Higher-Order Methods for Differential Inclusions

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Introduction

Differential inclusions are nondeterministic continuous-times systems with set-valued uncertainties $\dot{x}(t) \in F(x(t))$. A solution is an absolutely-continuous function satisfying the equation almost-everywhere. They arise from noisy systems $\dot{x}(t) = f(x(t), v(t))$ with $v(t) \in V$. We give a method for computing solution sets for systems with affine noise:

$$\dot{x}(t) = f(x(t)) + \textstyle\sum_{i=1}^{m} g_i(x(t))\, v_i(t);\ \ v_i(t) \in [-V_i, +V_i];\ \ x(t_0) = x_0.$$

Method

To over-approximate the set of solutions, we consider an auxiliary system of ordinary differential equations

$$\dot{y}(t) = f(y(t), w(t; a)), \quad y(0) = x_0,\ w(\cdot; a) \in W,\ a \in A$$

where A is a finite-dimensional compact set of parameters, and analytically determine a uniform error bound on the difference between solutions of the original system and its auxiliary counterpart.

We take constants $\Lambda \geq \lambda(Df(\cdot)),$, the logarithmic norm, and

$$\|f(z(t))\| \leq K, \quad \|g_i(z(t))\| \leq K_i \quad K' = \textstyle\sum_{i=1}^{m} V_i\, K_i,$$
$$\|Df(z(t))\| \leq L, \ \|Dg_i(z(t))\| \leq L_i, \quad L' = \textstyle\sum_{i=1}^{m} V_i\, L_i.$$

Then if f and g are C^1 and the w_i are measurable functions such that $\int_0^h v_i(\tau) - w_i(\tau)\, d\tau = 0$, the single-time-step error is

$$|x(h) - y(h)| \leq h^2\big((K + K')L'/3 + 2K'(L + L')\,(e^{\Lambda h} - 1)/\Lambda h\big) \tag{1}$$

By using two-parameter piecewise-constant or affine functions for each component $w_j(\cdot)$, we can obtain an error with terms of order $O(h^2)$ and $O(h^3)$, and full order of $O(h^3)$ is obtained in the cases of additive inputs $\dot{x} = f(x) + v$ and the one-input case.

The auxiliary system is solved using rigorous integration using polynomial models. The method has been implemented in the tool Ariadne, and performed competitively on the ARCH benchmarks [3].

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References

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