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Research Article

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Error Estimates for FE-BE Coupling of Scattering of Waves in the Time Domain

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Abstract: This article considers a coupled finite and boundary element method for an interface problem for the acoustic wave equation. Well-posedness, a priori and a posteriori error estimates are discussed for a symmetric space-time Galerkin discretization related to the energy. Numerical experiments in three dimensions illustrate the performance of the method in model problems.

Keywords: Time Domain Boundary Element Method, Wave Equation, FEM-BEM Coupling, A Priori Error Estimates, A Posteriori Error Estimates

MSC 2010: 65M38, 65M60

Dedicated to the memory of Francisco-Javier Sayas

1 Introduction

Coupled finite and boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of elliptic interface and contact problems, particularly in unbounded domains [19, 36]. On the other hand, much of the current interest in boundary element procedures focuses on hyperbolic problems in the time domain, both on Galerkin methods and convolution quadrature [4, 9, 11, 25, 27, 35].

The numerical solution of time-independent interface problems in unbounded domains by coupling finite elements (FEM) and boundary elements (BEM) is well understood [19, 36]. For time-dependent scattering problems in unbounded domains, boundary elements in the time domain have attracted much recent interest, either in the form of space-time Galerkin or convolution quadrature methods [4, 9, 11, 25, 35]. Much recent research has focused on convolution quadrature, with a fundamental contribution [6] followed by extensions to a variety of FEM-BEM coupling procedures in two dimensions, including fluid-structure interaction and nonlinear elastic and multi-physics problems [23, 24, 34], as well as efficient discretizations [22]. On the other hand, advanced Galerkin procedures for the coupling of finite elements and boundary elements in the time domain are much less understood. A first procedure for discontinuous finite elements coupled to Galerkin boundary elements was investigated in [1], while energetic formulations with continuous finite elements were studied in [3, 4]. In three dimensions, the authors recently considered the analysis and implementation for a fluid-structure interaction problem [16].

This article investigates a finite element and boundary element coupling based on a symmetric formulation using for the exterior problem the total Calderón projector consisting of the retarded single layer and double layer potentials and their normal derivatives on the interface boundary. For Ω a bounded Lipschitz

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domain, we consider the transmission problem between wave equations in the interior $\Omega =: \Omega^-$, respectively exterior $\Omega^+ := \Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$, with homogeneous initial conditions:

$$\rho_1 \frac{\partial^2 u}{\partial t^2} - \mu_1 \Delta u = 0 \quad (x, t) \in \Omega^- \times (0, \infty), \tag{1.1a}$$

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 \quad (x, t) \in \Omega^+ \times (0, \infty), \tag{1.1b}$$

$$u(x, 0) = \dot{u}(x, 0) = 0 \quad \text{in } \Omega^-, \tag{1.1c}$$

$$v(x, 0) = \dot{v}(x, 0) = 0 \quad \text{in } \Omega^+, \tag{1.1d}$$

$$\gamma^- u - \gamma^+ v = f \quad \text{on } \Gamma \times (0, \infty), \tag{1.1e}$$

$$\mu_1 \partial_n^- u - \partial_n^+ v = g \quad \text{on } \Gamma \times (0, \infty). \tag{1.1f}$$

Here $\mu_1, \rho_1 > 0$, and $n = n_x$ is the unit normal vector, pointing towards Ω^+. In the following we set $c_1^2 = \frac{\mu_1}{\rho_1}$. For $x \in \Gamma$, we define

$$\gamma^\pm v(x, t) := v_\pm(x, t) := \lim_{x' \in \Omega^\pm \to x} v(x', t)$$

and

$$\partial_n^\pm v(x, t) := \frac{\partial v_\pm}{\partial n}(x, t) := \lim_{x' \in \Omega^\pm \to x} n_x \cdot \nabla v(x', t).$$

We use retarded potentials as in [3] to formulate the interface problem (1.1) as a coupled domain/boundary integral equation, using a symmetric formulation of the Neumann-to-Dirichlet operator in terms of layer potentials in the exterior. We obtain wellposedness of the continuous domain/boundary integral formulations of system (1.1) in the time domain. The weak formulation is approximated using finite elements in Ω and Galerkin boundary elements on Γ, based on tensor products of piecewise polynomial functions in space and time. Well-posedness, a priori and a posteriori error estimates are obtained for the discretized problem.

The convergence rate of the error is determined by the singularities of the solution at nonsmooth points of the interface Γ. Near an edge or a vertex, we obtain a singular expansion of the solution into a leading part of explicit singular functions plus smoother remainder terms. The singular exponents are the same as for the time-independent Laplace equation, and our result builds on Plamenevskii's analysis for the wave equation in a domain with edges or cone points [33].

The results obtained in this article for the error analysis and singular expansions provide the key ingredients to adapt recent work on higher-order boundary element methods from boundary to interface problems. The singular expansion shows that graded meshes lead to quasi-optimal approximation of the solution by the *h* version as established in [12, 29]. The expansion also allows the analysis of *hp* versions as in [14, 29]. Similarly, the a posteriori error estimates in the current article are analogous to those established for boundary problems in [15, 17], and they are therefore expected to result in equally efficient space-time adaptive mesh refinement procedures.

We discuss the numerical implementation of the space-time Galerkin method. Numerical experiments illustrate the performance of the method in 3D model problems for the *h* version on both uniform and graded meshes.

The article is organized as follows. Section 2 reviews the functional analytic setting and properties of the boundary integral operators. Well-posedness of the symmetric coupling formulation is discussed in Section 3. There, also the regularity of solutions to the interface problem is studied. Section 4 presents an a priori error estimate as well as an a posteriori error estimate of residual type. Numerical experiments are presented in Section 5. The numerical implementation is discussed in an appendix.

2 Sobolev Spaces and Boundary Integral Operators

We review space-time anisotropic Sobolev spaces on the boundary Γ as a convenient functional analytic setting for the analysis of the time dependent boundary integral operators. A detailed exposition may be found in [13, 21]. In particular, we introduce the retarded potentials for the wave equation and list the mapping properties of the respective time dependent boundary integral operators (Theorem 1).

Let $\Gamma = \partial\Omega$ the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. We recall the scale of Sobolev spaces $H^s(\Gamma)$. Their norms are defined using a partition of unity α_i subordinate to a covering of Γ by open sets B_i. Given diffeomorphisms φ_i from B_i to the unit cube in $\mathbb{R}^n$, the Sobolev norms are induced from $\mathbb{R}^n$, with parameter $\omega \in \mathbb{C} \setminus \{0\}$:

$$\|u\|_{s,\omega,\Gamma} = \left(\sum_{i=1}^{p} \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}\{(\alpha_i u) \circ \varphi_i^{-1}\}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transform $\mathcal{F}\varphi(\xi) = \int e^{-ix\cdot\xi}\varphi(x)\,dx$. Because Γ is Lipschitz, these spaces are independent of the choice of α_i and φ_i when $|s| \leq 1$, as for standard Sobolev spaces. Different $\omega \in \mathbb{C} \setminus \{0\}$ induce equivalent norms on $H^s(\Gamma)$. When a specific ω is fixed, we write $H^s_\omega(\Gamma)$ for $H^s(\Gamma)$. We may now define a family of space-time anisotropic Sobolev spaces:

Definition 1. For $\sigma > 0$ and $r, s \in \mathbb{R}$ define

$$H^r_\sigma(\mathbb{R}^+, H^s(\Gamma)) = \{u \in \mathcal{D}'_+(H^s(\Gamma)) : e^{-\sigma t}u \in \mathcal{S}'_+(H^s(\Gamma)) \text{ and } \|u\|_{r,s,\Gamma} < \infty\}.$$

Here, $\mathcal{D}'_+(E)$ denotes the space of all distributions on $\mathbb{R}$ with support in $[0, \infty)$, taking values in the Hilbert space $E = H^s(\Gamma)$. $\mathcal{S}'_+(E) \subset \mathcal{D}'_+(E)$ denotes the subspace of tempered distributions. The spaces are endowed with the norms

$$\|u\|_{r,s} := \|u\|_{r,s,\Gamma\times\mathbb{R}^+} = \left(\int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2r} \|\hat{u}(\omega)\|^2_{s,\omega,\Gamma} \, d\omega \right)^{\frac{1}{2}},$$

where $\hat{u}(\omega) = \mathcal{F}_{t\mapsto\omega}u$ denotes the Fourier transform of u with respect to t.

The Sobolev spaces are Hilbert spaces. For $r = s = 0$ they correspond to the weighted L^2-space with scalar product $\int_0^\infty e^{-2\sigma t} \int_\Gamma u\bar{v}\, d\Gamma_{\mathbf{x}}\, dt$.

We shall also use the spaces $H^r([0, T], H^s(\Gamma))$ of restrictions to the finite time interval $[0, T]$. These spaces inherit the properties of $H^r_\sigma(\mathbb{R}^+, H^s(\Gamma))$, see the discussion in [8, Remark b on p. 214]. Note that a function $H^r_\sigma(\mathbb{R}^+, H^s(\Gamma))$, in particular, has $r + s$ t-derivatives and s spatial derivatives in L^2.

In the domain Ω we define space-time anisotropic Sobolev spaces $H^r_\sigma(\mathbb{R}^+, H^s(\Omega))$ and $H^r_\sigma([0, T], H^s(\Omega))$ analogously to above, starting from the Sobolev spaces $H^s(\Omega)$ with norm

$$\|u\|_{s,\Omega} = \inf_v \left(\int_{\mathbb{R}^d} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}v(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Here the infimum extends over all extensions $v \in H^s(\mathbb{R}^d)$ of $u \in H^s(\Omega)$, i.e. all v with $v|_\Omega = u$.

We recall the trace theorem in these spaces, see [20, Chapter 4, Lemma 5, resp. Lemma 7]: The trace $u \mapsto u|_{\Gamma\times[0,T]}$ extends to a continuous map

$$\|u|_{\Gamma\times[0,T]}\|_{0,1/2,\Gamma\times[0,T]} \leq C\|u\|_{0,1,\Omega\times[0,T]}. \tag{2.1}$$

Similarly, the Neumann trace $\frac{\partial u}{\partial n}$ of a solution u to the homogeneous wave equation extends to a continuous map

$$\left\|\frac{\partial u}{\partial n}\right\|_{0,-1/2,\Gamma\times[0,T]} \leq C\|u\|_{0,1,\Omega\times[0,T]}.$$

Let now Γ be the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. In $\mathbb{R}^3 \setminus \Gamma$, a solution v to the homogeneous wave equation may be represented in terms of the jump of the Dirichlet and Neumann data across Γ: $v = D\phi - S\lambda$. Here for $x \in \mathbb{R}^3 \setminus \Gamma$ and $t \in [0; T]$ for $T > 0$ fixed,

$$S\lambda(x, t) = \int_0^T \int_\Gamma G(t - \tau, x, y)\, \lambda(y, \tau)\, ds_y\, d\tau,$$

$$D\phi(x, t) = \int_0^T \int_\Gamma \frac{\partial G}{\partial n_y}(t - \tau, x, y)\, \phi(y, \tau)\, ds_y\, d\tau$$

are the single and double layer potentials for the wave equation defined from the fundamental solution

$$G(\tau, x, y) = \frac{\delta(\tau - |x-y|)}{4\pi|x-y|}.$$

Our coupling method uses the following boundary integral operators on $\Gamma \times [0,T]$. For $(x,t) \in \Gamma \times [0,T]$) we define the single layer operator V, the double layer operator K, the adjoint double layer operator K' and the hypersingular integral operator W:

$$V\phi(x,t) = \int_0^T \int_\Gamma G(t-\tau, x, y)\, \phi(y,\tau)\, ds_y\, d\tau\ ,$$

$$K\phi(x,t) = \int_0^T \int_\Gamma \frac{\partial G}{\partial n_y}(t-\tau, x, y)\phi(y,\tau)\, ds_y\, d\tau\ ,$$

$$K'\phi(x,t) = \int_0^T \int_\Gamma \frac{\partial G}{\partial n_x}(t-\tau, x, y)\phi(y,\tau)\, ds_y\, d\tau,$$

$$W\phi(x,t) = \int_0^T \int_\Gamma \frac{\partial^2 G}{\partial n_x \partial n_y}(t-\tau, x, y)\phi(y,\tau)\, ds_y\, d\tau.$$

For a Lipschitz boundary Γ, the following jump relations hold in the sense of distributions [21]:

$$(S\lambda)^-|_\Gamma = (S\lambda)^+|_\Gamma = V\lambda, \qquad \partial_n^-(D\phi) = \partial_n^+(D\phi) = W\phi,$$
$$\partial_n^+(S\lambda) = \Big(-\frac{1}{2}I + K'\Big)\lambda, \qquad \partial_n^-(S\lambda) = \Big(\frac{1}{2}I + K'\Big)\lambda,$$
$$(D\phi)^+|_\Gamma = \Big(\frac{1}{2}I + K\Big)\phi, \qquad (D\phi)^-|_\Gamma = \Big(-\frac{1}{2}I + K\Big)\phi.$$

We report from [9, 21] the mapping properties of the boundary integral operators for $T = \infty$. Following the discussion in Remark b on p. 214 in [8], they also hold in finite time intervals $[0,T]$.

Theorem 1. *The following operators are continuous for $r \in \mathbb{R}$:*

$$V : H^{r+1}([0,T], H^{-\frac{1}{2}}(\Gamma)) \to H^r([0,T], H^{\frac{1}{2}}(\Gamma)),$$
$$K' : H^{r+1}([0,T], H^{-\frac{1}{2}}(\Gamma)) \to H^r([0,T], H^{-\frac{1}{2}}(\Gamma)),$$
$$K : H^{r+1}([0,T], H^{\frac{1}{2}}(\Gamma)) \to H^r([0,T], H^{\frac{1}{2}}(\Gamma)),$$
$$W : H^{r+1}([0,T], H^{\frac{1}{2}}(\Gamma))) \to H^r([0,T], H^{-\frac{1}{2}}(\Gamma)).$$

with corresponding norms in space Ω, resp. Γ.

3 Symmetric Wave Coupling

We analyze a symmetric coupling method for the time-dependent interface problem (1.1). Our approach extends [2] and leads to the existence and uniqueness result in Theorem 2. We also discuss a regularity result, Theorem 3, for the behavior near singular points of the boundary.

The energy for the wave equation in Ω^+ resp. Ω^- is defined by

$$E_{\Omega^+}(t) = \frac{1}{2}\int_{\Omega^+} |\dot v(x,t)|^2 + |\nabla v(x,t)|^2\, dx, \quad E_{\Omega^-}(t) = \frac{1}{2}\int_{\Omega^-} |\dot u(x,t)|^2 + |\nabla u(x,t)|^2\, dx$$

for $t \in \mathbb{R}^+$. Using Green's formula for a test function $w \in H^1(\mathbb{R}_+, H^1(\Omega^\mp))$

$$\langle \partial_n^\mp v, \gamma^\mp w\rangle_{\Gamma\times\mathbb{R}_+} = \pm(\nabla v, \nabla w)_{\Omega^\mp\times\mathbb{R}_+} \pm (\Delta u, w)_{\Omega^\mp\times\mathbb{R}_+},$$

we obtain for the solution v to the wave equation in Ω^+:

$$\frac{\partial E_{\Omega^+}}{\partial t} = \int_{\Omega^+} \nabla v \nabla \dot{v} + \dot{v}\ddot{v}\,dx = \int_{\Omega^+} \dot{v}\ddot{v} - \Delta v \dot{v}\,dx - \int_{\Gamma} \partial_n^+ v \gamma^+ \dot{v}\,ds_x = -\int_{\Gamma} \partial_n^+ v \gamma^+ \dot{v}\,ds_x.$$

From the homogeneous initial conditions, we conclude that at time $t = T$

$$E_{\Omega^+}(T) = -\langle \partial_n^+ v, \gamma^+ \dot{v} \rangle_{\Gamma \times [0,T]}. \tag{3.1}$$

In the same way, we define the energy for the wave equation in Ω^-:

$$E_{\Omega^-}(T) = \int_0^T \int_{\Omega^-} \nabla u \nabla \dot{u} + \dot{u}\ddot{u}\,dx\,dt.$$

We remark that for $u \in H^0([0,T], H^1(\Omega^-))$ we can estimate the corresponding norm with the energy norm in the following way. First by using Cauchy–Schwarz inequality, we observe

$$|u(x,t)|^2 = \bigg|\int_0^t \dot{u}(x,r)\,dr\bigg|^2 = \bigg|\int_0^t 1 \cdot \dot{u}(x,r)\,dr\bigg|^2 \leq \bigg(\int_0^t 1\,dr\bigg)\bigg(\int_0^t |\dot{u}(x,r)|^2\,dr\bigg) = t\bigg(\int_0^t |\dot{u}(x,r)|^2\,dr\bigg).$$

Therefore, we get

$$\begin{aligned}
\|u\|^2_{0,1,\Omega^- \times [0,T]} &= \int_0^T \int_{\Omega^-} |u|^2 + |\nabla u|^2\,dx\,dt + \int_0^T \int_{\Omega^-} |\dot{u}|^2 + |u|^2\,dx\,dt \\
&= \int_0^T \int_{\Omega^-} |\dot{u}|^2 + |\nabla u|^2\,dx\,dt + 2\int_0^T \int_{\Omega^-} |u|^2\,dx\,dt \lesssim E_{\Omega^-}(T),
\end{aligned} \tag{3.2}$$

where the estimate depends on the time T on the parameters ρ_1 and μ_1. Analogously we achieve the same estimate for $v \in H^0([0,T], H^1(\Omega^+))$:

$$\|v\|^2_{0,1,\Omega^+ \times [0,T]} \lesssim E_{\Omega^+}(T). \tag{3.3}$$

Representing the solution of (1.1b) as $v = D\gamma^+ v - S\partial_n^+ v$, the jump relations yield

$$\frac{1}{2}(\gamma^+ v) = K(\gamma^+ v) - V(\partial_n^+ v), \quad \frac{1}{2}(\partial_n^+ v) = W(\gamma^+ v) - K'(\partial_n^+ v). \tag{3.4}$$

Testing now the first equation of (3.4) with $\dot{m} \in H^0([0,T], H^{-\frac{1}{2}}(\Gamma))$ and the second equation of (3.4) with $\dot{\omega} \in H^0([0,T], H^{\frac{1}{2}}(\Gamma))$ gives

$$0 = \langle (K(\gamma^+ v)), \dot{m} \rangle_{\Gamma \times [0,T]} - \langle (V(\partial_n^+ v)), \dot{m} \rangle_{\Gamma \times [0,T]} - \frac{1}{2}\langle \gamma^+ v, \dot{m} \rangle_{\Gamma \times [0,T]}, \tag{3.5}$$

$$0 = \langle W(\gamma^+ v), \dot{\omega} \rangle_{\Gamma \times [0,T]} - \langle (K'(\partial_n^+ v)), \dot{\omega} \rangle_{\Gamma \times [0,T]} - \frac{1}{2}\langle \partial_n^+ v, \dot{\omega} \rangle_{\Gamma \times [0,T]}. \tag{3.6}$$

Testing (1.1a) with $\dot{w} \in H^0([0,T], H^1(\Omega^-))$ gives

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega^-} \rho_1 \ddot{u}\dot{w}\,dx\,dt - \int_0^T \int_{\Omega^-} \mu_1 \Delta u \dot{w}\,dx\,dt \\
&= \int_0^T \int_{\Omega^-} \rho_1 \ddot{u}\dot{w}\,dx\,dt + \int_0^T \int_{\Omega^-} \mu_1 \nabla u \nabla \dot{w}\,dx\,dt - \int_0^T \int_{\Gamma} \mu_1 \partial_n^- u \gamma^- \dot{w}\,ds_x\,dt.
\end{aligned}$$

Thus, setting

$$a(u,w) := \int_0^T \int_{\Omega^-} \rho_1 \ddot{u}\dot{w}\,dx\,dt + \int_0^T \int_{\Omega^-} \mu_1 \nabla u \nabla \dot{w}\,dx\,dt$$

gives

$$E_{\Omega^-}(T) \lesssim a(u,u) \tag{3.7}$$

and hence

$$a(u,w) - \langle \mu_1 \partial_n^- u, \gamma^- \dot{w} \rangle_{\Gamma\times[0,T]} = 0. \tag{3.8}$$

Adding (3.5) and subtracting (3.6) from (3.8), yields

$$\begin{aligned} a(u,w) &- \langle g, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} - \langle \partial_n^+ v, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} \\ &+ \langle K(\gamma^+ v), \dot{m}\rangle_{\Gamma\times[0,T]} - \langle V(\partial_n^+ v), \dot{m}\rangle_{\Gamma\times[0,T]} - \frac{1}{2}\langle \gamma^+ v, \dot{m}\rangle_{\Gamma\times[0,T]} \\ &- \langle W(\gamma^+ v), \dot{\omega}\rangle_{\Gamma\times[0,T]} + \langle (K'(\partial_n^+ v)), \dot{\omega}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \partial_n^+ v, \dot{\omega}\rangle_{\Gamma\times[0,T]} = 0. \end{aligned}$$

Further adding $0 = -\frac{1}{2}\langle \gamma^+ v, \dot{m}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \gamma^+ v, \dot{m}\rangle_{\Gamma\times[0,T]}$ together with (1.1e),

$$\begin{aligned} a(u,w) &- \langle g, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} - \langle \partial_n^+ v, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} + \langle K(\gamma^+ v), \dot{m}\rangle_{\Gamma\times[0,T]} \\ &- \langle V(\partial_n^+ v), \dot{m}\rangle_{\Gamma\times[0,T]} - \langle \gamma^- u, \dot{m}\rangle_{\Gamma\times[0,T]} + \langle f, \dot{m}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \gamma^+ v, \dot{m}\rangle_{\Gamma\times[0,T]} \\ &- \langle W(\gamma^+ v), \dot{\omega}\rangle_{\Gamma\times[0,T]} + \langle (K'(\partial_n^+ v)), \dot{\omega}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \partial_n^+ v, \dot{\omega}\rangle_{\Gamma\times[0,T]} = 0. \end{aligned}$$

Define the bilinear form

$$\begin{aligned} \mathcal{A}((u,\phi,\lambda),(w,\omega,m)) := a(u,w) &- \langle \lambda, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} + \langle K\phi, \dot{m}\rangle_{\Gamma\times[0,T]} - \langle V\lambda, \dot{m}\rangle_{\Gamma\times[0,T]} \\ &- \langle \gamma^- u, \dot{m}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \phi, \dot{m}\rangle_{\Gamma\times[0,T]} \\ &- \langle W\phi, \dot{\omega}\rangle_{\Gamma\times[0,T]} + \langle K'\lambda, \dot{\omega}\rangle_{\Gamma\times[0,T]} + \frac{1}{2}\langle \lambda, \dot{\omega}\rangle_{\Gamma\times[0,T]} \end{aligned}$$

as well as

$$\mathcal{F}(w,\omega,m) := \langle g, \gamma^- \dot{w}\rangle_{\Gamma\times[0,T]} - \langle f, \dot{m}\rangle_{\Gamma\times[0,T]}.$$

Then the weak formulation of the transmission problem (1.1) reads as follows:

For $g \in H^2([0,T], H^{-\frac{1}{2}}(\Gamma))$ and $f \in H^2([0,T], H^{\frac{1}{2}}(\Gamma))$ find $u \in H^1([0,T], H^1(\Omega^-))$, $\phi \in H^1([0,T], H^{\frac{1}{2}}(\Gamma))$ and $\lambda \in H^1([0,T], H^{-\frac{1}{2}}(\Gamma))$ such that

$$\mathcal{A}((u,\phi,\lambda),(w,\omega,m)) = \mathcal{F}(w,\omega,m) \tag{3.9}$$

holds for all $w \in H^1([0,T], H^1(\Omega^-))$, $\omega \in H^1([0,T], H^{\frac{1}{2}}(\Gamma))$ and $m \in H^1([0,T], H^{-\frac{1}{2}}(\Gamma))$.

The solution to (3.9) is given by $(u,\phi,\lambda) = (u, \gamma^+ v, \partial_n^+ v)$.

Setting $w = u$, $\phi = \gamma^+ v = \omega$, $\lambda = \partial_n^+ v = m$ above gives with (3.1) and (3.7)

$$E_{\Omega^-}(T) + E_{\Omega^+}(T) \lesssim a(u,u) - \langle \partial_n^+ v, \gamma^+ \dot{v}\rangle_{\Gamma\times[0,T]} = \langle g, \gamma^- \dot{u}\rangle_{\Gamma\times([0,T]} - \langle f, \partial_n^+ \dot{v}\rangle_{\Gamma\times[0,T]}. \tag{3.10}$$

Therefore with the energy norm

$$|||(u,v)||| := (E_{\Omega^-}(T) + E_{\Omega^+}(T))^{\frac{1}{2}},$$

we get from (3.9) and (3.10) the coercivity

$$|||(u,v)|||^2 \lesssim \mathcal{A}((u,\phi,\lambda),(u,\phi,\lambda)). \tag{3.11}$$

From (3.2) and (3.3) and the coercivity and mapping properties of the integral operators we conclude

$$\|u\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\lambda\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]} \lesssim \mathcal{A}((u,\phi,\lambda),(u,\phi,\lambda))$$

and

$$\begin{aligned} \mathcal{A}((u,\phi,\lambda),(w,\omega,m)) &\lesssim \big(\|u\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\lambda\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]}\big)^{\frac{1}{2}} \\ &\quad \cdot \big(\|w\|^2_{1,1,\Omega^-\times[0,T]} + \|\omega\|^2_{1,\frac{1}{2},\Gamma\times[0,T]} + \|m\|^2_{1,-\frac{1}{2},\Gamma\times[0,T]}\big)^{\frac{1}{2}}. \end{aligned}$$

Standard arguments, as in [5, 13], now lead to the following existence and uniqueness result.

Theorem 2. *Let $r \geq 1$. For $g \in H^{1+r}([0,T], H^{-\frac{1}{2}}(\Gamma))$ and $f \in H^{1+r}([0,T], H^{\frac{1}{2}}(\Gamma))$, there exists a unique solution $u \in H^r([0,T], H^1(\Omega^-))$, $\phi \in H^r([0,T], H^{\frac{1}{2}}(\Gamma))$ and $\lambda \in H^r([0,T], H^{-\frac{1}{2}}(\Gamma))$ of (3.9).*

If the transmission problem (3.9) is posed in a polyhedral domain, the interior and exterior solutions u, respectively v, exhibit singularities at edges and corners of the interface. In the case of the wave equation in domains with conical or wedge singularities, the asymptotic behavior of solutions has been studied by Plamenevskii and collaborators since the late 1990s [26, 33]. Their results imply that at a fixed time t, the solution to the wave equation admits an explicit singular expansion with the same exponents as for elliptic equations. Recently, Müller and Schwab have used these results to obtain optimal convergence rates for a finite element method in polygonal domains in $\mathbb{R}^2$ (see [29]), which were later extended to $\mathbb{R}^3$ by the authors [12, 14].

To be specific, let $0 \leq d \leq 1$ and $K \subset \mathbb{R}^{3-d}$ an open cone with vertex at 0, smooth outside the vertex. We denote by $\mathcal{K} = K \times \mathbb{R}^d$ the wedge over K, with coordinates $x = (y, z)$. We consider the transmission problem (3.9) for the wave equation for $\Omega = \mathcal{K}$.

The analysis uses the Fourier–Laplace transform $\mathcal{F}_{t\to\omega}$ in time to reduce the problem to an interface problem for the Helmholtz equation

$$\begin{aligned} \rho_1\omega^2\hat{u} + \mu_1\Delta\hat{u} &= 0, && x \in \mathcal{K}, \\ \omega^2\hat{v} + \Delta\hat{v} &= 0, && x \in \mathcal{K}^c, \\ \gamma^-\hat{u} - \gamma^+\hat{v} &= \hat{f} && \text{on } \Gamma, \\ \mu_1\partial_n^-\hat{u} - \partial_n^+\hat{v} &= \hat{g} && \text{on } \Gamma \end{aligned}$$

with suitable radiation conditions at infinity. For the Helmholtz equation, the singular decomposition of the solution is known for every complex frequency ω, and the techniques introduced by Plamenevskii can be used to translate it into the time domain [33].

For now we set $\rho_1 = 1$. Singularities due to $\rho_1 \neq 1 = \rho_2$ lead to weaker singularities at the interface. We consider the model geometries $\mathcal{K}$ of a wedge ($d = 1$), respectively a corner ($d = 0$). For the wedge, we do separation of variables in cylindrical coordinates (r, Θ, z) near the edge of $\mathcal{K}$, with $r = |y|$ and $\Theta = \frac{y}{|y|}$, while for the corner spherical coordinates (r, Θ) are used. The separation of variables leads to a transmission problem in the angular variable Θ on $S^{3-d-1} \setminus \Gamma \cap S^{3-d-1}$, involving an operator $\mathfrak{A}(\lambda)$. With $p = \mu_1$ in $S^{3-d-1} \cap \mathcal{K}$ and $p = 1$ otherwise, it is given on functions w on S^{3-d-1} by

$$\mathfrak{A}(\lambda)w = p(i\lambda)^2 w + ip(n - d - 2)\lambda w - p\Delta_S w,$$

where Δ_S is the Laplace–Beltrami operator on S^{3-d-1} and on $S^{3-d-1} \cap \Gamma$ we have the following interface conditions:

$$[w] = 0, \quad [p\partial_n w] = 0.$$

For a wedge, $d = 1$, the nonzero eigenvalues $\lambda = \lambda_k$ of $\mathfrak{A}$ are determined by a Sturm–Liouville problem. If the opening angle of the wedge $\mathcal{K}$ is α,

$$(pw')' = \lambda_k^2 pw \quad \text{for } \theta \in S^1 \setminus \{0, \alpha\}, \qquad [w]_{\theta=0,\alpha} = 0, \qquad [pw']_{\theta=0,\alpha} = 0. \tag{3.12}$$

We here have $p = \mu_1$ for $\theta \in (0, \alpha)$, and $p = 1$ otherwise.

Costabel and Stephan [10] reduce the spectral problem (3.12) to an explicit transcendental equation for the eigenvalues $\lambda = -i\Lambda$. They are given by

$$\mu_1^{\pm 1} \tan\Big(\frac{\alpha\Lambda}{2}\Big) = -\tan\Big(\frac{(2\pi - \alpha)\Lambda}{2}\Big).$$

The associated eigenspaces are spanned by Φ_k, linear combinations of $\sin(\Lambda_k\theta)$ and $\cos(\Lambda_k\theta)$ in $(0, \alpha)$, respectively $(\alpha, 2\pi)$.

Theorem 3. *Let $\beta \leq 1$ and assume that the line $\operatorname{Im}\lambda = \beta - 1 + \frac{1-d}{2}$ does not intersect the spectrum of $\mathfrak{A}$. Further, for $d = 0$ define*

$$J_\beta = \Big\{k : \frac{1}{2} > \operatorname{Im}\lambda_k > \beta - \frac{1}{2}\Big\},$$

and for $d = 1$

$$J_\beta = \{k : 0 > \operatorname{Im} \lambda_k > \beta - 1\} \cup A,$$

with $A = \{0\}$ *for* $\beta \leq 0$ *and* $A = \emptyset$ *otherwise. If* (u, v) *is a strong solution to the inhomogeneous transmission problem for* $\mathcal{K}$ *near* $\{0\} \times \mathbb{R}^d$*, then u and v are of the form*

$$\sum_{j\in J_\beta} \Gamma(1+\nu_j)|y|^{i\lambda_j}\Phi_j(\Theta) \sum_{m=0}^{N_j} \frac{(\partial_t^2 - \Delta_z)^m (i|y|)^{2m}}{2^{2m} m! \Gamma(m+\nu_j+1)} \check{c}_j(y,t,z) + v_0(y,t,z),$$

assuming that $i\lambda_j \notin \mathbb{N}$*. If* $i\lambda_j \in \mathbb{N}$ *additional terms* $|y|^{i\lambda_j} \log(|y|)$ *appear. The remainder* v_0 *is less singular, with regularity depending on* β*.*

Corresponding results for Dirichlet and the Neumann problems have been shown in [14, 33], and the analysis extends to the transmission problem. See also [18, 30] for results in the time-independent case.

4 The FEM-BEM Coupling System

In this section the finite and boundary elements are introduced as tensor products of piecewise polynomials in space and time. Later we derive an a priori error estimate for the Galerkin error (Theorem 4) extending the estimates for the weakly singular and hypersingular time-dependent integral equations governing the exterior Dirichlet and Neumann problem for the wave equation from [5, 12], respectively to the coupling formulation (in Section 3) of the transmission problem. An adaptive Galerkin procedure based on residual type error estimators can be based on the a posteriori error estimate in Theorem 5 which also extends results for the exterior problems in [15, 17] to the interface problem.

We consider space-time discretizations based on tensor products of piecewise polynomials:

For simplicity, we assume that Ω is a polyhedral domain, with a tetrahedral mesh $\mathcal{T}_\Omega = \{\Omega_1, \dots, \Omega_{N_o}\}$ consisting of N_o tetrahedra. We assume that at most one face of each Ω_j is contained in Γ. The induced triangulation of the boundary Γ, $\mathcal{T}_\Gamma = \{T_1, \dots, T_{N_{s'}}\}$, should consist of closed triangular faces T_i such that each T_i is a face of a Ω_j.

We consider the space $V_h^q(\Omega)$ of piecewise polynomial functions on $\mathcal{T}_\Omega$ of degree $q \geq 0$ in space (continuous if $q \geq 1$); V_h^q consists of traces on Γ of functions in $V_h^q(\Omega)$. The parameter h denotes the maximal diameter of an element in $\mathcal{T}_\Omega$.

In this article we consider the approximation of (4.1) on both quasiuniform and β-graded meshes. To define β-graded meshes on the interval $[-1, 1]$, by symmetry it suffices to specify the nodes in $[-1, 0]$. There we let

$$x_k = -1 + \left(\frac{k}{N_l}\right)^\beta \quad \text{for } k = 1, \dots, N_l.$$

For a cube the nodes of the β-graded mesh are tuples of such points, (x_k, x_l, x_m), $k, l, m = 1, \dots, N_l$, by symmetry extended onto the whole cube. In a general polyhedral geometry a graded mesh is locally modeled on these examples, see Figures 1 and 8. For $\beta = 1$ one recovers a uniform mesh.

For the time discretization we choose an equidistant mesh on the positive half-line

$$\mathcal{T}_T = \{[0, t_1), [t_1, t_2), \dots\},$$

where $t_n = n(\Delta t)$; $V_{\Delta t}^p$ is the space of piecewise polynomial functions of degree p on $\mathcal{T}_T$ (continuous and vanishing at $t = 0$ if $p \geq 1$, C^1 if $p \geq 2$).

The space-time approximation spaces are given by tensor products of the approximation spaces in space and time, V_h^q and $V_{\Delta t}^p$, associated to the space-time meshes $\mathcal{T}_{\Omega,T} = \mathcal{T}_\Omega \times \mathcal{T}_T$, respectively $\mathcal{T}_{\Gamma,T} = \mathcal{T}_\Gamma \times \mathcal{T}_T$. We write

$$V_{h,\Delta t}^{q,p}(\Omega) := V_h^q(\Omega) \otimes V_{\Delta t}^p, \ V_{h,\Delta t}^{q,p}(\Gamma) := V_h^q(\Gamma) \otimes V_{\Delta t}^p.$$

Note that $V_{h,\Delta t}^{1,2}(\Omega^-) \subset H^1([0,T], H^1(\Omega^-))$, $V_{h,\Delta t}^{1,2}(\Gamma) \subset H^1([0,T], H^{\frac{1}{2}}(\Gamma))$ and $V_{h,\Delta t}^{0,1}(\Gamma) \subset H^1([0,T], H^{-\frac{1}{2}}(\Gamma))$.

The *Galerkin* discretization of (3.9) for $p \geq 2$, $q \geq 1$ then reads:

Find $(u_h, \phi_h, \lambda_h) \in V^{q,p}_{h,\Delta t}(\Omega^-) \times V^{q,p}_{h,\Delta t}(\Gamma) \times V^{q-1,p-1}_{h,\Delta t}(\Gamma) =: X_{h,\Delta t}$ *such that*

$$\mathcal{A}((u_h, \phi_h, \lambda_h), (w_h, \omega_h, m_h)) = \langle g, \gamma^- \dot{w}_h \rangle_{\Gamma\times[0,T]} - \langle f, \dot{m}_h \rangle_{\Gamma\times[0,T]} \tag{4.1}$$

hold for all $w_h \in X_{h,\Delta t}$.

We have the following a priori error estimate:

Theorem 4. *Let* $(u, \phi, \lambda) \in H^1([0, T], H^1(\Omega^-)) \times H^1([0, T], H^{\frac{1}{2}}(\Gamma)) \times H^1([0, T], H^{-\frac{1}{2}}(\Gamma))$ *solve problem* (3.9) *and let* $(u_h, \phi_h, \lambda_h) \in X_{h,\Delta t}$ *satisfy the corresponding Galerkin equation* (4.1). *Then there hold*

$$\begin{aligned}
&\|u - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi - \phi_h\|^2_{0,1/2,\Gamma\times[0,T]} + \|\lambda - \lambda_h\|^2_{0,-1/2,\Gamma\times[0,T]} \\
&\quad \lesssim \inf_{w_h,\psi_h,\mu_h \in X_{h,\Delta t}} \Big(1 + \frac{1}{(\Delta t)^2}\Big)\Big\{\|u - w_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi - \psi_h\|^2_{1,1/2,\Gamma\times[0,T]} + \|\lambda - \mu_h\|^2_{1,-1/2,\Gamma\times[0,T]}\Big\}.
\end{aligned}$$

Proof. Let $(u, \phi, \lambda) \in H^1([0, T], H^1(\Omega^-)) \times H^1([0, T], H^{\frac{1}{2}}(\Gamma)) \times H^1([0, T], H^{-\frac{1}{2}}(\Gamma))$ solve problem (3.9) and let $(u_h, \phi_h, \lambda_h) \in X_{h,\Delta t}$ satisfy the corresponding Galerkin equation (4.1). Then with $(\tilde{u}, \tilde{\phi}, \tilde{\lambda}) \in X_{h,\Delta t}$

$$\begin{aligned}
&\|u - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi - \phi_h\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\lambda - \lambda_h\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]} \\
&\quad \leq \|u - \tilde{u}\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{u} - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi - \tilde{\phi}\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\tilde{\phi} - \phi_h\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} \\
&\qquad + \|\lambda - \tilde{\lambda}\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]} + \|\tilde{\lambda} - \lambda_h\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]}.
\end{aligned}$$

We now estimate

$$\|\tilde{u} - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{\phi} - \phi_h\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\tilde{\lambda} - \lambda_h\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]}.$$

To do so, we use the coercivity, (3.11), (3.2) and (3.3) with $\tilde{r} := D\tilde{\phi} - S\tilde{\lambda}$ and $v_h := D\phi_h - S\lambda_h$, as well as $\gamma^+(\tilde{r} - v_h) = \tilde{\phi} - \phi_h$ and $\partial^+_n(\tilde{r} - v_h) = \tilde{\lambda} - \lambda_h$:

$$\begin{aligned}
&\|\tilde{u} - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{\phi} - \phi_h\|^2_{0,\frac{1}{2},\Gamma\times[0,T]} + \|\tilde{\lambda} - \lambda_h\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]} \\
&\quad \lesssim \|\tilde{u} - u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{r} - v_h\|^2_{0,1,\Omega^+\times[0,T]} \\
&\quad \lesssim |||\tilde{u} - u_h, \tilde{r} - v_h|||^2 \lesssim \mathcal{A}\left(\begin{pmatrix} \tilde{u} - u_h \\ \tilde{\phi} - \phi_h \\ \tilde{\lambda} - \lambda_h \end{pmatrix}^T, \begin{pmatrix} \tilde{u} - u_h \\ \tilde{\phi} - \phi_h \\ \tilde{\lambda} - \lambda_h \end{pmatrix}^T\right) \\
&\quad = \mathcal{A}\left(\begin{pmatrix} \tilde{u} - u \\ \tilde{\phi} - \phi \\ \tilde{\lambda} - \lambda \end{pmatrix}^T, \begin{pmatrix} \tilde{u} - u_h \\ \tilde{\phi} - \phi_h \\ \tilde{\lambda} - \lambda_h \end{pmatrix}^T\right) + \mathcal{A}\left(\begin{pmatrix} u - u_h \\ \phi - \phi_h \\ \lambda - \lambda_h \end{pmatrix}^T, \begin{pmatrix} \tilde{u} - u_h \\ \tilde{\phi} - \phi_h \\ \tilde{\lambda} - \lambda_h \end{pmatrix}^T\right) \\
&\quad = \mathcal{A}\left(\begin{pmatrix} \tilde{u} - u \\ \tilde{\phi} - \phi \\ \tilde{\lambda} - \lambda \end{pmatrix}^T, \begin{pmatrix} \tilde{u} - u_h \\ \tilde{\phi} - \phi_h \\ \tilde{\lambda} - \lambda_h \end{pmatrix}^T\right),
\end{aligned}$$

where we have used Galerkin orthogonality. Now consider every term separately: Using the continuity of the duality pairing and the continuous embeddings of the Sobolev spaces,

$$\begin{aligned}
&\int_0^T \int_{\Omega^-} \rho_1 \partial_t^2(\tilde{u} - u)\partial_t(\tilde{u} - u_h)\, dx\, dt + \int_0^T \int_{\Omega^-} \mu_1 \nabla(\tilde{u} - u)\nabla(\partial_t(\tilde{u} - u_h))\, dx\, dt \\
&\quad \lesssim \|\partial_t^2(\tilde{u} - u)\|_{-1,0,\Omega^-\times[0,T]}\|\partial_t(\tilde{u} - u_h)\|_{1,0,\Omega^-\times[0,T]} + \|\nabla(\tilde{u} - u)\|_{0,0,\Omega^-\times[0,T]}\|\nabla\partial_t(\tilde{u} - u_h)\|_{0,0,\Omega^-\times[0,T]} \\
&\quad \lesssim \|\tilde{u} - u\|_{1,0,\Omega^-\times[0,T]}\|\tilde{u} - u_h\|_{2,0,\Omega^-\times[0,T]} + \|\tilde{u} - u\|_{0,1,\Omega^-\times[0,T]}\|\tilde{u} - u_h\|_{1,1,\Omega^-\times[0,T]} \\
&\quad \lesssim \|\tilde{u} - u\|_{0,1,\Omega^-\times[0,T]}\|\tilde{u} - u_h\|_{1,1,\Omega^-\times[0,T]}.
\end{aligned}$$

With the inverse estimate and Young's inequality we get for small $\epsilon > 0$:

$$\|\tilde{u} - u\|_{0,1,\Omega^-\times[0,T]}\|\tilde{u} - u_h\|_{1,1,\Omega^-\times[0,T]} \lesssim \frac{1}{\epsilon(\Delta t)^2}\|\tilde{u} - u\|^2_{0,1,\Omega^-\times[0,T]} + \epsilon\|\tilde{u} - u_h\|^2_{0,1,\Omega^-\times[0,T]}.$$

For the coupling contributions we use [17, inverse estimate (3.182)]:

$$\|\tilde{\phi}\|_{1,1/2,\Gamma\times[0,T]} \lesssim \frac{1}{\Delta t}\|\tilde{\phi}\|_{0,1/2,\Gamma\times[0,T]}.$$

Combined with the trace theorem (2.1) and Young's inequality, we estimate

$$\begin{aligned}
-\langle \tilde{\lambda}-\lambda, \gamma^-(\partial_t(\tilde{u}-u_h))\rangle_{\Gamma\times[0,T]} &\lesssim \|\tilde{\lambda}-\lambda\|_{0,-1/2,\Gamma\times[0,T]}\|\gamma^-(\partial_t(\tilde{u}-u_h))\|_{0,1/2,\Gamma\times[0,T]}\\
&\lesssim \|\tilde{\lambda}-\lambda\|_{0,-1/2,\Gamma\times[0,T]}(\Delta t)^{-1}\|\tilde{u}-u_h\|_{0,1,\Omega^-\times[0,T]}\\
&\lesssim \frac{1}{\epsilon(\Delta t)^2}\|\tilde{\lambda}-\lambda\|^2_{0,-1/2,\Gamma\times[0,T]} + \epsilon\|\tilde{u}-u_h\|^2_{0,1,\Omega^-\times[0,T]}.
\end{aligned}$$

Now estimating the retarded single and double layer potential:

$$\begin{aligned}
&\Big\langle \Big(K+\frac{1}{2}I\Big)(\tilde{\phi}-\phi) - V(\tilde{\lambda}-\lambda), \partial_t(\tilde{\lambda}-\lambda_h)\Big\rangle_{\Gamma\times[0,T]}\\
&\quad\lesssim \Big(\Big\|\Big(K+\frac{1}{2}I\Big)(\tilde{\phi}-\phi)\Big\|_{0,1/2,\Gamma\times[0,T]} + \|V(\tilde{\lambda}-\lambda)\|_{0,1/2,\Gamma\times[0,T]}\Big)\|\tilde{\lambda}-\lambda_h\|_{1,-1/2,\Gamma\times[0,T]}\\
&\quad\lesssim (\|\tilde{\phi}-\phi\|_{1,1/2,\Gamma\times[0,T]} + \|\tilde{\lambda}-\lambda\|_{1,-1/2,\Gamma\times[0,T]})\|\tilde{\lambda}-\lambda_h\|_{1,-1/2,\Gamma\times[0,T]})\\
&\quad\lesssim \frac{1}{\epsilon(\Delta t)^2}\|\tilde{\phi}-\phi\|^2_{1,1/2,\Gamma\times[0,T]} + \frac{1}{\epsilon(\Delta t)^2}\|\tilde{\lambda}-\lambda\|^2_{1,-1/2,\Gamma\times[0,T]} + \epsilon\|\tilde{\lambda}-\lambda_h\|^2_{0,-1/2,\Gamma\times[0,T]},
\end{aligned}$$

where we used the mapping properties of V and K, the inverse estimate and Young's inequality. We continue with the other coupling contribution:

$$\begin{aligned}
-\langle \gamma^-(\tilde{u}-u), \partial_t(\tilde{\lambda}-\lambda_h)\rangle_{\Gamma\times[0,T]} &\lesssim \|\gamma^-(\tilde{u}-u)\|_{0,1/2,\Gamma\times[0,T]}\|\partial_t(\tilde{\lambda}-\lambda_h)\|_{0,-1/2,\Gamma\times[0,T]}\\
&\lesssim \|\tilde{u}-u\|_{0,1,\Omega\times[0,T]}\|\tilde{\lambda}-\lambda_h\|_{1,-1/2,\Gamma\times[0,T]}\\
&\lesssim \frac{1}{\epsilon(\Delta t)^2}\|\tilde{u}-u\|^2_{0,1,\Omega\times[0,T]} + \epsilon\|\tilde{\lambda}-\lambda_h\|^2_{0,-1/2,\Gamma\times[0,T]},
\end{aligned}$$

where we used the trace theorem (2.1), the inverse estimate and Young's inequality. Moving the terms with positive powers of ϵ to the left-hand side and choosing a fixed sufficiently small $\epsilon > 0$, we conclude

$$\begin{aligned}
&\|\tilde{u}-u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{\phi}-\phi_h\|^2_{0,1/2,\Gamma\times[0,T]} + \|\tilde{\lambda}-\lambda_h\|^2_{0,-1/2,\Gamma\times[0,T]}\\
&\quad\lesssim \frac{1}{(\Delta t)^2}\|\tilde{u}-u\|^2_{0,1,\Omega^-\times[0,T]} + \frac{1}{(\Delta t)^2}\|\tilde{\phi}-\phi\|^2_{1,1/2,\Gamma\times[0,T]} + \frac{1}{(\Delta t)^2}\|\tilde{\lambda}-\lambda\|^2_{1,-1/2,\Gamma\times[0,T]}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|u-u_h\|^2_{0,1,\Omega^-\times[0,T]} + \|\phi-\phi_h\|^2_{0,1/2,\Gamma\times[0,T]} + \|\lambda-\lambda_h\|^2_{0,-1/2,\Gamma\times[0,T]}\\
&\quad\lesssim \Big(1+\frac{1}{(\Delta t)^2}\Big)(\|\tilde{u}-u\|^2_{0,1,\Omega^-\times[0,T]} + \|\tilde{\phi}-\phi\|^2_{1,1/2,\Gamma\times[0,T]} + \|\tilde{\lambda}-\lambda\|^2_{1,-1/2,\Gamma\times[0,T]}).
\end{aligned}$$

Taking the infimum yields the assertion. □

Next we derive an a posteriori error estimate for the FEM-BEM coupled system. In the derivation we require an approximation result: Let $\Pi_{\Delta t}$ be the orthogonal projection from $L^2(\mathbb{R}_+)$ to $V^q_{\Delta t}$, Π_h the orthogonal projection from $L^2(\Gamma)$ to $V^p_h(\Gamma)$. Their approximation properties lead to corresponding properties of the composed operator $\Pi_h \circ \Pi_{\Delta t}$ for space-time, similarly to [13]):

Lemma 1. *Let $f \in H^s_\sigma(\mathbb{R}^+, H^m(\Gamma) \cap H^r(\Gamma))$, $0 < m \leq q+1$, $0 < s \leq p+1$, $r \leq s$, $|l| \leq \frac{1}{2}$ such that $l \cdot r \geq 0$. Then:*

- *If $l, r \leq 0$,*
$$\|f - \Pi_h \circ \Pi_{\Delta t} f\|_{r,l,\Gamma} \leq C(h^\alpha + (\Delta t)^\beta)\|f\|_{s,m,\Gamma},$$
where $\alpha = \min\{m-l, m-\frac{m(l+r)}{m+s}\}$, $\beta = \min\{m+s-(l+r), m+s-\frac{m+s}{m}l\}$.
- *If $l, r > 0$,*
$$\|f - \Pi_h \circ \Pi_{\Delta t} f\|_{r,l,\Gamma} \leq C(h^\alpha + (\Delta t)^\beta)\|f\|_{s,m,\Gamma},$$
where $\alpha = \min\{m-l, m-\frac{m(l+r)}{m+s}\}$, $\beta = m+s-(l+r)$.

The residual a posteriori error estimate reads as follows:

Theorem 5. *Let* $(u,\phi,\lambda)\in H^1([0,T],H^1(\Omega^-))\times H^1([0,T],H^{\frac{1}{2}}(\Gamma))\times H^1([0,T],H^{-\frac{1}{2}}(\Gamma))$ *and* $(u_h,\phi_h,\lambda_h)\in X_{h,\Delta t}$ *satisfy the wave coupling problem* (3.9) *respectively the corresponding Galerkin equation* (4.1)*. Let*

$$\bigcup_{j=1}^{N_s}\partial\Omega_j=\tilde{T}=\bigcup_{i=1}^{m}T_i,$$

where each T_i *is a face of one* Ω_j*. With* $[v]$*, a jump into face* T_i*, the following a posteriori error estimate holds:*

$$\|u_h-u\|^2_{0,1,\Omega^-\times[0,T]}+\|\phi_h-\phi\|^2_{0,1/2,\Gamma\times[0,T]}+\|\lambda_h-\lambda\|^2_{0,-1/2,\Gamma\times[0,T]}\lesssim\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2,$$

where

$$\eta_1^2=\sum_{\Omega_i}\|\rho_1\ddot{u}_h-\mu_1\Delta u_h\|^2_{0,0,\Omega_i\times[0,T]},\qquad \eta_2^2=\sum_{T_i\cap\Gamma=\emptyset}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|^2_{1,0,T_i\times[0,T]}\max\{\Delta t,h\},$$

$$\eta_3^2=\|-\lambda_h-g+\mu_1\partial_n^-u_h\|^2_{1,0,\Gamma\times[0,T]}\max\{\Delta t,h\},\quad \eta_4^2=\left\|\left(K+\frac{1}{2}I\right)\phi_h-V\lambda_h-\gamma^-(u_h)+f\right\|^2_{1,1/2,\Gamma\times[0,T]},$$

$$\eta_5^2=\left\|\left(K'+\frac{1}{2}I\right)\lambda_h-W\phi_h\right\|^2_{1,-1/2,\Gamma\times[0,T]}.$$

This a posteriori error estimate is analogous to those for boundary problems in [15, 17]. We therefore expect equally efficient space-time adaptive mesh refinement procedures as in these works.

Proof. Let $(u,\phi,\lambda)\in H^1([0,T],H^1(\Omega^c))\times H^1([0,T],H^{\frac{1}{2}}(\Gamma))\times H^1([0,T],H^{-\frac{1}{2}}(\Gamma))$ and let $(u_h,\phi_h,\lambda_h)\in X_{h,\Delta t}$ satisfy (3.9) respectively the corresponding Galerkin discretization (4.1). We define $v:=D\phi-S\lambda$ and $v_h:=D\phi_h-S\lambda_h$ with $\gamma^+(v-v_h)=\phi-\phi_h$ and $\partial_n^+(v-v_h)=\lambda-\lambda_h$. Then by using the trace theorem together with (3.11) and Galerkin orthogonality, we get

$$\begin{aligned}
&\|u_h-u\|^2_{0,1,\Omega^-\times[0,T]}+\|\phi_h-\phi\|^2_{0,\frac{1}{2},\Gamma\times[0,T]}+\|\lambda_h-\lambda\|^2_{0,-\frac{1}{2},\Gamma\times[0,T]}\\
&\quad\lesssim\|u_h-u\|^2_{0,1,\Omega^-\times[0,T]}+\|v_h-v\|^2_{0,1,\Omega^+\times[0,T]}\lesssim|||u_h-u,v_h-v|||^2\\
&\quad\lesssim\mathcal{A}\left(\begin{pmatrix}u_h-u\\ \phi_h-\phi\\ \lambda_h-\lambda\end{pmatrix}^T,\begin{pmatrix}u_h-u\\ \phi_h-\phi\\ \lambda_h-\lambda\end{pmatrix}^T\right)=\mathcal{A}\left(\begin{pmatrix}u_h\\ \phi_h\\ \lambda_h\end{pmatrix}^T,\begin{pmatrix}\tilde{u}-u\\ \tilde{\phi}-\phi\\ \tilde{\lambda}-\lambda\end{pmatrix}^T\right)-\mathcal{F}\left(\begin{pmatrix}\tilde{u}-u\\ \tilde{\phi}-\phi\\ \tilde{\lambda}-\lambda\end{pmatrix}^T\right)\\
&\quad=\int_0^T\bigg\{\int_{\Omega^-}\rho_1\ddot{u}_h\partial_t(\tilde{u}-u)\,dx+\int_{\Omega^-}\mu_1\nabla u_h\nabla(\partial_t(\tilde{u}-u))\,dx\\
&\qquad-\int_\Gamma\lambda_h\gamma^-(\partial_t(\tilde{u}-u))\,ds_x+\int_\Gamma\left(K+\frac{1}{2}I\right)\phi_h\partial_t(\tilde{\lambda}-\lambda)\,ds_x-\int_\Gamma V\lambda_h\partial_t(\tilde{\lambda}-\lambda)\,ds_x\\
&\qquad-\int_\Gamma\gamma^-(u_h)\partial_t(\tilde{\lambda}-\lambda)\,ds_x-\int_\Gamma W\phi_h\partial_t(\tilde{\phi}-\phi)\,ds_x+\int_\Gamma\left(K'+\frac{1}{2}I\right)\lambda_h\partial_t(\tilde{\phi}-\phi)\,ds_x\\
&\qquad-\int_\Gamma g\gamma^-(\partial_t(\tilde{u}-u))\,ds_x+\int_\Gamma f\partial_t(\tilde{\lambda}-\lambda)\,ds_x\bigg\}\,dt.
\end{aligned}$$

Next, using integration by parts on each element Ω_i,

$$\begin{aligned}
|||(u_h-u,v_h-v)|||^2\lesssim\int_0^T\bigg\{&\sum_{\Omega_i}\int_{\Omega_i}\rho_1\ddot{u}_h\partial_t(\tilde{u}-u)\,dx+\int_{\Omega_i}\mu_1\nabla u_h\nabla(\partial_t(\tilde{u}-u))\,dx-\int_\Gamma(\lambda_h+g)\partial_t(\tilde{u}-u)\,ds_x\\
&+\int_\Gamma\left(\left(K+\frac{1}{2}I\right)\phi_h-V\lambda_h-\gamma^-(u_h)+f\right)\partial_t(\tilde{\lambda}-\lambda)\,ds_x\\
&+\int_\Gamma\left(\left(K'+\frac{1}{2}I\right)\lambda_h-W\phi_h\right)\partial_t(\tilde{\phi}-\phi)\,ds_x\bigg\}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \Big\{ \sum_{\Omega_i} \int_{\Omega_i} (\rho_1 \ddot{u}_h - \mu_1 \Delta u_h) \partial_t(\tilde{u} - u)\, dx + \sum_{T_i \cap \Gamma = \emptyset} \int_{T_i} \mu_1 \Big[\frac{\partial^- u_h}{\partial n} \Big] (\partial_t(\tilde{u} - u))\, ds_x \\
&\qquad + \int_\Gamma \Big(-\lambda_h - g + \mu_1 \frac{\partial^- u_h}{\partial n} \Big) \gamma^-(\partial_t(\tilde{u} - u))\, ds_x \\
&\qquad + \int_\Gamma \Big(\Big(K + \frac{1}{2} I \Big) \phi_h - V\lambda_h - \gamma^-(u_h) + f \Big) \partial_t(\tilde{\lambda} - \lambda)\, ds_x \\
&\qquad + \int_\Gamma \Big(\Big(K' + \frac{1}{2} I \Big) \lambda_h - W\phi_h \Big) \partial_t(\tilde{\phi} - \phi)\, ds_x \Big\}\, dt \\
&\lesssim \sum_{\Omega_i} \|\rho_1 \ddot{u}_h - \mu_1 \Delta u_h\|_{0,0,\Omega_i \times [0,T]} \|\tilde{u} - u\|_{1,0,\Omega_i \times [0,T]} \\
&\qquad + \sum_{T_i \cap \Gamma = \emptyset} \Big\| \Big[\frac{\partial u_h}{\partial n} \Big] \Big\|_{1,0,T_i \times [0,T]} \|\tilde{u} - u\|_{0,0,T_i \times [0,T]} \\
&\qquad + \|-\lambda_h - g + \mu_1 \partial_n^- u_h\|_{1,0,\Gamma \times [0,T]} \|\gamma^-(\tilde{u} - u)\|_{0,0,\Gamma \times [0,T]} \\
&\qquad + \Big\| \Big(K + \frac{1}{2} I \Big) \phi_h - V\lambda_h - \gamma^-(u_h) + f \Big\|_{1,1/2,\Gamma \times [0,T]} \|\tilde{\lambda} - \lambda\|_{0,-1/2,\Gamma \times [0,T]} \\
&\qquad + \Big\| \Big(K' + \frac{1}{2} I \Big) \lambda_h - W\phi_h \Big\|_{1,-1/2,\Gamma \times [0,T]} \|\tilde{\phi} - \phi\|_{0,1/2,\Gamma \times [0,T]}.
\end{aligned}
$$

Here, in the final step we have also integrated by parts in t. In the following note that

$$
\|\tilde{u} - u\|_{1,0,\Omega_i \times [0,T]} \le \|\tilde{u} - u\|_{0,1,\Omega_i \times [0,T]}.
$$

We now choose

$$
\tilde{u} = u_h + \Pi_h \circ \Pi_{\Delta t}(u - u_h)
$$

for the second and third term in order to use Lemma 1. Further, choose $\tilde{u} = u_h$ for the first term and $\tilde{\phi} = \phi_h$ and $\tilde{\lambda} = \lambda_h$. Then

$$
\begin{aligned}
|||(u_h - u, v_h - v)|||^2 &\lesssim \sum_{\Omega_i} \|\rho_1 \ddot{u}_h - \mu_1 \Delta u_h\|_{0,0,\Omega_i \times [0,T]} \|u_h - u\|_{0,1,\Omega_i \times [0,T]} \\
&\qquad + \sum_{T_i \cap \Gamma = \emptyset} \Big\| \Big[\frac{\partial u_h}{\partial n} \Big] \Big\|_{1,0,T_i \times [0,T]} \|u_h - u\|_{0,1/2,T_i \times [0,T]} \max\{\Delta t, h\}^{\frac{1}{2}} \\
&\qquad + \|-\lambda_h - g + \mu_1 \partial_n^- u_h\|_{1,0,\Gamma \times [0,T]} \|\gamma^-(u_h - u)\|_{0,1/2,\Gamma \times [0,T]} \max\{\Delta t, h\}^{\frac{1}{2}} \\
&\qquad + \Big\| \Big(K + \frac{1}{2} I \Big) \phi_h - V\lambda_h - \gamma^-(u_h) + f \Big\|_{1,1/2,\Gamma \times [0,T]} \|\lambda_h - \lambda\|_{0,-1/2,\Gamma \times [0,T]} \\
&\qquad + \Big\| \Big(K' + \frac{1}{2} I \Big) \lambda_h - W\phi_h \Big\|_{1,-1/2,\Gamma \times [0,T]} \|\phi_h - \phi\|_{0,1/2,\Gamma \times [0,T]}.
\end{aligned}
$$

Using Young's inequality and the trace theorem,

$$
\begin{aligned}
|||(u_h - u, v_h - v)|||^2 &\lesssim \sum_{\Omega_i} \|\rho_1 \ddot{u}_h - \mu_1 \Delta u_h\|^2_{0,0,\Omega_i \times [0,T]} + \sum_{T_i \cap \Gamma = \emptyset} \Big\| \Big[\frac{\partial u_h}{\partial n} \Big] \Big\|^2_{1,0,T_i \times [0,T]} \max\{\Delta t, h\} \\
&\qquad + \|-\lambda_h - g + \mu_1 \partial_n^- u_h\|^2_{1,0,\Gamma \times [0,T]} \max\{\Delta t, h\} \\
&\qquad + \Big\| \Big(K + \frac{1}{2} I \Big) \phi_h - V\lambda_h - \gamma^-(u_h) + f \Big\|^2_{1,1/2,\Gamma \times [0,T]} \\
&\qquad + \Big\| \Big(K' + \frac{1}{2} I \Big) \lambda_h - W\phi_h \Big\|^2_{1,-1/2,\Gamma \times [0,T]} + \|\phi_h - \phi\|^2_{0,1/2,\Gamma \times [0,T]} \\
&\qquad + \|\lambda_h - \lambda\|^2_{0,-1/2,\Gamma \times [0,T]} + \|u_h - u\|^2_{0,1,\Omega^- \times [0,T]}.
\end{aligned}
$$

Combining the last three terms with the left-hand side yields the estimate. □

5 Numerical Results

In this section we present the numerical results obtained from the space-time Galerkin formulation in Section 4, using the discretization discussed in the appendix. We use a discretization of (4.1) by low-order finite and boundary elements, in particular with piecewise constant test functions in time. As shown in the appendix, one obtains a marching-on-in-time scheme, which solves a system of the following structure in time step $n \geq 3$:

$$\begin{pmatrix} \frac{\mu_1 \Delta t}{2} A + \frac{\rho_1}{\Delta t} M & [0, -\frac{\Delta t}{2} RI]^T & 0 \\ [0, RI] & -V^0 & K^0 - \frac{1}{2} I \\ 0 & (K')^0 + \frac{\Delta t}{4} I & -W^0 \end{pmatrix} \begin{pmatrix} u^n \\ \lambda^n \\ \phi^n \end{pmatrix}$$
$$= \begin{pmatrix} G^n + G^{n-1} - \frac{\mu_1 \Delta t}{2} A u^{n-1} + \frac{2\rho_1}{\Delta t} M u^{n-1} - \frac{\rho_1}{\Delta t} M u^{n-2} + [0, \frac{\Delta t}{2} RI \lambda^{n-1}]^T \\ F^n - F^{n-1} - RI u_\Gamma^{n-1} + \sum_{m=1}^{n-1} V^{n-m} \lambda^m - \sum_{m=1}^{n-1} K^{n-m} \phi^m - \frac{1}{2} I \phi^{n-1} \\ \sum_{m=1}^{n-1} W^{n-m} \phi^m - \sum_{m=1}^{n-1} (K')^{n-m} \lambda^m - \frac{1}{2} \frac{\Delta t}{2} I \lambda^{n-1} \end{pmatrix}. \tag{5.1}$$

See the appendix for the detailed expressions for the matrix entries in this system and their efficient computation.

The choice of the approximation spaces does not satisfy the conditions in Theorem 4. The numerical experiments show that in practice one may use an efficient implementation by a marching-on-in-time scheme, with much reduced memory requirements and without the need to solve a global space-time system.

All examples use $\mu_1 = 1$, so that the wave speed is given by $c_1 = \frac{1}{\sqrt{\rho_1}}$. The first examples investigate the proposed method on uniform meshes.

Example 1. For $\Omega = [-1, 1]^3$ we consider the solutions (u, v) to the coupled system (3.9) for times up to $T = 5$ and their numerical approximations $(u_{h,\Delta t}, v_{h,\Delta t})$ by (5.1). Wave speeds $c_1 = 1$ and $c_1 = \frac{1}{\sqrt{2}}$ are considered. We choose the data so that an exact solution is given by

$$u(x, t) = (\sin((tc_1 - x_1)\pi))^5 (H(-1 + tc_1 - x_1) - H(-3 + tc_1 - x_1)),$$
$$v(x, t) = \frac{|x| - t}{2|x|} \Big(1 + \cos\Big(\frac{\pi(|x| - t)}{0.9}\Big)\Big) H(0.9 - \||x| - t|).$$

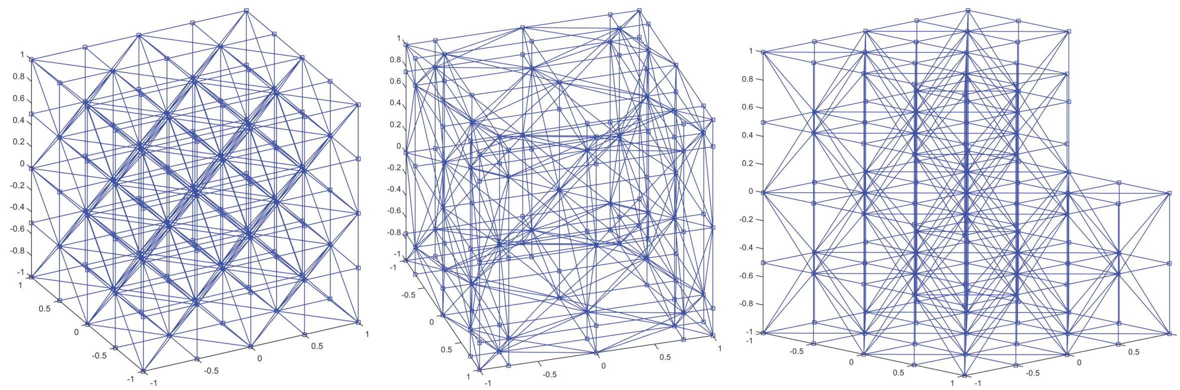
The discretized system (5.1) is studied on a sequence of uniform meshes as depicted in Figure 1 for uniform time steps Δt so that $\frac{\Delta t}{h_{\max}} \simeq 0.2828427$. Here $h_{\max}$ is the diameter of the largest triangular face in the mesh. Refinement levels $n = 4$ (125 nodes), 8 (729 nodes), 16 (4913 nodes) and 32 (35937 nodes) are used.

In the case $c_1 = 1$, Figures 2 (a) and 3 (a) illustrate the qualitative behavior of the exact and numerical solutions by plotting the norms $\|u\|_{L^2(\Omega)}$, $\|v\|_{L^2(\Gamma)}$, respectively $\|u_{h,\Delta t}\|_{L^2(\Omega)}$, $\|v_{h,\Delta t}\|_{L^2(\Gamma)}$, as a function of time. Figure 4 (a) plots the errors $\|u - u_{h,\Delta t}\|_{L^2(\Omega\times[0,T])}$, $\|v - v_{h,\Delta t}\|_{L^2(\Gamma\times[0,T])}$ as a function of the degrees of freedom of the space-time system. A convergence rate of approximately 0.2 is observed, corresponding to a convergence rate of 0.8 with respect to the mesh size h.

Note that we are using ansatz and test functions of lower polynomial degree than required in the a priori error estimate, Theorem 4, which requires piecewise quadratic polynomials in time for $u_{h,\Delta t}$.

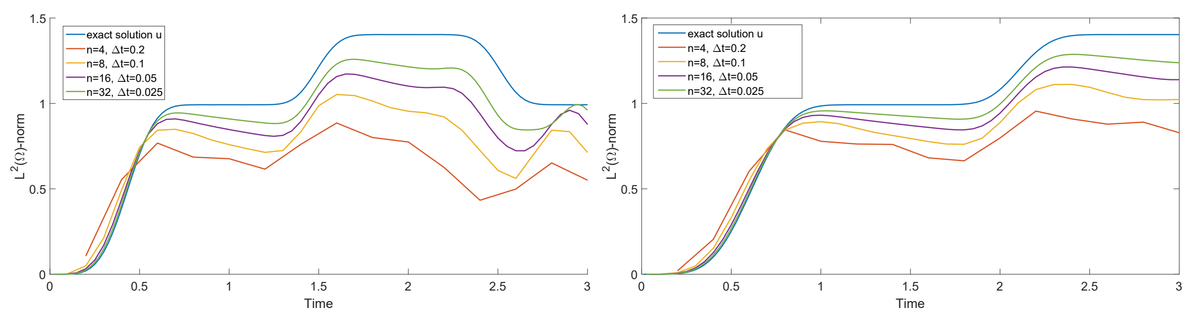
The obtained convergence rates are close to those observed for fluid-structure interaction in [16], compatible with linear convergence in h, Δt. Note that based on the a priori analysis and the trace theorem for Sobolev spaces, one might naively expect slower convergence for v than for u (by a difference of the rates 0.5) in the norms used here. However, as shown in [28] for time-independent FEM-BEM coupling, under mild regularity assumptions the BEM solution v converges at a rate 0.5 faster than predicted from a joint estimate for (u, v). This exactly cancels the above difference of rates and leads to identical convergences rates for u, v in the space-time L^2-norms on Ω, resp. Γ. The identical observed convergence rates for u, v are therefore expected and in line with those known for FEM-BEM coupling in time-independent problems [28] in the case of uniform meshes.

Analogous results are obtained for $c_1 = \frac{1}{\sqrt{2}}$. Figures 2 (b) and 3 (b) show the qualitative behavior of the norms $\|u\|_{L^2(\Omega)}$, $\|v\|_{L^2(\Gamma)}$, respectively $\|u_{h,\Delta t}\|_{L^2(\Omega)}$, $\|v_{h,\Delta t}\|_{L^2(\Gamma)}$, as a function of time. Figure 4 (b) depicts



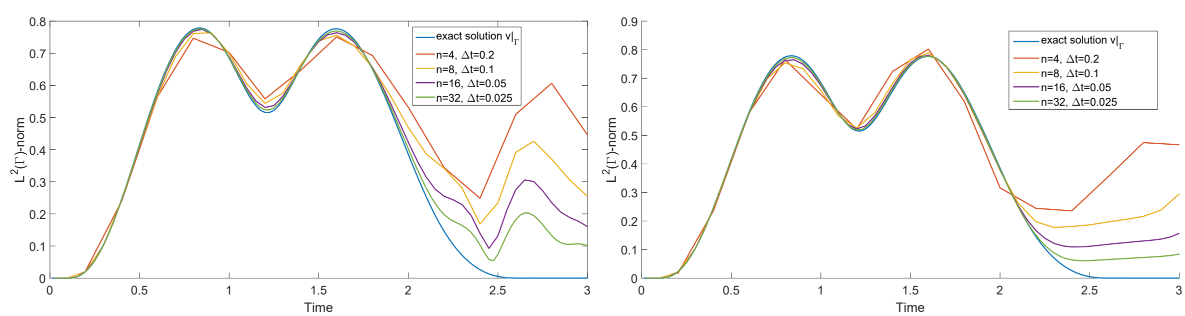
(a) Uniform, cube (b) Graded, cube (c) Uniform, Fichera cube

Figure 1: Uniform and graded meshes for cube and Fichera cube, refinement level $n = 4$.



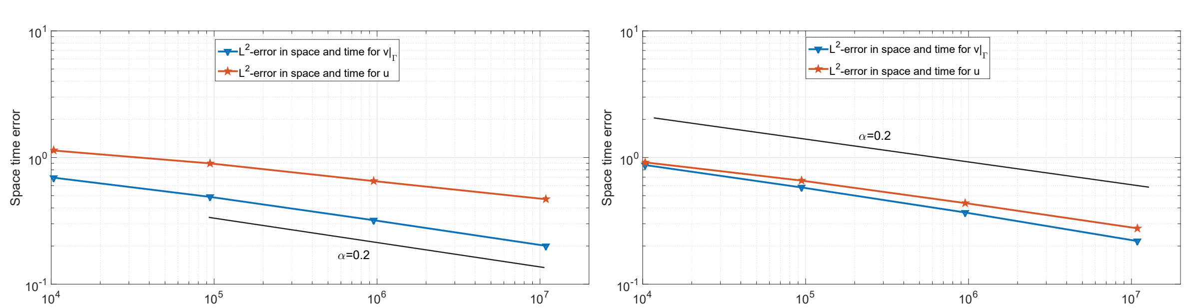
(a) $c_1 = 1$ (b) $c_1 = \frac{1}{\sqrt{2}}$

Figure 2: $L^2(\Omega)$ norm of u, Example 1.



(a) $c_1 = 1$ (b) $c_1 = \frac{1}{\sqrt{2}}$

Figure 3: $L^2(\Gamma)$ norm of $v|_\Gamma$, Example 1.



(a) $c_1 = 1$ (b) $c_1 = \frac{1}{\sqrt{2}}$

Figure 4: $L^2(\Omega \times [0, T])$ error of $u_{h,\Delta t}$, $L^2(\Gamma \times [0, T])$ error of $v_{h,\Delta t}|_\Gamma$ on uniform cube, Example 1.

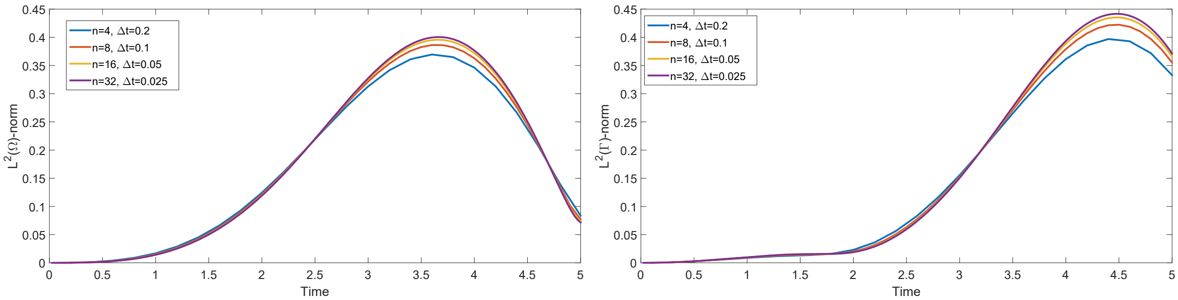


Figure 5: $L^2(\Omega)$ norm of $u_{h,\Delta t}$, $L^2(\Gamma)$ norm of $v_{h,\Delta t}|_\Gamma$, Example 2.

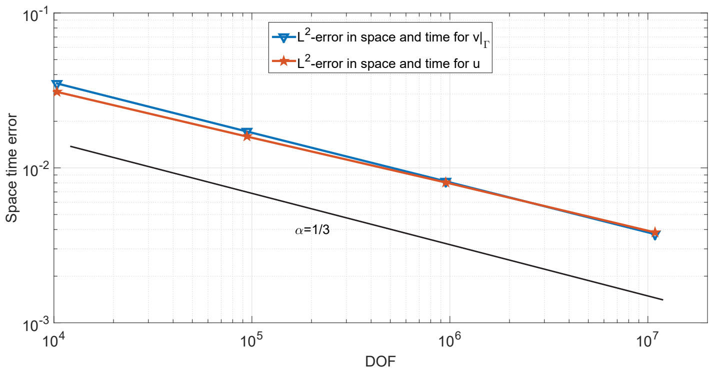


Figure 6: $L^2(\Omega \times [0, T])$ error of $u_{h,\Delta t}$, $L^2(\Gamma \times [0, T])$ error of $v_{h,\Delta t}|_\Gamma$ on uniform cube, Example 2.

the error $\|u - u_{h,\Delta t}\|_{L^2(\Omega\times[0,T])}$, $\|v - v_{h,\Delta t}\|_{L^2(\Gamma\times[0,T])}$ as a function of the degrees of freedom of the space-time system, compatible with the convergence rate of 0.2 (degrees of freedom), respectively 0.8 (h), observed for $c_1 = 1$.

A second example studies a scattering problem for which the exact solution is not known.

Example 2. For $\Omega = [-1, 1]^3$ we consider the solutions (u, v) to the coupled system (3.9) for times up to $T = 5$ and their numerical approximations $(u_{h,\Delta t}, v_{h,\Delta t})$ by (5.1). We choose the wave speed $c_1 = 1$ and data corresponding to an incoming plane wave with wave vector $k = 0.2 \cdot (1, 1, 1)$:

$$g(x, t) = \sin\Big(\frac{t}{10}\Big)^2 \cos(k \cdot x - \|k\|t), \tag{5.2}$$

$$f(x, t) = \frac{2}{10} \sin\Big(\frac{t}{10}\Big) \cos\Big(\frac{t}{10}\Big) \sin(k \cdot x - \|k\|t) - \|k\| \sin\Big(\frac{t}{10}\Big)^2 \cos(k \cdot x - \|k\|t). \tag{5.3}$$

The discretized system (5.1) is studied on the sequence of uniform meshes from Example 1. Figure 5 shows the behavior of $\|u_{h,\Delta t}\|_{L^2(\Omega)}$ and $\|v_{h,\Delta t}\|_{L^2(\Gamma)}$ as a function of time. Note the more slowly varying behavior of the solution compared to Example 1.

In Figure 6 we compare the results against a benchmark

$$\|u\|_{L^2(\Omega\times[0,T])} \simeq 0.52184 \quad \text{and} \quad \|v\|_{L^2(\Gamma\times[0,T])} \simeq 0.5205923,$$

obtained by extrapolation. We observe a convergence rate of approximately $\frac{1}{3}$ as the number of space-time degrees of freedom increases, corresponding to a rate $\frac{4}{3}$ with respect to h. The convergence rate is again compatible with linear convergence in h, Δt.

Example 3. For the Fichera cube $\Omega = [-1, 1]^3 \setminus [0, 1]^3$ shown in Figure 1 (c) we consider the solutions (u, v) to the coupled system (3.9) for times up to $T = 3$ and their numerical approximations $(u_{h,\Delta t}, v_{h,\Delta t})$ by (5.1). We choose the wave speed $c_1 = 1$ and data corresponding to the exact solution as in Example 1.

The discretized system (5.1) is studied on a sequence of uniform meshes, illustrated in Figure 1 (c). Refinement levels $n = 4$ (117 nodes), 8 (665 nodes), 16 (4401 nodes) and 32 (31841 nodes) are used.

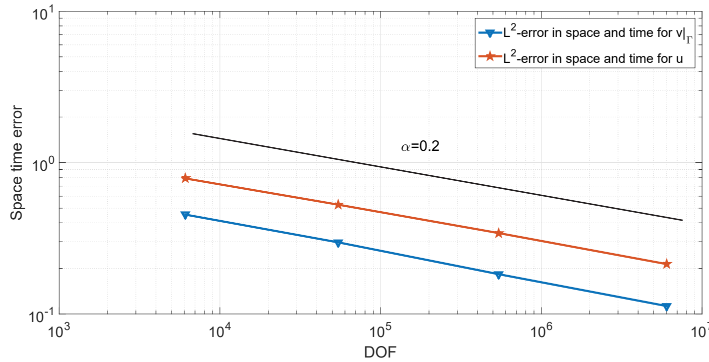


Figure 7: $L^2(\Omega \times [0, T])$ error of $u_{h,\Delta t}$, $L^2(\Gamma \times [0, T])$ error of $v_{h,\Delta t}|_\Gamma$ on Fichera cube, Example 3.

Figure 7 plots the error $\|u - u_{h,\Delta t}\|_{L^2(\Omega\times[0,T])}$, $\|v - v_{h,\Delta t}\|_{L^2(\Gamma\times[0,T])}$ as a function of the degrees of freedom of the space-time system. A convergence rate of approximately 0.2 is observed, corresponding to a rate of 0.8 with respect to the mesh size h. Note that the convergence rate agrees with the rate observed in Example 1, in spite of the more singular geometry, as the exact numerical solutions correspond to each other. Furthermore, for the transmission problems we consider the Fichera cube is not more challenging than a cube, as the reentrant corner in Ω corresponds to the reentrant corners of $\mathbb{R}^3 \setminus \Omega$ for the exterior problem.

The final example considers the numerical approximation on graded meshes:

Example 4. For $\Omega = [-1, 1]^3$ we consider the solutions (u, v) to the coupled system (3.9) for times up to $T = 5$ and their numerical approximations $(u_{h,\Delta t}, v_{h,\Delta t})$ by (5.1). We choose the wave speed $c_1 = \frac{1}{4}$ and the data (5.2), (5.3) corresponding to an incoming plane wave with wave vector $k = 0.2 \cdot (1, 1, 1)$, as in Example 2.

The discretized system (5.1) is studied on a sequence of 2-graded meshes, illustrated in Figure 1 (b), with the same number of degrees of freedom as for the uniform meshes in Examples 1 and 2.

The singular behavior of $\frac{\partial v}{\partial n}$ near edges and corners is depicted in Figure 8 at time $t = 3$. The leading singular exponent for $\frac{\partial v}{\partial n}$ at an edge is $> -\frac{1}{3}$, so that the asymptotic behavior is narrowly concentrated around the edges. This is in agreement with the regularity result, Theorem 3. In Figure 8 the narrow singularity is better resolved on the 2-graded mesh (left), and the numerical solution also takes on higher values near the edge than on the uniform mesh (right). This qualitative observation is quantitatively reflected in the convergence rate. Figure 9 compares the numerical results for $\|u_{h,\Delta t}\|_{L^2(\Omega\times[0,T])}$ and $\|v_{h,\Delta t}\|_{L^2(\Gamma\times[0,T])}$ against a benchmark, obtained by extrapolation. We observe a convergence rate for v of approximately 0.6 as the number of space-time degrees of freedom increases, corresponding to a rate > 2 with respect to h and Δt. The convergence rate on the 2-graded mesh is approximately twice the convergence rate observed on uniform meshes.

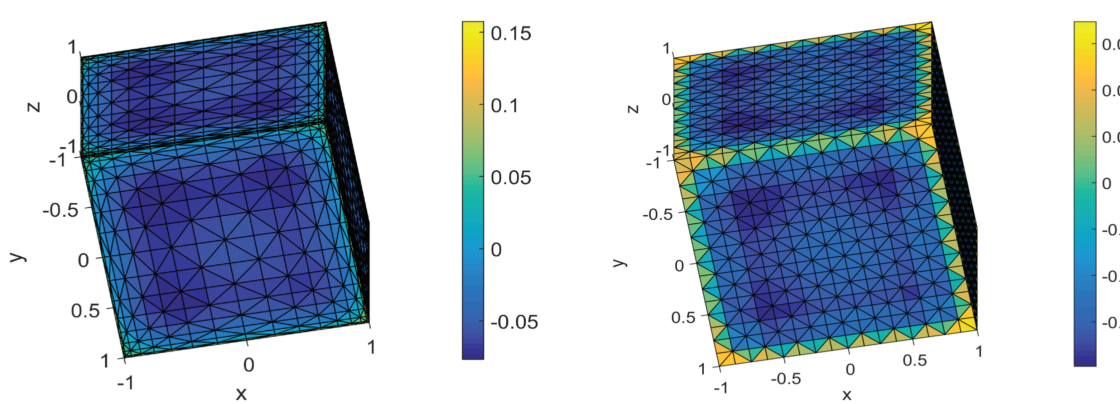


Figure 8: $(\frac{\partial v}{\partial n})_{h,\Delta t}$ on a graded (left), resp. uniform (right) cube at $t = 3$, Example 4.

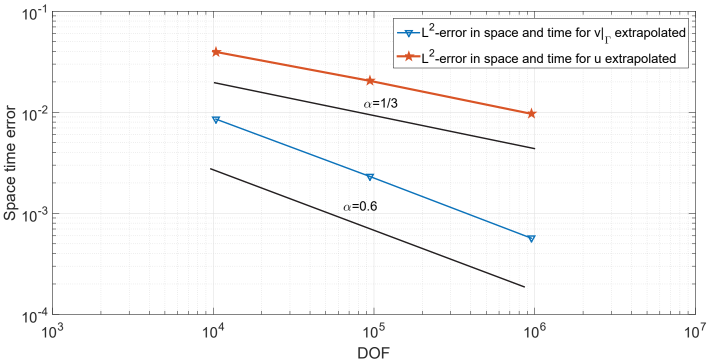


Figure 9: $L^2(\Omega \times [0,T])$ error of $u_{h,\Delta t}$, $L^2(\Gamma \times [0,T])$ error of $v_{h,\Delta t}|_\Gamma$ on graded cube, Example 4.

A Discretization and MOT-Algorithm

This appendix discusses the details of the discretization (4.1), which leads to the space-time Galerkin equations (5.1). Even though these matrices are sparse in each time step, their assembly is the most time consuming part in the MOT algorithm. The explicit formulas for their entries below reduces the computation to numerical quadratures over certain light cones E_l. This structure is crucial for the practical implementation in standard time-domain boundary element codes, see [7, 37]. Details of an efficient hp-composite Gauss-quadrature for the light cone integrals are discussed in [11, 31, 32].

The numerical solution in Section 5 uses the following ansatz and test functions:

- $u_{h,\Delta t} = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} u_i^m \beta_{\Delta t}^m(t) \eta_h^i(x) \in V_{h,\Delta t}^{1,1}(\overline{\Omega^-})$ piecewise linear in time and piecewise linear in the interior space $\overline{\Omega^-}$,
- $\dot{w}_{h,\Delta t} = \eta_h^l(x) \gamma_{\Delta t}^n(t) \in V_{h,\Delta t}^{1,0}(\Omega^-)$ piecewise constant in time and in piecewise linear in the interior space $\overline{\Omega^-}$ with $l = 1,\dots,N_s$ and $n = 1,\dots,N_t$,
- $\lambda_{h,\Delta t} = \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \beta_{\Delta t}^m(t) \rho_h^i(x) \in V_{h,\Delta t}^{0,1}(\Gamma)$ piecewise linear in time and piecewise constant in space Γ,
- $m_{h,\Delta t} = \gamma_{\Delta t}^n(t) \rho_h^i(x) \in V_{h,\Delta t}^{0,0}(\Gamma)$ piecewise constant in time and piecewise constant in space Γ for $n = 1,\dots,N_t$ and $i = 1,\dots,N_{s'}$,
- $\phi_{h,\Delta t} = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} \phi_i^m \beta_{\Delta t}^m(t) \xi_h^i(x) \in V_{h,\Delta t}^{1,1}(\Gamma)$ piecewise linear in time and piecewise linear in space Γ,
- $\dot{\omega}_{h,\Delta t} = \gamma_{\Delta t}^n(t) \xi_h^j(x) \in V_{h,\Delta t}^{1,0}(\Gamma)$ piecewise constant in time and piecewise linear in space in Γ for $n = 1,\dots,N_t$ and $j = 1,\dots,N_{s'}$.

For the discretization of the hypersingular operator note $\eta_h^l|_\Gamma = \xi_h^j$ for some j:

$$
\begin{aligned}
\langle W\phi_{h,\Delta t}, \dot{\omega}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} &= \frac{1}{4\pi}\int_0^T \iint_{\Gamma\times\Gamma} \bigg\{ \frac{-n_x\cdot n_y}{|x-y|}\dot{\phi}_{h,\Delta t}(y,t-|x-y|)\dot{\omega}_{h,\Delta t}(x,t) \\
&\qquad + \frac{(\operatorname{curl}_\Gamma \phi_{h,\Delta t})(y,t-|x-y|)\cdot(\operatorname{curl}_\Gamma \dot{\omega}_{h,\Delta t})(x,t)}{|x-y|}\bigg\}\, ds_y\, ds_x\, dt \\
&= \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \varphi_i^m \bigg[-\iint_{E_{n-m}} \frac{(n_x\cdot n_y)\xi_h^i(y)\xi_h^j(x)}{(\Delta t)|x-y|4\pi}\, ds_y\, ds_x + 2\iint_{E_{n-m-1}} \frac{(n_x\cdot n_y)\xi_h^i(y)\xi_h^j(x)}{(\Delta t)|x-y|4\pi}\, ds_y\, ds_x \\
&\qquad - \iint_{E_{n-m-2}} \frac{(n_x\cdot n_y)\xi_h^i(y)\xi_h^j(x)}{(\Delta t)|x-y|4\pi}\, ds_y\, ds_x \bigg] \\
&\quad + \sum_{m=1}^{N_t}\sum_{i=1}^{N_{\tilde{s}}} \varphi_i^m \iint_{\Gamma\times\Gamma} \frac{(\operatorname{curl}_\Gamma \xi_h^i)(y)\cdot(\operatorname{curl}_\Gamma \xi_h^j)(x))}{4\pi|x-y|}\gamma^{n-m}\, ds_y\, ds_x \\
&=: \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} W_{j,i}^{n-m}\varphi_i^m,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Y}^{n-m}(x,y) &= \int_0^T \beta_{\Delta t}^m(t-|x-y|)\gamma_{\Delta t}^n(t)\,dt \\
&= (2(\Delta t))^{-1}(|x-y|^2 - 2|x-y|(n-m+1)(\Delta t) + ((n-m+1)(\Delta t))^2)\chi_{E_{n-m}} \\
&\quad + (2(\Delta t))^{-1}(|x-y|^2 - 2|x-y|(n-m-2)(\Delta t) + ((n-m-2)(\Delta t))^2)\chi_{E_{n-m-2}} \\
&\quad + (2(\Delta t))^{-1}(-2|x-y|^2 + 2|x-y|((n-m-1)(\Delta t) + (n-m)(\Delta t)) \\
&\quad - (((n-m-1)(\Delta t))^2 + ((n-m)(\Delta t))^2) + 2(\Delta t)^2)\chi_{E_{n-m-1}}.
\end{aligned}
$$

For $l \in \mathbb{N}_0$,

$$E_l = \{(x,y) \in \Gamma\times\Gamma : t_l \le |x-y| \le t_{l+1}\} \subset \Gamma\times\Gamma$$

and

$$\chi_{E_l}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E_l, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix W^{n-m} is a sum of integrals over the three light cones E_{n-m}, E_{n-m-1} and E_{n-m-2}.

Similar formulas are obtained for the discretizations of the retarded single layer potential operator V^{n-m}, the retarded double layer potential operator K^{n-m}, and the adjoint discretized retarded double layer potential operator ${K'}^{n-m}$:

$$
\begin{aligned}
\langle K'\lambda_{h,\Delta t}, \dot{\omega}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} &= \int_0^T \iint_{\Gamma\times\Gamma} \frac{n_x\cdot(x-y)}{4\pi|x-y|}\Big(\frac{\lambda_{h,\Delta t}(y,t-|x-y|)}{|x-y|^2} + \frac{\dot{\lambda}_{h,\Delta t}(y,t-|x-y|)}{|x-y|}\Big)\dot{\omega}_{h,\Delta t}(x,t)\,ds_y\,ds_x\,dt \\
&= \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \lambda_i^m \iint_{\Gamma\times\Gamma} \frac{n_x\cdot(x-y)}{4\pi|x-y|^3}\rho_h^i(y)\xi_h^j(x)\mathcal{Y}^{n-m}(x,y)\,ds_y\,ds_x \\
&\quad + \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \lambda_i^m\Big[\iint_{E_{n-m}} n_x\cdot(x-y)\Big((n-m+1)\frac{\rho_h^i(y)\xi_h^j(x)}{4\pi|x-y|^2} - \frac{\rho_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-1}} n_x\cdot(x-y)\Big(-(2(n-m)-1)\frac{\rho_h^i(y)\xi_h^j(x)}{4\pi|x-y|^2} + 2\frac{\rho_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-2}} n_x\cdot(x-y)\Big((n-m-2)\frac{\rho_h^i(y)\xi_h^j(x)}{4\pi|x-y|^2} - \frac{\rho_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|}\Big)\,ds_y\,ds_x\Big] \\
&=: \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} (K')_{j,i}^{n-m}\lambda_i^m
\end{aligned}
$$

and

$$
\begin{aligned}
\langle V\lambda_{h,\Delta t}, \dot{m}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} &= \int_0^T\int_\Gamma\int_\Gamma \frac{1}{4\pi}\frac{\lambda_{h,\Delta t}(y,t-|x-y|)}{|x-y|}\dot{m}_{h,\Delta t}(x,t)\,ds_y ds_x\,dt \\
&= \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \lambda_i^m\Big[\iint_{E_{n-m}} \Big(-(n-m+1)\frac{\rho_h^i(y)\rho_h^j(x)}{4\pi|x-y|} + \frac{\rho_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-1}} \Big((2(n-m)-1)\frac{\rho_h^i(y)\rho_h^j(x)}{4\pi|x-y|} - 2\frac{\rho_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-2}} \Big(-(n-m-2)\frac{\rho_h^i(y)\rho_h^j(x)}{4\pi|x-y|} + \frac{\rho_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)}\Big)\,ds_y\,ds_x\Big] \\
&=: \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} V_{j,i}^{n-m}\lambda_i^m.
\end{aligned}
$$

Finally, we consider the discretized double layer potential operator

$$
\begin{aligned}
\langle K\phi_{h,\Delta t}, \dot{m}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} &= \int_0^T \iint_{\Gamma\times\Gamma} \frac{n_y\cdot(x-y)}{4\pi|x-y|}\Big(\frac{\phi_{h,\Delta t}(y,t-|x-y|)}{|x-y|^2} + \frac{\dot{\phi}_{h,\Delta t}(y,t-|x-y|)}{|x-y|}\Big)\dot{m}_{h,\Delta t}(x,t)\,ds_y\,ds_x\,dt \\
&= \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \varphi_i^m \Big[\iint_{E_{n-m}} n_y\cdot(x-y)\Big(-(n-m+1)\frac{\xi_h^i(y)\rho_h^j(x)}{4\pi|x-y|^3} + \frac{\xi_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)|x-y|^2}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-1}} n_y\cdot(x-y)\Big((2(n-m)-1)\frac{\xi_h^i(y)\rho_h^j(x)}{4\pi|x-y|^3} - 2\frac{\xi_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)|x-y|^2}\Big)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-2}} n_y\cdot(x-y)\Big(-(n-m-2)\frac{\xi_h^i(y)\rho_h^j(x)}{4\pi|x-y|^3} + \frac{\xi_h^i(y)\rho_h^j(x)}{4\pi(\Delta t)|x-y|^2}\Big)\,ds_y\,ds_x\Big] \\
&\quad + \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \varphi_i^m \Big[\iint_{E_{n-m}} \frac{-n_y\cdot(x-y)}{4\pi(\Delta t)|x-y|^2}\xi_h^i(y)\rho_h^j(x)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-1}} \frac{2n_y\cdot(x-y)}{4\pi(\Delta t)|x-y|^2}\xi_h^i(y)\rho_h^j(x)\,ds_y\,ds_x \\
&\qquad + \iint_{E_{n-m-2}} -\frac{n_y\cdot(x-y)}{4\pi(\Delta t)|x-y|^2}\xi_h^i(y)\rho_h^j(x)\,ds_y\,ds_x\Big] \\
&=: \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} K_{j,i}^{n-m}\varphi_i^m.
\end{aligned}
$$

The remaining matrices do not involve the retarded potentials and therefore do not require special quadratures. In particular, the mass matrices on the boundary Γ are given by

$$
\begin{aligned}
\langle \lambda_{h,\Delta t}, \dot{\omega}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} &= \sum_{i=1}^{N_{s'}}\Big(\int_\Gamma \rho_h^i(x)\xi_h^j(x)\,ds_x\Big)\frac{(\Delta t)}{2}\tilde{\lambda}_i^n \\
&=: \sum_{i=1}^{N_{s'}} I_{j,i}\tilde{\lambda}_i^n,
\end{aligned}
$$

where

$$
\tilde{\lambda}_i^n = \frac{(\Delta t)}{2}\begin{cases} \lambda^1, & n=1,\\ \lambda^n + \lambda^{n-1}, & n\geq 2,\end{cases}
$$

and

$$
\langle \phi_{h,\Delta t}, \dot{m}_{h,\Delta t}\rangle_{\Gamma\times\mathbb{R}_+} = \sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}}\varphi_i^m\Big(\int_\Gamma \xi_h^i(x)\rho_h^j(x)\,ds_x\Big)\tilde{\varphi}_i^n = \sum_{i=1}^{N_{s'}} I_{j,i}\tilde{\varphi}_i^n
$$

with

$$
\tilde{\varphi}_i^n = \begin{cases} -\varphi^1, & n=1,\\ -\varphi^n + \varphi^{n-1}, & n\geq 2.\end{cases}
$$

The mass and stiffness matrices M, respectively A, for the finite element discretization in Ω^- are standard and therefore omitted. We have

$$
\begin{aligned}
&\sum_{m=1}^{N_t}\sum_{i=1}^{N_s} u_i^m \int_\Gamma \eta_h^i|_\Gamma(x)\rho_h^j(x)\,ds_x \int_0^\infty \beta_{\Delta t}^m(t)\dot{\gamma}_{\Delta t}^n(t)\,dt =: RI(-u_\Gamma^n + u_\Gamma^{n-1}),\\
&\sum_{m=1}^{N_t}\sum_{i=1}^{N_{s'}} \lambda_i^m \int_\Gamma \rho_h^i(x)\eta_h^l|_\Gamma(x)\,ds_x \int_0^\infty \beta_{\Delta t}^m(t)\gamma_{\Delta t}^n(t)\,dt =: RI\frac{\Delta t}{2}(\lambda^n + \lambda^{n-1}).
\end{aligned}
$$

For the right-hand side:

$$\int_{\mathbb{R}_+}\int_{\Gamma} g\dot{w}_{h,\Delta t}\,ds_x\,dt = \frac{(\Delta t)}{2}\int_{\Gamma}(g^n + g^{n-1})\eta_h^l|_\Gamma(x)\,ds_x =: G^n + G^{n-1},$$

$$\int_{\mathbb{R}_+}\int_{\Gamma} f\dot{m}_{h,\Delta t}\,ds_x\,dt = \int_{\Gamma}(f^{n-1} - f^n)\rho_h^j(x)\,ds_x =: -F^n + F^{n-1},$$

where $g^n = g(x, t_n)$, $g \approx \sum_{m=1}^{N_t} g^m \beta_{\Delta t}^m(t)$ and $f^n = f(x, t_n)$, $f \approx \sum_{m=1}^{N_t} f^m \gamma_{\Delta t}^m(t)$.

Altogether we obtain a marching-on-in time scheme: For $n = 1$,

$$\begin{pmatrix} \frac{\mu_1\Delta t}{2}A + \frac{\rho_1}{\Delta t}M & [0, -\frac{\Delta t}{2}RI]^T & 0 \\ [0, RI] & -V^0 & K^0 - \frac{1}{2}I \\ 0 & (K')^0 + \frac{1}{2}\frac{\Delta t}{2}I & -W^0 \end{pmatrix}\begin{pmatrix} u^1 \\ \lambda^1 \\ \phi^1 \end{pmatrix} = \begin{pmatrix} G^1 + G^0 \\ F^1 - F^0 \\ 0 \end{pmatrix}.$$

We obtain for the second time step ($n = 2$),

$$\begin{pmatrix} \frac{\mu_1\Delta t}{2}A + \frac{\rho_1}{\Delta t}M & [0, -\frac{\Delta t}{2}RI]^T & 0 \\ [0, RI] & -V^0 & K^0 - \frac{1}{2}I \\ 0 & (K')^0 + \frac{1}{2}\frac{\Delta t}{2}I & -W^0 \end{pmatrix}\begin{pmatrix} u^2 \\ \lambda^2 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} G^2 + G^1 - \frac{\mu_1\Delta t}{2}Au^1 + \frac{2\rho}{\Delta t}Mu^1 + [0; \frac{\Delta t}{2}RI\lambda^1]^T \\ F^2 - F^1 - RIu_\Gamma^1 + V^1\lambda^1 - K^1\phi^1 - \frac{1}{2}I\phi^1 \\ W^1\phi^1 - (K')^1\lambda^1 - \frac{1}{2}\frac{\Delta t}{2}I\lambda^1 \end{pmatrix}.$$

For later time steps $n \geq 3$ we obtain (5.1):

$$\begin{pmatrix} \frac{\mu_1\Delta t}{2}A + \frac{\rho_1}{\Delta t}M & [0, -\frac{\Delta t}{2}RI]^T & 0 \\ [0, RI] & -V^0 & K^0 - \frac{1}{2}I \\ 0 & (K')^0 + \frac{1}{2}\frac{\Delta t}{2}I & -W^0 \end{pmatrix}\begin{pmatrix} u^n \\ \lambda^n \\ \phi^n \end{pmatrix}$$
$$= \begin{pmatrix} G^n + G^{n-1} - \frac{\mu_1\Delta t}{2}Au^{n-1} + \frac{2\rho_1}{\Delta t}Mu^{n-1} - \frac{\rho_1}{\Delta t}Mu^{n-2} + [0, \frac{\Delta t}{2}RI\lambda^{n-1}]^T \\ F^n - F^{n-1} - RIu_\Gamma^{n-1} + \sum_{m=1}^{n-1} V^{n-m}\lambda^m - \sum_{m=1}^{n-1} K^{n-m}\phi^m - \frac{1}{2}I\phi^{n-1} \\ \sum_{m=1}^{n-1} W^{n-m}\phi^m - \sum_{m=1}^{n-1}(K')^{n-m}\lambda^m - \frac{1}{2}\frac{\Delta t}{2}I\lambda^{n-1} \end{pmatrix}.$$

This system is solved repeatedly until reaching time step $N_t \geq 3$.

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