HM-bezogenen Mathematik-Diskurses zu rekonstruieren und ihre jeweils spezifischen Beziehungen darzustellen. Dabei beziehen wir uns unter anderem auf Vorarbeiten von Castela (2015). Wir adressieren dabei explizit zwei analytisch voneinander getrennte Mathematikdiskurse, die wir entsprechend ihren institutionellen Bezügen *Höhere-Mathematik-Diskurs* (HM) und *elektrotechnischer Mathematik-Diskurs* (ET) nennen. Zur genaueren Charakterisierung der Diskurse vergleiche Abschn. 6.2.1 und Abschn. 6.4. Dabei unterscheidet sich unser Zugang insbesondere von Ansätzen, die einen Mathematikdiskurs von einem (unmathematischen) Elektrotechnikdiskurs abgrenzen und die Verknüpfung zwischen Mathematik und unmathematischer Elektrotechnik untersuchen. Darüber hinaus werden Konstrukte wie der Modellierungskreislauf (z. B. Blum und Leiß 2007) von uns nicht verfolgt, da diese unter anderem nicht geeignet sind, die den mathematischen Praktiken in der Signaltheorie zugrunde liegenden komplexen Wechselbeziehungen zwischen Mathematik und Elektrotechnik zu erfassen, und damit insbesondere epistemologisch problematisch sind. Die von Biehler et al. (2015) vorgeschlagene Modifizierung des Modellierungskreislaufs für die Analyse bestimmter mathematikhaltiger Aufgaben aus elektrotechnischen Grundveranstaltungen erscheint uns für die komplexeren Aufgaben aus der Signaltheorie nicht ausreichend, da sich auch hier die von uns adressierten Beziehungen zwischen den verschiedenen mathematischen Praktiken und Orientierungen nicht adäquat abbilden lassen, diese aber unseres Erachtens ein nicht zu vernachlässigendes Charakteristikum der in diesem Beitrag untersuchten Aufgaben darstellen. Anschlussfähig scheinen unsere Überlegungen insbesondere an

[1]Vergleiche dazu auch die Arbeit von Barquero et al. (2011) zum *applicationism*.

den im Kompetenzrahmen des SEFI-Netzwerks (Alpers et al. 2013) formulierten Standpunkt zur Modellierung zu sein. Dort wird die Wahl des jeweils adäquaten Modells, das selbst schon immer eine Mischung aus Mathematik und mathematisch repräsentierter Ingenieurwissenschaft darstellt, hervorgehoben.

Wir werden zeigen, dass das im Folgenden von uns eingeführte praxeologische Modell ein für die Forschungs- und Lehrpraxis geeignetes Werkzeug darstellt, um in Aufgaben Beziehungen zwischen den verschiedenen mathematischen Praktiken mit Blick auf spezifische, im gewissen Sinne institutionelle Hürden bei deren Bearbeitung zu identifizieren. Zusätzlich machen wir einen Vorschlag zur grafischen Darstellung der Praxeologien und ihrer Blöcke. Die textförmige Darstellung der verschachtelten Struktur der verschiedenen mathematischen Diskurse stößt unseres Erachtens an Grenzen der Verständlichkeit und Handhabbarkeit. Die vorgeschlagene Art der Darstellung hebt die Verschlingung der verschiedenen mathematischen Diskurse, deren Übergänge bzw. das jeweilige Zueinander von Techniken und der darauf bezogenen Technologien für weitere, das Lernen der Studierenden und die Lehre betreffende Überlegungen hervor. Das eröffnet unter anderem erweiterte Möglichkeiten des Feedbacks an Studierende und für explizierende Bemerkungen in Vorlesungen und Tutorien.

Im Folgenden wird nun zunächst der theoretische Rahmen der ATD und dabei insbesondere unsere Ausdifferenzierung des praxeologischen 4T-Modells vorgestellt. Hierbei gehen wir auch auf Vorarbeiten (Castela 2015; Castela und Romo Vázquez 2011; Romo Vázquez 2009) ein. Nach einer fachlichen Einbettung der Aufgaben und einer Charakterisierung des elektrotechnischen Mathematik-Diskurses stellen wir die Analyse der Übungsaufgaben vor. Anschließend diskutieren wir unsere Ergebnisse und gehen dabei insbesondere auf unseren Vorschlag, praxeologische Analysen grafisch darzustellen, ein. Hier skizzieren wir schließlich einige für die Lehrpraxis relevante Aspekte.

6.2 Die Anthropologische Theorie der Didaktik und das erweiterte praxeologische Modell

Die Anthropologische Theorie der Didaktik (ATD) (Chevallard 1992; Bosch und Gascón 2014) steht in einer französischen Forschungstradition und hat ihre Ursprünge in der Theorie didaktischer Situationen, die hauptsächlich in den 1970er- und 1980er-Jahren entwickelt wurde (Brousseau 2002), und in der Theorie didaktischer Transpositionen (Chevallard 1985). Aus diesen Bezügen ergibt sich das allgemeine Didaktikverständnis der ATD. Beispielsweise schreiben Bosch und Gascón 2014:

> In the framework proposed by ATD, the institutional dimension of mathematical and didactic activities becomes much more explicit. Doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, are considered as human activities taking place in institutional settings. The science of *didactics* is thus concerned with the conditions governing these knowledge activities in society, as well as the restrictions hindering their development among social institutions. (p. 68).

Darauf beziehen sich die für unseren Beitrag zentralen theoretischen Konzepte der ATD: der institutionelle Standpunkt der ATD, nach dem menschliche Aktivitäten, wie beispielsweise das Mathematik-Betreiben, immer in Institutionen verortet ist und institutionelle Bedingungen bestimmen, welche Handlungen und Begründungen als adäquat gelten; die Konzeption von Wissen als Praxeologien, die abhängig von den je gegebenen institutionellen Bestimmungen existieren, und das Konzept der (didaktischen) Transposition, das es erlaubt, dynamische Aspekte wie Entwicklung, Veränderung und Verbreitung von Wissen über verschiedene Institutionen hinweg zu untersuchen.

Konzepte der ATD wurden bereits in verschiedensten Kontexten der mathematischen Hochschuldidaktik fruchtbar gemacht. Hervorzuheben sind insbesondere Arbeiten zu Problemen im Übergang Schule – Hochschule und zu mathematikbezogenen Übergängen innerhalb des Universitätsstudiums (Bosch 2014; Winsløw et al. 2014, 2018). Zur Rolle der Mathematik in den Ingenieurwissenschaften wären neben den bereits erwähnten Arbeiten von Castela und Romo Vázquez auch die Arbeiten von González-Martín und Hernandes-Gomes (2018, 2019) zu nennen. Letztere adressieren insbesondere die Frage der Passung von Praktiken bezüglich Aspekten des Integralbegriffs und der Integralverwendung in Calculus- und Mechanik-Lehrveranstaltungen. Diese im Wesentlichen curricularen Unterschiede wurden auch von Dammann (2016) beobachtet und untersucht. Ähnliche Differenzen bezüglich grundlegender Begriffe und deren Verwendung lassen sich auch in der Elektrotechnik im Kontext von theorieorientierten Grundlagenveranstaltungen finden, beispielsweise im Umfeld des Integralsatzes von Gauß (vgl. z. B. Henning et al. 2015). Unsere Analysen sind im Unterschied dazu auf Phänomene der Verknüpfung, der Integration und der jeweiligen Spezifik rechtfertigender Diskurse innerhalb einer fortgeschrittenen Lehrveranstaltung der Elektrotechnik gerichtet.

Nachdem im Folgenden der theoretische Rahmen – und dabei insbesondere das 4T-Modell – insoweit erläutert wird, wie es für das Verständnis unserer Aufgabenanalysen notwendig erscheint, gehen wir anschließend auf unseren Vorschlag zur Modifizierung des 4 T-Modells ein.

Mathematisches Wissen wird im Rahmen der ATD handlungstheoretisch aufgefasst und beinhaltet nicht nur Aspekte des „Know-why“, sondern auch praktisches Wissen im Sinne eines „Know-how“. Methodisch wird dies mittels des Konzepts der Praxeologie gefasst. Chevallard (2006) schreibt zum Begriff der Praxeologie:

> What exactly is a praxeology? [...] one can analyse any human doing into two main, interrelated components: praxis, i.e. the practical part, on the one hand, and logos, on the other hand. [Logos bezieht sich auf menschliches Denken, rationalen Diskurs, die Autoren]. How are P [Praxis, die Autoren] and L [Logos, die Autoren] interrelated within the praxeology [P/L], and how do they affect one another? The answer draws on one of the fundamental principle of ATD [...] according to which no human action can exist without being, at least partially, ‘explained’, made ‘intelligible’, ‘justified’, ‘accounted for’, in whatever style of ‘reasoning’ such as an explanation or justification may be cast. Praxis thus entails logos which in turn backs up praxis. For praxis needs support – just because, in the long run, no human doing goes unquestioned.[...] Following the French anthropologist Marcel Mauss (1872–1950), I will say that a praxeology is a ‘social idiosyncrasy’, that is, an organised way of doing and thinking contrived in a given society. (Chevallard 2006, S. 23, Hervorhebungen im Original)

Eine Praxeologie besteht also aus zwei zusammenhängenden, aufeinander bezogenen Blöcken: Der Praxisblock („Know-how“) besteht aus Aufgabentypen T und einer Reihe von relevanten zugehörigen Techniken τ zur Lösung der Aufgaben. Der Logosblock („Know-why“) wird durch die zwei Ebenen eines Begründungsdiskurses gebildet: Auf der ersten Ebene werden die Techniken des Praxisblocks durch Technologien θ unter anderem erklärt, gerechtfertigt, motiviert und begründet. Auf der zweiten Ebene organisiert und ordnet die Theorie Θ ihrerseits die Technologien. Insgesamt kann eine Praxeologie als 4 T-Modell dargestellt werden: $[T, \tau, \theta, \Theta]$.

Eine Kernposition der ATD ist, dass Praxeologien immer in Abhängigkeit spezifischer Institutionen existieren. Das Verständnis von Institution innerhalb der ATD geht deutlich über bürokratische Einrichtungen wie Schule, Universität, Gerichte usw. hinaus und lehnt sich an das von Douglas (1991) ausgearbeitete Verständnis an. Chevallard (2019) fasst darunter explizit (soziale) Entitäten und Strukturen, die eine gewisse formative Funktion erfüllen. Diese institutionelle Abhängigkeit bedeutet, dass in unterschiedlichen Institutionen je andere Aufgabentypen relevant, andere Lösungstechniken adäquat und andere Begründungsdiskurse akzeptabel sind. Fokussiert man also ein spezifisches Element mathematischen Wissens in unterschiedlichen Institutionen, ergeben sich unterschiedliche Praxeologien. Didaktische Fragestellungen sind zunächst auf dieser institutionellen Ebene angelegt, wobei der Fokus entsprechend jenseits individueller Merkmale und Eigenschaften der handelnden Menschen liegt. Damit einher geht ein Subjektverständnis als generisches Subjekt, das unter institutionellen Bedingungen stehend verstanden wird. Bosch (2015) charakterisiert das Subjektverständnis der ATD wie folgt:

> An institution lives through its actors, that is, the persons that are subjected to it – its subjects – and serve it, consciously or unconsciously. […] Freedom of people results from the power conferred by their institutional subjections, together with the capacity of choosing to play such or such subjection against a given institutional yoke. (Chevallard 2005, zitiert nach Bosch 2015, S. 52)

Dabei wird die Unterwerfung unter institutionelle Bedingungen nicht repressiv, sondern vor allem produktiv und konstitutiv verstanden. Diese institutionelle Abhängigkeit von Wissen reflektiert sich in unserer Erweiterung des praxeologischen Modells.

Während es Praxeologien erlauben, mathematisches Wissen in seiner institutionellen Konzeption eher statisch zu fassen, bietet die ATD mit dem Konzept der (didaktischen) Transposition die Möglichkeit, dynamische Aspekte der Produktion, Entwicklung, Veränderung und Verbreitung von Wissen zwischen Institutionen zu untersuchen und beschreiben. Grundlegend ist dabei der Gedanke, dass die Analyse von in Lehr-Lern-Kontexten relevanten Wissenselementen diese Prozesse berücksichtigen sollte. Das Basismodell des didaktischen Transpositionsprozesses geht dabei von einer Unterscheidung zwischen drei relevanten Institutionen aus: Zunächst wird das *scholarly mathematical knowledge* von Mathematikern oder anderen Experten in Universitäten oder Forschungsinstituten produziert. Das *mathematical knowledge to be taught* wird über offizielle Curricula festgelegt. In diesem Prozess sind Politiker, Wissenschaftler, Pädagogen und andere Mitglieder der *noosphere*[2] beteiligt. Daraus wird schließlich das *taught knowledge,* das sich wiederum über einen didaktischen Transpositionsprozess aus den curricularen Dokumenten ergibt (Bosch und Gascón 2006). Der Übergang vom scholarly mathematical knowledge zum knowledge to be taught wird als externe didaktische Transposition bezeichnet. Im Bereich hochschuldidaktischer Forschung sind hier Fragen der Organisation des Wissens in Module, Vorlesungen und in Form von Syllaby relevant. Der weitere Übergang zum taught knowledge wird interne didaktische Transposition genannt (vgl. Bosch et al. 2021).

Unser Beitrag berücksichtigt im Wesentlichen Aspekte interner didaktischer Transpositionen. Corine Castela und Avenilde Romo Vázquez rekurrieren in ihren Arbeiten (Castela 2015; Castela und Romo Vázquez 2011; Romo Vázquez 2009) insbesondere auf externe transpositive Effekte bzgl. der Produktion und Legitimation mathematischen Wissens, die beim Übergang von wissenschaftlicher Mathematik in berufsbezogene Domänen relevant sind. Dabei differenzieren sie das Modell des didaktischen Transpositionsprozesses im Hinblick auf unterschiedliche institutionelle Einflüsse und ihre Beziehungen untereinander aus und erweitern dabei schließlich das praxeologische 4 T-Modell, um diesen verschiedenen Einflüssen auf der Ebene praxeologischer Wissenselemente gerecht werden zu können[3]: Im Kontext der untersuchten Kurse differenzieren sie eine theoretische und eine praktische Komponente der Technologie. Diese Unterscheidung dient dort insbesondere dem Untersuchungsziel, bezüglich höherer Ebenen der Kodetermination die institutionelle Relativität technologischer Diskurse verschiedener Kurse zu rekonstruieren. Der Fokus liegt in den

[2]Mit *noosphere* wird in der ATD „[…] the sphere of those who ‘think’ (*noos*) about teaching-, its relationship to ‘scholarly knowledge’ which usually legitimates its introduction in educational institutions, and the specific form it takes when arriving in the classroom […].” (Bosch und Gascón 2014, S. 71) bezeichnet. Die *noosphere* umfasst alle Agenten, die am Prozess der didaktischen Transposition vom *scholarly mathematical knowledge* zum *knowledge to be taught* beteiligt sind. In diesem umfassenden Begriff drückt sich auch der Umstand aus, dass die an diesem Transpositionsprozess beteiligten Agenten und die zugehörigen historischen und institutionellen Bedingungen nicht immer einfach zu erkennen sind.

[3]Im Rahmen ihrer Untersuchungen bezieht sich Castela (2015) auf in Arbeiten mit Romo Vázquez rekonstruierte Funktionen der Technologie, nämlich Beschreiben, Motivieren, Fördern, Validieren, Erklären, Bewerten und Kontrollieren: „Drawing on the aforementioned textbooks, Romo Vázquez and I have differentiated six of them: *describing* the technique, *validating* it i.e. proving that this technique produces what is expected from it, *explaining* the reasons why this technique is efficient (knowing about causes), *motivating* the different gestures of the technique (knowing about objectives), *making* it *easier* to use the technique and *appraising* it (with regard to the field of efficiency, to the using comfort, relatively to other available techniques). […] This list should not be taken as exhaustive. For instance, […] I currently consider one more need: *controlling* the technique implementation.“ (S. 11) Wir schließen uns diesem erweiterten Verständnis der Funktionen von Technologie an.

Arbeiten von Castela und Romo Vázquez also unter anderem auch auf dem Nachspüren der Wirkung der komplexen äußeren didaktischen Transformation in verschiedenen Institutionalisierungen didaktischer Transformationen.

6.2.1 Das erweiterte praxeologische Modell

Wir stimmen mit der Position von Castela (2015) darin überein, dass eine Ausdifferenzierung des praxeologischen Modells dazu dienen kann, mathematische Aspekte menschlicher Aktivitäten in unterschiedlichen Kontexten zu untersuchen, ohne sich dabei im Wesentlichen ausschließlich auf die akademische Mathematik und deren spezifische Normen zu beziehen[4].

> When someone of this world [der akademischen Mathematik, die Autoren], that is, a mathematician, begins to investigate on mathematics education, especially but not only in vocational education, he needs tools to distance himself with the '*alma mater*'. Since the beginning, this has been Chevallard's objective with the anthropological theory of the didactic. I contend that the work I have presented here around the notion of praxeology provides a powerful tool to investigate the mathematics dimension of human social activities in any context, without referring to academic mathematics. […] This anthropology of the mathematics should investigate social practices without too narrow restrictions on what is an interesting object. […] It highlights dimensions of the institutional cognition that would be neglected otherwise, especially when the reference to acknowledged mathematics is too strong. Such a research program is directed towards epistemological and anthropological goals, intending to unearth the diversity of human mathematics praxeologies. (Castela 2015, S. 18, Hervorhebungen im Original)

Dies kann dazu beitragen, einer gegebenenfalls vor allem defizitorientierten Sicht auf mathematische Praktiken in der Elektrotechnik entgegenzuwirken. Als defizitär kann u. a. ein nicht vorhandener oder aus mathematischer Sicht nicht hinreichend ausgeführter Nachvollzug innermathematischer Begründungen von Techniken und deren ersatzweise nicht innermathematische Rechtfertigung interpretiert werden (siehe dazu auch unsere Bemerkungen in der Einleitung). Dabei ist unsere Fragestellung mit der von Castela und Romo Vázquez verwandt, aber doch verschieden. Wir untersuchen Aufgaben und zugehörige Dozenten-Musterlösungen aus einer Perspektive der inneren Strukturierung eines Elektrotechnik-Studiengangs an der Universität Kassel und der verschiedenen gelehrten Wissenselemente zum Zeitpunkt eines bestimmten Kurses. Im Kontext des Kurses „Signale und Systeme" (SST) wird ein mathematischer Diskurs identifiziert, in den Aspekte einer vorgängigen Höheren Mathematik, ggf. vorgängige einführende elektrotechnische und neue signaltheoretische Aspekte eingehen. Gestellte Aufgaben

[4] Wir verstehen dies insbesondere auch als eine zum epistemologischen Referenzmodell (siehe z. B. Bosch 2015) alternative Methode der Distanzierung vom eigenen institutionellen Standpunkt.

erfordern bei der Bearbeitung geeignete Wahlen und Anwendungen entsprechender Techniken und Technologien. Relevant ist hier unserer Ansicht nach insbesondere die unterschiedliche epistemologische Verfasstheit des mathematischen Wissens bezogen auf unterschiedliche Institutionen: das einer Institution HM zuordenbare Wissen auf der einen und das mathematische Wissen elektrotechnischer Institutionen, z. B. Lehrveranstaltungen wie „Grundlagen der Elektrotechnik“, „Signale und Systeme“ usw. auf der anderen Seite. Wir fassen diese beiden unterschiedlichen Arten von Mathematik als zwei verschiedene mathematische Diskurse, jeweils in Bezug auf die entsprechenden Institutionen, auf: einen auf die Höhere Mathematik bezogenen Diskurs (HM-Diskurs) und einen auf elektrotechnische Lehrveranstaltungen bezogenen elektrotechnischen Mathematik-Diskurs (ET-Diskurs). Der HM-Diskurs, wie er aus bisherigen Aufgabenanalysen von uns rekonstruiert wurde, zeichnet sich durch eine innermathematische Konzeption der Begriffe und Aussagen ohne konkrete Realitätsbezüge, eine Konzentration auf Rechenregeln und den Einbezug schulmathematischer Begriffe aus[5]. Im Gegensatz dazu weist der ET-Diskurs Realitätsbezüge auf. Darüber hinaus zeichnet er sich durch eine elektrotechnischtypische Art des Denkens und Sprechens über Mathematik und mathematische Praxen aus. Eine konkretere Charakterisierung des ET-Diskurses geben wir nachfolgend in Abschn. 6.4 im Rahmen der fachlichen Einordnung der von uns analysierten Aufgaben.

Um das Verhältnis der beiden auf epistemologischer Ebene unterschiedlich konstituierten mathematischen Diskurse im Rahmen der Lehrveranstaltung SST näher herausarbeiten zu können, verwenden wir ein erweitertes praxeologisches Modell:

$$\left[T, \begin{matrix}\tau_{\mathrm{HM}} \\ \tau_{\mathrm{ET}}\end{matrix}, \begin{matrix}\theta_{\mathrm{HM}} \\ \theta_{\mathrm{ET}}\end{matrix}, \Theta\right]_{\mathrm{SST}}$$

Dabei geht es uns nicht um eine Erweiterung der von Chevallard entwickelten Theorie als solcher, sondern um eine spezifische innere Ausdifferenzierung des 4 T-Modells im Hinblick auf unseren Fokus. Um unsere Unterscheidung eines HM-Diskurses und eines ET-Diskurses im Modell repräsentieren zu können, differenzieren wir zwischen Techniken τ_{HM} und τ_{ET} sowie Technologien θ_{HM} und θ_{ET}. Als HM-Techniken und -Technologien charakterisieren wir dabei diejenigen mathematischen Praxen, die dem oben beschriebenen HM-Diskurs zuordenbar sind. Mathematische ET-Techniken τ_{ET} und -Technologien θ_{ET} werden auf Basis des ET-Diskurses zugeordnet. Diese analytische Ausdifferenzierung der Techniken und Technologien nach den beiden Diskursen findet gewissermaßen innerhalb des praxeologischen 4 T-Modells statt. Insgesamt entstehen

[5]Die zugrunde gelegte Lehrbuchliteratur für die Vorlesungen zur Höheren Mathematik ist (Strampp 2012, 2015; Strampp et al. 1997a, b). Die an historisch-philosophischen Arbeiten orientierten Überlegungen zur unterschiedlichen epistemologischen Verfasstheit von Mathematik und Physik im Kontext der Einführung des Dirac-Impulses in (Hochmuth und Schreiber 2015; Hochmuth und Peters 2018) sind hier ebenfalls anschlussfähig.

im Rahmen der „Signale- und Systeme“-Vorlesung SST-Praxeologien (daher der Index in der grafischen Repräsentation des 4 T-Modells), in denen die Techniken und Technologien beider Diskurse in naheliegender Weise auch zusammen gedacht werden können, nämlich gemeinsam in ihrer Verschlingung als SST-Techniken und -Technologien.

Im Rahmen unserer bisherigen Untersuchungen hat sich gezeigt, dass die einzelnen Elemente des erweiterten praxeologischen Modells in vielfältigen Bezügen zueinander stehen können. In Abschn. 6.5 konkretisieren wir diese Überlegungen anhand zweier Analysen von Dozenten-Musterlösungen und rekonstruieren die Zusammenhänge, in denen die praxeologischen Elemente der unterschiedlichen Diskurse miteinander stehen.

6.3 Rahmenbedingungen der Veranstaltung „Signale und Systeme“ im Sommersemester 2013 in Kassel

Die Vorlesung „Signale und Systeme“ bildet zusammen mit der Vorlesung „Digitale Kommunikation“ das Modul „Signalübertragung“. Die Prüfungsleistung des Moduls besteht in einer vierstündigen Klausur über beide Lehrveranstaltungen. Die Vorlesung „Signale und Systeme“ wird dreistündig gehalten, wobei Übungen nach Bedarf in die Vorlesung integriert werden. Formulierungen der Übungsaufgaben finden sich sowohl auf den Vorlesungsfolien (dort aber teilweise in leicht abweichender Darstellung) und auf Übungsblättern. Die Übungsaufgaben wurden von den Studierenden selbstständig bearbeitet, abgegeben und korrigiert. Musterlösungen wurden dann später im Rahmen der Veranstaltung vom Dozenten präsentiert. Die von uns analysierten Dozenten-Musterlösungen gehören zu Teilaufgaben der Aufgabe 4 (Vorlesung S. 93) des Übungsblattes „Aufgaben zur Vorlesung Signalübertragung am 3.6.2013“. Dabei handelt es sich um das zweite von insgesamt fünf im Rahmen der Vorlesung behandelten Übungsblättern.

6.4 Fachlicher Kontext der Aufgaben und Charakteristika des elektrotechnischen Mathematik-Diskurses

Wie bei der Darstellung unseres erweiterten praxeologischen Modells bereits ausgeführt, unterscheiden wir neben einem HM-Diskurs auch einen elektrotechnischen Mathematik-Diskurs, den ET-Diskurs. Nachdem in Abschn. 6.2 die Eigenschaften eines solchen ET-Diskurses nur angedeutet wurden, soll in diesem Kapitel nun mit der fachlichen Einbettung der Aufgaben auch eine Herausarbeitung der Charakteristika dieses ET-Diskurses erfolgen. Dabei führen wir die fachliche Einbettung so weit aus, wie sie unserer Ansicht nach für das Verständnis der Analysen notwendig ist.

Die beiden Begriffe Signal und System sind nicht nur namensgebend für die Vorlesung, sondern auch zentral sowohl für die fachliche Einbettung der von uns analysierten Dozenten-Musterlösungen als auch für die Beschreibung des ET-Diskurses.

Unter einem Signal versteht das Handbuch der Elektrotechnik (Plaßmann und Schulz 2009) „die physikalische Realisierung der Nachricht (*wie* es mitgeteilt wird)" (S. 919, Hervorhebungen im Original), das Lehrbuch von Fettweis (1996) unterschiedet zwischen realen Signalen[6], die physikalische Größen sind, und idealisierten Signalen[7], die zur numerischen Berechnung und als Messsignale dienen (S. 4 ff.)[8], und das Lehrbuch von Frey und Bossert (2009) versteht unter einem Signal „eine abstrakte Beschreibung einer veränderlichen Größe" (S. 1) und liefert als Definition: „Ein (zeit-)kontinuierliches Signal wird durch eine reelle oder komplexe Funktion $x(t) \in \mathbb{R}(\mathbb{C})$ einer reellen Veränderlichen $t \in \mathbb{R}$ dargestellt. Der Wertebereich ist $\mathbb{R}(\mathbb{C})$ und der Definitionsbereich ist $\mathbb{R}$" (S. 2).

Diese drei Auffassungen des Begriffs Signal unterscheiden sich in einem ansteigenden Grad an Formalisierung und Abstraktion, erhalten aber alle den Bezug zu wirklichen Phänomenen aufrecht. Auch Frey und Bossert, deren Lehrbuch sich hier durch den höchsten Grad an Formalisierung auszeichnet, sprechen von einer Darstellung bzw. einer Beschreibung durch die reelle oder komplexe Funktion. Damit sind zwei Charakteristika des ET-Diskurses herausgearbeitet: zum einen der Bezug zur Realität, zum anderen eine sehr unterschiedlich starke Explikation dieses Realitätsbezugs, die einhergeht mit einer unterschiedlich stark ausgeprägten Formalisierung. Fettweis (1996) formuliert das Dilemma,

> … daß mit zunehmender Ausfeilung der zugrundeliegenden mathematischen Zusammenhänge das Verständnis für die physikalische Begründung der gewählten Vorgehensweise immer schwieriger wird. Was also auf der einen Seite an mathematischer Strenge gewonnen wird, geht auf der anderen Seite wieder verloren, wenn es um die Einsicht in die tatsächliche Anwendbarkeit auf physikalische Gegebenheiten geht. (S. iii)

Nach Fettweis erfordert also das Verständnis physikalischer Begründungen ein Abweichen von der „mathematischen Strenge", die andererseits als zentrale Orientierung und wesentlicher Maßstab für die mathematischen Praktiken fungiert. Letzteres

[6]Diese treten bei der Nachrichtenübertragung auf, sind von endlicher Dauer, stetig und ausreichend differenzierbar. Allerdings sind sie auch sehr unregelmäßig und unvorhersehbar (andernfalls wäre der Informationsgehalt der Nachricht auch sehr gering) (siehe Fettweis 1996, S. 4 ff.).

[7]Diese verletzen einige Eigenschaften realer Signale, lassen sich durch wenige Parameter beschreiben, können aber zu Schwierigkeiten beispielsweise hinsichtlich Konvergenz führen (siehe Fettweis 1996, S. 6).

[8]Vergleiche insbesondere auch unsere Überlegungen in Hochmuth und Peters (2018) zum Verhältnis realer und idealisierter Signale.

impliziert eine gewissermaßen defizitorientierte Sicht auf mathematische Ingenieurpraktiken, da diese dem hervorgehobenen Maßstab letztlich nicht genügen. Unsere Auffassung von mathematischen Ingenieurpraktiken grenzt sich davon in der Hinsicht ab, dass wir einen eigenen elektrotechnischen Mathematik-Diskurs identifizieren und charakterisieren sowie in unseren Analysen dessen Relevanz für das Verständnis mathematischer Ingenieurpraktiken aufzeigen.

Der System-Begriff, der in unterschiedlichen Quellen ebenfalls verschieden stark formalisiert präsentiert wird, verweist noch auf ein weiteres Charakteristikum des elektrotechnischen Mathematik-Diskurses: Unter einem System verstehen Frey und Bossert (2009, S. 3) „allgemein eine abstrahierte Anordnung, die mehrere Signale zueinander in Beziehung setzt. Dies entspricht der Abbildung eines oder mehrerer Eingangssignale auf ein oder mehrere Ausgangssignale.“ Sie führen zunächst einen mathematisch leicht handhabbaren Systemtyp ein und betrachten nur jeweils ein Eingangs- und ein Ausgangssignal, da „daher der Systemgedanke leichter zu erfassen ist“ (S. 6). Ein System kann als eine Blackbox aufgefasst werden, die auf ein konkretes Eingangssignal mit einem konkreten Ausgangssignal reagiert. Systeme werden dann zum Beispiel anhand ihrer Antwort auf ein pulsförmiges Eingangssignal charakterisiert.

Dieses Input–Output- oder Systemdenken geht einher mit einer Art des Sprechens über mathematische Praktiken, die sich von der Art der Mathematiker unterscheidet. Bissell und Dillon (2000, S. 7) illustrieren dies unter anderem anhand eines einfachen Beispiels: In einem einfachen elektrischen Schaltkreis sind Spannung U und Stromstärke I über einen Widerstand R mittels $U = R \cdot I$ zueinander in Beziehung gesetzt. Mathematisch handele es sich um einen linearen Zusammenhang mit R als Proportionalitätskonstante. Dieses mathematische Verständnis des Modells reiche aber nicht aus, um zu verstehen und zu erklären, wie sich Veränderungen von Stromstärke und Spannung in Stromkreisen auswirken. Hinzutreten müsse vielmehr ein Verständnis der Gleichung, dass es sich hier um physikalische Größen handle und das Modell das physikalische Verhalten eines Systems (hier eines einfachen elektrischen Stromkreises) beschreibe. Die mathematische Sicht blende diese für die Verwendung der Gleichung im elektrotechnischen Kontext im gewissen Sinne notwendige Sichtweise quasi aus. Aus mathematischer Sicht gelte die Beziehung zwischen Spannung und Stromstärke für jeden Zeitpunkt, die Veränderung des einen Wertes ziehe eine gleichzeitige Veränderung des anderen Wertes nach sich.

Dies entspricht im Wesentlichen der Kovariationsvorstellung eines funktionalen Zusammenhangs. Diese schließt die Vorstellung eines durchaus auch kausal verstandenen, aber im Wesentlichen quantitativen Zusammenhangs zwischen den involvierten Variablen ein. Die elektrotechnische Sichtweise ergänze diese Vorstellung aber nun wesentlich durch qualitative Aspekte, und zwar durch die spezifischen physikalischen Größen und mit diesen verknüpfte Vorstellungen und Bedeutungen: „This means that a change in the current causes the voltage to change.“ (Bissell und Dillon 2000, S. 7). Die elektrotechnische Rede über die Gleichung drückt also nicht nur einen funktionalen mathematischen Zusammenhang zwischen Variablen, sondern damit und darüber hinaus einen kausalen Zusammenhang zwischen elektrotechnischen Größen aus.

Nach Bissell und Dillon (2000, S. 10) handelt es sich dabei nicht nur um eine andere Art des Redens, sondern stellt eine eigene Art des Denkens dar[9]:

> Moreover, this linguistic shift is more than just jargon, and more than just a handy way of coping with the mathematics; the shift indicates a way of thinking about systems behaviour in which the features of the models are deeply linked to the systems they are describing.

Im Zusammenhang mit dieser an Kausalzusammenhängen orientierten Art, über die Gleichung des Schaltkreises zu reden und zu denken, steht die allgemeine, über das einfache Beispiel weit hinausgehende Entwicklung eines Systemdenkens, das es schließlich auch erlaube, grafische und piktorale Repräsentationen anstelle komplizierter mathematischer Ausdrücke zu manipulieren[10]. Nach Bissel (2004) war für diese Entwicklung die Einführung komplexer Größen in der Elektrotechnik, vorangetrieben u. a. von Steinmetz (1893), grundlegend. Er schlug vor, Größen wie Wechselstrom oder -spannung durch eine „komplex imaginäre Größe“ (S. 598) zu repräsentieren und in Polarkoordinaten als Phasor[11] darzustellen, da die Sinuswelle vollständig bestimmt ist durch Intensität und Phase. Dieser Ansatz führte zum einen zu einer wesentlichen Vereinfachung von Rechnungen:

> Wo wir früher mit periodischen Funktionen einer unabhängigen Variablen, ‚Zeit‘ zu thun hatten, gelangen wir jetzt durch einfache Addition, Subtraktion etc. konstanter Zahlengrössen zur Lösung. […] Selbst die Beschränkung der Methode auf Sinuswellen ist nicht wesentlich, da wir in der gewöhnlichen Weise die allgemeine periodische Funktion aus ihren Sinuswellenkomponenten zusammensetzen können. (Steinmetz 1893, S. 597)

[9] Im Rahmen dieses Beitrags sollen dieses einfache Beispiel und die folgenden Ausführungen genügen, um die eigene Art des Denkens zu illustrieren. Für eine ausführlichere Darstellung der Entwicklung verweisen wir zusätzlich auf die Arbeiten von Bissell und Dillon (2000) sowie von Bissell (2004, 2012).

[10] Hier verweisen wir auch auf unsere Analyse zu den rotierenden Zeigern in Abschn. 6.5.2 und auf die Arbeit von de Oliveira und Nunes (2014).

[11] Für eine sinusförmige Größe gilt $A \cdot \cos(\omega t + \varphi) = \Re\left(A \cdot \mathrm{e}^{i(\omega t+\varphi)}\right) = \Re\left(A \cdot \mathrm{e}^{i\omega t} \cdot \mathrm{e}^{i\varphi}\right)$, wobei A die Amplitude, ω die Kreisfrequenz und φ der Phasenwinkel ist (jeweils zeitunabhängig). Der Fakto $\underline{A} = A \cdot \mathrm{e}^{i\varphi}$ wird Phasor genannt. Bei der Analyse elektrischer Komponenten ist im Wesentlichen das Amplitudenverhältnis von Eingangssignal und Ausgangssignal sowie die Phasenverschiebung, die durch die Komponente verursacht wird, von Interesse. Die Funktion $\underline{A} \cdot \mathrm{e}^{i\omega t}$ kann als rotierender Zeiger in der komplexen Ebene dargestellt werden. Phasoren und rotierende Zeiger stellen wichtige grafische Mittel zur Interpretation und Analyse elektrotechnischer Vorgänge dar.

Daneben führte dieser Ansatz nach Bissell (2004) aber auch zum Systemdenken und zur Blackbox-Analyse:

> Second, it was indeed an important step towards the 'black box' concept. The defining equations for resistors, capacitors and inductors were all subsumed into a generalised, complex version of Ohm's relationship; and even if it would be premature to talk of 'implicit 2-terminal black boxes' at this time, such a representation of components as complex impedances was clearly a great conceptual step. (S. 309)

In der komplexen Version des Ohmschen Gesetzes wird der komplexe Widerstand $\underline{Z}$ bzw. die Impedanz als Verhältnis aus komplexer Spannung und komplexer Stromstärke aufgefasst[12]:

$$\underline{Z} = \frac{\underline{u}}{\underline{i}} = \frac{u \cdot e^{j\omega t} \cdot e^{j\varphi_u}}{i \cdot e^{j\omega t} \cdot e^{j\varphi_i}} = \frac{u}{i} \cdot e^{j(\varphi_u - \varphi_i)}$$

Die Impedanz kann auch dargestellt werden als $\underline{Z} = R + jX$ mit Wirkwiderstand R und Blindwiderstand X. Für einen Ohmschen Widerstand gilt nun $\underline{Z}_R = R$, für einen Kondensator $\underline{Z}_C = 1/j\omega C$ und für eine Spule $\underline{Z}_L = j\omega L$. Beispielsweise können so in elektrischen Schaltungen die Gesamtimpedanz und Phasenverschiebungen von Strömen und Spannungen berechnet werden. Relevant wurde das vor allem im Bereich Filteranalyse und -design im Rahmen einer technologischen Entwicklung im Hinblick auf das effektivere Ausnutzen von Bandbreite bei der Signalübertragung. Hier hat das Zusammenspiel von Mathematik, der Zusammenfassung von Schaltungskomponenten zu abstrakteren Zweipolen, die über Input–Output-Betrachtungen charakterisiert werden konnten, und Filter-Design zur effektiveren Gestaltung von Signalübertragungen geführt. Diese Aspekte eines anderen Denkens über mathematische Praxen in der Elektrotechnik sind insbesondere auch für die von uns in Abschn. 6.5 analysierten Aufgaben relevant.

Im Rahmen unseres ATD-Modells interpretieren wir nun diese eigene Art zu reden und das Systemdenken als eigenen mathematischen Diskurs – als ET-Diskurs –, der sich also zusammenfassend durch einen Bezug zur Realität, der sehr unterschiedlich stark expliziert wird, und durch ein Systemdenken auszeichnet.

Neben der Beschreibung und Charakterisierung von Signalen und Systemen ist in der Vorlesung auch die Signalübertragung ein zentrales Thema. Dabei spielt dann zusätzlich zur Beschreibung und Charakterisierung verschiedener Übertragungskanäle (Systeme) auch die Frage nach der Realisierung der Signalübertragung über einen bestimmten Kanal eine große Rolle. Ein wichtiges Kriterium ist hierbei die mögliche Mehrfach-

[12]Mit dem Unterstrich kennzeichnet man in der Elektrotechnik üblicherweise komplexe Größen. In der Elektrotechnik wird für die imaginäre Einheit der Buchstabe j verwendet, um Verwechslungen mit der zeitabhängigen Stromstärke i zu vermeiden.

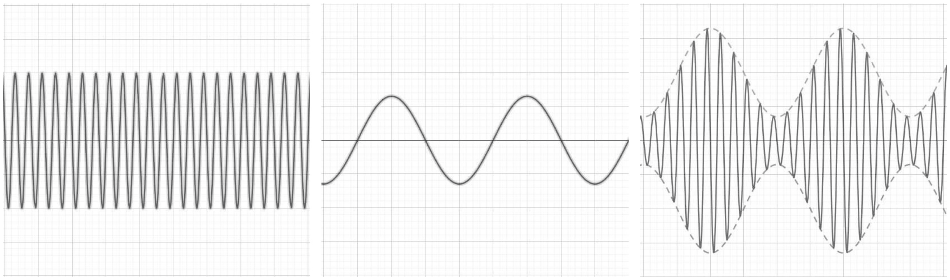

Abb. 6.1 Trägersignal (links), Primärsignal $s(t)$ (Mitte), AM-Signal mitgestrichelter Einhüllende (rechts)

ausnutzung des Übertragungskanals: Mehrere Signale sollen gleichzeitig übertragen werden, ohne dass es zum Übersprechen zwischen Signalen am Empfänger kommt. Ein klassisches Beispiel ist hier die Übertragung mehrerer Radiosender über Antenne oder Kabel. Ein einfaches und mit wenig technischem Aufwand durchführbares Verfahren ist die analoge Amplitudenmodulation und -demodulation[13], die schließlich in den von uns betrachteten Aufgaben thematisiert wird. Das Prinzip der Amplitudenmodulation wird in Abb. 6.1, veranschaulicht. Dabei wird die Amplitude eines hochfrequenten Trägersignals (Abb. 6.1 links) entsprechend dem Verlauf des niederfrequentes Primärsignals $s(t)$ (Abb. 6.1 Mitte) variiert. Das AM-Signal (Abb. 6.1 rechts) lässt sich darstellen als $x(t) = A[1 + m\, s(t)] \cos(2\pi f_0 t)$, wobei $\cos(2\pi f_0)$ das Trägersignal ist. Der Modulationsgrad m ist das Verhältnis aus Amplitude des Trägersignals und Amplitude des Primärsignals, außerdem gelten die Einschränkungen $\max_{t \in \mathbb{R}} |s(t)| = 1$ und $0 < m < 1$.

Mit der Amplitudenmodulation lassen sich mehrere Primärsignale (z. B. jeweils für verschiedene Radiosender) mit unterschiedlichen Trägerfrequenzen (den jeweiligen Senderfrequenzen) über Antenne übertragen und am Empfänger (Radiogerät) je nach eingestelltem Sender empfangen. Am Empfänger muss dann zur Rekonstruktion des Primärsignals eine Demodulation stattfinden. Eine einfache, auch technisch unaufwendig zu realisierende Methode der Demodulation ist der Enveloppendemodulator (oder Hüllkurvendetektor). Hier wird das amplitudenmodulierte Signal zunächst gleich gerichtet und das hochfrequente Trägersignal mit einem Tiefpassfilter entfernt (vgl. Fettweis 1996, S. 251)[14]. So wird die obere Hüllkurve (gestrichelt in Abb. 6.1 rechts) des amplitudenmodulierten Signals – und somit das Primärsignal – rekonstruiert.

Die in unserer Analyse betrachtete Aufgabe inklusive Dozenten-Musterlösung ist im Anhang abgebildet. Sie besteht aus zwei Punkten, die im Rahmen unserer Analyse als eigene Aufgaben aufgefasst werden:

[13] Ein Gerät, das sowohl moduliert als auch demoduliert, nennt man Modem.

[14] Im einfachsten Fall ist ein Tiefpassfilter eine Schaltung aus Widerstand und Kondensator, bei der die Ausgangsspannung gegenüber der Eingangsspannung um einen frequenzabhängigen Faktor geschwächt ist. Dabei ist die Abschwächung umso stärker, je höher die Frequenz ist.

- Im ersten Aufgabenteil soll gezeigt werden, dass unter bestimmten Voraussetzungen der Enveloppendemodulator ein Signal liefert, das proportional zur Amplitude des modulierten Trägersignals ist. Die Dozenten-Musterlösung zu dieser Aufgabe wird in Abschn. 6.5.1 analysiert.
- Der zweite Aufgabenteil ist in drei Unteraufgaben strukturiert. Es soll ein Primärsignal zuerst (1) amplitudenmoduliert, dann (2) als Summe dreier harmonischer Schwingungen aufgeschrieben und (3) das Ergebnis schließlich in der komplexen Ebene als rotierender Zeiger mit variierender Amplitude grafisch dargestellt werden. Die Dozenten-Musterlösung zu diesem dritten Teil wird in Abschn. 6.5.2 analysiert.

Die Aufgabe ist ebenfalls Bestandteil der Vorlesungsfolien und wird nach der Einführung des Enveloppendemodulators im Rahmen des Abschnitts zur Amplitudenmodulation gestellt.

6.5 Analyse

Die in diesem Abschnitt präsentierten ATD-Analysen beziehen sich auf die Dozenten-Musterlösungen der beiden Teilaufgaben von Aufgabe 4, die in Abschn. 6.4 inhaltlich vorgestellt und in den thematischen Zusammenhang der Vorlesung eingebettet worden sind. Alle Teile von Aufgabe 4 und die zugehörigen Dozenten-Musterlösungen sind im Anhang wiedergegeben.

Jede der beiden ATD-Analysen wird zur besseren Übersicht auch grafisch dargestellt. Dazu haben wir zunächst die Orientierung des 4 T-Modells verändert: Aufgabentypen, Techniken, Technologien und Theoriefacetten sind nicht mehr nebeneinander, sondern untereinander angeordnet. Im Rahmen unserer Analyse haben wir komplexe Techniken als neue Teilaufgaben aufgefasst und zu diesen wiederum die zugehörigen Lösungstechniken, Technologien und Theoriefacetten zugeordnet[15]. Der Übergang von komplexen Techniken zu neuen Teilaufgaben ist über entsprechend beschriftete Pfeile dargestellt. Zur besseren Unterscheidung von HM- und ET-Elementen der Praxeologie wurden unterschiedliche Farben und Formen verwendet (vgl. Legende in Abb. 6.2).

[15]Dieser Übergang von komplexen Techniken zu neuen (Teil-)Aufgaben kann als dialektisches Verhältnis zwischen Aufgaben und Techniken verstanden werden (vgl. Chevallard 2019, S. 85), das wir hier zur Strukturierung unserer Analyse nutzen.

6.5.1 Der Enveloppendemodulator

Diese Analyse bezieht sich auf die folgende Aufgabe (siehe erste Unterpunkte von Aufgabe 4 im Anhang).

> Unter der Annahme $0 < m < 1$ und somit $A(t) > 0$ (die Einhüllende oder Enveloppe des AM-Signals ist stets positiv) zeige man, dass der o.g. Enveloppendemodulator tatsächlich ein Signal proportional zu $A(t)$ liefert.

Die grafische Repräsentation der Analyseergebnisse zu dieser Aufgabe befindet sich in Abb. 6.2. Die Formulierung „…, dass der o.g. Enveloppendemodulator …“ bezieht sich dabei auf die entsprechende Vorlesungsfolie, auf der der Enveloppendemodulator eingeführt wurde.

Die Aufgabe (T_1) besteht darin zu zeigen, dass der Enveloppendemodulator unter den gegebenen Voraussetzungen eine bestimmte Eigenschaft, nämlich ein Signal proportional zur Einhüllenden $A(t)$ zu liefern, hat. Die Lösung der Aufgabe erfordert die Techniken Anwendung des Enveloppendemodulators (τ_{ET}) auf das gegebene Signal und Ablesen der Proportionalität (τ_{HM}). Technologieelemente sind hier einmal, als Voraussetzung der Anwendbarkeit dieses speziellen Detektors, das Vorliegen eines Signals mit Modulationsgrad $m < 1$ (θ_{ET}) und der Aspekt, dass elektrotechnische Größen auffassbar sind als Veränderliche im Kontext linearer Funktionen (θ_{HM}). Der Bezug zwischen linearen Funktionen und Proportionalität ist nicht notwendig Teil einer HM-Vorlesung[16]. Allerdings ist das Deuten von Parametern linearer Funktionen, im Spezialfall als Proportionalitätskonstante, Teil der Schulmathematik.

Die Anwendung des Enveloppendemodulators stellt eine komplexe Technik dar, die wir zu analytischen Zwecken als neue Teilaufgabe (T_2) aufgefasst haben. Diese erfordert die Techniken Gleichrichtung (τ_{ET}) und Anwendung des Tiefpassfilters (τ_{ET}). Diese beiden Techniken sind für die Rückgewinnung des modulierenden Signals zentral (θ_{ET}). Ohne Gleichrichtung würden sich positive und negative Amplitudenschwankungen im Mittel auslöschen (θ_{ET}). Mit dem Tiefpass lässt sich schließlich das höherfrequente Trägersignal eliminieren (θ_{ET}). Hier ist die Voraussetzung wichtig, dass die Trägerfrequenz größer ist als die Signalfrequenz (θ_{ET}).

Aus beiden Techniken lassen sich nun analytisch wieder jeweils eigene Teilaufgaben konstruieren. Die Gleichrichtung (T_3) erfordert die Anwendung der Betragsfunktion (τ_{HM}). Diese HM-Technik lernen die Studierenden im Rahmen der Vorlesung „Grundlagen der Elektrotechnik“ als mathematisches Modell des elektrotechnischen Vorgangs der Gleichrichtung (θ_{ET}) kennen. Dadurch erfährt eine HM-Technik eine neue, elektrotechnische Deutung. Die Anwendung des Tiefpassfilters (T_4) erfordert die Zerlegung des Signals in konstante und oszillierende Komponenten (τ_{ET}) und schließlich das Weglassen des Frequenzanteils (τ_{ET}) als mathematische Darstellung der Wirkung des Tiefpassfilters (θ_{ET}).

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[16] Beispielsweise nicht im zugrunde liegenden Lehrbuch von Strampp (2015).

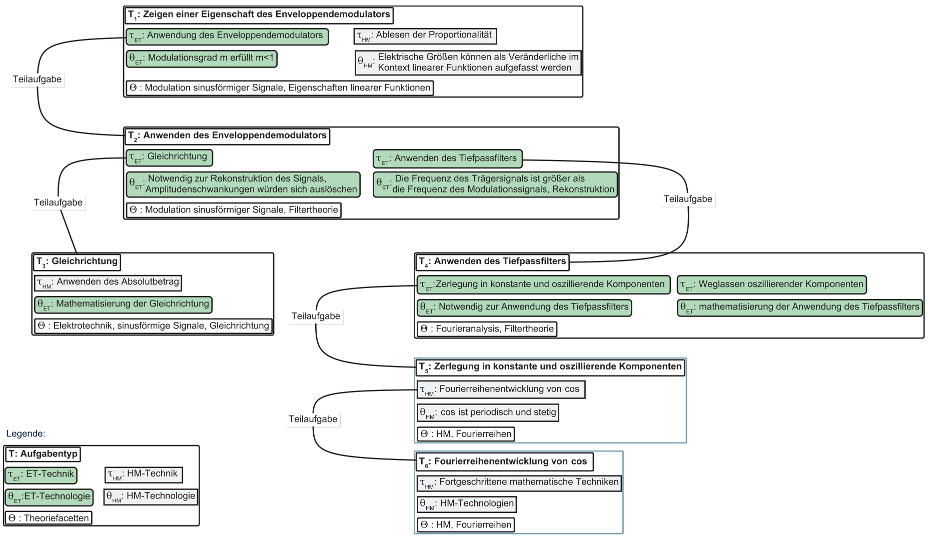

Abb. 6.2 Grafische Darstellung der Analyseergebnisse zum Enveloppendemodulator

Die Zerlegung des Signals in Gleich- und Frequenzanteile bildet eine weitere Teilaufgabe (T_5). Da |cos| eine periodische stetige Funktion ist (θ_{HM}), kann |cos| in eine Fourier-Reihe entwickelt werden (τ_{HM}). Diese HM-Technik kann wieder als neue Teilaufgabe (T_6) betrachtet werden, deren Lösung zum Teil anspruchsvolle HM-Techniken erfordert, die hier nicht näher ausdifferenziert werden. Elektrotechnische Bezüge werden im Verlauf der Lösung dieser Teilaufgabe nicht mehr hergestellt.

Zur Interpretation unserer Analyseergebnisse betrachten wir zunächst die rekonstruierten Praxeologien zu den jeweiligen Teilaufgaben einzeln: Insgesamt wurden sechs Teilaufgaben und zugehörige praxeologische Elemente rekonstruiert, die jeweils eine oder zwei Techniken und Technologien enthalten. Außer bei der Teilaufgabe zur Gleichrichtung (T_3) werden ET-Techniken durch ET-Technologien begründet und HM-Techniken durch HM-Technologien.

Bei der Teilaufgabe zur Gleichrichtung (T_3) ist zur Rechtfertigung der HM-Technik, den Absolutbetrag auf eine Funktion anzuwenden, eine elektrotechnische Deutung relevant. Der Absolutbetrag stellt die Mathematisierung der Wirkung des Gleichrichtens dar.

Zum Zeigen der spezifischen Eigenschaft des Enveloppendemodulators, Teilaufgabe (T_1), sind sowohl elektrotechnische Techniken mit elektrotechnischen Technologien als auch HM-Techniken mit zugehörigen HM-Technologien relevant. In allen weiteren Teilaufgaben treten entweder nur elektrotechnische Techniken und Technologien, (T_2) und (T_4), oder nur HM-Techniken und HM-Technologien, (T_5) und (T_6), auf. Bis auf die Teilaufgabe (T_3) finden die jeweiligen mathematischen Handlungen in ihren jeweils eigenen mathematischen Diskursen statt.

Bis hierher hat sich bereits gezeigt, dass Techniken und Technologien beider Diskurse vorkommen und in einem Fall, (T_3), auch eine Mischung, also die Rechtfertigung der Technik des einen Diskurses durch Elemente des anderen Diskurses, auftritt. Wenn wir nun bei der Interpretation berücksichtigen, dass die Trennung in Teilaufgaben ein analytischer Schritt ist, die sechs Praxeologien also integriert gedacht werden müssen, fallen weitere interdiskursive Beziehungen auf: Bei der Anwendung des Tiefpassfilters (T_4) muss das Signal in konstante und oszillierende Komponenten zerlegt werden, um schließlich zu zeigen, dass eine konstante positive Komponente existiert, nach Anwendung des Tiefpassfilters also ein positiver Proportionalitätsfaktor übrig bleibt. Dabei unterdrückt der Tiefpassfilter alle oszillierenden Komponenten. Die Rechtfertigung auf dieser Ebene ist elektrotechnischer Natur (vgl. hierzu auch die Ausführungen in Fettweis 1996, S. 236 ff.). In Teilaufgabe (T_5) und Teilaufgabe (T_6) wird diese Zerlegung nun als Entwicklung in eine Fourier-Reihe durchgeführt. Dabei ist das gesamte Vorgehen stark formalisiert ausgerichtet[17]. Hier findet im Verlauf der Aufgabenlösung ein Diskurswechsel statt, wobei die mathematische Strenge im weiteren Verlauf durch die Ansprüche der Aufgabenstellung nicht gerechtfertigt ist. Nach der Fourier-Reihen-Entwicklung findet eine Rückkehr in den elektrotechnischen Diskurs statt, in dem das Weglassen der oszillierenden Komponenten mit Bezug auf den Tiefpassfilter gerechtfertigt und somit auch das Bestimmen aller Fourier-Koeffizienten außer dem ersten gewissermaßen infrage gestellt wird.

6.5.2 Rotierende Zeiger

Diese Analyse bezieht sich auf die folgende Aufgabe (siehe dritte Aufgabe des zweiten Unterpunktes von Aufgabe 4 im Anhang):

> Stellen Sie $x(t)$ unter Ausnutzung der Beziehung $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ und des Ergebnisses unter Punkt 2. in der komplexen Ebene als rotierenden Zeiger mit variierender Amplitude grafisch dar.

Bevor wir zur Analyse der Dozenten-Musterlösung kommen, möchten wir zwei Bemerkungen einfügen: Zum einen wird in Punkt 2., auf den in der Aufgabenstellung Bezug genommen wird, das amplitudenmodulierte Signal $x(t) = A(1 + m\cos(\Omega t))\cos(2\pi f_0 t)$ als Summe dreier harmonischer Schwingungen dargestellt, wobei $\Omega \ll 2\pi f_0$ gilt. Das Ergebnis lautet:

[17]Möglich wäre hier auch eine pragmatische Lösung im Sinne der Aufgabenstellung, die nur den ersten Koeffizienten berechnet, um zu zeigen, dass der positiv ist. Das Berechnen weiterer Koeffizienten ist in gewissem Sinne unnötig, da die zugehörigen Signalanteile durch die anschließende Anwendung des Tiefpassfilters unterdrückt werden. Solche Varianten finden sich beispielsweise in Studierendenlösungen.

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$$

Damit wird in der von uns analysierten Aufgabe weitergearbeitet. Und zum anderen ist die Aufgabenstellung ungenau: Das Signal $x(t)$ ist der Realteil des rotierenden Zeigers, der in der komplexen Zahlenebene dargestellt werden soll, nicht der Zeiger selbst. In der Dozenten-Musterlösung wird zwischen dem Realteil des Zeigers und dem Zeiger selbst unterschieden.

Die Aufgabe (T_1) besteht nun also darin, den gegebenen Ausdruck als Realteil eines rotierenden Zeigers in der komplexen Ebene darzustellen. Insgesamt werden zur Lösung dieser Aufgabe vier Techniken rekonstruiert. Zwei davon werden wiederum jeweils aufgrund ihrer Komplexität als Teilaufgaben aufgefasst. Die grafische Darstellung der Analyseergebnisse zu dieser Aufgabe befindet sich in Abb. 6.3.

Zur Lösung von (T_1) muss das Signal $x(t)$ zunächst so umgeformt werden (τ_{HM}), dass es schließlich als Realteil eines sich drehenden Trägerzeigers mit zeitabhängiger Amplitude $A(t)$ interpretierbar ist. Begründet sind diese Umformungen in der Idee, Prinzipien der Amplitudenmodulation grafisch darzustellen (θ_{ET}). Insbesondere der Rechenschritt von.

$$x(t) = A\Re\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t - \Omega t))\},$$

in dem $x(t)$ als Realteil dreier im Ursprung gezeichneter rotierender Zeiger interpretierbar ist, hin zu.

$$x(t) = \Re\left\{\exp(j2\pi f_0 t)\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]\right\},$$

in dem $x(t)$ als rotierender Trägerzeiger mit zeitabhängiger Amplitude $A(t)$ interpretiert werden kann, ist hier zentral. Nur wenn $x(t)$ in dieser Form dargestellt wird, kann ein

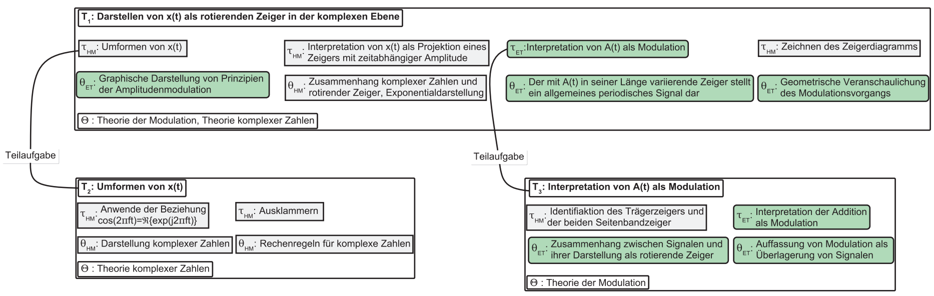

Abb. 6.3 Grafische Darstellung der Analyseergebnisse zu den rotierenden Zeigern

Zeigerdiagramm gezeichnet werden, in dem die Amplitudenmodulation des Signals $x(t)$ dargestellt werden kann[18].

Die Technik, $x(t)$ umzuformen, haben wir in unserer Analyse als Teilaufgabe (T_2) aufgefasst. Dazu wird zunächst die in der Aufgabenstellung gegebene Beziehung $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ auf $x(t)$ angewendet (τ_{HM}). Hier sind zur Begründung Zusammenhänge zwischen der Darstellung einer komplexen Zahl in Polarform und in Exponentialform relevant (θ_{HM}). Im weiteren Verlauf werden Rechenregeln für komplexe Zahlen angewendet (θ_{HM}), nämlich das Ausklammern des Realteils (τ_{HM}) und des Faktors $\exp(j2\pi f_0 t)$ (τ_{HM}). Als Ergebnis ergibt sich bis hierher:

$$x(t) = \Re\left\{\exp(j2\pi f_0 t)\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]\right\}$$

Im nächsten Schritt muss dieser Ausdruck als Projektion eines Zeigers mit zeitabhängiger Amplitude

$$A(t) = A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)$$

auf die reelle Achse interpretiert werden (τ_{HM}). Diese Technik ist dem HM-Diskurs zugeordnet, da auch in der Höheren Mathematik komplexe Zahlen als Zeiger aufgefasst, in Zeigerdiagrammen dargestellt und die Projektion des Zeigers auf die reelle Achse als Realteil der komplexen Zahl interpretiert werden (vgl. Strampp 2012). Gerechtfertigt wird dies im Rahmen der Behandlung komplexer Zahlen (θ_{HM}). Als Nächstes wird die Summe $A(t)$ als Modulation des Trägerzeigers interpretiert (τ_{ET}). Die Länge des Zeigers $\exp(j2\pi f_0 t)$ ändert sich zeitabhängig entsprechend $A(t)$. Somit korrespondiert dieser in seiner Länge variierende Zeiger mit einem allgemeinen periodischen Signal (θ_{ET}).

Diese Interpretation als Modulationsvorgang wird aufgrund ihrer Komplexität als Teilaufgabe (T_3) aufgefasst. Um den Ausdruck $A(t)$ als Modulation des Trägerzeigers interpretieren zu können, müssen zuerst der Trägerzeiger mit Amplitude A, der

[18]Die Abb. 6.1 rechts und das Zeigerdiagramm (siehe Abb. 4 in der Dozenten-Musterlösung im Anhang) stellen im Prinzip zwei Veranschaulichungen der Amplitudenmodulation dar. Das Zeigerdiagramm hat gegenüber der Darstellung in Abb. 6.1 rechts den Vorteil, dass sich damit einige Effekte, die bei der Amplitudenmodulation relevant sind, darstellen lassen. Ist beispielsweise der Modulationsgrad m größer als 1, kommt es zu einem Phasensprung: Zu dem Zeitpunkt, an dem die beiden Seitenbandzeiger genau entgegen der Richtung des Trägerzeigers liegen, sind die beiden Seitenbandzeiger dann zusammen länger als der Trägerzeiger. Insgesamt macht der Zeiger der Gesamtsumme dann einen Phasensprung. Ein zweites Beispiel ist eine ungleichmäßige Übertragung der beiden Seitenbänder. Wenn sich deren Amplituden unterscheiden, sind die beiden Seitenbandzeiger nicht mehr gleich lang. Ursprünglich zeigt die Summe der Seitenbandzeiger immer in oder genau entgegen der Richtung des Trägerzeigers. Schwanken die Amplituden der Seitenbänder, schwankt die Summe der Seitenbandzeiger um diese Mittellage und es kommt zu zusätzlicher Phasenmodulation.

sich mit Winkelgeschwindigkeit $\omega_0 = 2\pi f_0$ dreht, und die beiden Seitenbandzeiger mit Amplitude je $Am/2$, die sich mit Winkelgeschwindigkeit Ω. und $-\Omega$ den Trägerzeiger drehen, identifiziert werden (τ_{HM}). Diese Technik wurde dem HM-Diskurs zugeordnet, da es hier darum geht, Ausdrücke als komplexe Zahl mit der jeweils entsprechenden Darstellung als Zeiger zu interpretieren. Hier ist die Rechtfertigung dieser Interpretation durch die elektrotechnische Konzeption der Darstellung von Signalen durch rotierende Zeiger begründet (θ_{HM}). Anschließend wird die Addition der drei Zeiger als Modulation interpretiert (τ_{ET}), was sich durch die Auffassung von Modulation als Überlagerung von Signalen rechtfertigen lässt. Diese Überlagerung wird durch die Addition der entsprechenden Zeiger modelliert (θ_{ET}).

Die vierte Technik ist schließlich das Zeichnen des Zeigerdiagramms (τ_{HM}). Die Darstellung komplexer Zahlen als Zeiger in der komplexen Zahlenebene ist eine übliche HM-Technik. Hier jedoch bekommt diese HM-Technik wiederum eine elektrotechnische Deutung, da es sich um die geometrische Veranschaulichung eines Modulationsvorgangs handelt (θ_{ET}).

Insgesamt wurden zur Aufgabe T_1 vier Techniken mit zugehörigen Technologien identifiziert und rekonstruiert. Zwei Techniken wurden wiederum aufgrund ihrer Komplexität als Teilaufgaben T_2 und T_3, mit jeweils zwei Techniken und zugehörigen Technologien, aufgefasst. Im Vergleich zur Aufgabe zum Enveloppendemodulator kommen hier HM-Techniken, die im Rahmen eines mathematischen ET-Diskurses gerechtfertigt werden, häufiger vor: Das Umformen von $x(t)$ stellt an sich eine HM-Technik dar, und als Teilaufgabe T_2 steht sie komplett im HM-Diskurs, aber das Ziel dieser Umformungen ist eine ganz spezifische Form, die notwendig ist, um das Signal überhaupt als amplitudenmoduliert zu interpretieren. In Teilaufgabe T_3 müssen Trägerzeiger und die beiden Seitenbandzeiger identifiziert werden. Hier geht es prinzipiell darum, komplexe Zahlen in Polarform zu identifizieren und ihnen ihre entsprechende Darstellung als Zeiger zuzuordnen. Die technologische Ebene reflektiert dabei, dass es sich einmal um den Zeiger des Trägersignals und einmal um die beiden Zeiger der Seitenbänder im Kontext der Amplitudenmodulation handelt. Und schließlich wird in Aufgabe T_1 das Zeigerdiagramm gezeichnet. Auch diese Technik ist prinzipiell in der HM anzutreffen. In der vorliegenden Aufgabe liegt die Begründung aber in der Veranschaulichung eines elektrotechnischen Vorgangs. Man könnte auch alle drei Zeiger an den Ursprung zeichnen, entsprechend dem vorletzten Schritt in der Umformung. Diese Darstellung wäre aber nicht geeignet, relevante Aspekte der Amplitudenmodulation darzustellen.

Ein weiterer Aspekt, der uns im Rahmen der Analyse aufgefallen ist: HM-Techniken, die sich mit den Zeigern beschäftigen (Interpretation von $x(t)$ als Projektion eines Zeigers, Identifikation von Trägerzeiger und Seitenbandzeigern und Zeichnen des

Zeigerdiagramms), können vom Typ her in der HM vorkommen, treten aber in der vorliegenden Aufgabe in besonders komplexer Weise auf. Einen Ausdruck als Projektion eines Zeigers auf die reelle Achse zu interpretieren, ist HM-typisch. Hier kommt hinzu, dass der Zeiger eine zeitabhängige Amplitude besitzt. Ausdrücke als komplexe Zahl mit der jeweils entsprechenden Darstellung als Zeiger zu interpretieren, ist HM-typisch. Im Ausdruck für $A(t)$ die beiden Seitenbandzeiger zu identifizieren und A als Amplitude des Trägerzeigers zu erkennen, benötigt aber spezifisches Wissen über Prinzipien der Amplitudenmodulation. Und schließlich ist auch das Zeichnen von Zeigerdiagrammen üblich in der HM. Hier muss nun aber im Prinzip auf dem Trägerzeiger ein zweites Zeigerdiagramm gezeichnet werden (siehe Abb. 4 im Anhang). Nur so ist der Vorgang der Amplitudenmodulation überhaupt adäquat dargestellt und nur so ist es möglich, an dieser spezifischen Grafik weitergehende Aspekte (z. B. zum Modulationsgrad m oder zu Übertragungsverlusten in den Seitenbändern, vgl. Fußnote 18) zu untersuchen. Das Erkennen der Amplitudenmodulation in diesem Zeigerdiagramm und die Möglichkeit, dieses Zeigerdiagramm zum Studium bestimmter elektrotechnischer Effekte nutzen zu können, erfordert demnach gerade einen spezifischen „elektrotechnischen Blick", also den ET-Diskurs, und somit Aspekte, die mit den Überlegungen von Bissell und Dillon zum Systemdenken in Verbindung stehen.

6.6 Diskussion

Die Analysen der Aufgaben zeigen, dass sich in deren Lösungsschritten Techniken und Technologien unterscheiden lassen, die sich verschiedenen mathematischen Diskursen, dem HM-Diskurs oder dem ET-Diskurs zuordnen lassen. Dabei kann die Unterscheidung auf der Grundlage der vorher formulierten Charakterisierungen dieser beiden Diskurse vorgenommen werden. Im Hinblick auf die beiden Aufgaben und deren Lösungen adressieren die Charakterisierungen also identifizierbare Unterschiede und sind hinsichtlich dieses Ziels hinreichend präzise.

Insbesondere aus den grafischen Darstellungen der Analyse wird unmittelbar deutlich, an welchen Stellen in der Bearbeitung für Studierende eventuell problematische Übergänge zwischen mathematischem HM-Diskurs und mathematischem ET-Diskurs auftreten und explizit in der Lehre, etwa bei der Besprechung der Lösungen in der Lehrveranstaltung, angesprochen werden könnten. Gegebenenfalls in Aufgabenbearbeitungen auftretende Schwierigkeiten können anhand der Grafik diagnostisch hinsichtlich ihrer technischen oder technologischen Qualität beurteilt werden. Auf dieser Grundlage könnte darüber hinaus auch die Ebene der Rückmeldung an Studierende bedacht und geeignet gewählt werden. So könnte Feedback auf technische Probleme etwa nicht nur die jeweilige Technik adressieren, sondern auch technologische Aspekte und deren jeweilige Verortung im HM- bzw. ET-Diskurs der SST berücksichtigen. Sowohl die vorgenommene Charakterisierung der Diskurse als auch unser Vorschlag zur grafischen Darstellung von Ergebnissen der praxeologischen Analyse können also in fachbezogene Vor- und Nachbereitungsüberlegungen von Lehrveranstaltungen sowie in fachliches Feedback auf Aufgabenbearbeitungen als Werkzeug einbezogen werden.

Im Einzelnen und unter Berücksichtigung der jeweiligen Teilaufgabenebenen ergaben sich im Kontext der zwei Beispielaufgaben die folgenden drei praxeologisch zu unterscheidenden Typen von Verknüpfungen:

a. HM-Technik als Moment eines ET-Diskurses: z. B. in T_3 in der Enveloppen-Aufgabe (Gleichrichtung und Absolutbetrag) und in T_3 in der Aufgabe zum rotierenden Zeiger (Interpretation von $A(t)$ als Modulation). Hier muss man jeweils den mathematischen ET-Diskurs und dessen Realisierung kennen. Feedback zu einer problematischen Aufgabenbearbeitung an diesem Punkt könnte also sinnvollerweise dieses möglicherweise nicht vorhandene Verknüpfungswissen adressieren. Dies müsste gegebenenfalls im ET-Diskurs plausibel gemacht werden. Aktivierende Fragen könnten im Beispiel der Enveloppen-Aufgabe etwa sein: Was bedeutet Gleichrichtung? Wie wird das elektrotechnisch und mathematisch realisiert? Was macht das mit dem Signal?
b. ET-Technik als Moment des ET-Diskurses: z. B. in T_4 in der Enveloppen-Aufgabe (Anwenden des Tiefpassfilters)
c. HM-Technik als Moment des HM-Diskurses: z. B. in T_5 in der Enveloppen-Aufgabe (Fourier-Reihe)

Die Typen b. und c. sind jeweils, zumindest wenn man die Integriertheit der jeweiligen Unterebenen unberücksichtigt lässt, innerhalb der jeweiligen Diskurse angesiedelt. Inwieweit diese verschiedenen Typen tatsächlich praxeologisch zu unterscheidende Hürden für Studierende darstellen, ist natürlich eine empirisch weiter zu untersuchende Frage. Schließlich müssen strukturelle Hürden in Lösungen nicht notwendigerweise tatsächlich zu Hürden bei der Bearbeitung führen. Auch zur Verbreitung der jeweiligen Schwierigkeiten kann unsere Analyse keine Aussage machen.

Noch zwei kurze abschließende Bemerkungen: An den beiden in dieser Arbeit beispielhaft analysierten Aufgaben ließen sich Aspekte identifizieren, bei denen höhere Ebenen der Kodetermination einen Beitrag zur Aufklärung des Logos-Blocks leisten würden, analog zu unserer Analyse der Einführung des Dirac-Impulses (Hochmuth und Peters 2020). Im Hinblick auf die zentralen Fragestellungen dieses Beitrags schienen uns diese aber von untergeordneter Bedeutung. Unseres Erachtens zeigen die grafischen Darstellungen der Verschlingungen der mathematischen Diskurse auch noch einmal deutlich, dass sich im Kontext dieser eher fortgeschrittenen Aufgaben keine klare Unterscheidung in einen reinen Mathematikdiskurs und einen mathematikfreien Elektrotechnikdiskurs, was dem sogenannten Rest der Welt im Modellierungskreislauf entsprechen würde, treffen lässt. Das Verhältnis zwischen Mathematik und Elektrotechnik, das wir im Rahmen dieses Beitrags nicht explizit adressieren, sondern zusammen im mathematischen ET-Diskurs fassen, ist darüber hinaus mit der Anwendungsmetapher nur unzureichend erfasst. Damit stützen die Ergebnisse der beiden exemplarischen Aufgabenanalysen unsere methodischen und theoretischen Vorentscheidungen.

Anhang

Dieser Abschnitt gibt Übungsaufgaben und die zugehörigen, von uns analysierten Dozenten-Musterlösungen wieder, so wie sie auf dem Übungsblatt zur Vorlesung erscheinen.

AUFGABE 4 (Vorlesung S. 93)

- Unter der Annahme $0 < m < 1$ und somit $A(t) > 0$ (die Einhüllende oder *Enveloppe* des AM-Signals ist stets positiv) zeige man, dass der o.g. *Enveloppendemodulator* tatsächlich ein Signal proportional zu $A(t)$ liefert.

LÖSUNG

Der Enveloppendemodulator liefert zunächst das Signal $|x(t)| = |A(t)\cos(2\pi f_0 t)|$. Wegen $0 < m < 1$ und somit $A(t) > 0$ gilt somit $|x(t)| = A(t)\,|\cos(2\pi f_0 t)|$. Die periodische Funktion $|\cos(2\pi f_0 t)|$ lässt sich, wie nachfolgend gezeigt, in eine Fourierreihe entwickeln.

Wir betrachten hierzu die komplexe Fourierreihe, die in der Vorlesung verwendet wird, und setzen

$$s(t) = |\cos(2\pi f_0 t)| \stackrel{!}{=} \sum_{n=-\infty}^{\infty} S_n \mathrm{e}^{\jmath n 2\pi F t},$$

wobei die Periode T von $s(t)$ durch $T = 1/2f_0$ gegeben ist und somit $F = 1/T = 2f_0$ ist.

Die Koeffizienten der Fourierreihe sind definiert durch

$$S_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\mathrm{e}^{-\jmath n 2\pi F t}\mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2} s(t)[\cos(2\pi n F t) + \jmath\sin(2\pi n F t)]\mathrm{d}t.$$

Da $s(t)$ eine gerade Funktion ist, $\sin(2\pi n F t)$ jedoch eine ungerade, gilt

$$\begin{aligned}
S_n &= \frac{1}{T}\int\limits_{-T/2}^{T/2} s(t)\cos(2\pi nFt)\mathrm{d}t \\
&\overset{\text{Integrand gerade}}{=} \frac{2}{T}\int\limits_{0}^{T/2} s(t)\cos(2\pi nFt)\mathrm{d}t \\
&= \frac{2}{T}\int\limits_{0}^{T/2} |\cos(2\pi f_0 t)|\cos(2\pi nFt)\mathrm{d}t \\
&= \frac{2}{T}\int\limits_{0}^{T/2} \cos(2\pi f_0 t)\cos(2\pi nFt)\mathrm{d}t \\
&\overset{\text{Add.theorem}, F=2f_0}{=} \frac{1}{T}\int\limits_{0}^{T/2} [\cos(2\pi f_0(2n+1)t) + \cos(2\pi f_0(2n-1)t)]\mathrm{d}t \\
&= \frac{1}{2\pi T f_0}\left[\frac{1}{2n+1}\sin(2\pi f_0(2n+1)t)|_0^{T/2} + \frac{1}{2n-1}\sin(2\pi f_0(2n-1)t)|_0^{T/2}\right]. \\
&\overset{2f_0T=1}{=} \frac{1}{\pi}\left[\frac{1}{2n+1}\sin\left(\frac{\pi(2n+1)}{2}\right) + \frac{1}{2n-1}\sin\left(\frac{\pi(2n-1)}{2}\right)\right].
\end{aligned}$$

Wegen $\sin\left(\frac{\pi(2n+1)}{2}\right) = (-1)^n$ und $\sin\left(\frac{\pi(2n-1)}{2}\right) = (-1)^{n+1}$ für $n \in \mathbb{Z}$, ergibt sich

$$\begin{aligned}
S_n &= \frac{1}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right] \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right] \\
&= \frac{(-1)^n}{\pi(2n+1)(2n-1)}[(2n-1)-(2n+1)] \\
&= \frac{(-1)^n}{\pi(4n^2-1)}(-2) \\
&= \frac{2}{\pi}\frac{(-1)^{n+1}}{4n^2-1} \\
&= S_{-n}.
\end{aligned}$$

Somit ergibt sich schließlich

$$\begin{aligned}
s(t) &= \sum_{n=-\infty}^{\infty} S_n \mathrm{e}^{\jmath n 2\pi F t} \\
&\overset{S_n = S_{-n}}{=} S_0 + \sum_{n=1}^{\infty} S_n\left(\mathrm{e}^{\jmath n 2\pi F t} + \mathrm{e}^{-\jmath n 2\pi F t}\right) \\
&= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{4n^2-1}\cos(4\pi n f_0 t) \\
&= \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{4n^2-1}\cos(4\pi n f_0 t)\right] \\
&= \frac{2}{\pi}\left[1 + \frac{2}{3}\cos(4\pi f_0 t) - \frac{2}{15}\cos(8\pi f_0 t) \pm \ldots\right].
\end{aligned}$$

Wird nun das Signal

$$|x(t)| = A(t)s(t) = A(t)\frac{2}{\pi}\left[1 + \frac{2}{3}\cos(4\pi f_0 t) - \frac{2}{15}\cos(8\pi f_0 t) \pm \ldots\right]$$

einem Tiefpassfilter zugeführt, ergibt sich an dessen Ausgang

$$y_0(t) = \frac{2}{\pi}A(t)$$

- Gegeben sei ein Primärsignal $s_1(t) = \cos(\Omega t)$ mit $\Omega \ll 2\pi f_0$.

1. Wie lautet das resultierende AM-Signal (Zweiseitenbandmodulation mit Träger)?

LÖSUNG

Durch Einsetzen von $s_1(t) = \cos(\Omega t)$ ergibt sich sofort

$$x(t) = A(1 + m\cos(\Omega t))\cos(2\pi f_0 t).$$

2. Formen Sie $x(t)$ um, so dass sich das AM-Signal als Summe von drei harmonischen Schwingungen ergibt.

LÖSUNG

Wir erhalten mit dem bekannten Additionstheorem $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$$x(t) = A(1 + m\cos(\Omega t))\cos(2\pi f_0 t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t).$$

Das Spektrum von $x(t)$ ist in Abb. 3 dargestellt.

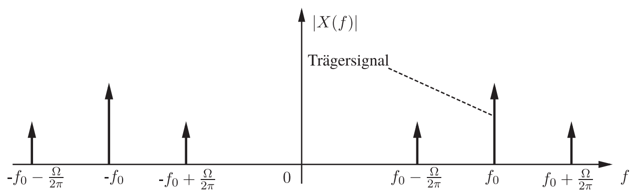

Abb. 3: Signalspektrum $X(f)$ des AM-Signals für $s_1(t) = \cos(\Omega t)$.

3. Stellen Sie $x(t)$ unter Ausnutzung der Beziehung $\cos(2\pi f t) = \Re\{\exp(\jmath 2\pi f t)\}$ und des Ergebnis unter Punkt 2. in der komplexen Ebene als rotierenden Zeiger mit variierender Amplitude grafisch dar.

LÖSUNG

Man schreibt zunächst

$$\begin{aligned}
x(t) &= A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \\
&= A\Re\{\exp(\jmath 2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(\jmath(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\Re\{\exp(\jmath(2\pi f_0 t - \Omega t))\} \\
&= \Re\left\{\exp(\jmath 2\pi f_0 t)\underbrace{\left[A + \frac{Am}{2}\exp(\jmath\Omega t) + \frac{Am}{2}\exp(-\jmath\Omega t)\right]}_{A(t)}\right\}
\end{aligned}$$

und interpretiert den Ausdruck in der eckigen Klammer als reellwertige zeitabhängige Amplitude $A(t)$, die den sich mit der Frequenz f_0 drehenden Trägerzeiger $\exp(\jmath 2\pi f_0 t)$ in Abb. 4 moduliert.

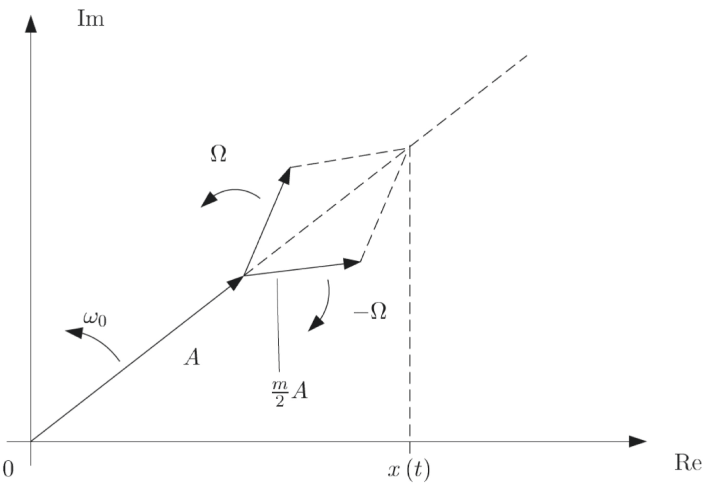

Abb. 4: Darstellung von $x(t) = x(t) = A(1 + m\cos(\Omega t))\cos(2\pi f_0 t)$ als Realteil eines sich drehenden Zeigers $A(t)\exp(\jmath 2\pi f_0 t)$ mit $\omega_0 = 2\pi f_0$.

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On the Analysis of Mathematical Practices in Signal Theory Courses

Reinhard Hochmuth[1] · Jana Peters[1]

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Abstract
The contribution aims at subject-specific analyses of student solutions of an exercise from an electrical engineering signal theory course. The basis for the analyses is provided by praxeological studies (in the sense of the Anthropological Theory of Didactics) and the identification of two institutional mathematical discourses, one related to higher mathematics for engineers and one related to electrical engineering. Regarding the relationship between institutional observations and analyses of students' solutions, we refer, among others, to Weber's (1904) concept of ideal types. In the subject-specific analyses of student solutions we address in particular transitions and interrelations within single processing steps that refer to the two mathematical discourses and different forms of embedding of mathematics into the electrical engineering context. Finally, we present a few ideas for teaching.

Keywords Mathematical practices · Student solutions · Ideal typical discourses · Institutions · Anthropological theory of the didactic

Introduction

The use of mathematics in engineering courses and its adequate conceptualisation is a frequently addressed issue (e.g. Alpers, 2017; Alpers et al., 2013; Barquero et al., 2011, 2013; Czocher, 2013; Harris et al., 2015; Rooch et al., 2016). In previous research we have dealt with this in the context of signal theory (Hochmuth & Peters, in press; Hochmuth & Schreiber, 2015). In particular, we have referred to Castela and Romo Vázquez (2011) in which institutional references of mathematical practices (in the sense of the Anthropological Theory of Didactics (ATD) (Bosch & Gascón, 2014; Chevallard, 1992)) were reconstructed. In (Peters & Hochmuth, in press) we have also

✉ Reinhard Hochmuth
hochmuth@idmp.uni–hannover.de

[1] Institute for Didactics of Mathematics and Physics, Leibniz University Hannover, Welfengarten 1, D-30167 Hannover, Germany

taken up their idea of an extended praxeological model, which they used to distinguish between practical and academic knowledge aspects in a vocational context. In contrast, we focused on investigating various elements of mathematical knowledge taught within a signal theory course in the context of an electrical engineering university program. We distinguished between a mathematical HM- and a mathematical ET-discourse[1] and showed, that the extended model and the identified discourses allow to figure out a complex interweaving of mathematical and engineering practices that cannot be conceptualised on the basis of a view that separates mathematics, the application of mathematics and engineering science. For characterising the mathematical discourses, we most notably used studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004, 2012).

In this study, too, the ATD serves as our theoretical framework. ATD is also used by González-Martín and Hernandes-Gomes (2018, 2019), where they address curricular differences between mathematics and engineering courses. In particular, they ask about the appropriateness of practices with regard to aspects of the integral concept and the use of integrals in calculus and mechanics courses. Similarly to Dammann (2016, p. 97), they notice that the mathematical requirements in statics lie primarily in the areas of basic arithmetic, the processing of linear equations and systems of equations, and that it is not necessary for students within statics to master the mathematical procedures of differential and integral calculus. By curricular differences we mean the phenomenon that there are mathematical topics or argumentation contexts that are dealt with in higher mathematics, but not in engineering courses, or that are substantially different in engineering courses and vice versa. This lack of fit is an important observation and certainly gives cause to think about appropriate adjustments and, if necessary, to incorporate them into the curriculum.

In contrast to these investigations, our research interest is directed towards refined analyses of mathematical practices in the engineering sciences and the interplay of different mathematical discourses within these practices. In (Peters & Hochmuth, in press) we analysed an exercise and could identify transitions and interrelations within single processing steps of a sample solution that goes beyond the vision of a pure application of mathematics in the engineering context, a vision which has been coined by Barquero, Bosch and Gascón (2011) as applicationism. Both the analysis of the exercise and the figured-out characteristics of the HM- and the ET-discourse referred to the institutional level in the sense of the ATD. If one now looks at solutions of the exercise by individual students, the question arise, whether the identified transitions and interrelations possibly appear as breaking points in solution processes, or, more generally, whether and how they can be found there. To answer this question is the main objective of this paper.

Since ATD distinguishes between institutional praxeologies and individual activities the question comes up how institutional analyses can be used to analyse student solutions. In our view, this question is not fully answered in ATD. For more details regarding the appraisal and argumentations in ATD studies we refer the reader to

[1] The acronyms HM and ET were introduced by Peters and Hochmuth (in press) to denote the two relevant contexts of “Höhere Mathematik” (HM, higher mathematics) and “Elektrotechnik” (ET, electrical engineering) and associated discourses. HM and ET are the standard German actronyms for these contexts. Although the English term electrical engineering requires the acronym EE, for reasons of consistency we stick here to the acronym ET.

section 3. To make progress on this issue, we need to clarify the meaning and analytical status of discourse more concretely than in previous contributions. Following our preceding work our use of discourse still focuses on subject-specific[2] aspects and is based on the concept of praxeology in ATD. Beyond that, however, it will prove fruitful in the following to link the notion of discourse with Weber's (1904) concept of ideal types. The ideal type concept turns out to be compatible with the ATD framework and from it methodical steps can be derived for analysing individual contributions on the basis of institutionally based models. From the point of view of the ATD Gascón and Nicolás (2017) have dealt with Weber: With reference to Weber's distinction between normative and scientific statements, the authors specified the position of the ATD with regard to responsibilities and objectives of didactics as a science. To our knowledge, the ideal type concept has not been discussed in the ATD so far.

We have structured our contribution as follows: In section 2 we characterise both the electrical engineering (ET) and the higher mathematics (HM) context representing important reference points for our analyses of the signal theory exercise and related student solutions. To illustrate the characterisations of two different mathematical discourses and to delimit the two contexts, we use the topic of complex numbers and their different subject-specific rationales in electrical engineering and higher mathematics. Finally, we give a short introduction to amplitude modulation, the specific subject of the exercise. In section 3, we introduce the ATD notions that we will subsequently use to grasp subject-specific aspects. Against the background of differences regarding the two course-contexts we introduce the two institutional mathematical discourses, the HM- and the ET-discourse and connect them to Weber's (1904) concept of ideal types. Additionally referring back to the theory of rational explanation (Schwemmer, 1976) and with the focus on subject-specific aspects we finally identify a methodical procedure with four steps for applying the institutional analysis to individual student solutions. Section 4 then starts by an institutional analysis of the exercise. The presented analysis is based on the analysis in (Peters & Hochmuth, in press), but develops it further with a view to the intended use in the current contribution. Applying the ideal typical mathematical discourses, we generate a praxeological model which we also present as a graphical scheme. The model is subsequently used to analyse the student solutions following the previously identified methodical steps. In particular we provide answers to the question of whether and how the institutionally identified transitions and interrelations regarding the mathematical discourses can be found there. In section 5 we finally discuss the obtained insights and present a few ideas for teaching based on them.

Context of the Study: Signal Theory and Amplitude Modulation

Focus of our analyses are student solutions of an exercise from a signal theory course. Signal theory courses are one of the first in-depth courses in electrical engineering studies at German universities. They are usually scheduled for the third or fourth

[2] In the institutional context, we usually skip the institutional and simply speak of subject-specific. This seems justified to us, since discipline-specificity is unthinkable without institutions. With regard to the individual level, we use the term individual subject-specific. The term subject-related, on the other hand, addresses aspects that consider the individual as subject, including societal and psychological moments.

semester, after students have attended courses on higher mathematics for engineering and introductory theory-orientated electrical engineering courses. Signal theory is considered to be very mathematical, while the extent to which formal mathematical concepts are elaborated varies. Sometimes it is offered as a Fourier analysis course, as found in mathematics studies, where electrical engineering terms or problems are hardly considered. In contrast our contribution deals with data from a signal theory course that is strongly oriented towards electrical engineering. The variation of mathematical formalism is also present in the definitions of electrical engineering concepts.[3] Engineering concepts are closely related to physical quantities. They are always related to measurement and to real phenomena.[4] These observations already indicate two characteristics of the electrical engineering context: on the one hand the reference to reality, and on the other hand a very different degree of explication of this reference to reality, which is accompanied by a different degree of mathematical formalisation. This is also mentioned by Fettweis (1996, p. i) and addressed as a dilemma: an increasing mathematical formalisation of concepts can make it increasingly difficult to understand their physical meaning and justification.

The notion of system refers to a third characteristic: Frey and Bossert (2009) generally understand a system to be "an abstracted arrangement that relates several signals to one another. This corresponds to the mapping of one or more input signals to one or more output signals". (p. 3). They first introduce a system type that is easy to handle mathematically and consider only one input and one output signal each, since "this makes the system thinking [Systemgedanke] easier to grasp." (p. 6). A system can be understood as a black box that responds to a specific input signal with a specific output signal. Studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004, 2012) show that system thinking and the electrical engineering way of doing and talking about mathematics, differs from the way of mathematicians. The authors argue that "this linguistic shift is more than just jargon, and more than just a handy way of coping with the mathematics" (Bissell & Dillon, 2000, p. 10).

To further illustrate the electrical engineering way of thinking and doing mathematics, we give a short overview on how complex numbers and relating concepts like phasors are treated in an introductory course on electrical engineering and in a course on higher mathematics for engineers respectively. The following observations are based on standard literature, lecture notes and students' notes for two consolidated standard courses which are held every year at the University of Kassel. Complex numbers play also an important role in the exercise (cf. Appendix) we examine in this paper. We will argue that in those courses the respective rationales of complex numbers and their justifications are different and that these variations constitute partly conflicting resp. complementary reference points for students in the signal theory course.

In Albach (2011), a standard textbook for introductory courses on electrical engineering, phasors are introduced with the purpose to graphically describe time-dependent sinusoidal[5] functions, see Fig. 1. The first introduction of phasors is without

[3] For example, compare definitions in the two books by Frey and Bossert (2009) and Fettweis (1996). Both books are recommended as standard literature for the signal theory course we are studying.

[4] For a more detailed discussion of such epistemological issues regarding the relationship of mathematics and empirical sciences we refer to Hochmuth und Peters (in press).

[5] Circuits are operated with sinusoidal current- and voltage forms in the power supply network as well as in many other important areas.

references to complex numbers: A phasor [Zeiger] is an arrow with a specific length and a specific angle with respect to a reference.

When analysing electrical components, the amplitude ratio of the input signal to the output signal and the phase shift caused by the component are of primary interest. Therefore, phasors are important graphic tools for interpretation and analysis of electrical engineering processes. Current- and voltage ratios in electrical networks can be displayed and analysed graphically in phasor diagrams without using complex numbers or differential equations. For the purpose of an algebraic description of phasors, the plane in which phasors are drawn, can be considered as the complex plane. The phasor can now be understood as a complex quantity that symbolically represents the time-dependent voltage (see Albach, 2011, p. 42). The compatibility of the rules for manipulating phasors and the calculation rules of complex numbers is justified via physical relations. Furthermore, for a sinusoidal quantity the following holds: $A\cos(\omega t+\varphi) = \Re\left(Ae^{j(\omega t+\varphi)}\right) = \Re\left(Ae^{j\omega t}e^{j\varphi}\right)$, where A is the amplitude, ω is angular velocity, φ is the phase angle (each independent of time) and j denotes the complex unit in electrical engineering. The factor $\underline{A} = Ae^{j\varphi}$ then is the mathematical representation of the phasor, graphically represented by an arrow with length A and angle φ with respect to a reference zero angle. The function $\underline{A}e^{j(\omega t)}$ is a representation of a rotating phasor in the phasor- or Argand diagram.

In the course on higher mathematics for engineers, complex numbers are considered in the first semester in the context of Linear Algebra (Strampp, 2012). Their introduction is motivated by the solvability of the equation $x^2 + 1 = 0$. For this purpose, real numbers are extended by a number i with the property $i^2 = -1$. This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations. This results in considerations about the general solution of algebraic equations, the fundamental theorem of algebra and Vieta's formula. Calculation rules for complex numbers are derived without introducing and proving formal concepts, but by stating that all rules which are relevant for calculating with real numbers should continue to be applicable (p. 59): Also, in further contexts it is pointed out that various terms are an extension of already known concepts from real numbers. For example, the complex exponential function $e^{i\phi}$, which is introduced to serve as an abbreviation for $\cos(\phi) + \sin(\phi)i$. Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, it is subject to an orientation towards the inner-mathematical, generalisation-oriented rational of academic mathematics.

In addition to the algebraic view on complex numbers, the chapter contains another, geometric, orientation: An analogy to vectors is established, but the vector concept is also distinguished from complex numbers: "We speak of phasors[6] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This

[6] We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrical engineering aspects play no role in the course and Strampp (2012) does not refer to them either. Another possible translation of Zeiger, without the connection to engineering concepts would be pointer. But we decided to use phasor for the following reason: In German, the term Zeiger is used both in electrical engineering and in mathematics courses for engineers, but with different meanings (reference to electrical engineering concepts vs. geometrical object with no further references). By using the term Zeiger instead of vector Strampp (2012) can thus establish a connection to the electrical engineering courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different course-contexts is in jeopardy of being lost through translation.

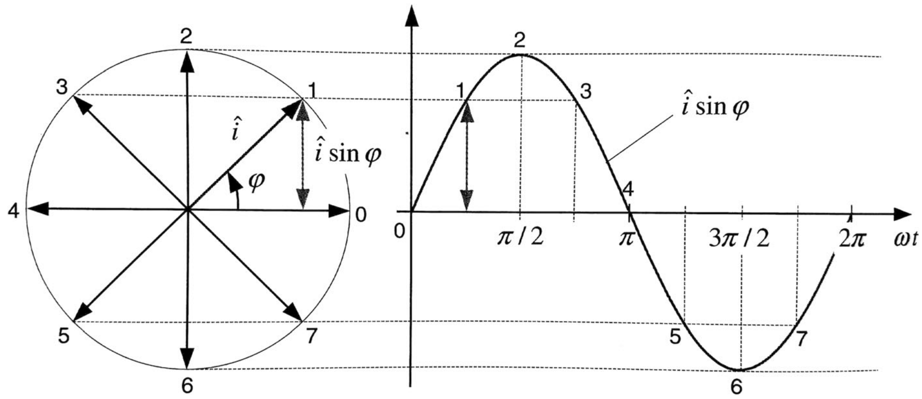


Fig. 1 Relationship between phasor and time-dependent function (Albach, 2011, p. 32)

multiplication extends the multiplication of real numbers." (p. 60). This phasor concept in the higher mathematics context differs from the phasor concept in the electrical engineering context: In HM the geometrical representation of complex numbers as arrows in the Argand diagram is used as a visualisation of calculation rules. In ET phasors are arrows that represent measurable, time-dependent quantities such as alternating voltages or currents. Complex numbers are then used for the convenient algebraic description of phasors.

In summary, whereas the electrical engineering context is notable for system thinking and for references to reality with different degrees of explication of this reference accompanied by different degrees of mathematical formalisation, the higher mathematics context is characterised by statements without references to reality and an inner-mathematical understanding and justification of concepts, in particular and, a generalisation-oriented rational following academic mathematics and a concentration on calculation rules.

The exercise we investigate in this paper belongs to *amplitude modulation* (AM), a central topic in signal theory. The principle of amplitude modulation is illustrated in Fig. 2:

The amplitude of a high-frequency carrier signal (Fig. 2, left) is varied corresponding to the course of the low-frequency message signal $s(t)$ (Fig. 2, middle). The AM signal (Fig. 2, right) can be represented as $x(t) = A[1 + m\ s(t)]\cos(2\pi f_0\ t)$ where $\cos(2\pi f_0)$ is the carrier signal. The modulation index m is the ratio between the amplitude of the carrier signal and the amplitude of the message signal, in addition the restrictions $\max_{t \in R} |s(t)| = 1$ und $0 < m < 1$ apply. With amplitude modulation, several message signals (e.g. for different radio stations) with different carrier frequencies can be transmitted via antenna and received without crosstalk between signals at the receiver (radio set) depending on the chosen frequency.

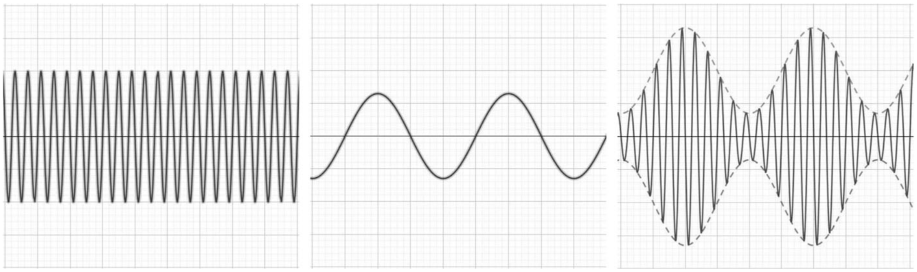


Fig. 2 Carrier signal (left), message signal s(t) (middle), and AM signal x(t) (right)

The Anthropological Theory of the Didactic and Ideal Typical Mathematical Discourses

ATD is a research program to study (mathematical) practices from an institutional perspective. The notion of *institution* within the ATD is related to the work by Mary Douglas (1986), who draws on ideas of Durkheim and Fleck. Her main point relevant for ATD is the elaboration of the idea that all knowledge depends on (social) institutions and conversely all institutions are based on shared knowledge (p. 45). In the following we will take a closer look of how observations regarding institutional practices and contexts can be referred to individual subject-specific analyses. In this respect, we will refer in particular to Weber's concept of the ideal type and, finally, propose a procedure in four steps. But first we will introduce some basic terms of the ATD. These constitute our starting point for linking the characterisations reported in section 2 with Weber's concept of ideal types.

The 4 T-Model and the Institutional Dependence of Knowledge

In ATD knowledge is related to human activities including not only aspects of know-why but also practical knowledge in the sense of know-how. This is subsumed under the term *praxeology*:

> What exactly is a praxeology? ... One can analyse any human doing into two main, interrelated components: *praxis*, i.e. the practical part, on the one hand, and

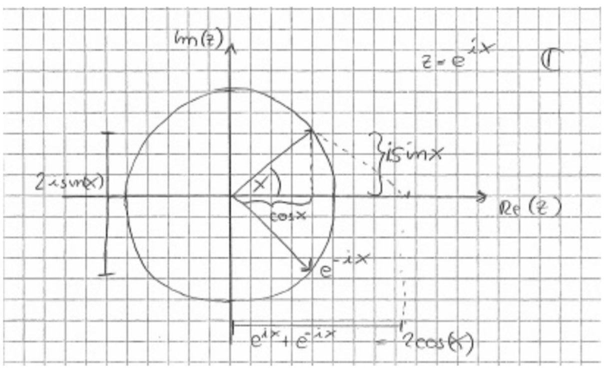


Fig. 3 Graphical representation of complex numbers. Students' lecture note

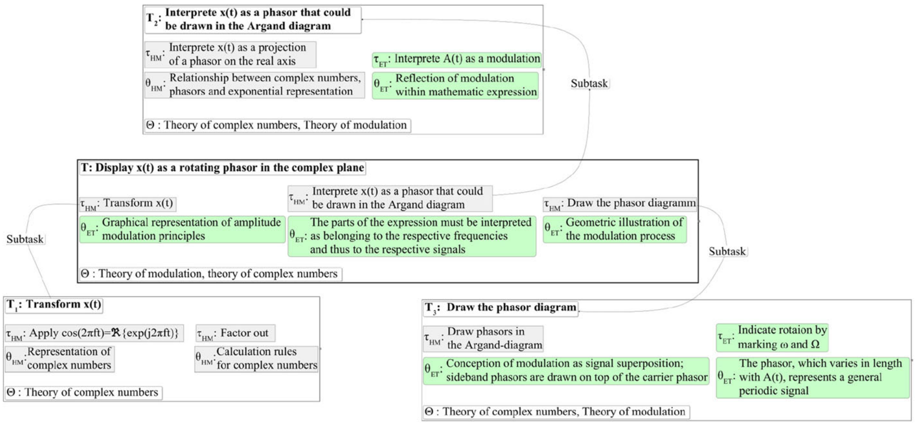


Fig. 4 Graphical representation of the institutional analysis

> *logos*, on the other hand. … How are *P* [Praxis] and *L* [Logos] interrelated within the praxeology [*P/L*], and how do they affect one another? The answer draws on one of the fundamental principles of ATD ... according to which no human action can exist without being, at least partially, 'explained', made 'intelligible', 'justified', 'accounted for', in whatever style of 'reasoning' such as an explanation or justification may be cast. *Praxis* thus entails *logos* which in turn backs up *praxis*. (Chevallard, 2006, p. 23)

A praxeology thus is a basic epistemological model to describe knowledge in the form of the two inseparable and interrelated blocks Praxis and Logos. Those two blocks can be differentiated further: the praxis block *P* (know-how) consists of problems or *tasks* T and a set of relevant *techniques* τ used to solve them. The logos block *L* (know-why) consists of a two-levelled reasoning discourse.[7] On the first level, the *technology* θ describes, justifies, explains etc. the techniques and on the second level the *theory* Θ organises, supports and explains the technology. Since praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is related to the discourse. In short praxeologies are denoted by the *standard 4T-model* $[T, \tau, \theta, \Theta]$.

In the Chevallard quote, the part "in whatever style of 'reasoning' such as an explanation or justification may be cast" refers to the idea that institutional conditions constitute the technological-theoretical discourse and the practices available. Regarding in particular the relationship of institutions and techniques, Chevallard (1999) writes:

> Finally, in a given institution *I*, with regard to a given type of task *T*, there is usually *only one* technique, or at least *a small number* of *institutionally recognised* techniques, to the exclusion of possible alternative techniques - which may actually exist, but then *in other institutions*. (p. 225, our translation)

Accordingly, we use the notion *scope of the technique* to address the set of tasks, which can be solved with the institutionally recognised technique. In considering one specific

[7] Within the ATD the term discourse, e.g. in expressions like "reasoning discourse" or "a discourse on praxis" is used in the etymological sense (e.g. Bosch & Gascón, 2014, p. 68).

piece of knowledge in different institutions, different praxeologies emerge: different types of tasks are relevant, different solution techniques are adequate, and different reasoning discourses are acceptable and constitutive. These relationships are addressed by the term *institutional dependence of knowledge*. In view of section 2, we can accordingly say that in the institution of electrical engineering, the ET-context, and in the institution of higher mathematics, the HM-context, there are different praxeologies concerning complex numbers. Furthermore, the characterisations of the two contexts can be understood as descriptions of institutional aspects that shape the logos block and thus, due to the dialectic of praxis and logos, also the practical part of praxeologies. In the following we understand the characterisations of the two contexts as characterisations of two different mathematical *discourses*[8] and associate praxeologies or praxeological elements to the mathematical *ET-discourse* or the mathematical *HM-discourse*, if they can be characterised according to the institutional ET- or the institutional HM context, respectively.[9] In order to articulate the references to the different institutional mathematical discourses, we use the further notations τ_{HM} and τ_{ET} as well as θ_{HM} and θ_{ET}.

The Difference between Institutional Praxeologies and Individual Activities

The institutional dependency of knowledge implies that human activities are constituted and located in institutions: A praxeology does not present itself as something individual, but as something institutional and societal. To stress the difference between institutional praxeologies and actual individual activities, Chevallard uses the notion of *relation to objects* (of knowledge): Institutions are based on shared knowledge, every object of knowledge O is in relation to the institution I, noted as $R_I(O)$. A praxeology is a concept to study the subject-specific content of those relations. Similarly, every person X, that acts with an object of knowledge O is in an individual relation to it, noted as $R(X, O)$.

> A person X becomes a good subject of I relative to the institutional object O when his personal relation $R(X, O)$ is judged to be *consistent* with the institutional relation $R_I(O)$. This person may also prove to be a bad subject, … and may, in the end, be expelled from I. Here is where a development relating to *intra-institutional evaluation* comes into play, relating to the mechanisms according to which I is led to pronounce, through some of its agents, a verdict of conformity (or non-conformity) of $R(X, O)$ to $R_I(O)$. . …. In particular, the institutional relation … is nobody's personal relation, … : conformity is not identity. (Chevallard, 1992, p. 146/7)

[8] This use of the term discourse goes beyond an etymological understanding (cf. footnote 6). We extend thereby a term which already exists within the ATD. Our extended understanding blends into the already existing concepts (e.g. institutional dependence of knowledge). We do not use the term discourse in the sense of discourse theory. Due to the limited word count, we refrain from further elaboration of possible connections and delimitations.

[9] The two mathematical discourses can also be connected to the work of Artaud (2020), where she describes two types of didactical transposition processes: An external didactical transposition process, originating in academic mathematics research institutions. Here one can locate the HM-discourse. And an endogenous didactical transposition process concerning processes within the engineering institution. Here one can locate the ET-discourse.

In summary, identified institutional praxeologies and discourses must be distinguished from individual actions and their products. But, although persons do not have to reproduce the specific institutional logos in a specific institution context, they provide *points of references* for their actual practices. Consequently, institutional praxeologies also provide important reference points for analyses of individual products, but cannot be directly and unmediatedly be related to them. Research in mathematics education sometimes neglect this difference[10] and rather identifies individual actions and institutional praxeologies. Regarding these arguments, it is important to stress once again that we are focusing on subject-specific aspects. For example, Hardy (2009) analyses students' interpretations of institutional praxeologies in view of political and educational issues. Thus, of course, differences between students' actions and institutional praxeologies are considered. But, focusing on the subject-specific, if students use the right symbols and write things down as they have been worked out in practice, both actions are identified with each other. Because of these implicit identifications the insightful explanations of the student's interpretation possess an hypothetical character also with respect to subject-specific aspects and not only with respect to the considered political and educational issues. To support the overall relevance of these latter issues Hardy proposed further studies (p. 357). However, it is not reflected in detail how this could actually contribute to clarify differences regarding subject-specific issues, particularly in view of general characteristics of discourses.

HM- and ET-Discourse and Weber's Notion of Ideal Type

To deal further with the issue of the difference between institutional praxeologies and individual products regarding the HM- and ET-discourse we will recur in the following to Max Weber's (1904) construct of ideal type. Its use in empirical research to explain individual actions has been investigated for many decades. We will refer to and adapt these diverse and in part in-depth investigations focusing on subject-specific aspects. This methodological considerations will enable us in the following to indicate a suitable approach (see the four steps at the end of this section) and to clarify its possibilities and limitations.

Weber (1904) introduces *ideal types* as a construction,

> which is obtained by *mental* enhancement of certain elements of reality. Its relationship to the empirically given facts of life consists merely in the fact that where connections of the kind represented abstractly in that construction ... are *determined* or *suspected* to be effective to some degree in reality, we can pragmatically *visualize* and understand the *peculiarity* of this connection on an *ideal type*. This possibility can be both heuristic and indispensable for the representation of value. … The ideal typical term … is not a 'hypothesis', but it wants to show the direction of hypothesis formation. It is not a *representation* of the real, but it wants to give the representation unambiguous means of expression. (p. 64/5, our translation)

[10] As expressed within ATD as difference between institutional and individual relations to objects of knowledge.

According to Weber's introduction, we interpret the two mathematical discourses as ideal types and the underlying characterisations of the discourses as a result of "mental enhancements" regarding aspects of mathematical practices within specified institutional contexts. Furthermore, we use the two ideal typical mathematical discourses as heuristics for our subject-specific analyses of the exercises and the student solutions as well as to formulate hypotheses. Thus, following Weber's formulation, we assume that the ideal typical discourses are to a certain extent effective and represent a means of expressing something real in sample solutions and students works on exercises. Thereby the relevance of this something in individual productions cannot be substantiated by the ideal typical discourses themselves, but has to be shown in the individual productions in a concrete and subject-specific way.

Weber's construct of the ideal type has been widely criticised and expanded in the social sciences (see Shubat, 2011, especially Chapter 1.4). There is no space here to elaborate on this in detail. Regarding our use, the following question adapted from Schwemmer (1984, p. 177) is particularly relevant to avoid circularity: In which way do the characterisations associated with the mathematical discourses enter into the empirical studies of individual productions and their results without withdrawing the characterisations from empirical criticism? Our standpoint on this is the following: Whether something is apparent in individual-related data whose concrete meaning can be demonstrated by means of one of the two mathematical discourses is empirically open with regard to the following two dimensions. (1) From a subject-specific point of view, the appearance must be shown in the specific context of exercises and related praxeologies. Thus, for example, it could be proven empirically that a subject-specific context cannot be reconstructed as a particular case of application of the mathematical discourses. (2) In subject-related respects, for example with regard to the question of whether a student experiences the identified aspects as such, corresponding claims are accessible to empirical criticism. Here, it could be proven empirically that the connections formulated by means of mathematical discourses do not contribute to an understanding of the students' thoughts about her actions. In this article we focus on (1), the subject-specific perspective. With regard to the rational explanation of concrete individual products, we adopt a further argument by Schwemmer (1976, p. 142) concerning the precondition of purpose-rationality as a methodological postulate: Our subject-specific analyses are based on the assumption that solutions and works on exercises follow the demand for a respective institutionally set subject-specific rationale. This is a methodological requirement, since otherwise (i.e., assuming that the students and their works do not follow any particular institutionalised disciplinary rationale) any relationship between institutional disciplinary analyses and the analysis of concrete exercises would be questioned from the outset.[11]

Methodological Consequences for Subject-Specific Analyses

Against the background of ATD concrete individual actions or their products are thus considered to be *explained* if their rational can be connected to praxeological analyses and

[11] This does not contradict the empirical openness discussed above with regard to the two mathematics discourses, since a concrete case could follow a different disciplinary rational, which might be completely independent of the ones we reconstructed.

the identified ideal typical discourses. In this respect, observations should be justified that certain mathematical actions can be substantiated on the basis of praxeologically reconstructed practices, including the consideration of ideal typical discourses. This implies formulating hypotheses in observation language [Beobachtungssprache][12] with a view to the respective subject-specific context. Specifically examined action situations are then to be validated with regard to the existence of corresponding observation correlates [Beobachtungskorrelate].[13] Hereby, the ideal typical discourses identified by us prove to be genetic concepts, i.e. they function as a guideline for hypothesis formation and allow the rational to be grasped in a concrete action situation with regard to its institutional meanings. The insights into practices gained through the discourses are of course not considered to be valid for all times. Thus, for example, the modification of institutionalised practices (for example, a modified treatment of complex numbers in higher mathematics courses) can change the role of the identified ideal typical discourses in the reconstruction of concrete practices.

In summary and with regard to our subject-specific focus, this results in the following sequence of methodical steps:

1. At the institutional level, praxeologies are to be identified and connected to ideal typical discourses.
2. Hypotheses in the specific context of signal theory or the exercises are to be formulated using observational language.
3. Concrete material (exercises, sample solutions and students' solutions) is to be validated with regard to corresponding observation correlates.
4. If these are available, concrete material can be explained on this basis.

In the next section we first present the institutional analysis of the lecturer's sample solution with a view to using it as a reference for the analyses of the students' solutions (step 1). This will also illustrate the ATD concepts introduced in this section. Thereby we will use the graphical method developed in (Peters & Hochmuth, in press) to present the result of the institutional analysis, see Fig. 4. In the next step we move on to the student work and formulate corresponding hypotheses (step 2) in observational language. With reference to these hypotheses, we will then identify observation correlates with respect to the rationales and, in particular, the ideal typical mathematical discourses (step 3). Finally, we explain the student solutions regarding their institutionalised subject-specific rationales (step 4). Here we use the graphical scheme, i.e. Figure 3 without text, to represent the analysis results of the students' solutions, see Figs. 5a, 6a, 7a, 8a, and 9a.

Analyses of the Student Solutions

The exercise and the sample solution are presented in the appendix: The complete exercise consists of three items. Results for items 1 and 2 and the complete sample

[12] Using observation language means to describe an observation without interpretations (Schwemmer, 1976, p. 165), whereas interpreting means to show actions as rational in purpose or sense (p. 168).
[13] The observation correlate of an action is the part of the action that is observed and described in observation language (Schwemmer, 1976, p. 168).

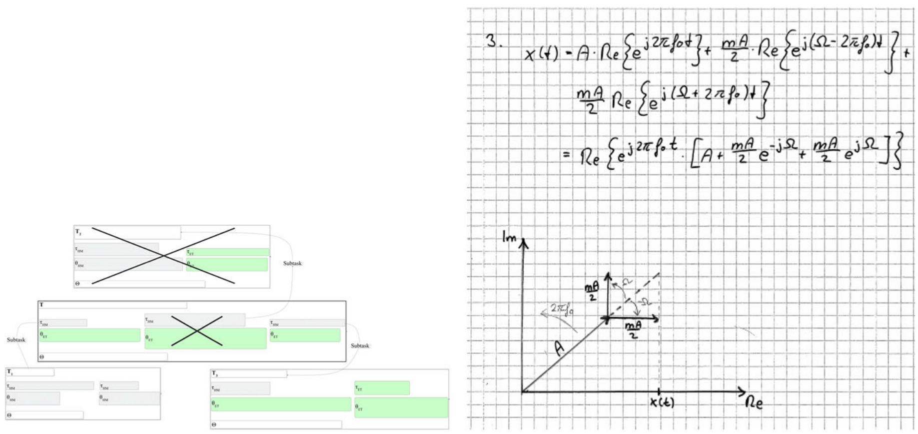
solution for item 3 is shown. In item 1, a given message signal should be amplitude modulated. For this purpose, the given term $s(t) = \cos(\Omega t)$ must be inserted in the formula for amplitude modulation: $x(t) = A[1 + m\, s(t)]\cos(2\pi f_0\, t)$. In item 2, this term should then be represented as an expression of three harmonic oscillations. And finally, in item 3, that is focussed in our analyses, $x(t)$ has to be graphically displayed in the complex plane as a rotating phasor with varying amplitude, using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ and the result of item 2.

Step 1: Institutional Analysis

Item 3 is solved in three steps: [1] Transforming mathematical expressions, [2] Interpreting mathematical expression to draw a diagram, and [3] drawing the phasor diagram. The main part of the exercise, to display $x(t)$ as a rotating phasor in the complex plane, is a task (T) in the sense of the ATD. We then assign technique and technology to each of the three solution steps [1] to [3], and only roughly summarise theoretical aspects (see the bold framed rectangle in Fig. 4). This assignment of techniques and technologies is then differentiated and refined in a second analysis step, in which the three techniques assigned to the steps [1] to [3], are considered as subtasks T_1 to T_3 (see the corresponding light framed rectangles in Fig. 4). All in all, we also present the result of the analysis in Fig. 4 graphically. This graphic representation will afterwards serve as a scheme for the analysis of the student solutions.

First we consider the three techniques that we assigned to the steps [1] to [3]: Each of the three techniques in T is located within the HM-discourse: [1] To transform mathematical expressions (τ_{HM}), [2] to interpret this expression for drawing a diagram (τ_{HM}), and finally [3] to draw a phasor diagram (τ_{HM}) are activities that are present in the corresponding higher mathematics course. For example, Fig. 3 shows a diagram from the first lecture of the signal theory course which dealt with repeating the properties of complex numbers according to the prior HM-course (definitions, calculation rules, phasors as visualisations of properties of complex numbers).

The mathematical expressions in the exercise and the diagram to be drawn are more complicated than corresponding content in the higher mathematics course, but the



a **b**

Fig. 5 **a** Analysis result for C1. **b** Student solution C1

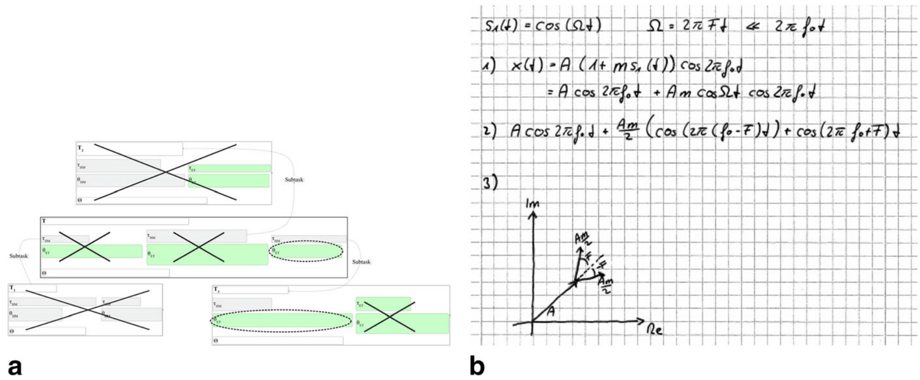


Fig. 6 **a** Analysis result for C2. **b** Student solution C2

techniques themselves are certainly HM-typical. Furthermore, there is no aspect here that suggests a location of the techniques within the ET-discourse.

Second, we consider the corresponding technologies: We have located the corresponding technologies within the ET-discourse. Here, aspects of scope and purpose and elements of justification arise that can no longer be located in the HM-discourse. In the following we will go through steps [1] to [3] and discuss the corresponding technologies in more detail:

[1] The signal $x(t)$ must first be transformed in lines (1) to (3) (τ_{HM}). Then $x(t)$ can be interpreted as a real part of a rotating carrier phasor with a time-dependent amplitude $A(t)$ and aspects of amplitude modulation could be visualised in the diagram. This scope of the technique is based on the idea of graphically representing the principles of amplitude modulation, therefore this first technique is justified within the ET-discourse (θ_{ET}). This justification, that is linked to the overall aim of the task and also already links to steps [2] and [3] of the solution process, can be further focused on the first step: In particular the calculation step

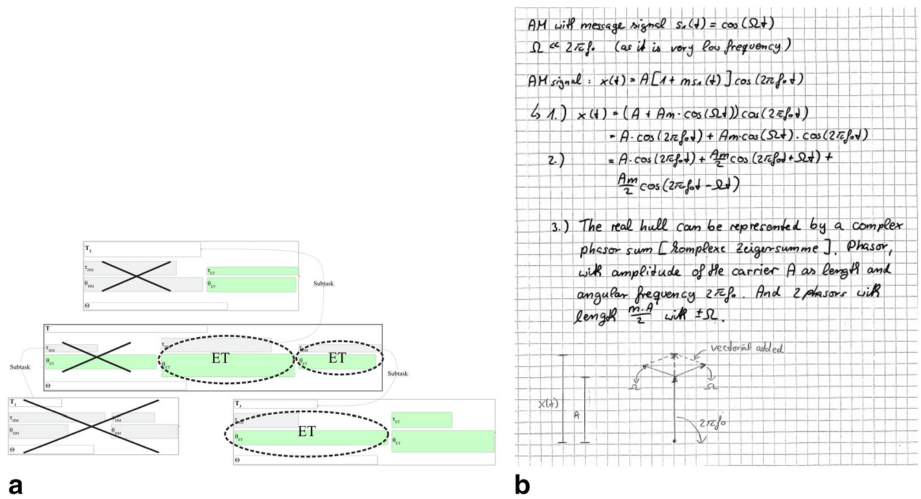
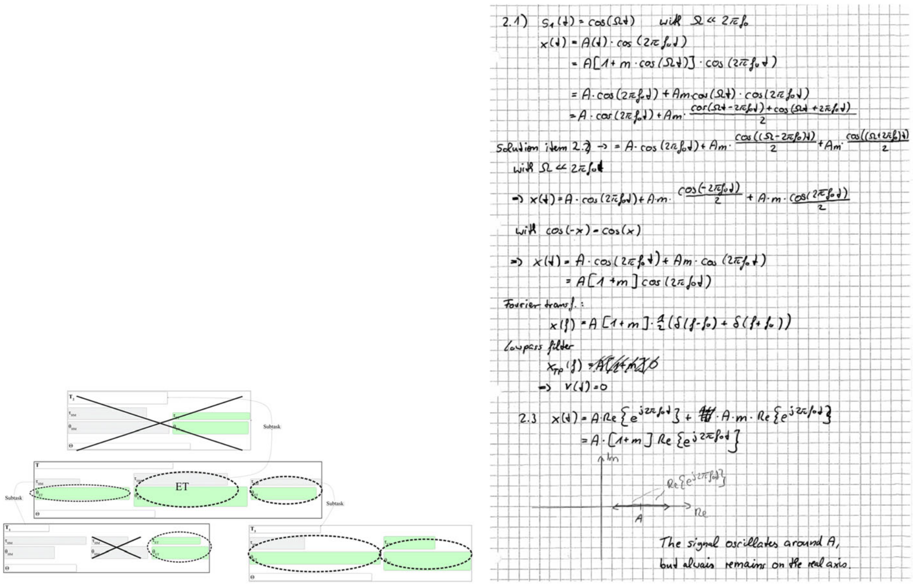


Fig. 7 **a** Analysis result for I1. **b** Student solution I1

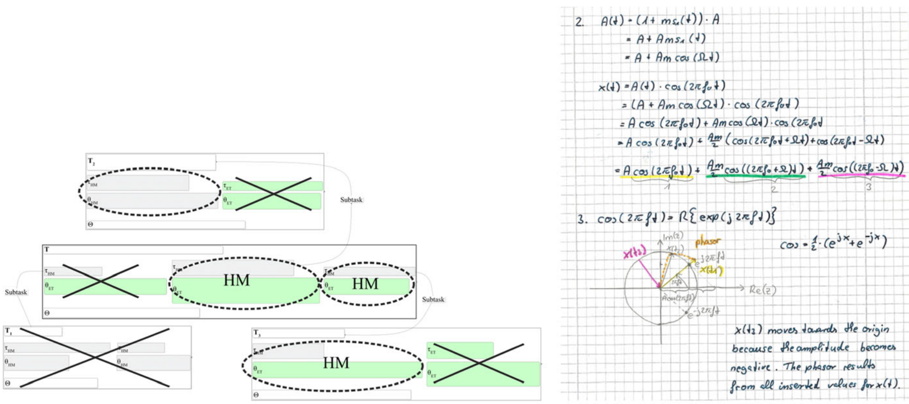


a **b**

Fig. 8 **a** Analysis result for I2. **b** Student solution I2

from line (2), in which $x(t)$ could be interpreted as a real part of three rotating phasors drawn in the origin, to line (3), in which $x(t)$ can be interpreted as a rotating carrier phasor with time-dependent amplitude $A(t)$, is central here. Only if $x(t)$ is represented as in line (3), $x(t)$ can be interpreted as amplitude modulated and a phasor diagram can be drawn in which the amplitude modulation of the signal $x(t)$ can be displayed graphically. There is no justification within our reconstructed HM-discourse, that justifies the step from line (2) to line (3).

[2] In the next step the mathematical expression must be interpreted in order to draw the diagram (τ_{HM}). The central point here is that the components of $x(t)$ must be



a **b**

Fig. 9 **a** Analysis result for I3. **b** Student solution I3

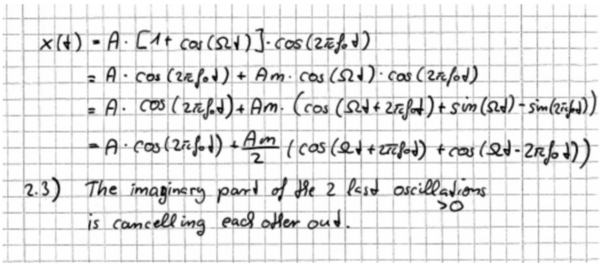


Fig. 10 Student solution N1

interpreted as belonging to the respective frequencies and thus to the respective signals (θ_{ET}).

[3] Finally, the third technique is the drawing of the phasor diagram (τ_{HM}). Here, however, this HM technique is embedded in the ET-discourse, since it is a geometric illustration of a modulation process (θ_{ET}). The phasor diagram is an alternative representation of the amplitude modulation in Fig. 2 (right). It has the advantage over the representation in Fig. 2 (right) that some effects relevant to amplitude modulation can be displayed.

In total, three techniques with corresponding technologies arise for steps [1] to [3], each of which is characterised by an embedding of HM-techniques in the ET-discourse. Relevant theoretical aspects come from theory of modulation and theory of complex numbers. In Fig. 4 this part of the analysis is illustrated in the bold framed rectangle. How the embeddings look like in each case will be clarified in a next step of analysis: Each of the three techniques will be considered as a separate subtask, T_1 to T_3, with its own techniques and technologies. In Fig. 4 these parts of the analysis are illustrated in the light framed rectangles.

In T_1, to transform $x(t)$, the identity $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ given in the problem definition must be applied (τ_{HM}). For the justification (θ_{HM}) of this step, the relation between the representation of a complex number in polar form and in exponential form is relevant. In the following, calculation rules for complex numbers (θ_{HM}) are applied, namely factoring out of the real part and of $\exp(j2\pi f_0 t)$ (τ_{HM}). These are techniques that occur in higher mathematics courses and are also correspondingly justified inner-mathematically. There are no references to ET aspects.

In T_2, to interpret $x(t)$ as a phasor that could be drawn in the Argand diagram, two relevant techniques play a role: First, the expression in line (3) must be interpreted as a

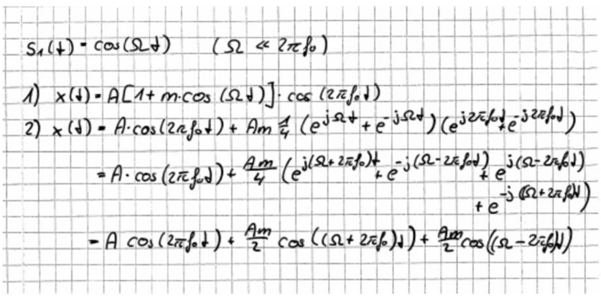


Fig. 11 Student solution N2

projection of a phasor onto the real axis (τ_{HM}). This technique is assigned to the HM-discourse, since also in higher mathematics complex numbers are understood as phasors, represented in Argand diagrams. The projection of the phasor on the real axis is interpreted as a real part of the complex number (cf. Fig. 3 and Strampp, 2012). This is justified in the introduction of complex numbers via the connection of complex numbers, phasors and exponential representation (θ_{HM}). And then the expression in the square brackets, $A(t)$, must be interpreted as modulation (τ_{ET}). Here knowledge about how the different frequencies are assigned to the different signal types, how carriers and message signals are related by the modulation process, and how this is reflected in the multiplication and the additions in the mathematical expression is relevant (θ_{ET}).

In T_3, the drawing of the phasor diagram, first the phasors must be drawn (τ_{HM}). This technique is located in the HM-discourse. But the concrete way in which the phasors have to be drawn, sideband phasors on top of carrier phasors, is explained by the fact that modulation is a superposition of signals (θ_{ET}). The corresponding technology is thus located in the ET-discourse. Here again a HM-technique embedded in the ET-discourse appears, which could be differentiated in a third step of analysis, which we will refrain from because it is not necessary for the analysis of the students' solutions. As a second technique, rotational aspects have to be marked by drawing curved arrows, which are labelled with the respective frequencies (τ_{ET}). This does not occur in higher mathematics courses and refers to electrical engineering aspects, hence our location in the ET-discourse. The markings also indicate that these are general periodic signals: This is based on the electrical engineering conception of the representation of signals by rotating phasors and that the length of the phasor $\exp(j2\pi f_0 t)$ changes time-dependently according to $A(t)$. So, the technology is also to be located in the ET-discourse.

Our analysis regarding the role of the two mathematical discourses in the three steps of the solution show that the embeddings take different forms in each case (see also Fig. 4): For the first step the embedding is formed only from HM-discourse aspects. In this case an interpretation in the sense of applying mathematics in the engineering context is plausible. For the second step the embedding is formed from both HM-discourse aspects as well as ET-discourse aspects. And for the third step the embedding contains another embedding of a HM-technique in the ET-discourse and its aspects. Except for step one, the view that mathematics is (simply) applied in electrical engineering is not adequate. Instead, diverse transitions between the two mathematical discourses appear. They constitute breaks in the sense that they each follow a different rational. These breaks often remain implicit, although they represent an important aspect. The breaks indicate places that are not accessible from a single discourse and its techniques, and thus mark something additional to be learned.

Steps 2 to 4: Analyses of Students' Solutions

Following the institutional analysis which showed that the shift of representation from the symbolic to the graphic form is the core of the solution and involves specific transitions between mathematical discourses, we will analyse the student solutions especially with regard to this shift of representation and focus our explanations on transitions between the mathematical discourses. We use the graphical representation of the institutional analysis as a reference for analysing the student solutions and as a tool to graphically represent the result of our analyses: Praxeological aspects that cannot be identified in the student solutions, because the data does not provide this, are crossed out in the corresponding diagram. Aspects that can be identified in the student solution and correspond to the aspects in the lecturer sample solution are displayed without specific marking. Aspects that are present but different are circled with dashes.

In total 15 students handed in their solutions for this exercise. According to the assistant's marking we categorised them as follows:

- **Correct diagram**: phasor diagram with no or minor corrections, e.g. added arrows indicating the direction of rotation or angle labels (4 solutions). See also C1 and C2 in Figs. 5b and 6b.
- **Incorrect diagram**: phasor diagram with major corrections, e.g. added arrows indicating phasors or adding a whole correct phasor diagram (5 solutions). See also I1 to I3 in Figs. 7b, 8b, and 9b.
- **No diagram**: student solutions, that contain calculations but no diagram (6 solutions). See also N1 and N2 in Figs. 10 and 11.

To protect the students' privacy, we have rewritten the solutions without reproducing the assistant's marking. All student solutions considered contain correct solutions for items 1 and 2 of the exercise, possibly with the exception of minor sign errors. In the remainder of this section we follow, thesis by thesis, steps 2 to 4.

Thesis 1: The Sample Solution Is Realised in Student Solutions

The student solution C1 in Fig. 5b largely follows the sample solution. Of the three steps in the solution process, steps [1] and [3] can be identified in C1. Step [2] is not identifiable, therefore it is crossed out in Fig. 5a. However, the exercise assignment did not ask for an explicit interpretation, so the absence is not a deficit.

In step [1] the transformation of $x(t)$ can be recognised as a HM-technique embedded in the ET-discourse. In particular, the step from line (2) of the sample solution to line (3), which is especially associated with the ET-discourse, is present. The change from line (2) to line (3) in itself represents, in the sense of our approach, a subject-specific observation correlate for the interplay of HM- and ET-discourse. In step 3 both aspects, the drawing of the phasors in their specific relations as well as the indication of rotational aspects are present. The validation of the observation correlates here essentially follow the institutional analysis; there is nothing in the student solution that suggests otherwise. With regard to the transitions between the mathematical discourses, we can state that, except for step two, the transitions from the institutional analysis occur. This supports our explanation of the student work C1 with regard to thesis 1 on the basis of the discourse-related observation correlate: The student solution C1 realises a correct solution in the sense of the sample solution. This does not imply that arguments and justifications can be found in the individual considerations of the students that realise further ideal typical aspects in the context of the change of presentation. However, the text-related analysis does at least point to this possibility.

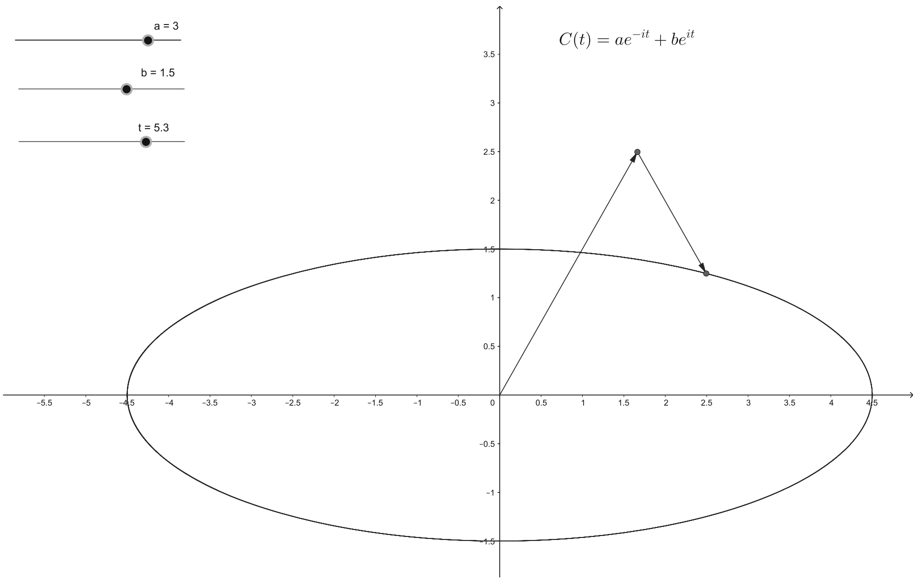


Fig. 12 Visualisation of curve and corresponding phasors with GeoGebra

Thesis 2. There Are (Almost) Correct Diagrams in Student Works, without the Step from Line (2) to Line (3) from the Sample Solution

The student solution C2 in Fig. 6b contains a correct diagram. Steps [1] and [2] are not present. The corresponding parts in Fig. 6a are crossed out. Especially the transformation step of the sample solution to line (3) is not realised. The diagram can be related to the solution for item 2: $A\cos 2\pi f_0 t + \frac{Am}{2}(\cos(2\pi(f_0 - F)\ t) + \cos(2\pi(f_0 + F)\ t))$. There, the

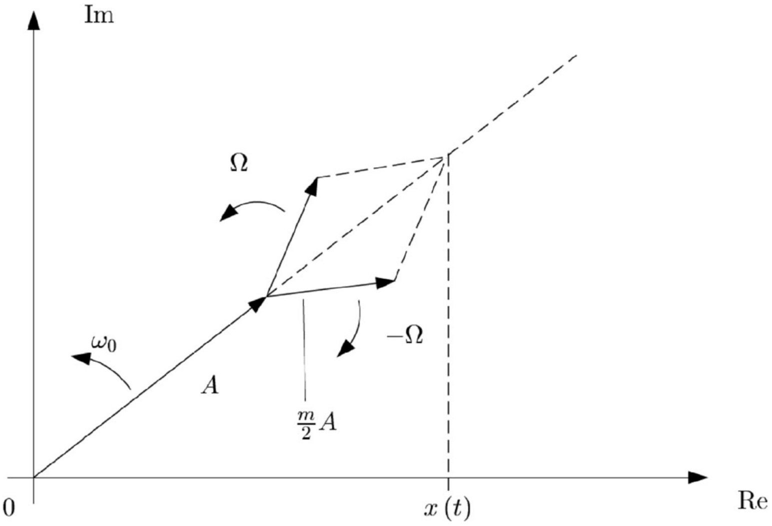


Fig. 13 Representation of $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0 = 2\pi f_0$

cosine representation does not allow to separate the different frequencies of carrier and message signal, which is, however, at the core of the representation in line (3) of the sample solution. This student work also does not contain any verbal justifications, the observation of which would allow further interpretations regarding the individual rationale. The correct drawing of the diagram, however, allows an institutionally supported interpretation possible within the ET-discourse: concepts from basic lectures in electrical engineering, which associate elementary signals and their cosine representations with a corresponding phasor representation (see Fig. 1), are used to create the diagram. This provides a justification within the ET-discourse that differs from the institutional analysis, thus the corresponding parts in the diagram are circled in dashed lines. It allows the correct diagram to be drawn without having to grasp what appears to be essential in the institutional context of amplitude modulation. This follows directly from the institutional analysis and is compatible with an explanation of the student solution, which identifies it as the realisation of the indicated ideal typical mathematical discourses: The drawing of the phasor diagram can be interpreted as HM-technique that is embedded in the ET-discourse, but with ET-aspects that differ from the institutional analysis and bridge the void that is formed by the missing amplitude modulation related ET-aspects. So, the step from line (2) to line (3) in the sample solution is not necessary to create a correct diagram since other ET-aspects, in particular the relationship between cosine and phasor (cf. Figure 1), could constitute a bridging.

Thesis 3: In Student Solutions that Do Not Contain a Diagram or a Wrong Diagram, both Obvious and no ET-Discourse Aspects Occur. In both Groups of Works, References to the HM-Discourse Occur. In Other Words, the Relationship between the ET-Discourse and the HM-Discourse Can Be Very Different in Student Solutions

We first deal with the solutions with incorrect diagrams (I1, I2 and I3 in Figs. 7b, 8b and 9b) and then move on to the student works without a diagram (N1 und N2 in Figs. 10 und 11). We do not give a diagram for the analyses results for N1 and N2.

In the student solution I1, see Fig. 7b, steps [2] and [3] can be identified. The result from item 2 is not transformed further, so the aspects that correspond with step [1] are crossed out in Fig. 7a. The student solution contains an interpretation with references to carrier phasor, two other phasors (representing the sidebands) and the respective frequencies but not to the corresponding mathematical expressions. Especially aspects, that focus the relationship of complex numbers, exponential representation and phasors are missing. Therefore, the HM-aspects in subtask T_2 are crossed out and the praxeological aspects in step [2] are solely located within the ET-discourse. In step [3] both aspects, drawing of phasors in their specific relation as well as the indication of rotation are present. But the phasors are not drawn in the Argand diagram. The student solution essentially contains correct and relevant aspects for graphically representing amplitude modulation. However, the assistant corrected this solution by providing the diagram from the sample solution. This indicates that the student's solution is not an adequate representation in terms of the institutional teaching-learning context. Student solution and sample solution differ in the following sense: the diagram in the sample solution contains phasors drawn into the Argand diagram and thus contains not only the link between amplitude modulated signal and phasors, but also the link to the mathematical description by complex numbers. This link to mathematisation is missing in the student solution I1. So, the HM-aspects in subtask T_3 are replaced by ET-aspects. Step [3], like step [2], is solely located in the ET-discourse.

This solution can thus be explained as a mathematically informal realisation of the ET-discourse: phasors are graphical representations of signals, which can be handled without a mathematical description (see also the introduction of phasors and complex numbers in electrical engineering in section 2). Transitions between discourses are not present.

The student solution I2 in Fig. 8b contains all three steps from the sample solution, but with significant deviations. Therefore, all three steps in T are circled in dashed lines, see Fig. 8a. In step [1] a variety of techniques are tried out: The given relationship $\Omega \ll 2\pi f_0$ is used to eliminate the Ω-part in the cosine. The attempts with Fourier transform and low-pass filter are represented in subtask T_1 by an additional ET-technique and –technology block that is not present in the schema of the institutional analysis. In the end the relation given in the problem definition is used to attempt finally a graphical representation of the term $A[1+m]\ Re\{e^{j2\pi f_0 t}\}$. All ET-discourse references appearing in this work, were covered in previous lectures of the course. The interpretation of the mathematical expression in step [2] focusses ET-discourse aspects like signals and oscillation. References to the Argand diagram, HM-discourse on complex numbers, or ET-aspects such as the connection of cosine or complex exponential function with the phasor representation are not present. Therefore step [2] is located solely in the ET-discourse. In I2 the Argand diagram is present, but no phasors. $x(t)$ is drawn as double arrow on the real axis. Therefore, the HM-technique embedded in an ET-discourse is present, but different from the institutional solution. Rotation resp. oscillation aspects are indicated, but these again differ from the sample solution. In summary both, HM-aspects and the ET-discourse, are clearly present, but not in a coherent and, in terms of the institutional solution, goal-oriented manner. In particular, the link between mathematical terms and their graphical representation in the spirit of modulation principles is missing.

The student solution I3 in Fig. 9b does not contain step [1]. Steps [2] and [3] are present, but differ from the institutional analysis. Therefore, the corresponding aspects in Fig. 9a are crossed out or circled in dashed lines.

Several elements in this solution indicate aspects of step [2]. First, each of the three cosine-terms is underlined with a different colour. Each term is thus individually interpreted as something to be drawn. These colours can also be found in the diagram, the respective phasors are marked accordingly. The ET-aspects from subtask T_2 are not present. The text explains the drawing, so it also gives hints on how to interpret the mathematical expression (in order to draw it). It contains functional aspects: “all values for x(t)”, insertion aspects, and “x(t1)”, “x(t2), x(t3)” for the three phasors. The Phasor diagram from the HM repetition in the first lecture of the signal theory course is reproduced (cf. Fig. 3). Step [2] is therefore located in the HM-discourse. The last aspect is also relevant for Step [3], that is also located in the HM-discourse. The three cosine terms are drawn as three different phasors each. In addition to the three phasors, the diagram also contains elementary properties of complex numbers: the connection between the cosine and the complex exponential function as phasor and the complex conjugate. Rotational aspects are missing. Subtask T_3 is solely located in the HM-discourse. The student solution I3 can be explained as a realisation of a pure HM-discourse. It mirrors the student solution I1, but with HM- instead of ET-discourse. Aspects indicating a connection to amplitude modulation are missing and transitions to the ET-discourse do not occur.

The solution to item 3 in N1 contains only an interpretation, that could be associated to step [2], cf. Fig. 10. The part of the calculation in item 2 to which this statement refers is "…$(\cos(\Omega t + 2\pi f_0 t) + \cos(\Omega t - 2\pi f_0 t))$". The arguments of the cosine terms are each interpreted as a complex number in Cartesian representation, whose imaginary parts cancel each other out due to different signs. This ignores the fact that there are no complex numbers and that the terms under consideration are arguments of cosine functions that cannot simply be added. We assign this part of the solution to the ET-discourse: the relation to oscillations and the relation between time-dependent sinusoidal functions, i.e. oscillations, and complex numbers is relevant in the ET-discourse. In the solution N1, the cosine is obviously also not interpreted as a time-dependent function, but as a somewhat unclear mixture of oscillation and complex number.

While the solution N1 has ET-discourse aspects, albeit incorrectly, the solution N2 does not contain ET-discourse aspects. In this work, the addition theorems are not used to transform the cosine terms. Instead, cosine terms are rewritten using the complex exponential function, the multiplication is performed according to the calculation rules, and then converted back into cosine terms. This corresponds to the rationale of the HM-discourse to use the complex exponential function to simplify calculations.

With regard to thesis 3, we can therefore conclude that here too, the student work could be explained on the basis of praxeological aspects and, in particular, the ideal typical mathematical discourses. In contrast to theses 1 and 2, the explanations are much more diverse, depending on the complexity of thesis 3. With regard to a transfer of these subject-specific explanations to individual and subjective justifications, analogous considerations apply, as they were formulated in the concluding discussion of thesis 1 and will be taken up again in the following final chapter.

Discussion

The praxeological approach enabled us to explain student solutions of an exercise in the context of amplitude modulation. The detailed analyses of the exercise and the student work were based on an identification of different ideal typical mathematical discourses within the signal theory course. Moreover, we have described a systematic sequence of methodical steps enabling a well-founded and productive connection between on the one hand institutional and ideal type analyses and on the other hand individual student work. The extensive use of a graphical tool for representing praxeological structures allow us to understand deeper possible relationships between the interrelated mathematical discourses and their effects, which transcends the vision of the role of mathematics in engineering as something essentially to be applied. This refers both to the sample solution and thus to institutionalised taught knowledge, as well as to individual student solutions with their very own discourse configurations. The latter results goes beyond analyses in which praxeological models are used as a reference to prove that student solutions differ from this reference.

In view of the (possibly non-existent) correspondence between subject-specific explanations and the subjective considerations of the students, we like to share the following remark: The ideas and considerations on which our analyses are based are compatible with actual-empirical research approaches in the field of subject science, which aim to reconstruct subject-related patterns of reasoning (Holzkamp, 1985, chapter 9). Studies concerning the extent to which the explanations presented in section 4 fit with students' individual considerations would of course require further empirical data, for example interviews. But their analyses presuppose, in terms of research logic, such praxeological analyses and subject-specific explanations that we have presented.

Our outcomes potentially enable HM- and ET-lecturers to make didactic decisions about whether or not to explicate various mathematical discourses that are effective in electrical engineering courses. Here, of course, appropriate didactic tools must be developed that do not use the research-related terms introduced in this paper. Simplified variants of the graphical representation of the praxeological analysis, see Fig. 3, could eventually be quite useful for lecturers to identify changes and breaks in mathematical discourses. Analogously, one could argue with regard to our analyses of the student work: Here, on the one hand, discourse-relevant observations or diagnoses would be possible and, on the other hand, these could of course form a basis for appropriate feedback to the students. Both suggestions could be useful and effective for teaching and learning, even if our explanations do not correspond one-to-one with the ideas of the teachers when creating the sample solution or of the students when working on the exercises, since the aspects we identified refer to some extent to the institutional knowledge to be taught.

Finally, we give an example of how our findings can be used to develop concrete suggestions for modifying exercises: In the course on higher mathematics, which our sample students have attended, nearly all exercises concerning complex numbers cover standard topics including change of representation, calculations with complex numbers and determining the roots of polynomials. The following exercise is an exception:

Which curves are described in the complex plane by

$$ae^{-ti} + be^{ti}, a, b \in \mathbb{R} \text{ constant}, t \in \mathbb{R}?$$

This exercise was hardly worked on by students of the course, and was labelled with "too difficult" in the students' lecture notes. The exercise immediately changes its character when software such as GeoGebra can and may be used. Then one can see, among other things, that circles and ellipses appear as curves (see Fig. 12).

The illustration of the terms as phasors gives rise to a connection between algebraic and geometric aspects of complex numbers (see our characterisation of the HM-discourse in section 2): By representing the two complex numbers as aligned phasors, the peak of their sum always moves on the curve. Moreover, this exercise can further be adapted so that a similar type of curves appears as the one relevant to the complex phasor diagram of the exercise we examine in this contribution.[14] In other words: By using software, an otherwise essentially unprocessed task can be extended in such a way that it becomes connectable to the mathematical ET-discourse of our considered exercise.

As far as we know, such an approach of modifying the teaching in higher mathematics has been little studied up to now. Generally, application problems from the main subjects are supplemented to show that mathematics can be applied in a meaningful way, which mainly addresses mathematics *for* engineering. In our suggestion of an exercise, the HM-discourse would be expanded with regard to the mathematical ET-discourse, or rather its practices, in order to establish connections to signal theory, which addresses elements of mathematics *in* engineering within HM.

[14] Here we also refer to the work of De Oliveira and Nunes (2014) who investigate rotating phasor pathways derived from different standard amplitude modulation systems.

Declarations

Conflict of Interest On behalf of both authors, the corresponding author states that there is no conflict of interest.

Appendix: Exercise and sample solution

The exercise under consideration is structured in three items:

1. A message signal $s(t) = \cos(\Omega t)$ has to be amplitude modulated. The result is $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$
2. The result of item 1. Has to be written as the sum of three harmonics. The result is

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$$

3. The result of item 2. Has then to be displayed graphically in the complex plane as a rotating phasor with varying amplitude.

Our analysis focusses item 3. of the exercise. The exact problem definition of item 3 is:

3. Graphically display $x(t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ and the result under item 2.

<u>Sample solution:</u>
One first writes

$$\begin{aligned} x(t) &= A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \\ &= A\Re\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t + \Omega t))\} \\ &\quad + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t - \Omega t))\} \\ &= \Re\left\{ \exp(j2\pi f_0 t) \underbrace{\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]}_{A(t)} \right\} \end{aligned} \tag{1}$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude $A(t)$, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Fig. 13.

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7. The epistemological and philosophical relationship between mathematics and electrical engineering

Study I: About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory

Study I was first presented at the *6th International Conference on the Anthropological Theory of the Didactic (citad6)* January 22-26, 2018 in Grenoble (France) and has been published as Hochmuth, R., & Peters, J. (2020). About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *22*(4), 454–471. https://doi.org/10.23925/1983-3156.2020v22i4p454-471.

In later studies we have also presented our analysis results graphically. We have not done so here. However, I have already worked with a graphical representation of the analysis result without publishing it. Since a graphical representation of the result is helpful for understanding the praxeological analysis, I present the graphical representation of the analysis result of Study I in Figure 7.1 below. This also represents a preliminary version to later representations and can thus serve as a comparison.

During the publication process, a number of editorial decisions were made that would have required an adjustment to the text. However, we did not have the opportunity to do this before publication. As some of these make the paper difficult to read, a list of important corrections and comments is attached in the Erratum below.

Erratum

- The first section heading is missing: Page 446 begins with the title of the paper. This should be the section "Introduction". The paper contains no numbering of sections and subsections but we refer to section- and subsection numbers in the text. The intended section- and subsection numbering is as follows:

  1. Introduction
  2. Context and focus
  3. Theoretical framework

  3.1. Anthropological theory of the didactic
  3.2. Epistemological-philosophical observations regarding physics

  4. Praxeological analysis

  4.1. General considerations on signals
  4.2. The Delta-impulse
  4.3. Further remarks related to epistemological-philosophical issues

5. Conclusion

- The three characterising properties on page 458 are

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t) = 1, \qquad \int_{-\infty}^{\infty} \varphi(t)\delta(t - t_0) = \varphi(t_0)$$

- The properties of real signals $x(t)$ on page 463 should be numbered. Those numbers are used later to refer to the properties respectively:
  (1) They are of finite duration, i.e. there exist t_0 and t_1 with $t_1 > t_0$, such that $x(t) = 0$ for $t < t_0$ and $t > t_1$.
  (2) They are continuous for all $t \in (-\infty, \infty)$.
  (3) They are sufficiently differentiable.

- The labels of the properties of $\delta(t) = \{f_n(t)\}$ on page 465 should be (i) and (ii) instead of (1) and (2).

- Last sentence on page 465 should be "For narrower and narrower pulses $f_n(t - t_0)$ the function $\varphi(t)$ could be replaced by the value $\varphi(t_0)$ (p_4).". The same error, the missing φ can be found on page 467: "The ever shorter durations of the pulsed signals justify the replacement of φ by $\varphi(t_0)$ (l_6)."

Study IV: About Two Epistemological Related Aspects in Mathematical Practices of Empirical Sciences

Study IV has been published as Hochmuth, R., & Peters, J. (2022). About two epistemological related aspects in mathematical practices of empirical sciences. In Y. Chevallard, B. B. Farràs, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & N. Ruiz-Munzón (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 327–342). Birkhäuser Basel. https://doi.org/10.1007/978-3-030-76791-4_26.

It is based on the lecture "About the Use of Mathematics in Other Sciences" by Reinhard Hochmuth and the Presentation "Mathematical discourses in practices of electrical engineering studies" by myself, both held at the Intensive Research Programme *Advances in the Anthropological Theory of the Didactic and their consequences in curriculum and in teacher education*, Course 4, 15 - 26 July, 2019 at the Centre de Recerca Matemàtica in Barcelona.

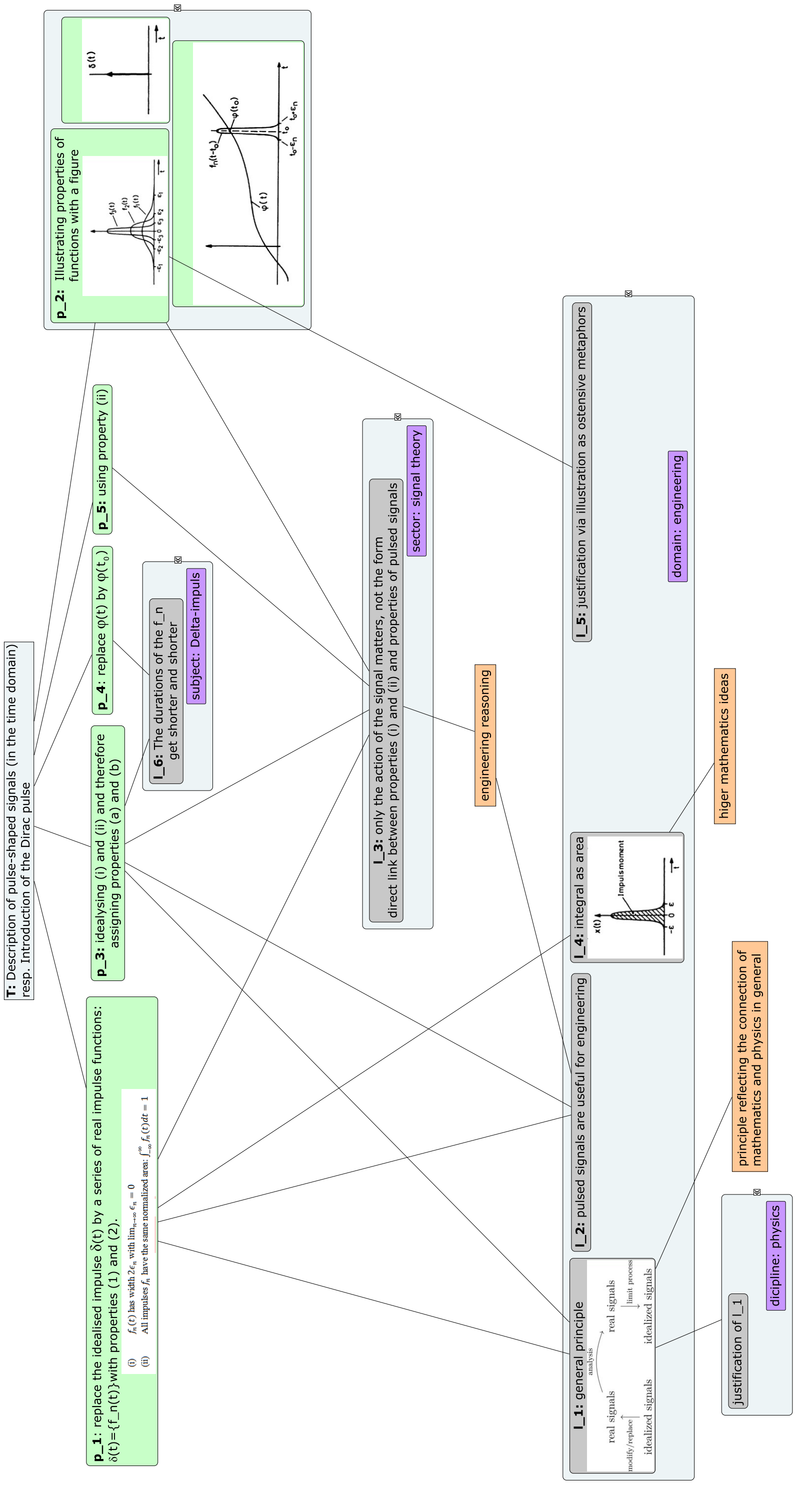

Figure 7.1.: Unpublished graphical representation of the analyses result from Study I.

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About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory

À propos du «mélange» de discours dans l'utilisation des mathématiques en théorie du signal

Reinhard Hochmuth[1]
Institute for Didactics of Mathematics and Physics, Leibniz Universität Hannover, Germany
https://orcid.org/0000-0002-4041-8706

Jana Peters[2]
Institute for Didactics of Mathematics and Physics, Leibniz Universität Hannover, Germany
https://orcid.org/0000-0003-0628-7105

Abstract

An important issue for research in university mathematics education is the use of mathematics in engineering. Here we focus on praxeologies in a course on system and signal theory (SST), which represents a typical module in electrical engineering studies in the third or fourth semester. In such courses, mathematics already studied in introductory mathematics courses will be applied, but also enriched by the introduction and development of new practices, in particular the so-called Dirac-impulse. We claim that the introduction and justification of the Dirac-impulse in SST is a convenient case where basic facets of epistemological relations between mathematics and engineering sciences might be illustrated and shown to be important for a detailed description and analysis of logos blocks of praxeologies. The background for our considerations regarding logos blocks of praxeologies that concern the introduction of the Dirac-impulse is given by philosophical studies by Wahsner and Borzeszkowski (1992, 2012) and a few illuminating remarks by Dirac.

Keywords: Signal Theory, Dirac impulse, Epistemology, ATD.

[1] hochmuth@idmp.uni-hannover.de
[2] peters@idmp.uni-hannover.de

Résumé

Une question importante pour la recherche en éducation mathématique universitaire est l'utilisation des mathématiques en ingénierie. Ici, nous nous concentrons sur les praxéologies dans un cours sur la théorie du système et du signal (SST), qui représente un module typique dans les études d'ingénierie électrique au troisième ou quatrième semestre. Dans ces cours, non seulement applique-t-on les mathématiques déjà enseignées et apprises dans les cours d'introduction à la mathématique, mais on introduit et utilise aussi de nouveaux concepts mathématiques, en particulier ce que l'on appelle l'impulsion de Dirac. Nous affirmons que l'introduction et la justification de l'impulsion de Dirac dans SST est un cas pratique par lequel les facettes fondamentales des relations épistémologiques entre mathématiques et ingénierie pourraient être illustrées et démontrées importantes pour la description détaillée et l'analyse des logos blocs de praxéologies. Le contexte de nos considérations au sujet des logos blocs de praxéologies concernant l'introduction de l'impulsion de Dirac est donné par des études philosophiques de Wahsner et Borzeszkowski (1992, 2012) et quelques remarques éclairantes de Dirac.

Mots-clés: Théorie du signal, impulsion de Dirac, épistémologie, TAD.

About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory

The use of mathematics in engineering and sciences is an important topic for research in university mathematics education. This is partly because of high dropout rates and the search for measures optimizing teaching and learning of mathematics in other study fields. Here we focus on praxeologies in a course on system and signal theory (SST) representing a typical module in electrical engineering studies in the third or fourth semester.

In recent years several papers have analyzed mathematical practices in engineering. Generally there are two interrelated foci: The first one is on aspects of modelling and application problems, where typically an engineering problem is prepared such that mathematics from introductory higher mathematics courses has to be applied to solve the task. In most cases it is obvious that the modelling cycle used for school mathematics, which separates the world in a mathematical world and the rest of the world (see for example Blum & Leiss (2005)), is not appropriate for describing and analyzing such activities since the engineering problem is a priori formulated in mathematical terms. Therefore, it has been suggested to use ATD for describing and analyzing the intertwined mathematical and engineering practices (see for example HOCHMUTH, BIEHLER AND SCHREIBER, 2014)). Moreover, Castela and Romo (2011) have introduced extended praxeological models, an idea which was adapted by Peters, Hochmuth & Schreiber (2017) to analyze tasks in a signal and system theory course. The institutional separation between mathematics and engineering in courses and curricula were also the starting point for investigations in (Barquero, Serrano, L. & Serrano V., 2013), where so called "study and research courses" are proposed for overcoming the dominant epistemology of "applicationism". Our research connects in particular the observations by Barquero, Bosch & Gascón (2011) regarding the "distinction between mathematics and the rest of natural sciences” and contributes to another application of the scale of level of codeterminations (Bosch & Gascón, 2006) studying conditions that frame the use of mathematics in other sciences.

The second focus is on the use of symbols: symbols are often both representations of mathematical variables and representations of physical or engineering quantities. It is often not clear to novices how they have to interpret symbols in view of a task and which argumentations are required or forbidden, see for example (Tuminaro & Redish, 2007; Hochmuth & Schreiber, 2015; Alpers, 2017; Peters, Hochmuth & Schreiber, 2017).

Here we adopt a slightly different position: We relate the intertwining of mathematical and engineering ideas to its historical genesis process and the dissolving of certain fundamental epistemological problems. Often it does not seem to be important for an understanding of actual teaching and learning contexts to enlighten such issues in detail. In our praxeological analyses of the use of mathematics in SST we came across those issues, as we tried to substantiate technological and theoretical issues: In the analysis of text-books besides clear arguments that are based on techniques and technologies developed in higher mathematics or electrical engineering, we observed vague argumentations bobbing up at certain steps. We had the impression, that the vague steps arise at points that are significant both for an understanding what it means that an engineering practice is pragmatic and for a better understanding of switching between mathematics and engineering.

Therefore, we began to think about incorporating basic observations from (Wahsner & Borzeszkowski, 1992; Borzeszkowski & Wahsner, 2012) that take into account the relation between mathematics and physics. They raise several epistemological issues which have to be resolved in any mathematically based theory intending to describe and calculate “nature”. These epistemological issues are also relevant for engineering sciences, since they can be interpreted in relation to those issues as concretizations in view of subject related aims embedded within culture-historical as well as socio-economical processes.

After clarifying the mathematical context and focus of our paper as well as the theoretical framework in sections 2 and 3, we exemplarily investigate passages from a SST text-book introducing the Dirac-impulse. A sketchy praxeological analysis allows linking vague passages to fundamental epistemological issues concerning the relation between mathematics and physics, respectively engineering sciences. We support our observations by a few illuminating remarks by Dirac.

Context and focus

We analyze the introduction of the Dirac-impulse in (Fettweis, 1996) with a focus on specific steps in the justification of some of its characterizing properties:

$$\delta(t) = \begin{cases} \infty \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}, \int_{-\infty}^{\infty} \delta(t)dt = 1, \int_{-\infty}^{\infty} \varphi(t)\delta(t - t_0) =$$

These properties are typically also introduced and used in quantum mechanics and go back to Dirac (1927), who was already aware of the problem that δ could not be a "normal" function and has to be interpreted in a specific way:

> Strictly, of course, δ(x) is not a proper function of x, but can be regarded only as a limit of a certain sequence of functions. All the same one can use δ(x) as though it were a proper function for practically all the purposes of quantum mechanics without getting incorrect results. (p. 625)

The mathematical knowledge of that time did not provide a consistent and well-defined framework for the Dirac-impulse, which was, by good reasons, not a real problem for Dirac, Heisenberg and Pauli in contrast to, e.g., von Neumann (Peters, 2004). Nowadays there are several possibilities to introduce the Dirac-impulse respecting the actual socio-mathematical norms in mathematics as a science. We remind of the following three possibilities:

a) In Functional analysis (see for example Schwartz (1947)) δ is considered as a distribution, that is a linear and continuous functional on so called test function spaces like $C_0^\infty(\mathbb{R})$or $S(\mathbb{R})$.

b) In Non-standard analysis (see for example Landers & Rogge (2013)) δ can be seen as a "normal" function from the hyperreal numbers $^*\mathbb{R}$ to $^*\mathbb{R}$. In the 19th and beginning 20th century there were some discussions (Purkert, 1990) about the usefulness of the ε-δ-calculus for engineering students and it was proposed, for example by Weisbach (1860), to teach instead Leibniz's infinitesimal calculus, which can be seen as a predecessor of non-standard analysis. Nowadays, non-standard analysis is typically not taught in mathematic courses for engineers in Germany.

c) Another possibility, which is partly adopted in (Fettweis, 1996), considers distributions as limits of sequences of functions, which converge in a specific way (Antosik, Mikusiński, & Sikorski, 1973). This approach can be elaborated on a level that is the most part compatible with higher mathematics taught in courses for engineers. Within this framework, integrals with respect to δ were introduced and interpreted in a symbolic way, as notions representing the result of limit processes.

Obviously the presentation in Fettweis is mathematically not complete and it could be argued whether and how it could be supplemented. In the following we do not want to discuss whether the introduction of more complete and formal mathematics would be useful from an engineering point of view. Instead we intend to demonstrate that certain appearing gaps can be linked to fundamental epistemological issues concerning the relation between mathematics and physics. This suggests that the gaps and their character are in the first instance not the deficit result of too little mathematics but the expression of a specific historic and institutional resolution of certain epistemological issues.

Theoretical framework

We combine a praxeological analysis with conclusions from historic-philosophical considerations based on a dialectic and materialistic point of view concerning the relation between mathematics and physics by Wahsner and Borzeszkowski (1992, 2012). We believe that those conclusions refer also to inherent characteristics of the relation between mathematics and engineering sciences. According to the status of our work in progress we use the praxeological approach for reconstructing the engineering content and inject philosophical considerations in the analyses of the logos-block focusing on the relation between "mathematics" (distribution theory) and "physical reality" (signals).

Anthropological theory of the didactic

In our analysis we address two concepts of ATD: First we outline a praxeological analysis of a SST-practice, where we focus the most elementary model of praxis/logos blocks. This praxeological model consists of the praxis block P containing tasks and techniques used to solve them and the logos block L containing the technological and theoretical discourse describing justifications, explanations and production of the elements of the praxis-block. This P/L-model could be refined into the so called 4T-model where the praxis block is differentiated into tasks T and techniques τ and the logos block is differentiated into technology θ and theory Θ, where theory forms a discourse on technology that is more elaborated and abstract (Chevallard, Bosch, & Kim, 2015). We forgo formulating tasks and techniques as well as technology and theory in detail because of limited space and since these details seem not necessary for representing the main point of this paper. Elements of the praxis block P, will be denoted by p_i, and elements of the logos block L, by l_i.

Second, for a more detailed understanding of technological-theoretical facets, that form the logos block, we give a rough allocation to higher levels of codetermination.

Epistemological-philosophical observations regarding physics

In philosophical and concrete historical studies, Wahsner and Borzeszkowski (1992, 2012) figure out those conceptual and experimental-objective preparations within physics that facilitate to use mathematics as mean for expressing, describing and analyzing dynamics in terms of laws and to link mathematics with measuring practices. The following both aspects are in particular important (Wahsner & Borzeszkowski, 1992, pp. 125-135):

a) Since only finite distances are measurable, conceptual contradictory identifications of infinite or infinitesimal quantities, which arise in mathematical structures, with finite quantities are enforced. The particular context dependent adequate but from a mathematical perspective inconsistent use of mathematical concepts is historically one of the most original achievements of physics.

b) Only effects of properties of objects are measurable and not dynamic interactional relations. This leads to the question, which behavior can be transformed to a property. Related answers could be found studying the complicated historical genesis of physical measured quantities.

Physical quantities are thinking-objects, which are constructed on the basis of real equalities, checked by specific instruments in specific experiments; they are tools for investigating real objects in contexts. Considering and treating physical quantities under the measurement aspect allows to formulate dynamics related laws in such a way that their assertions can empirically be proved.

In contrast to physics, quantities appear in mathematics merely within functional structured systems that presuppose their existence. This facilitates mathematics to be without inherent contradictions and formally consistent, but, at the same time, disable mathematics to make assertions about real objects and their behavior. Therefore mathematics needs physics (or another empirical science) to make statements about reality (p. 128). On the other hand, physics needs mathematics for measurements, calculations and expressing dynamic interactions by laws, that is: mathematics allows making basic relations calculable and measurable.

Praxeological analysis

In this section we present an ATD analysis of the introduction of the Delta-impulse in (FETTWEIS, 1996). The main ideas and results of this analysis are also relevant for related SST-books like Girod, Rabenstein and Stenger (2007).

General considerations on signals

Fettweis characterizes signal and system theory not as a technical but as a, in general, physical discipline. The author stresses the importance of physical understanding and argues that an increasing elaboration of mathematical concepts would not only go beyond the scope of the book, but would make the understanding of the physical reasoning of methodological issues increasingly harder. He constructs the relation between mathematics and physics as a dilemma between mathematical precision and the understanding of physical reasoning. This positioning between mathematical precision and the understanding of physical reasoning affects the praxeologies, especially their logos-blocks.

The focus on physical understanding leads Fettweis in particular to a general principle concerning two different but connected concepts of signals: “real signals x(t)" occur at communications transmission and are irregular. Furthermore, they have the following properties:

> They are of finite duration, i.e. there exist t_0 and t_1 with $t_1 > t_0$, such that $x(t)=0$ for $t < t_0$ and $t > t_1$.
> They are continuous for all $t \in (-\infty,\infty)$.
> They are sufficiently differentiable.

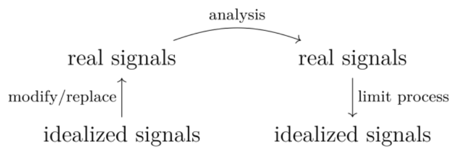
Following Fettweis (p. 5) real signals are characterized by irregularity and high diversity, so they are inappropriate for numerical and analytical calculations and are not usable as measurement signals. Therefore “idealized signals”, which will unavoidably violate some of the properties (1) to (3), are introduced. In spite of their simplicity, using idealized signals can cause difficulties, especially with respect to convergence. As an example, Fettweis (pp. 8) considers the unit step function that satisfies none of the properties (1) to (3)

$$u(t) = \begin{cases} 0 \text{ for } t < 0 \\ 1/2 \text{ for } t = 0 \\ 1 \text{ for } t > 0 \end{cases}$$

In such a case the idealized signal can be replaced by real signals, such that the specific difficulty does not arise any more. After an analysis using real signals the replacement can be reversed. This general principle is illustrated in Figure 1:

Figure 1

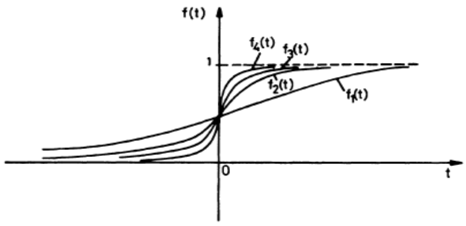
Illustration of the interplay between idealized and real signals



The unit step function $u(t)$, for example, could be replaced by continuous, in t = 0 rapidly increasing real signals $f_n(t)$, c.f. Figure 2.

Figure 2

Function series approximating the unit step function



The approximation is symbolically expressed by $u(t) = \{f_n(t)\}$ and also written as limit process: $u(t) = \lim_{n\to\infty} f_n(t)$.

The general principle is justified by referencing the compliance with approaches in other physical disciplines (p.6), by a need of physics to use function series for approximations and by claiming that an adequate application of the principle generates unambiguous and correct results (p. 12). In the same paragraph Fettweis refers also to Distribution Theory as a mathematical domain. He explicitly refers the work of L. Schwartz, that is presented in a "very abstract and physically less appealing form" (p. 12) and the work of Antosik, Mikusiński, and Sikorski (1973), that could be seen as an elaborated mathematical basis for the presentation in Fettweis.

The Delta-impulse

In this section we provide a sketchy praxeological analysis of the introduction of the Delta-impulse. First we summarize elements of the praxis blocks p_i. Then we describe the technological-theoretical discourse with regard to the considerations in 4.1. We specify the elements of the logos block by l_i.

The idealized impulse $\delta(t)$is defined by a series of real impulse-functions (p_1): $\delta(t) = \{f_n(t)\}$ with the following properties

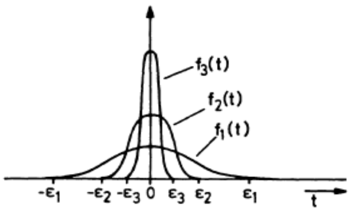
(1) $f_n(t)$ has width $2\epsilon_n$ with $\lim_{n\to\infty} \epsilon_n = 0$.

(2) All impulses f_n have the same normalized area: $\int_{-\infty}^{\infty} f_n(t)dt = 1$

This definition is illustrated by Figure 3:

Figure 3

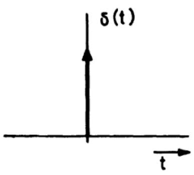
Function series representing the Delta-impulse



Illustrating properties of functions by graphical representations (p_2) is a common practice in engineering textbooks and in particular in SST. By idealizing (i) and (ii) the two properties a) $\delta(t) = \begin{cases} \infty \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}$ and b) $\int_{-\infty}^{\infty} \delta(t)dt = 1$ are assigned (p_3) and the idealized impulse $\delta(t)$) is visualized by Figure 4 (p_2):

Figure 4

Visualization of the idealized Delta-impulse

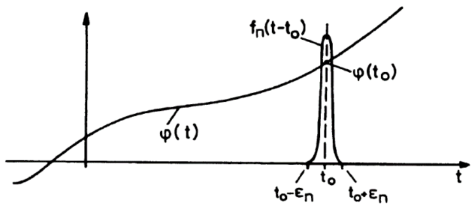


The third important property of the Delta-impulse, the sifting property $\int_{-\infty}^{\infty} \varphi(t)\delta(t - t_0) = \varphi(t_0)$, is deduced as follows: $\delta(t)$ is replaced by a function series $\{f_n\}$ (p_1) according to Figure 3 (p_2).For narrower and narrower pulses $f_n(t - t_0)$the function

$\varphi(t)$could be replaced by the value (t_0) (p_4). Using property (ii) (p_5) and referencing to Figure 5 (p_2), the sifting property follows.

Figure 5

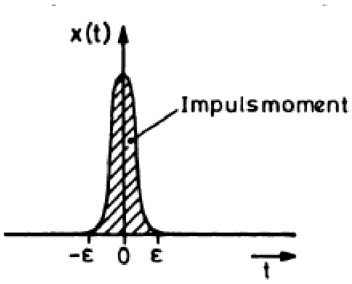
Sifting property



The technological-theoretical discourse is especially based on the general principle connecting idealized and real signals, illustrated in Figure 1. This provides justifications for p_1 and p_3 (l_1): The definition *via* a series of real impulses reflects the interplay between real and idealized signals. Furthermore, it is argued that pulsed signals are very useful for engineering (l_2) (p. 12). The properties (i) and (ii) are directly linked to properties of pulsed signals: only the action of the signal matters, not the specific form (l_2). This is fulfilled, if the duration of the signal is very short, i.e. 2ϵ. The action of the signal corresponds to the area, in Fettweis also denoted by "Impulsmoment", illustrated in Figure 6. This refers to the idea of the integral as area from mathematics courses (l_4).

Figure 6

Example of a real impulse



The element p_2 is produced by the justifying characteristic of visualizations (l_5): In Figure 3 a limit-process of real signals is visualized and the corresponding result is shown in Figure 4. So the visualizations reflect also the idealization process. The illustrations act as ostensive metaphors. Dirac (1963) claimed: “The delta function comes in just from picturing the infinity as something, which approximates to them.” Additionally the principle, that the exact course of the signal doesn’t matter, the essential is, that it is pulsed is important for all figures (l_3). The ever shorter durations of the pulsed signals justify the replacement of $\varphi(t)$ by (t_0) (l_6). The properties a) and b), which are assigned in p_3, are idealizations of properties (i) and (ii), properties of real signals, which are transferred to properties of an ideal signal. Especially property b), which contradicts the understanding of the integral in higher mathematics courses, is not mathematically justified yet. Fettweis (p. 14) discusses this point explicitly and justifies the integration by referring to real signals.

Summarizing, the technological-theoretical discourse is a mixture of higher mathematics ideas (l_4), engineering reasoning (i.e $l_2$and l_3) and a principle concerning the interplay between real and idealized signals reflecting the connection of mathematics and physics in general (l_1). The justification and explanation of practices considering idealized signals like $\delta(t)$ are done on the level of real signals. This correlates with Dirac’s (1958) hint, that one must exit the mathematical context for justification and do not interpret δ as mathematical symbol.

The reconstructed elements of the logos block and other justifications and explanations could be assigned to different levels of the scale of levels of codetermination: The justification of the principle in Figure 1 lies on the level of the discipline (physics). The technological-theoretical elements l_1, l_2, l_4 and l_5 could be assigned to domain (engineering), l_3 to sector (signal theory) and l_6 to the level of the subject (Delta-impulse). The local curriculum (level of university) and experience and propensity of the author of the textbook are also mentioned as a restriction for the content (Fettweis, p. iii). Finally we refer to a remark by Peters (2004, p. 99) that adapts a statement by Schwinger about Feynman-diagrams: "the δ-function 'was bringing computation to the masses'", which expresses the teaching and learning process related institutional aspect of the Dirac-impulse in a rather convincing way. Moreover, this statement indicates aspects on the level of society.

Further remarks related to epistemological-philosophical issues

The Dirac-impulse represents an idealized signal. *Via* approximation sequences this idealized signal, which is neither observable nor measurable, was linked to real signals, which are in principle observable and measurable. The scheme in figure 1 represented the basic consideration that underlies specific and, regarding the SST-context, adequate identifications. In particular theoretical relations including δ as well as δ itself gain empirical meaning: idealized signals like δ become physical quantities in the sense of 3.2, which allows formulating relations like the sifting property and measurements. These interrelations (e.g. allowing measurement, being element of a relation) imprint certain properties and induce techniques and technologies treating δ, which look purely mathematically and were historically important aspects for developing a systematic and axiomatic based mathematics for δ. From the physical point of view this might be helpful but is not necessary.

Furthermore, the mathematical theory as such does not allow injecting into δ physical meaning how it is enabled by, among others, the scheme in figure 1: For linking δ with measurable quantities, it has occasionally to be replaced in a SST adequate way, which necessarily transcends the formal mathematical context. In particular the inherent and specific identification of “finite” and “infinite” cannot mathematically be proved to be correct but could only be mathematically explored.

Moreover, the sifting property links a global continuous object φ to local values $\varphi(t_0)$. This is one of the issues of δ in equations, e.g. in transferring relations from discrete signals to continuous signals and *vice versa*. This gives the possibility for treating the dialectic between “point” and “continuum” in such a way that allows computation (δ appears in equations and calculus) and measuring.

Conclusion

We claim that a broad understanding of logos-blocks in praxeological reference models taking into account higher levels of codetermination is valuable, since it allows in particular identifying inherent issues, which have to be resolved in some way by any institutionalized didactical or pedagogical practice. Here the aim was amongst others to illustrate and identify the relevance of basic philosophical-epistemological ideas for enriching the logos-block of praxeologies in SST and how they contribute to a wider understanding of actual justifications of practices. We will move on in this direction.

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About Two Epistemological Related Aspects in Mathematical Practices of Empirical Sciences

Reinhard Hochmuth and Jana Peters

1 Introduction

Mathematical practices, techniques and algorithms play a significant role in many disciplines (Winsløw et al., 2018). Consequently, mathematical service courses have become part of many study programs. Beyond the service courses, mathematical practices are also developed, adapted, and taught in courses of other disciplines. There, mathematical concepts that are also taught in introductory service courses sometimes have different meanings. Moreover, advanced content, like for example the Gaussian theorem in basic electrical engineering courses, is justified and used long before it is taught in service courses. Another illustrative example is that concepts like the Dirac impulse in signal analysis are often not covered in service courses.

Although the use of mathematics in other disciplines and the issue of mathematical service courses have been discussed for a long time in mathematics education (see for example the third ICMI Study by Howson et al. (1988) and for an actual overview Hochmuth (2020)), it is only more recently that research on mathematical practices in service courses and beyond has been playing an increasing role at international conferences on university mathematics education like CERME (Winsløw et al., 2018), ICME (Biza et al., 2016), INDRUM (Durand-Guerrier et al., 2021), and RUME (Weinberg et al., 2017).

Whereas mathematical topics that are relevant in other disciplines, for example differentiation, integration or stochastic distributions, can easily be identified in curricula and textbooks, the respective discipline-related mathematical practices and their respective rationales are often not explicitly known in detail (Winsløw et al., 2018). Due to the differences between mathematical practices and rationales in

R. Hochmuth (✉) · J. Peters
Leibniz University Hannover, Hannover, Germany
e-mail: hochmuth@idmp.uni-hannover.de; peters@idmp.uni-hannover.de

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service courses and in major-subject courses, it is often not clear to students which activities and reasoning are allowed, required or forbidden and, in particular, how symbols have to be interpreted with regard to a specific task in major-subject courses (Hochmuth et al., 2014; Alpers, 2017).

In this contribution, we do not approach the study of mathematical practices in other disciplines by looking at students' or lecturers' concrete practices or lecturers' measures supporting students' learning in service courses. Instead, we start out from various historical-philosophical studies on the relationship of mathematics and empirical sciences and from there we explore two epistemological related aspects, that we have partly already investigated in detail in earlier research. One aspect deals with the identification of mathematical objects such as continuous variables and formal quantities with measurable, and therefore finite and discrete, quantities in empirical sciences (Hochmuth, 2019; Hochmuth & Peters, 2020). The second aspect concerns the characterisation of two different ideal-type (Weber, 1904) mathematical discourses and their roles in mathematical practices within empirical sciences (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021). The mathematical discourses connect to what is identified with each other in the sense of the first aspect. In this way they also refer in a certain way to the respective norms and rationales, on the one hand of mathematics and on the other hand of the respective empirical science. In this contribution we want to plausibly demonstrate that both aspects play a role in the use of mathematics in empirical sciences and illustrate this by examples from electrical engineering and psychology.

There is no place here to reflect in detail on the relationship between the two aspects. However, we want to emphasise that both aspects are essentially the result of institutional and societal processes. Each identification must have proved historically adequate and fruitful within the respective empirical science. And the various mathematical discourses are, among other things, the result of the historically specific organisation of the knowledge to be taught, taught and learned in educational institutions.

We have structured this article as follows: in the next section, we embed our research in the context of the ATD. Afterwards, we summarise some epistemological insights and observations from historical-philosophical and epistemological studies. These relate in particular to the two aspects outlined above. The quite abstract assertions are then exemplified for mathematical practices in electrical engineering and for psychology. A short outlook on subsequent research questions concludes our contribution.

2 ATD-Research on the Use of Mathematics in Other Sciences

Artigue et al. (1990) analysed different students' conceptions about differentials linked to mathematics and physics. Requirements regarding the institutional settings are figured out in essentially cognitively understood legitimations and validations of concepts and rituals assigned to mathematics or physics. They have observed, for example, that the idea of approximation works in mathematics as a constitutive moment of some notion and in physics as an excuse of loose reasoning, which reflects an "old conflict between rigorous mathematics and effectiveness in physics" (Artigue et al., 1990, p. 265). In view of teaching goals, the authors have especially suggested to make the various types of situations where differentials are needed more explicit.

Castela and Romo Vázquez (2011) applied and extended notions form ATD (Chevallard, 1992, 1999) in their analysis of mathematical praxeologies in signal and system theory courses. For studying the intrinsic intertwining of mathematics and its use they introduced a distinction between two technological components—a theoretical and a practical component—which reflects among others external didactical transpositions and different modalities of institutional validations. This idea of an internal differentiation of praxeological blocks is further extended and adapted in our analyses of signal and system theory-tasks (Hochmuth & Peters, 2021).

The institutionalised separation of teaching mathematics partially in service courses and in major-subject courses corresponds to a widespread understanding of the use of mathematics in other disciplines essentially as an application of previously constructed mathematical knowledge, an understanding which is coined by Barquero et al. (2013) as "applicationism". This understanding to some extent neglects the intrinsic dialectics between different mathematical practices and underlying needs, something we want to explore in this paper. In contrast, González-Martín and Hernandes-Gomes (2018) address curricular differences between practices, for example, regarding the integral notion and the integral use in calculus and mechanical engineering courses. Such curricular differences were also observed and investigated by Dammann (2016) and, for electrical engineering contexts, by Hennig et al. (2015).

In ATD there are the following two (in fact interrelated) options possible to make use of the idea of higher levels of codetermination: firstly, one could consider the impact of higher levels (for example societal dominant beliefs concerning the relationship between mathematics and other sciences like "applicationism") on the constitution of practices. Secondly, one could inform the analyses of mathematical practices and their institutionalisation by research results from history, sociology and/or philosophy of mathematics and sciences. In the following we mainly focus on the second option and outline a perspective with consequences for further analyses within ATD. Later, in addition to codetermination, we will also discuss the ATD principle of the institutional dependence of knowledge.

3 Epistemological Considerations Regarding the Relationship of Mathematics and Empirical Sciences

Practices in empirical sciences explicitly and/or implicitly claim to show an intrinsic relation to the world. This implies, for example, that their assertions cannot reasonably be justified and understood without recourse to the world. In our everyday life we show a realistic attitude, which means that we act under the premise, that there is conformity between mental images and reality. Philosophical reflections show that this view is highly problematic from an epistemological point of view[1] and, moreover, complicates understanding what the specific truth of empirical knowledge is. In our opinion, didactic studies are also at least occasionally based on the realistic position, such as the modelling cycle. In our contribution, we want to show that elaborate and epistemologically reflected positions are particularly helpful for a better understanding of mathematical practices in empirical sciences. On this basis, especially ATD-related concepts can be complemented and concretised in a suitable way in order to examine institutionalised mathematical practices.

With this in mind, pragmatic (Schlaudt, 2014) and historic-materialistic (Wahsner & von Borzeszkowski, 1992) views seem to be fruitful. According to the pragmatic position, empirical truth relies in "mastering objective means for the achievement of subjective purposes. It shows itself concretely in the agreement of will and ability in the action, in the performatively experienced resistance of the world" (Schlaudt, 2014, p. 11).[2] Accordingly, two readings of physics can be distinguished, a descriptive one "according to which the laws of physics tell us how certain objects behave, and a prescriptive one, according to which the laws are rather rules on how these objects can be manipulated" (Schlaudt, 2014, p. 123), The latter also means, that laws of nature have to be understood as instructions for action. Now an important point in our context is that the "resistance of the world" with regard to assertions from mathematics and empirical sciences like physics, engineering or psychology is quite different. Consequently, also the control of symbolic means for the achievement of purposes is quite different and subject to significantly different validity claims and related discourses. Such a position finally provides a basis for reconstructing the historic-specific societal-institutional constitution of practices and the generation of discourses, as well as for tracing its pedagogical and institutional reproduction.

According to the pragmatic position, and incorporating also historic-materialistic views, measurements in empirical sciences are not seen as the representation of a numerical determination of a property of things, but as something that informs about the behaviour of an object under certain norm conditions. Mathematics abstracts from behaviour and focuses on the pure quantity as well as presupposes the existence of objects in an axiomatic system of relations. But empirical sciences cannot "forget"

[1] See for example Schlaudt, pp. 42.

[2] All text passages originally in German were translated into English by the authors for this contribution.

these constituents. Instead, they are inherently reflected in the practices by constituting, incorporating and framing specific mathematical practices.

In historic specific transformations from dialectic interconnections to dualisms, contradictory conceptual identifications of, for example, infinite and infinitesimal quantities turn out to be particularly important (von Borzeszkowski & Wahsner, 2012). Especially with respect to metrological aspects Wahsner (1981) notes:

> However, natural science (at least this applies to a physical theory) does not make its statements directly about real objects, but about physical quantities and their relationships. These quantities are a means to recognise reality. They are finite determinations and must be, otherwise they cannot be measured. Natural science must therefore operate with these "objects of understanding". This is not metaphysics, but physics based on measurement theory. But these quantities, these objects of understanding are not natural, or are given directly in the imagination. They have to be produced through comparative work, through a comparative work that presupposes a human activity, but above all, it requires the development of a principle of scientific experience, the elaboration of a measurement theory (. . .), a theory that states how the contact between these quantities of the mind and the real objects is established. (p. 200)[3]

One aspect of mathematical knowledge is that a statement is true if it can be derived logically from true statements within the mathematical system. Truth (valid knowledge) is thus essentially determined inner-theoretical.[4] This is different in empirical sciences. Here, truth (valid knowledge) must always establish a reference beyond theory. Empirical sciences cannot be divided into an empirical (non-mathematical) part that regulates the relationship to reality and a mathematical part that is free of this relationship to reality. This phenomenon is made explicit in the investigations of Wahsner and von Borszeskowski on the relationship of mathematics and physics, but also by our investigations, especially with regard to the electrotechnical mathematics discourse. Such a decomposition, which in our opinion is ultimately not possible, would justify considering mathematical practices in empirical sciences as exclusively inner-mathematically justified actions. Accordingly, studies of mathematical practices in empirical sciences that ignore the empirical reference, including mathematics, to reality would imply this separation in an

[3]"Doch die Naturwissenschaft (wenigstens gilt dies für eine physikalische Theorie) trifft ihre Aussagen nicht unmittelbar über die wirklichen Gegenstände, sondern über Physikalische Größen und deren Beziehungen. Diese Größen sind ein Mittel, um die Wirklichkeit zu erkennen. Sie sind endliche Bestimmungen und müssen es sein, sonst kann man sie nicht messen. Die Naturwissenschaft muss daher mit diesen „Verstandesgegenständen" operieren. Es ist dies keine Metaphysik, sondern meßtheoretisch begründete Physik. Doch diese Größen, diese Verstandesgegenstände sind nicht naturgegeben, bzw. unmittelbar in der Vorstellung gegeben. Sie müssen durch Vergleichsarbeit erzeugt werden, durch eine Vergleichsarbeit, die die handelnde Tätigkeit des Menschen voraussetzt, vor allem aber die Entwicklung eines Prinzips wissenschaftlicher Erfahrung bedingt, die Ausarbeitung einer Meßtheorie (. . .), einer Theorie, die aussagt, wie der Kontakt zwischen diesen Verstandesgrößen und den realen Gegenständen hergestellt wird."

[4]Of course our description of mathematical knowledge is not comprehensive. But for our considerations, the point of inner-theoretical validation of knowledge is decisive. The same holds for validation of knowledge in empirical sciences.

empirical un-mathematical part and a mathematical part justified within mathematics. Examples are modelling cycles (Blum & Leiss, 2005) and "applicationsm" (Barquero et al., 2013). The fact that in empirical sciences mathematical practices also have a constitutive relation to reality have to be taken into account in didactic analyses. ATD enables this consideration, among other things, through the principle of institutional dependence on knowledge. In the institutions that can be assigned to the empirical sciences (e.g. lectures in the engineering sciences, research institutions, etc.), mathematical practices are justified, substantiated, validated and constituted differently than in academic mathematics. The core of this otherness is the empirical reference, which is constituted differently for each individual science.

In summary, we would like to point out that the epistemological questions outlined above have to be resolved in every mathematically based empirical science. Their relevance could be examined especially with regard to mathematical practices and, in the sense of ATD, to characterisations of technological-theoretical blocks of mathematical praxeologies in empirical sciences. Their consideration in electrical engineering is the focus of the next section. Afterwards, we briefly turn to mathematical practices in psychology.

4 About Mathematical Practices in Electrical Engineering

In various papers we have analysed mathematical practices in electrical engineering with a focus on the technological-theoretical block (Hochmuth et al., 2014; Hochmuth & Peters, 2020, 2021; Hochmuth & Schreiber, 2015a, 2015b, 2016; Peters et al., 2017; Peters & Hochmuth, 2021), and also in quantum mechanics (Hochmuth, 2019). A sociological and philosophical informed view has been crucial for the analyses of justifications, validations and their discursive constitution. In the following, we will elaborate on the two above mentioned epistemology related aspects. Firstly, the two different mathematical discourses and their roles in the use of mathematics within empirical sciences. And secondly, the identification of mathematical formal quantities with measurable quantities in empirical sciences.

The technological-theoretical blocks of mathematical practices in engineering and in mathematics differ in terms of general characteristics (empirical truth vs. deductive-logical truth, the ontology of objects etc.) and concrete contents. In previous studies we have been able to reconstruct two ideal-type mathematical discourses in relation to mathematical practices: an Electrotechnical Mathematics Discourse (ET) and a Higher Mathematics Discourse (HM) (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021). In ATD the logos is considered as a discourse on praxis, but as praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is also related to the institutional discourse. In the following, we will describe both mathematical discourses within the context of complex numbers. In the concrete studies just cited on mathematical practices in signal theory we have presented this in more detail, and in a broader context.

The mathematical knowledge associated with the HM-discourse is characterised by an inner-mathematical conception of terms and statements without concrete references to reality, a generalisation-oriented rational of academic mathematics and a concentration on calculation rules. To describe the mathematical knowledge concerning complex numbers we refer to the textbook by Strampp (2012). This book represents a standard approach to complex numbers. It is used as a course literature for a consolidated two-semester standard course on higher mathematics for engineers which is held every year at the University of Kassel and thus represents an important reference point for our previous analyses with regard to mathematical practices. Complex numbers are covered in the first semester in the context of Linear Algebra and are introduced as a field extension of real numbers, motivated by the solvability of the equation $x^2 + 1 = 0$. Field extension is not introduced as a formal algebraic concept. Strampp (2012) just states that the real numbers are extended by a number i with the property $i^2 = -1$ and that after the extension, all field axioms which are relevant for calculating with real numbers shall continue to exist (p. 59). This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations, resulting in considerations about the general solution of algebraic equations, the fundamental theorem of algebra and Vieta's formula. In doing so, however, no formal concepts are introduced and proven, but rather calculation rules for complex numbers are derived and presented. Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, an orientation towards the inner-mathematical, generalisation-oriented rational of academic mathematics can also be observed. In addition to the previously mentioned more algebraic view on complex numbers, the chapter contains another, geometric, orientation based on an analogy to vectors. However, the vector concept is also distinguished from complex numbers: "We speak of phasors[5] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This multiplication extends the multiplication of real numbers." (p. 60) This HM-phasor concept differs from the phasor concept in electrical engineering, described below, but refers to it.[6] The geometrical representation of complex

[5]We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrotechnical aspects play no role in the course and Stramp (2012) does not refer to them either. Another possible translation of Zeiger, without the connection to engineering concepts would be pointer. But we decided to use phasor for the following reason: In German, the term Zeiger is used both in electrical engineering and in mathematics courses for engineers, but with different meanings (reference to electrotechnical concepts vs. geometrical object with no further references to reality). By using the term Zeiger instead of vector Strampp (2012) can thus establish a connection to the electrotechnical courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different institutional contexts is in jeopardy of being lost through translation.

[6]The textbooks by Fettweis (1996) and Frey and Bossert (2009) cover signal and system theory, the context for our second example, the introduction of the Dirac impulse. Complex numbers are also very important in signal and system theory, especially in the context of amplitude modulation, see for example (Peters & Hochmuth, 2021).

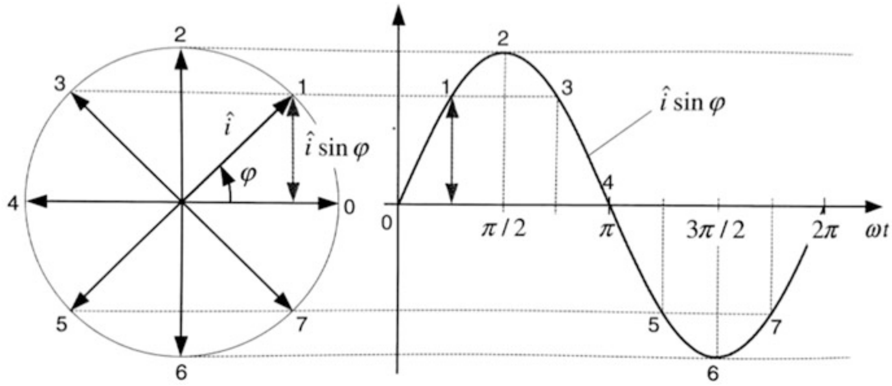

Fig. 1 Relationship between phasor and time-dependent function (Albach, 2011, p. 32)

numbers as arrows in the Argand diagram is used as a visualisation of calculation rules.

To characterise the ET-discourse we first note that electrical engineering as an empirical science inherently includes a reference to reality. However, a look at various textbooks, e.g. Fettweis (1996) and Frey and Bossert (2009) (see footnote 5), shows a large variance in the explication of this reference to reality, which is accompanied by a variance in the degree of mathematical formalisation. For a short overview of how complex numbers are treated in electrical engineering courses we refer to the standard textbook by Albach (2011). In Albach (2011) phasors [Zeiger] are introduced in the context of alternating currents and voltages. The first introduction is without reference to complex numbers: Here a phasor is an arrow with a specific length and a specific angle with respect to a reference angle. This arrow can be related to a time-dependent sinusoidal[7] function, see Fig. 1.

Current and voltage ratios in electrical networks can be displayed and analysed graphically in phasor diagrams without using differential equations. On the basis of Kirchhoff's rules for the analysis of electric circuits, geometric calculation rules for phasors are derived, which are analogous to the calculation rules for vectors. For the purpose of a mathematical description of phasors, the plane in which phasors are drawn, can be considered as the complex plane. The phasor is now understood as a complex quantity that symbolises the time-dependent voltage (see Albach, 2011, p. 42). Whereas in the HM-discourse phasors are used to graphically illustrate the properties of complex numbers, in electrical engineering phasors are arrows that represent measurable, time-dependent quantities such as alternating voltages or currents. Complex numbers are then used for the convenient mathematical description of phasors, justifying the compatibility of the rules for manipulating phasors and the calculation rules of complex numbers via physical relations.

[7]Circuits are operated with sinusoidal current- and voltage forms in the power supply network as well as in many other important areas.

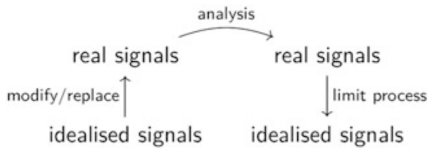

Fig. 2 Illustration of the interplay between idealised and real signals

With the above explanations we have shown how phasors and complex numbers are constituted as different epistemological objects in the mathematical discourses. In other publications we have shown how both discourses can also be reconstructed empirically on the basis of tasks, lecturer sample solutions (Peters & Hochmuth, 2021) and student work (Hochmuth & Peters, 2021).

In the following, we will use the introduction of the Dirac impulse in signal theory as an example for the principle of identifying infinitesimal formal mathematical quantities with finite measurable quantities. Furthermore, we show how this principle also interacts with the two mathematical discourses described above.

In previous work (Hochmuth & Peters, 2020) we analysed the introduction of the Dirac impulse in the signal theory textbook by Fettweis (1996): Thereby we have taken up Fettweis' distinction between idealised and real signals and highlighted a general principle (see diagram in Fig. 2), which plays an important role in justifications by Fettweis and is used by Dirac (1958) in a similar way.

Here we want to draw attention to the connection between the principle, illustrated in Fig. 2, and the two mathematical discourses outlined above. On this basis we will then show how this connection is also helpful for the reconstruction of a passage from the textbook by Frey and Bossert (2009).

In Fettweis's approach, the real signals represent irregular transmissions on the one hand, i.e. they refer to empirical objects, and on the other hand to functions with pleasant mathematical properties such as sufficient differentiability. Thus, they form a central link that enables further identifications, allow specific justifications in this context and connect discourses. The ideas associated with idealised signals, on the other hand, refer to irregular (according to Fettweis) mathematical objects, such as the Heaviside function or the Dirac impulse, as well as to signals which as such do not exist in reality, but only approximately. Here, too, references to mathematics and empirical sciences are brought together, and mathematical discourses can start from these. We now illustrate these general remarks with the example of a passage from the signal theory textbook by Frey and Bossert (2009).

Frey and Bossert (pp. 208) formulate the goal to differentiate the (in the usual sense) non-differentiable Heaviside function, which is defined by:

$$\varepsilon(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

The Heaviside function represents (according to Fettweis) an idealised signal. In order to apply HM-practices, the Heaviside function is represented approximately by the sequence of differentiable functions (real signals):

$$f_a(t) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{t}{a}\right) + \frac{\pi}{2}\right), a > 0.$$

The approximation can be interpreted in the sense of the HM-discourse as pointwise convergence. The derivation of the Heaviside function is then derived with the following steps:

$$\frac{d}{dt}\varepsilon(t) = \frac{d}{dt}\lim_{a\to 0} f_a(t) = \lim_{a\to 0}\frac{d}{dt} f_a(t) = \lim_{a\to 0}\frac{1}{\pi}\,\frac{a}{a^2+t^2} = \{ \begin{matrix} 0, & t \neq 0 \\ \infty, & t = 0 \end{matrix}.$$

The HM concept of pointwise convergence also allows to understand the equal sign in the last step. The central step in the argumentation is the transition from the second to the third term, which can be clarified with reference to the above principle and through this mediated interplay between the two mathematical discourses. From a mathematical point of view, the step from the second to the third term, i.e. the permutation of the limit values on the basis of the pointwise convergence of function sequences, is not permissible. This permutation can therefore not be justified in the HM discourse. This is where the principle comes into play: The derivative of the idealised signal cannot be calculated. Therefore, it is approximated and replaced by real signals. These are differentiated. Then finally the limit of the calculated derivatives is calculated. The above principle thus allows the step from the second to the third term to be broken up in such a way that HM discourse elements can be made effective. Here the principle proves to be an expression of the ET-discourse. Thus, in the step from the second to the third term an interesting interplay between the two mathematical discourses, mediated by the principle, results.

Of course, the discussed step could be rewritten in terms of distribution theory (also partially addressed by Frey and Bossert, pp. 110), so that it could be justified in this interpretation purely mathematically. But the central point in our argumentation is not that distribution theory is not used here as the basis for justification, which would correspond to a rather deficit-oriented view. Rather, our point is that the electrotechnical mathematics discourse, in its reference to empirical objects, not only allows for a justification of the step, but also establishes a reference of symbols and argumentation to empirical objects and contexts. A purely distribution-theoretical argumentation could not make this possible. To do so, it would have to be supplemented in a suitable way by electrotechnical means, which would of course be possible in principle.

5 About Mathematical Practices in Psychology

We consider psychology as another example of mathematical practices in an empirical science. Here we focus on psychometric tests, as used in diagnostics in the form of performance, intelligence, ability or development tests.[8] Such tests are also used or developed in empirical research projects in university mathematics education (see e.g. Kuklinski et al., 2018; Hochmuth et al., 2019). The aim of such tests is, in particular:

> to capture inter-individual differences in behaviour and experience as well as intra-individual characteristics and changes, including their relevant conditions, in such a way that sufficiently precise predictions of future behaviour and experience as well as possible changes in defined situations become possible.[9] (Amelang & Zielinski, 2002, Sect. 1.1)

In general, model assumptions underlying the tests can be distinguished in terms of traits and behaviour diagnostics. In the first case, the description of experience and behaviour in the form of traits is crucial, whereby traits are represented by hypothetical constructs that are derived from and refer to observable behaviour. In the second case, personality traits act as intervening variables, which are usually determined as the probability that a person with certain traits will exhibit certain behavioural tendencies. Theory and empirical knowledge are interdependent in test development and application: On the one hand, theories are available in the form of descriptions and conceptualisations of psychological constructs (e.g. motivation, self-efficacy, intelligence) and are usually embedded in broader theoretical contexts: they form the basis of quantitative models. On the other hand, modelling and testing also create opportunities for observation. These allow, for example, theoretically suggested hypotheses to be empirically confirmed or refuted. In theoretical preliminary considerations, the aim is to determine the situational test conditions as precisely and objectively as possible (e.g. also selection of suitable cohorts, suitable item formulations). The tests to be developed should ideally be sensitive to interesting factors and robust against interfering factors. Which factors and constructs come into view is determined by the underlying theories and their basic concepts. Each concrete test development, both in the traits and in behaviour diagnostics, now includes the transformation of theoretical constructs or factors into variables. This transformation is often referred to as operationalisation. The aim of

[8]Psychometric tests, of course, represent only a small section of psychology as an empirical science. It should at least be noted that throughout the history of psychology there have been repeated controversies about how to define the specific subject of psychology and what this means in terms of feasible and appropriate scientific methods. For example, the controversy of explanation-understanding is to be mentioned (see e.g. Riedel, 1978). There are, for example, many relationships between this controversy and our discussion in this section. For space reasons alone we cannot go into this here.

[9]"…interindividuelle Unterschiede im Verhalten und Erleben sowie intraindividuelle Merkmale und Veränderungen einschließlich ihrer jeweils relevanten Bedingungen so zu erfassen, hinlänglich präzise Vorhersagen künftigen Verhaltens und Erlebens sowie deren evtl. Veränderungen in definierten Situationen möglich werden".

operationalisation is then to formulate assumptions of interrelationships between variables, e.g. between independent and dependent ones. Based on variable-related data, the assumptions are tested by means of stochastic procedures. The goal of operationalisation is thus to enable statistically processable and assessable findings.

To make this possible, operationalisation must ensure that the events and reference variables that are considered quantifiable and measurable by variables meet conditions such as random variability:

> Only if a result could in principle also have occurred by chance does the statement that in the present case (according to agreed criteria) it is more frequently than simply random has empirical substance.[10] (Holzkamp, 1994, p. 85)

Random variability must therefore be ensured in the psychological design of experiments. Which constructs or factors in which situations are suitable for transformation into variables is a central specific contribution of the science of psychology and cannot be answered mathematically alone. In contrast to electrical engineering, for example, such transformations or "identifications" of psychological constructs and variables are controversial in psychology (cf. e.g. Echterhoff et al., 2013, pp. 39–42). From a historical point of view, operationalisation represents, among other things, a starting point for the formulation of a fundamental critique of the type of psychological research outlined here: For example, the assumptions of connections formulated in this way would often be "secondary constructions of abstract generality . . . that have very little to do with the real connections/contradictions that should actually be up for debate [in psychology; the authors], with which the research findings, because they `bypass the problem´, always seem somehow trivial, meaningless, indifferent" (Holzkamp, 1994, p. 82).[11] Currently, a large part of psychological research is oriented towards the methodological approach sketched up. With regard to this, it should have become plausible that it includes, in its operationalisation, an identification of mathematical objects with quantities that are considered measurable and quantitative.[12]

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[10]"Nur wenn ein Resultat prinzipiell auch zufällig zustande gekommen sein könnte, hat die Aussage, dass es im vorliegenden Fall (nach vereinbarten Kriterien) 'überzufällig' ist, einen empirischen Gehalt."

[11]"sekundäre Konstruktionen von abstrakter Allgemeinheit . . . die mit den wirklichen Zusammenhängen/Widersprüchen, um die es eigentlich [in der Psychology; the authors] gehen sollte, nicht viel zu tun haben, womit die Forschungsbefunde, weil sie `am Problem vorbei´ gehen, stets irgendwie als trivial, nichtssagend, gleichgültig anmuten". One can also compare Blumer's introduction of a theory of symbolic actionism, which was explicitly founded as an alternative to "variable psychology". ATD, with its focus on mathematics, also distinguishes itself to a certain extent from research in didactics that is essentially psychologically based. It goes without saying that this does not mean that psychometric tests in psychological or didactic research to investigate specific questions, such as the effects of interventions in the face of large cohorts, are rejected.

[12]In psychology, operationalisations often do not take place in a single step. Therefore, a sequence of such steps could be distinguished in detail. The last step, with regard to mathematical objects, would then be particularly close to considerations in electrical engineering. In a certain sense, the preceding steps would then be comprised in that step. Of course, the basic principle can only be roughly described here.

The second epistemology related aspect is the emergence of different mathematical discourses in mathematical practices in psychology, especially statistics, that can also be observed in many ways. This concerns the specific selection of stochastic models oriented to particular psychological research questions, but also their implementation in detail. This can be shown particularly well with path models and structural equation models (cf. e.g. Renner et al., 2012). These do not result only from quantitative calculations, but they are also usually based on theoretical psychological considerations. Only this subsequently enables the psychological interpretation of the calculated results.

The considerations about mathematical practices in psychology of this section cannot, of course, replace a praxeological study based on concrete empirical material. However, they point out that this could also be fruitful with regard to the two epistemology related aspects highlighted in this paper. Finally, we would like to briefly add that our reflections on psychology are also compatible with the pragmatic position referred to at the beginning: according to this position, psychological measurements (e.g. of an intelligence test) would not be understood as quantitative definitions of personal characteristics, but rather as something that makes statements about the behaviour of a person under certain conditions.

6 Outlook

In this contribution we examined a few ideas of how ATD-analyses could be informed by epistemological-philosophical insights. Aspects from historic-materialistic studies by Wahsner and von Borzeszkowski and philosophical-pragmatic considerations by Schlaudt were used to discuss relationship of mathematics and empirical sciences. Links to concepts of the ATD were drawn via the scale of levels of codetermination and the institutional dependence of knowledge. We have illustrated our considerations by examples of mathematical practices in electrical engineering and psychology.

The philosophical-epistemological reflections on mathematical practices indicate that they (partially) ground in major issues related to the interrelationships between empirical sciences and pure mathematics and their historic-specific manifestation in societal institutionalised teaching learning contexts. On this basis, especially ATD-related concepts can be complemented and concretised in a suitable way in order to examine institutionalised mathematical practices of teaching and learning in educational institutions and to draw conclusions for teaching innovations. In particular, implementations of well-meant measures might produce unsatisfactory or unintended effects as long as institutional, pedagogical and epistemological conditions are not sufficiently well understood. Beyond an analysis of technological-theoretical blocks of mathematical practices in empirical sciences like electrical engineering, physics but also psychology, sociology, etc., the philosophical-epistemological considerations further allow to question notions often used in mathematical education research that claim to make essential aspects of such mathematical practices didactically accessible. Examples of these questions are: Which aspects of those mathematical practices are covered and which are not covered by approaches applying modelling cycles (Blum & Leiss, 2005) or "Grundvorstellungen" (Greefrath et al., 2016)? How could these approaches be reinterpreted by ATD terms, if relevant, in order to complement them appropriately with regard to ignored aspects? And even more critically: How are those approaches and their deficits regarding the issue of mathematical practices in empirical sciences related to societal dominating reflections on teaching-learning issues? The latter question is (partially) further connected to questions regarding the level of external didactical transformations (see e.g. Bosch et al., 2021): How are the investigated issues reflected in the construction of study programs and module structures? And finally, regarding consequences for teaching: in addition to an emphasis on the explication of identifications and the rationales of different mathematical discourses in lectures and texts, the construction of suitable rich tasks with interesting opening questions for the establishment of SRPs could be an interesting and useful way. In connection with suitable initial problems for SRPs, the mathematical practices of the empirical sciences, which have historically been widely recognised as adequate, could and should indeed prove useful for their elaboration and solution by students.

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8. Revisiting the relationship between mathematics and electrical engineering

Study V: Sometimes mathematics is different in electrical engineering

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Study VI: Modifying exercises in Mathematics Service Courses for Engineers Based on Subject- Specific Analyses of Engineering Mathematical Practices

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SOMETIMES MATHEMATICS IS DIFFERENT IN ELECTRICAL ENGINEERING

Jana Peters[1] and Reinhard Hochmuth[1]
[1]Leibniz University Hannover

Abstract

In this contribution we will present an ongoing research project on mathematical practices in electrical engineering. Starting with interesting phenomena we have encountered in our research regarding the relationship of mathematics and engineering, we provide some general thoughts on the notions application and modelling. We then present our own vantage point: Using the Anthropological Theory of the Didactic (ATD), we take an institutional point of view on mathematical practices. This allows us to conceptualise two ideal type mathematical discourses corresponding to different epistemological constitutions of mathematical knowledge in mathematics courses for engineers and in advanced courses in electrical engineering, respectively. We will enrich our presentation with short vignettes of our latest research results to illustrate the kind of insights that the institutional point of view enables us to gain particularly regarding educational issues.

Key words: Anthropological theory of the didactic, mathematical practices, electrical engineering, application and modelling

INTRODUCTION

The study of engineering mathematical practices is an important topic in engineering mathematics education (Alpers, 2020; Winsløw et al., 2018). Explicitly focusing on the specific content related needs of engineering mathematics for didactic analyses enables a deeper understanding of practices and potential learning difficulties related to them. A deep analysis of teaching materials and students' works can then also open up new ideas for teaching design. In an ongoing research project on mathematical practices in Signal Theory, we refer to the Anthropological Theory of the Didactic (ATD) (see Bosch et al., 2019; Chevallard, 1992; Chevallard et al., 2022) and, besides other, its understanding of praxeology to model mathematical practices in electrical engineering to address those issues. Other recent ATD related studies on mathematical practices in engineering are done by Bartolomé et al. (2019), Florensa et al. (2018), González-Martín (2022), Palencia (2022), Rønning (2021) and Schmidt and Winsløw (2021). In our research project we developed three foci: First, with a focus on subject specific mathematical practices, we introduced an extended praxeological model to reconstruct the mathematical discourse that justifies the mathematical practices in signal theory (Peters & Hochmuth, 2021). In (Hochmuth & Peters, 2021) we show, how students' solutions to a signal theory exercise can be analysed and understood on the basis of our previous analyses. The second focus

Corresponding author: Jana Peters

considers the epistemological relationship between mathematics and electrical engineering (Hochmuth & Peters, 2020; 2022). We showed that epistemological relations between mathematics and engineering can be important for a detailed description and analysis of mathematical practices. Considering epistemological aspects of mathematical practices within the framework of the ATD makes those aspects accessible for didactical analyses and design. The third focus highlights the potential of our ATD analyses for teaching design (Peters, 2022). Here the emphasis is on possible connections between mathematics as taught in higher mathematics courses and mathematics in engineering courses. Based on previous work, we develop an idea for teaching design to foster such connections without the need for the introduction of application examples or the complete restructuring of the course. In an early phase of our research project, when we analysed teaching materials and students' works and had mainly the first focus in mind, we came across two interesting phenomena: First, we repeatedly encountered a deficit-orientation towards mathematical practices of engineers when discussing data and corresponding intermediate analysis results with colleagues. The mathematical practices we studied generally did not follow socio-mathematical norms of academic mathematics[1]. From the standpoint of academic mathematics, mathematical practices in electrical engineering courses like Signal and System Theory (SST) seemed to be sometimes wrong, incomplete and sketchy. This was ascribed to limitations in engineering studies, but was nevertheless seen as a (necessary) deficit. Second, from the perspective of academic mathematics some engineering mathematical practices were difficult or impossible to understand. For example, in our analyses, we could identify arithmetic transformations but could not explain their significance and reasons from the standpoint of academic mathematics (cf. our analysis vignette).

Both phenomena seemed also to be connected. Engineering mathematical practices, that were difficult or impossible to understand, were often simply framed as deficits from the mathematicians' point of view. As fruitful as analyses are that reveal such differences, it is not satisfactory to interpret them sweepingly as deficits. The question is also which deficits from the perspective of academic mathematics are part of an adequate electrotechnical mathematical practice and which are not? Which of the identified deviations from academic mathematics hinder the learning of mathematical concepts in engineering and which deviations are necessary for the adequate teaching and learning of engineering mathematical practices? However, the same question also arises for non-deficit mathematical practices in engineering. Is any access to a mathematical concept that is adequate from an academic mathematical point of view also beneficial for the learning of mathematical practices of engineers? We found these questions important specifically for our analyses, but also relevant to engineering mathematics education in general. Only a detachment from the deficit-oriented view of engineering mathematics practices makes these questions accessible. We could see that those mathematical practices were pragmatic and also necessary to solve specific epistemological problems related to physics or engineering, respectively. In our studies, we later, when we developed the second focus, were able to relate some apparent mathematical shortcomings of engineering mathematics to epistemological issues, which cannot be clarified by inner-mathematical considerations alone and which sometimes underlie

[1] We speak of academic mathematics when we refer to mathematical research institutes and institutes of mathematics at universities. We distinguish academic mathematics from other mathematical institutions, such as mathematics courses for engineers that are part of engineering study programs. In our studies we also use the ATD concept of didactical transposition to connect the relationship of different mathematical institutions with our analyses (cf. Peters, 2022).

the conceptualisation of mathematical knowledge in electrical engineering (Hochmuth & Peters, 2020; 2022). We realised that those positive aspects of the engineering way of doing mathematics were difficult to acknowledge and analyse from the perspective of academic mathematics alone.

In this contribution we want to proceed from this and move on to some general reflections on the concepts of *application* and *modelling*, which are often used in studies of engineering mathematics education to capture the relationship of mathematics and engineering. Thereby, the epistemology of this relationship generally remains implicit and unquestioned. On the other hand, applications of mathematics and modelling problems are often present as important design aspects to improve the teaching of mathematics to engineers (e.g. Alpers, 2020). After bringing up some critique on the standard concepts of application and modelling, we present our stance which enables us to avoid some of the difficulties inherent in application and modelling. After introducing central concepts of ATD and illustrating our stance with two vignettes from previous work, we come back to the notions of application and modelling and show further possibilities of alternative conceptualisations from the viewpoint of ATD.

ON THE CONCEPTS OF APPLICATION AND MODELLING AND AN ALTERNATIVE VANTAGE POINT TO BETTER UNDERSTAND ENGINEERING MATHEMATICAL PRACTICES

There are various definitions and understandings of application and modelling (e.g. Blum et al., 2007), most of which separate between an extra-mathematical world and mathematics. Often application of mathematics and mathematical modelling are seen as related to each other. Niss, Blum, and Galbraith (2007) summarises this relationship and their understanding as follows

> During the last one or two decades the term '*applications and modelling*' has been increasingly used to denote all kinds of relationships whatsoever between the real world and mathematics. The term 'modelling', on the one hand, tends to focus on the direction 'reality → mathematics' and, on the other hand and more generally, emphasises the *processes* involved. Simply put, with *modelling* we are standing outside mathematics looking in: 'Where can I find some mathematics to help me with this problem?' In contrast, the term 'application', on the one hand, tends to focus on the opposite direction 'mathematics → reality' and, more generally, emphasises the *objects* involved - in particular those parts of the real world which are (made) accessible to a mathematical treatment and to which corresponding mathematical models already exist. Again simply put, with *applications* we are standing inside mathematics looking out: 'Where can I use this particular piece of mathematical knowledge?' (p. 10f)

These widely held understandings of the relationship of mathematics and engineering can be very fruitful, especially in teaching design. But it is also regularly noted that the realisation and implementation in everyday teaching is problematic. Barquero et al. (2013) give a survey of literature illustrating the difficulties and barriers as a general problem for the dissemination of modelling activities. They use their own projects to identify and categorise difficulties and barriers. Besides other, they

> focus on describing some constraints related, in the first place, to what may be called the *dominant*

> *epistemology*, that is, the way our society, the university as an institution and, more particularly, the community of university teachers and students, understand what mathematics is and what its relation is to natural sciences. (p. 316)

Regarding the meaning of applications Barquero et al. (2013) reconstructed an epistemology of *applicationism* in the relationship of mathematics to other sciences and identify it as a restriction on the notion of mathematical modelling:

> One of the main characteristics of applicationism is that it greatly restricts the notion of *mathematical modelling*. Under its influence, modelling activity is understood and identified as a mere application of previously constructed mathematical knowledge or, in the extreme, as a simple exemplification of mathematical tools in some extra-mathematical context artificially build in advance to fit these tools. (p. 317)

Regarding the two worlds of mathematics and "the rest of natural sciences" they note that "it is furthermore supposed that both 'worlds' evolve with independent logic and without too many interactions" (p. 318). Also, they note that "in general, the mathematics taught present a highly stereotyped and crystallized structure that does not mingle with the systems that are modelled and, moreover, the mathematics taught are never 'modified' as a consequence of being applied." (p. 319). We can now ask whether an unquestioned academic mathematical perspective on mathematical practices in engineering, where mathematics is also seen as "never modified", can be linked to applicationism? From the applicationsm point of view, it is suggestive to understand mathematical practices in engineering only as more or less deficient applications of previously constructed academic mathematical knowledge.

Regarding the notion of mathematical modelling Bissell and Dillon (2000) note

> Mathematical modelling forms an important part of engineering education and practice. Yet precisely what is meant by the term 'modelling' is often extremely unclear - and, moreover, much of what students are told about the subject is considerably problematic from both a philosophical and a pedagogical point of view. (p. 3)

In their study they look at mathematical modelling from the practicing engineer's perspective. From this perspective the usual modelling cycles are too simplistic to capture mathematical activities of engineers. Also, they note that instead of creating mathematical models in engineering, it is much more important for practicing engineers to use already existing mathematical models (Bissell & Dillon, 2000, p. 4). This shift of perspective on mathematical modelling in engineering, enables an understanding of mathematical models without a necessary separation of mathematics and the rest of the world: here engineering is not only the context for application or the source of the modelling problem. Engineering itself is already mathematised. They also characterise necessary skills for using models: *manipulation* is "the ability to modify the form of the basic model, using algebraic and other skills; essentially 'mechanical'", *interpretation* is the "ability to interpret the modified form of the model in a way relevant to the situation; essentially 'reactive'", and application is the "ability to apply the interpretation and make appropriate recommendations; essentially 'proactive'" (p. 4). Note that they speak of applying the interpretation with respect to the relevance of the situation. Here the applied mathematics does not necessarily remain unchanged as it is the case in applicationism. By connecting their considerations about mathematical modelling also with the general question of "the position of mathematics in engineering" they state that "there is clearly a significant difference between what a mathematician calls ‘doing mathematics’ and what an engineer calls ‘doing mathematics’.” (p. 6).

In our research project we found that to better understand these different ways of doing mathematics and to analyse mathematical practices in engineering without restrictions to applicationism or a deficit-oriented view, we needed a different vantage point: A vantage point that enables us to understand the engineering specific justifications and explanations of mathematical practices and allows for deeper content specific analyses than the considerations by Bissell and Dillon. The ATD enables this, among other things, through the principle of *institutional dependence of knowledge*: In different subject specific institutional contexts mathematical practices are justified, substantiated, validated and constituted differently than in academic mathematics. Also, Castela (2015) emphasises[2] the advantages of an institutional approach to research on mathematical knowledge in different contexts that is fundamental to the ATD. This approach “provides a powerful tool to investigate the mathematics dimension of human social activities in any context, without referring to academic mathematics.” (Castela, 2015, p. 18). This can contribute to counteracting a deficit-oriented view of mathematical practices and provide a deeper understanding of mathematical practices in other siences.

INSTITUTIONAL POINT OF VIEW AND MATHEMATICAL DISCOURSES

Building on Castela’s work, we have introduced a specific extended[3] praxeological model (Peters & Hochmuth, 2021) that allows us to analyse mathematical practices in electrical engineering, particularly taking into account the engineering-specific institutional conceptualisation of mathematical practices. We also showed how institutional analyses of engineering mathematical practices can be related to and help understand individual students’ solutions to exercises (Hochmuth & Peters, 2021). In the following we will present vignettes from both studies to illustrate our approach.

Alongside the institutional dependence of knowledge, *praxeology* is another ATD concept that is important for our work. In ATD a praxeology is a basic epistemological model to describe institutional knowledge in the form of two inseparable, interrelated blocks: the praxis block (know-how) consists of types of problems or *tasks* (T) and a set of relevant *techniques* (τ) used to solve them. The logos block (know-why) consists of a two-levelled reasoning discourse. On the first level, the *technology* (θ) describes, justifies and explains the techniques and on the second level the *theory* (Θ) organises, supports and explains the technology. In short praxeologies are denoted by the 4T-Model $[T,\tau,\theta,\Theta]$.

We illustrate how the concept of praxeology, especially our extended praxeological model, is able to produce (in the sense of a phenomenotechnique, cf. Bosch et. al., 2019) an understanding of the engineering-specific conceptualisation of mathematical practices. We will introduce our extension to the praxeological model in the course of the following exemplary analysis at the step, where the need for an extension concretely arises. For this, we consider an exercise of an SST course at a German university that is taught in the second year

[2] She addresses the relationship of academic mathematics and mathematics in vocational contexts.

[3] With respect to the standard praxeological model of ATD. In our extension we differentiated techniques and technologies according to two mathematical discourses, see below.

of an electrical engineering study program. First, we focus on the lecturer's sample solution, i.e. the taught knowledge in SST. The context of this exercise is amplitude modulation. The exercise under consideration is[4]:

> Graphically display $x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$.

The lecturer sample solution is:

One first writes

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \quad (1)$$

$$= A\,\Re\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t - \Omega t))\} \quad (2)$$

$$= \Re\left\{ \exp(j2\pi f_0 t)\underbrace{\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]}_{A(t)} \right\} \quad (3)$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude $A(t)$, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Figure 1.

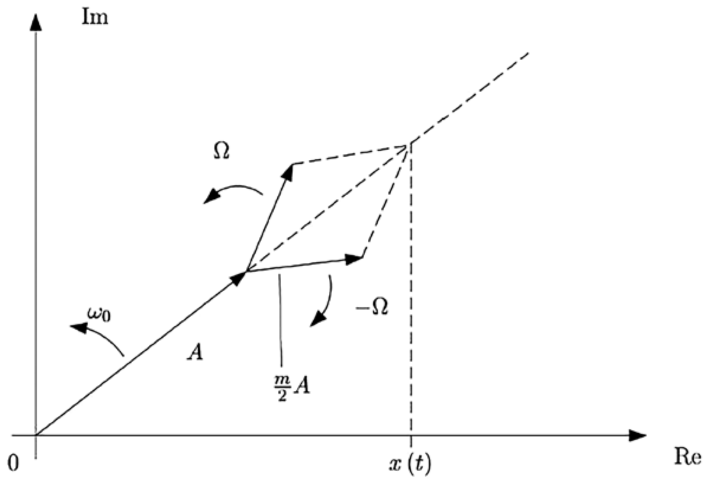

Figure 1: Representation of $x(t)=A[1+m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0=2\pi f_0$.

The sample solution of the exercise represents institutional knowledge of signal and system theory. We now can assign the praxeological components to the steps in the solution. Since this is one exercise, the reconstruction of types of tasks is not relevant here. In the full analysis, as presented in (maybe this is an

[4] Literal translation from the German exercise sheet by the author. The number of the figure is adjusted to the figure counter in this publication.

inconvenient pagebrake 2021) and (Hochmuth & Peters, 2021), we use the method of considering subtasks to further structure the analysis. For this analysis vignette, we focus on techniques as part of the praxis-part and on technologies as part of the logos-part of praxeologies. Tasks and theory will not be further considered and will therefore be understood as SST-tasks and SST-theory, i.e. tasks relevant in the institution SST and theory as the second level of the praxeological reasoning discourse according to the institution SST.

From line (1) to line (2), in the sample solution, using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ is a *technique* τ. Here, connections between the representations of a complex number in polar form and in exponential form are relevant for the justification, the *technology* θ . Both, technique and technology, for this step are known by the students from Higher Mathematics courses earlier in the study program. Here we can see, that for this SST-exercise techniques and technologies from Higher Mathematics courses are relevant. The institutional knowledge of signal and system theory therefore has connections to knowledge from a different institution.

Before we go deeper into details here, we come back to the already introduced idea of institutional dependence of knowledge to clarify what this means for our analysis in particular. Following (Castela, 2015), an institution is

> a stable social organisation that offers a framework in which some different groups of people carry out different groups of activities. These activities are subjected to a set of constraints, - rules, norms, rituals - which specifies the institutional expectations towards the individuals intending to act within the institution I. […] Institutions tend to constrain their subjects but conversely they provide the resources (material and cultural) necessary for activities to take place. (p. 7)

Institutional conditions, norms and aims constitute the technological-theoretical discourse and the practices available. This means that different types of tasks are relevant in different institutions, different solution techniques are adequate, different reasoning discourses are acceptable, and different reasons to study a subject occur. Thus, if one focuses on a specific mathematical subject in different institutions, different praxeologies could emerge. In the following we will show that in the context of our research, the institutional knowledge in SST shows references to other relevant institutions and corresponding institutional mathematical discourses[5]. Our analysis so far showed a praxeology concerning techniques and a technological discourse of dealing with complex numbers that can be assigned to an institution Higher Mathematics (HM). We denote this praxeology by $[T, \tau_{HM}, \theta_{HM}, \Theta]$. In our work, we used the textbook by Strampp (2012), students' lecture notes, and exercises from a course on Higher Mathematics for Engineers based on this textbook to characterise the mathematical knowledge associated with this institution, i.e. the HM-discourse: It is characterised by an internal mathematical conception without concrete references to reality, an orientation towards a generalising rational of academic mathematics, a concentration on calculation rules, and the inclusion of school mathematics concepts. The reason why complex numbers are studied in HM-courses is because they are useful for solving polynomial equations and they are important objects of calculation. Arrows in the Gauß-diagram are used to graphically illustrate calculation rules and properties.

[5] The term discourse refers to the logos part of praxeologies: In ATD, logos is considered as a discourse on praxis (reasoning discourse), but since praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is also related to the institutional discourse. Reasoning discourses are institutionally dependent, and so are the respective techniques and technologies. The notion of institutional discourse enables us to differentiate analytically between techniques and technologies that could be associated to different institutional discourses respectively.

When we now look at the solution step from line (2) to line (3) also techniques from the HM-discourse occur: The real parts of the summands are factored out, calculation rules for the exponential function are applied, and the resulting common factor $\exp(j2\pi f_0 t)$ is factored out. But it is difficult to understand the reasons for this transformation from the standpoint of Higher Mathematics. Why make a clearly structured expression more complicated? Also, from the way complex numbers are taught in the HM-course, drawing three phasors associated each with one of the summands in line (2), seems much more obvious than drawing phasors associated to the more complicated expression in line (3). To understand why this transformation is carried out, we have to look for the engineering reasoning that is not part of the HM-course: $x(t)$ is transformed in a specific way to graphically represent principles of amplitude modulation, that could not be represented by a graphical representation of line (1) or (2) (see also our second vignette of an analysis of a student solution to this exercise below). The cosine representation in line (1) does not allow to separate the different frequencies or angular velocities of the carrier-signal, $\omega_0 = 2\pi f_0$, and the message signal, Ω . This is, however, the core of both the representation in line (3) and the graphical representation in Figure 1 in the sample solution. There is no justification within our reconstructed HM-discourse, that gives the reason for the step from line (2) to line (3). So, the technological discourse underlying the step from line (2) to line (3) differs from the HM-discourse. This other mathematical discourse belongs to a different institution. In our analyses we denoted this other mathematical discourse as an electrotechnical mathematics-discourse (ET). In our work, we most notably use studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004; 2012) and the electrical engineering textbook by Albach (2011) to characterise this mathematical ET-discourse. In contrast to the HM-discourse, the ET-discourse has references to reality. The degree to which this reference to reality is made explicit can vary greatly, along with a different degree of formalisation and abstraction. In addition, it is characterised by a "linguistic shift" (Bissell & Dillon, 2000, p. 10) in the way of talking about mathematics and mathematical practices and an electrotechnical-typical way of "system-thinking". One reason why complex numbers are studied according to the ET-discourse is that they allow oscillating signals to be described algebraically in a very suitable way and visualised graphically as phasors. This visualisation does not serve to illustrate calculation rules or properties of complex numbers but represent important analysis tools, e.g. for AC circuits (cf. Albach, 2011) or amplitude modulation. For a more comprehensive description of the discourses see (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021; Peters, 2022).

The step from line (2) to line (3) can be associated to a praxeology $[T, \tau_{HM}, \theta_{ET}, \Theta]$. In the next step, the expression in line (3) has to be interpreted in order to draw the Gauß-diagram, cf. Figure 1. The part denoted by $A(t)$ must be interpreted as a modulation process (τ_{ET}). This is justyfied because the phasor which varies in length with $A(t)$ represents a general periodic signal (θ_{ET}). In this praxeology technique and technology are from the mathematical ET-discourse, $[T, \tau_{ET}, \theta_{ET}, \Theta]$.

This brief illustrative insight into our analysis of the sample solution shows that the solution of this task can be linked to different praxeological configurations ($[T, \tau_{HM}, \theta_{HM}, \Theta]$, $[T, \tau_{HM}, \theta_{ET}, \Theta]$, and $[T, \tau_{ET}, \theta_{ET}, \Theta]$) drawing on the two different institutional mathematical discourses. We could observe that both institutional discourses are interrelated and show up in different combinations of techniques and technologies. Transitions or shifts between the two mathematical discourses constitute epistemological ruptures in the sense that they each follow a different rational. These ruptures often remain implicit, although they represent important aspects. They indicate places that are not accessible from a single mathematical discourse and its techniques, and thus mark something additional to be learned. Neither the modelling and application-point of view nor the standard praxeological model are sufficient for this kind of analyses: Application and modelling both entail a conceptualisation of electrical engineering knowledge as consisting of inner-mathematically justified mathematical practices[6] and extra-mathematical engineering knowledge[7]. The praxeology $[T, \tau_{HM}, \theta_{HM}, \Theta]$ could be interpreted as a purely inner-mathematical, as the HM-discourse has strong relations to academic mathematics and no references to reality. So this part of the analysis could be related to the application- or modelling view. But there is no accompanying extra-mathematical engineering knowledge where this praxeology (i.e. the knowledge that is modelled within ATD with this praxeology) is applied to. The mixed praxeology $[T, \tau_{HM}, \theta_{ET}, \Theta]$, where a technique from the HM-discourse gets a new ET-discourse meaning, does not fit this viewpoint at all. The standard praxeological model that would allow to take different institutional origins of practices into account and to describe the knowledge in form of HM- and ET-praxeologies does not allow to shed light on the interrelatedness of the two mathematical discourses.

As a second vignette we present a short analysis of a student solution to this exercise[8] from (Hochmuth & Peters, 2021). This is an example of a solution, where the necessary shifts between the two mathematical discourses do not occur (cf. Figure 2).

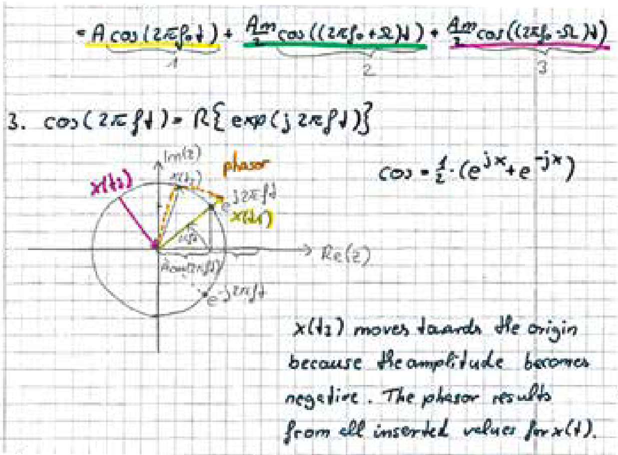

Figure 2 A student solution to the exercise (Hochmuth & Peters, 2021)

At the top we see each of the three cosine-terms separately underlined and each given a number. Each term is thus interpreted individually as something to be drawn. These numbers can also be found in the diagram; the respective phasors are marked accordingly ($x(t_1)$, $x(t_2)$, and $x(t_3)$). While underlining mathematical terms is a technique neither specific to the HM-discourse nor to the ET-discourse the idea represented in this technique, that each term is something to be drawn individually, is a technology of the HM-discourse (θ_{HM}): Each cosine-term stands for a complex number that could be drawn as an arrow starting at the origin of the Gauß-diagram. The sum of three complex numbers then could be drawn as the geometric sum of the

[6] Learnt in mathematical courses and applied later in different contexts.

[7] Providing the context for the application of the mathematical knowledge or the modelling problem.

[8] To protect the student's privacy, we have rewritten the student solution. We omitted the assistant's marking.

respective arrows. The student tried to graphically add the three arrows (dashed line in the diagram in Figure 2). Additionally, the diagram also contains elementary properties of complex numbers: the connection between cosine and the complex exponential function and the complex conjugate. This student solution reproduces the HM-discourse by drawing a diagram similar to diagrams from the mathematics service course where the Gauß-diagram and the unit circle are used to illustrate properties of complex numbers (τ_{HM}). Aspects indicating a connection to amplitude modulation are missing and transitions to the ET-discourse do not occur. This student solution does not produce a diagram that is capable of illustrating aspects of amplitude modulation.

SOMETIMES MATHEMATICS IS DIFFERENT: A DISCUSSION

This short analysis vignettes show the relevance of taking into account the specific mathematical ET-discourse for a deeper understanding of mathematical practices of electrical engineers. Reconstructed mathematical practices from a lecturer sample solution of a signal and system theory-task contain aspects of both discourses, the HM-discourse and the mathematical ET-discourse. An analysis vignette of a student solution to this exercise showed that referring only to the HM-discourse is both a possible student action and not sufficient to solve the task.

We already argued that the standard conceptualisations of the relationship of engineering and mathematics, modelling and application, are not able to capture the complex nature of engineering mathematical practices in this way. From our ATD perspective, we see this relationship not as a relation between independent fields of knowledge. At the core of our approach is the acknowledgement of mathematical practices of engineers as institutional mathematical practices in their own right and with engineering specific conceptualisations of mathematical knowledge. Relations to academic mathematics are present, e.g. in our analysis in the HM-discourse. Also, relations to mathematics developed within the engineering institutions are present, e.g. in our analysis in the ET-discourse. In (Peters, 2022), these relations are discussed in more detail. The mathematical discourses interact in complex ways and are not understandable from the standard modelling and application point of view.

Nevertheless, there are conceptualisations, that are connectable to our stance. Concerning an understanding of application we would like to mention the work by Schmidt and Winsløw (2021). Using an analysis of didactical transpositions between institutions Mathematics and Engineering, they develop a method to design Authentic Problems from Engineering (APE). In their approach the idea of applying mathematical knowledge to engineering starts with engineering. From there they look for possibilities to let the engineering knowledge interact with the mathematical concepts. This approach is specifically capable of counteracting the problem of applicationism. Concerning modelling, we already mentioned the perspective of Bissell and Dillon (2000) who shift the focus to the *use* of mathematical models and show how in the historical process engineers developed mathematical practices specific to their needs and aims (see also Bissell, 2004; 2012). Another important line of development is the reformulation of modelling from an ATD perspective as it is presented for example can we change this to: by Garcia et al. (2006). This reformulation seems particularly appropriate to us here, of course, because we share the same framework of ATD. But apart from that, we also consider the approach fruitful because it takes a decidedly epistemological and institutional perspective on modelling. A first important interpretation of modelling within the ATD is "that modelling is [not] just one more aspect or dimension of mathematics, but that mathematical activity is essentially a modelling activity in itself" (p. 232). Two statements are then important. First mathematical modelling is not restricted to "'mathematization' of non-mathematical issues" (p. 232). Also, inner-mathematical activities are understandable as modelling activities. Second, they highlight the meaning of modelling activity from the standpoint of ATD:

> In the framework of the ATD, what is relevant is not the specific problem situation proposed to be solved (except in 'life or death' situations), but what can be done with the solution obtained –that is, with the constructed praxeology–. The only interesting problems are those that can be reproduced and developed into wider and more complex types of problems. The study of those *fertile problems* provokes the necessity of building new techniques and new technologies to explain these techniques. In other words, the research should focus on those *crucial questions* that can give rise to a rich and wide set of mathematical organizations. Sometimes, those *crucial questions* have an extra-mathematical origin, sometimes they have not. (p. 233)

They summarise the proposed understanding of the modelling process as

> a process of reconstruction and interconnection of praxeologies of increasing complexity (*specific → local → regional*). This process should emerge from an initial question that constitutes the rationale of the sequence of praxeologies. From this questioning, some *crucial questions* to be answered by the *community of study* should arise. (p. 233)

Within ATD this approach was further developed under the notion of study and research paths (SRP) (e.g. Bartolomé et al., 2019; Chevallard, 2006; Florensa et al., 2018). From our research perspective especially, the focus on a crucial question that guides the research or learning process is relevant here. This question is not only a question from a specific context but a question that also has the potential for questioning the content specific institutional rationales. In our analysis, in the context of complex numbers, two different rationales, each within a specific institutional mathematical discourse, occurred. We can connect those rationales with the idea of different ways of doing mathematics from Bissell and Dillon (2000) and shed more light on the meaning of the "significant difference between what a mathematician calls 'doing mathematics' and what an engineer calls 'doing mathematics'." (p. 6).

So, sometimes mathematics is different but the question appearing now from our perspective is: when is which discourse relevant? Our analyses show that both discourses show up in electrical engineering exercise solutions: The HM-discourse with its orientation towards academic mathematics is important, as well as the engineering specific mathematical ET-discourse. To be able to successfully solve exercises as our example, students have to know when which discourse is adequate and when to switch. However, this switching itself is often not made explicit in teaching. Analyses of student solutions of this exercise, like the one above, show that students difficulties could be connected to this question of switching between mathematical discourses (Hochmuth & Peters, 2021). Our analyses can therefore help to identify hurdles for students and to achieve subject-related clarifications. Lecturers can use this to explicitly address difficulties when discussing sample solutions.

In addition, the acknowledgement of mathematical practices of electrical engineers as a justified institutional discourse in itself can prevent a deficit-oriented view and open up new possibilities for teaching, task design and student guidance. In our analyses, for example, we refer to different reasons for studying complex numbers. In the Higher Mathematics course, complex numbers are important because they allow to solve every polynomial equation. In electrical engineering, the general problem of the solvability of polynomial equations is not the main interest. Here, complex numbers are particularly important because they allow to describe oscillating signals. Such reasons to study a mathematical concept are part of the logos block, thus especially part of the institutional mathematical discourses, and may not be explicitly addressed in teaching. So, students may implicitly learn one reason to study complex numbers in one course and a different reason in other courses. The different reasons may then not fit together or even contradict each other. This could convey the impression that mathematics as taught in Higher Mathematics courses is not useful for or disconnected from engineering. In (Peters, 2022) we present a concrete teaching development idea for mathematics service courses based on our research findings. In doing so, we illustrate how the difference between the two identified mathematical discourses can be used constructively in teaching development.

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Jana Peters
Leibniz University Hannover
E-mail:peters@idmp.uni-hannover.de
https://orcid.org/0000-0003-0628-7105

Reinhard Hochmuth
Leibniz University Hannover
E-mail:hochmuth@idmp.uni-hannover.de
https://orcid.org/0000-0002-4041-8706

Chapter 28
Modifying Exercises in Mathematics Service Courses for Engineers Based on Subject-Specific Analyses of Engineering Mathematical Practices

Jana Peters

Abstract This contribution presents the idea of modifying exercises from a mathematics service course on the basis of analyses (in the sense of the Anthropological Theory of the Didactic) of mathematical practices from electrical engineering. The core of this small-scale approach is to use the respective specific conceptualisation of mathematical knowledge in electrical engineering and in mathematics service courses for teaching design. In earlier work, this specifically conceptualised mathematical knowledge could be methodologically grasped with two different institutional mathematical discourses. The example shows how an existing exercise of a mathematics service course can be modified to support connections to mathematical practices from the engineering mathematics discourse. This illustrates exemplarily the importance of recognising the subject specificity of institutional mathematical practices in electrical engineering.

Keywords ATD · Mathematical practices of engineers · Modifying exercises · Connecting engineering and mathematics · Mathematical discourses

28.1 Introduction

Mathematics has at least two locations in engineering study programs: Firstly, in mathematical service courses for engineers, usually students from different engineering study programs learn basic mathematical practices, often taught by mathematicians. Secondly, mathematical practices are also taught in specific engineering courses (e.g. Signal Theory), usually by lecturers of the engineering faculty. Research on university mathematics education shows, that both, mathematics in service courses *for* engineers, and mathematics *in* engineering courses (see also

J. Peters (✉)
Leibniz Universität Hannover, Hannover, Germany
e-mail: peters@idmp.uni-hannover.de

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Pepin et al., 2021), are different and often not connected (Hochmuth, 2020; Winsløw et al., 2018, section 2.5). For example, Gueudet and Quere (2018) show differences between mathematics in service courses for engineers and mathematics in engineering courses in terms of connections within the subject matter. With a focus on trigonometry, they show that while in engineering courses multiple connections are made between contexts, representations and concepts, hardly any of those connections are found in the mathematics courses studied. Schmidt and Winsløw (2018) show that both types of courses are separated on an institutional level. In this regard, they note that "the selection of mathematical contents to be taught may be based on needs and priorities from the engineering disciplines, while the actual teaching [in mathematics service courses] is carried out according to generic standards and methods for teaching mathematics." (p. 165). In a study on the views of engineers and mathematicians on the concept of continuity, Alpers (2018) shows that this separation is also reflected in the different views of mathematicians teaching service courses and lecturers of the engineering faculty.

Most attempts to establish and support connections in mathematics service courses to mathematics in engineering courses[1] are based on the introduction of application examples from the engineering sciences in mathematical service courses (e.g. Härterich et al., 2012; Schmidt & Winsløw, 2018) or on more innovative course structures for mathematical service courses, such as project work (e.g. Alpers, 2002) and study and research paths (e.g. Barquero et al., 2008).

Both approaches can be problematic: The teaching and learning of mathematics, like any other subject, is situated in societal and institutional conditions that constitute the possibilities and restrictions of action. From the standpoint of the Anthropological Theory of the Didactic (ATD), Barquero et al. (2013) study institutional constraints and limitations within the educational system that hinder the large-scale dissemination of modelling activities.[2] In addition to a survey of literature, which shows that difficulties and barriers in this respect are a general problem, they systematically identify problems at different levels in a detailed study of one of their own projects. They categorise the constraints under the headings of monumentalism, individualism, and protectionism (Barquero et al., 2013, p. 322 ff). Furthermore, consolidated course structures (time tables, distribution of working hours and teaching, weekly assignments) are not always changeable and teachers are usually not empowered to change the traditional organisation of the mathematical content.

[1] Other approaches that attempt to establish connections to mathematics within engineering courses are not considered here.

[2] Within the ATD mathematical modelling is understood as a specific mathematical activity, more precisely as "processes of reconstruction and connection of praxeologies of increasing complexity . . . that should emerge from the questioning of the rationale of the praxeologies that are to be reconstructed and connected." (García et al., 2006, p. 243). This includes both intra-mathematical modelling as well as processes starting from extra-mathematical questions.

In addition to these large-scale approaches, the inclusion of isolated application examples from engineering in mathematics service courses represents a small-scale approach that enables changes in teaching under existing conditions. But this can be problematic in an epistemological sense:[3] Using isolated application examples could promote the image of engineering science as application of mathematics in specific extra-mathematical contexts (Barquero et al., 2011). This can promote the view of engineering per se as extra-mathematical and thereby establish a distinction between a mathematical and an engineering world[4] that somewhat contradicts the attempt to connect: Application examples can provide connections between mathematics and the engineering context. But those connections must explicitly take the specificity of mathematics *in* engineering into account. Otherwise they presuppose an understanding of the relationship of mathematics and engineering as per se disconnected. However, in everyday teaching, i.e. outside of larger teaching development projects, recourse to isolated application examples without considering the different conceptualisation of mathematics *in* engineering may appear to be the only option, especially in view of institutional constraints.

In this contribution I present a third approach that focusses on establishing and promoting connections within mathematical practices. The idea of modifying exercises from a mathematics service course according to reconstructed aspects from engineering mathematics practices is a small-scale approach that is based on the research perspective and results from an ongoing research project with Reinhard Hochmuth (Hochmuth et al., 2014; Hochmuth & Peters, 2020, 2021b; Hochmuth & Schreiber, 2015; Peters & Hochmuth, 2021). One aspect that distinguishes this approach from the approach of using application examples (the other small-scale approach) is that identified important aspects from engineering mathematical practices are brought into the mathematics service course without also introducing the engineering context.

In the following, I will build on our theoretical conceptions and the results of our research (Sect. 28.2); show by means of a detailed example (Sect. 28.3) how an existing exercise in a mathematical service course can be modified in such a way that the mathematical discourse related to service courses could be internally expanded with regard to the engineering mathematical discourse; and finally discuss (Sect. 28.4) the connections to aspects of mathematics *in* engineering, that could possibly be established and supported by this small-scale approach as well as further considerations.

[3] See our considerations in (Hochmuth & Peters, 2021a). I would like to note that such epistemological considerations are also relevant for large-scale projects. In addition, epistemological aspects are also part of institutional and societal conditions. No teaching development approach is free from possibly coming into conflict with existing conditions.

[4] Barquero et al. (2011) refer to this phenomenon as "applicationism". That the separation of the non-mathematical engineering context and the mathematical world is not adequate is also observed by Biehler et al. (2015) in the context of modelling cycles.

28.2 Theoretical Perspective and Previous Research

The idea of modifying existing exercises in mathematics service courses is based on the one hand on the general research perspective of ATD on mathematical practices and on the other hand on concrete study results on mathematical practices in Signal Theory by Hochmuth and Peters (2021b) and Peters and Hochmuth (2021). Therefore, I will first give a brief overview of ATD concepts relevant here. I will then summarise the research context and findings relevant to the exercise modification from our studies that follows in Sect. 28.3. This then not only provides the background for exercise modification but also shows how through the process of analysing materials, connections can be reconstructed that each in itself hold potential for change.

28.2.1 Concepts of the Anthropological Theory of the Didactic

ATD is a research programme to study human practices from an institutional perspective.[5] The concept of institution in ATD is based on the work by Douglas (1986). She elaborates the idea that all knowledge is dependent on (social) institutions and, conversely, that all institutions are based on shared knowledge (p. 45). Castela (2015) defines an institution *I* as "a stable social organisation that offers a framework in which some different groups of people carry out different groups of activities. These activities are subjected to a set of constraints, – rules, norms, rituals – which specifies the institutional expectations towards the individuals intending to act within the institution *I*." (p. 7). Any form of knowledge, and thus also actions in relation to this knowledge, is thus located in institutions and subject to institutional conditions.

Praxeology is the concept for the detailed subject-specific specification of institutional knowledge. In ATD praxeology is used to describe knowledge in terms of two inseparable, interconnected blocks: The praxis block consists of types of tasks (T) and relevant techniques (τ) used to solve them. The logos block consists of a two-level reasoning discourse. At the first level, technology (θ) describes, justifies and explains the techniques and at the second level, theory (Θ) organises, supports and explains the technique. A praxeology is usually represented in short as the 4 T-model $[\mathrm{T}, \tau, \theta, \Theta]$. An important aspect of technology, i.e. part of the logos block, is the raison d'être of a body of knowledge. This is the reason why it exists in an institution, what it is good for, and why it is studied. When considering a particular topic in different institutions, different praxeologies emerge: different types of tasks

[5] Fundamental elaborations on ATD can be found, for example, in Bosch and Gascón (2014) and Chevallard (1992, 2019); in addition, Bosch et al. (2011) and Bosch et al. (2019) provide insight into typical studies in this research programme.

are relevant, different solution techniques are adequate, different raisons d'être exists and different reasoning discourses are acceptable and constitutive. This is referred to as the institutional dependence of knowledge.

While praxeologies allow mathematical knowledge to be grasped rather statically in its institutional conception, the concept of (didactic) transposition offers the possibility to investigate and describe dynamic aspects of the production, development, change and dissemination of knowledge between institutions (e.g. Bosch & Gascón, 2014). The basic model of the didactic transposition process is based on a distinction between three relevant institutions: First, scholarly knowledge is produced by experts in universities or research institutes. The knowledge to be taught is determined by official curricula. Finally, this becomes the taught knowledge that is taught in courses. The transition from scholarly knowledge to knowledge to be taught is also referred to as external didactic transposition, the transition to taught knowledge as internal didactic transposition. Schmidt and Winsløw (2018) refer to these concepts and show in particular that the specific institutional conditions of engineering thus enter into the external didactic transposition, but not into the internal one. They call this "the parallel model for didactic transposition in engineering education" (p. 165).

28.2.2 Mathematical Practices in Signal Theory

Schmidt and Winsløw (2018) focus on mathematical knowledge for engineering students that is provided through mathematics service courses. In our own studies, though, we point out that mathematical practices especially in higher-level engineering courses, such as Signal Theory, are rather a mixture of practices of mathematics from service courses, mathematics as developed and used in basic electrical engineering courses, and specific signal theory content (Hochmuth & Peters, 2020, 2021b; Peters & Hochmuth, 2021). The various combinations of dark- and light grey techniques and technologies in Fig. 28.3 are an example of such a mixture. Moreover, our analyses show that mathematical practices in these courses cannot be understood solely as the application of mathematical concepts taught in mathematics service courses.

To grasp this mixture of mathematical practices in Signal Theory, we introduced an extended praxeological 4 T-model and two corresponding mathematical discourses[6] (Peters & Hochmuth, 2021). The starting point for the idea of exercise modification presented in Sect. 28.3 are then analyses of an exercise with a lecturer's sample solution in the context of amplitude modulation and associated student solutions (Hochmuth & Peters, 2021b). The exercise and the lecturer's sample solution are presented in the Appendix.

[6] Our understanding of discourse, i.e. its meaning and analytical status, is clarified and linked to Weber's (1904) concept of ideal types in (Hochmuth & Peters, 2021b).

The amplitude modulation context will provide us with interesting insights into the role of complex numbers in electrical engineering, which will eventually be used in the exercise modification. To this end, I will first introduce the context and contrast the role of complex numbers in electrical engineering with the role of complex numbers in the mathematics service course.[7] Those different roles are grasp within our work as different mathematical discourses on complex numbers. The analysis of the roles of complex numbers in electrical engineering and in mathematics service courses is based on standard literature, lecture notes, and students' notes for consolidated standard courses which are held at the University of Kassel. Both, the mathematics service course and the introductory course on electrical engineering are courses that students attend before attending the course on signal theory. The described mathematics service course is also the setting for the exercise modification in Sect. 28.3. Secondly I will summarise the results of the analysis of the lecturer's sample solution and address some of the results of the analyses of the student solutions.

28.2.2.1 Amplitude Modulation and the Role of Complex Numbers in Electrical Engineering and in Mathematics Service Courses

Amplitude modulation (AM) is a central topic in signal theory. With amplitude modulation, several message signals (e.g. for different radio stations) with different carrier frequencies can be transmitted (e.g. via antenna) and received without crosstalk between signals at the receiver (e.g. radio set) depending on the chosen carrier frequency. The principle of amplitude modulation is illustrated in Fig. 28.1: The amplitude of a high-frequency carrier signal $\cos(2\pi f_0 t)$ (left) is varied in relation to that of the low-frequency message signal $s(t) = \cos(\Omega t)$ (centre). The AM signal can then be represented as $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ (right).

In Fig. 28.1 amplitude modulation is visualised using waveforms of the corresponding signals. In the exercise we have analysed, an AM signal is to be represented as a rotating phasor in the complex plane. This change to phasor-representation (see Fig. 28.6 in the appendix) makes it possible to study properties of AM that are not apparent in the waveform representation. The connection between waveform, phasor-representation and algebraic description with complex numbers of a periodic signal is also a basic topic of introductory courses on electrical engineering. Albach (2011), a standard textbook for introductory courses on electrical engineering, first introduces phasors with the purpose to graphically describe time-dependent sinusoidal functions. The relationship between phasor- and waveform-representation is shown in Fig. 28.2, left side: The phasor with length $\hat{\mathrm{i}}$,

[7] The subject-specific context is also relevant in other publications within the research project on mathematical practices in Signal Theory. These or similar presentations of amplitude modulation and connections to complex numbers can therefore also be found in (Hochmuth & Peters, 2021a, b; Peters & Hochmuth, 2021).

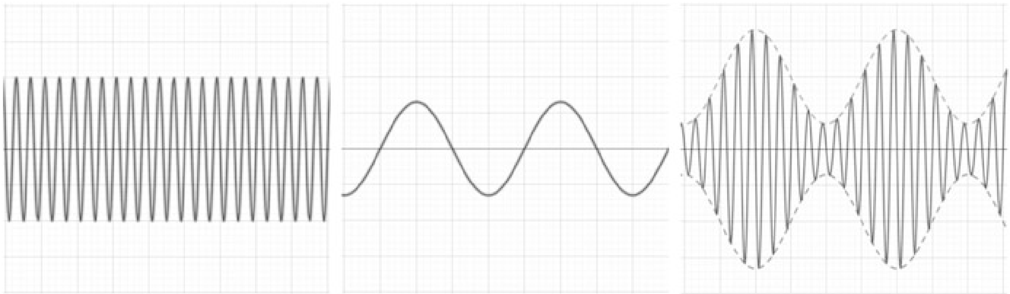

Fig. 28.1 Visualisation of amplitude modulation: high-frequency carrier signal (left), low-frequency message signal (centre), and AM signal (right), created with GeoGebra

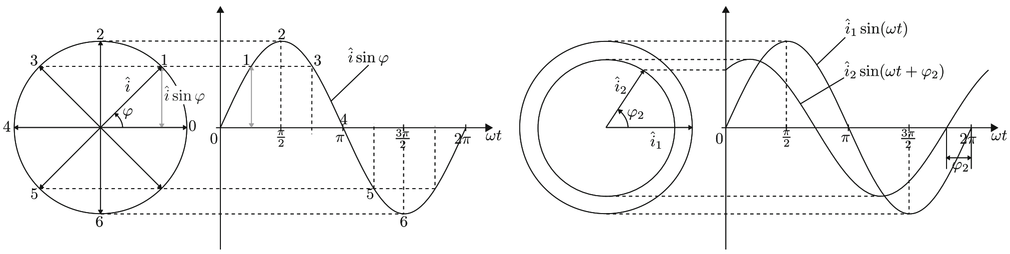

Fig. 28.2 Relationship between phasor and time-dependent functions. (Redrawn similar to Albach, 2011, p. 32)

rotates constantly counterclockwise with angular velocity ω. The projection onto the vertical axis provides the waveform of $\widehat{i}\sin\varphi$. At the right side of Fig. 28.2 two sinusoidal currents with different amplitudes $\widehat{i}_1$ and $\widehat{i}_2$ and phase difference φ_2 are shown.

As both phasors (Fig. 28.2, right side) rotate with the same angular velocity ω, the rotation of the phasors can be neglected. When analysing electrical components, the amplitude ratio of the input signal to the output signal and the phase shift between input and output caused by the components are of primary interest. Therefore, phasors are important graphic tools for interpretation and analysis of electrical engineering processes.[8] Current- and voltage ratios in electrical networks can be displayed and analysed graphically in static phasor diagrams (or Argand diagrams). For the purpose of an algebraic description of phasors, the plane in which phasors are drawn, is considered as the complex plane. The phasor can now be understood as a complex quantity that symbolically represents the time-dependent periodic signal. The compatibility of the geometric rules for manipulating phasors and the calculation rules of complex numbers is justified via physical relations (e.g. Kirchhoff's laws).

[8] In this contribution I focus on the role of complex numbers in the electrical engineering conceptualisation of phasor. The introduction of complex numbers in electrical engineering had a much bigger significance (e.g. Bissell, 2004; Bissell & Dillon, 2000, 2012).

Furthermore, for a sinusoidal quantity the following holds: $A\cos(\omega t + \varphi) = \Re(Ae^{j(\omega t + \varphi)}) = \Re(Ae^{j\omega t}e^{j\varphi})$, where A is the amplitude and j denotes the complex unit in electrical engineering. The algebraic representation of the rotating phasor is $Ae^{j\omega t}e^{j\varphi}$. In circuits where all quantities change with the same angular velocity, the time dependent factor ejωt can be factored out, i.e. the rotation of the phasor can be neglected. Here the Euler representation is important for separating rotational and constant components.

In the case of amplitude modulation, rotational aspects can no longer be neglected because the carrier signal and the message signal have different angular velocities. The algebraic representation of the phasor for amplitude modulation is given in line (3) of the sample solution in the appendix. Here, also the Euler representation is important for separating the different frequencies of carrier- and message signal.

In the mathematics service course, complex numbers are considered in the first part of the course in the context of Linear Algebra (Strampp, 2012). Their introduction is motivated by the solvability of the equation $x^2 + 1 = 0$. For this purpose, real numbers are extended by a number i with the property $i^2 = -1$. This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations. Calculation rules for complex numbers are derived without introducing and proving formal concepts, but by stating that all rules which are relevant for calculating with real numbers should continue to be applicable (p. 59). Also, it is pointed out that various terms are an extension of already known concepts from real numbers. For example, the complex exponential function $e^{\phi i}$, which is introduced to serve as a pointwise convenient abbreviation for $\cos(\phi) + \sin(\phi)i$ (p. 74). Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, it is subject to an orientation towards the inner-mathematical, generalisation-oriented formal rational of academic mathematics. In addition to the algebraic view on complex numbers, the chapter also contains a geometric view: An analogy to vectors is established, but the vector concept is also distinguished from complex numbers: "We speak of phasors[9] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This multiplication extends the multiplication of real numbers." (p. 60, translated by author). Phasors provide an illustrative justification for the formal conceptualisations of complex numbers. As Felix Klein (1967) notes, this is a view of complex numbers that was already held by Gauss. Klein states that Gauss "justifies the legitimacy of operating with complex numbers by the fact that one can give them and the operations with them that illustrative geometric interpretation. . ." (p. 64, translated by author). This meaning

[9]We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrical engineering aspects play no role in the course and Strampp (2012) does not refer to them either. In German the term Zeiger is used both in electrical engineering and in mathematics service courses, but with different meanings (reference to electrical engineering concepts vs. geometrical object with no further references). By using the term Zeiger Strampp (2012) can thus establish a connection to the electrical engineering courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different course-contexts is in jeopardy of being lost through translation.

of complex numbers in the mathematics service course differs from the meaning of complex numbers in electrical engineering (see also the work of Steinmetz (1893) who first introduced complex numbers to electrical engineering). Furthermore, the compatibility of the rules for graphically manipulating phasors and the calculation rules of complex numbers are justified via physical laws.

28.2.2.2 ATD Analyses of the Lecturer's Sample Solution and Student Solutions

From the institutional standpoint of the ATD, courses of a study program can be understood as institutions. In the following we will differentiate two institutions: an institution HM associated with the mathematics service course and an institution ET associated with electrical engineering.[10] According to the institutional dependence of knowledge, the different institutions give rise to different conceptualisations (praxeologies) of complex numbers. The two different characterisations of complex numbers in the previous section can be understood as descriptions of institutional aspects that shape the logos blocks of the respective institutional praxeologies and thus, due to the dialectic of praxis and logos, also the practical part, i.e. they can each be understood as part of two associated institutional mathematical discourses: one associated with the institution HM, i.e. the HM-discourse, and one associated with the institution ET, i.e. the ET-discourse. An important difference between the two institutional discourses is the difference between the respective raisons d'être: In electrical engineering the raison d'être for complex numbers is to describe periodic signals, together with strong connections to phasors and waveforms. In the mathematics service course the raison d'être for complex numbers is to serve for generalising concepts from real numbers, to solve equations, and as formal objects of calculation. There is also a connection to phasors but the phasor concept is different and usually serves to visualise properties of complex numbers.[11]

In our research on mathematical practices within a signal theory course, we used the notion of institutional discourses to capture the mixture of mathematical practices that occurred in a praxeological analysis of the lecturer sample solution of an exercise in the context of amplitude modulation. We identified the two mathematical discourses and associated praxeological elements to the HM-discourse (τ_{HM} and θ_{HM}) or the ET-discourse (τ_{ET} and θ_{ET}) depending on the respective institutional orientation within the solution steps.

[10] The acronyms HM and ET were introduced by Peters and Hochmuth (2021) to denote the two relevant contexts of "Höhere Mathematik" (HM, mathematics service course) and "Elektrotechnik" (ET, electrical engineering) and the associated discourses. HM and ET are the standard German acronyms for these contexts.

[11] Nevertheless, this visualisation aspect is important because it contributes to the logos block, i.e. the reasoning discourse, e.g. with regard to abstract calculation rules.

In addition, our approach refers to the work of Artaud (2020) that allows to connect the two mathematical discourses with two different didactic transposition processes: Artaud has considered two different ways how mathematical knowledge arise in fields such as electrical engineering: (1) Either the mathematical knowledge required in electrical engineering institutions is already elaborated and developed in other institutions, for example academic mathematical research institutes. This knowledge then enters the electrical engineering institution via didactic transposition processes, so to speak externally, and serves the mathematical education of future electrical engineers. Here, one can localise the HM-discourse and the idea of mathematics *for* engineering. Through the didactic transposition process, however, the academic mathematical knowledge is changed and adapted especially to the needs of electrical engineering institutions for the education of future engineers, but maintains the orientation towards academic mathematics. Schmidt and Winsløw (2018) also note this and it is to this aspect that their parallel model for didactic transposition refers. (2) Or, the relevant mathematical knowledge has been developed in the course of a historical process by actors specialising in electrical engineering. In this case, the mathematical knowledge entered the electrical engineering institution a long time ago via an institutional transposition process to be put to use. Bissell's (2004) investigation of the introduction of complex quantities in electrical engineering, driven by Steinmetz (1893) among others, that allows to manipulate graphical and pictorial representations instead of complicated mathematical expressions and also led to systems thinking and black box analysis (p. 309), give a glimpse on such an institutional transposition process. In the course of time this knowledge was used in electrical engineering and was didactically transformed in order to be taught. This didactic transposition process is endogenous. Here the ET-discourse and the idea of mathematics *in* engineering can be situated.

A graphical representation of our analysis result of the lecturer's sample solution is shown in Fig. 28.3, see also the detailed analysis in (Hochmuth & Peters, 2021b).

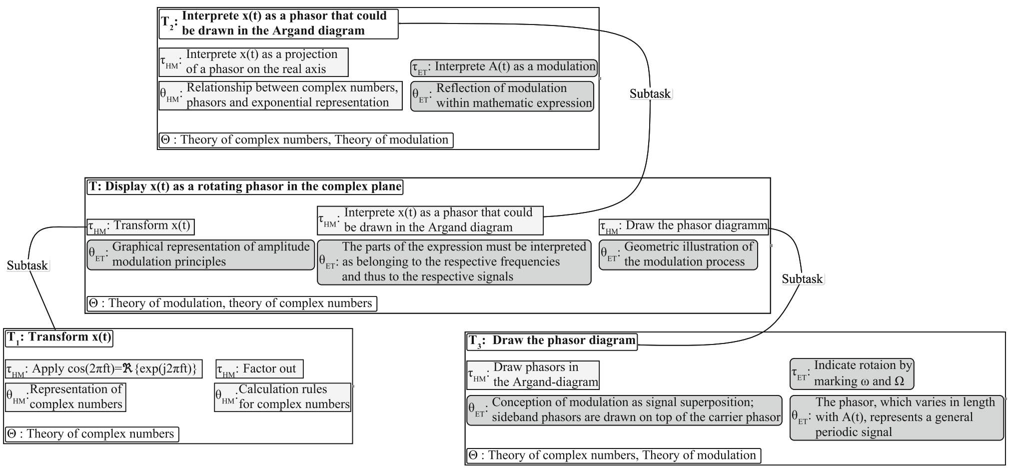

Fig. 28.3 Graphical representation of the ATD Analysis (Hochmuth & Peters, 2021b)

The exercise is solved in three steps (see Appendix): Transforming mathematical expressions, interpreting the mathematical expression to draw a diagram, and drawing the phasor diagram. The main part of the exercise, to display $x(t)$ as a rotating phasor in the complex plane, is a task (T) in the sense of the ATD. We then assigned techniques and technologies (τ_{HM}, θ_{HM} in light grey, τ_{ET}, θ_{ET} in dark grey) to each of the three solution steps. This is shown in the bold framed rectangle in Fig. 28.3: Without focussing on the detailed analysis one can see, that for each solution step HM-techniques are accompanied with ET-technologies. We characterised this as "an embedding of HM-techniques in the ET-discourse" (Hochmuth & Peters, 2021b). We further refined the analysis in a second analysis step, in which the three techniques assigned to the three solution steps are considered as subtasks T_1 to T_3, see the corresponding light framed rectangles in Fig. 28.3: In this step we were able to further enlight the nature of the respective embeddings.

Although we will not go into the details of the analysis here, it is clear from Fig. 28.3, that the two mathematical discourses, the ET-discourse (dark grey) and the HM-discourse (light grey), interact in various ways. The view that mathematics is simply applied in electrical engineering is not adequate, practices in Signal Theory contain aspects of both mathematical discourses. Dealing adequately with both discourses is therefore a requirement for engineering students.

In Hochmuth and Peters (2021b) we used this result as a reference for analyses of students solutions. In the following I will show clips from two student solutions[12] in which the adequate switching between the discourses was not present and the correct phasor diagram was not produced (Fig. 28.4). Two decisive steps in the course of the sample solution are firstly the calculation step from line (2), in which $x(t)$ could be interpreted as a real part of three rotating phasors drawn in the origin, to line (3), in which $x(t)$ can be interpreted as a rotating carrier phasor with time-dependent amplitude $A(t)$. And secondly the change of representation from the algebraic expression in line (3) to the phasor diagram.

The student solution on the left side of Fig. 28.4 does not contain the step from line (2) to line (3) from the sample solution. Instead this student reproduces the HM-discourse by drawing a diagram similar to diagrams from the mathematics service course where the Argand diagram and the unit circle are used to illustrate properties of complex numbers. This student solution also contains further information on properties of complex numbers like the complex conjugate $e^{-j2\pi ft}$ that are not relevant for the solution of the exercise. The three terms from line (2) are drawn as three separate phasors. Important aspects of amplitude modulation and references to the ET-discourse are not present.

The student solution on the right side of Fig. 28.4 mainly contains ET-discourse aspects but significantly deviates from the sample solution. References are made to previous topics in the lecture (Fourier transform and low pass filter), but these are not

[12] In order to protect the privacy of the students, the student solutions are translated from German and rewritten by the author without correction marks. In the detailed analyses in (Hochmuth & Peters, 2021b) those two student solutions are labelled I2 and I3.

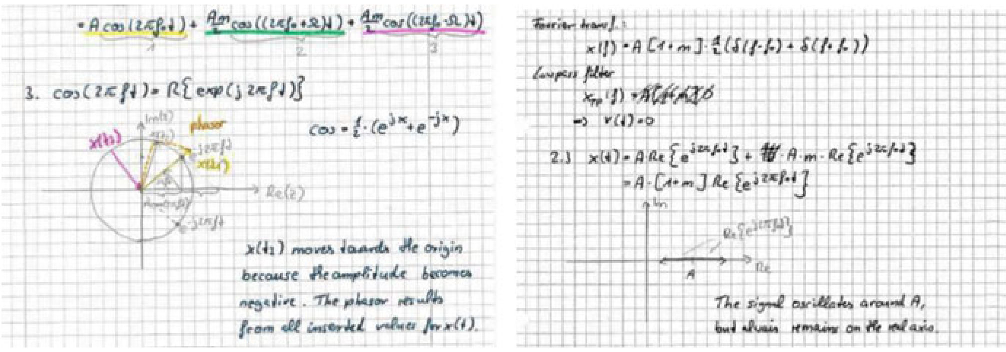

Fig. 28.4 Left: student solution solely within the HM-discourse. Right: student solution mainly within the ET-discourse (Hochmuth & Peters, 2021b)

goal-oriented and appropriate. Although this solution maintains an orientation towards the ET-discourse and some rotational aspects are present, the connection between the mathematical concepts and their graphic representation in terms of modulation principles is missing.

28.3 From Analyses of Engineering Mathematical Practices to Modifying Exercises in Mathematics Service Courses

The summary of the analyses in the preceding section showed that adequately working with two different mathematical discourses is a requirement for engineering students. Furthermore, some difficulties of students, that were not able to flexibly switch between the two discourses are shown. Subject-specific aspects, that also seem to be at the core of those difficulties are: dynamic aspects cannot be neglected; more than one rotating phasor is relevant and phasors have to be drawn in a specific way; a complex algebraic expression must be represented graphically; time-dependent exponential function.

Those analyses results now shall serve to inform an exercise modification in the mathematics service course. The mathematics service course under consideration is a two-semester consolidated course that is regularly held at the university of Kassel. Material from this course consists of student lecture notes, standard literature (Strampp, 2012), and exercise sheets with sample solutions from 2013. It consists of two lectures, one exercise session and one special exercise session where selected, important topics are presented, per week. Students are expected to individually work on weekly exercise sheets, that are eventually handed in and graded. Application examples are not present in the material. With focus on the chapter of complex numbers we characterised the mathematical discourse, i.e. the HM-discourse, as orientated towards the inner-mathematical, generalisation-oriented formal rational of academic mathematics. The raison d'être for complex numbers is that they allow for generalisation, they are useful to solve equations, and they are formal objects of calculation. We also noted that in the chapter of complex numbers no connections to phasor representations other than for illustrative reasons are made. Furthermore, the Euler equation, with which the internal relationship of the exponential function, sine and cosine could be recognised, only serves a useful shortcut to simplify calculations. Important connections between complex numbers and trigonometric functions, that go beyond this convenient calculation tool, are not present.

The basic idea behind the proposal for exercise modification is now to modify an existing exercise from the mathematics service course such that the HM-discourse on complex numbers could be enriched or expanded towards the ET-discourse in order to establish connections to mathematical practices that are relevant in electrical engineering. However, neither the course organisation nor the general orientation towards academic mathematics is to be changed.

To demonstrate this idea with an example, it is first noted that the exercises in the chapter of complex numbers are mainly standard exercises: change between Euler- and Cartesian representation, training of basic calculations, and determining the roots of polynomials. The following exercise is an exception.[13] It is the only exercise in which a time-dependent exponential function occurs:

Which curves are described in the complex plane by

$$ae^{-ti} + be^{ti},\ a, b \in \mathbb{R} \text{ constant},\ t \in \mathbb{R}?$$

This exercise was marked “too difficult” in the student’s notes.

The sample solution from the student’s notes is:

$$\begin{aligned} ae^{-ti} + be^{ti} &= a(\cos(-t) + \sin(-t)i) + b(\cos(t) + \sin(t)i) \\ &= a(\cos(t) - \sin(t)i) + b(\cos(t) + \sin(t)i) \\ &= (a+b)\cos(t) + (-a+b)\sin(t)i \\ &= x + yi \\ &= \left(\frac{x}{a+b}\right)^2 + \left(\frac{y}{-a+b}\right)^2 = 1 \end{aligned}$$

This then is recognised as the equation of the ellipse, that was introduced in the preceding special exercise session. The cases $a + b = 0$ and $-\mathrm{a} + b = 0$ are treated separately, in which the ellipse becomes a straight line: for $a + b = 0$ we get $2b \sin(t)i$, a straight line on the imaginary axis between $-2bi$ and $2bi$. For $-a + b = 0$ we get $2a \cos(t)$, a straight line on the real axis between $-2a$ and $2a$.

The question of the exercise already points to the raison d’être of the ET-discourse (complex numbers are useful to describe periodic signals). But in the sample solution, only elementary relations such as Euler’s equation and Pythagorean identity are used to give the expression a form that could be recognised from

[13] Exercise and sample solution are translated from German by the author.

previous lectures. Why $ae^{-ti} + be^{ti}$ describes an ellipse, or why the special cases generate straight lines is not explained, periodic or rotational aspects are not present.

This changes when software like GeoGebra (Hohenwarter et al., 2018) is used. With digital tools like GeoGebra, dynamic aspects can be visualised and explored and an otherwise too difficult exercise becomes accessible. Since students may be inexperienced in using GeoGebra, I will present the modified exercise with a step-by-step construction of the ellipse with phasors below:[14]

Which curves are described in the complex plane by

$$C(t) = ae^{-ti} + be^{ti},\ a, b \in \mathbb{R} \text{ constant},\ t \in \mathbb{R}?$$

- Plot the corresponding locus curve in GeoGebra by following the steps below:
  - Write C = a e^(−i t) + b e^(i t) in the Input and confirm.
  - Create sliders for a, b, and t.
  - Write Locus in the Input and chose
    Locus (<Point Creating Locus Line>, <Slider>).
  - Replace <Point Creating Locus Line> with C and <Slider> with t.

  Try different values for a and b and describe the curves. Use the slider for t to explore how the point C moves on the curve. What happens for $a = 0$, $b = 0$, $a = b$ and $a = -b$?
- In the next step we construct phasors for ae^{-ti} and be^{ti}:
  - Write P= a e^(−i t) in the Input and confirm.
  - Write u = Vector in the Input, chose Vector (<Point>) and replace <Point> with P.
  - Write v = Vector in the Input, chose Vector (<Start Point>, <End Point>). and replace <Start Point> with P and <End Point> with C.

  We have now represented the point C as the sum of the two phasors. Again, try different values for a and b and explore how the rotating phasors construct the curve. What is the consequence of the different signs in the exponents?

By using software like GeoGebra, dynamic aspects can be visualised (see also Fig. 28.5). In addition, connections can be made to the phasor representation, which now goes beyond only serving to visualise properties. In this exercise, it can be explored how the combination of two rotating phasors describes a closed curve and how the algebraic representation of a complex number is connected to its phasor-representation. The cases $a = 0$ or $b = 0$ result in circles with one phasor

[14] For more experienced students, this exercise could be formulated in less detail:

(a) Create the locus curve in GeoGebra including sliders for a, b and t. How does the locus curve change depending on values for a and b?
(b) Represent the components of the equation ae^{-ti} and be^{ti} as phasors respectively (use the GeoGebra Vector function) and represent $C(t)$ as the result of adding the two phasors so that point C moves on the locus as you vary t. What is the consequence of the different signs in the exponents?

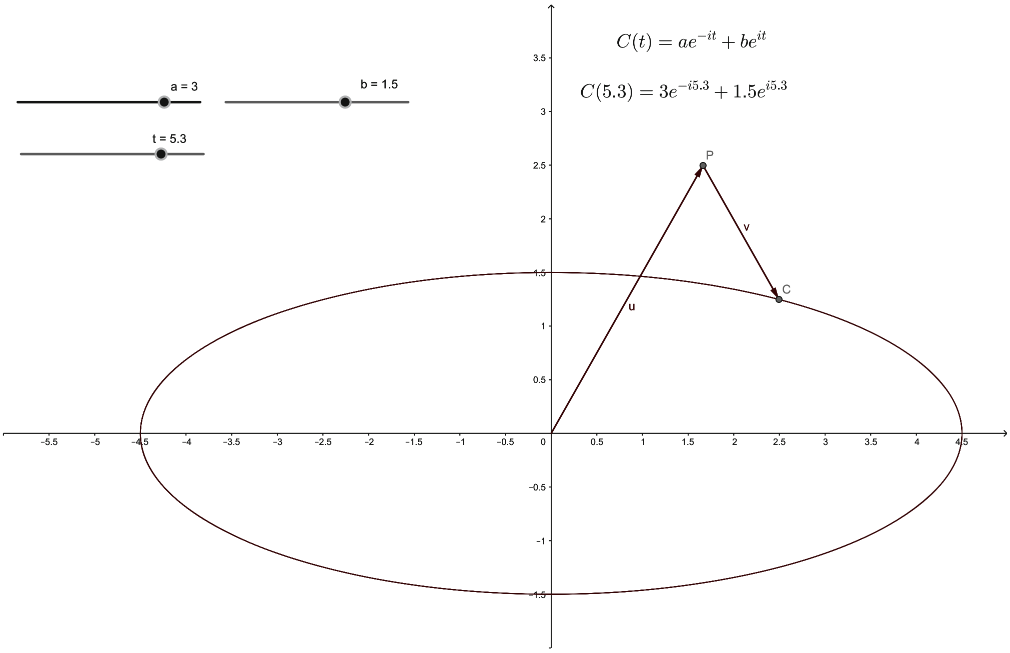

Fig. 28.5 Ellipse with corresponding phasors, created with GeoGebra

each. This is familiar from the lecture, as properties of complex numbers, in particular the introduction of Euler's formula, are illustrated with the unit circle. So, connections to previous aspects of the lecture are established. Furthermore, the raison d'être for complex numbers in the HM-discourse can be extended by the aspect that complex numbers are suitable to describe periodic functions or closed curves.

This modified exercise also fits in the course structure, e.g. this exercise can be part of a weekly exercise sheet. The exercise does not violate the orientation towards the rational of academic mathematics of the HM-discourse. It is also possible to embed the task in a mathematical-historical context: The method of constructing the ellipse by two rotating phasors is very similar to historical conic section drawers (e.g. Van Maanen, 1992).

28.4 Discussion

The aim of this contribution is to show, and illustrate with an example, how subject-specific analyses of mathematical practices from signal theory can serve to modify exercises from mathematics service courses, even within restricted institutional conditions. The focus is to support connections between mathematical practices from service courses and from electrical engineering within the mathematics. The approach presented here thus represents, in addition to large-scale development projects and the inclusion of application examples, a further possibility for changes in teaching.

The ATD concept of the institutional dependence of knowledge is at the core of this approach: The same mathematic topic, e.g. complex numbers, is conceptualised differently in different institutions, the subject-specific rationales and meanings, the raisons d'être, overall, the mathematical discourses are different. This is associated with a specific research stance: Within this approach, mathematical practices of engineers are acknowledged as institutional mathematical practices in their own right. This stance is not compatible with the introduction of application examples in the sense of applicationism, i.e. without taking the engineering specific conceptualisations of mathematical knowledge into account. From this stance, it is possible to reconstruct engineering-specific mathematical discourse aspects like, besides other, the engineering-specific raison d'être of a mathematical concept. These aspects are mathematical aspects, and not aspects from an extra-mathematical engineering context, that have been endogenously developed and modified over time within engineering institutions (cf. Artaud, 2020). Therefore, they can be included in the mathematical discourse of mathematics service courses, that are often oriented towards academic mathematics. At this point changes are necessary: The ET-context should be removed but the discourse aspect kept. The analysis of the AM exercise shows that the orientation towards academic mathematics is important in engineering courses such as Signal Theory. This is also one of the reasons for maintaining this orientation for the HM-discourse in this approach for exercise modification. In the example, the raison d'être for complex numbers in the ET-discourse was, among other things, describing periodic signals. In the HM-discourse, this can be changed to: describing periodic functions or curves. If the students encounter complex numbers in engineering the mathematical discourse on complex numbers (i.e. the raison d'être) in the ET-discourse, to describe periodic signals, is not entirely different from the extended mathematical HM-discourse. Therefore this approach can contribute to reduce the metaphorical distance between the mathematical discourse of the mathematics service course and the engineering mathematical discourse inner-mathematically. This concerns both the internal didactic transposition (cf. Schmidt & Winsløw, 2018) and the establishment of connections within the subject matter (cf. Gueudet & Quere, 2018), e.g. connections of HM-techniques with ET-technologies. Many of the differences in the views of mathematicians and engineers addressed by Alpers (2018) can also be understood as aspects of a respective institutional mathematical discourse. From the perspective presented here, however, it is not enough for a mathematician to read engineering books and talk to engineers, for example. To really take the engineering view seriously, it is necessary to take it seriously in its own institutional conception. Under this precondition, however, discussions with engineers, textbooks by engineers, but also historical and philosophical studies are useful in order to characterise mathematical discourses specific to engineering.

Of course, this small-scale approach presented here is not free from problems and of coming into conflict with societal and institutional conditions either. I have shown how, in the process of analysing institutional mathematical practices, potential for change in teaching can be identified within existing conditions. This is double-edged, as it can also support the position of not needing to change social and institutional conditions, and thus act as a counter-argument for approaches that aim precisely to such changes. On the other hand, while acknowledging this criticism, it can be stated that the ATD-specific research stance is also relevant for lecturers and entails that retreating to the position that changes in teaching are entirely possible without conflict with and change in social and institutional conditions is short-sighted. The approach presented here presupposes lecturers to question their own institutional standpoint, their own mathematical discourse. But this stance does not solve contradictions and possible conflicts. There is no clear solution for societal and institutional conflicts. Societal struggles cannot be solved on the basis of ATD[15] analyses. However, such an analysis, as presented here, offers a differentiated view of what is possible at the exercise level and what is not.

ATD focuses specifically on institutional and subject-specific conditions. In order to take social struggles and contradictions into account, a research perspective that addresses a more general level is necessary. A promising approach in this direction is the subject-scientific approach from the field of critical psychology (e.g. Holzkamp, 1985; Schraube & Osterkamp, 2013). Various studies have already shown that this approach is compatible with ATD (Hochmuth, 2018; Hochmuth & Schreiber, 2015; Ruge et al., 2019).

Consideration of the relationship between lecturers of mathematics service courses and teaching approaches developed in research brings the focus to the sustainability of teaching development research and therefore also to professional development. Ruge and Peters (2021) develop an understanding of professional growth based on the subject-scientific approach which, besides other things, adopts a view of professional development that goes beyond deriving practical and applicable tools from research. In this sense, the approach to exercise modification presented here also does not provide a directly applicable tool, but shows how there is potential for teaching development within the process of analysing respective institutional mathematical discourses and reflecting the institutional situatedness of mathematical practices.

Appendix: Exercise with Lecturer Sample Solution

The exercise under consideration is structured in three items:

1. A message signal $s(t) = \cos(\Omega t)$ has to be amplitude modulated. The result is $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$
2. The result of item 1. Has to be written as the sum of three harmonics. The result is $x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$
3. The result of item 2. Has then to be displayed graphically in the complex plane as a rotating phasor with varying amplitude.

[15] Or any other theoretical approach.

The ATD analysis focusses item 3. of the exercise. The exact problem definition of item 3 is (my translation):

> Graphically display $x(t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi ft)\Re\{\exp(j2\pi ft)\}$ and the result under item 2.

Lecturer Sample Solution
One first writes

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \quad (1)$$

$$= A\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\{\exp(j(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\{\exp(j(2\pi f_0 t - \Omega t))\} \quad (2)$$

$$= \{\exp(j2\pi f_0 t)\underbrace{[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)]\}}_{A(t)} \quad (3)$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude $A(t)$, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Fig. 28.6.

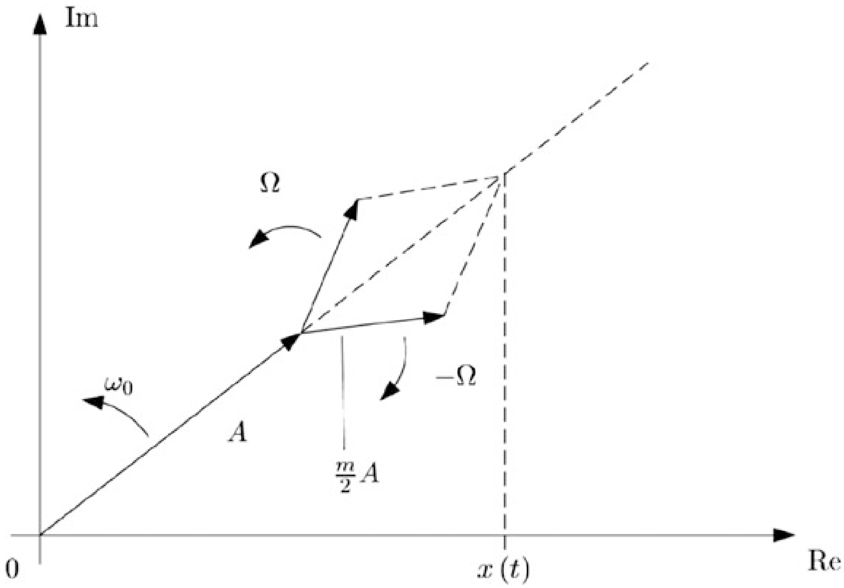

Fig. 28.6 Representation of $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0 = 2\pi f_0$

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Curriculum Vitae

- June 2023 -

Jana Peters

Seumestraße 12
30161 Hannover
Germany

E-Mail: peters@idmp.uni-hannover.de

https://orcid.org/0000-0003-0628-7105

Birthday & place:	December 14, 1981 in Neubrandenburg
Nationality:	German
Marital Status:	Married, 2 Children (* 2010, * 2016)

Professional Experience

11/2021 – 07/2024	Research associate, University of Kassel Institute of Mathematics, Didactics of Mathematics
10/2016 – 04/2023	Research associate, Leibniz University Hannover Institute of Mathematics and Physics Education
10/2014 – 09/2016	Research assistant, Leibniz University Hannover Institute of Mathematics and Physics Education
05/2009 – 05/2013	Research associate, PhD student, University Hamburg Faculty of Business, Economics and Social Sciences, in cooperation with Carl Friedrich von Weizsäcker-Centre for Science and Peace Research (ZNF) at the University of Hamburg (12/2010 – 02/2012 parental leave)
04/2006 – 04/2009	Student assistant at the Carl Friedrich von Weizsäcker-Centre for Science and Peace Research (ZNF) at the University of Hamburg (including participation in the preparation of a peer-reviewed publication (Kalinowski et al., 2010) and presentations at DPG- and EGU-conferences (Peters & Kalinowski, 2008a, 2008b))

Education

since 10/2021	PhD Program in Mathematics (Mathematics Education)
03/2021	**Master of Education**
02/2015 – 03/2021	Master's Teacher Training Programm for Grammar Schools (M.Ed.) Mathematics/Physics, Leibniz University Hannover
02/2015	**Bachelor of Science**

04/2013 – 02/2015	Interdisciplinary Bachelor's Degree Programme, Mathematics/Physics Leibniz University Hannover
04/2009	**Diploma in Mathematics**
2005 – 2009	Diploma degree study program for Mathematics, University of Hamburg
2001 – 2005	Diploma degree study program for Mathematics, University of Siegen
06/2001	**Abitur**
1998 – 2001	Berufskolleg für Technik mit gymnasialer Oberstufe (Technical Gymnasium), Siegen
1992 – 1998	Realschule Freudenberg
1988 – 1992	7. Polytechnische Oberschule, Neubrandenburg

Publications

Gómez-Chacón, I. M., Hochmuth, R., & Peters, J. (2022). Promoting Inquiry in Mathematics: Professional Development of University Lecturers. In M. Trigueros, B. Barquero, R. Hochmuth & J. Peters (Hrsg.), *INDRUM 2022 PROCEEDGINGS* (S. 498–507). University of Hannover; INDRUM.

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