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Hyperkähler geometry of rational curves in twistor spaces

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Abstract: We investigate the pseudo-hyperkähler geometry of higher degree rational curves in the twistor space of a hyperkähler 4-manifold.

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The twistor space of a hypercomplex or a hyperkähler manifold M is a complex manifold Z equipped with a holomorphic submersion $\pi : Z \rightarrow \mathbb{P}^1$ and an antiholomorphic involution σ covering the antipodal map. The manifold M is then recovered as (a component of) the Kodaira moduli space of σ -invariant sections of π with normal bundle splitting as $\oplus \mathcal{O}(1)$.

In [3] the first author observed that, if $\dim M = 4$, then we also obtain a hypercomplex or pseudo-hyperkähler structure on a subset of the Douady space consisting of σ -invariant curves of degree d , $d > 1$, which are cohomologically stable, i.e. satisfy $h^1(N_{C/Z}(-2)) = 0$.

In [5] C. Peternell and the first author showed that in the case of curves of genus 0 in $\mathbb{P}^3 \setminus \mathbb{P}^1$ (i.e. in the twistor space of the flat \mathbb{R}^4) this pseudo-hyperkähler structure can be obtained as a pseudo-hyperkähler quotient of a flat space by a non-reductive Lie group. Even in that case, however, we had been unable to determine the signature of the metric for $d > 3$.

In this work we investigate the pseudo-hyperkähler geometry of higher degree \mathbb{P}^1 's embedded in the twistor space of an arbitrary 4-dimensional hyperkähler manifold. First of all, if such a \mathbb{P}^1 of degree d is to satisfy reality conditions, then d must be odd. This has been proved in [5, Prop. 5.9], but it also follows from the observation that a rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d can commute with the antipodal map only for odd d . With this restriction, let us denote by M_d the subset of the Douady space consisting of σ -invariant cohomologically stable $\mathbb{P}^1 \subset Z$ of degree d . M_d is hypercomplex, resp. pseudo-hyperkähler, if M is hypercomplex, resp. hyperkähler. We remark that in this situation “cohomologically stable” is equivalent to $N_{C/Z} \simeq \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1)$.

Our main result is:

Theorem A. *Let M be a 4-dimensional hyperkähler manifold. Suppose that $d \in \mathbb{N}$ is odd and M_d is nonempty. Then:*

- (i) *the signature of the pseudo-hyperkähler metric on M_d is $(2d+2, 2d-2)$;*
- (ii) *there exists a natural submersion $\rho : M_d \rightarrow \mathbb{R}\mathbb{P}^{2d-2}$ and an open dense subset U of M_d such each fibre of $\rho|_U$ has a natural d -hypercomplex¹ structure.*

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¹ A definition of a d -hypercomplex manifold may be found in §3.

Part (ii) holds also if M is only hypercomplex. The subset U consists of C such that the restricted vertical tangent bundle $(\text{Ker } d\pi)|_C$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$.

The structure of the paper is as follows. In the next section we recall facts about the geometry of degree d curves (of arbitrary arithmetic genus) in Z . We also interpret the $\mathcal{O}_{\mathbb{P}^1}(2)$ -valued symplectic form on the fibres of

the twistor space of such curves directly in terms of the normal bundles of curves (as long as they are local complete intersections). In §2 we define and study the map ρ without any reality assumptions. These are imposed in §3, where we prove Theorem A. Finally, we discuss in detail the case of degree $d \mathbb{P}^1$'s embedded in the twistor space of an ALE or ALF gravitational instanton of type A_k . In the ALE case we can actually view M_d as an open subset of the real locus of the Hilbert scheme of degree d rational curves on a singular Fano 3-fold - a hypersurface in a weighted projective 4-space (cf. Remark 4.2).

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1 Geometry of Douady spaces of curves in a twistor space

Let Z be a complex 3-dimensional manifold with a holomorphic submersion $\pi : Z \rightarrow \mathbb{P}^1$. We write $\mathcal{O}_Z(i)$ for $\pi^* \mathcal{O}_{\mathbb{P}^1}(i)$ and $\mathcal{F}(i)$ for $\mathcal{F} \otimes \mathcal{O}_Z(i)$ for any sheaf \mathcal{F} on Z . We denote by T_F the vertical tangent bundle $\text{Ker } d\pi$ of Z . From the exact sequence

$$0 \longrightarrow T_F \longrightarrow TZ \longrightarrow \pi^* T\mathbb{P}^1 \longrightarrow 0,$$

we conclude that $K_Z \simeq \Lambda^2 T_F^*(-2)$. In particular, an $\mathcal{O}(2)$ -valued symplectic form ω along the fibres of π , i.e. a trivialisation of $\Lambda^2 T_F^*(2)$, can be viewed as a nowhere vanishing section of $K_Z(4)$.

We now consider the subset X_d of the Douady space of 1-dimensional compact subspaces of Z consisting of subschemes C such that $\pi|_C : C \rightarrow \mathbb{P}^1$ is flat of degree d . In particular, each such C is pure-dimensional and Cohen-Macaulay. We denote by $X_d^{(i)}$, $i = 0, 1, 2$, the subset of X_d consisting of C , the normal sheaf $\mathcal{N}_{C/Z}$ of which satisfies $h^1(\mathcal{N}(-i)) = 0$. We summarize the main properties of $X_d^{(i)}$ as follows:

Proposition 1.1. *In each statement below suppose that the corresponding $X_d^{(i)}$, $i = 0, 1, 2$, is nonempty.*

- (i) $X_d^{(0)}$ a smooth $4d$ -dimensional manifold with a canonical isomorphism $T_C X_d^{(0)} \simeq H^0(C, \mathcal{N}_{C/Z})$ for each C .
- (ii) $X_d^{(1)}$ is equipped with a natural integrable 2-Kronecker structure, i.e. a holomorphic vector bundle E , $E_C = H^0(C, \mathcal{N}_{C/Z}(-1))$, and a bundle map $\alpha : E \otimes \mathbb{C}^2 \rightarrow TX_d^{(1)}$ such that $\alpha(E \otimes v)$ is an integrable rank 2d distribution for any nonzero $v \in \mathbb{C}^2$.
- (iii) $X_d^{(2)}$ is a \mathbb{C} -hypercomplex manifold, i.e. the map α is an isomorphism everywhere. Consequently $X_d^{(2)}$ is equipped with a holomorphic Obata connection, i.e. a torsion-free holomorphic connection with holonomy in $GL(d, \mathbb{C}) \simeq GL(E)$.
- (iv) If Z is also equipped with an $\mathcal{O}(2)$ -valued symplectic form along the fibres of π , then $X_d^{(2)}$ is a \mathbb{C} -hyperkahler manifold, i.e. it has a nowhere degenerate \mathbb{C} -valued symmetric bilinear form g , such that the corresponding holomorphic Levi-Civita connection coincides with the Obata connection.

Proof. Part (i) is easy in the case when Z is quasiprojective. It follows then from the fact that codimension 2 Cohen-Macaulay subspaces are locally unobstructed [9, §2.8]. In the general case we have to proceed differently. We consider the relative Hilbert scheme $Z_\pi^{[d]}$ of d points along the fibres of π . It is a smooth $(2d + 1)$ -dimensional manifold with a holomorphic submersion $\pi^{[d]} : Z_\pi^{[d]} \rightarrow \mathbb{P}^1$, and the same argument as in [5, Prop. 3.1] shows that X_d is isomorphic to the Douady space of sections of $\pi^{[d]}$. Furthermore, [5, Lemma 3.2]

remains true, so that the normal bundle N_s of a section s corresponding to a curve C is isomorphic to $\pi^*\mathcal{N}_{C/Z}$. Hence, if $H^1(C, \mathcal{N}_{C/Z}) = 0$, then $h^1(N_s) = 0$, and this means that the Douady space of sections is smooth at s . This proves (i).

Parts (ii)-(iv) have been proved in [3, 4]. □

In the case when the curve C is a *local complete intersection* (lci), we can say more:

Proposition 1.2. (i) *If $C \in X_d^{(0)}$, resp. $C \in X_d^{(1)}$, is lci, then there is a canonical isomorphism*

$$T_C^*X_d^{(0)} \simeq H^1(C, \mathcal{N}_{C/Z} \otimes K_Z), \quad \text{resp. } E_C^* \simeq H^1(C, \mathcal{N}_{C/Z} \otimes K_Z(1)).$$

(ii) *If $C \in X_d^{(2)}$ is lci, then there are additional canonical isomorphisms*

$$E_C \simeq H^1(C, \mathcal{N}_{C/Z}(-3)), \quad E_C^* \simeq H^0(C, \mathcal{N}_{C/Z} \otimes K_Z(3)).$$

Proof. Write N for the locally free sheaf $\mathcal{N}_{C/Z}$. The adjunction formula holds for lci subschemes [15, Ch. 6, Thm. 4.9], and hence:

$$K_C \simeq K_Z|_C \otimes \Lambda^2 N. \tag{1.1}$$

Consequently:

$$T_C^*X_d^{(0)} \simeq H^0(C, N)^* \simeq H^1(C, K_C \otimes N^*) \simeq H^1(C, K_Z \otimes N).$$

The second isomorphism in (i) follows completely analogously, given that $E_C \simeq H^0(C, N(-1))$.

For (ii) observe that we have a short exact sequence

$$0 \longrightarrow N(-3) \longrightarrow N(-2) \oplus N(-2) \longrightarrow N(-1) \longrightarrow 0,$$

from which the first isomorphism follows immediately, since $N(-2)$ has trivial cohomology. Since $N^* \otimes K_C(2)$ also has trivial cohomology, the same argument, using the exact sequence

$$0 \longrightarrow N^* \otimes K_C(1) \longrightarrow (N^* \otimes K_C(2))^{\oplus 2} \longrightarrow N^* \otimes K_C(3) \longrightarrow 0$$

and (1.1), shows the second isomorphism. □

It follows that a nowhere vanishing section ω of $\Lambda^2 T_F^*(2) \simeq K_Z(4)$ defines an isomorphism

$$E_C \simeq H^0(C, \mathcal{N}_{C/Z}(-1)) \xrightarrow{\omega} H^0(C, \mathcal{N}_{C/Z} \otimes K_Z(3)) \simeq E_C^* \tag{1.2}$$

for any lci curve $C \in X_d^{(2)}$. Write $\omega^{[d]}$ for the corresponding nondegenerate bilinear form on E given by $(s, t) \mapsto (s\omega)(t)$.

Remark 1.3. This construction of a symplectic form on E is due to Nash [14], who showed that the hyperkähler structure on a moduli space of framed Euclidean $SU(2)$ -monopoles can be obtained this way.

Let $\zeta \in \mathbb{P}^1$. For a $C \in X_d^{(2)}$, sections of $\mathcal{N}_{C/Z}(-1)$ can be identified with the tangent space to $C_\zeta = C \cap \pi^{-1}(\zeta)$ in the Hilbert scheme of d points in the fibre $\pi^{-1}(\zeta)$. Formula (1.2) implies that $\omega^{[d]}$ coincides with the induced symplectic form [1] on the Hilbert scheme of points (this is obvious on the subset where C_ζ consists of distinct points, and hence, by continuity, everywhere).

Therefore (cf. [3]) the symplectic form $\omega^{[d]}$ induces a \mathbb{C} -hyperkähler structure on $X_d^{(2)}$, and a pseudo-hyperkähler structure on the σ -invariant subset of $X_d^{(2)}$, if Z is equipped with an antiholomorphic involution σ covering the antipodal map.

Following Nash [14], we are going to give another proof of the skew-symmetry of $\omega^{[d]}$, since the argument will be helpful when proving Theorem A(i).

Proposition 1.4. $\omega^{[d]}$ is skew-symmetric.

Proof. We can express $\omega^{[d]}$ as the composition of the natural skew-symmetric map

$$H^0(C, N(-1)) \times H^0(C, N(-1)) \rightarrow H^0(C, (\Lambda^2 N)(-2))$$

with

$$H^0(C, (\Lambda^2 N)(-2)) \simeq H^0(C, K_Z^*(-2) \otimes K_C) \xrightarrow{(\omega\lambda)} H^1(C, K_C) \simeq \mathbb{C},$$

where $\lambda \in H^1(C, \mathcal{O}_C(-2))$ is the pullback of the extension class of

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0. \quad \square$$

2 Rational curves

Let $\pi : Z \rightarrow \mathbb{P}^1$ be as in the previous section. We denote by $X_{d,0}$ the component of X_d consisting of smooth rational curves $C \simeq \mathbb{P}^1$, and write $X_{d,0}^{(i)} = X_d^{(i)} \cap X_{d,0}$, $i = 0, 1, 2$. We remark that $C \in X_{d,0}^{(2)}$ if and only if its normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1)$.

For a $C \in X_{d,0}$, let R_C , resp. B_C , be the ramification divisor, resp. the branch divisor, of $\pi|_C$. These are 0-dimensional subschemes of \mathbb{P}^1 and we obtain a holomorphic map

$$\rho : X_{d,0} \rightarrow \mathbb{P}^{2d-2}, \quad C \mapsto B_C. \tag{2.1}$$

2.1 Covers of \mathbb{P}^1 and their parametrisations

In order to understand the map ρ , we make a brief detour. The map ρ can be viewed abstractly as associating to a degree cover $\pi : C \rightarrow \mathbb{P}^1$ its branch divisor B_C . On the other hand, we can also parameterise $C, f : \mathbb{P}^1 \rightarrow C$, and obtain a degree d rational map $\phi = \pi \circ f$. Let Rat_d denote the space of degree d rational maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. The quotient of Rat_d by $PGL(2, \mathbb{C})$ can be viewed as the moduli space of abstract degree d covers of \mathbb{P}^1 , but since the action of $PGL(2, \mathbb{C})$ has fixed points, this quotient is not manifold.

On the other hand, we can associate to ϕ its branch divisor. Classical Hurwitz conditions [12] imply that, given an effective divisor B of degree $2d-2$ on \mathbb{P}^1 , there exist, up to automorphisms, only finitely many rational maps $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d with branch divisor B .

Let $\phi \in \text{Rat}_d$ and consider the induced sequence

$$0 \rightarrow T\mathbb{P}^1 \xrightarrow{d\phi} \phi^* T\mathbb{P}^1 \rightarrow \mathcal{F}_\phi \rightarrow 0, \tag{2.2}$$

where \mathcal{F}_ϕ is supported on the ramification divisor of ϕ . The space of global sections of the middle term is naturally isomorphic to $T_\phi \text{Rat}_d$, while global sections of $T\mathbb{P}^1$ correspond to infinitesimal automorphisms of \mathbb{P}^1 . Thus we can identify global sections of \mathcal{F}_ϕ with deformations of the branch divisor B of ϕ , i.e. locally on Rat_d we have a natural isomorphism $H^0(\mathbb{P}^1, \mathcal{F}_\phi) \simeq T\mathbb{P}^{2d-2}$.

2.2 The geometry of the map ρ

With these preparations, we can prove:

Proposition 2.1. *The map ρ is a submersion on an open subset of $X_{d,0}^{(0)}$ where $h^1(T_F|_C) = 0$. This open subset contains $X_{d,0}^{(2)}$.*

Proof. Let $C \in X_{d,0}^{(0)}$. We have an analogue of (2.2):

$$0 \rightarrow TC \xrightarrow{d\pi} \pi^* T\mathbb{P}^1 \rightarrow \mathcal{F} \rightarrow 0. \tag{2.3}$$

The sheaf \mathcal{F} is supported on the ramification divisor R_C and is isomorphic to \mathcal{F}_ϕ for any parameterisation of C . Owing to the above discussion we have a natural isomorphism $T_{B_C} \mathbb{P}^{2d-2} \simeq H^0(C, \mathcal{F})$.

We also have the following two short exact sequences:

$$0 \longrightarrow TC \longrightarrow TZ|_C \longrightarrow N_{C/Z} \longrightarrow 0, \quad (2.4)$$

$$0 \longrightarrow T_F \longrightarrow TZ \xrightarrow{d\pi} \pi^* T\mathbb{P}^1 \longrightarrow 0, \quad (2.5)$$

where T_F is the vertical tangent bundle.

Observe that the composition

$$TZ|_C \xrightarrow{d\pi} \pi^* T\mathbb{P}^1 \longrightarrow \mathcal{F}$$

factors through $N_{C/Z}$, and we obtain the following short exact sequence of sheaves on C :

$$0 \longrightarrow T_F|_C \longrightarrow N_{C/Z} \longrightarrow \mathcal{F} \longrightarrow 0. \quad (2.6)$$

The induced map $H^0(C, N_{C/Z}) \rightarrow H^0(C, \mathcal{F})$ is $d\rho|_C$, and the first statement follows. If $C \in X_{d,2}^{(2)}$, then $N_{C/Z} \simeq \mathcal{O}_{\mathbb{P}^1}(2d-1)^{\oplus 2}$. Sequence (2.4) implies then that the direct summands of $TX|_C$ have degree at most $2d-1$. Sequence (2.5), restricted to C , implies now that the direct summands of $T_F|_C$ have degree at most $2d-1$.

Since $c_1(T_F|_C) = 2d$, it follows that the direct summands of $T_F|_C$ have positive degree. \square

We now consider the structure of the fibres of ρ . As discussed in §2.1, the connected components of $\rho^{-1}(B)$ correspond to $PGL(2, \mathbb{C})$ -orbits of rational maps with branch divisor B . Let us fix such a rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and suppose that there exists a $C_0 \in X_{d,0}^{(0)}$ with a parameterisation $f_0 : \mathbb{P}^1 \rightarrow C_0$ such that $\pi \circ f_0 = \phi$. Then the connected component X_ϕ of $\rho^{-1}(B)$ containing C is isomorphic to the space of embeddings $f : \mathbb{P}^1 \rightarrow Z$ such that $\pi \circ f = \phi$. Let ϕ^*Z denote the fibred product

$$\phi^*Z = \{(t, z) \in \mathbb{P}^1 \times Z; \phi(t) = \pi(z)\}, \quad (2.7)$$

and $\tilde{\phi} : \phi^*Z \rightarrow Z$ the projection on the second coordinate.

We conclude that X_ϕ is isomorphic to the open subset of the Kodaira moduli space of sections s of $\phi^*Z \rightarrow \mathbb{P}^1$ such that $\tilde{\phi} \circ s$ is an isomorphism.

The tangent space to X_ϕ at C is canonically isomorphic to $H^0(C, T_F|_C)$ (on the open subset where $h^1(T_F|_C) = 0$).

Remark 2.2. Let $s : \mathbb{P}^1 \rightarrow Z$ be a section of π with normal bundle isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then $s \circ \phi$ is a section of ϕ^*Z with normal bundle $\mathcal{O}(d) \oplus \mathcal{O}(d)$. Hence ϕ^*Z has a $(2d+2)$ -dimensional smooth family of sections with this normal bundle. A generic element of this family will map to an embedded \mathbb{P}^1 in Z . Sequence (2.6) implies then that the normal bundle N of this \mathbb{P}^1 satisfies $h^1(N(-1)) = 0$. Consequently, each fibre of ρ contains elements of $X_{d,0}^{(1)}$.

Example 2.3. Let Z be the twistor space of the flat \mathbb{R}^4 , i.e. the total space of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Equivalently $Z = \mathbb{P}^3 \setminus \mathbb{P}^1$, where $\mathbb{P}^1 = \{[z_0, z_1, 0, 0]\}$. The map $\pi : Z \rightarrow \mathbb{P}^1$ is then the projection onto the last two coordinates. Let C be a degree d rational curve in Z , parameterized by $[f_0(u, v), \dots, f_3(u, v)]$, where $f_i(u, v)$ are homogeneous polynomials of degree d , $i = 0, \dots, 3$. The normal bundle of C is then the cokernel of $Df : \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus 4}$, where Df is the Jacobian matrix of (f_0, \dots, f_3) [8]. The sheaf \mathcal{F} is the cokernel of $D\phi : \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d) \simeq T_F|_C$ where $\phi = (f_2, f_3)$, and $T_F = \{(a, b, 0, 0) \in TZ\}$. If $N_{C/Z} \simeq \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1)$, then we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{Df} \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus 4} \xrightarrow{(\alpha_1, \alpha_2)} \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1) \rightarrow 0,$$

where α_1 and α_2 are 2×2 matrices of degree $d-1$ homogeneous polynomials in u, v . If we write $\phi = (f_2, f_3)$ and $\psi = (f_0, f_1)$, then the exactness of the above sequence implies $\alpha_1 D\psi + \alpha_2 D\phi = 0$. The sequence (2.6) is then

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d) \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The connected component X_ϕ of $\rho^{-1}(B)$ is an open subset of \mathbb{C}^{2d+2} consisting of pairs $(f_0(u, v), f_1(u, v))$ of homogeneous polynomials of degree d such that $[f_0, f_1, f_2, f_3]$ is an embedding.

3 Real manifolds

We now suppose, in addition, that Z is equipped with an antiholomorphic involution σ covering the antipodal map on \mathbb{P}^1 . We denote by $M_d^{(i)}$, $i = 0, 1, 2$, the σ -invariant part of the corresponding $X_d^{(i)}$, and by $M_{d,0}^{(i)}$ the σ -invariant part of $X_{d,0}^{(i)}$. The manifolds $M_d^{(1)}$ and $M_d^{(2)}$ are equipped with the real versions of the geometry stated in Proposition 1.1, i.e. an integrable quaternionic 2-Kronecker structure in the case of $M_d^{(1)}$, and a hypercomplex or pseudo-hyperkähler structure on $M_d^{(2)}$. In the case of rational curves we have the following restriction on d :

Lemma 3.1. [5, Prop. 5.9] *Let C be a connected projective curve of arithmetic genus 0 equipped with a flat projection $\pi : C \rightarrow \mathbb{P}^1$ of degree d . If C admits an antiholomorphic involution covering the antipodal map on \mathbb{P}^1 , then d is odd.* □

We assume, therefore, that d is odd. The restriction of (2.1) to $M_{d,0}^{(0)}$ yields a smooth map

$$\rho : M_{d,0}^{(0)} \longrightarrow \mathbb{R}\mathbb{P}^{2d-2}. \tag{3.1}$$

The connected components of its fibres correspond to $SO(3)$ -orbits of rational maps $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d which commute with the antipodal map (up to automorphisms). We recall the notion of a d -hypercomplex manifold [2, 6, 7]:

Definition 3.2. Let $d \in \mathbb{N}$ be odd. An almost d -hypercomplex structure on a smooth manifold M is given by an isomorphism $T^{\mathbb{C}}M \simeq E \otimes \mathbb{C}^{d+1}$, where E is a quaternionic vector bundle. Moreover, this isomorphism is required to intertwine the complex conjugation on $T^{\mathbb{C}}M$ and the tensor product of the quaternionic structure on E and the standard quaternionic structure on \mathbb{C}^{d+1} .

An almost d -hypercomplex structure is integrable, i.e. a d -hypercomplex structure, if, for each Borel subgroup $B_\zeta \subset SL(2, \mathbb{C})$, $\zeta \in \mathbb{P}^1$, the subbundle $E \otimes K_\zeta$ is involutive, where K_ζ is the direct sum of all, except the lowest, weight subspaces of \mathbb{C}^{d+1} for the standard irreducible representation of $SL(2, \mathbb{C})$.

As discussed in [2], this is the natural geometry on the space of sections of a holomorphic submersion $\pi : Z \rightarrow \mathbb{P}^1$, the normal bundle of which splits as $\bigoplus \mathcal{O}(d)$.

Proposition 3.3. *The fibres of the map ρ restricted to the open subset of $M_{d,0}^{(0)}$ where $T_F|_C \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ have a natural d -hypercomplex structure.*

Proof. Let M_ϕ be a connected component of a fibre of ρ determined by a rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d , commuting with the antipodal map. The arguments of the previous section imply that M_ϕ is an open subset of the σ -invariant part of the Kodaira moduli space of sections of ϕ^*Z . As observed in the previous section, the normal bundle of such a section corresponding to C in the open subset of the statement is isomorphic to $T_F|_C \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$. □

3.1 Signature of the metric

Suppose now that Z is also equipped with an $\mathcal{O}(2)$ -valued symplectic 2-form along fibres of π , which is compatible with the real structure. Then the manifold $M_{d,0}^{(2)}$ has a natural pseudo-hyperkähler metric g . We shall now use the description of the induced symplectic form $\omega^{[d]}$ on the bundle of E over $M_{d,0}^{(2)}$, given in §1, to determine the signature of the metric. A real tangent vector in $TM_{d,0}^{(2)} \simeq E \otimes \mathbb{C}^2$ can be written as $x = (e, -je)$, where j is the quaternionic structure of E , and the metric is then (cf., e.g., [11, (3.103)])

$$g(x, x) = -2\omega^{[d]}(e, je). \tag{3.2}$$

Let now $C \in M_{d,0}^{(2)}$. We fix a parametrisation $r : \mathbb{P}^1 \rightarrow C$ such that $\sigma \circ r = r \circ \sigma$ (where $\sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ denotes the antipodal map) and consider its composition with $\pi|_C$. This is a degree d rational map, commuting with

the antipodal map. Without loss of generality we can assume that neither 0 nor ∞ is mapped to ∞ . We can then write this rational map as $p(t)/q(t)$, where p and q are relatively prime degree d polynomials in the affine coordinate t on \mathbb{P}^1 .

Since $C \in M_{d,0}^{(2)}$, its normal bundle N is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2d - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - 1)$. Then $N(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(d - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(d - 1)$, and the isomorphism

$$E_C \otimes \mathbb{C}^2 \simeq H^0(C, N(-1)) \otimes \mathbb{C}^2 \longrightarrow H^0(C, N) \simeq T_C M_{d,0}^{(2)}$$

can be written as

$$H^0(\mathbb{P}^1, \mathcal{O}(d - 1) \oplus \mathcal{O}(d - 1)) \otimes \mathbb{C}^2 \ni ((f_1, g_1), (f_2, g_2)) \longmapsto (pf_1 + qg_1, pf_2 + qg_2).$$

We can also assume that the quaternionic structure of E_C is the standard one on $H^0(\mathbb{P}^1, \mathcal{O}(d - 1) \oplus \mathcal{O}(d - 1))$, i.e.

$$j(f(t), g(t)) = t^{d-1} \left(\overline{-g(-1/\bar{t})}, \overline{f(-1/\bar{t})} \right).$$

We now unravel the description of $\omega^{|d|}$, given in the proof of Proposition 1.4.

Let $(f_i, g_i) \in H^0(C, N(-1))$, $i = 1, 2$, be two sections, consisting of pairs of polynomials of degree $d - 1$. Then $f_1g_2 - g_1f_2 \in H^0(C, (\Lambda^2 N)(-2))$, which we view (using $\omega \in H^0(C, K_Z(4)|_C)$) as a section of $H^0(C, K_C(2))$. The corresponding meromorphic 1-form is $(f_1g_2 - g_1f_2)(q^2(t))^{-1} dt$, and it has poles bounded by $2(q(t))$. The extension class in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ can be viewed as the Laurent tail $\zeta^{-1} \cdot \infty$, and its pullback is then

$$\sum_{i=1}^d (\text{linear term of } q(t)/p(t) \text{ at } t = t_i) \cdot t_i,$$

where t_i are the zeros of q (since $\zeta = p(t)/q(t)$). The pairing of $H^0(C, K_C(2))$ and $H^1(C, \mathcal{O}(-2))$ is given by the residue map

$$\sum_{i=1}^d \text{Res}_{t=t_i} \frac{q(t)}{p(t)} (f_1g_2 - g_1f_2)(q^2(t))^{-1} dt = \sum_{i=1}^d \text{Res}_{t=t_i} \frac{f_1g_2 - g_1f_2}{p(t)q(t)} dt.$$

Let us write, for a polynomial of degree k ,

$$\tau(f)(t) = (-t)^k \overline{f(-1/\bar{t})}.$$

The square of this map is Id if k is even, and $-\text{Id}$ if k is odd. The fact that p/q commutes with the antipodal map means that $p = -\tau(q)$.

Let γ be a simple contour in \mathbb{C} separating the roots from q from the roots of $\tau(q) = -p$. It follows from the above and from (3.2) that the metric on $T_C M_{d,0}^{(2)}$ is equal to

$$\|x\|^2 = \|((f, g), -j(f, g))\|^2 = \frac{1}{\pi i} \oint_{\gamma} \frac{f\tau(f) + g\tau(g)}{q\tau(q)} dt. \tag{3.3}$$

We want to determine the signature of the right-hand side on pairs (f, g) of polynomials of degree $d - 1$. Owing to the continuity, it is enough to compute the signature for one particular q , say $q(t) = t^d$. Then $\tau(q) = 1$ (since d is odd) and the right-hand side is the middle degree term of $2f\tau(f) + 2g\tau(g)$, i.e.:

$$2 \sum_{i=0}^{d-1} (-1)^{d-1-i} |f_i|^2 + 2 \sum_{i=0}^{d-1} (-1)^{d-1-i} |g_i|^2.$$

Therefore the signature of the metric g is $(2d + 2, 2d - 2)$.

4 Example: gravitational instantons of type A_k

Consider an ALE or an ALF gravitational instanton M of type A_k . We recall, after Hitchin [10], its construction using twistor methods. The twistor space of M has a singular model given as a hypersurface \bar{Z} in the total space of a vector bundle over $T\mathbb{P}^1$. If ζ is the affine coordinate on \mathbb{P}^1 and η is the corresponding fibre coordinate on $T\mathbb{P}^1$, we denote by L^c , $c \in \mathbb{C}$, the line bundle on $T\mathbb{P}^1$ with transition function $\exp(-c\eta/\zeta)$ from $\zeta \neq \infty$ to $\zeta \neq 0$. Then \bar{Z} is given by

$$\{(x, y, z) \in L^c(k) \oplus L^{-c}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(2); xy = \prod_{i=1}^k (z - a_i(\zeta))\}$$

where c is real and a_i are quadratic polynomials satisfying reality conditions. M is then the space of real sections of $\pi : \bar{Z} \rightarrow \mathbb{P}^1$ obtained by choosing an arbitrary real section $z(\zeta) = (x_2 + ix_3) + 2x_1\zeta - (x_2 - ix_3)\zeta^2$ of $\mathcal{O}_{\mathbb{P}^1}(2)$, and dividing the set of all zeros of $z(\zeta) - a_i(\zeta)$, $i = 1, \dots, k$, into two subsets Δ_1, Δ_2 , interchanged by the antipodal map. This can be done consistently as shown in [10]. The sections of $\pi : \bar{Z} \rightarrow \mathbb{P}^1$ are then

$$x(\zeta) = Ae^{c(x_1 - (x_2 - ix_3)\zeta)} \prod_{\zeta_i \in \Delta_1} (\zeta - \zeta_i), \quad y(\zeta) = Be^{-c(x_1 - (x_2 - ix_3)\zeta)} \prod_{\zeta_i \in \Delta_2} (\zeta - \zeta_i), \tag{4.1}$$

over $\zeta \neq \infty$.

The nonzero scalars A, B are determined up to a circle action, which yields an isometric S^1 -action on M . We remark that resolving the singularities of \bar{Z} is not necessary for computing M and its metric.

We now discuss the geometry of real (i.e. σ -invariant) \mathbb{P}^1 's of degree d , d - odd, in \bar{Z} .

Let $\phi = p(t)/q(t)$ be a rational map of degree d , commuting with the antipodal map, and assume for simplicity that $t = \infty$ is not a pole of ϕ .

The function ϕ can be viewed as the transition function for the bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ from $U_0 = \{q \neq 0\}$ to $U_1 = \{p \neq 0\}$. A section of $\mathcal{O}_{\mathbb{P}^1}(kd)$ is then represented by b/q^k on U_0 and b/p^k on U_1 , where b is a polynomial of degree kd . Let $z = b/q^2$ be a section of $\mathcal{O}_{\mathbb{P}^1}(2d)$ and write $b = b_0p + b_1q$. We get a section of the line bundle L_ϕ^c with transition function $\exp(cz/\phi)$ by setting

$$(s_0, s_1) = (\exp(-cb_0/q), \exp(cb_1/p))$$

in U_0 and U_1 respectively. If we now consider the fibred product $\phi^*\bar{Z}$, as in §2.2, then its sections, and hence the fibre of the map $\rho : M_{d,0}^{(2)} \rightarrow \mathbb{R}\mathbb{P}^{2d-2}$, are obtained in the same way as for $d = 1$: choose an arbitrary real section $z(t) = b(t)/q(t)^2$ of $\mathcal{O}_{\mathbb{P}^1}(2d)$, divide the zeros of all $z(t) - a_i(\phi(t))$ into two sets, and obtain $x(t), y(t)$ as in (4.1), replacing the exponential factors by $(\exp(-cb_0/q)$ over $q(t) \neq 0$ and by $\exp(cb_1/p)$ over $p(t) \neq 0$. The space of real sections of $\phi^*\bar{Z}$ with normal bundle $\mathcal{O}(d) \oplus \mathcal{O}(d)$ is nonempty owing to Remark 2.2, and is a d -hypercomplex analogue of the original gravitational instanton, as introduced in [7, §3.1.2].

A generic section s of $\phi^*\bar{Z}$ will yield an embedded \mathbb{P}^1 in \bar{Z} , and hence, by varying ϕ , we obtain a $4d$ -dimensional space of embedded real \mathbb{P}^1 's of degree d in \bar{Z} .

We claim that, for generic c and a_i , $i = 1, \dots, k$, the normal bundle of a generic such curve is $\mathcal{O}(2d - 1) \oplus \mathcal{O}(2d - 1)$, i.e. $M_{d,0}^{(2)}$ is nonempty (and hence of dimension $4d$). Indeed, were this not the case, the normal bundle of every degree d rational curve (flat over \mathbb{P}^1) in the twistor space of $(\mathbb{R}^4 \setminus \{0\})/\mathbb{Z}_k$ would also be different from $\mathcal{O}(2d - 1) \oplus \mathcal{O}(2d - 1)$. This twistor space Z_0 is the quotient by \mathbb{Z}_k of the total space W of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ with the zero section removed. In particular a generic degree d rational curve in W descends to a rational curve of degree d curve in Z_0 with isomorphic normal bundle. Since W is an open subset of \mathbb{P}^3 , a generic degree d \mathbb{P}^1 in W has normal bundle isomorphic to $\mathcal{O}(2d - 1) \oplus \mathcal{O}(2d - 1)$ [8]. This contradiction proves our claim.

We can say more in the case $c = 0$, i.e. when Z is the twistor space of an ALE manifold.

The fibred product $\phi^*\bar{Z}$ is then a hypersurface in the total space of the vector bundle $E_d = \mathcal{O}(kd) \oplus \mathcal{O}(kd) \oplus \mathcal{O}(2d)$.

If s is a section of $\phi^*\bar{Z}$, given by homogeneous polynomials $x(u, v), y(u, v), z(u, v)$ of degrees kd, kd , and $2d$, then its normal bundle fits into a short exact sequence

$$0 \rightarrow N_{s/\phi^*\bar{Z}} \xrightarrow{j} E_d \rightarrow \mathcal{O}(2kd) \rightarrow 0, \tag{4.2}$$

since $N_{s/E_d} \simeq E_d$. The projection $E_d \rightarrow \mathcal{O}(2kd)$ is given by

$$\left[y, x, -\sum_i \prod_{j \neq i} (z - a_j) \right]^T,$$

from which one can compute N_{s/ϕ^*Z} .

The original twistor space Z is a hypersurface in the total space of the bundle E_1 . We can view the curve C given by ϕ and s as being embedded in E_1 . Its normal bundle fits then into a short exact sequence (cf. Ex. 2.3):

$$0 \rightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{\Psi} \mathcal{O}(kd)^{\oplus 2} \oplus \mathcal{O}(2d) \oplus \mathcal{O}(d)^{\oplus 2} \rightarrow N_{C/E_1} \rightarrow 0, \tag{4.3}$$

where Ψ is the Jacobi matrix of $[x(u, v), y(u, v), z(u, v), p(u, v), q(u, v)]^T$. We can extend (4.2) and (4.3) to a commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & N_{s/\phi^*Z} \oplus \mathcal{O}(d)^{\oplus 2} & \xrightarrow{\nu} & N_{C/Z} & \longrightarrow & 0 \\
 & \nearrow \lambda & \downarrow j \oplus \text{Id} & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\Psi} & E_d \oplus \mathcal{O}(d)^{\oplus 2} & \longrightarrow & N_{C/E_1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathcal{O}(2kd) & \xlongequal{\quad} & \mathcal{O}(2kd) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

It is now the matter of (a complicated) linear algebra to compute the map λ from Ψ and from the vertical projection $E_d \rightarrow \mathcal{O}(2kd)$. Assuming that $N_{s/\phi^*Z} \simeq \mathcal{O}(d) \oplus \mathcal{O}(d)$, λ will be a 4×2 matrix of degree $d - 1$ polynomials in u, v , the coefficients of which depend on the polynomials x, y, z, p, q . The map ν can then be computed from λ , and the condition that $N_{C/Z} \simeq \mathcal{O}(2d - 1) \oplus \mathcal{O}(2d - 1)$ can be written as a determinant of a matrix given by coefficients of λ . The polynomials x and y depend (up to scale) algebraically on z, p, q and, hence, $X_{d,0}^{(2)}$, and its real part $M_{d,0}^{(2)}$, are described by this algebraic relation between the coefficients of arbitrary polynomials z, p, q of degrees $2d, d, d$.

Remark 4.1. In principle, once the maps λ and ν are determined, the pseudo-hyperkähler metric on $M_{d,0}^{(2)}$ can be computed using the method of the previous section.

Remark 4.2. We should like to point out that in the ALE case, the singular model \bar{Z} compactifies to a hypersurface in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 2, k, k)$. For more details on this compactification see [13]. Since the degree of the hypersurface is $2k$, the adjunction formula implies that its canonical sheaf is isomorphic to $\mathcal{O}_{\mathbb{P}}(2k - 1 - 1 - 2 - k - k) \simeq \mathcal{O}_{\mathbb{P}}(-4)$, and hence the compactification of \bar{Z} is Fano.

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