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An efficient reduced-order method for stochastic eigenvalue analysis

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Abstract

This article presents an efficient numerical algorithm to compute eigenvalues of stochastic problems. The proposed method represents stochastic eigenvectors by a sum of the products of unknown random variables and deterministic vectors. Stochastic eigenproblems are thus decoupled into deterministic and stochastic analyses. Deterministic vectors are computed efficiently via a few number of deterministic eigenvalue problems. Corresponding random variables and stochastic eigenvalues are solved by a reduced-order stochastic eigenvalue problem that is built by deterministic vectors. The computational effort and storage of the proposed algorithm increase slightly as the stochastic dimension increases. It can solve high-dimensional stochastic problems with low computational effort, thus the proposed method avoids the curse of dimensionality with great success. Numerical examples compared to existing methods are given to demonstrate the good accuracy and high efficiency of the proposed method.

KEYWORDS

curse of dimensionality, high-dimensional problems, reduced-order equations, stochastic finite element method, structural stochastic eigenvalues

1 | INTRODUCTION

Developments in numerical techniques and computing hardware have made it possible to solve high-resolution models in various computational physics problems. The considerable influence of uncertainties on system behavior has led to the development of dedicated numerical methods for uncertainty analysis. Predicting uncertainty propagation on the physical models has become an essential part of the analysis and design of practical engineering systems.

As an important part of structural dynamics analysis, the eigenvalue problem has received extensive attention and is well understood for deterministic problems.[1,2] Its extension, known as the stochastic eigenvalue problem, is introduced to consider influence of uncertainties on system dynamics analysis. There are several kinds of methods for solving stochastic eigenvalue problems. The first kind of method is Monte Carlo simulation (MCS).[3] As a most powerful tool for stochastic analysis, MCS can be applied to almost all kinds of stochastic problems. MCS is very easy to implement by use of the already existing deterministic solvers but has poor convergence. A large number of deterministic eigenvalue problems are solved in order to compute high-accuracy stochastic eigenvalues, which are computationally expensive, especially for large-scale problems. Several improvements, for example, quasi-MCS and multilevel MCS,[4] are used to save computational costs of MCS. The second kind of method is perturbation methods.[5] The perturbation approach expands the random variables around their mean values to first/second order by use of Taylor series. It is easy to implement and has high computational efficiency. However, being limited to small variability and only giving statistical moments make the method unsuitable for complex stochastic problems. Several improvements of perturbation methods are given in References 6 and 7. The third kind of method is the Polynomial chaos-based (PC) method.[8] In this kind of method, the stochastic matrices, the stochastic eigenvectors, and the stochastic eigenvalues are projected onto a stochastic space spanned by (generalized) PC basis. Stochastic Galerkin method is then used to transform the original stochastic eigenvalue problem into a nonlinear system of coupled deterministic equations whose size is much larger than that of the original problem. The Newton–Raphson method is adopted to solve the nonlinear system. PC-based methods have rigorous mathematical foundation and good convergence. It is general-purpose and can be applied to both real- and complex-valued stochastic eigenvalue problems. However, the computational efficiency of the PC-based method needs to be further improved. Several improvements[7,9] are proposed to reduce computational costs. Other extensions of PC-based methods are to solve stochastic eigenvalue problems by combing PC expansion and deterministic numerical techniques, for example, power method,[10] inverse power method,[11-13] subspace iteration.[14-16] Other methods are also proposed to solve stochastic eigenvalue problems. The stochastic collocation method[17] is to approximate stochastic eigenvalues and eigenvectors via an interpolation approach. It is non-intrusive and only uses deterministic solvers to solve a set of random samples of the solutions. The methods in References 18 and 19 transform the stochastic eigenvalue problem into initial equations with a pseudo time parameter, which is then solved by the PC-based method. The homotopy approach is proposed in the Reference 20, which expands stochastic eigenvalues and eigenvectors by an infinite multivariate series and adopts homotopy analysis to compute expanded coefficients. Stochastic eigenvalues and eigenvectors are solved by combining PC expansion and dedicatedly iterative algorithms for low-rank approximations in References 21 and 22.

We mention another method, known as reduced-order method (ROM). In this method, stochastic eigenvalues and eigenvectors are solved by a reduced-order (normally small-scale) eigenvalue problem, which is obtained by projection subspace of the large-scale problem. Solutions of the reduced-order eigenvalue problem are very close to the exact solutions if the subspace is similar to the space of stochastic eigenvectors of the original problem.[1] The key of this kind of method is to construct a good projection approximation subspace. Several methods are proposed for this purpose, for example, perturbation-based subspace,[23] optimization-based subspace,[24,25] stochastic Krylov subspace,[26] and subspace of mean matrices.[27] Another point that needs to pay more attention is the stochastic dimension, which has significant influence on the computational accuracy and efficiency of numerical methods for stochastic eigenvalue problems. There are usually a large number of uncertain parameters in many applications, for example, the input is approximated by random fields with a large number of random variables, which leads to the curse of dimensionality in high-dimensional stochastic spaces. MCS and its extensions[4] can overcome curse of dimensionality but a large number of deterministic simulations are needed. Stochastic collocation method[17] needs to construct high-dimensional interpolation formulas of the solutions based on a certain amount of deterministic simulations. The computational effort of PC-based methods increases dramatically as the number of stochastic dimensions and/or the order of PC basis increase, which is prohibitively expensive. Sparse PC method is an available tool to reduce the computational effort.[28]

We also highlight another kind of method called polynomial dimension decomposition (PDD).[29] Similar to PC-based methods, orthogonal polynomial bases are used in the PDD method to approximate stochastic responses. The PDD method develops a dimensional hierarchy of the stochastic response, which alleviates the curse of dimensionality suffered from PC-based methods. The application of the PDD method in stochastic eigenvalue problems is given in References 30 and 31, which allows lower-variate approximations of stochastic eigenvalues and lower-dimensional numerical integration for the statistical moments. To avoid the difficulty that the polynomial basis does not work well in capturing strongly local variations of solutions, for example, nonsmooth and discontinuous stochastic solutions, a spline chaos expansion[32]

is also proposed, which represents stochastic solutions by using orthogonal B-spline bases. Combining and generalizing ideas of the dimension decomposition and the spline chaos expansion, the spline dimensional decomposition method is developed to solve the high-dimensional and dynamical stochastic problems,[33,34] which avoids the curse of dimensionality to a great extent. The above methods have become powerful methods in uncertainty quantification and have great potential for very complex stochastic problems.

Although a lot of excellent work have been discussed above, the development of numerical methods for stochastic eigenvalue problems is still an attractive topic, especially robust, efficient, and accurate methods for solving large-scale, high-dimensional stochastic eigenvalue problems. In this article, we focus on developing efficient numerical algorithms to compute first several maximum/minimum stochastic eigenvalues of problems with high stochastic dimensions and large scale. The stochastic eigenvectors are first approximated by summing a set of products of random variables and deterministic vectors. The deterministic vectors (also considered as a set of reduced basis) are computed via a few number of decoupled deterministic eigenvalue problems and a dedicated iterative algorithm, where the deterministic eigenvalue problems are obtained by combining an approximation of the stochastic eigenvector and stochastic Galerkin method. A reduced-order stochastic eigenvalue problem based on the obtained reduced basis is used to solve stochastic eigenvalues of the original problem and random coefficients of the reduced basis. All stochastic analysis, including solving reduced basis and solving the reduced-order stochastic eigenvalue problem, are implemented by non-intrusive sampling methods, which has less computational effort and is almost independent of the stochastic dimension. The curse of dimensionality induced by the high-dimensional stochastic problem is thus avoided, which is demonstrated by using a numerical example with up to a hundred stochastic dimensions. Also, compared to PC and PDD methods, the non-intrusiveness of the proposed method is highlighted, which does not require dedicated approximation structures of the solutions in stochastic spaces. The proposed method is easy to implement and existing solvers can be embedded into solving procedure. It combines the fast convergence of intrusive methods and the weak dimensionality dependency of non-intrusive methods. Another advantage is that the proposed method obtains sample representations of stochastic eigenvalues and semi-explicit representations of stochastic eigenvectors, which provides a pathway to describe the probability density function of the quantity of interest and can be readily applied to structural stochastic dynamical analysis.[35-37]

The article is organized as follows: Section 2 gives the basic setting of the stochastic eigenvalue problem and a brief description of the PC method for solving stochastic eigenvalue problems. In Section 3, we propose a new reduced-order method to solve stochastic eigenvalue problems, including the construction of stochastic eigenvectors, the solution of reduced basis, the applicability to high-dimensional stochastic problems and the proofs of convergence and optimal approximation. The algorithm implementation of the proposed method is elaborated in Section 4. Following that, several numerical examples of low- and high-dimensional cases are given to demonstrate the performance of the proposed method in Section 5, and conclusions and discussions follow in Section 6.

2 | STOCHASTIC EIGENVALUE PROBLEMS

In this article, let $(\Theta, \Sigma, \mathcal{P})$ be a complete probability space, where Θ denotes the space of elementary events, Σ is the σ-algebra defined on Θ and $\mathcal{P}$ is the probability measure. We consider the following stochastic eigenvalue equation,

$$K(\theta)\, u(\theta) = \lambda(\theta)\, u(\theta), \tag{1}$$

where $\theta \in \Theta$, $K(\theta) \in \mathbb{R}^{n\times n}$ is the real symmetric and positive definite stochastic stiffness matrix related to stochastic properties of the physical models, $\lambda(\theta) \in \mathbb{R}$ and $u(\theta) \in \mathbb{R}^n$ are stochastic eigenvalues and stochastic eigenvectors, respectively. The orthonormal condition

$$u_i^T(\theta)\, u_j(\theta) = \delta_{ij}, \ \theta \in \Theta, \tag{2}$$

is met for the eigenvectors $u_i(\theta)$ and $u_j(\theta)$, where δ_{ij} is the Kronecker delta function. In a general setting, the matrix $K(\theta)$ depends on a finite set of real valued random variables reduced from the infinite-dimensional probability space. When the inputs are random fields represented with a set of random variables, the number of random variables may be large, which possibly induces the so called curse of dimensionality.

2.1 | Polynomial Chaos expansion for solving stochastic eigenvalue problems

Polynomial Chaos expansion is a powerful tool for stochastic analysis and has been applied to many kinds of stochastic problems. A PC-based method[8] is proposed in order to solve Equation (1). In this method, $K(\theta)$, $\lambda(\theta)$, and $u(\theta)$ are represented by PC basis in the form,

$$K(\theta)=\sum_{i=0}^{l-1}\psi_i(\theta)K_i,\ u_m(\theta)=\sum_{i=0}^{p-1}\psi_i(\theta)u_m^{(i)},\ \lambda_m(\theta)=\sum_{i=0}^{p-1}\psi_i(\theta)\lambda_m^{(i)}, \tag{3}$$

where $\{K_i\}_{i=0}^{l-1}\in\mathbb{R}^{n\times n}$ are deterministic matrices, $\left\{u_m^{(i)}\right\}_{i=0}^{p-1}\in\mathbb{R}^n$ are deterministic vectors and $\left\{\lambda_m^{(i)}\right\}_{i=0}^{p-1}\in\mathbb{R}$ are expanded coefficients. All of them are unknown and need to be computed by a coupled system of equations. Moreover, the orthonormal condition Equation (2) is written as

$$u_k^T(\theta)u_m(\theta)=\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\psi_i(\theta)\psi_j(\theta)u_k^{(i)T}u_m^{(j)}=\delta_{km}. \tag{4}$$

Substituting the expansion Equation (3) into Equation (1) and applying stochastic Galerkin method[38,39] to Equations (1) and (4) yield a deterministic system of equations

$$\sum_{i=0}^{l-1}\sum_{j=0}^{p-1}c_{ijq}K_iu^{(j)}=\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}c_{ijq}\lambda_iu^{(j)}, \tag{5}$$

and the equation of the orthonormal condition

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}c_{ijq}u_k^{(i)T}u_m^{(j)}=\delta_{km}\delta_{q0}, \tag{6}$$

where the coefficients is given by $c_{ijq}=\mathbb{E}\left\{\psi_i(\theta)\psi_j(\theta)\psi_q(\theta)\right\}$, $q=0,\ \dots\ ,p-1$. Equations (5) and (6) can be rewritten as a compact system of nonlinear equations,

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}(\mathbf{B}_i\mathbf{\Gamma}_i-\lambda_i\mathbf{\Gamma}_i)\mathbf{u}=0,\ \mathbf{B}_i=0,\ i>l, \tag{7}$$

and

$$\mathbf{u}_k^T\mathbf{\Gamma}_q\mathbf{u}_m=\delta_{km}\delta_{q0}, \tag{8}$$

where the matrices $\mathbf{B}_i$ and $\mathbf{\Gamma}_i$ are given by

$$\mathbf{B}_i=\begin{bmatrix}K_i & & 0\\ & \ddots & \\ 0 & & K_i\end{bmatrix}\in\mathbb{R}^{np\times np},\ \mathbf{\Gamma}_i=\left[c_{ijq}I_n\right]_{j,q=0}^{p-1}\in\mathbb{R}^{np\times np}, \tag{9}$$

$I_n\in\mathbb{R}^{n\times n}$ is the identity matrix.

The above PC-based method gives a powerful tool to solve stochastic eigenvalue problems. It has a rigorous mathematical foundation and is of high accuracy. The method can provide probability density descriptions for eigenvalues and eigenvectors instead of moments. However, it needs to be further improved in terms of computational efficiency. On one hand, it requires to solve the augmented nonlinear system Equation (7) for each eigenpair $\{\lambda(\theta),u(\theta)\}$. On the other hand, the size of Equation (7) is prohibitively large as the stochastic dimension, the order of PC basis and the degree of freedom (DOF) of the physical model increase, whose solution is time-consuming. We remark that the dimension

decomposition method, including PDD[29] and the spline dimension decomposition,[34] can be used to reduce the computational cost and capture high-accuracy stochastic solutions. Similar to the PC method, the method adopts orthogonal polynomial/spline bases to approximate stochastic solutions and develops a hierarchical decomposition to overcome some difficulties arising in PC methods, for example, the curse of dimensionality, capturing discontinuous stochastic solutions. An exhaustively comparative study of the two methods of solving stochastic eigenvalue problems can be found in the Reference 31. In this article, we will explore a non-intrusive method instead of intrusive solution approximations in stochastic spaces.

3 | A REDUCED-ORDER METHOD FOR STOCHASTIC EIGENVALUE ANALYSIS

In this section, we develop an efficient method to solve the stochastic eigenproblem Equation (1). A new expansion similar to the PC expansion is used to approximate stochastic eigenvectors, but the random coefficients and deterministic vectors are not known a priori. A numerical algorithm is proposed to compute corresponding deterministic vectors and a reduced-order eigenequation is then used to solve stochastic eigenvalues of the original eigenproblem.

3.1 | Reduced-order stochastic eigenvalue equation

Similar to the expansion Equation (3), we decompose the stochastic eigenvector into the deterministic and stochastic spaces and consider the expansion of the stochastic eigenvector $u_m(\theta)$ in a form

$$u_m(\theta) = \sum_{i=1}^{k} \phi_m^{(i)}(\theta)\, d_i = D\phi_m(\theta) \in \mathbb{R}^n, \tag{10}$$

where the deterministic vector $d_i \in \mathbb{R}^n$, the random variable $\phi_m^{(i)}(\theta) \in \mathbb{R}$, the deterministic matrix $D = \left[d_i\right]_{i=1}^{k} \in \mathbb{R}^{n\times k}$, the random vector $\phi_m(\theta) = \left[\phi_m^{(i)}(\theta)\right]_{i=1}^{k} \in \mathbb{R}^k$. All of them are not known a priori and need to be solved. Moreover, we let the orthonormal condition $D^TD = I_k$ hold, that is, the vectors $\{d_i\}_{i=1}^{k}$ are orthogonal $d_i^T d_j = \delta_{ij}$. The original eigenproblem Equation (1) thus becomes as

$$K(\theta) D\phi_m(\theta) = \lambda_m(\theta) D\phi_m(\theta). \tag{11}$$

It is noted that Equation (11) is insoluble since both the matrix D and the random vector $\phi_m(\theta)$ are unknown. If one of them has been known, the other is readily computable. In this way, an available way is to fix one of them and then to solve the other. Inspired by the classical subspace iteration method[1,2] for solving deterministic eigenvalue problems and reduced basis methods, we assume the matrix D has been known and then solve the unknown random vector $\phi_m(\theta)$. By use of the matrix D, a reduced-order eigenvalue problem can be obtained,

$$\tilde{K}_k(\theta)\, \phi_m(\theta) = \lambda_m(\theta)\, \phi_m(\theta), \tag{12}$$

where the reduced-order stochastic matrix $\tilde{K}_k(\theta)$ is given by $\tilde{K}_k(\theta) = D^T K(\theta) D \in \mathbb{R}^{k\times k}$ and $\{\phi_m(\theta)\}_m$ meets the orthonormal condition $\phi_i^T(\theta)\,\phi_m(\theta) = \delta_{im}$ almost everywhere (a.e.) since they are eigenvectors of the eigenequation (12). In this way, we recall the orthonormal condition Equation (2) of the original eigenproblem,

$$u_i^T(\theta)\, u_j(\theta) = \phi_i^T(\theta)\, D^T D\phi_j(\theta) = \delta_{ij}, \tag{13}$$

holds naturally, thus extra equations (like Equation (4)) are not needed to impose the orthonormal condition. The size of the reduced-order problem Equation (12) is k, which is much lower than the size of the original eigenproblem Equation (1). In our experience, a small number k normally achieves a good approximation of the stochastic eigenvectors of the original eigenproblem. Several methods can be used to solve Equation (12) efficiently and accurately, for example,

the Monte Carlo simulation and the PC-based method described in Section 2.1. In order to enable the proposed method to solve high-dimensional stochastic eigenproblems, we adopt a non-intrusive sampling method to solve Equation (12) in this article, which is easy to implement and has high accuracy and high efficiency thanking to the small size k. By Equation (12), the randomness of the matrix $K(\theta)$ is transferred to the reduced-order matrix $\tilde{K}_k(\theta)$. Thus the input random variables are propagated through a reduced-order eigenvalue system to the stochastic eigenvalues of both reduced-order and original eigenequations and the stochastic eigenvectors of the reduced-order eigenequations. However, compared to the PC method, the proposed method cannot provide an explicit representation between the input random variables and the stochastic eigenvalues/eigenvectors. To avoid this point, an available method is to represent the obtained $\lambda_m(\theta)$ and $\phi_m(\theta)$ using the PC basis.

The key part of the proposed method is to determine the matrix $D = \left[d_i\right]_{i=1}^{k}$. For this purpose, we solve the vector d_i one by one by considering the following approximation of the stochastic eigenvector $u(\theta)$,

$$\min_{d\in\mathbb{R}^n} \|u(\theta) - d\|^2, \tag{14}$$

where $\|\cdot\|^2$ is defined as $\|u(\theta)\|^2 = \mathbb{E}\left\{u(\theta)^T u(\theta)\right\}$ and $d \in \mathbb{R}^n$ is an unknown deterministic vector. From a vector approximation point of view, we approximate the stochastic eigenvector $u(\theta)$ by using the deterministic vector d. The approximation accuracy is very low due to the loss of randomness in the vector d, thus it is wrong to some extent. To illustrate the reasonableness of the approximation, we consider Equation (14) from a subspace point of view. In Section 3.5, we will show that the stochastic eigenvalues of Equation (12) converge to the stochastic eigenvalues of Equation (1) if the stochastic vector $u(\theta)$ is nearly in the subspace obtained by deterministic vectors $\{d_i\}_i$. In this sense, Equation (14) is used to construct a subspace that nearly includes the stochastic vector $u(\theta)$. In practice, the stochastic eigenvector $u(\theta)$ is not known a priori, thus the vector d cannot be computed directly by using Equation (14). To avoid this difficulty, we substitute the stochastic vector $u(\theta) = d$ into the eigenproblem Equation (1) and thus obtain the following stochastic residual

$$\mathcal{R}(\theta) = [K(\theta) - \lambda(\theta) I_n] d \in \mathbb{R}^n, \tag{15}$$

only the random variable $\lambda(\theta)$ and the deterministic vector d in which are unknown. Thus the problem is to find $\lambda(\theta)$ and d to minimize $\|\mathcal{R}(\theta)\|^2$,

$$\min_{\lambda(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|\mathcal{R}(\theta)\|^2 = \min_{\lambda(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|K(\theta) d - \lambda(\theta) d\|^2. \tag{16}$$

Remark 1. From a vector approximation point of view, a better approximation of the stochastic eigenvector $u(\theta)$ is given as follows

$$\min_{\varphi(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|u(\theta) - \varphi(\theta) d\|^2, \tag{17}$$

where $\varphi(\theta) \in \mathbb{R}$ is a unknown random variable. Equation (17) can be considered as a kind of rank-1 random singular value decomposition (SVD) of $u(\theta)$. Specifically, giving the sample representation $u(\theta) \in \mathbb{R}^{n\times n_s}$ (n_s is the number of random samples) of $u(\theta)$ we have the following rank-1 SVD,

$$u(\theta) \approx d\varphi(\theta),\ \varphi(\theta) \in \mathbb{R}^{1\times n_s}. \tag{18}$$

In the sense of sample representation, Equation (17) is the optimal rank-1 approximation[40] of $u(\theta)$. But it is not easy to solve the couple $\{\varphi(\theta), d\}$ since both of them are unknown. The solution of Equation (17) needs further study and we only consider the approximation Equation (14) in this article.

We now focus on computing the random variable $\lambda(\theta)$ and the deterministic vector d in Equation (16). To this end, we develop an alternating minimization iteration, the idea of which is to fix one of $\lambda(\theta)$ and d to solve the other and then to update the fixed one according to the solution. Specifically, for a known random variable $\lambda(\theta)$ (or a given initial value), we apply the stochastic Galerkin method[38,39] to the stochastic residual Equation (15) and transform it as

$$\mathbb{E}\{\lambda(\theta)\mathcal{R}(\theta)\} = 0, \tag{19}$$

which is used to solve the vector d and can be rewritten as a compact form,

$$K^* d = \lambda^* d, \tag{20}$$

where the deterministic matrix $K^* = \mathbb{E}\{\lambda(\theta) K(\theta)\} \in \mathbb{R}^{n\times n}$ and $\lambda^* = \mathbb{E}\left\{\lambda^2(\theta)\right\} \in \mathbb{R}$. Equation (20) is a classically deterministic eigenvalue equation, which can be solved by use of existing methods, for example, power method, Lanczos method and QR method.[2] The details will be discussed in the next subsection. After solving the vector d by Equation (20), the random variable $\lambda(\theta)$ is updated by the Galerkin procedure,

$$d^T \mathcal{R}(\theta) = 0, \tag{21}$$

equivalently,

$$\lambda(\theta) = \frac{d^T K(\theta) d}{d^T d} \in \mathbb{R}, \tag{22}$$

which can be simplified as $\lambda(\theta) = d^T K(\theta) d$ by considering the normalization $d^T d = 1$.

There are stochastic computations involved in Equations (20) and (22), that is, the expectation $\mathbb{E}\{\lambda(\theta) K(\theta)\}$ and the deterministic vector-stochastic matrix multiplication $d^T K(\theta) d$. A common method is to approximate $K(\theta)$ and $\lambda(\theta)$ by use of an Equation (3)-like expansion. In this method, the size of the equation for solving expanded coefficients of $\lambda(\theta)$ is $(m+p)!/(m!p!)$, where $(\cdot)!$ represents the factorial operator, m and p are the number of random variables and the order of PC basis. It increases dramatically as the stochastic dimension increases, for instance, the size is about 1×10^3 when $m = 10$, $p = 4$ and 4.6×10^6 when $m = 100$, $p = 4$. An available method to avoid the difficulty is sparse PC expansion.[28] In order to overcome the dependence on stochastic dimensions, we adopt a non-intrusive sampling method,

$$\mathbb{E}\{\lambda(\theta) K(\theta)\} = \mathbb{E}\{\lambda(\boldsymbol{\theta}) K(\boldsymbol{\theta})\}, \ \lambda(\boldsymbol{\theta}) = d^T K(\boldsymbol{\theta}) d \in \mathbb{R}^{n_s}, \tag{23}$$

where $\lambda(\boldsymbol{\theta}) \in \mathbb{R}^{n_s}$ and $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ represent samples of the random variable $\lambda(\theta)$ and the matrix $K(\theta)$, respectively. It is noted that Equation (23) has low computational effort even for very high stochastic dimensions, the applicability of which to high-dimensional problems will be discussed in Section 3.3.

It is seen from Equations (20) and (22) that $d^T K(\theta) d \neq 0$ needs to be hold. In other words, the proposed method does not work on the case $\lambda(\theta) = 0$. If there are zero eigenvalues in the problem, we can adopt the frequency-shifting strategy[2] to move the eigenvalues away from zero. We remark that although the proposed method is derived from real symmetric and positive definite stochastic matrices, it can be extended to more general stochastic matrices, for example, the non-symmetric complex stochastic matrices, which is simply illustrated in Section 5.1.3. Also, nonlinear stochastic eigenvalue problems arise in many practical problems.[31,33] The current version of the proposed method cannot solve nonlinear stochastic eigenvalue problems well. The proposed method is possible to be extended to the nonlinear eigenvalue problems by combining the idea in this article and deterministic nonlinear eigenvalue methods,[41,42] which is out of the scope of this article and an exhaustive study of which will be presented in following-up studies. In addition, the proposed method does not require the correlation of input random variables. As shown in Equation (23), we adopt a non-intrusive way to perform stochastic computations. After generating samples of correlated/independent input random variables, all computations of the proposed method are the same. Thus the proposed method can be applied to both correlated and independent input random variables.

3.2 | Solution of deterministic eigenvectors

In this section, we simply discuss the method for solving the deterministic eigenproblem Equation (20). For each vector d, the single vector iteration method is enough for our purpose. In this article, we adopt the power iteration for the explanation of our method. For complex and large-scale problems in practice, other numerical methods can be found in the Reference 2 and they can be readily used as an alternative to the power method in this article.

By adopting the power iteration to compute the maximum eigenvalue of Equation (20), a new solution $d^{(j)}$ is computed based on a known approximation $d^{(j-1)}$,

$$d^{(j)} = K^* d^{(j-1)}, \tag{24}$$

where the deterministic matrix is inherited from Equation (20) and d is the deterministic vector to be solved. The iteration Equation (24) is stopped until d converges.

Although Equation (24) is only used to solve a single vector d, it can be readily extended to solve a set of vectors $d_1, \ldots, d_k$. To illustrate this point, assuming that the first $k-1$ vectors $d_1, \ldots, d_{k-1}$ have been known, we calculate the kth vector d_k. Equation (24) is still adopted

$$d_k^{(j)} = K^* d_k^{(j-1)}, \tag{25}$$

which is stopped until the vector d_k converges. It is noted that d_k is the eigenvector of the different matrix $K^* = \mathbb{E}\{\lambda_k(\theta) K(\theta)\}$ since $\lambda_k(\theta)$ are different random variables that vary with k. To speed up the computation and to avoid the overlapping eigenmodes, we let the vector $d_k^{(j)}$ orthogonal to the already obtained vectors $d_1, \ldots, d_{k-1}$. Here we utilize Gram–Schmidt orthonormalization

$$d_k^{(j)} = d_k^{(j)} - \sum_{i=1}^{k-1} \frac{d_k^{(j)T} d_i}{d_i^T d_i} d_i, \; d_k^{(j)} = d_k^{(j)} \Big/ \left(d_k^{(j)T} d_k^{(j)} \right), \tag{26}$$

which needs to hold along whole iterative process of Equation (25) until d_k converges. The iterations Equations (25) and (26) are very similar to the classical power method for solving the eigenvectors of a deterministic matrix, but the randomness is embedded into the matrix K^* in this article, which allows the subspace obtained by $[d_1, \ldots, d_k]$ to be a good approximation of the space of the first few stochastic eigenvectors. In the same way, we can compute new vector $d_{k+1}, d_{k+2}, \ldots$ by using Equation (25) until the specified number of items is calculated. Similarly, the inverse power method can be used to compute deterministic vectors $\{d_i\}_i$ that are used to approximate the first several minimum stochastic eigenvectors.

We remark that it is suggested to only use the proposed method to compute the first few maximum and minimum stochastic eigenvalues and eigenvectors, although it can be used to solve more eigenvalues and eigenvectors. Much more reduced basis $\{d_i\}_i$ are required when a large number of stochastic eigenvalues and stochastic eigenvectors are considered. Although the calculation of deterministic vectors $\{d_i\}_i$ is readily implemented and can be accelerated and improved by other methods, for example, the Lanczos method and the QR method, the size of the reduced-order eigenequation (12) increases as the number of reduced basis $\{d_i\}_i$. More computational effort is needed for the solution of Equation (12). Hence, although the proposed method can be applied to calculate a large number of eigenvalues and eigenvectors, it only speeds up the computation to a lesser extent. An extreme case is that when we consider all eigenvalues and eigenvectors of the matrix $K(\theta)$, the size of reduced-order matrix D is the same as the stochastic matrix $K(\theta)$ and the size of the reduced-order Equation (12) is the same as the original problem, which makes the "reduced-order" no sense.

3.3 | High-dimensional stochastic eigenvalue problems

In this section, we show that the proposed method can be applied to high-dimensional stochastic eigenvalue problems without any modification and extra computational effort. We assume that the stochastic matrix $K(\theta)$ can be represented in a series expansion form

$$K(\theta) = \sum_{j=1}^{r} \xi_j(\theta) K_j, \tag{27}$$

where $\{\xi_j(\theta)\}_{j=1}^r$ are random variables described by probability distributions, random samples or PC approximation, $\{K_j\}_{j=1}^r \in \mathbb{R}^{n \times n}$ are deterministic matrices. High-dimensional cases are induced by a large value of r. For non-separated

stochastic matrices, Equation (27) can be obtained by PC expansion in Equation (3), or a third-order tensor of random samples $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ is generated by using non-intrusive methods.

We introduce the following sample matrix of random variables $\left\{\xi_j(\theta)\right\}_{j=1}^{r}$,

$$\xi(\boldsymbol{\theta}) = \begin{bmatrix} \xi_1\left(\theta^{(1)}\right) & \cdots & \xi_r\left(\theta^{(1)}\right) \\ \vdots & \ddots & \vdots \\ \xi_1\left(\theta^{(n_s)}\right) & \cdots & \xi_r\left(\theta^{(n_s)}\right) \end{bmatrix} \in \mathbb{R}^{n_s\times r}, \tag{28}$$

and the sample vector of $\lambda(\theta)$ is $\lambda(\boldsymbol{\theta}) = \left[\lambda\left(\theta^{(1)}\right), \ldots, \lambda\left(\theta^{(n_s)}\right)\right]^T \in \mathbb{R}^{n_s}$. Thus stochastic computations in Equation (23) are reformulated as

$$\mathbb{E}\left\{\lambda(\boldsymbol{\theta})K(\boldsymbol{\theta})\right\} = \sum_{j=1}^{r}\mathbb{E}\left\{\lambda(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}K_j,\ \ \lambda(\boldsymbol{\theta}) = \sum_{j=1}^{r}\xi_j(\boldsymbol{\theta})\,d^TK_jd = \xi(\boldsymbol{\theta})\,\mathbf{c} \in \mathbb{R}^{n_s}, \tag{29}$$

where the coefficient vector $\mathbf{c} = \left[d^TK_1d, \ldots, d^TK_rd\right]^T \in \mathbb{R}^r$ and the expectation $\mathbb{E}\left\{\lambda(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}$ are calculated efficiently in a non-intrusive way

$$\left[\mathbb{E}\left\{\lambda(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}\right]_{j=1}^{r} = \frac{1}{n_s}\lambda(\boldsymbol{\theta})^T\xi(\boldsymbol{\theta}) \in \mathbb{R}^r. \tag{30}$$

In this way, we use the same method for solving both low- and high-dimensional stochastic problems and do not need to design dedicated algorithms for high-dimensional cases. The computational effort increases slightly as the dimension increases since only extra memories for storing $\xi(\boldsymbol{\theta}) \in \mathbb{R}^{n_s\times r}$ and $\left\{K_j\right\}_{j=1}^{r}$ are needed.

Remark 2. Combining Equations (20) and (22) we have

$$\mathbb{E}\left\{\left[d^TK(\boldsymbol{\theta})d\right]K(\boldsymbol{\theta})\right\}d = \lambda^*d, \tag{31}$$

or the separated form

$$\left(\sum_{j=1}^{r}g_j(d)\,K_j\right)d = \lambda^*d, \tag{32}$$

where the scalar function $g_j(d) = \sum_{i=1}^{r}\mathbb{E}\left\{\xi_i(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}\left(d^TK_id\right) \in \mathbb{R}$ and $\mathbb{E}\left\{\xi_i(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}$ are given by

$$\left[\mathbb{E}\left\{\xi_i(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}\right]_{i,j=1}^{r} = \frac{1}{n_s}\xi(\boldsymbol{\theta})^T\xi(\boldsymbol{\theta}) \in \mathbb{R}^{r\times r}. \tag{33}$$

Equations (31) and (32) are deterministic nonlinear eigenvalue problems. The vector d can be considered as the eigenvector of the matrix obtained by the combination of $\left\{K_j\right\}_{j=1}^{r}$. In this article, we solve stochastic eigenproblems by iteratively solving linear eigenvalue problems. Compared to the proposed method, only one stochastic computation Equation (33) is required for solving Equation (32) and the vector d is solved only by deterministic problems. However, nonlinear eigenvalue problems are beyond the scope of this article and can be found in References 41 and 42.

3.4 | Maximum and minimum stochastic eigenvalues

The proposed method utilizes a two-step strategy to compute eigenvalues of the original eigenproblem Equation (1), that is, the first step is to solve deterministic eigenvalue problems to generate a set of reduced basis and the second step is to solve a reduced-order stochastic eigenvalue problem. In this section we discuss computing original maximum and minimum stochastic eigenvalues by combining two-step eigenvalue analysis.

The maximum and minimum eigenvalues of Equation (1) are given by

$$\lambda_{\max}(\theta) = \max_{u(\theta)\neq 0} \frac{u^T(\theta) K(\theta) u(\theta)}{u^T(\theta) u(\theta)},\ \lambda_{\min}(\theta) = \min_{u(\theta)\neq 0} \frac{u^T(\theta) K(\theta) u(\theta)}{u^T(\theta) u(\theta)}. \tag{34}$$

In the first step, maximum/minimum stochastic eigenvectors $u(\theta)$ are approximated by the vector d that is solved by the minimization Equation (16),

$$\max_{d\neq 0} \frac{d^T K^* d}{d^T d} \to d_{\max},\ \min_{d\neq 0} \frac{d^T K^* d}{d^T d} \to d_{\min}, \tag{35}$$

which indicates that $d_{\max}/d_{\min}$ are the maximum/minimum eigenvectors of K^*. Further, maximum/minimum stochastic eigenvectors of the original eigenproblem are solved by the eigenvalues problem Equation (12) in the second step,

$$\phi_{\max}(\theta) = \max_{\phi(\theta)\neq 0} \frac{\phi^T(\theta) \tilde{K}_k(\theta) \phi(\theta)}{\phi^T(\theta) \phi(\theta)},\ \phi_{\min}(\theta) = \min_{\phi(\theta)\neq 0} \frac{\phi^T(\theta) \tilde{K}_k(\theta) \phi(\theta)}{\phi^T(\theta) \phi(\theta)}. \tag{36}$$

which indicates that $\phi_{\max}(\theta)/\phi_{\min}(\theta)$ are the maximum/minimum eigenvectors of the reduced-order stochastic eigenvalue problem.

Thus the maximum/minimum stochastic eigenvalues and eigenvectors of the original eigenproblems are obtained by two-step maximum/minimum eigenvalue problems,

$$u_{\max,i}(\theta) = D_{\max}\phi_{\max,i}(\theta),\ u_{\min,i}(\theta) = D_{\min}\phi_{\min,i}(\theta),\ i = 1, 2, \ldots \tag{37}$$

where $D_{\max} = \left[d_{\max,1}, \ldots, d_{\max,k}\right] \in \mathbb{R}^{n\times k}$, $\phi_{\max,i}(\theta) \in \mathbb{R}^k$ and $D_{\min} = \left[d_{\min,1}, \ldots, d_{\min,k}\right] \in \mathbb{R}^{n\times k}$, $\phi_{\min,i}(\theta) \in \mathbb{R}^k$ are the first several maximum/minimum eigenvectors obtained by Equations (20) and (12).

3.5 | Convergence analysis of stochastic eigenvalues

In this section, we extend the analysis of deterministic eigenvalue problems to stochastic cases. We will demonstrate that if an eigenvector of the matrix $K(\theta)$ is nearly in the subspace $\mathcal{K}(D_k)$ consisting of the reduced-order matrix D_k (we denote the matrix D in Equation (10) as D_k in this section), the corresponding stochastic eigenvalue of the reduced-order eigenproblem Equation (12) converges to the stochastic eigenvalue of the full-order eigenproblem Equation (1).

We adopt some concepts from the perturbation theory of eigenspaces.[2,43] The acute angle between a vector v and a subspace $\mathcal{K}$ is defined as the smallest acute angle between v and all vectors $w \in \mathcal{K}$

$$\angle(v, \mathcal{K}) = \min_{w\in\mathcal{K}} \angle(v, w), \tag{38}$$

where $\angle(v, w)$ is defined as the acute angle between the nonzero vectors v and w and

$$\cos\angle(v, w) = \frac{|(v, w)|}{\|v\|_2 \|w\|_2},\ 0 \le \angle(v, w) \le \frac{\pi}{2}. \tag{39}$$

In this way, recalling Equations (1) and (12), the angle between the eigenvector $u(\theta)$ of the full-order eigenproblem and the subspace $\mathcal{K}(D_k)$ is given by

$$\alpha(\theta) = \angle(u(\theta), \mathcal{K}(D_k)). \tag{40}$$

As illustrated in Figure 1, we decompose the eigenvector $u(\theta)$ into two parts

$$u(\theta) = u_{\in}(\theta) + u_{\perp}(\theta) \qquad \text{a.e.}, \tag{41}$$

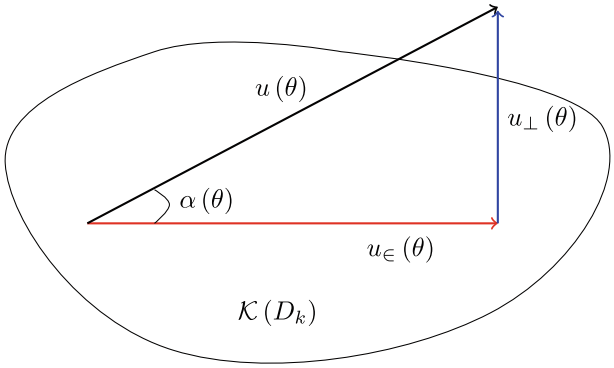


FIGURE 1 Decomposition of the eigenvector $u(\theta)$

where the components are

$$u_\in(\theta) = D_k D_k^T u(\theta) \in \mathcal{K}(D_k),\ u_\perp(\theta) = \left(I - D_k D_k^T\right) u(\theta) \perp \mathcal{K}(D_k). \tag{42}$$

According to Equations (38) and (39), the relationships between the L^2 norms of $u_\in(\theta)$, $u_\perp(\theta)$ and the angle $\alpha(\theta)$ are given by

$$\|u_\in(\theta)\|_2 = \cos\alpha(\theta),\ \|u_\perp(\theta)\|_2 = \sin\alpha(\theta). \tag{43}$$

Considering $K(\theta)u(\theta)$ by Equation (41) we have

$$K(\theta)u(\theta) = K(\theta)u_\in(\theta) + K(\theta)u_\perp(\theta) = K(\theta)D_k D_k^T u(\theta) + K(\theta)u_\perp(\theta), \tag{44}$$

Substituting which into Equation (1) and multiplying by D_k^T from left yields

$$D_k^T K(\theta)u(\theta) = \widetilde{K}_k(\theta)D_k^T u(\theta) + D_k^T K(\theta)u_\perp(\theta) = \lambda(\theta)D_k^T u(\theta), \tag{45}$$

which is equivalent to

$$\left(\lambda(\theta)I - \widetilde{K}_k(\theta)\right) D_k^T u(\theta) = D_k^T K(\theta)u_\perp(\theta) \qquad \text{a.e.} \tag{46}$$

Further, the eigendecomposition of the matrix $\widetilde{K}_k(\theta)$ is

$$\widetilde{K}_k(\theta) = \widetilde{U}(\theta)\widetilde{\Lambda}(\theta)\widetilde{U}(\theta)^T \qquad \text{a.e.}, \tag{47}$$

where $\widetilde{\Lambda}(\theta) \in \mathbb{R}^{k\times k}$ is a diagonal matrix consisting of the eigenvalues $\widetilde{\lambda}(\theta)$ of the reduced-order matrix $\widetilde{K}_k(\theta)$. Substituting Equation (47) into Equation (46) we have

$$\widetilde{U}(\theta)\left(\lambda(\theta)I - \widetilde{\Lambda}(\theta)\right)\widetilde{U}(\theta)^T D_k^T u(\theta) = D_k^T K(\theta)u_\perp(\theta), \tag{48}$$

multiplying which by $\widetilde{U}(\theta)^T$ from left we have

$$\left(\lambda(\theta)I - \widetilde{\Lambda}(\theta)\right)\widetilde{U}(\theta)^T D_k^T u(\theta) = \widetilde{U}(\theta)^T D_k^T K(\theta)u_\perp(\theta) \qquad \text{a.e.} \tag{49}$$

Thus the following inequality holds

$$\min_{\lambda(\theta)\in eig(K(\theta))} \left|\lambda(\theta) - \widetilde{\lambda}(\theta)\right| \left\|\widetilde{U}(\theta)^T D_k^T u(\theta)\right\|_2 \le \left\|\left(\lambda(\theta)I - \widetilde{\Lambda}(\theta)\right)\widetilde{U}(\theta)^T D_k^T u(\theta)\right\|_2 = \left\|\widetilde{U}(\theta)^T D_k^T K(\theta)u_\perp(\theta)\right\|_2, \tag{50}$$

equivalently,

$$\min_{\lambda(\theta)\in eig(K(\theta))} \left|\widetilde{\lambda}(\theta)-\lambda(\theta)\right| \le \frac{\left\|\widetilde{U}(\theta)^T D_k^T K(\theta)\, u_\perp(\theta)\right\|_2}{\left\|\widetilde{U}(\theta)^T \left(D_k^T D_k\right) D_k^T u(\theta)\right\|_2} \le \frac{\|K(\theta)\|_2 \|u_\perp(\theta)\|_2}{\|u_\in(\theta)\|_2} = \|K(\theta)\|_2 \tan\alpha(\theta) \qquad \text{a.e.} \tag{51}$$

The right side of Equation (51) only depends on the values $\|K(\theta)\|_2$ and $\tan\alpha(\theta)$. The value $\|K(\theta)\|_2$ is fixed for a given matrix $K(\theta)$. Hence we can conclude $\left|\widetilde{\lambda}(\theta)-\lambda(\theta)\right| \to 0$ as the angle $\alpha(\theta) \to 0$. In other words, if the eigenvector $u(\theta)$ of the matrix $K(\theta)$ is nearly in $\mathcal{K}(D_k)$, the stochastic eigenvalue $\widetilde{\lambda}(\theta)$ of the reduced-order matrix $\widetilde{K}_k(\theta)$ converges to the stochastic eigenvalue $\lambda(\theta)$ of the full-order matrix $K(\theta)$. It is noted that we need to make sure that the subspace $\mathcal{K}(D_k)$ is "good" enough such that the target eigenvector is nearly in it. A simple way for this purpose is to increase the dimension of the subspace. Furthermore, only symmetrically positive definite matrix $K(\theta)$ is considered in this article. Equation (51)-like bound estimations for more generally deterministic matrices can be found in the References 43 and 44 and we can extend these theories to the corresponding stochastic cases.

3.6 | Optimal approximation

In this section, we will demonstrate that the stochastic eigenvalues and eigenvectors of the reduced-order matrix $\widetilde{K}_k(\theta)$ are considered optimal approximations to the stochastic eigenvalues and eigenvectors of the full-order matrix $K(\theta)$ from the given subspace $\mathcal{K}(D_k)$. For this purpose, we will show that

$$\min_{Q(\theta)\in\mathbb{R}^{k\times k}} \|K(\theta)D_k - D_k Q(\theta)\|_2, \tag{52}$$

reaches its minimum value when $Q(\theta) = D_k^T K(\theta) D_k$ (i.e., the reduced-order stochastic matrix in Equation (12)).

Let $\mathbf{D} = [D_k, D_r] \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, where the reduced-order matrix $D_k \in \mathbb{R}^{n\times k}$, the supplementary matrix $D_r \in \mathbb{R}^{n\times r}$, $r = n-k$. Thus we have

$$\widetilde{K}(\theta) = \mathbf{D}^T K(\theta)\mathbf{D} = [D_k, D_r]^T K(\theta)[D_k, D_r] = \begin{bmatrix} D_k^T K(\theta) D_k & D_k^T K(\theta) D_r \\ D_r^T K(\theta) D_k & D_r^T K(\theta) D_r \end{bmatrix} = \begin{bmatrix} \widetilde{K}_{kk}(\theta) & \widetilde{K}_{kr}(\theta) \\ \widetilde{K}_{rk}(\theta) & \widetilde{K}_{rr}(\theta) \end{bmatrix}. \tag{53}$$

To proof Equation (52), we let

$$Q(\theta) = \widetilde{K}_{kk}(\theta) + S(\theta), \ \forall S(\theta) \in \mathbb{R}^{k\times k}. \tag{54}$$

Hence we have

$$\begin{aligned}
&[K(\theta)D_k - D_k Q(\theta)]^T [K(\theta)D_k - D_k Q(\theta)] \\
&= \left[K(\theta)D_k - D_k\left(\widetilde{K}_{kk}(\theta) + S(\theta)\right)\right]^T \left[K(\theta)D_k - D_k\left(\widetilde{K}_{kk}(\theta) + S(\theta)\right)\right] \\
&= \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] - \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T (D_k S(\theta)) \\
&\quad - (D_k S(\theta))^T \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] + (D_k S(\theta))^T (D_k S(\theta)) \\
&= \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] - \left(D_k^T K(\theta)^T D_k - \widetilde{K}_{kk}(\theta)^T\right) S(\theta) \\
&\quad - S(\theta)^T \left(D_k^T K(\theta) D_k - \widetilde{K}_{kk}(\theta)\right) + S(\theta)^T S(\theta) \\
&= \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T \left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] + S(\theta)^T S(\theta) \qquad \text{a.e.},
\end{aligned} \tag{55}$$

which reaches the minimum when $S(\theta) = 0$, that is, $Q(\theta) = \widetilde{K}_{kk}(\theta)$, thus Equation (52) is proved. Furthermore, the minimum value is given by

$$\begin{aligned}
&\left\|K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right\|_2 \\
&= \left\|[D_k, D_r][D_k, D_r]^T K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right\|_2
\end{aligned}$$

$$= \left\| \left[D_k \widetilde{K}_{kk}(\theta) + D_r \widetilde{K}_{rk}(\theta) \right] - D_k \widetilde{K}_{kk}(\theta) \right\|_2$$
$$= \left\| \widetilde{K}_{rk}(\theta) \right\|_2 \qquad \text{a.e.} \tag{56}$$

4 | ALGORITHM IMPLEMENTATION

The proposed method for solving the stochastic eigenvalue problem Equation (1) is summarized in Algorithm 1, which includes two parts in turn. The first part is from step 1 to 17, which is to compute the reduced-order matrix D and includes a triple-loop iteration. The innermost loop, which is from step 4 to 10, is used to solve the vector $d_k^{(j)}$ from a given random variable $\lambda_k^{(j)}(\theta)$, where the subscript k represents the kth reduced basis, the superscript j represents the jth iteration $\left\{ \lambda_k^{(j)}(\theta), d_k^{(j)} \right\}$ and the superscript q only locally works on the power iteration for deterministic eigenproblems. The maximum/minimum eigenvector $d_k^{(j)}$ is solved by power/inverse power methods in step 6 and the orthonormalization is processed in step 7. The convergence error in step 8 is defined as

$$\varepsilon_{d,k,j,q} = \frac{\left\| d_k^{(j,q)} - d_k^{(j,q-1)} \right\|_2}{\left\| d_k^{(j,q-1)} \right\|_2} = \left\| d_k^{(j,q)} - d_k^{(j,q-1)} \right\|_2, \tag{57}$$

which measures the convergence of the eigenvector $d_k^{(j)}$ of the deterministic eigenproblem Equation (20). The middle loop from step 2 to 14 corresponds to computing the kth couple $\{\lambda_k(\theta), d_k\}$. The random variable $\lambda_k(\theta)$ is initialized by n_s random samples in step 2 and is updated in step 11 based on the vector $d_k^{(j)}$ obtained by the innermost loop. The convergence error in step 12 is defined as

$$\varepsilon_{l,k,j} = \frac{\left\| d_k^{(j)} - d_k^{(j-1)} \right\|_2}{\left\| d_k^{(j-1)} \right\|_2} = \left\| d_k^{(j)} - d_k^{(j-1)} \right\|_2, \tag{58}$$

ALGORITHM 1. Algorithm for solving stochastic eigenvalue problems

1: **while** $k \le k_{\max}$ **do**
2: Initialize $\lambda_k^{(0)}(\boldsymbol{\theta}) = \left\{ \lambda_k^{(0)}\left(\theta^{(i)}\right) \right\}_{i=1}^{n_s} \in \mathbb{R}^{n_s}$
3: **while** $\varepsilon_{l,k,j} > \varepsilon_l$ **do**
4: Initialize $d_k^{(j,0)} \in \mathbb{R}^n$
5: **while** $\varepsilon_{d,k,j,q} > \varepsilon_d$ **do**
6: Compute $d_k^{(j,q)}$ by Equation∼(25) (inverse power method for the minimum eigenvectors)
7: Orthogonalize $d_k^{(j,q)} \perp d_i, \sim i = 1, \ldots, k-1$ and normalize $\left\| d_k^{(j,q)} \right\|_2 = 1$
8: Compute the iterative error $\varepsilon_{d,k,j,q}$
9: $q \leftarrow q + 1$
10: **end while**
11: Compute $\lambda_k^{(j)}(\theta)$ by Equation (23)
12: Compute the iterative error $\varepsilon_{l,k,j}$
13: $j \leftarrow j + 1$
14: **end while**
15: Update the matrix $D = \left[D, d_k\right] \in \mathbb{R}^{n \times k}$
16: $k \leftarrow k + 1$
17: **end while**
18: Compute eigenpairs $\{\lambda_m(\theta), \phi_m(\theta)\}_m$ by Equation∼(12)
19: Compute the mth stochastic eigenvector $u_m(\theta) = D\phi_m(\theta)$

which measures the difference between $d_k^{(j)}$ and $d_k^{(j-1)}$. The iteration is stopped when $d_k^{(j)}$ is almost the same as $d_k^{(j-1)}$. The outermost loop from step 1 to 17 is used to compute the deterministic vector d_k one after another, where $k_{\max}$ is the specified number of retained terms. For the second part, the stochastic eigenvalues of both reduced-order and original eigenequations and the stochastic eigenvectors of the reduced-order eigenequation (12) are solved in step 18. The stochastic eigenvectors of the original eigenproblem Equation (1) are then calculated in step 19 by combining reduced basis and stochastic eigenvectors of the reduced-order eigenproblem. Although only the standard eigenvalue problem is considered, Algorithm 1 can be readily extended to generalized eigenvalue equations, which will be demonstrated in numerical examples.

5 | NUMERICAL EXAMPLES

Numerical implementations of the proposed method are illustrated with the aid of five examples. For all considered examples, convergence errors in steps 3 and 5 of Algorithm 1 are set as $\varepsilon_l = \varepsilon_d = 1 \times 10^{-4}$. 1×10^4 random samples are adopted in step 2 of Algorithm 1. Reference solutions are computed by adopting 1×10^4 MCS. All examples are tested on a laptop (dual-core, Intel Core i7, 2.40 GHz).

5.1 | Eigenvalues of stochastic matrices

In this example, we test the proposed method using separated and non-separated stochastic matrices and compare the performance of the proposed method, MCS and PC method.

5.1.1 | Separated stochastic matrix

In this case, we consider the stochastic eigenvalue problem Equation (1) and compute stochastic eigenvalues of the following stochastic matrix,

$$K(\theta) = \sum_{i=1}^{r} \xi_i(\theta) K_i \in \mathbb{R}^{n\times n}, \tag{59}$$

where $\{\xi_i(\theta)\}_{i=1}^r$ are mutually independent uniform random variables on $[1, 2]$ and deterministic matrices $\{K_i\}_{i=1}^r \in \mathbb{R}^{n\times n}$ are given by

$$K_i = U_i^T D_i U_i \in \mathbb{R}^{n\times n}. \tag{60}$$

We set the matrix size $n = 50$, the stochastic dimension $r = 5$ and the retained number of Algorithm 1 $k_{\max} = 10$ in this case. Deterministic matrices D_i and U_i are a sample realization of stochastic matrices $D_i(\theta)$ and $U_i(\theta)$, where the diagonal matrix $D_i(\theta) = \mathrm{diag}([\eta_{i1}(\theta), \dots, \eta_{in}(\theta)]) \in \mathbb{R}^{n\times n}$, $\{\eta_{ij}(\theta)\}_{i=1,\dots,r,j=1,\dots,n}$ are mutually independent uniform random variables on [10,100] and $U_i(\theta) \in \mathbb{R}^{n\times n}$ is a matrix formed by the orthonormal basis of the stochastic matrix $[\varsigma_{ikl}(\theta)]_{k,l=1}^n \in \mathbb{R}^{n\times n}$, $\{\varsigma_{ikl}(\theta)\}_{i=1,\dots,r;\ k,l=1,\dots,n}$ are mutually independent uniform random variables on $[0, 1]$.

Figure 2 shows probability density functions (PDFs) of first four minimum and maximum stochastic eigenvalues obtained by the proposed ROM and MCS. For both minimum and maximum stochastic eigenvalues, PDFs obtained by ROM have good agreements with MCS, which demonstrate good accuracy of the proposed ROM. It is seen from Figure 2B that the proposed method works well even for highly close eigenvalues, which allows the proposed method to deal with the problems with close eigenmodes. It is noted that the rank order of each sample realization of stochastic eigenvalues is fixed, for instance, $\lambda_1\left(\theta^{(i)}\right) > \lambda_2\left(\theta^{(i)}\right)$, $i = 1, \dots, n_s$ must hold if we consider $\lambda_1(\theta)$ and $\lambda_2(\theta)$ as the largest and second largest stochastic eigenvalues, which means that there is strong correlations between stochastic eigenvalues. We compute the correlations of eight stochastic eigenvalues (including the first four minimum eigenvalues shown in Figure 2A and the first four maximum eigenvalues shown in Figure 2B) and the Pearson correlation coefficient

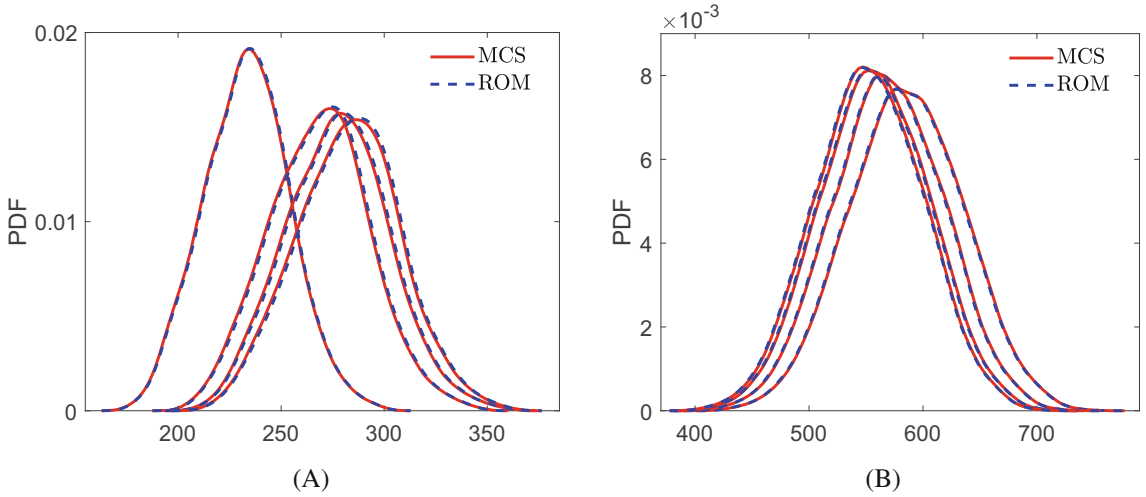


FIGURE 2 PDFs of first four minimum and maximum eigenvalues. (A) PDFs of first four minimum eigenvalues; (B) PDFs of first four maximum eigenvalues

TABLE 1 Correlations between stochastic eigenvalues

Eigenvalues	$\lambda_1^{\mathrm{MinEv}}(\theta)$	$\lambda_2^{\mathrm{MinEv}}(\theta)$	$\lambda_3^{\mathrm{MinEv}}(\theta)$	$\lambda_4^{\mathrm{MinEv}}(\theta)$	$\lambda_1^{\mathrm{MaxEv}}(\theta)$	$\lambda_2^{\mathrm{MaxEv}}(\theta)$	$\lambda_3^{\mathrm{MaxEv}}(\theta)$	$\lambda_4^{\mathrm{MaxEv}}(\theta)$
$\lambda_1^{\mathrm{MinEv}}(\theta)$	1.0000							
$\lambda_2^{\mathrm{MinEv}}(\theta)$	0.9755	1.0000						
$\lambda_3^{\mathrm{MinEv}}(\theta)$	0.9973	0.9825	1.0000				sym.	
$\lambda_4^{\mathrm{MinEv}}(\theta)$	0.9952	0.9819	0.9939	1.0000				
$\lambda_1^{\mathrm{MaxEv}}(\theta)$	0.9969	0.9749	0.9937	0.9916	1.0000			
$\lambda_2^{\mathrm{MaxEv}}(\theta)$	0.9935	0.9750	0.9891	0.9894	0.9975	1.0000		
$\lambda_3^{\mathrm{MaxEv}}(\theta)$	0.9954	0.9816	0.9935	0.9947	0.9984	0.9975	1.0000	
$\lambda_4^{\mathrm{MaxEv}}(\theta)$	0.9955	0.9804	0.9932	0.9947	0.9977	0.9983	0.9994	1.0000

Abbreviations: $\lambda_i^{\mathrm{MinEv}}(\theta)$, the ith minimum eigenvalues; $\lambda_i^{\mathrm{MaxEv}}(\theta)$, the ith maximum eigenvalues.

$$\rho_{ij} = \frac{\mathbb{E}\left\{\lambda_i(\theta)\,\lambda_j(\theta)\right\} - \mathbb{E}\left\{\lambda_i(\theta)\right\}\mathbb{E}\left\{\lambda_j(\theta)\right\}}{\sqrt{\mathbb{E}\left\{\lambda_i^2(\theta)\right\} - \left[\mathbb{E}\left\{\lambda_i(\theta)\right\}\right]^2}\sqrt{\mathbb{E}\left\{\lambda_j^2(\theta)\right\} - \left[\mathbb{E}\left\{\lambda_j(\theta)\right\}\right]^2}}, \tag{61}$$

is adopted. As listed in Table 1, minimum stochastic eigenvalues are strongly correlative to maximum stochastic eigenvalues, which verifies the correlations between stochastic eigenvalues. The correlation decreases if the values of two stochastic eigenvalues are far away.

As a comparison, we solve the problem by use of the PC method described in Section 2.1. Two-order Legendre PC basis of five uniform random variables are adopted. The number of PC basis is 21 and the size of the derived nonlinear system of equations is 1050. It is seen from Figure 3A that both ROM and PC method are in very good accordance with MCS. Relative errors (defined by $\left|\frac{\mathrm{PDF}-\mathrm{PDF}_{\mathrm{MCS}}}{\mathrm{PDF}_{\mathrm{MCS}}}\right| \times 100\%$) depicted in Figure 3B indicate both of their errors are small enough, but PC method has a bit better accuracy than ROM.

We test computational efficiencies of the proposed method, MCS and PC method. Table 2 shows computational times of minimum and maximum eigenvalues obtained by ROM, MCS, and PC methods. The computational costs of ROM are obviously less than MCS for both minimum and maximum eigenvalues, which demonstrate the high efficiency of the proposed ROM. Compared to ROM, the PC method needs more effort since the size of the derived nonlinear system of equations is larger than that of the original problem and Newton–Raphson is used to solve the nonlinear system.

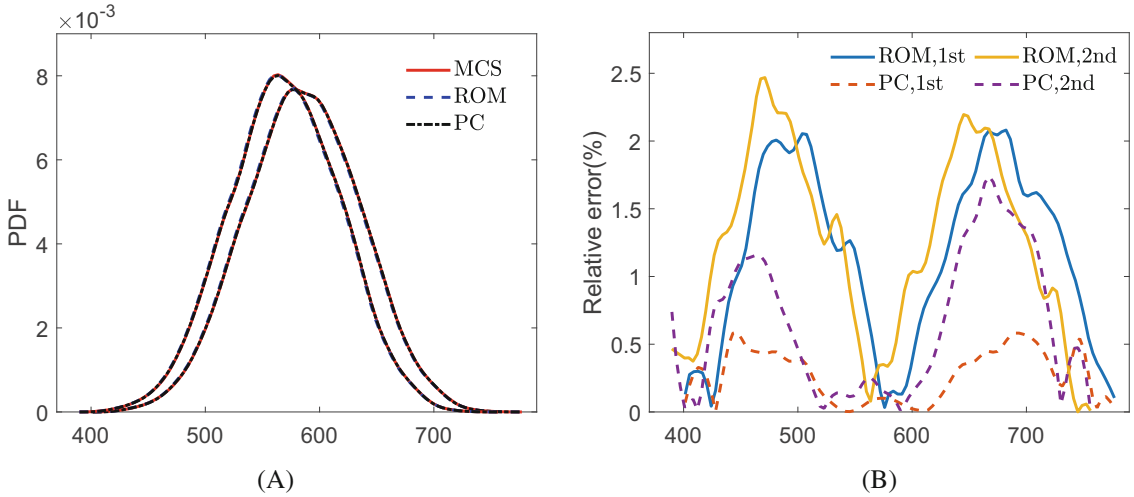


FIGURE 3 Comparisons of PDFs of first two maximum eigenvalues obtained by MCS, ROM, and PC method and corresponding relative errors. (A) Comparisons of PDFs; (B) errors of PDFs of ROM and PC relative to MCS

TABLE 2 Computational costs of minimum and maximum eigenvalues.

	MinEv	MinEv (MC)	MaxEv	MaxEv (MC)	MaxEv (PC)
Solving costs (D)	1.70		1.37		
Solving costs (ROM)	6.98		6.62		
Total costs (s)	8.68	91.05	8.99	93.40	46.28

Abbreviations: MaxEv, maximum eigenvalues; MinEv, minimum eigenvalues.

This difficulty will be more pronounced for large-scale and high-dimensional stochastic problems. More efficient methods are necessary to reduce the computational effort of the PC method. Total computational times of ROM consist of the cost for computing the matrix D and the cost for solving the reduced-order stochastic eigenvalue problem and the former is normally much higher than the latter. It is the opposite in this example since only a small matrix size is tested.

5.1.2 | Non-separated stochastic matrix

In this case, we consider a non-separated stochastic matrix

$$K_{ij}(\theta) = \exp\left(-\frac{|x_i - x_j|}{l_x(\theta)}\right) \in \mathbb{R}^{n\times n},\ x_i, x_j \in [0,1], \tag{62}$$

which is discretized with $n = 100$ DOFs, where $l_x(\theta)$ is the uniform random variable on $[0.5, 1]$.

Expansion methods are available to approximate $K_{ij}(\theta)$, for instance, Equation (3)-like can be obtained by adopting PC method to expand $K(\theta)$. It is noted that the proposed method can be applied to non-separated stochastic matrices without any modification. In this case, we use $n_s = 1\times 10^4$ random samples $K(\theta) \in \mathbb{R}^{n\times n\times n_s}$ instead of the approximation of $K(\theta)$ and set $k_{\max} = 10$. PDFs of first four maximum eigenvalues compared to MCS are shown in Figure 4, which indicates our method has good accuracy for non-separated stochastic matrices. The computational time are 103.19 s for MCS and 21.08 s for ROM, including 13.72 s for computing the matrix D and 7.36 s for solving the reduced-order eigenvalue problem. Compared to separated cases, more effort are needed to compute $\mathbb{E}\{\lambda(\theta)K(\theta)\} \in \mathbb{R}^{n\times n}$ and $d^T K(\theta) d \in \mathbb{R}^{n_s}$ for non-separated stochastic matrices.

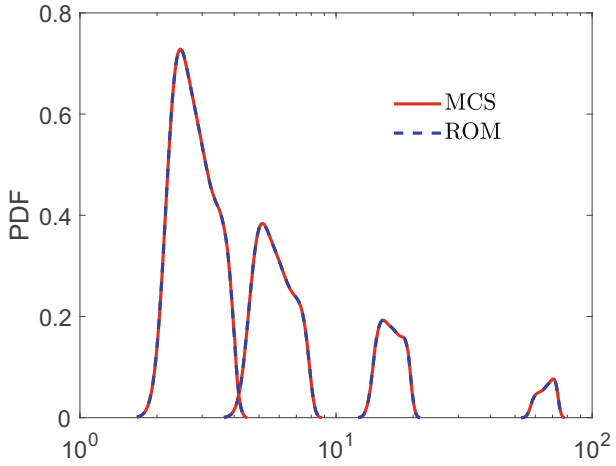


FIGURE 4 PDFs of first four maximum eigenvalues

5.1.3 | Non-symmetric complex stochastic matrix

In this case, we consider a non-symmetric complex stochastic matrix

$$K_{ij}(\theta) = \exp\left(-\frac{|x_i - x_j|}{l_{x,1}(\theta)}\right) + \exp\left(-\frac{\sqrt{x_i - x_j}}{l_{x,2}(\theta)}\right) \in \mathbb{C}^{n\times n},\ x_i, x_j \in [0,1], \tag{63}$$

which is discretized with $n = 100$ DOFs, where $l_{x,1}(\theta)$ and $l_{x,2}(\theta)$ are uniform random variables on $[0.5, 1]$ and $[1, 1.5]$, respectively, and their Pearson correlation coefficient is 0.5.

In this case, we adopt $n_s = 1 \times 10^4$ random samples $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ to describe the stochastic matrix and set $k_{\max} = 10$. The complex stochastic eigenvalue has the form $\lambda(\theta) = \lambda_{\text{real}}(\theta) + i_\lambda \cdot \lambda_{\text{imag}}(\theta)$, where $i_\lambda = \sqrt{-1}$ is the imaginary unit, $\lambda_{\text{real}}(\theta)$ and $\lambda_{\text{imag}}(\theta)$ are real-part and imaginary-part random variables. PDFs of first four maximum eigenvalues compared to MCS are shown in Figure 5, where PDFs of the first four real-part random variables $\left\{\lambda_{j,\text{real}}(\theta)\right\}_{j=1}^4$ are depicted in Figure 5A and PDFs of the first four imaginary-part random variables $\left\{\lambda_{j,\text{imag}}(\theta)\right\}_{j=1}^4$ are depicted in Figure 5B. The PDFs of both real-part and imaginary-part random variables have good agreements with that of MCS, which demonstrates the proposed method still works well in this case. The computational times are 143.96 s for MCS and 31.27 s for ROM, including 27.17 s for computing the matrix D and 4.10 s for solving the reduced-order eigenvalue problem. The proposed method saves a lot of costs compared with MCS.

5.2 | Stochastic vibration modes of a membrane

This example is to calculate the stochastic vibration modes of a membrane, which requires the solution of the following eigenvalue partial differential equation,

$$-\nabla \cdot [c(x,y,\theta)\nabla u(x,y,\theta)] = \lambda(\theta)\, u(x,y,\theta), \tag{64}$$

defined on the domain shown in Figure 6A and $u(x,y,\theta) = 0$ holds on all boundaries (including inner and outer boundaries). The finite element mesh is depicted in Figure 6B, including $n_p = 1157$ nodes $n_e = 1874$ triangular elements. The coefficient $c(x,y,\theta)$ is a Gaussian random field with mean function $c_0(x,y) = 2$ and covariance function

$$C_{cc}(x_1,y_1;x_2,y_2) = \sigma_c^2 \exp\left(-\frac{|x_1 - x_2|}{l_x} - \frac{|y_1 - y_2|}{l_y}\right), \tag{65}$$

where the variance $\sigma_c = 0.1$ and correlation lengths $l_i = \max(i) - \min(i)$, $i = x, y$.

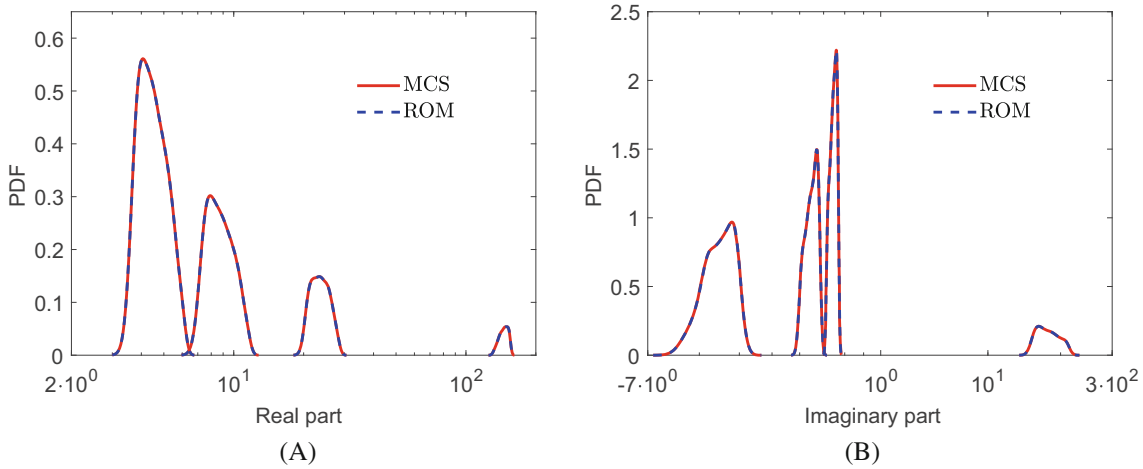


FIGURE 5 PDFs of first four maximum eigenvalues. (A) PDFs of the real parts of first four maximum eigenvalues; (B) PDFs of the imaginary parts of first four maximum eigenvalues

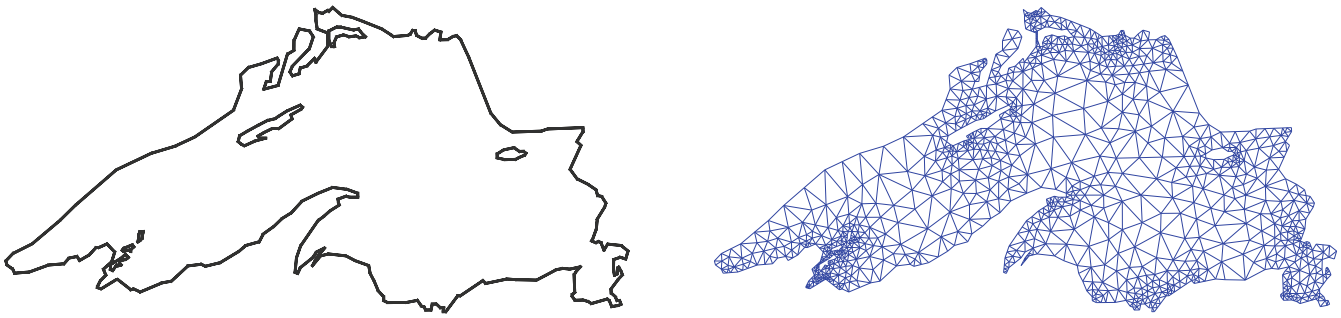


FIGURE 6 Model of the membrane and its finite element mesh. (A) Geometry; (B) finite element mesh

By use of Karhunen–Loève expansion,[45,46] the random field $c(\theta, x, y)$ is approximated as

$$c(x, y, \theta) = \sum_{j=0}^{r} \xi_j(\theta) \sqrt{\kappa_j} c_j(x, y), \tag{66}$$

where $\xi_0(\theta) \equiv 1$, $\kappa_0 \equiv 1$, r is the truncated number, $\{\xi_j(\theta)\}_{j=1}^{r}$ are mutually independent standard Gaussian random variables and $\left\{\sqrt{\kappa_j}, c_j(x, y)\right\}_{j=1}^{r}$ are solved by the following homogeneous Fredholm integral equation of the second kind,

$$\int_{\Omega} C_{cc}(x_1, y_1; x_2, y_2) c_j(x_1, y_1) dx_1 dy_1 = \kappa_j c_j(x_2, y_2). \tag{67}$$

In order to ensure the well-posedness of Equation (64), we need to keep $\min_{x,y \in \Omega}(c(x, y, \theta)) > 0$, $\forall \theta \in \Theta$ in the practical numerical implementation. For this purpose, the sample realization $\theta^{(i)}$ such that $\min_{x,y \in \Omega}\left(c\left(x, y, \theta^{(i)}\right)\right) < 1 \times 10^{-3}$ will be dropped out. Thus $c(x, y, \theta)$ is considered as a truncated Gaussian random field in the numerical processing. In this example, we truncate Equation (66) at $r = 10$ and the first six $\{c_j(x, y)\}_{j=1}^{6}$ obtained by Equation (67) are depicted in Figure 7.

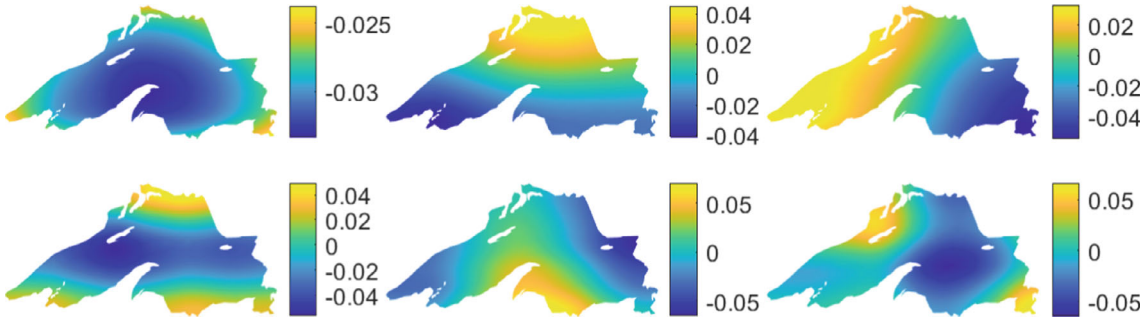


FIGURE 7 First six eigenvectors $\{c_j(x,y)\}_{j=1}^{6}$

Considering the weak form of Equation (64) we have

$$\lambda(\theta)\int_\Omega u(x,y,\theta)\,v(x,y)\,dxdy = \int_\Omega c(x,y,\theta)\,\nabla u(x,y,\theta)\,\nabla v(x,y)\,dxdy \tag{68}$$

$$= \sum_{j=0}^{r}\xi_j(\theta)\sqrt{\kappa_j}\int_\Omega c_j(x,y)\,\nabla u(x,y,\theta)\,\nabla v(x,y)\,dxdy, \tag{69}$$

where $v(x,y)$ is the test function. A stochastic eigenequation is thus generated as

$$K(\theta)\,u(\theta) = \lambda(\theta)\,Mu(\theta), \tag{70}$$

where $K(\theta) = \sum_{j=0}^{r}\xi_j(\theta)\,K_j \in \mathbb{R}^{n_p\times n_p}$, matrices $\{K_j\}_{j=0}^{r}$ and M are computed by using shape functions $\{\varpi_k(x,y)\}$ of triangular elements,

$$K_{j,kl} = \sqrt{\kappa_j}\int_\Omega c_j(x,y)\,\nabla\varpi_k(x,y)\,\nabla\varpi_l(x,y)\,dxdy, \tag{71}$$

$$M_{kl} = \int_\Omega \varpi_k(x,y)\,\varpi_l(x,y)\,dxdy,\ \ k,l = 1,\ \ldots\ ,n. \tag{72}$$

From the perspective of practical engineering applications, it is more concerned about the first several minimum eigenvalues since they are related to structural frequencies and natural vibration modes. We only compute the first five minimum eigenvalues in this example. By use of the proposed ROM, the first nine vectors $\{d_i\}_{i=1}^{9}$ of the matrix D are shown in Figure 8. It is seen that the mode of d_i becomes less important as the number i increases. In other words, we can use fewer $\{d_i\}$ to approximate stochastic eigenvectors in this example. Different retained terms $k_{\max} = 5, 10, 20$ are thus tested. It is seen from Figure 9A that all PDFs of three cases are good enough to approximate the reference solutions and they have similar approximation errors (seen from Figure 9B), which demonstrate five retained terms $\{d_i\}_{i=1}^{5}$ are enough for this example. However, we cannot determine the retained term $k_{\max}$ a priori for practical problems. The preselection of $k_{\max}$ as small as possible is still an open problem for our method and needs further study. Also, the computational time 18.73 s (including 9.97 s for computing the matrix D and 8.76 s for solving reduced-order eigenvalue problem) of ROM is much lower than 4.03×10^3 s for 1×10^4 MCS, which verifies the high efficiency of the proposed ROM again.

5.3 | Stochastic eigenvalue analysis for a single part of robotic arm

This example considers a linear elastic robotic arm shown in Figure 10A, stochastic eigenvalue analysis of only single part of which is proceed. The finite mesh is depicted in Figure 10B, including $n_p = 2062$ nodes and $n_e = 6729$ tetrahedron elements. Material properties of the arm are Poisson's ratio $\nu = 0.30$ and mass density $\rho = 2000$ kg/m^3. The Young's modulus $E(x,y,\theta)$ is considered as a Gaussian random field with mean function $E_0(x,y) = 1.50\times10^{11}$ Pa and covariance function

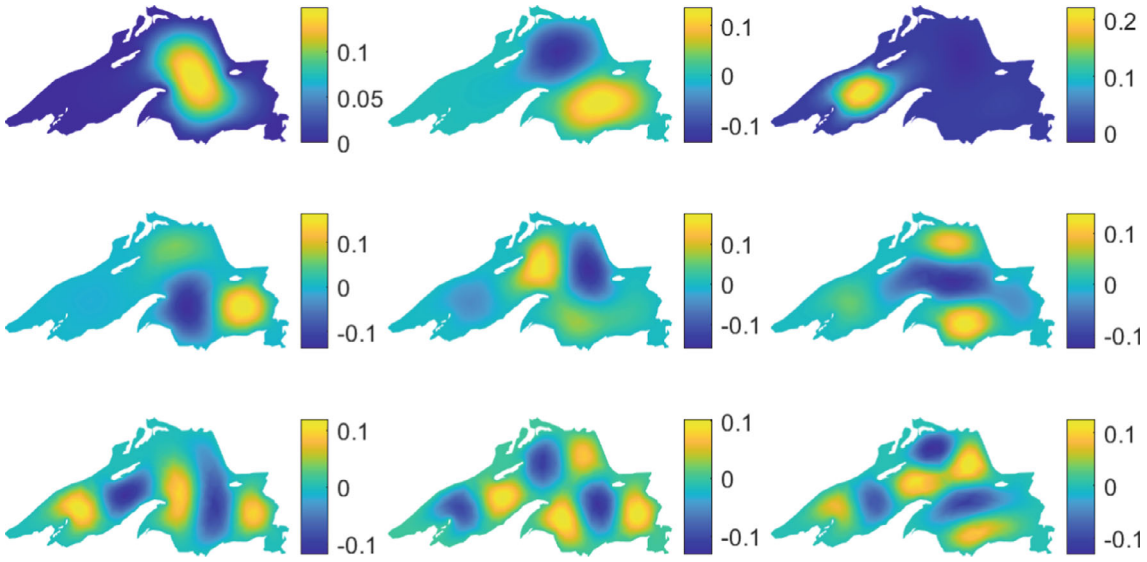


FIGURE 8 First nine reduced basis $\{d_i\}_{i=1}^{9}$

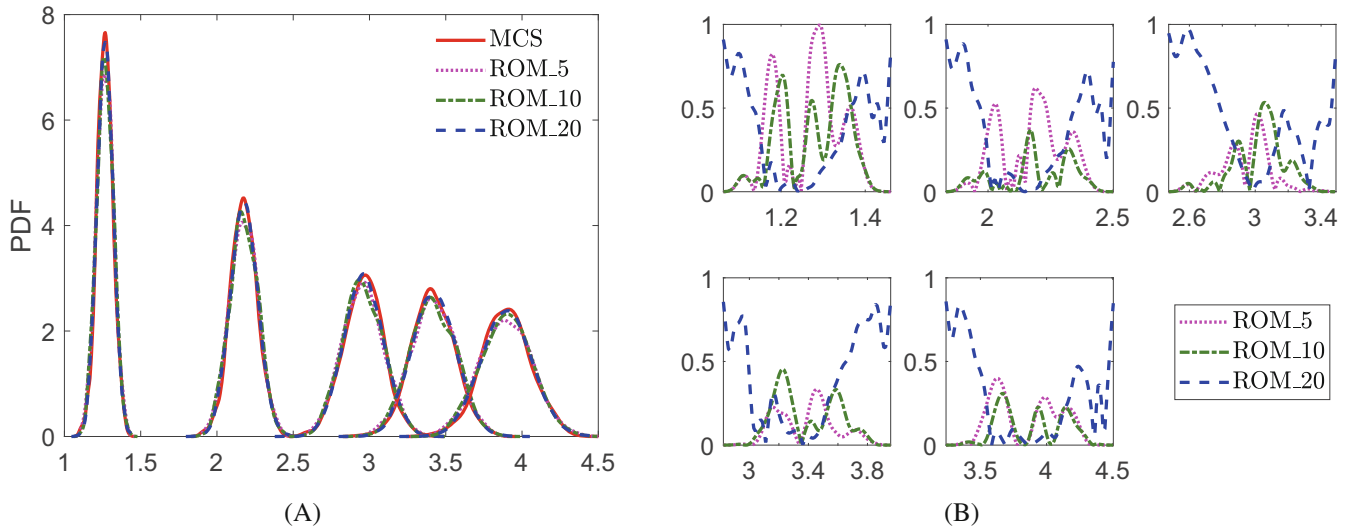


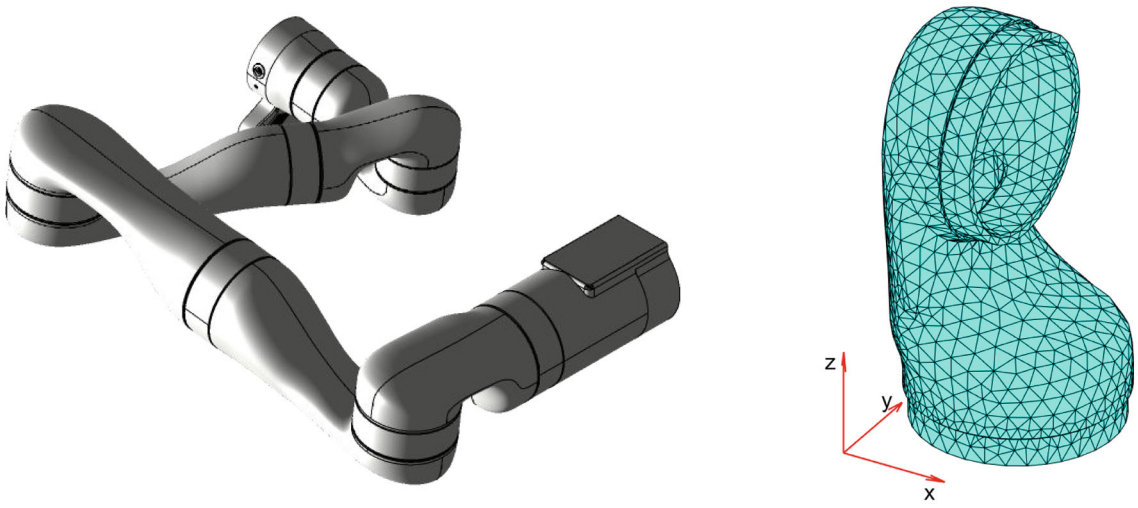
FIGURE 9 PDFs of first five maximum eigenvalues obtained by different numbers of reduced basis and corresponding relative errors. (A) PDFs of first five minimum eigenvalues from MCS and 5, 10, 20 reduced basis; (B) errors of PDFs of 5-, 10-, 20-term ROM relative to MCS

$$C_{EE}(x_1,y_1,z_1;x_2,y_2,z_2)=\sigma_E^2\exp\left(-\frac{|x_1-x_2|}{l_x}-\frac{|y_1-y_2|}{l_y}-\frac{|z_1-z_2|}{l_z}\right), \tag{73}$$

where the variance $\sigma_E=1.50\times10^{10}$ and correlation lengths $l_i=\max(i)-\min(i)$, $i=x,y,z$.

Similar to Section 5.2, we expand the random filed $E(x,y,\theta)$ by use of Equation (66) and obtain a same stochastic eigenequation as Equation (70). Similar to the random field $c(x,y,\theta)$ considered in Equation (66), the sample realization $\theta^{(i)}$ such that $\min_{x,y,z\in\Omega}\left(E\left(x,y,z,\theta^{(i)}\right)\right)<1\times10^{-3}$ is discarded to ensure that the Young's modulus is positive. The retained number is set as $k_{\max}=20$ in this case and the first six minimum eigenvalues are computed. A low-dimensional case $r=10$ is first considered. It is seen from Figure 11 that PDFs of the first six minimum eigenvalues are still in very good accordance with MCS.

The computational time of the case $r=10$ is 73.12 s (seen from Table 3) and corresponding MCS cost is 1.52×10^4 s. For large-scale problems, ROM is more efficient since it only solves a few number of large-scale deterministic eigenequations.



(A) (B)

FIGURE 10 Arm model and its finite element mesh. (A) Model; (B) finite element mesh

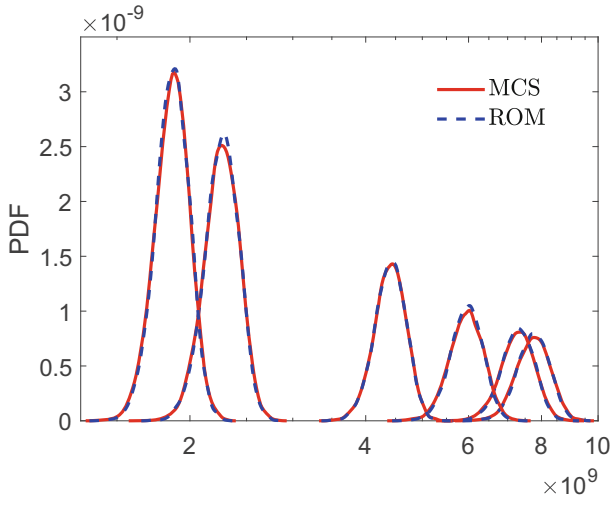


FIGURE 11 PDFs of first six minimum eigenvalues

TABLE 3 Computational costs of different stochastic dimensions *r*.

Dimensions	10	30	60	100
Solving costs (*D*)	63.21	101.63	225.97	251.94
Solving costs (ROM)	9.91	8.09	9.80	10.59
Total costs (s)	73.12	109.72	235.27	262.53

To verify the validity of the proposed method for high-dimensional stochastic problems, different stochastic dimensions $r = 10, 30, 60, 100$ are tested and their computational times are listed in Table 3, which indicate that the proposed ROM is efficient even for a stochastic dimension up to 100. As the stochastic dimension increases, computational times for computing the matrix D increase since extra effort and storage are needed to a large number of matrices $\{K_j\}_{j=1}^{r}$. Computational times for solving reduced-order stochastic eigenequations are almost changeless since the size $k_{\max}$ is chosen to be fixed.

6 | CONCLUSIONS

This article proposes an efficient reduced-order algorithm for solving stochastic eigenvalue problems and certifies its accuracy and efficiency with the aid of several numerical examples. By constructing an approximation of stochastic eigenvectors and developing a dedicated iterative algorithm, solutions of reduced basis are transformed into a few number of deterministic eigenproblems. Existing solvers can be readily incorporated into the computational procedure. Based on the obtained reduced basis, the original eigenequation is transformed into a reduced-order eigenvalue problem, whose solution is solved by use of a non-intrusive sampling method. The proposed method has low computational effort even for very high-dimensional stochastic problems. The curse of dimensionality is thus avoided with great success, which has been illustrated by the numerical example of up to 100 dimensions. In these senses, the proposed method is particularly appropriate for large-scale and high-dimensional stochastic eigenvalue analysis of practical interests.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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