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$\mathcal{H}$-matrix approximability of inverses of FEM matrices for the time-harmonic Maxwell equations

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Abstract
The inverse of the stiffness matrix of the time-harmonic Maxwell equation with perfectly conducting boundary conditions is approximated in the blockwise low-rank format of $\mathcal{H}$-matrices. Under a technical assumption on the mesh, we prove that root exponential convergence in the block rank can be achieved, if the block structure conforms to a standard admissibility criterion.

Keywords Maxwell equations · Hierarchical matrices · Finite element method · Helmholtz decompositions

1 Introduction

A backbone of computational electromagnetics is the solution of the time-harmonic Maxwell equations. Since the discovery of Nédélec's edge elements (and their higher order generalizations) finite element methods (FEMs) have become an important discretization technique for these equations with an established convergence theory [35]. While the resulting linear system is sparse, a direct solver cannot

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achieve linear complexity as one has to expect, already for the case of quasi-uniform meshes with problem size N, a complexity $O(N^{4/3})$ for the memory requirement and $O(N^2)$ for the solution time of a multifrontal solver [33]. Iterative solvers such as multigrid or preconditioned Schwarz methods can lead to optimal (or near optimal) complexity for the numerical solution of the time-harmonic Maxwell equations, at least in the low-frequency regime [2, 21, 27]. For the design and analysis of these methods, a key insight was the appropriate treatment of the gradient part of the Nédélec space and thus Helmholtz decompositions play an important role. The analysis of fast solvers for Maxwell's equations, however, is less developed in areas such as high-frequency applications.

An alternative to classical direct solvers and iterative solvers came with the introduction of $\mathcal{H}$-matrices in [25]. This class of matrices consists of blockwise low-rank matrices of rank r, where the blocks are organized in a tree $\mathbb{T}_{\mathcal{I}}$ so that the memory requirement is typically $O(rN\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}}))$, where N is the problem size. This format comes with an (approximate) arithmetic that allows for addition, multiplication, inversion, and LU-factorization in logarithmic-linear complexity. Therefore, computing an (approximate) inverse in the $\mathcal{H}$-format can be considered a serious alternative to a direct solver or it can be used as a "black box" preconditioner in iterative solvers. We refer to the works [10, 20, 23, 26] for a more detailed discussion of analytical and algorithmic aspects of $\mathcal{H}$-matrices.

A basic question in connection with the $\mathcal{H}$-matrix arithmetic is whether matrices and their inverses or factors in an LU-factorization can be represented well in the chosen format. While stiffness matrices arising from differential operators are sparse and are thus easily represented exactly in the standard $\mathcal{H}$-matrix formats, the situation is more involved for the inverse. A first proof that inverses can be represented in the $\mathcal{H}$-matrix format harks back to [5, 6] for scalar elliptic problems and [9] for the $\mathrm{curl}\mu^{-1}\mathrm{curl}$ operator; a generalization to pseudodifferential operators is done in [14]. These proofs rely on locally separable approximations of the continuous Green's function and a final projection of these approximations into discrete spaces. The final projection step limits, at least formally, the achievable accuracy of the matrix approximation by the discretization error. To circumvent this, a fully discrete approach was taken for FEM discretizations of various scalar elliptic operators in [1, 16] to show that the inverse of the FEM-matrix can be approximated at a root exponential rate in the block rank. The works [17, 18] extend these results to the boundary element method (BEM) and [19] to a FEM-BEM coupling setting. The underlying mechanism in these works is that ellipticity of the operator allows one to prove a discrete Caccioppoli inequality, where a higher order norm (e.g., the H^1-norm) is controlled by a lower order norm (e.g., the L^2-norm) on a slightly larger region. This gain in regularity can be exploited for approximation purposes, and an exponential approximation can be obtained by iterating the argument. The present setting of Maxwell's equations is different since the corresponding Caccioppoli inequality (Lemma 4.1) controls only the $\mathbf{H}(\mathrm{curl})$-norm by the $\mathbf{L}^2$-norm. Since $\mathbf{H}(\mathrm{curl})$ is not compactly embedded in $\mathbf{L}^2$, this Caccioppoli inequality is insufficient for approximation purposes. We therefore combine this Caccioppoli inequality with a local discrete Helmholtz-type decomposition. The gradient part can be treated with techniques established in [16] for Poisson problems, whereas the remaining part can, up to a small perturbation, be controlled in $\mathbf{H}^1$ so that approximation becomes feasible and one may proceed structurally similarly to the scalar case. The local discrete Helmholtz-type decomposition (Lemma 3.11) may also be of independent interest.

This paper is organized as follows. In Section 2, we introduce the time-harmonic Maxwell equations and their discretization with Nédélec's curl-conforming elements. We state the main result of this paper, namely, the existence of $\mathcal{H}$-matrix approximations to the inverse stiffness matrix that converge root exponentially in the block rank. We hasten to add that we do not track the dependence on the frequency ω in our analysis and focus on the case of fixed wave number κ. As in the case of the Helmholtz equation, the high-frequency case of $\omega \to \infty$ would require specialized matrix formats such as directional $\mathcal{H}^2$-matrices ($\mathcal{DH}^2$) or the butterfly format; we refer to the literature discussions in [4, 7, 8]. To prove the approximability result of Section 2, we present in Section 3 a local discrete Helmholtz decomposition and prove stability and approximation properties of this decomposition under a certain technical assumption on the mesh. In Section 4, we present a Caccioppoli-type inequality for discrete $\boldsymbol{\mathcal{L}}$-harmonic functions with $\boldsymbol{\mathcal{L}}$ being the Maxwell operator. Furthermore, we obtain exponentially convergent approximations to discrete $\boldsymbol{\mathcal{L}}$-harmonic functions. Section 5 is concerned with the proof of the main result of this paper.

Concerning notation: Constants $C > 0$ may differ in different occurrences but are independent of critical parameters such as the mesh size. $a \lesssim b$ indicates the existence of a constant $C > 0$ such that $a \leq Cb$. For a set $A \subset \mathbb{R}^3$, we denote by $|A|$ its Lebesgue measure. For finite sets B, the cardinality of B is also denoted by $|B|$. We employ standard Sobolev spaces as described in [34]. We also denote $\Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$.

2 Main results

2.1 Model problem

Maxwell's equations are a system of first-order partial differential equations that connect the temporal and spatial rates of change of the electric and magnetic fields possibly in the presence of additional source terms. Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma := \partial\Omega$ that, in physical terms, is filled with a homogeneous isotropic material. Maxwell's equations then connect the electric field $\mathcal{E}$ to the magnetic field $\mathcal{H}$ by

$$\left(\varepsilon \frac{\partial}{\partial t} + \sigma\right) \mathcal{E} - \nabla \times \mathcal{H} = \mathcal{G} \qquad \text{in } \Omega, \tag{2.1a}$$

$$\mu \frac{\partial}{\partial t} \mathcal{H} + \nabla \times \mathcal{E} = 0 \qquad \text{in } \Omega, \tag{2.1b}$$

where $\mathcal{G}$ is a given function representing the applied current. Homogeneous isotropic materials can be characterized by a positive dielectric constant $\varepsilon > 0$, a positive permeability constant $\mu > 0$, and a non-negative electric conductivity constant $\sigma \geq 0$. In this paper, we consider perfectly conducting boundary conditions for $\mathcal{E}$, i.e.,

$$\mathbf{n} \times \mathcal{E} = 0 \qquad \text{on} \quad \Gamma,$$

where $\mathbf{n}$ is the unit outward normal vector on Γ.

We assume the arising fields to be time-harmonic, i.e.,

$$\mathcal{E}(x,t) = e^{-i\omega t}\mathbf{E}(x), \qquad \mathcal{H}(x,t) = e^{-i\omega t}\mathbf{H}(x), \qquad \mathcal{G}(x,t) = e^{-i\omega t}\mathbf{J}(x) \tag{2.2}$$

for some given frequency ω. Substituting (2.2) into (2.1a) and (2.1b), we get

$$-\nabla \times \mathbf{H} - i\omega\eta\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \tag{2.3a}$$

$$\nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0 \quad \text{in } \Omega, \tag{2.3b}$$

where $\eta := \varepsilon + i\sigma/\omega$. Finally, the first-order system (2.3a) can be reduced to a second-order equation by eliminating $\mathbf{H}$

$$\mathcal{L}\mathbf{E} := \nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \kappa\mathbf{E} = \mathbf{F} \qquad \text{in } \Omega, \tag{2.4}$$

where $\kappa := \omega^2\eta$ and $\mathbf{F} := -i\omega\mathbf{J}$. For the sake of simplicity, we also assume $\mu = 1$ in the following.

With $\mathbf{L}^2(\Omega) := L^2(\Omega)^3$, we define the space $\mathbf{H}(\text{curl},\Omega) := \left\{\mathbf{U} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{U} \in \mathbf{L}^2(\Omega)\right\}$, equipped with the norm

$$\|\mathbf{U}\|^2_{\mathbf{H}(\text{curl},\Omega)} := \|\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{U}\|^2_{\mathbf{L}^2(\Omega)},$$

and the subspace $\mathbf{H}_0(\text{curl},\Omega) \subset \mathbf{H}(\text{curl},\Omega)$ with zero boundary conditions

$$\mathbf{H}_0(\text{curl},\Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{U} \in \mathbf{L}^2(\Omega), \mathbf{n} \times \mathbf{U} = 0 \text{ on } \Gamma\}.$$

The following lemma asserts that the tangential trace operator for functions in $\mathbf{H}(\text{curl},\Omega)$ is indeed well-defined:

Lemma 2.1 [35, Thm. 3.29] Let Ω be a bounded Lipschitz domain. Then, the *trace operator*

$$\gamma_T : \mathbf{C}^\infty(\overline{\Omega}) \to \mathbf{C}^0(\Gamma), \quad \mathbf{U} \mapsto \mathbf{n} \times \mathbf{U}|_\Gamma$$

can be uniquely extended to a bounded linear operator $\gamma_T : \mathbf{H}(\text{curl},\Omega) \to \mathbf{H}^{-1/2}(\Gamma)$.

Multiplying both sides of (2.4) with $\Psi \in \mathbf{H}_0(\text{curl},\Omega)$ and integrating by parts, we obtain the weak formulation: Find $\mathbf{E} \in \mathbf{H}_0(\text{curl},\Omega)$ such that

$$a(\mathbf{E},\Psi) := \langle \nabla \times \mathbf{E}, \nabla \times \Psi\rangle_{\mathbf{L}^2(\Omega)} - \kappa\,\langle \mathbf{E},\Psi\rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{F},\Psi\rangle_{\mathbf{L}^2(\Omega)} \quad \forall \Psi \in \mathbf{H}_0(\text{curl},\Omega), \tag{2.5}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{L}^2(\Omega)}$ is the $\mathbf{L}^2(\Omega)$-inner product. We assume that κ is not an eigenvalue of the operator $\nabla \times \nabla \times$, see, e.g., [35, Sec. 4]. This implies in particular that $\kappa \neq 0$ since $\nabla H^1_0(\Omega)$ is contained in the kernel of the operator $\nabla \times \nabla \times$. Then, the Fredholm alternative provides the existence of a unique solution to the variational problem, and we have the a priori estimate

$$\|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C_{\mathrm{stab}}\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \tag{2.6}$$

for a constant C_{stab} that depends on Ω and κ, see, e.g., [28, Thm. 5.2].

2.2 Discretization by edge elements

Let $\mathcal{T}_h = \{T_1, \dots, T_{N_\mathcal{T}}\}$ be a quasi-uniform triangulation of Ω with the mesh width $h := \max_{T_j \in \mathcal{T}_h} \mathrm{diam}(T_j)$, where the elements $T_j \in \mathcal{T}_h$ are open tetrahedra. The mesh $\mathcal{T}_h$ is assumed to be regular in the sense of Ciarlet, i.e., there are no hanging nodes. The assumption of quasi-uniformity includes the assumption of γ-shape regularity, i.e., there is $\gamma > 0$ such that $\mathrm{diam}(T_j) \leq \gamma |T_j|^{1/3}$ for all $T_j \in \mathcal{T}_h$. For the Galerkin discretization of (2.5), we use lowest order Nédélec's $\mathbf{H}(\mathrm{curl},\Omega)$-conforming elements of the first kind, see, e.g., [35, Sec. 5]. That is, on $T \in \mathcal{T}_h$, we introduce the lowest order local Nédélec space

$$\mathcal{N}_0(T) = \{\mathbf{a} + \mathbf{b} \times \mathbf{x} : \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \quad \mathbf{x} \in T\},$$

and set

$$\begin{aligned} \mathbf{X}_h(\mathcal{T}_h, \Omega) &:= \{\mathbf{U}_h \in \mathbf{H}(\mathrm{curl}, \Omega) : \ \mathbf{U}_h|_T \in \mathcal{N}_0(T) \quad \forall T \in \mathcal{T}_h\}, \\ \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega) &:= \mathbf{X}_h(\mathcal{T}_h, \Omega) \cap \mathbf{H}_0(\mathrm{curl}, \Omega). \end{aligned}$$

The standard degrees of freedom of $\mathbf{X}_h(\mathcal{T}_h, \Omega)$ are the line integrals of the tangential component of $\mathbf{U}_h$ on the edges of $\mathcal{T}_h$, see, e.g., [35, Sec. 5.5.1], [3, Sec. 2.3.2]. Hence, the dimension of $\mathbf{X}_h(\mathcal{T}_h, \Omega)$ is the number of edges of $\mathcal{T}_h$. The standard basis $\mathcal{X}_h := \{\Psi_e\}$ of $\mathbf{X}_h(\mathcal{T}_h, \Omega)$ consists of the so-called (lowest order) edge elements, where the function $\Psi_e \in \mathbf{X}_h(\mathcal{T}_h, \Omega)$ is associated with the edge e of $\mathcal{T}_h$ and is supported by the union of the tetrahedra sharing the edge e. More specifically, for an edge e with endpoints V_1, V_2 and a tetrahedron T with edge e, one has $\Psi_e|_T = \lambda_{V_1} \nabla \lambda_{V_2} - \lambda_{V_2} \nabla \lambda_{V_1}$, where λ_{V_i} is the standard hat function associated with vertex V_i.

A basis $\mathcal{X}_{h,0} := \{\Psi_1, \dots, \Psi_N\}$ of $\mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ with $N := \dim \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ is obtained by taking the $\Psi_e \in \mathcal{X}_h$, whose edge e satisfies $e \subset \Omega$; that is, $\mathcal{X}_{h,0}$ is obtained from $\mathcal{X}_h$ by removing the shape functions associated with edges lying on Γ.

Using $\mathbf{X}_{h,0}(\mathcal{T}_h, \Omega) \subseteq \mathbf{H}_0(\mathrm{curl}, \Omega)$ as ansatz and test space in (2.5), we arrive at the Galerkin discretization of finding $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ such that

$$a(\mathbf{E}_h, \Psi_h) = \langle \mathbf{F}, \Psi_h \rangle_{\mathbf{L}^2(\Omega)} \qquad \forall \Psi_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega). \tag{2.7}$$

Using the basis $\mathcal{X}_{h,0}$, the Galerkin discretization (2.7) can be formulated as a linear system of equations, where the system matrix $\mathbf{A} \in \mathbb{C}^{N \times N}$ is given by

$$\mathbf{A}_{ij} := a(\Psi_j, \Psi_i), \qquad \Psi_j, \Psi_i \in \mathcal{X}_{h,0}. \tag{2.8}$$

For unique solvability of the discrete problem (2.7) or, equivalently, the invertibility of $\mathbf{A}$, we recall the following Lemma 2.2. In that result and throughout the paper, we denote by

$$\mathbf{\Pi}_h^{L^2} : \mathbf{L}^2(\Omega) \to \mathbf{X}_h(\mathcal{T}_h, \Omega) \tag{2.9}$$

the $\mathbf{L}^2(\Omega)$-orthogonal projection onto $\mathbf{X}_h(\mathcal{T}_h, \Omega)$.

Lemma 2.2 [28, Thm. 5.7] Assume (2.6). There exists $h_0 > 0$ depending on the parameters of the continuous problem and the γ-shape regularity of $\mathcal{T}_h$, such that, for $h < h_0$, the discrete problem (2.7) has a unique solution, and there holds the stability estimate

$$\|\mathbf{E}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C \left\|\mathbf{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}.$$

Here, $C > 0$ is a constant depending solely on the γ-shape regularity of $\mathcal{T}_h$ and the parameters of the continuous problem.

2.3 Hierarchical matrices

The goal of this paper is to obtain an $\mathcal{H}$-matrix approximation of the inverse matrix $\mathbf{B} := \mathbf{A}^{-1}$. An $\mathcal{H}$-matrix is a blockwise low-rank matrix, where suitable blocks for low-rank approximation are chosen by the concept of *admissibility*, which is defined in the following.

Definition 2.3 (bounding boxes and η-admissibility) A cluster τ is a subset of the index set $\mathcal{I} = \{1, 2, \dots, N\}$. For a cluster $\tau \subset \mathcal{I}$, an axis-parallel box $B_{R_\tau} \subseteq \mathbb{R}^3$ is called a *bounding box*, if B_{R_τ} is a cube with side length R_τ and $\cup_{i\in\tau} \operatorname{supp}\Psi_i \subseteq B_{R_\tau}$. Let $\eta > 0$. Then, a pair of clusters is called *η-admissible*, if there exist bounding boxes B_{R_τ} and B_{R_σ} of τ and σ such that

$$\min\{\operatorname{diam}(B_{R_\tau}), \operatorname{diam}(B_{R_\sigma})\} \leq \eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}). \tag{2.10}$$

Definition 2.4 (Concentric boxes) Axis-parallel boxes B_R of side length R are called *boxes*. Two boxes B_R and $B_{R'}$ of side length R and R' are said to be *concentric*, if they have the same barycenter and B_R can be obtained by a stretching of $B_{R'}$ by the factor R/R' taking their common barycenter as the origin.

Definition 2.5 (cluster tree) A *cluster tree* with *leaf size* $n_{\mathrm{leaf}} \in \mathbb{N}$ is a binary tree $\mathbb{T}_{\mathcal{I}}$ with root $\mathcal{I}$ such that each cluster $\tau \in \mathbb{T}_{\mathcal{I}}$ is either a leaf of the tree and satisfies $|\tau| \leq n_{\mathrm{leaf}}$, or there exist disjoint subsets τ', $\tau'' \in \mathbb{T}_{\mathcal{I}}$ of τ, called *sons*, with $\tau = \tau' \dot{\cup} \tau''$. The *level function* $\mathrm{level} : \mathbb{T}_{\mathcal{I}} \to \mathbb{N}_0$ is inductively defined by $\mathrm{level}(\mathcal{I}) = 0$ and $\mathrm{level}(\tau') := \mathrm{level}(\tau) + 1$ for τ' a son of τ. Furthermore, $\mathrm{depth}(\mathbb{T}_{\mathcal{I}}) := \max_{\tau\in\mathbb{T}_{\mathcal{I}}} \mathrm{level}(\tau)$ is called the *depth* of a cluster tree.

Definition 2.6 (block cluster tree, sparsity constant and partition) Let $\mathbb{T}_{\mathcal{I}}$ be a cluster tree with root $\mathcal{I}$ and $\eta > 0$ be a fixed admissibility parameter. The block cluster tree $\mathbb{T}_{\mathcal{I}\times\mathcal{I}}$ is a tree constructed recursively from the root $\mathcal{I}\times\mathcal{I}$ such that for each block $\tau\times\sigma \in \mathbb{T}_{\mathcal{I}\times\mathcal{I}}$ with $\tau, \sigma \in \mathbb{T}_{\mathcal{I}}$, the set of sons of $\tau\times\sigma$ is defined as

$$\mathcal{S}(\tau\times\sigma) := \begin{cases} \emptyset & \text{if } \tau\times\sigma \text{ is } \eta\text{-admissible or } \mathcal{S}(\tau)=\emptyset \text{ or } \mathcal{S}(\sigma)=\emptyset,\\ \mathcal{S}(\tau)\times\mathcal{S}(\sigma) & \text{else.}\end{cases}$$

The *sparsity constant* $C_{\rm sp}$ of a block cluster tree, see, e.g., [23, 30], is given as

$$C_{\rm sp} := \max\left\{\max_{\tau\in\mathbb{T}_{\mathcal{I}}}\left|\{\sigma\in\mathbb{T}_{\mathcal{I}} : \tau\times\sigma\in\mathbb{T}_{\mathcal{I}\times\mathcal{I}}\}\right|, \max_{\sigma\in\mathbb{T}_{\mathcal{I}}}\left|\{\tau\in\mathbb{T}_{\mathcal{I}} : \tau\times\sigma\in\mathbb{T}_{\mathcal{I}\times\mathcal{I}}\}\right|\right\}. \tag{2.11}$$

The leaves of the block cluster tree induce a partition P of the set $\mathcal{I}\times\mathcal{I}$, which we call a partition based on $\mathbb{T}_{\mathcal{I}}$. For such a partition P and a fixed admissibility parameter $\eta > 0$, we define the *far field* and the *near field* as

$$P_{\rm far} := \{(\tau,\sigma)\in P : (\tau,\sigma) \text{ is } \eta\text{-admissible}\}, \quad P_{\rm near} := P\setminus P_{\rm far}. \tag{2.12}$$

For clusters $\tau, \sigma \subset \mathcal{I}$, we adopt the notation

$$\begin{aligned} \mathbb{C}^{\tau} &:= \left\{\mathbf{x}\in\mathbb{C}^N : \mathbf{x}_i = 0 \quad \text{if } i\notin\tau\right\},\\ \mathbb{C}^{\tau\times\sigma} &:= \{\mathbf{A}\in\mathbb{C}^{N\times N} : \mathbf{A}_{ij}=0 \quad \text{if } i\notin\tau \text{ or } j\notin\sigma\}.\end{aligned}$$

For $\mathbf{x}\in\mathbb{C}^N$ and $\mathbf{A}\in\mathbb{C}^{N\times N}$, the restrictions $\mathbf{x}|_\tau$ and $\mathbf{A}|_{\tau\times\sigma}$ are understood as $(\mathbf{x}|_\tau)_i = \chi_\tau(i)\mathbf{x}_i$ and $(\mathbf{A}|_{\tau\times\sigma})_{ij} = \chi_\tau(i)\chi_\sigma(j)\mathbf{A}_{ij}$, where χ_τ and χ_σ are the characteristic functions of the sets τ, σ. For integers $r\in\mathbb{N}$, matrices $\mathbb{C}^{\tau\times r}$ are understood as matrices in $\mathbb{C}^{N\times r}$ such that each column is in $\mathbb{C}^\tau$.

Definition 2.7 ($\mathcal{H}$-matrices) Let P be a partition of $\mathcal{I}\times\mathcal{I}$ based on a cluster tree $\mathbb{T}_{\mathcal{I}}$ and admissibility parameter $\eta > 0$. A matrix $\mathbf{A}\in\mathbb{C}^{N\times N}$ is an $\mathcal{H}$-matrix, if, for every admissible pair $(\tau,\sigma)\in P_{\rm far}$, we have a rank r factorization

$$\mathbf{A}|_{\tau\times\sigma} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H,$$

where $\mathbf{X}_{\tau\sigma}\in\mathbb{C}^{\tau\times r}$ and $\mathbf{Y}_{\tau\sigma}\in\mathbb{C}^{\sigma\times r}$.

2.4 Main result

The following theorem is the main result of this paper. It states that the inverse of the Galerkin matrix $\mathbf{A}$ from (2.8) can be approximated at an exponential rate in the block rank by an $\mathcal{H}$-matrix.

Theorem 2.8 Let $\eta > 0$ be a fixed admissibility parameter and P be a partition of $\mathcal{I}\times\mathcal{I}$ based on the cluster tree $\mathbb{T}_{\mathcal{I}}$ and η. Let the mesh $\mathcal{T}_h$ be such that Assumption 3.4 holds true for any box. Let $h < h_0$ with h_0 given by

Proposition 2.2, and let $\mathbf{A}$ be the stiffness matrix given by (2.8). Then, there exists an $\mathcal{H}$-matrix $\mathbf{B}_{\mathcal{H}}$ with blockwise rank r such that

$$\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}\|_2 \leq C_{\text{apx}} C_{\text{sp}} \text{depth}(\mathbb{T}_{\mathcal{I}}) h^{-1} e^{-b(r^{1/4}/\ln r)}.$$

The constants C_{apx}, $b > 0$ depend only on κ, Ω, η, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$. The constant C_{sp} (defined in (2.11)) depends only on the partition η, γ, d, and Ω.

Remark 2.9 The low-rank structure of the far-field blocks allow for efficient storage of $\mathcal{H}$-matrices as the memory requirement to store an $\mathcal{H}$-matrix is $\mathcal{O}(C_{\text{sp}} \text{depth}(\mathbb{T}_{\mathcal{I}}) rN)$. Standard clustering methods such as the geometric clustering for quasi-uniform meshes (see, e.g., [26, Sec. 5.4.2]) lead to balanced cluster trees, i.e., $\text{depth}(\mathbb{T}_{\mathcal{I}}) \sim \log(N)$ and a uniformly (in the mesh size h) bounded sparsity constant. In total this gives a storage complexity of $\mathcal{O}(rN \log(N))$ for the matrix $\mathbf{B}_{\mathcal{H}}$ instead of the $\mathcal{O}(N^2)$ for the fully populated inverse $\mathbf{A}^{-1}$.

3 Helmholtz decompositions: continuous and localized discrete

Helmholtz decompositions, i.e., writing a vector field as a sum of a divergence-free field and a gradient field, play a key role in our analysis. In fact, we use two different decompositions, the regular decomposition (see, e.g., [28, Lem. 2.4] and [29, Thm. 11]) and a localized discrete version (Definition 3.6).

Lemma 3.1 (Regular decomposition) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, there is a constant $C > 0$ depending only on Ω such that any $\mathbf{E} \in \mathbf{H}_0(\text{curl},\Omega)$ can be written as $\mathbf{E} = \mathbf{z} + \nabla p$ with $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ and

$$\|\mathbf{z}\|_{\mathbf{H}_0^1(\Omega)} \leq C\|\mathbf{E}\|_{\mathbf{H}(\text{curl},\Omega)}, \quad \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}.$$

Proof Regular decompositions are available in the literature, see, e.g., [28, Lem. 2.4] and [29, Thm. 11]. The statement that $\|\mathbf{z}\|_{\mathbf{L}^2(\Omega)}$ and $\|\nabla p\|_{\mathbf{L}^2(\Omega)}$ are controlled by $\|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$ is a variation of these estimates. For a proof, see [32] or the appendix. □

The function $\mathbf{z}$ of the regular decomposition provided by Lemma 3.1 is not necessarily divergence-free. This can be corrected by subtracting a gradient. To that end, we introduce, for a given open set $\widetilde{D} \subseteq \Omega$ and a chosen $\widetilde{\eta} \in L^\infty(\Omega)$ with $\widetilde{\eta} \equiv 1$ on $\widetilde{D}$, the mapping $\mathbf{L}^2(\Omega) \to H_0^1(\Omega) : \mathbf{z} \mapsto \varphi_{\mathbf{z}}$ by

$$\langle \nabla \varphi_{\mathbf{z}}, \nabla v \rangle_{L^2(\Omega)} = \langle \widetilde{\eta}\mathbf{z}, \nabla v \rangle_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega). \tag{3.1}$$

Lemma 3.2 The mapping $\mathbf{L}^2(\Omega) \ni \mathbf{z} \mapsto \varphi_{\mathbf{z}} \in H_0^1(\Omega)$ has the following properties:

(i) $\|\varphi_{\mathbf{z}}\|_{H^1(\Omega)} \leq C\|\widetilde{\eta}\|_{L^\infty(\Omega)}\|\mathbf{z}\|_{\mathbf{L}^2(\text{supp}\,\widetilde{\eta})}$, where the constant depends only on Ω.
(ii) $\langle \mathbf{z} - \nabla\varphi_{\mathbf{z}}, \nabla v \rangle_{L^2(\widetilde{D})} = 0$ for all $v \in H_0^1(\Omega)$.

Proof By construction, we have $\|\nabla\varphi_z\|_{\mathbf{L}^2(\Omega)} \leq \|\widetilde{\eta}\mathbf{z}\|_{\mathbf{L}^2(\Omega)}$. The constant C in statement (i) reflects the Poincaré constant of the simply connected domain Ω. The property (ii) follows by construction. □

Remark 3.3 **(classical Helmholtz decomposition)** Selecting $\widetilde{D} = \Omega$ and correspondingly $\widetilde{\eta} \equiv 1$ yields the decomposition $\mathbf{E} = (\mathbf{z} - \nabla\varphi_{\mathbf{z}}) + \nabla(p + \varphi_{\mathbf{z}})$ with the orthogonality $\langle \mathbf{z} - \nabla\varphi_{\mathbf{z}}, \nabla(p + \varphi_{\mathbf{z}})\rangle_{L^2(\Omega)} = 0$ and $\|\mathbf{z} - \nabla\varphi_{\mathbf{z}}\|_{\mathbf{H}(\text{curl},\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\text{curl},\Omega)}$, $\|\mathbf{z} - \nabla\varphi_{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$, $\|\nabla(p + \varphi_{\mathbf{z}})\|_{L^2(\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$.

Regular decompositions as in Lemma 3.1 can also be done locally for discrete functions. Let $\mathcal{P}_1(T)$ denote the space of polynomials of degree at most 1 on $T \in \mathcal{T}_h$. We introduce spaces of globally continuous, piecewise linear polynomials by

$$S^{1,1}(\mathcal{T}_h) := \{p_h \in H^1(\Omega) : p_h|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h\}, \tag{3.2}$$

$$S_0^{1,1}(\mathcal{T}_h) := S^{1,1}(\mathcal{T}_h) \cap H_0^1(\Omega). \tag{3.3}$$

We will require the following assumption on the meshes $\mathcal{T}_h$:

Assumption 3.4 For a simply connected domain $D \subset \mathbb{R}^3$, define the sets of elements touching D as

$$\mathcal{T}_h(D) := \{T \in \mathcal{T}_h : |T \cap D| > 0\},$$
$$\widehat{D} := \operatorname{int}\left(\bigcup_{T\in\mathcal{T}_h(D)} \overline{T}\right).$$

For any box $D \subset \mathbb{R}^3$, there is a set $\widetilde{D}$ which is a union of elements in $\mathcal{T}_h$ such that

1. $\widehat{D} \subset \widetilde{D}$,
2. $\operatorname{dist}(\partial\widetilde{D}, D) \leq 2h$,
3. $\widetilde{D}$ is simply connected.

We call $\widetilde{D}$ a *mesh-conforming region* for D. If a box D has more than one mesh-conforming region $\widetilde{D}$, one is selected as "the" mesh-conforming one.

Remark 3.5 The reason behind Assumption 3.4 is that the region $\widehat{D}$ may not be simply connected, but by adding elements of the mesh holes may be filled to obtain a simply connected set $\widetilde{D}$.

The spaces *localized* to a mesh-conforming region $\widetilde{D}$ are given by

$$S^{1,1}(\mathcal{T}_h, \widetilde{D}) := \left\{p_h|_{\widetilde{D}} : p_h \in S_0^{1,1}(\mathcal{T}_h)\right\}, \tag{3.4}$$

$$\mathbf{X}_h(\mathcal{T}_h, \widetilde{D}) := \left\{\mathbf{E}_h|_{\widetilde{D}} : \mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)\right\}. \tag{3.5}$$

Definition 3.6 (Local discrete regular decomposition) Let $D \subset \mathbb{R}^3$ be a box and $\widetilde{D}$ be the corresponding mesh-conforming region from Assumption 3.4. We denote by $\Pi^{\nabla}_{\widetilde{D}} : \mathbf{L}^2(\widetilde{D}) \to \nabla S^{1,1}(\mathcal{T}_h, \widetilde{D})$ the $\mathbf{L}^2(\widetilde{D})$-projection onto $\nabla S^{1,1}(\mathcal{T}_h, \widetilde{D})$ given by

$$\langle \mathbf{p} - \Pi^{\nabla}_{\widetilde{D}} \mathbf{p}, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{D})} = 0 \qquad \forall v_h \in S^{1,1}(\mathcal{T}_h, \widetilde{D}). \tag{3.6}$$

Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\widetilde{D}$. Let $\mathbf{E}_h$ be such that $\eta \mathbf{E}_h \in \mathbf{H}_0(\text{curl},\Omega)$ as well as $\mathbf{E}_h\big|_{\widetilde{D}} \in \mathbf{X}_h(\mathcal{T}_h, \widetilde{D})$. Decompose $\eta \mathbf{E}_h \in \mathbf{H}_0(\text{curl},\Omega)$ as $\eta \mathbf{E}_h = \mathbf{z} + \nabla p$, where $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ and $p \in H^1_0(\Omega)$ are given by Lemma 3.1.

Then, the *local discrete regular decomposition* is given by $\mathbf{E}_h = \mathbf{z}_h + \Pi^{\nabla}_{\widetilde{D}} \nabla p$ on $\widetilde{D}$ with $\mathbf{z}_h := \mathbf{E}_h - \Pi^{\nabla}_{\widetilde{D}} \nabla p$. We write $\nabla p_h = \Pi^{\nabla}_{\widetilde{D}} \nabla p$ for some $p_h \in S^{1,1}(\mathcal{T}_h, \widetilde{D})$.

For future reference, we note that

$$\left\| \Pi^{\nabla}_{\widetilde{D}} \mathbf{p} \right\|_{\mathbf{L}^2(\widetilde{D})} \le \|\mathbf{p}\|_{\mathbf{L}^2(\widetilde{D})}. \tag{3.7}$$

Remark 3.7

1. The function $p_h \in S^{1,1}(\mathcal{T}_h, \widetilde{D})$ that satisfies $\nabla p_h = \Pi^{\nabla}_{\widetilde{D}} \mathbf{p}$, is not unique. However, its gradient ∇p_h is unique.
2. Due to the cut-off function η, the decomposition depends on $\mathbf{E}_h$ on supp η only, which is quantified in the stability assertions of Lemma 3.11.
3. The local regular decomposition provides, for a function $\mathbf{E}_h$ that is a discrete function on $\widetilde{D}$, two representations in view of $\eta \equiv 1$ on $\widetilde{D}$, namely, $\mathbf{E}_h|_{\widetilde{D}} = (\mathbf{z} + \nabla p)|_{\widetilde{D}} = \mathbf{z}_h + \nabla p_h$.
4. For $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$, the decomposition $\mathbf{E}_h = (\mathbf{z} - \nabla \varphi_{\mathbf{z}}) + \nabla(p + \varphi_{\mathbf{z}})$ of Remark 3.3 yields upon setting $\nabla p_h := \Pi^{\nabla}_{\Omega} \nabla(p + \varphi_{\mathbf{z}}) \in \nabla S^{1,1}_0(\mathcal{T}_h, \Omega) \subset \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ and $\mathbf{z}_h := \mathbf{E}_h - \nabla p_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ the decomposition $\mathbf{E}_h = \mathbf{z}_h + \nabla p_h$ with

$$\langle \mathbf{z}_h, \nabla p_h \rangle_{\mathbf{L}^2(\Omega)} = 0, \quad \|\mathbf{z}_h\|_{\mathbf{L}^2(\Omega)} + \|\nabla p_h\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)},$$
$$\|\mathbf{z}_h\|_{\mathbf{H}(\text{curl},\Omega)} \lesssim \|\mathbf{E}_h\|_{\mathbf{H}(\text{curl},\Omega)},$$

   which is a discrete Helmholtz decomposition as described in, e.g., [22, Cor. 5.1] and [35, Sec. 7.2.1].

The following lemma formulates a local exact sequence property.

Lemma 3.8 Let $D \subset \mathbb{R}^3$ be a box such that $D \cap \Omega$ is a simply connected Lipschitz domain and $\widetilde{D}$ be given according to Assumption 3.4. Assume that $\overline{\widetilde{D}} \cap \partial\Omega$ is connected. (In particular, the empty set is connected.) Then, for all $\mathbf{v}_h \in \mathbf{X}_h(\mathcal{T}_h, \widetilde{D})$ with $\nabla \times \mathbf{v}_h = 0$ on $\widetilde{D}$, we can find a $\widetilde{\varphi}_h \in S^{1,1}(\mathcal{T}_h, \widetilde{D})$ such that $\mathbf{v}_h = \nabla \widetilde{\varphi}_h$.

Proof We recall from, e.g., [35, Thm. 3.37] the following commuting diagram property: for a simply connected Lipschitz domain ω the condition $\nabla \times \mathfrak{w} = 0$ implies

$\mathfrak{w} = \nabla\psi$ for some $\psi \in H^1(\omega)$; furthermore, ψ is unique up to a constant. The discrete commuting diagram property for a tetrahedron T is: if $\mathfrak{w} \in \mathcal{N}_0(T)$ satisfies $\nabla \times \mathfrak{w} = 0$, then there is $\psi_h \in \mathcal{P}_1(T)$ with $\mathfrak{w} = \nabla\psi_h$.

The condition $\nabla \times \mathbf{v}_h = 0$ on $\widetilde{D}$ implies $\mathbf{v}_h = \nabla\varphi_h$ for some $\varphi_h \in H^1(\widetilde{D})$. The function φ_h is unique up to a constant, which we fix, for example, by the condition $\int_{\widetilde{D}} \varphi_h = 0$. For each $T \in \mathcal{T}_h(D)$, the condition $\nabla \times \mathbf{v}_h = 0$ on T implies the existence of $\widetilde{\varphi}_{h,T} \in \mathcal{P}_1(T)$ with $\mathbf{v}_h = \nabla\widetilde{\varphi}_{h,T}$ on T. The polynomial $\widetilde{\varphi}_{h,T}$ is unique up to a constant, which we fix by requiring $\int_T \widetilde{\varphi}_{h,T} = \int_T \varphi_h$. By the uniqueness assertion we have $\varphi_h|_T = \widetilde{\varphi}_{h,T}|_T$. Define $\widetilde{\varphi}_h \in S^{1,0}(\mathcal{T}_h, \widetilde{D})$ elementwise by $\widetilde{\varphi}_h|_T = \widetilde{\varphi}_{h,T}$. Since $\varphi_h \in H^1(\widetilde{D})$ we directly obtain $\widetilde{\varphi}_h \in S^{1,1}(\mathcal{T}_h, \widetilde{D})$. □

In order to prove the following lemmas, we need to introduce some projections and their properties. Let $D \subset \mathbb{R}^3$ be a box and $\widetilde{D}$ be defined according to Assumption 3.4. We define the space

$$\mathbf{H}(\mathrm{div}, \widetilde{D}) := \left\{ \mathbf{U} \in \mathbf{L}^2(\widetilde{D}) : \nabla \cdot \mathbf{U} \in L^2(\widetilde{D}) \right\}.$$

Let $\mathbf{RT}_0(T) := \{\mathbf{a} + b\mathbf{x} : \mathbf{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$ be the classical lowest order Raviart-Thomas element defined on T. Introduce

$$\mathbf{V}_h(\mathcal{T}_h, \widetilde{D}) := \{\mathbf{U}_h \in \mathbf{H}(\mathrm{div}, \widetilde{D}) : \mathbf{U}_h|_T \in \mathbf{RT}_0(T) \quad \forall T \in \mathcal{T}_h(D)\}. \tag{3.8}$$

On $\widetilde{D}$ the Raviart-Thomas interpolation operator $\mathfrak{w}_{\widetilde{D}} : \mathbf{H}^1(\widetilde{D}) \to \mathbf{V}_h(\mathcal{T}_h, \widetilde{D})$ is defined elementwise by $\mathfrak{w}_{\widetilde{D}}\mathbf{U}|_T := \mathfrak{w}_T\mathbf{U}$, where the elemental interpolation operator $\mathfrak{w}_T : \mathbf{H}^1(T) \to \mathbf{RT}_0(T)$ is characterized by the vanishing of certain moments of $\mathbf{U} - \mathfrak{w}_T\mathbf{U}$, viz.,

$$\int_f (\mathbf{U} - \mathfrak{w}_T\mathbf{U}) \cdot \nu q dA = 0 \quad \forall q \in \mathcal{P}_0(f) \; \forall \text{ faces} f \text{ of } T \in \mathcal{T}_h,$$

where ν is the unit normal to f and dA denotes the surface measure on f. Define the space

$$\mathbf{D}_h(\mathcal{T}_h, \widetilde{D}) := \{\mathbf{U} \in \mathbf{H}^1(\widetilde{D}) : \nabla \times \mathbf{U} \in \mathbf{H}^1(T) \quad \forall T \in \mathcal{T}_h(D)\} \tag{3.9}$$

and the Nédélec interpolation operator $\mathfrak{r}_{\widetilde{D}} : \mathbf{D}_h(\mathcal{T}_h, \widetilde{D}) \to \mathbf{X}_h(\mathcal{T}_h, \widetilde{D})$ elementwise by $\mathfrak{r}_{\widetilde{D}}\mathbf{U}|_T := \mathfrak{r}_T\mathbf{U}$, where the elemental interpolant $\mathfrak{r}_T\mathbf{U} \in \mathcal{N}_0(T)$ is characterized by the vanishing of certain moments of $\mathbf{U} - \mathfrak{r}_T\mathbf{U}$, viz.,

$$\int_e \left(\mathbf{U} - \mathfrak{r}_T\mathbf{U}\right) \cdot \boldsymbol{\tau} de = 0 \quad \forall \text{ edges } e \text{ of } T \in \mathcal{T}_h;$$

here, $\boldsymbol{\tau}$ is a unit vector parallel to the edge e. A key property of the operators $\mathfrak{r}_{\widetilde{D}}$ and $\mathfrak{w}_{\widetilde{D}}$ is that they commute, i.e., (see, e.g., [35, (5.59)])

$$\mathfrak{w}_{\widetilde{D}} \nabla \times \mathbf{U} = \nabla \times \mathfrak{r}_{\widetilde{D}}\mathbf{U} \qquad \forall \mathbf{U} \in \mathbf{D}_h(\mathcal{T}_h, \widetilde{D}). \tag{3.10}$$

Moreover, the lowest order elemental Nédélec interpolants have first-order approximation properties.

Lemma 3.9 ([35, Thm. 5.41]) Let $T \in \mathcal{T}_h$. Then, for $\mathbf{U} \in \mathbf{H}^1(T)$ with $\nabla\times\mathbf{U} \in \mathbf{H}^1(T)$, we have

$$\begin{aligned}\left\|\mathbf{U} - \mathfrak{r}_T\mathbf{U}\right\|_{\mathbf{L}^2(T)} &\lesssim h\big(|\mathbf{U}|_{\mathbf{H}^1(T)} + \|\nabla\times\mathbf{U}\|_{\mathbf{H}^1(T)}\big),\\ \left\|\nabla\times(\mathbf{U} - \mathfrak{r}_T\mathbf{U})\right\|_{\mathbf{L}^2(T)} &\lesssim h\|\nabla\times\mathbf{U}\|_{\mathbf{H}^1(T)}.\end{aligned}$$

In the following, we show local stability and approximation properties for the local discrete regular decomposition of Definition 3.6. This will be based on Lemma 3.8 with $D = B_R$, where B_R is a box with side length R. It is an important geometric observation that, due to the assumption that Ω is a Lipschitz polyhedron, the intersection $B_R \cap \Omega$ is a Lipschitz domain and the intersection $B_R \cap \Omega^c$ is connected provided R is sufficiently small. Then, the additional assumptions on $D \cap \Omega = B_R \cap \Omega$ in Lemma 3.8 can be satisfied. We formulate this as an assumption on R in terms of a number $R_{\max} > 0$ that depends on Ω:

Definition 3.10 ($R_{\max}$)$R_{\max} > 0$ is such that for any $R \in (0, R_{\max}]$ and any box B_R with $\left|B_R \cap \Omega\right| > 0$, the intersection $B_R \cap \Omega$ is a Lipschitz domain and $B_R \cap \Omega^c$ is connected.

Lemma 3.11 (stability of local discrete regular decomposition) Let $\varepsilon \in (0,1)$, $R \in (0, R_{\max}]$ be such that $\frac{h}{R} < \frac{\varepsilon}{4}$, and let B_R and $B_{(1+\varepsilon)R}$ be concentric boxes. Define $\widetilde{B}_R$ according to Assumption 3.4. Let $\eta \in W^{1,\infty}(\Omega)$ be a cut-off function with supp $\eta \subseteq \overline{B_{(1+\varepsilon)R} \cap \Omega}$, $\eta \equiv 1$ on $\widetilde{B}_R$, $0 \le \eta \le 1$, and $\|\nabla\eta\|_{L^\infty(\Omega)} \le C_\eta \frac{1}{\varepsilon R}$. Let $\mathbf{E}_h \in \mathbf{H}(\mathrm{curl}, B_{(1+\varepsilon)R} \cap \Omega)$ be such that $\eta\mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ as well as $\mathbf{E}_h \in \mathbf{X}_h(\mathcal{T}_h, \widetilde{B}_R)$. Let $\eta\mathbf{E}_h = \mathbf{z} + \nabla p$ be the regular decomposition of $\eta\mathbf{E}_h$ given by Lemma 3.1 and let $\mathbf{z}_h$ and ∇p_h be the contributions of the local discrete regular decomposition of Definition 3.6 with $D = B_R$ and $\widetilde{D} = \widetilde{B}_R$ there. Then, $\mathbf{E}_h = \mathbf{z}_h + \nabla p_h$ on $\widetilde{B}_R \cap \Omega$, and the following local stability and approximation results hold:

$$\begin{aligned}\left\|\nabla p_h\right\|_{\mathbf{L}^2(B_R\cap\Omega)} + \left\|\mathbf{z}_h\right\|_{\mathbf{H}(\mathrm{curl}, B_R\cap\Omega)} &\le C\Big(\left\|\nabla\times\mathbf{E}_h\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \tfrac{1}{\varepsilon R}\left\|\mathbf{E}_h\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\Big),\\ \left\|\mathbf{z} - \mathbf{z}_h\right\|_{\mathbf{L}^2(B_R\cap\Omega)} &\le Ch\|\mathbf{z}\|_{\mathbf{H}^1(B_{(1+\varepsilon)R}\cap\Omega)}\\ &\le Ch\Big(\left\|\nabla\times\mathbf{E}_h\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \tfrac{1}{\varepsilon R}\left\|\mathbf{E}_h\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\Big),\end{aligned}$$

where the constant $C > 0$ depends only on Ω, the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$, and C_η.

Proof The proof is done in two steps. We note that the condition on the parameter ε and the assumption on the mesh-conforming region (Assumption 3.4) ensures that $\widetilde{B}_R \subseteq B_{(1+\varepsilon)R}$.

Step 1: In this step we provide a proof of the stability estimate. Recalling the stability estimate Lemma 3.1 and using the product rule for the curl operator, it follows that

$$
\begin{aligned}
&\|\mathbf{z}\|_{\mathbf{H}_0^1(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \lesssim \big\|\eta \mathbf{E}_h\big\|_{\mathbf{H}(\mathrm{curl},\Omega)} \\
\lesssim\ & \big\|\nabla \times \mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\nabla \eta\|_{L^\infty(B_{(1+\varepsilon)R}\cap\Omega)}\big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \\
\overset{\varepsilon R \lesssim 1}{\lesssim}\ & \big\|\nabla \times \mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.
\end{aligned} \tag{3.11}
$$

Since ∇p_h satisfies (3.6), we get with (3.7) and the aid of (3.11)

$$
\big\|\nabla p_h\big\|_{L^2(B_R\cap\Omega)} \leq \|\nabla p\|_{L^2(\widetilde{B}_R)} \leq \|\nabla p\|_{L^2(\Omega)} \lesssim \big\|\nabla \times \mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.
$$

The definition of $\mathbf{z}_h$ gives

$$
\big\|\mathbf{z}_h\big\|_{\mathbf{H}(\mathrm{curl},B_R\cap\Omega)} \lesssim \big\|\nabla \times \mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.
$$

The combination of the above inequalities provides the desired local stability result.

Step 2: To prove the approximation property, we first need to ascertain the existence of $\varphi_h \in S^{1,1}(\mathcal{T}_h, \widetilde{B}_R)$ such that $\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} = \nabla\varphi_h$ on $\widetilde{B}_R$. To that end, we note that $\mathbf{z}_h \in \mathbf{D}_h(\mathcal{T}_h, \widetilde{B}_R)$, use the commuting diagram property (3.10) of $\mathfrak{r}_{\widetilde{B}_R}$ and $\mathfrak{w}_{\widetilde{B}_R}$, and the fact that $\mathfrak{r}_{\widetilde{B}_R}$ is a projection operator to compute on $\widetilde{B}_R$:

$$
\begin{aligned}
\nabla \times (\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}) &= \nabla \times \mathbf{z}_h - \mathfrak{w}_{\widetilde{B}_R}\nabla \times \mathbf{z} = \nabla \times \left(\mathbf{E}_h|_{\widetilde{B}_R}\right) - \mathfrak{w}_{\widetilde{B}_R}\nabla \times \left(\mathbf{E}_h|_{\widetilde{B}_R}\right) \\
&= \nabla \times \left(\mathbf{E}_h|_{\widetilde{B}_R}\right) - \nabla \times \mathfrak{r}_{\widetilde{B}_R}\left(\mathbf{E}_h|_{\widetilde{B}_R}\right) = 0.
\end{aligned}
$$

Lemma 3.8 then provides the existence of $\varphi_h \in S^{1,1}(\mathcal{T}_h, \widetilde{B}_R)$ such that $\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} = \nabla\varphi_h$ on $\widetilde{B}_R$. Since p_h satisfies (3.6), we get from $\mathbf{z} + \nabla p = \mathbf{E}_h = \mathbf{z}_h + \nabla p_h$ on $\widetilde{B}_R$ and the approximation property of $\mathfrak{r}_{\widetilde{B}_R}$ given in Lemma 3.9

$$
\begin{aligned}
\big\|\mathbf{z} - \mathbf{z}_h\big\|^2_{\mathbf{L}^2(\widetilde{B}_R)} &= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z} - \mathbf{z}_h \right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} + \left\langle \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} - \mathbf{z}_h, \mathbf{z} - \mathbf{z}_h \right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} \\
&= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z} - \mathbf{z}_h \right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} - \langle \nabla\varphi_h, \nabla(p_h - p)\rangle_{\mathbf{L}^2(\widetilde{B}_R)} \\
&= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z} - \mathbf{z}_h \right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} \lesssim \left\|\mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}\right\|_{\mathbf{L}^2(\widetilde{B}_R)} \big\|\mathbf{z} - \mathbf{z}_h\big\|_{\mathbf{L}^2(\widetilde{B}_R)} \\
&\lesssim h\|\mathbf{z}\|_{\mathbf{H}^1(B_{(1+\varepsilon)R}\cap\Omega)} \big\|\mathbf{z} - \mathbf{z}_h\big\|_{\mathbf{L}^2(\widetilde{B}_R)}.
\end{aligned}
$$

The combination of the above inequality and (3.11) implies

$$
\begin{aligned}
\big\|\mathbf{z} - \mathbf{z}_h\big\|_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \big\|\mathbf{z} - \mathbf{z}_h\big\|_{\mathbf{L}^2(\widetilde{B}_R)} \lesssim h\|\mathbf{z}\|_{\mathbf{H}^1(B_{(1+\varepsilon)R}\cap\Omega)} \\
&\lesssim h\Big(\big\|\nabla \times \mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\big\|\mathbf{E}_h\big\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\Big),
\end{aligned}
$$

which finishes the proof. □

4 Low-dimensional approximation of discrete $\mathcal{L}$-harmonic functions

We say that $\mathbf{E}_h \in \mathbf{X}_h(\mathcal{T}_h, \widetilde{D})$ is *discrete $\mathcal{L}$-harmonic* on $\widetilde{D}$, if $a(\mathbf{E}_h, \mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ with $\operatorname{supp} \mathbf{v}_h \subset \overline{\widetilde{D}}$; such a space will be formally introduced as $\mathcal{H}_{c,h}(\widetilde{D})$ below. In this section, we show that discrete $\mathcal{L}$-harmonic functions can be approximated from low-dimensional spaces on compact subsets of $\widetilde{D}$. Discrete interior regularity estimates, introduced in the following, play a key role.

4.1 The Caccioppoli-type inequalities

Caccioppoli inequalities usually estimate higher order derivatives by lower order derivatives on (slightly) enlarged regions. The following discrete Caccioppoli-type inequalities are formulated with an h-weighted $\mathbf{H}(\mathrm{curl})$-norm and an h-weighted H^1-norm. For a box B_R of side length $R > 0$, we define the norms $|||\cdot|||_{c,h,R}$ and $|||\cdot|||_{g,h,R}$ (the subscripts c and g abbreviate "curl" and "gradient") as follows:

$$|||\mathbf{U}|||^2_{c,h,R} := \frac{h^2}{R^2}\|\nabla \times \mathbf{U}\|^2_{\mathbf{L}^2(B_R\cap\Omega)} + \frac{1}{R^2}\|\mathbf{U}\|^2_{\mathbf{L}^2(B_R\cap\Omega)} \quad \forall \mathbf{U} \in \mathbf{H}(\mathrm{curl}, B_R \cap \Omega), \tag{4.1}$$

$$|||u|||^2_{g,h,R} := \frac{h^2}{R^2}\|\nabla u\|^2_{\mathbf{L}^2(B_R\cap\Omega)} + \frac{1}{R^2}\|u\|^2_{L^2(B_R\cap\Omega)} \qquad \forall u \in H^1(B_R \cap \Omega). \tag{4.2}$$

For any bounded open set $B \subset \mathbb{R}^3$, we define

$$\mathcal{H}_{c,h}(B\cap\Omega) := \{\mathbf{U}_h \in \mathbf{H}(\mathrm{curl}, B\cap\Omega) \ : \ \exists \widetilde{\mathbf{U}}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega) \text{ s.t. } \mathbf{U}_h|_{B\cap\Omega} = \widetilde{\mathbf{U}}_h|_{B\cap\Omega},$$
$$a(\mathbf{U}_h, \boldsymbol{\Psi}_h) = 0 \quad \forall \boldsymbol{\Psi}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega), \operatorname{supp} \boldsymbol{\Psi}_h \subset \overline{B\cap\Omega}\}$$

and

$$\mathcal{H}_{g,h}(B\cap\Omega) := \{p_h \in H^1(B\cap\Omega) \ : \ \exists \widetilde{p}_h \in S_0^{1,1}(\mathcal{T}_h) \text{ s.t. } p_h|_{B\cap\Omega} = \widetilde{p}_h|_{B\cap\Omega},$$
$$\langle \nabla p_h, \nabla \psi_h \rangle_{\mathbf{L}^2(B\cap\Omega)} = 0 \forall \psi_h \in S_0^{1,1}(\mathcal{T}_h), \operatorname{supp} \psi_h \subset \overline{B\cap\Omega}\}.$$

The following lemma provides a discrete Caccioppoli-type estimate for functions in $\mathcal{H}_{c,h}(B_{(1+\varepsilon)R} \cap \Omega)$.

Lemma 4.1 Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{4}$. Let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes and $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+\varepsilon)R} \cap \Omega)$. Then, there exists a constant C depending only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$ such that

$$\|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \le C\frac{1+\varepsilon}{\varepsilon}|||\mathbf{E}_h|||_{c,h,(1+\varepsilon)R}.$$

Proof Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $\operatorname{supp} \eta \subseteq B_{(1+\varepsilon/2)R}$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_R \cap \Omega$, and $\|\nabla^j \eta\|_{L^\infty(\Omega)} \lesssim (\varepsilon R)^{-j}$ for $j \in \{0,1,2\}$. We notice supp $(\eta^2\mathbf{E}_h) \subseteq \overline{B_{(1+\varepsilon/2)R} \cap \Omega}$ and since $4h \leq \varepsilon R$ we have supp$\mathfrak{r}_\Omega(\eta^2\mathbf{E}_h) \subseteq \overline{B_{(1+\varepsilon)R} \cap \Omega}$. The proof is done in two steps.

Step 1: Using the vector identity

$$\begin{aligned}\eta^2(\nabla\times\mathbf{E}_h)\cdot(\nabla\times\mathbf{E}_h) &= \nabla\times\mathbf{E}_h\cdot\left(\nabla\times\left(\eta^2\mathbf{E}_h\right)-\nabla\eta^2\times\mathbf{E}_h\right)\\ &= (\nabla\times\mathbf{E}_h)\cdot\nabla\times\left(\eta^2\mathbf{E}_h\right)-2\eta(\nabla\times\mathbf{E}_h)\cdot(\nabla\eta\times\mathbf{E}_h),\end{aligned}$$

we get

$$\begin{aligned}&\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} \leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}\\ &= \operatorname{Re}\left(a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h\right)+\kappa\langle\eta\mathbf{E}_h,\eta\mathbf{E}_h\rangle_{\mathbf{L}^2(B_R\cap\Omega)}-2\langle\eta\nabla\times\mathbf{E}_h,\nabla\eta\times\mathbf{E}_h\rangle_{\mathbf{L}^2(B_R\cap\Omega)}\right)\\ &\leq \operatorname{Re}a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h\right)+|\kappa|\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}+2\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)}\|\nabla\eta\times\mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)}.\end{aligned}$$

Young's inequality then gives

$$\begin{aligned}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}\\ &\leq \operatorname{Re}a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h\right)+|\kappa|\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\\ &\quad+\tfrac{1}{2}\|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)}+2\|\nabla\eta\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)}.\end{aligned} \tag{4.3}$$

Kicking back the term $\frac{1}{2}\|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)}$ to the left-hand side, we arrive at

$$\begin{aligned}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}\\ &\leq 2\operatorname{Re}a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h\right)+2\left(|\kappa|+2\|\nabla\eta\|^2_{L^\infty}\right)\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.\end{aligned} \tag{4.4}$$

Since $|\kappa|+\|\nabla\eta\|^2_{L^\infty}\lesssim(\varepsilon R)^{-2}$ with implied constant depending on κ, we are left with estimating Re $a(\eta\mathbf{E}_h,\eta\mathbf{E}_h)$.

Step 2: Using the orthogonality relation in the definition of the space $\mathcal{H}_{c,h}(B_{(1+\varepsilon)R}\cap\Omega)$, we get

$$\begin{aligned}\operatorname{Re}a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h\right) &= \operatorname{Re}a\left(\mathbf{E}_h,\eta^2\mathbf{E}_h-\mathfrak{r}_\Omega\left(\eta^2\mathbf{E}_h\right)\right)\\ &\lesssim \|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\left\|\nabla\times\left(\eta^2\mathbf{E}_h-\mathfrak{r}_\Omega\left(\eta^2\mathbf{E}_h\right)\right)\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\\ &\quad+\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\left\|\eta^2\mathbf{E}_h-\mathfrak{r}_\Omega\left(\eta^2\mathbf{E}_h\right)\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.\end{aligned} \tag{4.5}$$

For each element $T\in\mathcal{T}_h$, Lemma 3.9 yields

$$\begin{aligned}&\left\|\eta^2\mathbf{E}_h-\mathfrak{r}_\Omega\left(\eta^2\mathbf{E}_h\right)\right\|^2_{\mathbf{L}^2(T)}+\left\|\nabla\times\left(\eta^2\mathbf{E}_h-\mathfrak{r}_\Omega\left(\eta^2\mathbf{E}_h\right)\right)\right\|^2_{\mathbf{L}^2(T)}\\ &\qquad\lesssim h^2\left(|\eta^2\mathbf{E}_h|^2_{\mathbf{H}^1(T)}+|\nabla\times\left(\eta^2\mathbf{E}_h\right)|^2_{\mathbf{H}^1(T)}\right).\end{aligned} \tag{4.6}$$

To proceed further, we observe that $\mathbf{E}_h|_T\in\mathcal{N}_0(T)$ has the form $\mathbf{E}_h=\mathbf{a}+\mathbf{b}\times\mathbf{x}$ so that curl $\mathbf{E}_h|_T=2\mathbf{b}$ and hence $\sum_{j=1}^3|\partial_{x_j}\mathbf{E}_h|\lesssim|\nabla\times\mathbf{E}_h|$ pointwise on T so that we get with an implied constant independent of the function η

$$\sum_{j=1}^{3} \left\|\eta \partial_{x_j} \mathbf{E}_h\right\|_{\mathbf{L}^2(T)} \lesssim \|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(T)}. \tag{4.7}$$

Using (4.7) we obtain

$$|\eta^2 \mathbf{E}_h|_{\mathbf{H}^1(T)} \lesssim \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(T)} + \|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(T)}. \tag{4.8}$$

Computing $\nabla \times \left(\eta^2 \mathbf{E}_h\right) = \nabla \eta^2 \times \mathbf{E}_h + \eta^2 \nabla \times \mathbf{E}_h$, using the product rule and the fact that $\partial_{x_j}(\nabla \times \mathbf{E}_h) = 0$ since $\nabla \times \mathbf{E}_h$ is constant gives again in view of (4.7) and $\varepsilon R \lesssim 1$

$$|\nabla \times \left(\eta^2 \mathbf{E}_h\right)|_{\mathbf{H}^1(T)} \lesssim \frac{1}{(\varepsilon R)^2}\|\mathbf{E}_h\|_{\mathbf{L}^2(T)} + \frac{1}{\varepsilon R}\|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(T)}. \tag{4.9}$$

Summing the squares of (4.8), (4.9) over all elements T with $T \cap \operatorname{supp} \eta \neq \emptyset$, which is ensured if we sum over all T with $T \subset B_{(1+\varepsilon)R} \cap \Omega$, and inserting the result in (4.6) yields

$$\begin{aligned}\operatorname{Re} a\left(\mathbf{E}_h, \eta^2 \mathbf{E}_h - \mathfrak{r}_\Omega\left(\eta^2 \mathbf{E}_h\right)\right) \lesssim &\left(\|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}\right) \\ &\times \frac{h}{\varepsilon R}\left(\frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \|\eta(\nabla \times \mathbf{E}_h)\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}\right).\end{aligned}$$

Using Young's inequality, $h \lesssim 1$ and $0 \leq \eta \leq 1$ as well as the definition of the norm $|||\cdot|||_{c,h,R}$, we obtain

$$\begin{aligned}\operatorname{Re} a\left(\mathbf{E}_h, \eta^2 \mathbf{E}_h - \mathfrak{r}_\Omega\left(\eta^2 \mathbf{E}_h\right)\right) \lesssim &\frac{h^2}{(\varepsilon R)^2}\|\nabla \times \mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{(\varepsilon R)^2}\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} \\ &+ \frac{h}{\varepsilon R}\|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}\|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} \\ \lesssim \varepsilon^{-2}|||\mathbf{E}_h|||^2_{c,h,(1+\varepsilon)R} + \varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+\varepsilon)R}&\|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}.\end{aligned}$$

Inserting this in (4.4) produces

$$\begin{aligned}\|\nabla \times \mathbf{E}_h\|^2_{\mathbf{L}^2(B_R \cap \Omega)} &\leq \|\eta \nabla \times \mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} \\ &\lesssim \varepsilon^{-2}|||\mathbf{E}_h|||^2_{c,h,(1+\varepsilon)R} + \varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+\varepsilon)R}\|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}.\end{aligned}$$

Using again Young's inequality to kick the term $\|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}$ of the right-hand side back to the left-hand side produces the desired estimate. □

For functions in $\mathcal{H}_{g,h}(B_{(1+\varepsilon)R} \cap \Omega)$, a discrete Caccioppoli-type estimate has already been established in [16, Lem. 2], which we state in the following for the sake of completeness.

Lemma 4.2 ([16, Lem. 2]) Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{4}$. Let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes and $p_h \in \mathcal{H}_{g,h}(B_{(1+\varepsilon)R} \cap \Omega)$. Then, there exists a constant $C > 0$ depending only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$ such that

$$\|\nabla p_h\|_{\mathbf{L}^2(B_R \cap \Omega)} \leq C\frac{1+\varepsilon}{\varepsilon}|||p_h|||_{g,h,(1+\varepsilon)R}.$$

4.2 Low-dimensional approximation in $\mathcal{H}_{c,h}(B_R \cap \Omega)$.

In this subsection, we apply the Caccioppoli-type estimates from Lemmas 4.1 and 4.2 to find approximations of the Galerkin solutions from low-dimensional spaces. We will need a Poincaré inequality as given in [24, (7.45)]: for open sets $D \subset \omega$ with $|D| > 0$ and $u \in H^1(\omega)$, we have

$$\left\| u - \frac{1}{|D|}\int_D u dx \right\|_{L^2(\omega)} \lesssim |D|^{-2/3}(\mathrm{diam}(D))^3 \|\nabla u\|_{L^2(\omega)}. \tag{4.10}$$

In the following, we consider low-dimensional approximations of discrete harmonic functions in Lemma 4.3 that generalizes [16, Lem. 4].

Lemma 4.3 Let $\varepsilon \in (0,1)$, $q \in (0,1)$, $R \in (0, 2\mathrm{diam}(\Omega))$, and $m \in \mathbb{N}$ satisfy

$$\frac{h}{R} \le \frac{q\varepsilon}{8m \max\left\{1, C_{\mathrm{app}}\right\}}, \tag{4.11}$$

where the constant C_{app} is given in [16, Lem. 3, Lem. 4] and depends only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$. Let B_R, $B_{(1+\varepsilon)R}$, $B_{(1+2\varepsilon)R}$ be concentric boxes. Then, there exists a subspace W_m of $\mathcal{H}_{g,h}(B_R \cap \Omega)$ of dimension

$$\dim W_m \le C'_{\dim}\left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4$$

with the following approximation properties:

(i) If $u_h \in \mathcal{H}_{g,h}(B_{(1+\varepsilon)R} \cap \Omega)$ and $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c = \emptyset$, then

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \le C'_{\mathrm{app}} q^m \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)}.$$

(ii) If $u_h \in \mathcal{H}_{g,h}(B_{(1+2\varepsilon)R} \cap \Omega)$ and $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c \ne \emptyset$, then

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \le C'_{\mathrm{app}} q^m \varepsilon^{-2} \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R} \cap \Omega)}.$$

Here, $C'_{\dim}$, C'_{app} depend only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$.

Proof We start with the case of boxes not entirely contained in Ω.

Case 1: Let $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c \ne \emptyset$. For the Lipschitz domain Ω, [37, Chap. VI, Sec. 3, Thm. 5'] asserts the existence of a bounded linear extension operator $\mathcal{E}_{\Omega^c} : H^1(\Omega^c) \to H^1(\mathbb{R}^3)$ such that $\mathcal{E}_{\Omega^c} v|_{\Omega^c} = v$ for each $v \in H^1(\Omega^c)$. The fact that Ω^c is Lipschitz (see [31, Thm. 2] for details) implies the existence of a constant $c > 0$ depending only on Ω such that for all $x \in \Omega^c$ and all $r \in (0,1)$ we have $|B_r(x) \cap \Omega^c| \ge cr^3$, where $B_r(x)$ denotes the ball of radius r centered at x. Selecting an $x \in B_{(1+\varepsilon)R} \cap \Omega^c$ and noting that $B_{\varepsilon R/2}(x) \subset B_{(1+2\varepsilon)R}$, we conclude

$$|B_{(1+2\varepsilon)R} \cap \Omega^c| \ge |B_{\varepsilon R/2}(x) \cap \Omega^c| \ge c(\varepsilon R/2)^3.$$

Due to (4.11), [16, Lem. 4] provides a subspace W_m of $\mathcal{H}_{g,h}(B_R \cap \Omega)$ such that

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \leq q^m |||u_h|||_{g,h,(1+\varepsilon)R}, \tag{4.12}$$

$$\dim W_m \leq C_{\dim} \left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4, \tag{4.13}$$

where $C_{\dim}$ depends only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$. We denote by $\widehat{u}_h$ the extension by zero of u_h to Ω^c. It follows from the Poincaré inequality (4.10) and $\left|B_{(1+2\varepsilon)R} \cap \Omega^c\right| \gtrsim (\varepsilon R)^3$ that

$$\begin{aligned} \frac{1}{R}\|u_h\|_{L^2(B_{(1+\varepsilon)R}\cap\Omega)} &\leq \frac{1}{R}\|u_h\|_{L^2(B_{(1+2\varepsilon)R}\cap\Omega)} = \frac{1}{R}\|\widehat{u}_h\|_{L^2(B_{(1+2\varepsilon)R})} \\ &\lesssim \frac{\left|B_{(1+2\varepsilon)R}\right|}{R\left|B_{(1+2\varepsilon)R}\cap\Omega^c\right|^{2/3}} \|\nabla\widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})} \\ &\lesssim \frac{(1+2\varepsilon)^3R^3}{\varepsilon^2R^3}\|\nabla\widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})} \lesssim \varepsilon^{-2}\|\nabla\widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})}. \end{aligned} \tag{4.14}$$

Combining (4.14) and (4.12) leads to

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \lesssim \varepsilon^{-2} q^m \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R}\cap\Omega)}. \tag{4.15}$$

Case 2: Let $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c = \emptyset$. We note that constant functions are in $\mathcal{H}_{g,h}(B_R \cap \Omega)$. Hence, by [16, Lem. 4] there is a subspace $W_m \subset \mathcal{H}_{g,h}(B_R \cap \Omega)$ such that $1 \in W_m$ and

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} = \min_{\widetilde{u}_m \in W_m, c\in\mathbb{R}} |||u_h - \widetilde{u}_m + c|||_{g,h,R} \leq q^m \min_{c\in\mathbb{R}} |||u_h - c|||_{g,h,(1+\varepsilon)R} \tag{4.16}$$

with dimension

$$\dim W_m \leq C_{\dim}\left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4 + 1 \lesssim \left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4.$$

A standard Poincaré inequality (i.e., (4.10) with $D = B_{(1+\varepsilon)R}$) implies

$$\begin{aligned} \min_{c\in\mathbb{R}} |||u_h - c|||_{g,h,(1+\varepsilon)R} &\leq \left|\left|\left| u_h - \frac{1}{\left|B_{(1+\varepsilon)R}\right|}\int_{B_{(1+\varepsilon)R}} u_h \right|\right|\right|_{g,h,(1+\varepsilon)R} \\ &\lesssim \frac{\left|B_{(1+\varepsilon)R}\right|}{R\left|B_{(1+\varepsilon)R}\right|^{2/3}}\|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{h}{(1+\varepsilon)R}\|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \\ &\lesssim \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}. \end{aligned} \tag{4.17}$$

Combining (4.17) and (4.16) completes the proof. □

Remark 4.4 The factor ε^{-2} instead of ε^{-0} for boxes B_R near the boundary is a consequence of not assuming a relation between the orientation of the boxes and the boundary. Aligning boxes with the boundary allows one to better exploit boundary conditions and improve the factor ε^{-2}.

In the following, we will need a simplified version of Lemma 4.3:

Corollary 4.5 Let $R \in (0,2\mathrm{diam}(\Omega))$, $\varepsilon \in (0,1)$, $q \in (0,1)$. There are constants C''_{dim} and C''_{app} depending only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$ such that, for any concentric boxes B_R, $B_{(1+2\varepsilon)R}$ and any $m \in \mathbb{N}$, there exists a subspace $W_m \subset \mathcal{H}_{g,h}(B_R \cap \Omega)$ of dimension

$$\dim W_m \le C''_{\mathrm{dim}}(\varepsilon q)^{-3} m^4$$

such that for any $u_h \in \mathcal{H}_{g,h}(B_{(1+2\varepsilon)R} \cap \Omega)$ there holds

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \le C''_{\mathrm{app}} q^m \varepsilon^{-2} \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R} \cap \Omega)}. \tag{4.18}$$

Proof The case that the parameters satisfy (4.11) is covered by Lemma 4.3. For the converse case $h/R > q\varepsilon/(8m\max\{1, C_{\mathrm{app}}\})$, we take $W_m := \mathcal{H}_{g,h}(B_R \cap \Omega)$ so that the minimum in (4.18) is zero and observe in view of the quasi-uniformity of $\mathcal{T}_h$

$$\dim \mathcal{H}_{g,h}(B_R \cap \Omega) \lesssim \left(\frac{R}{h}\right)^3 \lesssim \left(\frac{m}{\varepsilon q}\right)^3 = (\varepsilon q)^{-3} m^3 \le (\varepsilon q)^{-3} m^4,$$

which finishes the proof. □

If $\mathbf{E}_h$ is locally discrete divergence-free, then the function $\nabla(p + \varphi_{\mathbf{z}})$ in the decomposition $\mathbf{E}_h = \mathbf{z} - \nabla\varphi_{\mathbf{z}} + \nabla(p + \varphi_{\mathbf{z}})$ given by Definition 3.6 is also locally discrete divergence-free since $\mathbf{z} - \nabla\varphi_{\mathbf{z}}$ is divergence-free. The following lemma shows that $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p + \varphi_{\mathbf{z}})$ is discrete divergence-free as well:

Lemma 4.6 Let $\varepsilon \in (0,1)$, $R \in (0,2\mathrm{diam}(\Omega))$, and let $B_{(1+j\varepsilon)R}$, $j \in \{0,1,2\}$, be concentric boxes. Introduce $\mathcal{T}_h(B_{(1+2\varepsilon)R} \cap \Omega)$ and $\widetilde{B}_{(1+2\varepsilon)R}$ according to Assumption 3.4. Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $\eta \equiv 1$ on $B_{(1+2\varepsilon)R}$. Let $\mathbf{E}_h$ be such that $\eta \mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ and $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+2\varepsilon)R} \cap \Omega)$. Decompose $\eta\mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl},\Omega)$ as $\eta\mathbf{E}_h = \mathbf{z} + \nabla p$ with $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ and $p \in H^1_0(\Omega)$ according to Lemma 3.1. Let the mapping $\varphi_{\mathbf{z}} : \mathbf{H}^1_0(\Omega) \to H^1_0(\Omega)$ be defined according to (3.1) taking $\widetilde{\eta} \equiv \eta$ there. Then, $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p + \varphi_{\mathbf{z}})$ is discrete divergence-free on $\widetilde{B}_{(1+2\varepsilon)R}$, i.e.,

$$\langle \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p + \varphi_{\mathbf{z}}), \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} = 0 \quad \forall v_h \in S^{1,1}(\mathcal{T}_h, \widetilde{B}_{(1+2\varepsilon)R}), \quad \mathrm{supp}\, v_h \subset \overline{\widetilde{B}_{(1+2\varepsilon)R}}. \tag{4.19}$$

Proof We use $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+2\varepsilon)R} \cap \Omega)$ and (3.6) so that, for $v_h \in S^{1,1}(\mathcal{T}_h, \widetilde{B}_{(1+2\varepsilon)R})$ with $\mathrm{supp} v_h \subset \overline{\widetilde{B}_{(1+2\varepsilon)R}}$, we have

$$\begin{aligned}
0 &= a(\mathbf{E}_h, \nabla v_h) = \langle \nabla \times \mathbf{E}_h, \nabla \times \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} - \kappa \langle \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} = -\kappa \langle \eta \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle \mathbf{z} + \nabla p, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle \mathbf{z} - \nabla\varphi_{\mathbf{z}} + \nabla\varphi_{\mathbf{z}} + \nabla p, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle (\mathbf{z} - \nabla\varphi_{\mathbf{z}}) + \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} (\nabla\varphi_{\mathbf{z}} + \nabla p), \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&\overset{\text{Lem. 3.2}}{=} -\kappa \langle \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} (\nabla\varphi_{\mathbf{z}} + \nabla p), \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})},
\end{aligned}$$

which finishes the proof. □

We will make use of the orthogonal projection

$$\Pi_{B_R} : (\mathbf{H}(\mathrm{curl}, B_R \cap \Omega), |||\cdot|||_{c,h,R}) \to (\mathcal{H}_{c,h}(B_R \cap \Omega), |||\cdot|||_{c,h,R}), \tag{4.20}$$

where orthogonality is defined in terms of the inner product associated with the weighted norm $|||\cdot|||_{c,h,R}$.

Lemma 4.7 (single-step approximation) Let $\varepsilon \in (0,1)$, $R > 0$ be such that $(1+4\varepsilon)R \in (0, R_{\max}]$, and $q \in (0,1)$. Let $B_{(1+j\varepsilon)R}$, $j = 0,\ldots,4$, be concentric boxes. Then, there exists a family of linear spaces $\mathbf{V}_{H,m} \subset \mathcal{H}_{c,h}(B_R \cap \Omega)$ (parametrized by $H > 0$, $m \in \mathbb{N}$) with the following approximation properties: For each $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+4\varepsilon)R} \cap \Omega)$ there is a $\mathbf{E}_{1,h} \in \mathbf{V}_{H,m} \subset \mathcal{H}_{c,h}(B_R \cap \Omega)$ with

(i) $(\mathbf{E}_h - \mathbf{E}_{1,h})|_{B_R \cap \Omega} \in \mathcal{H}_{c,h}(B_R \cap \Omega)$,
(ii) $|||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,R} \le C''_{\mathrm{app}} \left(\frac{H}{R}\varepsilon^{-1} + q^m \varepsilon^{-3} \right) |||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}$,
(iii) $\dim \mathbf{V}_{H,m} \le C''_{\dim} \left[\left(\frac{R}{H} \right)^3 + (\varepsilon q)^{-3} m^4 \right]$,

where the constants C''_{app} and $C''_{\dim}$ depend only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$. Furthermore,

(iv) if $h \ge H$ or $h/R \ge \varepsilon/4$, one may actually take $\mathbf{V}_{H,m} = \mathcal{H}_{c,h}(B_R \cap \Omega)$ and $\mathbf{E}_{1,h}$ may be taken as $\mathbf{E}_{1,h} = \mathbf{E}_h|_{B_R \cap \Omega}$.

Proof **Step 1:** (reduction to $h < H$) As a preliminary step, we show (iv) so that afterwards we may restrict our attention to the case $h < H$ together with $h/R < \varepsilon/4$. If $h \ge H$ or $h/R \ge \varepsilon/4$, we take $\mathbf{V}_{H,m} := \mathcal{H}_{c,h}(B_R \cap \Omega)$, which implies that the choice $\mathbf{E}_{1,h} = \mathbf{E}_h|_{B_R \cap \Omega}$ is admissible so that $\mathbf{E}_h - \mathbf{E}_{1,h} = 0$. Since either $h \ge H$ or $h/R \ge \varepsilon/4$, we have

$$\dim \mathcal{H}_{c,h}(B_R \cap \Omega) \lesssim \left(\tfrac{R}{h}\right)^3 \lesssim \left(\tfrac{R}{H}\right)^3 + \varepsilon^{-3} \lesssim \left(\tfrac{R}{H}\right)^3 + (\varepsilon q)^{-3}, \tag{4.21}$$

which shows that the complexity bound in (iii) is satisfied. We have thus shown (iv) and will assume $h < H$ and $h/R < \varepsilon/4$ for the remainder of the proof.

Step 2: (reduction to $H/R \le \varepsilon/4$) For $\frac{H}{R} > \frac{\varepsilon}{4}$, we may take the space constructed below with the choice $\frac{H}{R} = \frac{\varepsilon}{4}$ since then, the approximation property (ii) and the complexity estimate (iii) are still satisfied. Therefore, we assume in the remainder that $\frac{H}{R} \le \frac{\varepsilon}{4}$.

Step 3: (Scott-Zhang approximation on $\mathbb{R}^3$) Let $\mathcal{M}_H$ be a quasi-uniform infinite triangulation of $\mathbb{R}^3$ with mesh width H. Define further $\mathbf{S}^{1,1}(\mathcal{M}_H) := \{\mathbf{p}_H \in \mathbf{H}^1(\mathbb{R}^3) : \mathbf{p}_H|_M \in (\mathcal{P}_1(M))^3 \quad \forall M \in \mathcal{M}_H\}$. We will use the Scott–Zhang projection operator $\mathbf{I}_H^{SZ} : \mathbf{H}^1(\mathbb{R}^3) \to \mathbf{S}^{1,1}(\mathcal{M}_H)$ introduced in [38]. Denoting by ω_M the element patch of $M \in \mathcal{M}_H$, this operator has the local approximation property

$$\left\|\mathbf{U}-\mathbf{I}_H^{\mathrm{SZ}}\mathbf{U}\right\|_{\mathbf{L}^2(M)}^2 \leq CH^2\|\mathbf{U}\|_{\mathbf{H}^1(\omega_M)}^2 \qquad \forall \mathbf{U}\in\mathbf{H}^1(\omega_M) \tag{4.22}$$

with a constant C depending only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathscr{M}_H$. Let $\boldsymbol{\mathcal{E}}:\ \mathbf{H}^1(\Omega)\to\mathbf{H}^1(\mathbb{R}^3)$ be an $\mathbf{H}^1$-stable extension operator such as the one from [37, Chap. VI, Sec. 3, Thm. 5'].

Step 4: Let $\mathcal{T}_h(B_{(1+2\varepsilon)R}\cap\Omega)$ and $\widetilde{B}_{(1+2\varepsilon)R}$ be given according to Assumption 3.4. Let $\eta\in C^\infty(\overline{\Omega})$ be a cut-off function with $\operatorname{supp}\eta\subseteq\overline{B_{(1+3\varepsilon)R}\cap\Omega}$, $\eta\equiv 1$ on $\widetilde{B}_{(1+2\varepsilon)R}$, $0\leq\eta\leq 1$ and $\|\nabla^\ell\eta\|_{L^\infty(\Omega)}\lesssim\frac{1}{(\varepsilon R)^\ell}$ for $\ell\in\{0,1,2\}$. Note that $\eta\mathbf{E}_h\in\mathbf{H}_0(\mathrm{curl},\Omega)$. Decompose $\eta\mathbf{E}_h\in\mathbf{H}_0(\mathrm{curl},\Omega)$ as $\eta\mathbf{E}_h=\mathbf{z}+\nabla p$ with $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$ and $p\in H_0^1(\Omega)$ according to Lemma 3.1. Let $\varphi_{\mathbf{z}}$ be given by (3.1) taking $\widetilde{\eta}=\eta$ there. Select representers $p_h,\ \varphi_{\mathbf{z},h}\in S_0^{1,1}(\mathcal{T}_h)$ such that $\nabla p_h=\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla p$ and $\nabla\varphi_{\mathbf{z},h}=\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}}$ on $\widetilde{B}_{(1+2\varepsilon)R}$. By Lemma 4.6, we have that $\nabla(p_h+\varphi_{\mathbf{z},h})$ is discrete divergence-free on $\widetilde{B}_{(1+2\varepsilon)R}$ so that $(p_h+\varphi_{\mathbf{z},h})\in\mathcal{H}_{g,h}(B_{(1+2\varepsilon)R}\cap\Omega)$. We apply Corollary 4.5 with the pair (R,ε) replaced with $(\widetilde{R},\widetilde{\varepsilon})=(R(1+\varepsilon),\frac{\varepsilon}{2(1+\varepsilon)})$ to get a subspace $W_m\subset\mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$ for the box $B_{(1+\varepsilon)R}\cap\Omega$ and an $w_m\in W_m$ such that

$$|||p_h+\varphi_{\mathbf{z},h}-w_m|||_{g,h,(1+\varepsilon)R}\lesssim q^m\varepsilon^{-2}\|\nabla(p_h+\varphi_{\mathbf{z},h})\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R}\cap\Omega)}. \tag{4.23}$$

Step 5: Define $\mathbf{z}_H:=(\mathbf{I}_H^{SZ}\boldsymbol{\mathcal{E}}\mathbf{z})|_{B_{(1+4\varepsilon)R}\cap\Omega}$. Using Definition 3.6 and with the function $\varphi_{\mathbf{z}_H}$ given by (3.1) (again, with $\widetilde{\eta}=\eta$ there) we have the representation

$$\begin{aligned}\mathbf{E}_h|_{\widetilde{B}_{(1+2\varepsilon)R}} &= \mathbf{z}_h+\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla p=(\mathbf{z}_h-\mathbf{z})+\mathbf{z}-\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}}+\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(\varphi_{\mathbf{z}}+p)\\ &= (\mathbf{z}_h-\mathbf{z})+(\mathbf{z}-\mathbf{z}_H)-\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla\varphi_{\mathbf{z}}-\nabla\varphi_{\mathbf{z}_H})\\ &\quad -\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H}+\mathbf{z}_H+\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(\varphi_{\mathbf{z}}+p).\end{aligned}$$

Of these 6 terms, the first three terms are shown to be small, the next two terms are from a low-dimensional space, and the last term is exponentially (in m) close to ∇w_m by (4.23), which is also from a low-dimensional space, namely, ∇W_m. As the approximation of $\mathbf{E}_h$, we thus take

$$\mathbf{E}_{1,h}:=\boldsymbol{\Pi}_{B_R}\left(-\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H}+\mathbf{z}_H+\nabla w_m\right), \tag{4.24}$$

with the $|||\cdot|||_{c,h,R}$-orthogonal projection $\boldsymbol{\Pi}_{B_R}$ of (4.20). Property (i) is then satisfied by construction. In order to prove (ii), we compute

$$\begin{aligned}|||\mathbf{E}_h-\mathbf{E}_{1,h}|||_{c,h,R} &= \left|\left|\left|\boldsymbol{\Pi}_{B_R}\left(\mathbf{E}_h+\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H}-\mathbf{z}_H-\nabla w_m\right)\right|\right|\right|_{c,h,R}\\ &\leq \left|\left|\left|\mathbf{E}_h+\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H}-\mathbf{z}_H-\nabla w_m\right|\right|\right|_{c,h,R}\\ &\leq |||\mathbf{z}_h-\mathbf{z}|||_{c,h,R}+|||\mathbf{z}-\mathbf{z}_H|||_{c,h,R}+\left|\left|\left|\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla\varphi_{\mathbf{z}}-\nabla\varphi_{\mathbf{z}_H})\right|\right|\right|_{c,h,R}\\ &\quad+\left|\left|\left|\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_{\mathbf{z}})-\nabla w_m\right|\right|\right|_{c,h,R}.\end{aligned} \tag{4.25}$$

Step 6: (stability estimates) The stability estimate (3.7) for p_h in the local discrete regular decomposition implies together with Lemma 3.1

$$\|\nabla p_h\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} + \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \overset{(3.7)}{\lesssim} \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \lesssim \|\eta \mathbf{E}\|_{\mathbf{L}^2(\Omega)}. \tag{4.26}$$

By Lemma 3.11 and the Caccioppoli-type estimate of Lemma 4.1 (replacing the pairs (R,ε) there with suitably adjusted $(\widetilde{R},\widetilde{\varepsilon})$ as needed), we have

$$\begin{aligned}\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},B_R)} + \|\mathbf{z}\|_{\mathbf{H}^1_0(\Omega)} &\lesssim \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} \\ &\underset{\text{Lem. 4.1}}{\lesssim} \varepsilon^{-1} |||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}.\end{aligned} \tag{4.27}$$

Finally, combining Lemmas 3.1, 3.2, and estimate (3.7) leads to

$$\|\nabla \varphi_{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \varphi_{\mathbf{z},h}\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \lesssim \|\mathbf{z}\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} \lesssim \|\eta \mathbf{E}\|_{\mathbf{L}^2(\Omega)} \tag{4.28}$$

as well as

$$\left\|\nabla(\varphi_{\mathbf{z}} - \varphi_{\mathbf{z}_H})\right\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{z} - \mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)}. \tag{4.29}$$

Step 7: (controlling $\mathbf{z} - \mathbf{z}_h$) By Lemma 3.11 and (4.27), we have

$$\frac{1}{R}\|\mathbf{z} - \mathbf{z}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \lesssim \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \tag{4.30}$$

Noting $\nabla \times \mathbf{z} = \nabla \times (\eta \mathbf{E}_h)$ together with the definition of $|||\cdot|||_{c,h,R}$ and the estimate (4.30), we obtain

$$\begin{aligned}|||\mathbf{z} - \mathbf{z}_h|||_{c,h,R} &\leq \frac{h}{R}\left(\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},B_R)} + \|\nabla \times \mathbf{z}\|_{\mathbf{L}^2(B_R\cap\Omega)}\right) + \frac{1}{R}\|\mathbf{z} - \mathbf{z}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \\ &\leq \frac{h}{R}\left(\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},B_R)} + \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\eta \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right) \\ &\quad + \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}.\end{aligned} \tag{4.31}$$

Combining this with Lemma 4.1 and the stability estimate (4.27) leads to

$$|||\mathbf{z} - \mathbf{z}_h|||_{c,h,R} \lesssim \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \tag{4.32}$$

Step 8: (controlling $\mathbf{z} - \mathbf{z}_H$ and $\nabla(\varphi_{\mathbf{z}} - \varphi_{\mathbf{z}_H})$) For $\mathbf{z}_H = (\mathbf{I}_H^{SZ}\mathcal{E}\mathbf{z})|_{B_{(1+4\varepsilon)R}\cap\Omega}$, we have by the approximation result (4.22), the assumption $H/R \leq \varepsilon/4$, and the stability properties of $\mathbf{I}_H^{SZ}$

$$\frac{1}{R}\|\mathbf{z} - \mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+j\varepsilon)R}\cap\Omega)} \lesssim \frac{H}{R}\|\mathcal{E}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})}, \qquad j = 0, \dots, 3, \tag{4.33}$$

$$\frac{h}{R}\|\mathbf{z} - \mathbf{z}_H\|_{\mathbf{H}^1(B_{(1+j\varepsilon)R}\cap\Omega)} \lesssim \frac{h}{R}\|\mathcal{E}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})}, \qquad j = 0, \dots, 3, \tag{4.34}$$

so that, using $\|\mathcal{E}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+4\varepsilon)R})} \lesssim \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)}$, we obtain for $j=0,\dots,3$

$$\|\!|\mathbf{z}-\mathbf{z}_H|\!\|_{c,h,(1+j\varepsilon)R} \lesssim \left(\frac{h}{R}+\frac{H}{R}\right)\|\mathcal{E}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})} \overset{(4.27)}{\lesssim} \left(\frac{h}{R}+\frac{H}{R}\right)\varepsilon^{-1}\|\!|\mathbf{E}_h|\!\|_{c,h,(1+4\varepsilon)R}. \tag{4.35}$$

By the stability properties of the operator $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}$ given in (3.7) and (4.29), we infer

$$\begin{aligned}\left\|\!\left|\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(\varphi_{\mathbf{z}}-\varphi_{\mathbf{z}_H})\right|\!\right\|_{c,h,R} &\leq \frac{1}{R}\left\|\nabla(\varphi_{\mathbf{z}}-\varphi_{\mathbf{z}_H})\right\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \overset{(4.29)}{\lesssim} \frac{1}{R}\|\mathbf{z}-\mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)}\\ &\overset{(4.35)}{\lesssim} \left(\tfrac{h}{R}+\tfrac{H}{R}\right)\varepsilon^{-1}\|\!|\mathbf{E}_h|\!\|_{c,h,(1+4\varepsilon)R}.\end{aligned} \tag{4.36}$$

Step 9: (Estimate of $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_{\mathbf{z}})-\nabla w_m$) By Step 4, we have $p_h+\varphi_{\mathbf{z},h}-w_m \in \mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$.
Noting $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_{\mathbf{z}})-\nabla w_m = \nabla(p_h+\varphi_{\mathbf{z},h}-w_m)$ on $B_{(1+\varepsilon)R}\cap\Omega$, we get

$$\begin{aligned}\left\|\!\left|\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_{\mathbf{z}})-\nabla w_m\right|\!\right\|_{c,h,R} &= \tfrac{1}{R}\left\|\nabla(p_h+\varphi_{\mathbf{z},h}-w_m)\right\|_{\mathbf{L}^2(B_R\cap\Omega)}\\ &\overset{\text{Lem. 4.2}}{\lesssim} \frac{1+\varepsilon}{\varepsilon R}\|\!|(p_h+\varphi_{\mathbf{z},h})-w_m|\!\|_{g,h,(1+\varepsilon)R}\\ &\overset{(4.23)}{\lesssim} \frac{q^m\varepsilon^{-2}(1+\varepsilon)}{\varepsilon R}\left\|\nabla(p_h+\varphi_{\mathbf{z},h})\right\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R}\cap\Omega)}\\ &\overset{(4.26),(4.28)}{\lesssim} \frac{q^m\varepsilon^{-2}}{\varepsilon R}\left\|\eta\mathbf{E}_h\right\|_{\mathbf{L}^2(\Omega)} \lesssim \frac{q^m\varepsilon^{-2}}{\varepsilon R}\left\|\mathbf{E}_h\right\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)}\\ &\lesssim q^m\varepsilon^{-3}\|\!|\mathbf{E}_h|\!\|_{c,h,(1+3\varepsilon)R}.\end{aligned} \tag{4.37}$$

Substituting (4.32), (4.35), (4.36) and (4.37) into (4.25) concludes the proof of (ii).

Step 10: By construction, the approximation $\mathbf{E}_{1,h}$ of (4.24) is from the space

$$\mathbf{V}_{H,m} := \left\{\mathbf{\Pi}_{B_R}\left(-\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H}+\mathbf{z}_H+\nabla w_m\right) : \mathbf{z}_H \in \left(\mathbf{I}^{\mathrm{SZ}}_H\mathbf{H}^1(\mathbb{R}^3)\right)|_{B_{(1+4\varepsilon)R}\cap\Omega}, w_m \in \nabla W_m\right\}.$$

By the linearity of the maps $\mathbf{\Pi}_{B_R}$, $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}$, and $\mathbf{z}\mapsto\varphi_{\mathbf{z}}$, the space $\mathbf{V}_{H,m}$ is a linear space. In view of $\dim W_m \lesssim (\varepsilon q)^{-3}m^4$ from Corollary 4.5 and $\dim \mathbf{I}^{SZ}_H\mathcal{E}(\mathbf{H}^1(\Omega))|_{B_{(1+4\mathcal{E})R}\cap\Omega} \lesssim \left(\frac{(1+4\mathcal{E})R}{H}\right)^3$ we get (iii). ☐

Lemma 4.8 (multi-step approximation) Let $\zeta \in (0,1)$, $q' \in (0,1)$, $R \in (0, R_{\max}/2]$. Then, for each $k \in \mathbb{N}$, there exists a subspace $\mathbf{V}_k$ of $\mathcal{H}_{c,h}(B_R\cap\Omega)$ of dimension

$$\dim \mathbf{V}_k \leq C'''_{\dim}k\left(\tfrac{k}{\zeta}\right)^3\left(q'^{-3}+\ln^4\tfrac{k}{\zeta}\right), \tag{4.38}$$

such that for $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+\zeta)R}\cap\Omega)$ there holds

$$\min_{\widetilde{\mathbf{E}}_k\in\mathbf{V}_k}\left\|\!\left|\mathbf{E}_h-\widetilde{\mathbf{E}}_k\right|\!\right\|_{c,h,R} \leq q'^k\|\!|\mathbf{E}_h|\!\|_{h,(1+\zeta)R}. \tag{4.39}$$

Here, $C'''_{\dim}$ depends only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$.

Proof The proof relies on iterating the approximation result of Lemma 4.7 on boxes $B_{(1+\varepsilon_j)R}$, where $\varepsilon_j = \zeta(1-\frac{j}{k})$ for $j=0,\dots,k$. We note that $\zeta = \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_k = 0$. Define

$$\widetilde{R}_j := R(1+\varepsilon_j), \qquad \widetilde{\varepsilon}_j := \frac{\zeta}{4k(1+\varepsilon_j)} < \frac{1}{4}$$

and note the relationship $B_{(1+4\widetilde{\varepsilon}_j)\widetilde{R}_j} = B_{\widetilde{R}_{j-1}} = B_{R(1+\varepsilon_{j-1})}$ as well as $B_{\widetilde{R}_k} = B_R$ and $B_{\widetilde{R}_0} = B_{R(1+\zeta)}$. Also note

$$\frac{\zeta}{8k} \le \frac{\zeta}{4k(1+\zeta)} \le \widetilde{\varepsilon}_j \le \frac{\zeta}{4k}, \qquad R \le \widetilde{R}_j \le (1+\zeta)R, \qquad j = 0,\dots,k.$$

Select $q \in (0,1)$. With the constant C''_{app} of Lemma 4.7 choose

$$H := \frac{q' R\zeta}{8k\max\{1, C''_{\mathrm{app}}\}}, \qquad m := \left\lceil \frac{3\ln(\zeta/(4k)) - \ln\max\{1, C''_{\mathrm{app}}\} + \ln(q'/2)}{\ln q} \right\rceil.$$

These constants are chosen such that

$$C''_{\mathrm{app}} \frac{H}{\widetilde{\varepsilon}_j \widetilde{R}_j} \le \frac{1}{2} q' \qquad \text{and} \qquad C''_{\mathrm{app}} \widetilde{\varepsilon}_j^{-3} q^m \le \frac{1}{2} q'. \tag{4.40}$$

Moreover, the assumption $R \le R_{\max}/2$ implies that $(1+4\widetilde{\varepsilon}_j)\widetilde{R}_j = R(1+\varepsilon_{j-1}) \le R_{\max}$. Therefore, Lemma 4.7 provides a space $\mathbf{V}^1_{H,m} \subset \mathcal{H}_{c,h}(B_{\widetilde{R}_1} \cap \Omega)$ and an approximation $\mathbf{E}_{1,h} \in \mathbf{V}^1_{H,m}$ with

$$\begin{aligned} |||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,\widetilde{R}_1} &\le C''_{\mathrm{app}}\left(\frac{H}{\widetilde{\varepsilon}_1\widetilde{R}_1} + \widetilde{\varepsilon}_1^{-3} q^m\right) |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0} \overset{(4.40)}{\le} q' |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0}, \\ \dim \mathbf{V}^1_{H,m} &\lesssim \left(\frac{\widetilde{R}_1}{H}\right)^3 + \left(\widetilde{\varepsilon}_1 q\right)^{-3} m^4 \le C\left(\frac{k}{\zeta}\right)^3 \left(q'^{-3} + \ln^4(k/\zeta)\right), \end{aligned} \tag{4.41}$$

where the constant $C > 0$ is independent of $j \in \{0,\dots,k\}$, ζ, k, and q'. Since $\mathbf{E}_h - \mathbf{E}_{1,h} \in \mathcal{H}_{c,h}(B_{\widetilde{R}_1} \cap \Omega)$, we may apply Lemma 4.7 again to find a space $\mathbf{V}^2_{H,m} \subset \mathcal{H}_{c,h}(B_{\widetilde{R}_2} \cap \Omega)$ and an approximation $\mathbf{E}_h \in \mathbf{V}^2_{H,m}$ with $\dim \mathbf{V}^2_{H,m} \le C(k/\zeta)^3\left(q'^{-3} + \ln^4(k/\zeta)\right)$ such that

$$|||\mathbf{E}_h - \mathbf{E}_{1,h} - \mathbf{E}_{2,h}|||_{c,h,\widetilde{R}_2} \le q' |||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,\widetilde{R}_1} \le q'^2 |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0}.$$

Repeating this process $k-2$ times leads to the approximation $\widetilde{\mathbf{E}}_k = \sum_{i=1}^k \mathbf{E}_{i,h}$ in the space $\mathbf{V}_k := \sum_{i=1}^k \mathbf{V}^i_{H,m}$ of dimension

$$\dim \mathbf{V}_k \le Ck(k/\zeta)^3\left(q'^{-3} + \ln^4(k/\zeta)\right),$$

which concludes the proof. □

5 Proof of main results

The results of the preceding Section 4 allow us to show that the Galerkin approximation $\mathbf{E}_h$ of (2.7) can be approximated from low-dimensional spaces in regions B_{R_τ} away from the support of the right-hand side $\mathbf{F}$.

Theorem 5.1 Let $h_0 > 0$ be given by Lemma 2.2, and let $\mathcal{T}_h$ be a quasi-uniform mesh with mesh size $h \le h_0$. Fix $q \in (0,1)$ and $\eta > 0$. Set $\zeta = 1/(1+\eta)$. For every cluster pair (τ,σ) with bounding boxes B_{R_τ} and B_{R_σ} with $\eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \ge \operatorname{diam}(B_{R_\tau})$ and each $k \in \mathbb{N}$, there exists a space $\mathbf{V}_k \subset \mathbf{L}^2(B_{R_\tau} \cap \Omega)$ with

$$\dim \mathbf{V}_k \le \widetilde{C}_{\dim} k (k/\zeta)^3 \left(q^{-3} + \ln^4(k/\zeta)\right), \tag{5.1}$$

such that for an arbitrary right-hand side $\mathbf{F} \in \mathbf{L}^2(\Omega)$ with $\operatorname{supp} \mathbf{F} \subset B_{R_\sigma} \cap \overline{\Omega}$, the corresponding Galerkin solution $\mathbf{E}_h$ of (2.7) can be approximated from $\mathbf{V}_k$ such that

$$\min_{\widetilde{\mathbf{E}}_k \in \mathbf{V}_k} \left\| \mathbf{E}_h - \widetilde{\mathbf{E}}_k \right\|_{\mathbf{L}^2(B_{R_\tau} \cap \Omega)} \le C_{\text{box}} q^k \left\| \boldsymbol{\Pi}_h^{L^2} \mathbf{F} \right\|_{\mathbf{L}^2(\Omega)} \le C_{\text{box}} q^k \|\mathbf{F}\|_{\mathbf{L}^2(B_{R_\sigma} \cap \Omega)}.$$

Here, $\boldsymbol{\Pi}_h^{L^2}$ is the $\mathbf{L}^2$-orthogonal projection onto $\mathbf{X}_h(\mathcal{T}_h, \Omega)$ and C_{box}, $\widetilde{C}_{\dim}$ are constants depending only on κ, Ω, and the shape regularity of $\mathcal{T}_h$.

Proof From Lemma 2.2, we have the a priori estimate

$$\|\mathbf{E}_h\|_{\mathbf{H}(\text{curl},\Omega)} \le C \|\boldsymbol{\Pi}_h^{L^2} \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le C \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} = C \|\mathbf{F}\|_{\mathbf{L}^2(B_\sigma \cap \Omega)}.$$

From $\operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \ge \eta^{-1} \operatorname{diam} B_{R_\tau}$, the choice $\zeta = 1/(1+\eta)$ implies

$$\operatorname{dist}(B_{(1+\zeta)R_\tau}, B_{R_\sigma}) \ge \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) - \zeta R_\tau \sqrt{3} \ge \sqrt{3} R_\tau (\eta^{-1} - \zeta) = \sqrt{3} R_\tau \tfrac{1}{\eta(\eta+1)} > 0.$$

Hence, the Galerkin solution $\mathbf{E}_h$ satisfies $\mathbf{E}_h|_{B_{(1+\zeta)R_\tau} \cap \Omega} \in \mathcal{H}_{c,h}(B_{(1+\zeta)R_\tau} \cap \Omega)$. Since $\frac{h}{R_\tau} \lesssim 1$, it is immediate that

$$|||\mathbf{E}_h|||_{h,(1+\zeta)R} \lesssim \left(1 + \tfrac{1}{R_\tau}\right) \|\mathbf{E}_h\|_{\mathbf{H}(\text{curl},\Omega)} \lesssim \left(1 + \tfrac{1}{R_\tau}\right) \left\| \boldsymbol{\Pi}_h^{L^2} \mathbf{F} \right\|_{\mathbf{L}^2(\Omega)}. \tag{5.2}$$

In the following, we employ Lemma 4.8. In order to do so, boxes have to have side length smaller than $R_{\max}/2$, which may not hold for general bounding boxes B_{R_τ}. However, as bounding boxes can always be chosen to satisfy $R_\tau < 2 \operatorname{diam}(\Omega)$, there exists a constant $L \in \mathbb{N}$ independent of R_τ such that $R_\tau / L \le R_{\max}/2$ with $R_{\max}$ given in Definition 3.10. Consequently, we can decompose a box $B_{R_\tau} = \operatorname{int}\left(\overline{\bigcup_{\ell=1}^{C_L} B_{R_{\tau_\ell}}}\right)$ into $C_L \in \mathbb{N}$ subboxes $\left\{B_{R_{\tau_\ell}}\right\}_{\ell=1}^{C_L}$ of side length R_{τ_ℓ} such that $R_{\tau_\ell} \le R_{\max}/2$, where C_L does only depend on L. Then, for each $B_{R_{\tau_\ell}}$, Lemma 4.8 provides a space $\mathbf{V}_{k,\ell} \subset \mathcal{H}_{c,h}(B_{R_{\tau_\ell}} \cap \Omega)$, whose dimension is bounded by (4.38) such that

$$\min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}}\left\|\mathbf{E}_h-\widetilde{\mathbf{E}}_{k,\ell}\right\|_{\mathbf{L}^2(B_{R_{\tau_\ell}}\cap\Omega)}\leq R_{\tau_\ell}\min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}}\left|\left|\left|\mathbf{E}_h-\widetilde{\mathbf{E}}_{k,\ell}\right|\right|\right|_{c,h,R_{\tau_\ell}}\leq Cq^k(R_{\tau_\ell}+1)\left\|\mathbf{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}$$
$$\lesssim \operatorname{diam}(\Omega)q^k\left\|\mathbf{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}.$$

Now, we define the space $\mathbf{V}_k$ as a subspace of $\mathbf{L}^2(B_{R_\tau}\cap\Omega)$ by simply combining all the spaces $\mathbf{V}_{k,\ell}$ of the subboxes, i.e., we extend functions in $\mathbf{V}_{k,\ell}$ by zero to the larger box B_{R_τ} and write $\widehat{\mathbf{V}}_{k,\ell}$ for this space. Then, we can define $\mathbf{V}_k:=\sum_{\ell=1}^{C_L}\widehat{\mathbf{V}}_{k,\ell}$ and set $\widetilde{\mathbf{E}}_k|_{B_{R_{\tau_\ell}}}:=\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}$ for $\widetilde{\mathbf{E}}_k\in\mathbf{V}_k$. This gives

$$\min_{\widetilde{\mathbf{E}}_k\in\mathbf{V}_k}\left\|\mathbf{E}_h-\widetilde{\mathbf{E}}_k\right\|_{\mathbf{L}^2(B_{R_\tau}\cap\Omega)}\leq\sum_{\ell=1}^{C_L}\min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}}\left\|\mathbf{E}_h-\widetilde{\mathbf{E}}_{k,\ell}\right\|_{\mathbf{L}^2(B_{R_{\tau_\ell}}\cap\Omega)}$$
$$\lesssim C_Lq^k\left\|\mathbf{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}.$$

The dimension of $\mathbf{V}_k$ is bounded by

$$\dim\mathbf{V}_k\leq C_LC_{\dim}'''k\left(\frac{k}{\zeta}\right)^3\left(q'^{-3}+\ln^4\frac{k}{\zeta}\right),$$

which concludes the proof. □

The following result allows us to transfer the approximation result of Theorem 5.1 to the matrix level. We recall that the system matrix $\mathbf{A}$ is given by (2.8).

Lemma 5.2 Let $h\leq h_0$ with h_0 given by Lemma 2.2. Then, there are constants $\widetilde{C}_{\dim}$, $\widehat{C}_{\mathrm{app}}$ that depend only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}_h$ such that for $\eta>0$, $q\in(0,1)$, $k\in\mathbb{N}$, and η-admissible cluster pairs (τ,σ) there exist matrices $\mathbf{X}_{\tau\sigma}\in\mathbb{C}^{\tau\times r}$, $\mathbf{Y}_{\tau\sigma}\in\mathbb{C}^{\sigma\times r}$ of rank $r\leq\widetilde{C}_{\dim}(1+\eta)^3k^4\left(q^{-3}+\ln^4(k(1+\eta))\right)$ such that

$$\left\|\mathbf{A}^{-1}|_{\tau\times\sigma}-\mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H\right\|_2\leq\widehat{C}_{\mathrm{app}}h^{-1}q^k.$$

Proof As a preliminary step, we show that we can reduce the consideration to the case $\operatorname{diam}B_{R_\tau}\leq\eta\operatorname{dist}(B_{R_\tau},B_{R_\sigma})$. Indeed, as $\mathbf{A}$ is symmetric also $\mathbf{A}^{-1}$ is symmetric so that $\mathbf{A}^{-1}|_{\tau\times\sigma}=\mathbf{A}^{-1}|_{\sigma\times\tau}$ and one may approximate either $\mathbf{A}^{-1}|_{\tau\times\sigma}$ or $\mathbf{A}^{-1}|_{\sigma\times\tau}$ by a low-rank matrix. In view of the definition of the admissibility condition (2.10), we may therefore assume $\operatorname{diam}B_{R_\tau}\leq\eta\operatorname{dist}(B_{R_\tau},B_{R_\sigma})$.

The matrices $\mathbf{X}_{\tau\sigma}$ and $\mathbf{Y}_{\tau\sigma}$ will be constructed with the aid of Theorem 5.1. In particular, let the constant $\widetilde{C}_{\dim}$ be given from Theorem 5.1. We distinguish between the cases of “small” blocks and “large” blocks.

Case 1. If $\widetilde{C}_{\dim}(1+\eta)^3k^4\left(q^{-3}+\ln^4(k(1+\eta))\right)\geq\min(|\tau|,|\sigma|)$, we use the exact matrix block $\mathbf{X}_{\tau\sigma}=\mathbf{A}^{-1}|_{\tau\times\sigma}$ and we put $\mathbf{Y}_{\tau\sigma}=\mathbf{I}|_{\sigma\times\sigma}$ with $\mathbf{I}\in\mathbb{C}^{N\times N}$ being the identity matrix.

Case 2. If $\widetilde{C}_{\dim}(1+\eta)^3k^4\left(q^{-3}+\ln^4(k(1+\eta))\right)<\min(|\tau|,|\sigma|)$, let $\mathbf{V}_k$ be the space constructed in Theorem 5.1. From $\mathbf{V}_k$ we construct $\mathbf{X}_{\tau\sigma}$ and $\mathbf{Y}_{\tau\sigma}$ in the following two steps.

Step 1. Let functions $\lambda_i\in\mathbf{L}^2(\Omega)$, $i=1,\ldots,N$, satisfy

$$\operatorname{supp}\lambda_i \subset \operatorname{supp}\Psi_i, \qquad i = 1,\dots,N, \tag{5.3a}$$

$$\langle \lambda_i, \Psi_j \rangle_{\mathbf{L}^2(\Omega)} = \delta_{ij}, \qquad i,j = 1,\dots,N, \tag{5.3b}$$

$$\|\lambda_i\|_{\mathbf{L}^2(\Omega)} \le Ch^{-1/2}, \qquad i = 1,\dots,N. \tag{5.3c}$$

Such a dual basis of $\{\Psi_i\colon i = 1,\dots,N\}$ can be constructed as (discontinuous) piecewise polynomials of degree 1 as described in, e.g., [11, Sec. 4.8] for classical Lagrange elements. In fact, $\operatorname{supp}\boldsymbol{\lambda}_i$ can be taken to be a single tetrahedron in $\operatorname{supp}\Psi_i$. The constant C depends solely on the γ-shape regularity of $\mathcal{T}_h$. We emphasize that our choice of scaling of the functions Ψ_i is responsible for the factor $h^{-1/2}$.

For clusters τ', define the mappings

$$\Lambda_{\tau'}: \mathbf{L}^2(\Omega) \to \mathbb{C}^{\tau'}, \quad \mathbf{v} \mapsto \left(\chi_{\tau'}(i)\langle \lambda_i, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}\right)_{i\in\mathcal{I}},$$

where $\chi_{\tau'}$ is the characteristic function of τ'. For $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and a cluster τ' with bounding box $B_{R_{\tau'}}$, we observe for the ℓ^2-norm $\|\cdot\|_2$ on $\mathbb{C}^{\tau'}$ that

$$\|\Lambda_{\tau'}\mathbf{v}\|_2^2 = \sum_{i\in\tau'} |\langle \lambda_i, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}|^2 \le \sum_{i\in\tau'} \|\lambda_i\|^2_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|^2_{\mathbf{L}^2(\operatorname{supp}\lambda_i)} \overset{(5.3c)}{\lesssim} h^{-1}\|\mathbf{v}\|^2_{\mathbf{L}^2(B_{R_{\tau'}}\cap\Omega)}. \tag{5.4}$$

We observe that, for $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ expanded as $\mathbf{E}_h = \sum_{i\in\mathcal{I}} \mu_i \Psi_i$, we have $\mu_i = (\Lambda_{\mathcal{I}}(\mathbf{E}_h))_i$. In particular, we have for the coefficients μ_i with $i \in \tau'$

$$\mu_i = (\Lambda_{\tau'}(\mathbf{E}_h))_i \qquad \forall i \in \tau'. \tag{5.5}$$

Step 2: Let $\mathbf{V}_k$ be the space given by Theorem 5.1 for the boxes B_{R_τ}, B_{R_σ}. For arbitrary $\mathbf{b} \in \mathbb{C}^\sigma$, define the function $\mathbf{f_b} := \sum_{i\in\sigma} \mathbf{b}_i \lambda_i$ and observe:

$$\operatorname{supp}\mathbf{f_b} \overset{(5.3a)}{\subset} B_{R_\sigma}, \tag{5.6a}$$

$$\|\mathbf{f_b}\|_{\mathbf{L}^2(\Omega)} \overset{(5.4)}{\lesssim} h^{-1/2}\|\mathbf{b}\|_2, \tag{5.6b}$$

$$\langle \mathbf{f_b}, \Psi_i \rangle_{\mathbf{L}^2(\Omega)} \overset{(5.3b)}{=} \mathbf{b}_i, \qquad i = 1,\dots,N. \tag{5.6c}$$

Let $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}_h, \Omega)$ be the Galerkin solution corresponding to the right-hand side $\mathbf{f_b}$ and $\widetilde{\mathbf{E}}_h \in \mathbf{V}_k$ be the approximation to $\mathbf{E}_h$ asserted in Theorem 5.1. Then,

$$\begin{aligned}\left\|\Lambda_\tau \mathbf{E}_h - \Lambda_\tau \widetilde{\mathbf{E}}_k\right\|_2 &\overset{(5.4)}{\lesssim} h^{-1/2}\left\|\mathbf{E}_h - \widetilde{\mathbf{E}}_h\right\|_{\mathbf{L}^2(B_{R_\tau}\cap\Omega)} \\ &\overset{\text{Thm. 5.1}}{\lesssim} h^{-1/2} q^k \left\|\mathbf{f_b}\right\|_{\mathbf{L}^2(\Omega)} \overset{(5.6b)}{\lesssim} h^{-1} q^k \|\mathbf{b}\|_2.\end{aligned}$$

We define the low-rank factor $\mathbf{X}_{\tau\sigma}$ as an orthogonal basis of the space $\mathcal{V}_\tau := \{\Lambda_\tau(\widetilde{\mathbf{E}}_k) : \widetilde{\mathbf{E}}_k \in \mathbf{V}_k\}$ and set $\mathbf{Y}_{\tau\sigma} := \mathbf{A}^{-1}|^H_{\tau\times\sigma}\mathbf{X}_{\tau\sigma}$. Then, the rank of $X_{\tau\sigma}$ is bounded by $\dim \mathbf{V}_k \leq \widetilde{C}_{\dim}(1+\eta)^3 k^4 \big(q^{-3} + \ln^4(k(1+\eta))\big)$. Since $\mathbf{X}_{\tau\sigma}\mathbf{X}_{\tau\sigma}^H$ is the orthogonal projection from $\mathbb{C}^N$ onto $\mathcal{V}_\tau$, we conclude that $\mathbf{z} := \mathbf{X}_{\tau\sigma}\mathbf{X}_{\tau\sigma}^H(\Lambda_\tau \mathbf{E}_h)$ is the $\|\cdot\|_2$-best approximation of the Galerkin solution in $\mathcal{V}_\tau$, which results in

$$\left\|\Lambda_\tau \mathbf{E}_h - \mathbf{z}\right\|_2 \lesssim \left\|\Lambda_\tau \mathbf{E}_h - \Lambda_\tau \widetilde{\mathbf{E}}_h\right\|_2 \lesssim h^{-1} q^k \|\mathbf{b}\|_2.$$

By (5.5) and $\mathbf{b} \in \mathbb{C}^\sigma$, we have

$$\Lambda_\tau \mathbf{E}_h \overset{(5.5)}{=} (\Lambda_{\mathcal{I}} \mathbf{E}_h)|_\tau = (\mathbf{A}^{-1}\mathbf{b})|_\tau \overset{\mathbf{b}\in\mathbb{C}^\sigma}{=} (\mathbf{A}^{-1}|_{\tau\times\sigma})\mathbf{b}.$$

Since $\mathbf{z} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H \mathbf{b}$, we conclude

$$\left\|\left(\mathbf{A}^{-1}|_{\tau\times\sigma} - \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H\right)\mathbf{b}\right\|_2 = \left\|\Lambda_\tau \mathbf{E}_h - \mathbf{z}\right\|_2 \lesssim h^{-1} q^k \|\mathbf{b}\|_2.$$

As $\mathbf{b}$ was arbitrary, we obtain the stated norm bound. □

Proof (Proof of Theorem 2.8) For each admissible cluster pair (τ,σ), let the matrices $\mathbf{X}_{\tau\sigma}$, $\mathbf{Y}_{\tau\sigma}$ be given by Lemma 5.2. Define the $\mathcal{H}$-matrix approximation $\mathbf{B}_{\mathcal{H}}$ by the conditions

$$\mathbf{B}_{\mathcal{H}}|_{\tau\times\sigma} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H \quad \text{if } (\tau,\sigma) \in P_{\text{far}}, \qquad \mathbf{B}_{\mathcal{H}}|_{\tau\times\sigma} = \mathbf{A}^{-1}|_{\tau\times\sigma} \quad \text{if } (\tau,\sigma) \in P_{\text{near}}.$$

The blockwise estimate of Lemma 5.2 for $q \in (0,1)$ and [10, Lemma 6] yield

$$\begin{aligned}\left\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 &\leq C_{\text{sp}}\left(\sum_{\ell=0}^{\infty} \max\left\{\left\|(\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}})|_{\tau\times\sigma}\right\|_2 : (\tau,\sigma) \in P, \text{level}(\tau) = \ell\right\}\right)\\ &\leq \widehat{C}_{\text{app}} C_{\text{sp}} \text{depth}(\mathbb{T}_{\mathcal{I}}) h^{-1} q^k.\end{aligned}$$

We next relate k to the blockwise rank r. For $y \geq 0$, the unique (positive) solution k of $k \ln k = y$ has the form

$$k = \frac{y}{\log y}(1 + o(1)) \qquad \text{as } y \to \infty \tag{5.7}$$

by, e.g., [36, Ex. 5.7, Chap. 1]. In passing, we mention that even higher order asymptotics can directly be inferred from the asymptotics of Lambert's W-function as described in [13, p. 25–27] or [15, Eq. (4.13.10)]. The asymptotics (5.7) implies that the solution k of $k^4 \ln^4 k = y$ satisfies $k = y^{1/4} / \ln(y^{1/4})(1 + o(1))$ as $y \to \infty$.

From Lemma 5.2 we have the rank bound $r \leq \widetilde{C}_{\dim}(1+\eta)^3 k^4 \big(q^{-3} + \ln^4(k(1+\eta))\big) \leq \widetilde{C}_{\dim}\big((1+\eta)q^{-1}\big)^3 k^4 \ln^4 k$, so that, for suitable b, $C > 0$ independent of r, we get $q^k \leq C \exp(-br^{1/4} / \ln r)$. Consequently, we have

$$\left\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq C_{\text{apx}} C_{\text{sp}} \text{depth}(\mathbb{T}_{\mathcal{I}}) h^{-1} e^{-b\left(r^{1/4} / \ln r\right)},$$

which concludes the proof. □

Appendix: Regular decompositions

The following lemma follows from the seminal paper [12]. The notation follows [12] in that $H^s_{\underline{\Omega}}(\mathbb{R}^3)$, $s \in \mathbb{R}$ denotes the spaces of distributions in $H^s(\mathbb{R}^3)$ supported by $\overline{\Omega}$, and that $C^\infty_{\underline{\Omega}}(\mathbb{R}^3)$ is the space of $C^\infty(\mathbb{R}^3)$-functions supported by $\overline{\Omega}$.

We introduce the space

$$\mathbf{H}^s_{\underline{\Omega}}(\mathrm{curl}) := \left\{ \mathbf{E} \in \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) : \nabla \times \mathbf{E} \in \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) \right\}$$

equipped with the norm $\|\mathbf{E}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathrm{curl})} := \|\mathbf{E}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} + \|\nabla \times \mathbf{E}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)}$.

Remark A.1 From [12, p. 301], for any $s \in \mathbb{R}$, the space $\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)$ is naturally isomorphic to the dual space of $\mathbf{H}^{-s}(\Omega)$. Hence, for $s \geq 0$, we have the alternative norm equivalence $\|\mathbf{v}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} \sim \|\mathbf{v}\|_{\widetilde{\mathbf{H}}^s(\Omega)} = \|\mathbf{v}^\star\|_{\mathbf{H}^s(\mathbb{R}^3)}$, where $\mathbf{v}^\star$ is the zero extension of a function $\mathbf{v}$ defined on Ω.

Lemma A.2 Let Ω be a bounded Lipschitz domain. There exist pseudodifferential operators T_1 and $\mathbf{T}_2$ of order -1 and a pseudodifferential operator $\mathbf{L}$ of order $-\infty$ on $\mathbb{R}^3$ with the following properties: For each $s \in \mathbb{R}$, they have the mapping properties $T_1 : \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) \to H^{s+1}_{\underline{\Omega}}(\mathbb{R}^3)$, $\mathbf{T}_2 : \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s+1}_{\underline{\Omega}}(\mathbb{R}^3)$, and $\mathbf{L} : \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) \to \mathbf{C}^\infty_{\underline{\Omega}}(\mathbb{R}^3)$ and for any $\mathbf{u} \in \mathbf{H}^s_{\underline{\Omega}}(\mathrm{curl})$, there holds the representation

$$\mathbf{u} = \nabla T_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big) + \mathbf{T}_2(\nabla \times \mathbf{u}) + \mathbf{L}\mathbf{u}. \tag{A.1}$$

Proof In [12, Theorem 4.6], operators T_1, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{L}_1$, $\mathbf{L}_2$ with the mapping properties

$$\begin{aligned} T_1 &: \mathbf{H}^s_{\underline{\Omega}}\big(\mathbb{R}^3\big) \to H^{s+1}_{\underline{\Omega}}\big(\mathbb{R}^3\big), \\ \mathbf{T}_2 &: \mathbf{H}^s_{\underline{\Omega}}\big(\mathbb{R}^3\big) \to \mathbf{H}^{s+1}_{\underline{\Omega}}\big(\mathbb{R}^3\big), \\ \mathbf{T}_3 &: H^s_{\underline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s+1}_{\underline{\Omega}}\big(\mathbb{R}^3\big), \\ \mathbf{L}_\ell &: \mathbf{H}^s_{\underline{\Omega}}\big(\mathbb{R}^3\big) \to \mathbf{C}^\infty_{\underline{\Omega}}\big(\mathbb{R}^3\big), \quad \ell = 1, 2, \end{aligned}$$

are defined, and it is shown that

$$\nabla T_1 \mathbf{v} + \mathbf{T}_2(\nabla \times \mathbf{v}) = \mathbf{v} - \mathbf{L}_1 \mathbf{v}, \tag{A.2a}$$

$$\nabla \times \mathbf{T}_2 \mathbf{v} + \mathbf{T}_3(\nabla \cdot \mathbf{v}) = \mathbf{v} - \mathbf{L}_2 \mathbf{v}. \tag{A.2b}$$

Taking $\mathbf{v} = \mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})$ in (A.2a), we obtain

$$\nabla T_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big) + \mathbf{T}_2\big(\nabla \times \big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big)\big) = \mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u}) - \mathbf{L}_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big). \tag{A.3}$$

Since $\nabla \times \mathbf{u}$ is divergence-free, we obtain from (A.2b) with the choice $\mathbf{v} = \nabla \times \mathbf{u}$

$$\mathbf{T}_2\big(\nabla \times \big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big)\big) = \mathbf{T}_2(\nabla \times \mathbf{u}) - \mathbf{T}_2\big(\nabla \times \mathbf{u} - \mathbf{L}_2\nabla \times \mathbf{u}\big)$$
$$= \mathbf{T}_2\big(\mathbf{L}_2(\nabla \times \mathbf{u})\big) =: \mathbf{L}_3\mathbf{u},$$

where, again, $\mathbf{L}_3$ is a smoothing operator of order $-\infty$ mapping into $\mathbf{C}^\infty_{\underline{\Omega}}(\mathbb{R}^3)$. Inserting this into (A.3) leads to

$$\nabla T_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big) + \mathbf{T}_2(\nabla \times \mathbf{u}) = \mathbf{u} - \mathbf{L}_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big) - \mathbf{L}_3\mathbf{u}.$$

Choosing $\mathbf{Lu} := \big(\mathbf{L}_1\big(\mathbf{u} - \mathbf{T}_2\nabla \times \mathbf{u}\big)\big) + \mathbf{L}_3\mathbf{u}$, we arrive at the representation (A.1).

□

Corollary A.3 Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, for every $s \geq 0$, there is a constant C (depending only on Ω and s) such that every $\mathbf{u} \in \mathbf{H}^s_0(\mathrm{curl}, \Omega)$ can be decomposed as $\mathbf{u} = \mathbf{z} + \nabla p$ with $\mathbf{z} \in \mathbf{H}^{s+1}_{\underline{\Omega}}(\mathbb{R}^3)$ and $p \in H^{s+1}_{\underline{\Omega}}(\mathbb{R}^3)$ together with

$$\|\mathbf{z}\|_{\mathbf{H}^{s+1}_{\underline{\Omega}}(\mathbb{R}^3)} \leq C\|\mathbf{u}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathrm{curl})}, \qquad \|\nabla p\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} \leq C\|\mathbf{u}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)}. \tag{A.4}$$

Proof From Lemma A.2 we can write $\mathbf{u} = \mathbf{z} + \nabla p$ with

$$\mathbf{z} := \mathbf{T}_2(\nabla \times \mathbf{u}) + \mathbf{Lu}, \qquad p := T_1\big(\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\big).$$

The stability estimate for $\mathbf{z}$ follows from the mapping properties of the operators $\mathbf{T}_2$ and $\mathbf{L}$. The mapping properties of $\mathbf{T}_1$ yield

$$\|\nabla p\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} \lesssim \|\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)} + \|\nabla \times \mathbf{u}\|_{\mathbf{H}^{s-1}_{\underline{\Omega}}(\mathbb{R}^3)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3)},$$

where the last step follows from the mapping property $\nabla\times : \mathbf{H}^s_{\underline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s-1}_{\underline{\Omega}}(\mathbb{R}^3)$.

□

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Declarations

Conflict of interest The authors declare no competing interests.

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