Validated Model Predictive Control based on Exponential Enclosures

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Introduction

Guaranteed numerical integration is a fundamental tool to solve initial value problems of ordinary differential equations (IVP-ODEs) with uncertain initial conditions and parameters in a reliable and validated way. Providing guaranteed solution enclosures to these IVP-ODEs is essential for designing and verifying linear and nonlinear feedback controllers, mainly for predictive control approaches. In the literature, many solvers have been developed, such as the DynIbex library, allowing for the computation of enclosures which are guaranteed to contain all possible system states. The DynIbex library is based on Runge-Kutta schemes to obtain tight state enclosures [1]. Nevertheless, it has been shown that — due to the computational complexity of Runge-Kutta methods [2] — fast convergence and high accuracy of the computed enclosures are not always guaranteed for finitely long integration time spans, possibly leading to an excessive duration to get the IVP-ODEs' solutions. To overcome these issues, exponential enclosure techniques for IVP-ODE problems seem to be attractive to remarkably reduce the computing time of validated methods and to approach real-time capability [3,4].

The time aspect is especially crucial, because at each sampling instant, a validated nonlinear model predictive controller (NMPC) needs to compute optimal and guaranteed system inputs along a receding horizon that minimize some interval cost function and ensure compatibility constraints (such as actuator saturations or safety constraints on the state trajectories) [2]. Our motivation is to interface exponential enclosure techniques with the validated NMPC to remarkably speed up the solution.

Guaranteed Nonlinear Model Predictive Control

Consider a dynamic system defined by the following IVP-ODEs :

$$\begin{cases} \dot{\mathbf{x}}_t = \mathbf{f}(t, \mathbf{x}_t, \mathbf{u}, \mathbf{p}) \\ \mathbf{x}_0 \in [\mathbf{x}_0] \subseteq \mathbb{IR}^n \\ \mathbf{u} \in [\mathbf{u}] \subseteq \mathbb{IR}^m \\ \mathbf{p} \in [\mathbf{p}] \subseteq \mathbb{IR}^p, \end{cases} \tag{1}$$

where the state vector is denoted by $\mathbf{x}_t$, the vector of parameters by $\mathbf{p}$, and the control vector by $\mathbf{u}$. The sets $[\mathbf{x}_0] = \left[[x_{10}] \;\; \ldots \;\; [x_{n0}]\right]^T$, $[\mathbf{u}] = \left[[u_1] \;\; \ldots \;\; [u_m]\right]^T$, and $[\mathbf{p}] = \left[[p_1] \;\; \ldots \;\; [p_p]\right]^T$, expressed as interval boxes, are respectively the initial condition of the state vector, the interval-bounded input, and the set of feasible dynamic parameters. The proposed guaranteed NMPC encompasses two stages [2]:

1. **Filtering and branching:** The first step provides a sequence of guaranteed input interval boxes at each time-step k over the prediction horizon N_p, denoted as $[\mathbf{U}]_k = [\mathbf{u}]_k \times [\mathbf{u}]_{k+1} \times \ldots \times [\mathbf{u}]_{k+N_\mathrm{p}-1}$. Branching and filtering procedures allow the computation of safe input intervals along the receding time horizon that satisfy the state constraints (i.e., $\forall j, [x_j] \subseteq [x_{\min,j}, x_{\max,j}]$, where $x_{\min,j}$ and $x_{\max,j}$ are the bounds for the admissible domain for each state variable) and ensure convergence to the reference interval (i.e., $[\mathbf{x}_k] \to [\mathbf{x}_{\mathrm{r},k}]$).

2. **Interval optimization:** Since safe inputs are computed over a finite time horizon, the optimization algorithm is launched to com-

pute the optimal inputs $[\mathbf{U}]_k^\star$ by minimizing as much as as possible a newly formulated interval objective function to reduce the error between predicted and reference outputs as well as the norm of the input intervals.

Exponential Enclosure Technique

Guaranteed numerical integration methods aim at computing the state enclosure sequences $(t_j, [\mathbf{x}_j])_{j\in\mathbb{N}}$, assuming that the input and parameter boxes $[\mathbf{u}]$ and $[\mathbf{p}]$, respectively, are piecewise constant and known for each validated simulation. Here, the exponential enclosure technique will be applied to approximate the IVP-ODEs' solutions, given in (1). It has been shown that this method improves the accuracy of the computed state enclosures and reduces the required computation time for asymptotically stable systems [3]. The dynamic model (1) can be reformulated by considering that the dynamic parameters are represented by constant intervals, and the input variables are assumed to be included in an augmented state vector, i.e., $\left[\mathbf{x}_t^T \quad \mathbf{u}^T(\mathbf{x}_t)\right]^T$, denoted for brevity again as $\mathbf{x}_t$ with

$$\dot{\mathbf{x}}_t = \mathbf{f}(\mathbf{x}_t). \tag{2}$$

To ensure the (local) asymptotic stability of the system model in the neighborhood of a desired terminal state, we assume — as a prerequisite for the exponential enclosure approach — that a feedback controller is included in a cascaded manner in the control law $\mathbf{u}(\mathbf{x}_t)$ so that the NMPC effectively computes a kind of feedforward control sequence.

To prevent the growth of the diameters of the intervals $(t_j, [\mathbf{x}_j])_{j\in\mathbb{N}}$ for asymptotically stable systems with a minimum computational capacity, the exact solution $\mathbf{x}_t^\star$ can be bracketed into the following exponential state enclosures

$$\mathbf{x}_t^\star \in [\mathbf{x}_e](t) = \exp\Big([\mathbf{\Lambda}] \cdot t\Big) \cdot [\mathbf{x}_e](0) \ , \quad [\mathbf{x}_e](0) = [\mathbf{x}_0], \tag{3}$$

where $\mathbf{\Lambda}$ represents a yet unknown dynamics matrix. By choosing $[\mathbf{\Lambda}] = \mathrm{diag}\{[\lambda_i]\}$, $i = 1, \ldots, n$, as a diagonal matrix, its elements λ_i

need to have negative real parts to describe contracting state enclosures.

Using the exponential state enclosures (3) and a Picard iteration with the iteration index κ, we obtain

$$\begin{aligned}\mathbf{x}_t^\star \in [\mathbf{x}_e]_{(\kappa+1)} &= \exp\Big([\mathbf{\Lambda}]_{(\kappa+1)} \cdot t\Big) \cdot [\mathbf{x}_e](0) \\ &= [\mathbf{x}_e](0) + \int_0^t \mathbf{f}\Big(\exp\Big([\mathbf{\Lambda}]_{(\kappa)} \cdot s\Big) \cdot [\mathbf{x}_e](0)\Big)\,\mathrm{d}s.\end{aligned} \tag{4}$$

The differentiation of (4) with respect to time, belonging to the integration interval $t \in [t]$, leads to

$$\dot{\mathbf{x}}_{[t]}^\star \in [\mathbf{\Lambda}]_{(\kappa+1)} \cdot \exp\Big([\mathbf{\Lambda}]_{(\kappa+1)} \cdot [t]\Big) \cdot [\mathbf{x}_e](0) = \mathbf{f}\Big(\exp\Big([\mathbf{\Lambda}]_{(\kappa)} \cdot [t]\Big) \cdot [\mathbf{x}_e](0)\Big). \tag{5}$$

Assuming a converging iteration with $[\mathbf{\Lambda}]_{(\kappa+1)} \subseteq [\mathbf{\Lambda}]_{(\kappa)}$ and, thus, $[\lambda_i]_{(\kappa+1)} \subseteq [\lambda_i]_{(\kappa)}$, the iteration formula for $[\lambda_i]_{(\kappa+1)}$ can be expressed as

$$[\lambda_i]_{(\kappa+1)} = \frac{f_i\Big(\exp\Big([\mathbf{\Lambda}]_{(\kappa)} \cdot [t]\Big) \cdot [\mathbf{x}_{e,i}](0)\Big)}{\exp\Big([\mathbf{\Lambda}]_{(\kappa)} \cdot [t]\Big).[\mathbf{x}_{e,i}](0)}, \quad i = 1, \ldots, n. \tag{6}$$

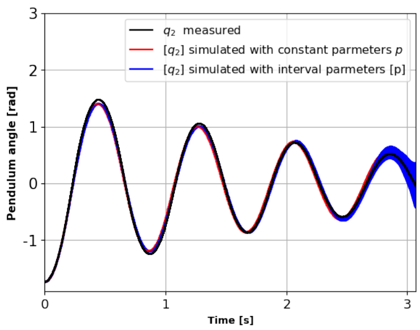
The guaranteed state enclosure at the time instant $t = T = \sup([t])$ is given by

$$\mathbf{x}_t^\star \in [\mathbf{x}_e](t) = \exp\Big([\mathbf{\Lambda}] \cdot T\Big) \cdot [\mathbf{x}_e](0), \tag{7}$$

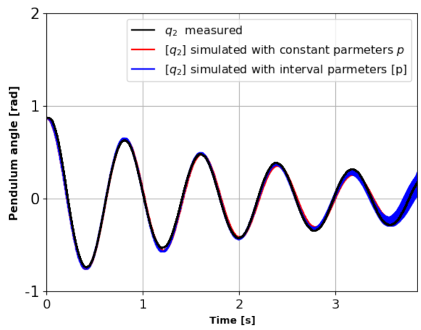
where $[\mathbf{\Lambda}]$ is the final result of the iteration (6).

Preliminary Results using DynIbex

The NMPC strategy is applied to stabilize a nonlinear inverted pendulum with two serial joints, actuated by a DC motor whose angular speed is the input variable. To evaluate the dynamic model of the inverted pendulum, we can solve the IVP-ODEs in a validated way using the DynIbex library. Figs. 1a and 1b show the measured pendulum angle (black lines) with the computed enclosures by DynIbex using point-valued parameters $\mathbf{p}$ (red enclosures) and interval parameters $[\mathbf{p}]$ (blue enclosures). We can notice that the simulated tubes of the pendulum angle are close to the real measured signal. Moreover, we have calculated the coverage ratios between the measurements and the simulated tubes as recapped in Tab. 1. The coverage ratios confirm that the model is identified with high precision when the dynamic parameters are considered as intervals that account for different uncertainties related to the measurements and dynamic modeling. However, the accuracy of the validated simulation should be enhanced because the widths of the computed Rung-Kutta enclosures enlarge with time, and the coverage ratios are not quite satisfactory.



(a) Initial conditions: $[\mathbf{u}_0] = [0\,\mathrm{Nm}, 0\,\mathrm{Nm}]$ and $[\mathbf{x}_0] = \underbrace{[0^\circ, 0^\circ]}_{[x_{10}]} \times \underbrace{[0, 0]}_{[x_{20}]} \times \underbrace{[-98^\circ, -100^\circ]}_{[x_{30}]} \times \underbrace{[0, 0]}_{[x_{40}]}$.



(b) Initial conditions: $[\mathbf{u}_0] = [0.15\,\mathrm{Nm}, 0.18\,\mathrm{Nm}]$ and $[\mathbf{x}_0] = \underbrace{[0^\circ, 0^\circ]}_{[x_{10}]} \times \underbrace{[0, 0]}_{[x_{20}]} \times \underbrace{[46^\circ, 50^\circ]}_{[x_{30}]} \times \underbrace{[0, 0]}_{[x_{40}]}$.

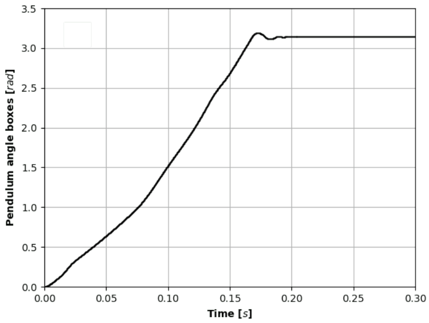
Figure 1: The validation of the dynamic model of a nonlinear inverted pendulum using the DynIbex library. Comparison between the actual and simulated pendulum angles at different initial conditions with point-valued and interval parameters.

Figs. 2a and 2b display the simulation results of the validated NMPC approach. As it can be seen in Fig. 2a, the pendulum arm starts from the downward position, and it is stabilized via the vali-

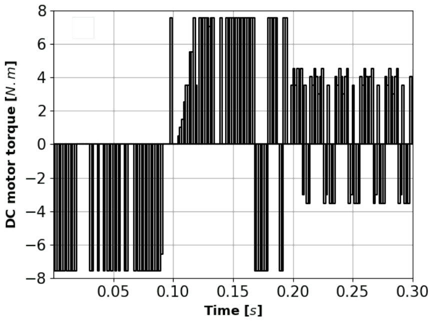
Table 1: Coverage rates between the model and physical reality.

Scenario	(a)	(b)
With interval-valued dynamic parameters $[\mathbf{p}]$	51%	61%
With point-valued dynamic parameters $\mathbf{p}$	38%	34%

dated NMPC in its vertical upright position interval $[\mathbf{x}_{\mathrm{r}}]$ with a small settling-time (around $t_{\mathrm{r}5\%} \approx 0.18\,\mathrm{s}$). Despite its proven effectiveness in making the system output converge to the desired reference interval, it still has some drawbacks. The main ones are related globally to the computation time, which depends mainly on a large number of bisections of the initial input domain $[\mathbf{u}_k]$ preventing the validation of this approach in real-time. This issue can be reduced by using exponential enclosure techniques in combination with an underlying feedback controller.



(a) Tube for the pendulum angle.



(b) Input Intervals.

Figure 2: Validated NMPC results starting from the downward position.

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