



# Harmonic maps from surfaces of arbitrary genus into spheres

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## Abstract

We relate the existence problem of harmonic maps into  $S^2$  to the convex geometry of  $S^2$ . On one hand, this allows us to construct new examples of harmonic maps of degree 0 from compact surfaces of arbitrary genus into  $S^2$ . On the other hand, we produce new examples of regions that do not contain closed geodesics (that is, harmonic maps from  $S^1$ ) but do contain images of harmonic maps from other domains. These regions can therefore not support a strictly convex functions. Our construction uses M. Struwe's heat flow approach for the existence of harmonic maps from surfaces.

## 1 Introduction

M. Emery [3] conjectured that a region in a Riemannian manifold that does not contain a closed geodesic supports a strictly convex function. W. Kendall [10] gave a counterexample to that conjecture. Having such a counterexample is important to understand the relation between convexity of domains and convexity of functions in Riemannian geometry.

Here, we refine the analysis of that counterexample and connect it with the existence theory of harmonic maps into  $S^2$ . The counterexample is  $S^2$ , equipped with its standard metric, minus three equally spaced subarcs of an equator of length  $\pi/3$  each. That is, we subdivide the equator of  $S^2$  into 6 arcs of equal length and remove every second of them. We construct nontrivial harmonic maps from compact Riemann surfaces whose images are compactly contained in that region. This excludes the existence of a strictly convex function on our region. The reason is that the composition of a harmonic map with a convex function is a subharmonic function on the domain of the map, and therefore has to be constant since our domain is compact. But this is not possible if both the map and the function are nontrivial.

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The main technical achievement of this paper is the construction of the harmonic maps. The existence theory of harmonic maps into surfaces was developed in important papers of J. Sacks and K. Uhlenbeck [15], and L. Lemaire [11]. Since then, various other existence schemes have been discovered (see for instance [7]). While our construction will be more explicit than the general existence results, we also need arguments from the general theory. We use results from the heat flow approach of M. Struwe [16]

Let us now describe our methods in more technical terms. Let  $(N, h)$  be a complete connected Riemannian manifold and let  $V \subsetneq N$  be an open connected subset of  $N$ . Suppose there exists a  $C^2$ -function  $f : V \rightarrow \mathbb{R}$  which is strictly geodesically convex. That is,

$$(f \circ c)'' > 0$$

for every geodesic  $c : (-\epsilon, \epsilon) \rightarrow V$ . We say that such  $V$  is a *convex supporting domain*.

**Remark 1.1** While any subdomain of a convex supporting domain is convex supporting, we can not say much about their topology. But there are geometric constraints. For instance, for  $\epsilon > 0$  there exists no strictly convex function supported in  $\Omega_2^\epsilon := \{x \in S^n \mid d(x, (0, 0, 1)) \leq \frac{\pi}{2} + \epsilon\}$ .

We shall address the question of the relation between the existence of strictly convex functions and the absence of closed geodesics by refining it. For that purpose, let us consider the following properties of an open connected subset  $V \subsetneq N$ .

- (i) There exists a strictly convex function  $f : V \rightarrow \mathbb{R}$
- (ii) Any harmonic map from a closed manifold to  $N$  whose image is contained in  $V$  is constant
- (iii) Any harmonic map from  $S^n$  to  $N$ ,  $n \geq 1$ , whose image is contained in  $V$  is constant
- (iv) There are no closed geodesic in  $V$ .

The maximum principle guarantees that if  $V$  has property (i), then it has property (ii), and it is obvious that (ii) implies (iii) which implies (iv). The question then is which of the converse implications hold. M. Emery [3] conjectured that, if a subset  $V \subseteq N$  admits no closed geodesics, that is, if it has property (iv), then there exists a strictly convex function  $f : V \rightarrow \mathbb{R}$ ; in other words, (iv) should imply (i). This conjecture was refuted by W. Kendall [10], who gave an example of a two-dimensional domain of the shape of a propeller without closed geodesics, where one cannot define a strictly convex function. In the present paper we obtain a stronger result regarding the existence of harmonic maps into a region that is simpler than Kendall's (but has the same symmetries), where we can easily see that it can neither support a strictly convex function nor a closed geodesic. To analyze the above refined question, we then construct non-constant harmonic maps into this region which proves that property (iv) does not imply (ii).

We use the harmonic map heat flow to construct a smooth harmonic map

$$u_\infty : \Sigma_2 \rightarrow S^2$$

where  $\Sigma_2$  is a closed Riemann surface of genus 2, and whose image shares the symmetries of the set

$$\Omega := S^2 \setminus (\Gamma^1, \Gamma^2, \Gamma^3).$$

A detailed explanation of this region is present in the next sections, see Fig. 2. Our construction works for Riemannian surfaces of higher genus, with its image now contained in

subsets of the sphere with more equally spaced portions of the equator cut out, and it gives a general method to construct harmonic maps from compact Riemann surfaces of any genus into symmetric surfaces: We first construct harmonic maps from graphs and then fatten the graphs to become Riemann surfaces. To construct the appropriate harmonic maps is the main technical achievement. In our construction, image symmetries under the harmonic map flow are preserved by the solution at time  $t = \infty$ . The result can be summarized as follows.

**Theorem** *There exist a genus two Riemann surface  $\Sigma_2$  and a smooth harmonic map  $\phi : \Sigma_2 \rightarrow S^2$  whose image is contained in  $S^2$  with three equally spaced regions of the equator cut out. In particular,  $\phi$  is a degree zero harmonic map into  $S^2$  with no closed geodesics on its image.*

As mentioned above, we can adapt this result to higher genus surfaces with values in a corresponding region of  $S^2$  in such a way that the above theorem holds true. The image of the constructed harmonic map will have no closed geodesics if and only if the respective region has an odd number of line segments removed.

What about other reverse implications for (i)-(iv)? In the last section of the present paper we show that the first reverse implication essentially holds following our work [1], and add remarks and strategies to decide whether the other implications might be true.

A complete answer to all possible implications between (i)-(iv) would be a powerful result. In particular, since we have given abstract ways of showing that a region has property (ii) [1], this would yield a conceptual way of verifying whether a region does admit a strictly convex function. A similar question had already been raised by W. Kendall from a martingale perspective [9].

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## 2 Preliminaries

### 2.1 Harmonic maps

In this section, we introduce basic notions and properties of harmonic maps; references are [13, 17] or [8].

Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds without boundary, of dimension  $m$  and  $n$ , respectively. By Nash’s theorem there exists an isometric embedding  $N \hookrightarrow \mathbb{R}^L$ .

**Definition 2.1** A map  $u \in W^{1,2}(M, N)$  is called harmonic if and only if it is a critical point of the energy functional

$$E(u) := \frac{1}{2} \int_M \|du\|^2 dvol_g \tag{2.1}$$

where  $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$  is the metric over the bundle  $T^*M \otimes u^{-1}TN$  induced by  $g$  and  $h$ .

Recall that the Sobolev space  $W^{1,2}(M, N)$  is defined as

$$W^{1,2}(M, N) = \left\{ v : M \rightarrow \mathbb{R}^L; \|v\|_{W^{1,2}(M)}^2 = \int_M (|v|^2 + \|dv\|^2) dvol_g < +\infty \text{ and} \right.$$

$$v(x) \in N \text{ for almost every } x \in M \Big\}$$

**Remark 2.2** If  $M$  is a Riemann surface,  $u \in W^{1,2}(M, N)$  is a harmonic map, and  $\phi : M \rightarrow M$  is a conformal diffeomorphism, i.e.  $\phi^*g = \lambda^2g$  where  $\lambda$  is a smooth function, we have, by (2.1), that

$$E(u \circ \phi) = \int_M \|du\|^2 dvol_g, \tag{2.2}$$

that is, the energy of harmonic maps from surfaces is conformally invariant. Therefore, on a surface, we need not specify a Riemannian metric, but only a conformal class, that is the structure of a Riemann surface, to define harmonic maps.

For  $u \in W^{1,2}(M, N)$ , define the map  $du : \Gamma(TM) \rightarrow u^{-1}(TN)$ , given by  $X \mapsto u_*X$ . We denote by  $\nabla du$  the gradient of  $du$  over the induced bundle  $T^*M \otimes u^{-1}(TN)$ , that is,  $\nabla du$  satisfies  $\nabla_Y du \in \Gamma(T^*M \otimes u^{-1}(TN))$ , for each  $Y \in \Gamma(TM)$ .

**Definition 2.3** (*Second fundamental form*) The second fundamental form of the map  $u : M \rightarrow N$  is the map defined by

$$B_{XY}(u) = (\nabla_X du)(Y) \in \Gamma(u^{-1}TN). \tag{2.3}$$

$B$  is bilinear and symmetric in  $X, Y \in \Gamma(TM)$ . It can also be seen as

$$B(u) \in \Gamma(Hom(TM \odot TM, u^{-1}TN)).$$

**Definition 2.4** The tension field of a map  $u : M \rightarrow N$  is the trace of the second fundamental form

$$\tau(u) = B_{e_i e_i}(u) = (\nabla_{e_i} du)(e_i) \tag{2.4}$$

seen as a cross-section of the bundle  $u^{-1}TN$ .

Taking a  $C^1$  variation  $(u(\cdot, t))_{\|t\| < \epsilon}$  yields

$$\frac{d}{dt} E(u(\cdot, t)) = - \left\langle \tau(u(x, t)), \frac{\partial u}{\partial t} \right\rangle_{L^2}. \tag{2.5}$$

By (2.5), the Euler-Lagrange equations for the energy functional are

$$\tau(u) = \left( \Delta_g u^\alpha + g^{ij} \Gamma_{\beta\gamma}^\alpha \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} h_{\beta\gamma} \right) \frac{\partial}{\partial u^\alpha} = 0, \tag{2.6}$$

where the  $\Gamma_{\beta\gamma}^\alpha$  denote the Christoffel symbols of  $N$ .

Hence we can equivalently define (weakly) harmonic maps as maps that (weakly) satisfy the *harmonic map equation* (2.6).

**Definition 2.5** (*Totally geodesic maps*) A map  $u \in W^{1,2}(M, N)$  is called totally geodesic if and only if its second fundamental form vanishes identically, i.e.,  $B(u) \equiv 0$ .

**Lemma 2.6** (*Composition formulas*) Let  $u : M \rightarrow N$  and  $f : N \rightarrow P$ , where  $(P, i)$  is another Riemannian manifold. Then

$$\nabla d(f \circ u) = df \circ \nabla du + \nabla df(du, du), \tag{2.7}$$

$$\tau(f \circ u) = df \circ \tau(u) + \text{tr} \nabla df(du, du). \tag{2.8}$$

**Example 2.7** (L. Lemaire [12])

For every integers  $p$  and  $\mathcal{D}$  such that  $|\mathcal{D}| \leq |p - 1|$ , there exists a Riemann surface of genus  $p$  and a harmonic nonholomorphic map of degree  $\mathcal{D}$  from that surface to the sphere. For the case of  $\mathcal{D} = 0$ , one considers a 1-form as a harmonic map  $\tilde{\phi} : \Sigma_g \rightarrow S^1$  and a totally geodesic embedding  $\iota : S^1 \hookrightarrow S^2$ . The composition

$$\phi := \iota \circ \tilde{\phi}$$

is harmonic by Equation (2.8). Here, the image

$$\phi(\Sigma_g) \subset S^2$$

is contained in the image of a closed geodesic of  $S^2$ , and therefore, is geometrically trivial. The main construction of this paper yields harmonic maps from Riemann surfaces of even genus into  $S^2$  that do not contain closed geodesics in their images.

### 2.2 The harmonic map flow

Let  $M$  be a compact Riemann surface and  $N$  a compact Riemannian manifold. M. Struwe [16] (see also Chang [2]) showed that the heat flow

$$\begin{cases} \partial_t u(x, t) = \tau(u(x, t)) \\ u(\cdot, 0) = u_0. \end{cases} \tag{2.9}$$

has a global weak solution. By (2.5), we have

$$\begin{aligned} \frac{d}{dt} E(u(x, t)) &= - \int_M \langle \tau(u(x, t)), \partial_t u(x, t) \rangle_{u^*TN} dvol_g \\ &= - \int_M |\tau(u(x, t))|^2 dvol_g, \end{aligned} \tag{2.10}$$

and since  $|\tau(u(x, t))|^2 \geq 0$ , the energy does not increase along the flow.

M. Struwe’s solution is smooth with the possible exception of finitely many singular points. A. Freire [4] proved uniqueness among the maps for which  $E(\cdot, t)$  is decreasing on  $t$ . We shall summarize both results in the following theorem.

**Theorem 2.8** (M. Struwe, A. Freire) *For any initial value  $u_0 \in W^{1,2}(M, N)$ , there exists a weak solution  $u$  of the equation (2.9) in  $W^{1,2}(M \times [0, +\infty), N)$ , and this solution is unique among the maps for which  $E(\cdot, t)$  is decreasing in  $t$ . Moreover, in  $M \times [0, +\infty)$ ,  $u$  is smooth with the exception of finitely many points.*

Struwe’s results on the harmonic map flow will be the main tools to construct the desired harmonic map.

## 3 The main construction

### 3.1 The construction of the Riemann surface

We start by describing the region on  $S^2$  which will contain the image of our harmonic map, see Fig. 1.

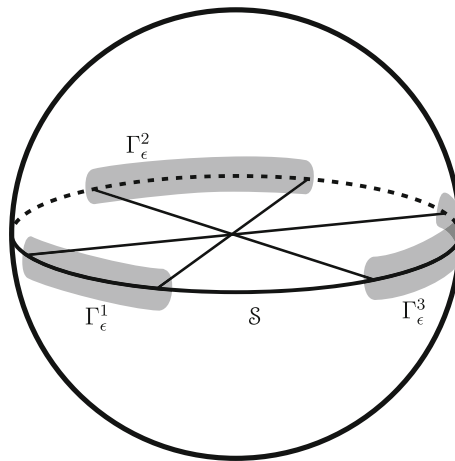


Fig. 1 The subset  $\Omega_\epsilon := S^2 \setminus (\Gamma_\epsilon^1 \cup \Gamma_\epsilon^2 \cup \Gamma_\epsilon^3)$  has no closed geodesics

**Notation 3.1** We denote by  $S$  the horizontal equator of  $S^2$ . More precisely,

$$S := \{(x, y, z) \in S^2 \mid z = 0\}$$

**Definition 3.2** Divide  $S$  in six equal pieces and pick  $\Gamma^1, \Gamma^2, \Gamma^3$  as three non-intersecting, equal lengths, equidistant sub-arcs of  $S$ , as in Fig. 1. Thus, each of these arcs has length  $\frac{\pi}{3}$ , and they are separated from each other by arcs of the same length. We consider the fattened sets

$$\Gamma_\epsilon^i := \{x \in S^2 \mid \text{dist}(x, \Gamma_i) \leq \epsilon\},$$

for each  $i = 1, 2, 3$ . We put

$$\Omega_\epsilon := S^2 \setminus (\Gamma_\epsilon^1 \cup \Gamma_\epsilon^2 \cup \Gamma_\epsilon^3)$$

.

For a point  $p \in S \setminus (\Gamma^1 \cup \Gamma^2 \cup \Gamma^3)$ , it is obvious that its antipodal point  $A(p)$  fulfills

$$A(p) \in \Gamma^1 \cup \Gamma^2 \cup \Gamma^3,$$

therefore  $\Omega_\epsilon$  does not admit closed geodesics.

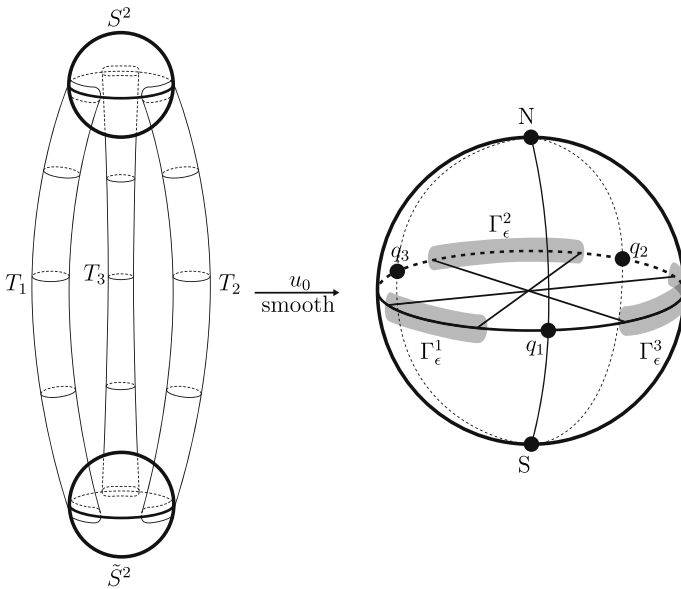
Now we proceed taking inspiration from the symmetries of  $\Omega_\epsilon$  to define a compact Riemann surface  $\Sigma_2$  that will serve us as the domain of our harmonic map as follows, see Fig. 2.

For a number  $R > 1$ , consider the poles  $S = (0, 0, R - 1)$  and  $N = (0, 0, R + 1)$  in

$$S^2(R, 1) := \left\{ (x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2 + (z - R)^2} = 1 \right\}$$

and denote by  $\tilde{S}^2$  the sphere below

$$\tilde{S}^2 := S^2(-R, 1) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2 + (z + R)^2} = 1 \right\}$$



**Fig. 2** The Riemann surface  $\Sigma_2$  and the map  $u_0$

where we will also use the tilde sign to denote the points  $\tilde{N} = (0, 0, -R + 1)$ , and  $\tilde{S} = (0, 0, -R - 1)$ . In  $S^2(R, 1)$  and  $\tilde{S}^2$ , consider also segments  $\Gamma^i, \tilde{\Gamma}^i$ , and the sets  $\Gamma_\epsilon^i, \tilde{\Gamma}_\epsilon^i$  defined as in Definition 3.2.

To build the Riemann surface out of these sets, we connect the two spheres by three cylinders, called  $T^1, T^2, T^3$  such that

$$\partial T^i = \Gamma_\epsilon^i \cup \tilde{\Gamma}_\epsilon^i,$$

i.e., the cylinders connect the sets  $\Gamma_\epsilon^i$  and  $\tilde{\Gamma}_\epsilon^i$  pairwise and smoothly.

We shall finally denote by  $\Sigma_2$  the smooth Riemann surface resulting from the union of the sets defined above. More precisely,

$$\Sigma_2 := \left( S^2(R, 1) \setminus (\Gamma_\epsilon^1 \cup \Gamma_\epsilon^2 \cup \Gamma_\epsilon^3) \right) \cup T^1 \cup T^2 \cup T^3 \cup \left( \tilde{S}^2 \setminus (\tilde{\Gamma}_\epsilon^1 \cup \tilde{\Gamma}_\epsilon^2 \cup \tilde{\Gamma}_\epsilon^3) \right) \quad (3.1)$$

Note that  $\Sigma_2$  has genus two. The following two symmetries of  $\Sigma_2$  will be of central importance.

- (a) A  $\mathbb{Z}_3$  symmetry around the  $z$ -axis is preserved, i.e.,  $\Sigma_2$  is invariant under a rotation by a multiple of  $\frac{2\pi}{3}$ .
- (b) A  $\mathbb{Z}_2$  symmetry around the midpoint between  $\tilde{N}$  and  $S$  is preserved, i.e.,  $\Sigma_2$  is invariant under an upside-down rotation.

### 3.2 The initial condition for the harmonic map heat flow

To find a non-trivial harmonic map from  $\Sigma_2$  into  $S^2$  with its image contained in  $\Omega_\epsilon$ , we define an initial condition  $u_0 \in C^\infty(\Sigma_2, S^2)$ , and explore the symmetry preserving properties of the harmonic map flow. We construct  $u_0$  systematically in the next paragraphs.

Let  $\gamma$  be the geodesic in  $S^2$  connecting  $(0, 0, 1) \in S^2$  to the point  $(0, 0, -1) \in S^2$  given by the equation below.

$$\begin{aligned} \gamma : [-R, R] &\longrightarrow S^2 \\ t &\longmapsto \left( \sin \left( \frac{\pi(t+R)}{2R} \right), 0, -\cos \left( \frac{\pi(t+R)}{2R} \right) \right), \end{aligned}$$

where  $2R$  is the height of the sets  $T^i, i = 1, 2,$  or  $3,$  defined in the previous section. In other words,  $2R = \text{dist}(\Gamma_\epsilon^i, \tilde{\Gamma}_\epsilon^i),$  where  $\text{dist}$  denotes obviously the distance in  $\mathbb{R}^3.$

Let  $\varphi : S^2 \longrightarrow S^2$  be the  $\mathbb{Z}_3$ -action on  $S^2$  given by the anti-clockwise rotation of  $\frac{2\pi}{3}$  degrees over the  $z$ -axis in  $\mathbb{R}^3.$  Using this action and the above curve, we define three distinct geodesics  $\gamma^1, \gamma^2,$  and  $\gamma^3$  in  $S^2$  by  $\gamma^1 := \gamma, \gamma^2 := \varphi(\gamma),$  and  $\gamma^3 := \varphi^2(\gamma).$

Moreover, consider the parametrization of  $T^i$  given by the following map.

$$\begin{aligned} \Gamma^i : S^1 \times [-R, R] &\longrightarrow T^i \\ (\theta, t) &\longmapsto (r \cos \theta, r \sin \theta, t), \end{aligned}$$

where  $r$  is the radius of the cylinders  $T^i.$

Up to the diffeomorphism that makes the cylinders of  $\Sigma_2$  straight, that is,  $T^i = S^1 \times [-R, R],$  we pick  $u_0 \in C^\infty(\Sigma_2, S^2)$  as a map satisfying

$$u_0 : \Sigma_2 \longrightarrow S^2 \tag{3.2}$$

$$x \mapsto \begin{cases} (0, 0, 1) & \text{if } x \in S^2(R, 1) \setminus (\Gamma_\epsilon^1 \cup \Gamma_\epsilon^2 \cup \Gamma_\epsilon^3) \\ (0, 0, -1) & \text{if } x \in \tilde{S}^2 \setminus (\Gamma_\epsilon^1 \cup \Gamma_\epsilon^2 \cup \Gamma_\epsilon^3) \\ \gamma^i(t) & \text{if } x = (\theta, t) \in T^i \end{cases} \tag{3.3}$$

Note that we are omitting a geometrically trivial step: Near the connection of the cylinders to the spheres, one needs to smooth out  $\Sigma_2$  and  $u_0.$  This can be obviously done via writing the equations for  $u_0$  under a suitable perturbation of the radius of each  $T^i$  near the spheres, but that would add a tedious equation to a rather trivial geometrical idea.

It is a very important step in the construction that the image of  $\Sigma_2$  under  $u_0$  preserves the  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  symmetries of  $\Sigma_2,$  and the harmonic map flow will preserve these symmetries of the initial condition, as we explicit in the lemma below.

**Lemma 3.3** (*Symmetry property of the flow*) *Let  $\varphi$  be the  $\mathbb{Z}_3$ -action on  $\Sigma_2$  given by the anti-clockwise rotation of  $\frac{2\pi}{3}$  over the  $z$ -axis in  $\mathbb{R}^3.$  Let  $\psi$  be the corresponding  $\mathbb{Z}_3$  action over the sphere  $S^2.$  Let  $u$  be the Struwe solution for (2.9) with initial condition  $u_0 \in C^\infty(\Sigma_2, S^2).$  Then  $(\psi \circ u)$  is the Struwe solution for the initial value problem*

$$\begin{cases} \partial_t(\psi \circ u) = \tau(\psi \circ u) \\ \psi \circ u(\cdot, t) = u_0 \circ \varphi. \end{cases}$$

Analogously,  $(u \circ \varphi)$  solves

$$\begin{cases} \partial_t(u \circ \varphi) = \tau(u \circ \varphi) \\ u \circ \varphi(\cdot, t) = \psi \circ u_0. \end{cases}$$

**Proof** Since  $\psi$  is a totally geodesic map, we have

$$\begin{aligned} \partial_t(\psi \circ u) - \tau(\psi \circ u) &= d\psi(u(\cdot, t))\partial_t u - d\psi(u(\cdot, t))\tau(u) \\ &= d\psi(u(\cdot, t))(\partial_t u - \tau(u)), \end{aligned}$$



and the latter is zero, since  $u$  is a solution to (2.9). The lemma now follows from Freire’s uniqueness of the Struwe solution.  $\square$

An analogous result holds for the  $\mathbb{Z}_2$  action  $(a, b, c) \mapsto (a, b, -c)$ . Therefore the harmonic map  $u_\infty : \Sigma_2 \rightarrow S^2$  given by Struwe’s solution of (2.9) with initial condition  $u_0$ , if it develops no singularities, is  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$  invariant with respect to the actions defined above.

Another important remark about the initial condition  $u_0$  is that we can control its energy by changing the length and radius of the tubes  $T^i$ .

**Lemma 3.4** *Let  $u_0 \in C^\infty(\Sigma_2, S^2)$  be the map given by the Definition 3.2. Let  $r$  be the radius of the cylinder  $T^i$  and  $2R$  its length. Then the following inequality holds.*

$$0 < E(u_0) \leq \frac{\pi^3 r^2}{4R} \tag{3.4}$$

**Proof** Since  $\|\dot{\gamma}(t)\| = \frac{\pi}{2R}$ , we have

$$0 < \|du_0\| \leq \frac{\pi}{2R}$$

and therefore

$$\begin{aligned} 0 < E(u_0) &\leq \frac{1}{2} \int_{T^i} \left(\frac{\pi}{2R}\right)^2 \, d\text{vol}_{T^i} \\ &= \frac{\pi^2}{8R^2} \int_{T^i} d\text{vol}_{T^i} \\ &= \frac{\pi^3 r^2}{4R}. \end{aligned}$$

$\square$

By making the cylinders connecting  $S^2(R, 1)$  and  $\tilde{S}^2$  in  $\Sigma_2$  thinner and longer, we can make the energy of  $u_0$  arbitrarily small. More precisely, given any  $\eta > 0$ , we can pick  $r \in (0, 1)$  and  $R \in (1, +\infty)$  such that  $E(u_0) \leq \eta$ . By Equation (2.10), we know that the energy decreases along the flow and therefore the above lemma gives a control on the energy of  $u(\cdot, t)$  for every  $t \in \mathbb{R}_+$ . Once again, after such a procedure of varying the cylinders forming  $\Sigma_2$ , one needs to change the definition of  $u_0$  accordingly. Note that the image of the adapted  $u_0$  will omit the desired set in  $S^2$ .

### 3.3 Controlling the image of $u_\infty$

Theorem 2.8 above roughly tells us that given any initial condition  $u_0 \in W^{1,2}(M, N)$ , there exist a smooth solution to the harmonic map flow with the exception of some finite points on which the energy is controlled. In other words, we have a harmonic map

$$u_\infty : M \setminus \{q_1, \dots, q_l\} \rightarrow N$$

and around the singularities this map can be extended to a harmonic sphere  $h : S^2 \rightarrow N$ .

Since in our case the target is  $S^2$  and each harmonic two-sphere which bubbles out carries at least the energy  $4\pi$  and every energy loss during the flow is due to the formation of a bubble (see [14]), we avoid the formation of bubbles by taking  $\Sigma_2$  as the compact Riemann

surface with cylinders  $T^i$  of length  $R \in (1, +\infty)$  and radius  $r \in (0, 1)$ , such that for a given  $\eta_0 > 0$  we have

$$E(u_0) \leq \eta_0 < 4\pi.$$

Putting it more simply, a bubble has to cover all of  $S^2$ , but our maps cannot do that as their energy is too small for that.

With this initial condition  $u_0$ , we have a unique global smooth solution  $u$  of the harmonic map flow (2.9). This gives us the smooth harmonic map

$$u_\infty : \Sigma_2 \rightarrow S^2. \tag{3.5}$$

with the desired  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$  symmetries. But to obtain the harmonic map whose image is properly contained in the subset  $S^2 \setminus (\Gamma^1 \cup \Gamma^2 \cup \Gamma^3)$ , we need the following lemma due to R. Courant, together with the maximum principle and a perturbation argument of harmonic replacement type, see for instance [7].

**Lemma 3.5** (Courant-Lebesgue) *Let  $f \in W^{1,2}(D, \mathbb{R}^d)$ ,  $E(f) \leq C$ ,  $\delta < 1$  and  $p \in D = \{(x, y) \in \mathbb{C}; x^2 + y^2 = 1\}$ . Then there exists some  $r \in (\delta, \sqrt{\delta})$  for which  $f|_{\partial B(p,r) \cap D}$  is absolutely continuous and*

$$|f(x_1) - f(x_2)| \leq (8\pi C)^{\frac{1}{2}} \left( \log \frac{1}{\delta} \right)^{-\frac{1}{2}} \tag{3.6}$$

for all  $x_1, x_2 \in \partial B(p, r) \cap D$ .

The  $\mathbb{Z}_2$  symmetry of  $u_\infty$  implies that there are two antipodal points, which we can say without loss of generality are  $(0, 0, 1)$  and  $(0, 0, -1)$ , on the image  $u_\infty(\Sigma_2)$ . More than that, by the  $\mathbb{Z}_3$  symmetry and the fact that the image is connected, there exist three points  $p_i \in T_i$ ,  $i = 1, 2, 3$ , each of them  $\mathbb{Z}_2$ -invariant, with  $u_\infty(p_i) \in \mathcal{S} := \{(x, y, z) \in S^2 \mid z = 0\}$ . Obviously,  $\varphi(u_\infty(p_1)) = u_\infty(p_2)$ ,  $\varphi(u_\infty(p_2)) = u_\infty(p_3)$  and  $\varphi(u_\infty(p_3)) = u_\infty(p_1)$ , since  $u_\infty$  has  $\mathbb{Z}_3$ -symmetry.

With the help of (Courant-Lebesgue) Lemma 3.5 and the maximum principle, we shall see that a small cylindrical neighborhood of  $T^i$  around  $p_i$  is mapped into a small neighborhood of  $u_\infty(p_i)$  of controlled size. In fact, since

$$E(u_\infty) \leq E(u_0) < \frac{r^2}{R},$$

where  $r$  is the radius of the sets  $T^i$  and  $R$  their heights, we start taking  $r \in (0, 1)$  and  $R \in (1, +\infty)$  such that  $u_\infty$  has no bubbles and  $r\pi < \pi/6$ . As previously remarked, this is possible since the minimum energy for a bubble in our setting is  $4\pi$  and the energy is controlled by equation (3.4).

Therefore, taking  $\delta < r^2$ , the Courant-Lebesgue lemma implies that there exists some  $s \in (r^2, r)$  such that

$$|u_\infty(x_1) - u_\infty(x_2)| \leq \left( \frac{8\pi \cdot r^2}{R} \right)^{\frac{1}{2}} \left( \log \left( \frac{1}{\delta} \right) \right)^{-\frac{1}{2}} \tag{3.7}$$

for every  $x_1, x_2 \in \partial B_g(p_i, s) \cap B_g(p_i, 1)$ . But since  $\delta < r^2$  and  $\frac{1}{\delta} > \frac{1}{s}$ , we get

$$\left( \log \left( \frac{1}{s} \right) \right)^{-\frac{1}{2}} > \left( \log \left( \frac{1}{\delta} \right) \right)^{-\frac{1}{2}},$$

and therefore by equation (3.7) we have

$$\begin{aligned}
 |u_\infty(x_1) - u_\infty(x_2)| &\leq \left(\frac{8\pi \cdot s}{R}\right)^{\frac{1}{2}} \left(\log\left(\frac{1}{s}\right)\right)^{-\frac{1}{2}} \\
 &= \frac{\sqrt{s}}{\left(\log\left(\frac{1}{s}\right)\right)^{\frac{1}{2}}} \cdot \sqrt{\frac{8\pi}{R}} < \frac{\pi}{6}.
 \end{aligned}$$

In particular, if we call  $S_i^1$  the circle given by the intersection of the  $x, y$ -plane in  $\mathbb{R}^3$  with  $T^i$ , the above argument based on the Courant-Lebesgue lemma shows that the image of the boundary of a  $\delta$ -neighborhood of  $S_i^1$  under  $u_\infty$  is contained in a neighborhood of  $u_\infty(p_i)$  that does not intersect any of the removed sets  $\Gamma_\epsilon^i$ . But since  $u_\infty$  is harmonic, the maximum principle (see [6]) implies that the image of the entire  $\delta$ -neighborhood of  $S_i^1$  under  $u_\infty$  is in the same neighborhood of  $u_\infty(S_i^1)$ .

To finally obtain that the image of the harmonic map under construction is contained in the desired set, we could proceed via taking a  $C^1$ -variation of  $u_\infty$  and showing that once  $u_\infty$  intersects  $\mathcal{S}$ , the equator in Notation 3.1, without touching the removed sets, any variation of it that does intersect any of the  $\Gamma^i$ 's would locally increase energy, or we can simply perform a harmonic replacement on  $u_\infty$  as follows.

Define the set of Sobolev maps that are in the homotopy class

$$\mathbb{B} := \left\{ f \in W^{1,2}(\Sigma_2, S^2) \cap C^0(\Sigma_2, S^2) \mid f \circ \varphi = \psi \circ f, f_3 = -f_3, f(N) = u_\infty(N) \right. \\
 \left. f(\tilde{S}) = u_\infty(\tilde{S}), \text{ and } f(p_i) = u_\infty(p_i) \right\}$$

where  $\varphi, \psi$  are the actions on  $\Sigma_2$  and  $S^2$  as in the Lemma 3.3, and we have written  $f = (f_1, f_2, f_3)$ . We replace  $u_\infty$  by a harmonic map  $\phi : \Sigma_2 \rightarrow S^2$  in  $\mathbb{B}$  constructed from  $u_\infty$  in the following way: For  $i = 1, 2$ , or  $3$ , denote by  $D_1^i$  the subset of  $\partial B(u_\infty(p_i), s_1) \cap u_\infty(T^i)$  contained in the north hemisphere and by  $D_2^i$  the connected component of  $\partial B((0, 0, 1), s_2) \cap u_\infty(T^i)$  whose inverse image through  $u_\infty$  has the biggest  $z$ -component in  $T^i$  (that is, take a continuous curve  $c$  going from a point in  $S^2(R, 1)$  to a point in  $\tilde{S}^2$  passing through  $T^i$ , then  $D_2^i$  is the connected component containing the first point of intersection between  $u_\infty(c)$  and  $\partial B((0, 0, 1), s_2)$ ). Note that the above sets are contained in one regular ball of  $S^2$  with radius smaller than  $\pi/4$ , provided  $s_1, s_2 < \pi/6$  being large enough so that

$$\left(\partial B((0, 0, 1), s_2) \cap u_\infty(T^i)\right) \cap \left(\partial B((0, 0, 1), s_2) \cap u_\infty(T^j)\right) = \emptyset, \text{ when } j \neq i,$$

where  $B(\cdot, \cdot)$  denote geodesic balls in  $S^2$ . Now, one considers the small solution of the Dirichlet problem with  $D_1^i$  and  $D_2^i$  as boundary conditions in the sense of S. Hildebrandt (see [5] for details on regular balls). Then one reflects this solution, called  $\phi$ , using the  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  symmetries of  $u_\infty$ , and make  $\phi \equiv u_\infty$  in  $B((0, 0, 1), s_2)$ ,  $B((0, 0, -1), s_2)$ , and  $B(u_\infty(p_i), s_1)$  for every  $i = 1, 2, 3$ . The resulting map  $\phi$  is therefore continuous, satisfies the harmonic map equation since it is a harmonic replacement of  $u_\infty$ , it belongs to  $\mathbb{B}$ , and it does not intersect the removed sets by construction. Furthermore, since our domain has dimension two and  $\phi \in W^{1,2}(\Sigma_2, S^2) \cap C^0(\Sigma_2, S^2)$  is harmonic, regularity tells us that  $\phi$  is indeed smooth, as we wanted. The above constructions can be summarized in the following theorem.

**Theorem 3.6** *There exist a genus two Riemann surface  $\Sigma_2$  and a smooth harmonic map  $\phi : \Sigma_2 \rightarrow S^2$  whose image stays away from the three equally spaced subarcs of the equator of length  $\frac{\pi}{3}$ . In particular,  $\phi$  is a degree zero harmonic map into  $S^2$  with no closed geodesics on its image.*

**Remark 3.7** Using analogous constructions, we could construct harmonic maps from compact Riemann surfaces of genus  $p$  for any  $p > 1$ , replacing the  $\mathbb{Z}_3$  symmetry by a  $\mathbb{Z}_{p+1}$  symmetry. Such maps will have no closed geodesics in their images only if  $p$  is even. More interestingly, if we replace the target by another 2-dimensional surface with the appropriate symmetries, we could try to construct a  $u_0$ -type initial condition respecting the necessary symmetries.

### 4 Remarks and open questions

As pointed out in the introduction, it is an interesting question whether the implications between properties (i) to (iv) of a given subset  $V \subset (N, h)$  hold true. Here, we make a couple of remarks on this problem to illustrate some applications and possible consequences.

We start by considering the equivalence between properties (i) and (ii). Since (i) implies (ii) by the maximum principle, the question is whether (ii) implies (i). More precisely, we are assuming that a subset  $V \subseteq (N, h)$  admits no image of non-constant harmonic maps  $\phi : (M, g) \rightarrow (N, h)$ , where  $M$  is a closed Riemannian manifold, and we ask if this implies the existence of a strictly convex function  $f : V \rightarrow \mathbb{R}$ .

We start by remarking that, if one takes a geodesic ball  $B(p, r)$  in a complete manifold  $N$  such that  $r$  is smaller than the convexity radius of  $N$  at  $p$ , then  $\partial B(p, r)$  is a hypersurface of  $N$  with definite second fundamental form for every point  $q \in \partial B(p, r)$ .

A partial answer to the above question was obtained in [1] where the following theorem was proven.

**Theorem 4.1** *Let  $(N, h)$  be a complete Riemannian manifold and  $\gamma : [a, b] \rightarrow N$  a smooth embedded curve. Consider a smooth function  $r : [a, b] \rightarrow \mathbb{R}_+$  and a region*

$$\mathcal{R} := \bigcup_{t \in [a, b]} B(\Gamma(t), r(t)), \tag{4.1}$$

where  $B(\cdot, \cdot)$  is the geodesic ball and  $r(t)$  is smaller than the convexity radius of  $N$  for any  $t$ . If, for each  $t_0 \in (a, b)$ , the set  $\mathcal{R} \setminus B(\gamma(t_0), r(t_0))$  is the union of two disjoint connected sets, namely the connected component of  $\gamma(a)$  and the one of  $\gamma(b)$ , then there exists no compact manifold  $(M, g)$  and non-constant harmonic map  $\phi : M \rightarrow N$  such that  $\phi(M) \subseteq \mathcal{R}$ .

According to the above definitions, the region  $\mathcal{R}$  of theorem 4.1 has property (ii). As a direct corollary of the proof, which is based on an application of a maximum principle, we conclude that, if a subset  $V$  of  $N$  admits a sweep-out by convex hypersurfaces with the additional property that the absence of each leave of this sweep-out divides  $V$  in two connected components like the region  $\mathcal{R}$  above, then  $V$  has property (ii).

This corollary gives us a geometric way of checking whether a certain subset of a manifold has property (ii). If  $f : V \rightarrow \mathbb{R}$  is strictly convex, then the level sets of  $f$  naturally give us the desired sweepout in 4.1. On the other hand, a function  $f : V \rightarrow \mathbb{R}$  whose level sets yield a convex sweepout, is not necessarily strictly convex. We may have to reparametrize  $f$  to make it strictly convex: It is enough to use the parameter  $t \in [a, b]$  of the 1-parameter family of convex leaves in the above theorem and control the growth of the gradient of  $f$  while it walks through the convex leaves of the sweep-out.

The above argument is a strategy for the proof that when  $V$  satisfies the properties of the region  $\mathcal{R}$  in theorem 4.1, then  $V$  has property (i). In other words, this gives us a method to obtain strictly convex functions on some subset  $V$  of  $(N, h)$  based on its geometry.

It is quite plausible that our target  $\Omega_\epsilon$  does not contain any nontrivial harmonic map from a sphere, which would mean that (iii) does not imply (ii).

Another question is whether (iv) implies (iii). Suppose  $V$  does not have closed geodesics. Does that imply that there are no harmonic maps  $\phi : (S^k, \mathring{g}) \rightarrow (N, h)$  with  $\phi(S^k) \subset V$ ? In particular, does the absence of closed geodesics imply no bubbles in  $V$ ?

In the previous sections we have proven that property (iv) does not imply property (ii). It may be that with similar techniques, and a symmetric space different from  $S^2$  as target, one can build a counterexample for the implication (iv)  $\implies$  (iii) as well, allowing the energy of the initial condition to be big enough to form a bubble, while still controlling the image of the final map preserving symmetries, as in Sect. 2.

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