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Commutative Toeplitz Algebras and Their Gelfand Theory: Old and New Results

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Abstract

We present a survey and new results on the construction and Gelfand theory of commutative Toeplitz algebras over the standard weighted Bergman and Hardy spaces over the unit ball in $\mathbb{C}^n$. As an application we discuss semi-simplicity and the spectral invariance of these algebras. The different function Hilbert spaces are dealt with in parallel in successive chapters so that a direct comparison of the results is possible. As a new aspect of the theory we define commutative Toeplitz algebras over spaces of functions in infinitely many variables and present some structural results. The paper concludes with a short list of open problems in this area of research.

Keywords Bergman and Hardy space · Gaussian measure in infinite dimensions · Fock space of functions in infinitely many variables · Commutative Banach algebras

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1 Introduction

During the last years there has been an intensive study of commutative Banach and C^* algebras generated by Toeplitz operators acting on different function Hilbert spaces

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In memory of Jörg Eschmeier.

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such as Bergman, Hardy or Fock spaces, see [3–7, 12, 13, 15, 20, 24–28, 30–34]. We call them *Toeplitz algebras* and remark that interesting examples already appear if the underlying domain is complex one dimensional. As is well known, Toeplitz operators acting on the Hardy space over the unit circle only commute in rare cases (see [10] for a precise statement). However, when passing to the standard weighted Bergman space over the unit disc $\mathbb{D}$ in $\mathbb{C}$ one discovers a variety of commutative C^* Toeplitz algebras. When assuming some "richness" of the symbol class and commutativity of the generating Toeplitz operators simultaneously in the weight parameter one even obtains a complete classification of such algebras based on the structure of geodesic pencils in the Poincaré hyperbolic disc, see [33]. One source of commutativity of a C^* Toeplitz algebra over $\mathbb{D}$ is the invariance of the symbols of the generating operators under the action of a maximal commutative subgroup of the automorphism group $\text{Aut}(\mathbb{D})$. In [27] the authors have extended these results to complex domains of dimension $n > 1$ and studied commutative C^* algebras generated by Toeplitz operators over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. Again, families of such algebras subordinate to the maximal commutative subgroups of the automorphism group of the domain could be constructed and a spectral decomposition of Toeplitz operators in each of these algebras was derived in a rather explicit form, see [27, 32, 34]. However, a classification result as was mentioned in dimension $n = 1$ is still missing. In the higher dimensional setting $n > 1$ the dependence of the symbol functions on different (groups of) coordinates is another source of commutativity and provides additional flexibility in the construction of commutative Toeplitz C^* algebras. In [12] the authors extend the results in [27, 33] replacing the unit ball by a general bounded symmetric domain of higher rank. Based on tools from representation theory this paper characterizes subgroups of the automorphism group that induce commutative C^* Toeplitz algebras generated by operators having symbols that are constant along the orbits.

Another line of research is concerned with the structure of commutative Toeplitz Banach algebras which are not C^*. More precisely, symbol classes subordinate to a given abelian subgroup of $\text{Aut}(\mathbb{B}^n)$ having the property that the corresponding Toeplitz operators generate a commutative Banach algebra while the generated C^* algebra is non-commutative have been considered [3–7, 32, 34]. Typical problems concern the structural properties of these algebras, such as a description of their maximal ideal spaces and Gelfand transform, semi-simplicity or spectral invariance inside the full algebra of bounded operators.

It is natural to study the existence and structure of the above commutative Toeplitz algebras for different function Hilbert spaces such as the Hardy space over the unit sphere [24, 30] or the Fock space of Gaussian square integrable entire functions [4, 13]. In fact, there is an interesting interplay between these cases and the analysis of one of these spaces may be useful in the study of another.

The aim of this paper is twofold. First, we present a survey of the recent research on commutative Toeplitz Banach and C^* algebras for the weighted Bergman spaces and the Hardy space over the unit ball and the unit sphere in $\mathbb{C}^n$, respectively. By adding some new results we will complement this survey. However, due to the constantly growing literature on the subject, not all aspects of the theory can be dealt with in detail. Secondly, we introduce a new set up in the construction of operator algebras by considering Toeplitz operators over the Fock space in infinitely many variables.

Such operators have been introduced in [21, 22] and generalize the notion of Toeplitz operators on the Fock-Segal-Bargmann space $H^2(\mathbb{C}^n, \mu)$ of Gaussian square integrable entire functions in $\mathbb{C}^n$. New effects in the analysis of Toeplitz operators can be observed and in parts are a consequence of the infinite dimensional measure theory of the underlying Hilbert space $H = \ell^2(\mathbb{N})$. Another new feature from a topological point of view is the non-nuclearity of the compact open topology on the space of entire functions over H. In the classical setting of the space $H^2(\mathbb{C}^n, \mu)$ commutative C^* and Banach algebras generated by Toeplitz operators were considered in [4, 13, 15].

The main Sects. 3–5 of the paper discuss commutative Toeplitz algebras for the above mentioned function Hilbert spaces and all sections have the same structure. This allows to easily compare the results and methods in the different cases.

The structural analysis of commutative Toeplitz algebras is a current topic of interest in Operator Theory and Complex Analysis. A number of open problems intends to stimulate further research in the area. Therefore, in the last section we have collected a (certainly incomplete) list of questions which arose from the analysis presented in this work.

We now describe the structure of the paper. In Sect. 2 we explain some notation that are standard in the literature and will be used throughout. Commutative algebras generated by Toeplitz operators acting on the (weighted) Bergman space over the unit ball have been studied most intensively (see [3–7, 20, 26, 27, 31–34] and the literature cited therein) and will be treated in Sect. 3. We introduce various symbol classes which either are invariant or have a homogeneity property under a torus action on $\mathbb{B}^n$. The induced Toeplitz operators generate commutative Banach algebras. The main result concerns the Gelfand theory of such algebras. As an application we discuss semi-simplicity, description of the radical and spectral invariance in the algebra of all bounded operators on the Bergman space.

Section 4 carries out the corresponding analysis in case of the classical Hardy space $H^2(S^{2n-1})$ over the unit sphere S^{2n-1} in $\mathbb{C}^n$, see [1, 10, 24, 30]. An useful ingredient to the proofs is a decomposition of Toeplitz operators (with certain symbols) as an infinite sum of Bergman space Toeplitz operators over $\mathbb{B}^{n-1}$ with integer weights. This provides a link to the results in Sect. 3. We present some original results, which extend and generalize for general dimensions the results regarding the case $n = 3$, which was presented in [24].

Section 5 starts with a reminder on the construction of Gaussian measures on an infinite dimensional separable Hilbert space H [9, 14, 18]. Without restriction we assume that $H = \ell^2(\mathbb{N})$ is the space of square summable sequences over the natural numbers $\mathbb{N}$. We introduce the corresponding Fock space, Toeplitz operators and consider commutative Toeplitz algebras in this set up, see [21, 22]. The structural results on these algebras are new and therefore we have added some proofs.

Various open questions remain unsolved. In the concluding Sect. 6 we compare some of the results on operator algebras over different function Hilbert spaces. Moreover, a list of open problems will be mentioned.

2 Notations

We will often divide n-tuples of numbers in groups as follows. Let $m \in \mathbb{N}$ and $k = (k_1, \dots, k_m) \in \mathbb{N}^m$ such that $n = k_1 + \cdots + k_m$. Given a tuple of complex numbers $u = (u_1, \dots, u_n)$, we define $u_{(j)}$, $j = 1, \dots, m$, by

$$\begin{aligned} u_{(1)} &= (u_1, \dots, u_{k_1}) \\ &\dots \\ u_{(j)} &= (u_{k_1+\cdots+k_{j-1}+1}, \dots, u_{k_1+\cdots+k_j}), \quad j > 1 \\ &\dots \\ u_{(m)} &= (u_{k_1+\cdots+k_{m-1}+1}, \dots, u_n). \end{aligned}$$

Hence we have $u = (u_{(1)}, \dots, u_{(m)})$. We will write $u_{(j)} = (u_{j,1}, \dots, u_{j,k_j})$ for the entries of $u_{(j)}$. Throughout the paper we will denote by $\mathbb{Z}_+$ the set $\mathbb{N} \cup \{0\}$ and, for any integer $m > 1$, we will repeatedly use the usual multi-index notations for the elements of $\mathbb{Z}_+^m$:

$$\begin{aligned} \alpha! &= \alpha_1! \cdots \alpha_m!, \\ |\alpha| &= \alpha_1 + \cdots + \alpha_m, \\ z^\alpha &= z_1^{\alpha_1} \cdots z_m^{\alpha_m}, \quad z \in \mathbb{C}^m. \end{aligned}$$

We denote by $\mathbb{B}^n$ the open unit ball of $\mathbb{C}^n$, by $S^{2n-1} = \partial\mathbb{B}^n$ the unit sphere in $\mathbb{C}^n$ and by $\mathbb{T} = S^1$ the unit circle. Moreover, $\tau(\mathbb{B}^n)$ will denote the basis of $\mathbb{B}^n$ as a Reinhardt domain:

$$\tau(\mathbb{B}^n) = \{(r_1, \dots, r_n) \in \mathbb{R}_+^n : r_1^2 + \cdots + r_n^2 < 1\}.$$

As usual, we write $\mathcal{L}(X)$ for the space of bounded operators acting on a Banach space X.

2.1 On the Spectrum of an Operator

Since we will constantly deal with different notions of spectrum, we set up the following notations. Given an operator $S \in \mathcal{L}(X)$, we denote by $\mathrm{sp}_{\mathrm{pt}}(S)$ the point-spectrum, understood as the set of all eigenvalues of S. If S belongs to an algebra $\mathcal{A}$, then $\sigma_{\mathcal{A}}(S)$ means the spectrum of S as an element of $\mathcal{A}$ and $\mathrm{sp}(S)$ denotes the spectrum of S as an operator with respect to the algebra $\mathcal{L}(X)$. Furthermore, ess-sp(S) denotes the essential spectrum of S.

If U is a bounded subset of $\mathbb{C}^n$, the *polynomially convex hull* of U is defined as the set $\widehat{U}$ of all $z \in \mathbb{C}^n$ such that for every (analytic) polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ we have

$$|p(z)| \leq \sup_{w \in U} |p(w)|.$$

Recall that a set U is called *polynomially convex* if $U = \widehat{U}$.

As is well-known, a set is polynomially convex if and only if it is the maximal ideal space of some finitely generated commutative unital Banach algebra. As a consequence, if $\mathcal{A}$ is the commutative unital Banach algebra generated by an operator S, then $\sigma_{\mathcal{A}}(S) = \widehat{\operatorname{sp} S}$. For more details consult [16, 23].

3 Bergman Space Over the Unit Ball

3.1 The Bergman Space

For $\lambda > -1$ define the (probability) measure dv_λ on $\mathbb{B}^n$ by

$$dv_\lambda(z) = c_\lambda(1 - |z|^2)^\lambda dv(z),$$

where dv denotes the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and c_λ is a normalizing constant given by

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}.$$

We denote by $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ the *weighted Bergman space with weight parameter* λ, being the space of all holomorphic functions from $L^2(\mathbb{B}^n, dv_\lambda)$.

As is well known, the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is a reproducing kernel Hilbert space, with a standard orthonormal basis $(e_\alpha)_{\alpha \in \mathbb{Z}^n_+}$ consisting of normalized monomials:

$$e_\alpha(z) = \sqrt{\frac{\Gamma(n + |\alpha| + \lambda + 1)}{\alpha!\Gamma(n + \lambda + 1)}} z^\alpha, \quad \alpha \in \mathbb{Z}^n_+. \tag{3.1}$$

We denote by P the orthogonal projection from $L^2(\mathbb{B}^n, dv_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Given m and k as in the previous section, we can decompose the Bergman space with respect to these data into an orthogonal sum. So, for any $\kappa \in \mathbb{Z}^m_+$ consider the finite dimensional subspace

$$H_\kappa = \operatorname{span}\{e_\alpha : |\alpha_{(j)}| = \kappa_j, \quad j = 1, \ldots, m\}. \tag{3.2}$$

Then we have

$$\mathcal{A}^2_\lambda(\mathbb{B}^n) = \bigoplus_{\kappa \in \mathbb{Z}^m_+} H_\kappa. \tag{3.3}$$

Given a bounded measurable function $\varphi \in L^\infty(\mathbb{B}^n)$ we define the *Toeplitz operator with symbol* φ as

$$T_\varphi : \mathcal{A}^2_\lambda(\mathbb{B}^n) \longrightarrow \mathcal{A}^2_\lambda(\mathbb{B}^n) : T_\varphi(f) = P(\varphi f).$$

Our main task is to study certain operator algebras generated by Toeplitz operators with special symbols. In particular, as it was indicated in the introduction, some symbol classes subordinate to the maximal abelian subgroups of automorphisms of $\mathbb{B}^n$ lead to Banach algebras that are commutative for each weight parameter $\lambda > -1$, but such that the C^* algebra generated by them is no longer commutative.

In this work, we present operator algebras that appear when applying such ideas in case of different function spaces. Since the best understood cases are algebras associated with the so-called quasi-radial symbols, we begin our study with this setting and use it as a model case to introduce some of the main ideas.

3.2 Quasi-radial Symbols

A bounded measurable function $\varphi = \varphi(z)$ defined on $\mathbb{B}^n$ is called *k-quasi-radial* if it only depends on the groupal radii $|z_{(j)}|$, $j = 1, \ldots, m$. That is, if there is a bounded function $\widetilde{\varphi}$ defined on $\tau(\mathbb{R}^m_+) = \{(r_1, \ldots, r_m) \in \mathbb{R}^m_+ \ : \ r_1^2 + \ldots + r_m^2 < 1\}$ such that

$$\varphi(z) = \widetilde{\varphi}(|z_{(1)}|, \ldots, |z_{(m)}|), \quad \forall z \in \mathbb{B}^n.$$

Alternatively, a bounded measurable function φ is k-quasi-radial if and only if it is invariant under the action of the Cartesian product of unitary groups $U_{(1)} \times \cdots \times U_{(m)}$, where $U_{(j)} = \mathcal{U}(\mathbb{C}^{k_j})$ denotes the group of unitary $k_j \times k_j$-matrices and each $U_{(j)}$ acts on $\mathbb{C}^{k_j}$. This notion interpolates between some well known special cases: φ is *radial* when $m = 1$ and *separately radial* when $m = n$.

Let $L^\infty_{k-qr}(\mathbb{B}^n)$ denote the space of all k-quasi-radial functions and $\mathcal{T}_{\text{k-qr}}$ the corresponding C^* algebra generated by Toeplitz operators with symbols in this set.

Lemma 3.1 [32, Lemma 3.1] *Given a k-quasi-radial function $a = a(r_1, \ldots, r_m)$, where $r_j := |z_{(j)}|$, the Toeplitz operator T_a is diagonal with respect to the standard orthonormal basis* (3.1). *More precisely, we have*

$$T_a z^\alpha = \gamma_{a,k,\lambda}(\alpha) z^\alpha, \quad \alpha \in \mathbb{Z}^n_+,$$

where

$$\begin{aligned}\gamma_{a,k,\lambda}(\alpha) &= \widetilde{\gamma_{a,k,\lambda}}(|\alpha_{(1)}|, \ldots, |\alpha_{(m)}|) \\ &= \frac{2^m \Gamma(n + |\alpha| + \lambda + 1)}{\Gamma(\lambda + 1) \prod_{j=1}^m (k_j - 1 + |\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)(1 - |r|^2)^\lambda \\ &\quad \times \prod_{j=1}^m r_j^{2|\alpha_{(j)}| + 2k_j - 1} dr_j. \end{aligned} \tag{3.4}$$

Since the above formula depends only on the quantities $|\alpha_{(j)}|$, the functions $\widetilde{\gamma_{a,k,\lambda}}$ generate an algebra of bounded functions on the set $\mathbb{Z}^m_+$. Moreover, as was shown in [32], this algebra separates points of $\mathbb{Z}^m_+$ and thus we can identify $\mathcal{T}_{\text{k-qr}}$ with a C^* algebra of continuous functions on a suitable compactification $M(\mathcal{T}_{\text{k-qr}})$ of $\mathbb{Z}^m_+$.

As was observed in [5], the algebra $\mathcal{T}_{\text{k-qr}}$ contains the C^* algebra of all functions on $\mathbb{Z}_+^m$ having limits at infinity. In particular, it contains all orthogonal projections P_κ from $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ onto H_κ, $\kappa \in \mathbb{Z}_+^m$. Furthermore, as a non-trivial fact, it turns out to contain also all orthogonal projections $Q_d^{(j)}$ mapping $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ onto the infinite dimensional spaces

$$H_d^{(j)} = \overline{\text{span}}\{e_\alpha : |\alpha_{(j)}| = d\}, \qquad (d \in \mathbb{Z}_+, \quad j \in \{1, \dots, m\}) \tag{3.5}$$

that is, the projections

$$Q_d^{(j)} = \bigoplus_{\kappa \in \mathbb{Z}_+^m, \kappa_j = d} P_\kappa.$$

Note that for each fixed $j \in \{1, \dots, m\}$ one obtains an orthogonal decomposition of the Bergman space:

$$\mathcal{A}_\lambda^2(\mathbb{B}^n) = \bigoplus_{d \in \mathbb{Z}_+} H_d^{(j)}. \tag{3.6}$$

Finally, we recall a useful fibration of the compact set $M(\mathcal{T}_{\text{k-qr}})$ of maximal ideals which was presented in [3]. Consider the set $\Theta = \{0, 1\}^m$ and for each $\theta \in \Theta$ define $J_\theta = \{j : \theta_j = 1\}$. We set

$$\mathbb{Z}_+^\theta := \bigoplus_{j \in J_\theta} \mathbb{Z}_+(j) \quad \text{and} \quad \kappa_\theta := \{(\kappa_{j_1}, \dots, \kappa_{j_{|\theta|}}) : j_p \in J_\theta\},$$

where $\mathbb{Z}_+(j)$ denotes a copy of $\mathbb{Z}_+$.

Given $\theta \in \Theta$ we set

$$M_\theta := \left\{ \mu \in M(\mathcal{T}_{\text{k-qr}}) : \mu(Q_d^{(j)}) = \begin{cases} 0 \text{ for all } d \in \mathbb{Z}_+, & \text{if } \theta_j = 0 \\ 1 \text{ for some } d \in \mathbb{Z}_+, & \text{if } \theta_j = 1 \end{cases} \right\}.$$

Thus the set M_θ consists of all points of the maximal ideal space that are reached by nets (κ_α) in $\mathbb{Z}_+^m$ such that the coordinate $(\kappa_\alpha)_j$ tends to infinity if and only if $\theta_j = 0$. Since the other entries $(\kappa_\alpha)_j$ are essentially constant, we can further decompose the set M_θ as

$$M_\theta = \bigcup_{\kappa_\theta \in \mathbb{Z}_+^\theta} M_\theta(\kappa_\theta),$$

where $M_\theta(\kappa_\theta) := \{\mu \in M_\theta : \mu(Q_{\kappa_j}^{(j)}) = 1 \text{ for all } j \in J_\theta\}$. In particular, we have $M_{\mathbf{1}} = \mathbb{Z}_+^m$ and $M_{\mathbf{1}}(\kappa) = \{\kappa\}$ for any $\kappa \in \mathbb{Z}_+^m$. Here and subsequently we use the notation $\mathbf{1} := (1, \dots, 1) \in \{0, 1\}^m$.

By construction we obtain the following result.

Lemma 3.2 [3, Lemma 3.13] *The compact space of maximal ideals of* $\mathcal{T}_{\text{k-qr}}$ *admits the following decomposition into mutually disjoints sets*

$$M(\mathcal{T}_{\text{k-qr}}) = \bigcup_{\theta\in\Theta} M_\theta = \bigcup_{\theta\in\Theta} \bigcup_{\kappa_\theta\in\mathbb{Z}_+^\theta} M_\theta(\kappa_\theta). \tag{3.7}$$

The elements of $\mathcal{T}_{\text{k-qr}}$, being diagonal with respect to the canonical basis, will customary be written as D_γ, where γ denotes the corresponding eigenvalue function defined on $\mathbb{Z}_+^m$. Furthermore, given an element $\mu \in M(\mathcal{T}_{\text{k-qr}})$, we will usually write $\gamma(\mu)$ for $\mu(D_\gamma)$, i.e. the evaluation of the multiplicative functional μ in D_γ.

3.3 Quasi- and Pseudo-homogeneous Symbols

The next ingredient to the construction of the algebras we are studying is a family of Toeplitz operators whose symbols are subordinated to the aforementioned group $U_{(1)} \times \cdots \times U_{(m)}$ (see [25]).

To introduce these symbols we need some notations. Let $z \in \mathbb{B}^n$ and recall the decomposition $z = (z_{(1)}, \ldots, z_{(m)})$. We write $r_j = |z_{(j)}|$, for each $j = 1, \ldots, m$, and express $z_{(j)}$ as:

$$z_{(j)} = r_j \xi_{(j)}, \quad \xi_{(j)} \in S^{2k_j-1} := \partial\mathbb{B}^{k_j}, \quad r_j \in \mathbb{R}_+.$$

Furthermore, we decompose the tuples $\xi_{(j)}$ into polar coordinates as

$$\xi_{(j)} = (\xi_{j,1}, \ldots, \xi_{j,k_j}) = (s_{j,1}t_{j,1}, \ldots, s_{j,k_j}t_{j,k_j}),$$

where

$$s_{(j)} = (s_{j,1}, \ldots, s_{j,k_j}) \in S_+^{k_j-1} := S^{k_j-1} \cap \mathbb{R}_+^{k_j}$$

and

$$t_{(j)} = (t_{j,1}, \ldots, t_{j,k_j}) \in \mathbb{T}^{k_j}.$$

The first version of these symbols was introduced in [32] and developed in [3–5]. It leads to Toeplitz operators with so-called *quasi-homogeneous* symbols. These functions are defined by

$$\phi_j(z) = \xi_{(j)}^{p_{(j)}} \overline{\xi_{(j)}^{q_{(j)}}}, \tag{3.8}$$

where $j \in \{1, \ldots, m\}$ and $p_{(j)}, q_{(j)} \in \mathbb{Z}_+^{k_j}$ are such that $p_{(j)} \cdot q_{(j)} = 0$ and $|p_{(j)}| = |q_{(j)}|$.

This class of functions was generalized afterwards to the family of *pseudo-homogeneous* symbols (see [17, 34]), which are defined as

$$\phi_j(z) = b(s_{(j)})t_{(j)}^{p_{(j)}}, \tag{3.9}$$

where $b \in L^\infty(S_+^{k_j-1})$ and $p_{(j)} \in \mathbb{Z}_+^{k_j}$ is such that $|p_{(j)}| = 0$. We remark that a Toeplitz operator T_ϕ with symbol ϕ of the form (3.8) or (3.9) leaves H_κ invariant and that the restriction $T_\phi|_{H_\kappa}$ is nilpotent for every $\kappa \in \mathbb{Z}^m$. In particular, $\mathrm{sp}(T_\phi|_{H_\kappa}) = \{0\}$. Both classes of functions are invariant under the action of $\mathbb{T}$ on $\mathbb{C}^{k_j}$ given by

$$(t, z_{(j)}) \in \mathbb{T} \times \mathbb{C}^{k_j} \longmapsto (tz_{j,1}, \dots, tz_{j,k_j}). \tag{3.10}$$

It turns out that this condition suffices for much of the previous results to hold and so this was the approach in [30], where the so-called *generalized pseudo-homogeneous* symbols were introduced. These symbols are defined by

$$\phi_j(z) = c(s_{(j)}, t_{(j)}), \tag{3.11}$$

where $c \in L^\infty(S_+^{k_j-1} \times \mathbb{T}^{k_j})$ is invariant under the aforementioned action of $\mathbb{T}$ (restricted to $\mathbb{T}^{k_j}$) on its second component.

Finally, as was shown in [25], the most general form of these kind of symbols is given by adjoining a quasi-radial dependence to (3.11). That is, symbols of the form

$$\phi_j(z) = g(|z_{(1)}|, \dots, |z_{(m)}|, s_{(j)}, t_{(j)}). \tag{3.12}$$

Whatever the case may be, it can be shown that the corresponding Toeplitz operator T_{ϕ_j} also leaves all subspaces H_κ invariant and so it can be decomposed as

$$T_{\phi_j} = \bigoplus_{\kappa \in \mathbb{Z}_+^m} T_{\phi_j}|_{H_\kappa}. \tag{3.13}$$

We note that the symbols of the form (3.11) (and its above particular cases) can be regarded as functions defined on $\mathbb{B}^n$, $\mathbb{B}^{k_j}$ or even $\mathbb{C}^n$ and $\mathbb{C}^{k_j}$. To avoid lengthy notation we will use the same symbol to represent any of these functions.

As an immediate consequence of the decomposition (3.13), the operator T_{ϕ_j} commutes with all operators from the C^* algebra $\mathcal{T}_{\text{k-qr}}$. Moreover, summarizing the results from [25], the action of T_{ϕ_j} is given by the following proposition.

Proposition 3.3 [25, Corollary 7.7] *Let $j \in \{1, \dots, m\}$, ϕ_j as in (3.12) and $\alpha, \beta \in \mathbb{Z}_+^n$. If $\kappa_j := |\alpha_{(j)}| \neq |\beta_{(j)}|$ or $\alpha_{(l)} \neq \beta_{(l)}$ for some $l \neq j$, then $\langle T_{\phi_j} e_\alpha, e_\beta \rangle = 0$. Otherwise*

$$\langle T_{\phi_j} e_\alpha, e_\beta \rangle = \frac{2^{m-1}\Gamma(n+\lambda+|\alpha|+1)}{\pi^{k_j}\Gamma(\lambda+1)\sqrt{\alpha_{(j)}!\beta_{(j)}!}\prod_{l\neq j}(k_l+\kappa_l-1)!}$$
$$\times \int_{\tau(\mathbb{B}^m)\times S_+^{k_j-1}\times\mathbb{T}^{k_j}} g(r, s_{(j)}, t_{(j)}) s_{(j)}^{\alpha_{(j)}+\beta_{(j)}+\mathbf{1}_{k_j}} t_{(j)}^{\alpha_{(j)}-\beta_{(j)}}$$
$$\times (1-|r|^2)^\lambda \prod_{l=1}^m r_l^{2k_l+2\kappa_l-1} dr_l ds_{(j)} dt_{(j)}, \tag{3.14}$$

where $\mathbf{1}_{k_j} := (1, \dots, 1) \in \mathbb{Z}_+^{k_j}$.

It can be shown that the restriction of such an operator T_{ϕ_j} to the spaces H_κ is naturally unitarily equivalent to a tensor product of operators where all factors except the one at the j^{th} position equal the identity operator. So, for different indices $j_1, j_2 \in \{1, \dots, m\}$ the operators $T_{\phi_{j_1}}$ and $T_{\phi_{j_2}}$, acting on different "positions" with respect to the tensor product, commute. (See Section 7 from [25] for more details).

As is known, the Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ does not carry a natural tensor product structure such as the Fock space, and so in general the phenomenon described above has to be studied locally on H_κ instead of on the whole space. Nevertheless, when we restrict ourselves to symbols of the form (3.11), we obtain a useful representation for these operators.

Consider the tensor product of weightless Bergman spaces $\mathcal{A}_0^2(\mathbb{B}^{k_1}) \otimes \cdots \otimes \mathcal{A}_0^2(\mathbb{B}^{k_m})$. This is a Hilbert space with a canonical orthonormal basis given by

$$e_{\alpha_{(1)}}^{(1)} \otimes \cdots \otimes e_{\alpha_{(m)}}^{(m)}, \quad \alpha \in \mathbb{Z}_+^n,$$

where $e_{\alpha_{(j)}}^{(j)}$ is the canonical basic monomial of the space $\mathcal{A}_0^2(\mathbb{B}^{k_j})$, see (3.1).

We note that, by identifying the corresponding orthonormal basis, the space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ is isomorphic to this tensor product of Bergman spaces. More specifically, we denote by $U\colon \mathcal{A}_\lambda^2(\mathbb{B}^n) \to \mathcal{A}_0^2(\mathbb{B}^{k_1}) \otimes \cdots \otimes \mathcal{A}_0^2(\mathbb{B}^{k_m})$ the unique unitary operator such that

$$U(e_\alpha(z)) = e_{\alpha_{(1)}}^{(1)} \otimes \cdots \otimes e_{\alpha_{(m)}}^{(m)}, \quad \alpha \in \mathbb{Z}_+^n.$$

Lemma 3.4 *Let* $j \in \{1, \dots, m\}$ *and let* ϕ_j *be a symbol of the form* (3.11). *We have*

$$T_{\phi_j} = U^*(I \otimes \cdots \otimes T_{\phi_j}^{(j)} \otimes \cdots \otimes I)U, \tag{3.15}$$

where $T_{\phi_j}^{(j)}$ *is the Toeplitz operator with symbol* ϕ_j *acting on the Bergman space* $\mathcal{A}_0^2(\mathbb{B}^{k_j})$.

Although the proof of this lemma is straightforward, one way of understanding why this representation holds consists of analyzing the action of the corresponding Toeplitz operators on the Fock space with symbols given by (3.11) (changing trivially its domain from $\mathbb{B}^{k_j}$ to $\mathbb{C}^{k_j}$). Indeed, since those symbols ϕ_j do not depend on the quantities $|z_{(j)}|$, the formulas appearing on the Fock space turn out to be the same as the corresponding ones on the Bergman space, so that one can naturally identify these operators. Then using the inherent tensor product structure of the Fock space one easily shows that the corresponding Toeplitz operators can be viewed as some tensor products of operators as in the above lemma. This approach was followed, for example, in [4] and [33].

We introduce some notations suggested by the tensor product structure mentioned above. For $j \in \{1, \dots, m\}$ and $d \in \mathbb{Z}_+$ let $\widetilde{H}_d^{(j)}$ be the finite-dimensional subspace of $\mathcal{A}_0^2(\mathbb{B}^{k_j})$ defined by

$$\widetilde{H}_d^{(j)} = \operatorname{span}\{e_{\alpha_{(j)}}^{(j)} : |\alpha_{(j)}| = d, \alpha_{(j)} \in \mathbb{Z}_+^{k_j}\}.$$

Note that U^* maps

$$\mathcal{A}_0^2(\mathbb{B}^{k_1}) \otimes \cdots \otimes \widetilde{H}_d^{(j)} \otimes \cdots \otimes \mathcal{A}_0^2(\mathbb{B}^{k_m}),$$

being $\widetilde{H}_d^{(j)}$ on the j^{th} position, onto the space $H_d^{(j)}$ given by (3.5).

3.4 Commutative Banach Algebras

We will present the most general setting and explain the particular cases. However, it is worth mentioning that, although the quasi-homogeneous case is the simplest one regarding the symbols, it has the interesting property that for each portion j we can take several generators, which leads to some non-trivial properties of the algebra.

Example 3.5 Suppose $n > 1$ and consider the case $m = 1$. Let $h \in \{1, \dots, n-1\}$ and let $\mathcal{P}$ be the subset of all tuples (p, q) from $\mathbb{Z}_+^n \times \mathbb{Z}_+^n$ such that

$$p_{h+1} = \cdots = p_{k_1} = q_1 = \cdots = q_h = 0 \text{ and } |p| = |q|.$$

For each $(p, q) \in \mathcal{P}$, let $\psi_{(p,q)}$ denote the quasi-homogeneous function given by

$$\psi_{(p,q)}(z) = \xi_{(1)}^p \overline{\xi_{(1)}^q}, \qquad z \neq 0.$$

Then $\{T_{\psi_{(p,q)}} : (p, q) \in \mathcal{P}\}$ is a commuting set of operators and so the Banach algebra generated by them together with $\mathcal{T}_{\text{k-qr}}$ is a commutative unital Banach algebra. Furthermore, it can be shown that the infinite set of generators $T_{\psi_{(p,q)}}$ can be reduced to a finite set. Although the maximal ideal space of the unital Banach algebra generated by the operators from $\{T_{\psi_{(p,q)}} : (p, q) \in \mathcal{P}\}$ is known, it is still an open question to find an explicit description of this set, being the polynomially convex hull of a subset of $\mathbb{C}^{h(n-h)}$. As a consequence of this, not much can be said about the spectral invariance of the algebras studied in this context (compare with Proposition 3.12 below). See [3, 5, 32] for more details.

The action of a Toeplitz operator with arbitrary pseudo-homogeneous symbol on H_κ could be more complicated, as (3.14) indicates. In general, it is no longer possible to take more than one generator for each portion j without losing commutativity.

Since much of the results of this section still hold for quasi-homogeneous symbols, we proceed to describe the Gelfand theory of the corresponding algebras when the symbols are of the more general form (3.11). Thus fix for each $j = 1, \dots, m$ a generalized pseudo-homogeneous symbol ϕ_j of the form (3.11) and let $\mathcal{T}_{\mathrm{ph}}$ be the (commutative) unital Banach algebra generated by the Toeplitz operators T_{ϕ_j}, $j = 1, \dots, m$.

3.5 Gelfand Theory

We recall that for $j \in \{1, \dots, m\}$ one has a decomposition

$$T^{(j)}_{\phi_j} = \bigoplus_{d \in \mathbb{Z}_+} T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_d},$$

where $T^{(j)}_{\phi_j}$ is the operator from (3.15). Since each $T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_d}$ is an operator on a finite-dimensional space one has the following result.

Lemma 3.6 (See [30, Section 3.3]). *Let $j \in \{1, \dots, m\}$ and ϕ_j be as in (3.11), considered as a function on $\mathbb{B}^{k_j}$, and $\zeta_j \in \mathrm{sp}(T^{(j)}_{\phi_j})$. Then there is a sequence of unimodular functions $(f^{(j)}_n)_n$ such that, for all n, $f^{(j)}_n \in \widetilde{H}^{(j)}_{d(n)}$ for some sequence $(d(n))_n$, and*

$$\lim_{n \to \infty} \|(T^{(j)}_{\phi_j} - \zeta_j) f^{(j)}_n\| = 0. \tag{3.16}$$

Moreover, if $\zeta_j \in \mathrm{sp}_{\mathrm{pt}}(T^{(j)}_{\phi_j})$ then one can choose $d(n) = d$ for any fixed d such that $\zeta_j \in \mathrm{sp}_{\mathrm{pt}}(T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_d})$ and if $\zeta_j \in \text{ess-sp}(T^{(j)}_{\phi_j})$, then one can suppose that $d(n) \to \infty$.

Using a suitable sequence of tensor products of the above functions $f^{(j)}_n$ one obtains the following characterization for the maximal ideal space of the algebra $\mathcal{T}_{\mathrm{ph}}$.

Proposition 3.7 [29, Proposition 3.5]. *The compact set $M(\mathcal{T}_{\mathrm{ph}})$ of maximal ideals of $\mathcal{T}_{\mathrm{ph}}$ can be identified with the set*

$$M(\mathcal{T}_{\mathrm{ph}}) = \widehat{\mathrm{sp}(T^{(1)}_{\phi_1})} \times \cdots \times \widehat{\mathrm{sp}(T^{(m)}_{\phi_m})}.$$

We introduce now our main object of study. Let $\mathcal{T}_{\text{k-qr,ph}}$ be the Banach algebra generated by both algebras $\mathcal{T}_{\text{k-qr}}$ and $\mathcal{T}_{\mathrm{ph}}$. By our previous discussion, $\mathcal{T}_{\text{k-qr,ph}}$ is a commutative Banach algebra for each weight parameter $\lambda > -1$ and, in general, it is no longer commutative when it is extended to a C^* algebra. Denote by $\widetilde{\mathcal{T}_{\text{k-qr,ph}}}$ the non-closed dense subalgebra of $\mathcal{T}_{\text{k-qr,ph}}$ generated by all finite sums of finite products of the generators.

Now we proceed to develop the Gelfand theory of this algebra, presenting the main results and ideas. Since some of the proofs are non-trivial and lenghty, we refer to [29] for a detailed discussion.

First, we determine the maximal ideal space of $\mathcal{T}_{\text{k-qr,ph}}$. By restricting multiplicative functionals to the generating subalgebras and applying Proposition 3.7 we obtain the following inclusion:

$$M(\mathcal{T}_{\text{k-qr,ph}}) \subset M(\mathcal{T}_{\text{k-qr}}) \times M(\mathcal{T}_{\text{ph}}) = M(\mathcal{T}_{\text{k-qr}}) \times \widehat{\text{sp}(T_{\phi_1})} \times \cdots \times \widehat{\text{sp}(T_{\phi_m})}.$$

Thus one needs to determine all points from the Cartesian product that lie in the maximal ideal space of $\mathcal{T}_{\text{k-qr,ph}}$. For the following remarks and results, we recall decomposition (3.7) and the notations introduced before it. Furthermore, we will constantly write ζ for the tuple $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$.

Roughly speaking (and under some additional conditions) it turns out that the compact set of maximal ideals $M(\mathcal{T}_{\text{k-qr,ph}})$ consists of (the polynomially convex hull of) those ordered pairs (μ, ζ) from the above Cartesian product such that $\mu \in M_\theta(\kappa_\theta)$ for $\theta \in \{0, 1\}^m$ and $\kappa \in \mathbb{Z}_+^\theta$, and

$$\zeta_j \in \begin{cases} \text{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_{\kappa_j}^{(j)}}), & \text{if } j \in J_\theta \\ \text{ess-sp}(T_{\phi_j}^{(j)}), & \text{otherwise.} \end{cases}$$

Although the first condition holds in general, we have to assume special properties of the symbols ϕ_j for the second one to hold. As is well known (see for example [11]), the C^* algebra $\mathcal{T}_0(C(\overline{\mathbb{B}^{k_j}}))$ generated by Toeplitz operators with continuous symbols on $\overline{\mathbb{B}^{k_j}}$ contains the ideal of compact operators $\mathcal{K}(\mathcal{A}_0^2(\mathbb{B}^{k_j}))$ and the quotient $\mathcal{T}_0(C(\overline{\mathbb{B}^{k_j}}))/\mathcal{K}(\mathcal{A}_0^2(\mathbb{B}^{k_j}))$ is isometrically isomorphic to the C^* algebra $C(\partial\mathbb{B}^{k_j})$ via the isomorphism generated by the mapping

$$T_a + \mathcal{K}(\mathcal{A}_0^2(\mathbb{B}^{k_j})) \longmapsto a|_{\partial\mathbb{B}^{k_j}}.$$

In particular, for a Toeplitz operator $T_a \in \mathcal{T}_0(C(\overline{\mathbb{B}^{k_j}}))$, we have

$$\text{ess-sp}(T_a) = a(\partial\mathbb{B}^{k_j}).$$

The compactness of the semi-commutators $T_{|\phi_j|^2} - T_{\phi_j} T_{\phi_j}^*$ will be an essential ingredient to our analysis. Therefore, for the rest of the section, we will assume that the symbols ϕ_j, considered as functions on $\mathbb{B}^{k_j}$, extend continuously to the boundary of $\mathbb{B}^{k_j}$. This condition was assumed for symbols of the form (3.9) in [17].

Proposition 3.8 [29, Theorem 4.4] *The maximal ideal space of $\mathcal{T}_{\text{k-qr,ph}}$ can be identified with the set*

$$\bigcup_{\theta \in \Theta} \bigcup_{\kappa_\theta \in \mathbb{Z}_+^\theta} M_\theta(\kappa_\theta) \times M_{\theta,\kappa_\theta,1} \times \cdots \times M_{\theta,\kappa_\theta,m}, \tag{3.17}$$

where with the previous notation $\kappa_\theta = (\kappa_{j_1}, \dots, \kappa_{j_{|\theta|}})$, *and* $j_l \in J_\theta$

$$M_{\theta,\kappa_\theta,j} = \begin{cases} \mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}^{(j)}_{\kappa_j}}), & \text{if } j \in J_\theta \\ \widehat{\text{ess-sp}}(T_{\phi_j}^{(j)}), & \text{otherwise}. \end{cases}$$

Furthermore, the Gelfand transform is generated by the following map on the generators of the algebra:

$$\sum_{\rho \in F} D_{\gamma_\rho} T^\rho \longmapsto \sum_{\rho \in F} \gamma_\rho(\mu)\zeta^\rho, \quad (\mu, \zeta) \in M_\theta(\kappa_\theta) \times M_{\theta,\kappa_\theta,1} \times \cdots \times M_{\theta,\kappa_\theta,m}, \tag{3.18}$$

where $F \subset \mathbb{Z}_+^m$ *is a finite subset,* $T^\rho := T_{\phi_1}^{\rho_1} \cdots T_{\phi_m}^{\rho_m}$ *and the operators* D_{γ_ρ} *are diagonal operators in* $\mathcal{T}_{\text{k-qr}}$ *with eigenvalue sequence* γ_ρ.

It is not difficult to recover the known particular cases. If $n = 2$ (and then necessarily $m = 1$), we get

$$M(\mathcal{T}_{\text{k-qr}}) = M_{(1)} \cup M_{(0)} = \mathbb{Z}_+ \cup M_\infty,$$

where M_∞ represents the points at infinity. Since for the quasi-homogeneous and pseudo-homogeneous cases, all matrices $T_{\phi_1}^{(j)}|_{\widetilde{H}^{(1)}_d}$ are nilpotent and thus $\mathrm{sp}(T_{\phi_1}^{(1)}|_{\widetilde{H}^{(1)}_d}) = \{0\}$, we have

$$M(\mathcal{T}_{\text{k-qr,ph}}) = (\mathbb{Z}_+ \times \{0\}) \cup \big(M_\infty \times \widehat{\text{ess-sp}}(T_{\phi_1}^{(1)})\big),$$

which coincides with the previous results from [5, 17].

3.6 Applications

We make use of the above results to obtain structural information about the algebra $\mathcal{T}_{\text{k-qr,ph}}$. A non-trivial problem concerns the characterization of its radical. Since the first works on this topic, one usually starts the study of the radical by analyzing the non-closed subalgebra $\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) \cap \widetilde{\mathcal{T}_{\text{k-qr,ph}}}$.

After a short examination one can detect some typical elements of this algebra. These operators have the form

$$D_\gamma \bigoplus_{d \in F_L} Q_d^{(j)} p_d^{(j)}(T_{\phi_j}), \tag{3.19}$$

according to the orthogonal decomposition (3.6). Here, $p_d^{(j)}$ is the polynomial

$$p_d^{(j)}(z) = (z - \zeta_1) \cdots (z - \zeta_M),$$

where $\mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}) = \{\zeta_1, \dots, \zeta_M\}$, and $D_\gamma \in \mathcal{T}_{\text{k-qr}}$ is such that $\gamma(\mu) = 0$ for any $\mu \in M_\theta$ with $\theta_j = 0$. Moreover, $F_L = \{d \in \mathbb{Z}_+ : |\,\mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}})| \leq L\}$, $L \in \mathbb{Z}_+$.

Indeed, provided such an element belongs to the algebra $\mathcal{T}_{\text{k-qr,ph}}$, one can easily see that it is mapped to zero by all multiplicative functionals in Propositon 3.8.

Nevertheless it is still not clear whether such an operator always belong to $\mathcal{T}_{\text{k-qr,ph}}$. A complete answer to this (and a generalization of Proposition 3.10 below) could probably be obtained by analyzing the (asymptotical) behaviour of the cardinality of the finite sets $\mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}})$ in the parameter d. In particular, it would be useful to know under which conditions on the symbol ϕ_j the quantity $|\,\mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}})|$ tends to infinity as $d \to \infty$.

We remark that if ϕ_j is of the form (3.8) or (3.9), then all matrices $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$ are nilpotent and thus $p_d^{(j)}(z) = z$ for every $d \in \mathbb{Z}_+$. Hence we have

$$\bigoplus_{d \in F_L} Q_d^{(j)} p_d^{(j)}(T_{\phi_j}) = T_{\phi_j},$$

so that (3.19) can be written as $D_\gamma T_{\phi_j}$, recovering the corresponding operators described in [4, 5].

As we can see, the polynomials $p_d^{(j)}$ above turn out to play an essential role. One can furthermore characterize semi-simplicity by means of these polynomials:

Proposition 3.9 [29, Proposition 3.9]. *The Banach algebra $\mathcal{T}_{\text{k-qr,ph}}$ is semi-simple, i.e.* $\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) = \{0\}$, *if and only if all matrices $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$ satisfy the condition* $p_d^{(j)}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}) = 0$, *that is, if and only if all such matrices are diagonalizable in the sense that their Jordan canonical form is diagonal.*

Although a complete characterization of $\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) \cap \widetilde{\mathcal{T}}_{k-qr,qh}$ is still missing for the most general case, in all examples we know, the operators of the form (3.19) generate this algebra. We present some important cases:

Proposition 3.10 [29, Proposition 7.4] *Assume that one of the following conditions holds:*

(1) $m = 1$,
(2) *All matrices $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$ are nilpotent,*
(3) *For each $j \in \{1, \dots, m\}$, the number of distinct eigenvalues of the matrix $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$ tends to infinity as $d \to \infty$.*

Then $\widetilde{\mathcal{R}} :=$ $\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) \cap \widetilde{\mathcal{T}}_{k-qr,qh}$ is the non-closed algebra generated by all operators of the form (3.19).

As a non-trivial fact, it turns out that the closure of $\widetilde{\mathcal{R}}$ coincides with the whole radical:

Proposition 3.11 [29, Corollary 6.9]. *Suppose that ϕ_j is continuous up to the boundary of $\mathbb{B}^{k_j}$ for every $j \in \{1, \dots, m\}$. Then*

$$\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) = \mathrm{clos}\left(\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) \cap \widetilde{\mathcal{T}_{\text{k-qr,ph}}}\right). \tag{3.20}$$

This result was partially proved for the case $n = 2$ in [5] and for the case $m = 1$ (and any positive integer n) in [4]. We remark that it is still valid for the quasi-homogeneous case taking more than one generator in each division. (See Section 6 from [29]).

Finally, we state an interesting result regarding the spectral invariance, which can be completely characterized in this setting. We observe a different behaviour from what was obtained in the quasi-homogeneous case. Recall that an algebra $\mathcal{A} \subset \mathcal{L}(X)$ of bounded operators acting on some Banach space X, is called *spectral invariant* or *closed under inversion* if the inverse of every invertible operator $A \in \mathcal{A}$ (invertible with respect to algebra of all bounded operators $\mathcal{L}(X)$) belongs to $\mathcal{A}$. We may shortly express this statement in the form

$$\mathcal{A} \cap \mathcal{L}(X)^{-1} = \mathcal{A}^{-1},$$

where $\mathcal{A}^{-1}$ denotes the group of invertible elements in $\mathcal{A}$. For a commutative unital Banach algebra $\mathcal{A}$, spectral invariance is not automatic and can be studied in terms of its multiplicative functionals. Indeed, given an operator $S \in \mathcal{A}$, invertible with respect to the algebra of bounded operators, its inverse S^{-1} will also belong to $\mathcal{A}$ if and only if the Gelfand transform of S is nowhere zero.

Proposition 3.12 [29, Proposition 5.1] *The algebra $\mathcal{T}_{\text{k-qr,qh}}$ is spectral invariant if and only if the sets* $\text{ess-sp}(T^{(j)}_{\phi_j})$ *are polynomially convex for $j = 1, \dots, m$.*

4 Hardy Space Over the Unit Ball

In this section we construct and analyze families of commutative Banach algebras generated by Toeplitz operators on the Hardy space over the unit sphere.

One of the first approaches to the study of such algebras is [1]. Partially by adapting methods in [27] the authors described commutative Toeplitz C^* algebras by studying symbols invariant under the action of maximal abelian groups of automorphisms. However, there is a different approach based on the known Bergman space theory. In [24], the Hardy space is decomposed into a direct sum of Bergman spaces (with different integer weight parameters). Then results obtained in the Bergman space setting can be applied. This approach permits also the study of commutative Toeplitz Banach algebras as in the preceding section. We recall the ideas and further extend the analysis in [24].

4.1 The Hardy Space

For the study of the Hardy space we follow the notations and the constructions from [24].

Let $S^{2n-1} = \partial\mathbb{B}^n$ denote the unit sphere in $\mathbb{C}^n$ and by $d\sigma$ denote the normalized surface measure of S^{2n-1}. Denote the points z of S^{2n-1} as $z = (z', z_n)$, where $z' \in \mathbb{B}^{n-1}$ and $z_n \in \mathbb{C}$ with $|z_n| = \sqrt{1 - |z'|^2}$.

As usual we define the Hardy space $H^2(S^{2n-1})$ as the (closed) subspace of the Hilbert space $L^2(S^{2n-1}, d\sigma)$ consisting of functions f satisfying the tangential Cauchy-Riemann equations:

$$L_{k,j} f = \left(z_k \frac{\partial}{\partial \overline{z_j}} - z_j \frac{\partial}{\partial \overline{z_k}} \right) f = 0, \quad 1 \le k < j \le n.$$

The orthogonal projection from $L^2(S^{2n-1}, d\sigma)$ onto $H^2(S^{2n-1})$ is called the *Szegö projection* and will be denoted by P.

There is an interesting relation between the Hardy space and Bergman spaces over $\mathbb{B}^{n-1}$ with integer weights:

Proposition 4.1 [24, Theorem 2.1, Corollary 2.2] *There is a unitary operator U from $L^2(S^{2n-1}, d\sigma)$ onto*

$$\bigoplus_{p \in \mathbb{Z}_+} L^2(\mathbb{B}^{n-1}, dv_p)$$

under which $H^2(S^{2n-1})$ is mapped onto $\bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_p^2(\mathbb{B}^{n-1})$.

Given a bounded measurable function $\boldsymbol{\varphi}$ defined on S^{2n-1} we define the Toeplitz operator $\boldsymbol{T}_{\boldsymbol{\varphi}}$ acting on the Hardy space $H^2(S^{2n-1})$ as

$$\boldsymbol{T}_{\boldsymbol{\varphi}} f = P(\boldsymbol{\varphi} f), \quad f \in H^2(S^{2n-1}).$$

To avoid confusion we will denote by T_φ^p the Toeplitz operator with symbol $\varphi \in L^\infty(\mathbb{B}^{n-1})$ acting on the Bergman space $\mathcal{A}_p^2(\mathbb{B}^{n-1})$. Now let $\boldsymbol{\varphi} \in L^\infty(S^{2n-1})$ be a function of the form

$$\boldsymbol{\varphi}(z) = \boldsymbol{\varphi}(z', |z_n|) = \boldsymbol{\varphi}(z', \sqrt{1 - |z'|^2}). \tag{4.1}$$

Note that to $\boldsymbol{\varphi}$ we can associate a unique function $\varphi \in L^\infty(\mathbb{B}^{n-1})$ such that

$$\varphi(z_1, \ldots, z_{n-1}) = \boldsymbol{\varphi}(z_1, \ldots, z_{n-1}, \sqrt{1 - |z'|^2}). \tag{4.2}$$

Toeplitz operators with that kind of symbols behave well with respect to the decomposition from Theorem 4.1 as the following result shows.

Proposition 4.2 [24, Theorem 3.1] *Let $\boldsymbol{\varphi}(z', |z_n|)$ be a bounded measurable symbol defined on S^{2n-1}. Under the above isomorphism U, the Toeplitz operator $\boldsymbol{T}_{\boldsymbol{\varphi}}$, acting on the Hardy soace $H^2(S^{2n-1})$ is unitarily equivalent to the operator $\bigoplus_{p \in \mathbb{Z}_+} T_\varphi^p$, acting on*

$$\bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_p^2(\mathbb{B}^{n-1}),$$

where $\varphi = \varphi(z')$ is of the form (4.2).

By means of Propositions 4.2 and 4.1, one can easily apply our construction in the setting of the Bergman space to the Hardy space.

Given a weight parameter $p \in \mathbb{Z}_+$ and $\kappa \in \mathbb{Z}_+^m$, let H_κ^p denote the subspace H_κ of the Bergman space $\mathcal{A}_p^2(\mathbb{B}^{n-1})$ as defined in (3.2). Then by Proposition 4.1, we can further decompose the Hardy space as the direct sum

$$H^2(S^{2n-1}) = \bigoplus_{p\in\mathbb{Z}_+,\kappa\in\mathbb{Z}_+^m} H_\kappa^p.$$

Consider a positive integer m and a tuple of positive integers $k = (k_1, \dots, k_m)$ such that $k_1 + \cdots + k_m = n - 1$. Taking into account our representation of points in S^{2n-1} as $z = (z', z_n)$, with $z' \in \mathbb{B}^{n-1}$, we divide the tuple z' into m groups as before (see Sect. 2). Thus we will write $z' = (z_{(1)}, \dots, z_{(m)})$.

4.2 Quasi-radial Symbols

Let $L^\infty_{k-qr}(S^{2n-1})$ be the set of all symbols $\boldsymbol{a}$ of the form (4.1) such that the associated symbol a belongs to $L^\infty_{k-qr}(\mathbb{B}^{n-1})$. By Proposition 4.2 and Lemma 3.1, a Toeplitz operator $\boldsymbol{T_a}$ with $\boldsymbol{a} \in L^\infty_{k-qr}(S^{2n-1})$ acts as a constant multiple of the identity on all subspaces H_κ^p,

$$\boldsymbol{T_a} = \bigoplus_{p\in\mathbb{Z}_+,\kappa\in\mathbb{Z}_+^m} \gamma_{a,k}(p,\kappa)I.$$

Here $\gamma_{a,k}$ is the function given by $\gamma_{a,k}(p,\kappa) := \gamma_{a,k,p}(\kappa)$, where $\gamma_{a,k,p}$ was defined in (3.4). Thus $\boldsymbol{T_a}$ is a diagonal operator whose eigenvalue set depends on κ as well as on the weight parameter $p \in \mathbb{Z}_+$. Again, we denote by $\mathcal{T}_{\text{k-qr}}$ the C^* algebra generated by Toeplitz operators with symbols from $L^\infty_{k-qr}(S^{2n-1})$.

First, we present some facts in the particular case $m = 1$ and $k = (k_1) = (n-1)$. That is, we consider the space $H^2(S^{2k_1+1})$. In this case, for a k-quasi-radial function $a \in L^\infty(0,1)$, the eigenvalues $\gamma_{a,k}$ of the corresponding Toeplitz operator have the form

$$\gamma_{a,k}(\kappa,p) = \frac{2^m(n+\kappa+p)!}{p!(k_1-1+\kappa)!}\int_0^1 a(r)(1-r^2)^p r^{2\kappa+2k_1-1}dr, \quad (\kappa,p) \in \mathbb{Z}_+^2.$$

So, the algebra $\mathcal{T}_{\text{k-qr}}$ is isomorphic to an algebra of bounded functions on $\mathbb{Z}_+^2$. We remark that

$$\gamma_{a,k}(\kappa,p) = \gamma_{a,(2)}(\kappa+k_1-2,p),$$

where $\gamma_{a,(2)}$ represents the corresponding function for the particular case $n = 3$ (corresponding to operators acting on $H^2(S^5)$). This case was studied in [24, Section 6] and, due to the above relation, one can imitate the analysis done there.

In particular, we conclude that the algebra $\mathcal{T}_{\text{k-qr}}$ separates the points of $\mathbb{Z}_+^2$ and, furthermore, it contains the C^* algebra $C(\widehat{\mathbb{Z}_+^2})$, where $\widehat{\mathbb{Z}_+^2} = \mathbb{Z}_+^2 \cup S_\infty$ is the compactification of $\mathbb{Z}_+^2$ by the *infinitely far* quarter-circle $S_\infty \cong \{e_\infty^{i\theta} : \theta \in [0, \pi/2]\}$. Here $e_\infty^{i\theta}$ is the point reached by sequences $((k_\ell, p_\ell))_\ell$ with

$$\tan\theta = \lim_{\ell\to\infty} \frac{k_\ell}{p_\ell}.$$

Finally, we consider the symbol $g \in L^\infty(0, 1)$ (see Section 6 in [24]) given by

$$g(r) = \frac{1}{\Gamma(1+i)}(\ln(1 - r^{-2}))^{-i}. \tag{4.3}$$

As was shown in the aforementioned work, given $(\kappa, p) \in \mathbb{Z}_+^2$, we have

$$\gamma_{g,(2)}(\kappa, p) = (p+1)^{-i}\left(d_{\kappa+1} + O\left(\frac{1}{p+1}\right)\right),$$

where $d_{\kappa+1} = d_\kappa(1 + \frac{i}{\kappa+1})$ and $d_1 = 1 + i$. By some careful analysis one can show that indeed $O(\frac{1}{p+1})$ is a function of (κ, p) that converges uniformly to zero whenever $\kappa + p \to \infty$ with $\frac{\kappa}{p} \le 1$. Moreover, we have $\lim_{\kappa\to\infty} d_\kappa = \Gamma(1+i)^{-1}$.

As a consequence, we can separate some of the points of $M(\mathcal{T}_{\text{k-qr}})\backslash\mathbb{Z}_+^2$ for different integer coordinates κ: If $(\kappa_\alpha, p_\alpha)$ is a net convergent to an element μ in $M(\mathcal{T}_{\text{k-qr}})$ and $\frac{\kappa_\alpha}{p_\alpha} \le 1$ for all α, then $\mu \in d_{\kappa_0+1}\mathbb{T}$ if and only if, for some subnet $((\kappa_{\alpha_\beta}, p_{\alpha_\beta}))_\beta$, we have $\kappa_{\alpha_\beta} = \kappa_0$.

Now we proceed to examine the general case $m \ge 1$ by means of the above remarks. Given $j \in \{1, \dots, m\}$ and a function $a \in L^\infty(0, 1)$, we denote by $\boldsymbol{a}^{(j)}$ the function defined on S^{2n-1} by

$$\boldsymbol{a}^{(j)}(z) = a(|z_{(j)}|). \tag{4.4}$$

Lemma 4.3 *Let $j \in \{1, \dots, m\}$ and $a \in L^\infty(0, 1)$. Then*

$$\gamma_{\boldsymbol{a}^{(j)},k}(\kappa, p) = \gamma_{a,(k_j)}(\kappa_j, p + |\tau_j(\kappa)|), \quad \kappa \in \mathbb{Z}_+^m,$$

where $\gamma_{a,(k_j)}$ is the eigenvalue sequence corresponding to the Toeplitz operator with radial symbol a acting on the space $H^2(S^{2k_j+1})$.

Proof It follows directly from Fubini's Theorem applied to the representation (3.4) of eigenvalues and well-known properties of the Beta function. □

Given $(\kappa, p) \in \mathbb{Z}_+^m \times \mathbb{Z}_+$ consider the following subspaces of the Hardy space

$$H_\kappa^p = \operatorname{span}\{e_\alpha^p : |\alpha_{(j)}| = \kappa_j, \quad j = 1, \dots, m\},$$

where e_α^p denote the normalized monomials from the Bergman space $\mathcal{A}_p^2(\mathbb{B}^{n-1})$ given by (3.1). We denote by $P_{(\kappa,p)}$ the orthogonal projection from $H^2(S^{2n-1})$ onto H_κ^p.

Corollary 4.4 *The algebra $\mathcal{T}_{\text{k-qr}}$ contains all projections $P_{(\kappa,p)}$, $(\kappa, p) \in \mathbb{Z}_+^{m+1}$.*

Proof This is clear for the case $k = (k_1)$, since all such projections belong to $C(\widehat{\mathbb{Z}_+^2})$. The general case then follows from Lemma 4.3 by multiplying adequate projections for each coordinate κ_j. $\square$

According to our previous remarks about g in (4.3), the eigenvalue function $\gamma_{\boldsymbol{g}^{(j)},k}$, defined on $\mathbb{Z}_+^{m+1}$ and corresponding to an Toeplitz operator on $H^2(S^{2n-1})$, has the property that for every net $((\kappa^\alpha, p^\alpha))_\alpha$ in $\mathbb{Z}_+^{m+1}$ with $\frac{\kappa_j^\alpha}{p^\alpha+|\tau_j(\kappa^\alpha)|} \leq 1$ and convergent in $M(\mathcal{T}_{\text{k-qr}})$ to $\mu \in M(\mathcal{T}_{\text{k-qr}})\backslash\mathbb{Z}_+^{m+1}$, we have

$$\gamma_{\boldsymbol{g}^{(j)},k}(\mu) \in d_{\rho+k_j-1}\mathbb{T} \quad \textit{if and only if} \quad \lim_\alpha \kappa_j^\alpha = \rho.$$

We can use this to separate the points of $M(\mathcal{T}_{\text{k-qr}})\backslash\mathbb{Z}_+^{m+1}$ with different "finite" coordinates:

Corollary 4.5 *Let $j \in \{1, \dots, m\}$ and $\rho \in \mathbb{Z}_+$. Then there is an operator $D_{h_{j,\rho}} \in \mathcal{T}_{\text{k-qr}}$ with $h_{j,\rho}(\kappa, p) \in [0, 1]$ for all $(\kappa, p) \in \mathbb{Z}_+^{m+1}$ and such that, for any net $((\kappa^\alpha, p^\alpha))_\alpha$ in $\mathbb{Z}_+^{m+1}$ convergent to $\mu \in M(\mathcal{T}_{\text{k-qr}})\backslash\mathbb{Z}_+^{m+1}$,*

$$h_{j,\rho}(\mu) = \begin{cases} 1 & \textit{if } \lim_\alpha \kappa_j^\alpha = \rho \\ 0 & \textit{otherwise.} \end{cases}$$

Proof Let $f_1 \colon \mathbb{C} \to [0, 1]$ be a bump function equal to 1 on the set $d_{\rho+k_j-1}\mathbb{T}$ and equal to 0 outside a sufficiently small neighborhood of this set. Let $f_2 \colon \widehat{\mathbb{Z}_+^2} \to [0, 1]$ be a bump function with $f_2(e_\infty^{0\cdot i}) = 1$ and equal to 0 outside a sufficiently small neighborhood of this point.

Then the function $f_1(D_{\gamma_{\boldsymbol{g}^{(j)},k}})D_{f_2\circ\iota} \in \mathcal{T}_{\text{k-qr}}$ has the required properties, where

$$\iota \colon M(\mathcal{T}_{\text{k-qr}}) \to \widehat{\mathbb{Z}_+^2}$$

is the inclusion obtained from the inclusion of C^* algebras $C(\widehat{\mathbb{Z}_+^2}) \subset \mathcal{T}_{\text{k-qr}}$. $\square$

For the sake of simplicity, in the following analysis we will denote the weight parameter p as κ_{m+1}, so that the point $(\kappa, p) \in \mathbb{Z}_+^m \times \mathbb{Z}_+$ will be also written as $(\kappa, p) = (\kappa_1, \dots, \kappa_m, \kappa_{m+1}) \in \mathbb{Z}_+^{m+1}$. As in the previous section, let $\Theta = \{0, 1\}^{m+1}$ and for every $\theta \in \Theta$ consider the sets $J_\theta = \{j : \theta_j = 1\}$ and $\mathbb{Z}_+^\theta = \bigoplus_{j \in J_\theta} \mathbb{Z}_+(j)$.

We denote by M_θ the set of all points μ from $M(\mathcal{T}_{\text{k-qr}})$ that are limits of a net $((\kappa_1^\alpha, \dots, \kappa_{m+1}^\alpha))_\alpha$ in $\mathbb{Z}_+^{m+1}$ such that $\kappa_j^\alpha \to \infty$ if and only if $\theta_j = 0$. Furthermore, for each $\kappa_\theta = (\kappa_{j_1}, \ldots, \kappa_{j_{|\theta|}}) \in \mathbb{Z}_+^\theta$ we define the sets $M_\theta(\kappa_\theta)$ by

$$M_\theta(\kappa_\theta) = \left\{\mu \in M_\theta : \mu = \lim_\alpha (\kappa_1^\alpha, \ldots, \kappa_{m+1}^\alpha) \text{ with } \kappa_j^\alpha = \kappa_j, \text{ for all } j \in J_\theta\right\}.$$

Corollary 4.6 *The compact set of maximal ideals of $\mathcal{T}_{\text{k-qr}}$ admits the following decomposition into disjoint sets*

$$M(\mathcal{T}_{\text{k-qr}}) = \bigcup_{\theta\in\Theta} \bigcup_{\kappa_\theta \in \mathbb{Z}_+^\theta} M_\theta(\kappa_\theta).$$

Proof It follows from separating the points by means of the operators from Corollary 4.5 and the projections $P_{(\kappa,p)}$, $(\kappa, p) \in \mathbb{Z}_+^{m+1}$, similarly as in the Bergman space setting. □

4.3 Commutative Banach Algebras

Note that we can reproduce the construction from Sect. 3.3 for the z'-component of the tuples $(z', z_n) \in S^{2n-1}$. Thus we can naturally define symbols $\boldsymbol{\phi}_j \in L^\infty(S^{2n-1})$ such that the corresponding functions $\phi_j \in L^\infty(\mathbb{B}^{n-1})$ are of the form (3.12).

Moreover, such symbols do not depend on z_n, and with respect to the decomposition in Proposition 4.2 the corresponding Toeplitz operators $\boldsymbol{T}_{\boldsymbol{\phi}_j}$ decompose as

$$\boldsymbol{T}_{\boldsymbol{\phi}_j} = \bigoplus_{p\in\mathbb{Z}_+} T^p_{\phi_j}.$$

Here $T^p_{\phi_j}$ is the Toeplitz operator with symbol ϕ_j acting on the Bergman space $\mathcal{A}^2_p(\mathbb{B}^{n-1})$, already studied in Sect. 3.

From the integral expression (3.14) one observes that when the functions ϕ_j are of the form (3.11), the action of $T^p_{\phi_j}$ does not depend on p so that the operator is essentially the same for different weight parameters. This observation simplifies some of the calculations. We will consider only this case, leaving the study of symbols of the form (3.12) for a future work.

4.4 Gelfand Theory and Applications

Fix for each $j \in \{1, \ldots, m\}$ a symbol ϕ_j of the form (3.11) and let $\mathcal{T}_{\text{ph}}$ be the Banach algebra generated by the operators $\boldsymbol{T}_{\boldsymbol{\phi}_1}, \ldots, \boldsymbol{T}_{\boldsymbol{\phi}_m}$. By the analysis of the previous section, we note that this is a commutative Banach algebra, which in general is non-commutative, when extended to a C^* algebra.

To avoid notation, we will use the Toeplitz operators $T^{(j)}_{\phi_j}$ acting on the Bergman spaces $\mathcal{A}^2_0(\mathbb{B}^{k_j})$ from Sect. 3.3. Since the action of the operators $\boldsymbol{T}_{\boldsymbol{\phi}_j}$ does not depend on the weight parameter p, one can see that the maximal ideal space remains the same as in the Bergman space case:

Proposition 4.7 *The compact space of maximal ideals of the algebra $\mathcal{T}_{\mathrm{ph}}$ coincides with the set*

$$M(\mathcal{T}_{\mathrm{ph}}) = \widehat{\mathrm{sp}(T^{(1)}_{\phi_1})} \times \cdots \times \widehat{\mathrm{sp}(T^{(m)}_{\phi_m})}.$$

Proof A similar argument as in the Bergman space case. □

Let $\mathcal{T}_{\text{k-qr,ph}}$ be the Banach algebra generated by the algebra $\mathcal{T}_{\text{k-qr}}$ and the Toeplitz operators $\boldsymbol{T}_{\boldsymbol{\phi}_1}, \ldots, \boldsymbol{T}_{\boldsymbol{\phi}_m}$. As usual, we denote by $M(\mathcal{T}_{\text{k-qr,ph}})$ its maximal ideal space.

Reasoning as before, we can identify $M(\mathcal{T}_{\text{k-qr,ph}})$ with a subset of the Cartesian product of the maximal ideal spaces of the generating algebras:

$$M(\mathcal{T}_{\text{k-qr,ph}}) \subset M(\mathcal{T}_{\text{k-qr}}) \times M(\mathcal{T}_{\mathrm{ph}}) = M(\mathcal{T}_{\text{k-qr}}) \times \widehat{\mathrm{sp}(T^{(1)}_{\phi_1})} \times \cdots \times \widehat{\mathrm{sp}(T^{(m)}_{\phi_m})}.$$

Using the same arguments as in the Bergman space case, one can detect almost all points belonging to the maximal ideal space. There is, however, an interesting difference compared to the Bergman space setting given by the following proposition.

Proposition 4.8 *If $(\mu, \zeta) \in M(\mathcal{T}_{\text{k-qr,ph}})$, $\theta \in \Theta$ and $\mu \in M_\theta(\kappa_\theta)$ (see (3.7)), then $\zeta_j \in \mathrm{sp}(T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_{\kappa_j}})$ for every $j \in \{1, \ldots, m\}$ with $\theta_j = 1$.*

Proof Let q be the characteristic polynomial of the matrix $(T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_{\kappa_j}})$. We assume that $\theta \neq 0$ and distinguish two cases:

(1) If $\theta = \mathbf{1}$, then $P_{(\kappa,p)} q(\boldsymbol{T}_{\boldsymbol{\phi}_j}) = 0$ and thus, after evaluating the functional μ, we get $q(\zeta_j) = 0$.

(2) Otherwise, let $j \in \{1, \ldots, m\}$ with $\theta_j = 1$ and write $\rho := \kappa_j$.
Let $\mathcal{I}$ be the closed ideal in $\mathcal{T}_{\text{k-qr,ph}}$ generated by the projections $P_{(\kappa,p)}$, $(\kappa, p) \in \mathbb{Z}^{m+1}_+$, and let $\pi : \mathcal{T}_{\text{k-qr,ph}} \longrightarrow \mathcal{T}_{\text{k-qr,ph}}/\mathcal{I}$ be the canonical projection. One easily sees that the multiplicative functional $(\mu, \zeta) =: \psi_{(\mu,\zeta)}$ can be factorized as

$$\psi_{(\mu,\zeta)} = \widetilde{\psi}_{(\mu,\zeta)} \circ \pi,$$

where $\widetilde{\psi}_{(\mu,\zeta)}$ is a multiplicative functional defined on $\mathcal{T}_{\text{k-qr,ph}}/\mathcal{I}$.
Consider now the operator $S = D_{h_{j,\rho}} q(\boldsymbol{T}_{\boldsymbol{\phi}_j})$, where $D_{h_{j,\rho}}$ is given by Corollary 4.5. By construction, one can check that $S + \mathcal{I} = 0$ and thus

$$1 \cdot q(\zeta_j) = h_{j,\rho}(\mu) q(\zeta_j) = \psi_{(\mu,\zeta)}(S) = \widetilde{\psi}_{(\mu,\zeta)}(S + \mathcal{I}) = 0.$$

In both cases (1) and (2) we have $q(\zeta_j) = 0$ and thus $\zeta_j \in \mathrm{sp}(T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_{\kappa_j}})$. □

As a consequence, we see that for a multiplicative functional $(\mu, \zeta) \in M(\mathcal{T}_{\text{k-qr,ph}})$ we have $\zeta_j \in \text{ess-sp}(T^{(j)}_{\phi_j})$ only when $\mu \in M_\theta$ with $\theta_j = 0$, that is, roughly speaking, when μ has an "infinite j^{th} coordinate". We note that this fact is independent of the behaviour of the coordinate $\kappa_{m+1} = p$, which plays the role of the weight parameter in the space decomposition in Proposition 4.1.

Proposition 4.9 *The maximal ideal space of $\mathcal{T}_{\text{k-qr,ph}}$ coincides with the set*

$$\bigcup_{\theta \in \Theta} \bigcup_{\kappa_\theta \in \mathbb{Z}_+^\theta} M_\theta(\kappa_\theta) \times M_{\theta,\kappa_\theta,1} \times \cdots \times M_{\theta,\kappa_\theta,m}.$$

where

$$M_{\theta,\kappa_\theta,j} = \begin{cases} \mathrm{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_{\kappa_j}^{(j)}}), & \text{if } j \in J_\theta \backslash \{m+1\} \\ \widehat{\text{ess-sp}}(T_{\phi_j}^{(j)}), & \text{otherwise}. \end{cases}$$

Furthermore, the Gelfand transform is generated by the following map on the generators of the algebra $\mathcal{T}_{\text{k-qr,ph}}$:

$$\sum_{\rho \in F} D_{\gamma_\rho} \boldsymbol{T}^\rho \longmapsto \sum_{\rho \in F} \gamma_\rho(\mu)\zeta^\rho, \quad (\mu, \zeta) \in M_\theta(\kappa_\theta) \times M_{\theta,\kappa_\theta,1} \times \cdots \times M_{\theta,\kappa_\theta,m}, \tag{4.5}$$

where $F \subset \mathbb{Z}_+^m$ is a finite subset and $\boldsymbol{T}^\rho := \boldsymbol{T}_{\phi_1}^{\rho_1} \cdots \boldsymbol{T}_{\phi_m}^{\rho_m}$.

Proof It follows from Proposition 4.8 and similar arguments as in the Bergman space setting. □

Compare with [24] for the case $n = 3$. Although the maximal ideal space has almost the same form as in the Bergman space setting, it is worth mentioning that, according to our remarks before the proposition, we have the above set $J_\theta \backslash \{m + 1\}$, instead of just J_θ, as in the Bergman space. This will probably change for symbols of the form (3.12).

We can apply this result to obtain some structural information on the algebra $\mathcal{T}_{\text{k-qr,ph}}$ as in the previous section. However, there is still much work to do in this direction.

As a matter of example, we can search for typical elements of the radical of $\mathcal{T}_{\text{k-qr,ph}}$. However, such operators seem to be more complicated compared to those found in the Bergman space case. Indeed, one would consider elements similar to the operator (3.19), replacing the projections $Q_d^{(j)}$ by operators appearing in Corollary 4.5.

Nevertheless, one can still characterize semi-simplicity, having the same phenomenon as in the Bergman space. (We use for simplicity once again the notations introduced in that case).

Proposition 4.10 *The algebra $\mathcal{T}_{\text{k-qr,ph}}$ is semi-simple, i.e.* $\mathrm{Rad}(\mathcal{T}_{\text{k-qr,ph}}) = \{0\}$, *if and only if all matrices $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$ satisfy the condition $p_d^{(j)}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}) = 0$, that is, if and only if all such matrices are diagonalizable in the sense that their Jordan canonical form is diagonal.*

Proof This follows from the same arguments as in the Bergman space setting. □

5 Fock Space Over Infinite Dimensional Hilbert Spaces

In this section we consider Toeplitz operators acting on the Segal-Bargmann space over an infinite dimensional Hilbert space (see [21, 22]) and commutative Banach algebras generated by such operators.

We start by recalling the notion of Gaussian measures on a real separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$ (see [18] for further details and proofs). Let $(\cdot, \cdot)$ be a continuous scalar product (positive definite bilinear form) in H and by $H' \cong H$ we denote the dual space consisting of all continuous linear functionals. Then there exists a bounded self-adjoint, injective and positive operator B on H such

$$(x, y) = \langle Bx, y \rangle \quad \textit{for all} \quad x, y \in H. \tag{5.1}$$

Let $F \subset H$ be a subspace of finite dimension $n \in \mathbb{N}$. Bochner's Theorem implies that on F there is a unique Radon measure ν_F with Fourier transform $\chi_{\nu_F}(y)$ satisfying:

$$\chi_{\nu_F}(y) := \int_F e^{i(x,y)} d\nu_F(x) = e^{-\frac{(y,y)}{4}} \qquad \textit{for all} \;\; y \in F.$$

We call ν_F the *Gaussian measure* on F associated with the inner product $(\cdot, \cdot)$. Define the annihilator space

$$F^\circ = \left\{ \varphi \in H' \; : \; \varphi(x) = 0 \;\; \textit{for} \;\; x \in F \right\}$$

and consider the finite dimensional quotient H'/F°. Let

$$\pi : H \cong H' \to H'/F^\circ$$

denote the canonical projection. Given any Borel subset $X \subset H'/F^\circ$ we can consider the preimage $N_X := \pi^{-1}(X) \subset H$. We call N_X a *cylinder set* with *base* X and we refer to H'/F° as its *generating space*. When F runs through the collection of all finite dimensional subspaces of H then the corresponding cylinder sets form an algebra denoted by $\mathcal{C}(H)$. Moreover, the σ-algebra generated by $\mathcal{C}(H)$ coincides with the Borel σ-algebra $\mathcal{B}(H)$ of H. Consider $F \subset H$ equipped with the restriction of $(\cdot, \cdot)$ as a Hilbert space. There is a canonical isomorphism

$$I_F : F \cong F' \longrightarrow H'/F^\circ : v \mapsto (\cdot, v) + F^\circ,$$

which can be applied to define a Radon measure μ_F on $\mathcal{B}(H'/F^\circ)$ by

$$\mu_F(X) := \nu_F\big(I_F^{-1}(X)\big).$$

It can be verified that the family of measures $(\mu_F)_F$, where F runs through the finite dimensional subspaces of H fulfills the following *compatibility condition*: let $G \subset H$ be finite dimensional with $F \subset G$ and consider the canonical map

$$p : H'/G^\circ \to H'/F^\circ : \varphi + G^\circ \mapsto \varphi + F^\circ.$$

Then, for each Borel set $X \in H'/F^{\circ}$ we have:

$$\mu_F(X) = \mu_G\big(p^{-1}(X)\big). \tag{5.2}$$

Condition (5.2) implies that there is a well-defined real valued set function μ on $\mathcal{C}(H)$ such that

$$\mu(N_X) = \mu_F(X) \ \ where \ \ X \in \mathcal{B}(H'/F^{\circ}).$$

Clearly $0 \leq \mu(N) \leq 1$ for all cylinder sets $N \in \mathcal{C}(H)$ and $\mu(H) = 1$. We call μ a *Gaussian cylinder measure*. It is important to note that a cylinder measure in general is not σ-additive. It only is defined on the algebra of cylinder sets and may not extend to a measure on the full Borel σ-algebra $\mathcal{B}(H)$. A characterization of the σ-additivity of cylinder measures in terms of the embedding

$$(H, \langle\cdot,\cdot\rangle) \subset (H, (\cdot,\cdot)) \tag{5.3}$$

is given by the following result:

Proposition 5.1 [18] *The Gaussian cylinder measure μ on $\mathcal{C}(H)$ above extends to an σ-additive Borel measure on $\mathcal{B}(H)$ if and only if the operator B in* (5.1) *is trace class. In this case the embedding* (5.3) *is of Hilbert-Schmidt type.*

In this section we consider complex separable Hilbert space H. Since H can as well be seen as a real Hilbert space the above construction of Gaussian measures applies. The operator B appearing there will be assumed to be complex linear.

To simplify the setting we consider the model case $H = \ell^2(\mathbb{N})$ of all square-summable complex valued sequences equipped with the standard inner product. Let $[\varepsilon_j = (\delta_{\ell j})_\ell \in H \ : \ j \in \mathbb{N}]$ denote the canonical orthonormal basis of H and assume that B is a positive diagonal nuclear (trace class) operator, i.e.

$$B\varepsilon_j = \lambda_j \varepsilon_j \quad where \quad \lambda_j > 0 \quad and \quad \sum_{j\in\mathbb{N}} \lambda_j = \operatorname{tr}(B) < \infty.$$

We write $\mathcal{Z}_0 \subset \mathbb{Z}_+^{\mathbb{N}}$ for the set of all sequences $\alpha = (\alpha_n)_n$ with non-negative integer entries such that $\alpha_n = 0$ for all but finitely many $n \in \mathbb{N}$. Let $z = (z_1, z_2, \ldots)$ be the coordinates of H. With $\alpha \in \mathcal{Z}_0$ we use the usual notation:

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots \quad \alpha! = \alpha_1!\alpha_2!\alpha_3! \ldots \quad \lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3} \ldots .$$

According to Proposition 5.1, the induced Gaussian cylinder measure is a σ-additive measure on $\mathcal{B}(H)$ and will be denoted by μ_B. Note that the space $L_B^2 := L^2(H, d\mu_B)$ contains all complex polynomials in H.

Definition 5.2 We define the *Fock space over H* induced from B by (see [21, 22]):

$$\mathcal{A}_B^2(H) = \overline{\text{span}}\Big\{ e_\alpha(z) = \sqrt{\frac{1}{\lambda^\alpha \alpha!}} z^\alpha \ : \ \alpha \in \mathcal{Z}_0 \Big\} \subset L_B^2,$$

where the closure is taken in L_B^2.

It can be checked that the monomials $[e_\alpha \ : \ \alpha \in \mathcal{Z}_0]$ form an orthonormal basis of $\mathcal{A}_B^2(H)$. In the following we call it the standard orthonormal basis.

Remark 5.3 In the classical case where we deal with a finite number of variables (i.e. $H = \mathbb{C}^n$ for some $n \in \mathbb{N}$) it is well-known that $\mathcal{A}_B^2(\mathbb{C}^n)$ can be interpreted as a Hilbert space of (pointwisely defined) entire functions with reproducing kernel. In fact, well-known apriori estimates (see [35]) imply that L^2-convergence in $\mathcal{A}_B^2(\mathbb{C}^n)$ implies uniform convergence on compact subsets of $\mathbb{C}^n$. In the setting of functions in infinitely many variables, i.e. $H = \ell^2(\mathbb{N})$, we only have the following weaker property: let $V \subset (H, \langle \cdot, \cdot \rangle)$ be an open set and assume that $z_0 \in V \cap B^{\frac{1}{2}} H$. We equip the range $H_{\frac{1}{2}} := B^{\frac{1}{2}} H$ with the Hilbert space norm $\| \cdot \|_{\frac{1}{2}} := \| B^{-\frac{1}{2}} \cdot \|$. Then it can be shown that there is an open neighbourhood $W_{z_0} \subset (H_{\frac{1}{2}}, \| \cdot \|_{\frac{1}{2}})$ of z_0 (open in the topology of $H_{\frac{1}{2}}$) and a constant $C_{z_0} > 0$ such that for any holomorphic function f in V:

$$\sup \big\{ |f(z)| \ : \ z \in W_{z_0} \big\} \leq C_{z_0} \left(\int_V |f(w)|^2 d\mu_B(w) \right)^{\frac{1}{2}}.$$

Hence L_B^2-convergence in $\mathcal{A}_B^2(H)$ implies uniform compact convergence in the subspace $(H_{\frac{1}{2}}, \| \cdot \|_{\frac{1}{2}})$ only. In conclusion we may consider elements in the Fock space $\mathcal{A}_B^2(H)$ as holomorphic functions on $H_{\frac{1}{2}} \subset H$. In general they do not extend to holomorphic functions on H. Also note that $H_{\frac{1}{2}}$ is dense in H of measure $\mu_B(H_{\frac{1}{2}}) = 0$.

Let $P : L_B^2 \to \mathcal{A}_B^2(H)$ denote the orthogonal projection. Given any bounded measurable function $\varphi \in L^\infty(H, \mu_B)$ we define the *Toeplitz operator* T_φ with symbol φ in the usual way:

$$T_\varphi : \mathcal{A}_B^2(H) \to \mathcal{A}_B^2(H), \qquad T_\varphi(f) = P(\varphi f), \qquad f \in \mathcal{A}_B^2(H).$$

Clearly $\|T_\varphi\| \leq \|\varphi\|_\infty$ such that T_φ defines a bounded operator on the Fock space. Now we pass to operator symbols with additional structure such that the corresponding Toeplitz operators commute. We are interested in the generated commutative Banach algebras and, in particular, in extensions of the results in [3–5, 13, 32, 34] we consider the infinite dimensional setting $H = \ell^2(\mathbb{N})$.

5.1 Quasi-radial Symbols

Let $k = (k_j)_{j\in\mathbb{N}}$ be a fixed integer sequence. We use k to subdivide any $z = (z_1, z_2, \ldots) \in H$ into groups as follows:

$$z = (z_{(1)}, z_{(2)} \ldots) \textit{ where } z_{(j)} = (z_{k_1+\ldots+k_{j-1}+1}, \ldots, z_{k_1+\ldots+k_j}) \in \mathbb{C}^{k_j}, \quad j \in \mathbb{N}.$$

Furthermore, we assume that the eigenvalue sequence $\lambda = (\lambda_n)_n$ of the operator B in the above construction is compatible with k. By this we mean that there is a sequence $(u_j)_{j\in\mathbb{N}}$ of positive real numbers such that

$$\lambda_{(j)} = (u_j, u_j, \ldots, u_j) \in \mathbb{R}_+^{k_j}, \qquad (j \in \mathbb{N}).$$

Clearly, in the case where $k = (1, 1, 1, \ldots)$ there is no additional assumption on the sequence $\lambda \in \ell^1(\mathbb{N})$.

Let $a : \ell^2(\mathbb{N}) \to \mathbb{C}$ be a k-quasi-radial measurable and bounded symbol, i.e. $a = a(z)$ only depends on the infinite vector $(|z_{(j)}|)_{j\in\mathbb{N}}$. Standard arguments from representation theory show that the Toeplitz operator T_a acts on the standard orthonormal basis $\mathcal{E}_B := [e_\alpha : \alpha \in \mathcal{Z}_0]$ as a diagonal operator with eigenvalues γ_a. More precisely:

$$T_a e_\alpha = \gamma_a(\kappa) e_\alpha \quad \textit{ where } \quad \kappa = \kappa(\alpha) = (|\alpha_{(j)}|)_j \in \mathcal{Z}_0. \tag{5.4}$$

In the following we denote by $\mathcal{T}_{\text{k-qr}}$ the C^* algebra in $\mathcal{L}(\mathcal{A}^2_B(H))$ generated by all Toeplitz operators with bounded k-quasi-radial symbols. According to (5.4) we can identify $\mathcal{T}_{\text{k-qr}}$ with a C^* subalgebra of the bounded complex-valued functions $C_b(\mathcal{Z}_0)$ on the discrete set $\mathcal{Z}_0$. Since the system $\mathcal{E}_B$ forms an orthonormal basis of $\mathcal{A}^2_B(H)$ we obtain:

$$\gamma_a(\kappa) = \langle T_a e_\alpha, e_\alpha \rangle = \langle a e_\alpha, e_\alpha \rangle = \int_H a|e_\alpha|^2 d\mu_B.$$

If, in addition, a is cylindrical, *k-quasi-radial*, i.e. a is of the form

$$a(z) = \tilde{a}\big(|z_{(1)}|, \ldots, |z_{(L)}|\big) \tag{5.5}$$

with some $L \in \mathbb{N}$ and $\tilde{a} \in L^\infty(\mathbb{R}^L_+)$, then we can evaluate $\gamma_a(\kappa)$ more explicitly in form of a finite dimensional integral:

$$\gamma_a(\kappa) = \prod_{j=1}^{L} \frac{1}{(k_j - 1 + \kappa_j)!} \int_{\mathbb{R}^L_+} \tilde{a}\big(\sqrt{u_1 r_1}, \ldots, \sqrt{u_L r_L}\big) r^{k-\mathbf{1}+\tilde{\kappa}} e^{-|r|} dr_1 \ldots dr_L, \tag{5.6}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^L_+$, $\tilde{\kappa} = (\kappa_1, \ldots, \kappa_L)$ and $|r| = r_1 + \ldots + r_L$.

Remark 5.4 If $a \geq 0$, then the eigenvalue sequence $\gamma_a(\kappa)$ is a *moment sequence*. Note that the corresponding moment problem which asks for a characterization of moment sequences in infinitely many variables via representing measures has been studied in the literature (cf. [2, 19]). In particular, there is an infinite dimensional generalization of *Haviland's Theorem*.

The *final sections* of the algebra $\mathcal{T}_{\text{k-qr}}$ are well understood (see [13]). More precisely, consider the natural inclusions $\mathbb{Z}_+^n \subset \mathcal{Z}_0$ with $n \in \mathbb{N}$ defined via extension by zero, i.e. $\iota_n : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0, 0 \ldots)$. Let $n = k_1 + \ldots + k_L$ where $L \in \mathbb{N}$ and consider the algebra:

$$\mathcal{T}_{\text{k-qr}}^{(L)} := C^*\Big(\gamma_a(\kappa) \ : \ a \text{ as in (5.5) and } \kappa = \iota_L(\kappa) \in \mathcal{Z}_0\Big). \tag{5.7}$$

Clearly, for all $n \in \mathbb{N}$ as above:

$$\mathcal{T}_{\text{k-qr}}^{(L)} \subset \mathcal{T}_{\text{k-qr}} \subset C_b(\mathcal{Z}_0).$$

Note that the eigenvalues $\gamma_a(\kappa)$ in (5.7) only depend on $\tilde{\kappa} = (\kappa_1, \ldots, \kappa_L) \in \mathbb{Z}_+^L$. Hence, we can identity $\mathcal{T}_{\text{k-qr}}^{(L)}$ with a subalgebra of $C_b(\mathbb{Z}_+^L)$.

In [13] the authors define the *square root metric* ρ_L on $\mathbb{Z}_+^L$ and consider the C^* subalgebra $C_{b,u}(\mathbb{Z}_+^L, \rho_L)$ in $C_b(\mathbb{Z}_+^L)$ of all bounded functions on $\mathbb{Z}_+^L$ that are uniformly continuous with respect to ρ_L. It is an interesting observation in [13] that the algebra $C_{b,u}(\mathbb{Z}_+^L, \rho_L)$ in general is strictly larger than the L-fold tensor product of $C_{b,u}(\mathbb{Z}_+, \rho_1)$. The main result of [13] (which generalizes [15] to the case $L > 1$) immediately implies:

Theorem 5.5 [13, 15] *The algebra $\mathcal{T}_{\text{k-qr}}^{(L)}$ is isometrically isomorphic to $C_{b,u}(\mathbb{Z}_+^L, \rho_L)$.*

For each $L \in \mathbb{N}$ we can consider the C^* algebra

$$\mathcal{T}_L := \Big\{\gamma(\tilde{\kappa}, 0, 0 \ldots) \ : \ \gamma \in \mathcal{T}_{\text{k-qr}}, \ \tilde{\kappa} = (\kappa_1, \ldots, \kappa_L) \in \mathbb{Z}_+^L\Big\} \subset C_b(\mathbb{Z}_+^L).$$

Then, clearly, $\mathcal{T}_{\text{k-qr}}^{(L)} \subset \mathcal{T}_L$. However, one can show more:

Proposition 5.6 *Both C^* algebras above coincide, i.e. $\mathcal{T}_{\text{k-qr}}^{(L)} = \mathcal{T}_L$.*

Proof Let $\gamma \in \mathcal{T}_{\text{k-qr}}$. Then $\kappa \mapsto \gamma(\kappa)$ uniformly can be approximated by a sequence $(\Sigma_\ell)_\ell$, where Σ_ℓ is a finite sum of finite products of eigenvalues of the form:

$$\gamma_{a_1}(\kappa)\gamma_{a_2}(\kappa) \ldots \gamma_{a_k}(\kappa), \qquad (k \in \mathbb{N}).$$

Each a_j for $j = 1, \ldots, k$ is a bounded k-quasi-radial function on $H = \ell^2(\mathbb{N})$. Let $L \in \mathbb{N}$ be fixed and pick $\tilde{\kappa} = (\kappa_1, \ldots, \kappa_L) \in \mathbb{Z}_+^L$. Choose $\alpha \in \mathcal{Z}_0$ such that $\kappa = (\tilde{\kappa}, 0, 0, \ldots) = \kappa(\alpha)$. Now, we replace a_j by a cylindrical function without changing the eigenvalue $\gamma_{a_j}(\kappa)$. Consider the orthogonal projection

$$\Pi_n : H \to \operatorname{span}\big\{\varepsilon_j \ : \ 1 \leq j \leq n\big\} \subset H,$$

where $n = k_1 + \ldots + k_L$. By Fubini's Theorem we obtain:

$$\begin{aligned}\gamma_{a_j}(\kappa) &= \int_H a_j |e_\alpha|^2 d\mu_B \\ &= \int_{\Pi_n H}\int_{(I-\Pi_n)H} a_j(x+y)|e_\alpha(x+y)|^2 d\mu_{(I-\Pi_n)B}(y) d\mu_{\Pi_n B}(x) \\ &= \int_{\Pi_n H} \tilde{a}_j(x)|e_\alpha(x)|^2 d\mu_{\Pi_n B}(x) = \gamma_{\tilde{a}_j}(\kappa).\end{aligned}$$

In the last equality we used the definition:

$$\tilde{a}_j(x) := \int_{(I-\Pi_n)H} a_j\big(\Pi_n x + y\big) d\mu_{(I-\Pi_n)B}(y).$$

Note that $\tilde{a}_j$ is a bounded cylindrical k-quasi-radial function. Hence, for all $\kappa = (\tilde{\kappa}, 0, \ldots)$ of the above form we can replace Σ_ℓ by an element in $\mathcal{T}^{(L)}_{\text{k-qr}}$ which proves the assertion. □

Consider now the C^* algebra:

$$\mathcal{A} := \Big\{\gamma \in C_b(\mathcal{Z}_0) \;:\; \gamma(\tilde{\kappa}, 0, 0, \ldots) \in \mathcal{T}^{(L)}_{\text{k-qr}} \textit{ for all } \tilde{\kappa} \in \mathbb{Z}^L_+ \textit{ for all } L \in \mathbb{N}\Big\}. \tag{5.8}$$

From Proposition 5.6 it follows that

$$\mathcal{T}_{\text{k-qr}} \subset \mathcal{A}. \tag{5.9}$$

Furthermore, consider the C^* subalgebra $\mathcal{T}_{\text{c-k-qr}} \subset \mathcal{T}_{\text{k-qr}}$ generated by Toeplitz operators having cylindrical bounded k-quasi radial symbols, i.e. symbols of the type (5.5). With $L \in \mathbb{N}$ define the projections:

$$\pi_L : \mathcal{Z}_0 \to \mathcal{Z}_0 : \pi_n(\kappa) := (\kappa_1, \ldots, \kappa_L, 0, 0 \ldots).$$

Lemma 5.7 *We have the equality*

$$\mathcal{T}_{\text{c-k-qr}} = \{\gamma \in \mathcal{A} \;:\; \lim_{L\to\infty} \gamma \circ \pi_L = \gamma \quad \textit{uniformly on } \mathcal{Z}_0\big\}.$$

Proof Let $\gamma \in \mathcal{T}_{\text{c-k-qr}}$. From $\mathcal{T}_{\text{c-k-qr}} \subset \mathcal{T}_{\text{k-qr}} \subset \mathcal{A}$ we conclude that $\gamma \in \mathcal{A}$. Let $\tilde{\gamma}$ be a finite sum of finite products of eigenvalues of Toeplitz operators with symbols of the form (5.5). From the expression (5.6) of the eigenvalues we conclude that $\tilde{\gamma} \circ \pi_L = \tilde{\gamma}$ for $L \in \mathbb{N}$ sufficiently large. Hence

$$\lim_{L\to\infty} \tilde{\gamma} \circ \pi_L = \tilde{\gamma} \tag{5.10}$$

uniformly on $\mathcal{Z}_0$. Since such sequences are uniformly dense in $\mathcal{T}_{\text{c-k-qr}}$ we conclude that γ fulfills (5.10) as well.

Conversely, let $\gamma \in \mathcal{A}$ such that $\lim_{L\to\infty} \gamma \circ \pi_L = \gamma$ uniformly on $\mathcal{Z}_0$. By definition of $\mathcal{A}$ we have $\gamma \circ \pi_L \in \mathcal{T}^{(L)}_{\text{k-qr}} \subset \mathcal{T}_{\text{c-k-qr}}$. Hence $\gamma \in \mathcal{T}_{\text{c-k-qr}}$. □

Example 5.8 We show that the algebra $\mathcal{T}_{\text{c-k-qr}}$ contains Toeplitz operators with non-cylindrical symbols, i.e. functions that depend on infinitely many variables. For simplicity we only consider the case $k = (1, 1, \dots)$. Let $\rho \in \ell^1(\mathbb{N})$ with $1 > \rho_j > 0$ for all $j \in \mathbb{N}$. Consider a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ of non-constant continuous functions $f_\ell : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$1 - \rho_\ell \leq f_\ell(y) \leq 1 + \rho_\ell \quad \textit{for all} \quad y \in \mathbb{C},\ \ell \in \mathbb{N}.$$

Define a k-quasi-radial (non-cylindrical) function $F : \ell^2(\mathbb{N}) \to \mathbb{R}_+$ as the infinite product:

$$F(z) = F(z_1, z_2, \dots) = \prod_{\ell=1}^{\infty} f_\ell(|z_\ell|).$$

For each $L \in \mathbb{N}$ put $F_L = F \circ \Pi_L = \prod_{\ell=0}^{L} f_\ell(|z_\ell|)$. Then

$$\begin{aligned} \big|F(z) - F_L(z)| &= |F_L(z)| \Big| 1 - \prod_{\ell=L+1}^{\infty} f_\ell(|z_\ell|) \Big| \\ &\leq \exp\Big(\sum_{\ell=1}^{\infty} \log(1+\rho_\ell) \Big) \Big| 1 - \exp\Big(\sum_{\ell=L+1}^{\infty} \log f_\ell\big(|z_\ell|\big) \Big) \Big|. \end{aligned}$$

Note that for all $z \in \ell^2(\mathbb{N})$:

$$\begin{aligned} \sum_{\ell=L+1}^{\infty} \frac{\rho_\ell}{\rho_\ell - 1} &\leq \sum_{\ell=L+1}^{\infty} \log(1-\rho_\ell) \\ &\leq R_L(z) := \sum_{\ell=L+1}^{\infty} \log f_\ell\big(|z_\ell|\big) \leq \sum_{\ell=L+1}^{\infty} \log(1+\rho_\ell) \leq \sum_{\ell=L+1}^{\infty} \rho_\ell. \end{aligned}$$

Since $\rho \in \ell^1(\mathbb{N})$ we conclude that $\lim_{L\to\infty} R_L(z) = 0$ uniformly in $\ell^2(\mathbb{N})$. Therefore $\lim_{L\to\infty} F_L = F$ uniformly. It follows that $\gamma_F = \lim_{L\to\infty} \gamma_{F_L} \in \mathcal{T}_{\text{c-k-qr}}$.

Example 5.9 Let $k = (1, 1, \dots)$ and $\gamma \in C_b(\mathcal{Z}_0)$. We define the *generating function* G_γ of γ by:

$$G_\gamma(x) = \sum_{\kappa \in \mathcal{Z}_0} \gamma(\kappa) x^\kappa \quad \textit{for} \quad x \in \Omega \subset H = \ell^2(\mathbb{N}), \tag{5.11}$$

where Ω denotes the domain of convergence of G_γ. Since γ is bounded on $\mathcal{Z}_0$ it is easy to see that

$$\left\{x = (x_1, x_2, \ldots) \in \ell^1(\mathbb{N}) \ : \ |x_j| < 1, \quad \forall \ j \in \mathbb{N}\right\} \subset \Omega.$$

Let $\gamma \in C_b(\mathcal{Z}_0)$ be defined by

$$\gamma(\kappa) := \begin{cases} 1 & \textit{if} \ \ \kappa = \varepsilon_j \textit{ for some } \ j \in \mathbb{N}, \\ 0 & \textit{else}. \end{cases}$$

Note that actually $\gamma \in \mathcal{A}$. Then

$$G_\gamma(x) = \sum_{j=0}^{\infty} x_j \quad \textit{where} \quad x = (x_1, x_2, \ldots) \in \Omega = \ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N}).$$

In particular, let $\eta \in \mathbb{C}$ with $|\eta| < 1$ and consider $x_\eta := -(\eta, \eta^2, \eta^3, \ldots) \in \Omega$. Then

$$G_\gamma(x_\eta) = -\sum_{j=1}^{\infty} \eta^j = \frac{\eta}{\eta - 1}$$

such that the map $\mathbb{D} := \{y \in \mathbb{C} \ : \ |y| < 1\} \ni \eta \to G_\gamma(x_\eta)$ extends to a meromorphic function with a simple pole at $\eta = 1$.

We now consider the generating function of an eigenvalue sequences. Let $a \in L^\infty(H, \mu_B)$ be a k-quasi-radial function. Let $\delta \in (0, 1)$ and consider

$$\Omega_\delta := \left\{x \in \ell^1(\mathbb{Z}_+) \ : \ |x_j| < \delta\lambda_j\right\}.$$

From $\exp\{\delta\|z\|^2\} \in L^1(H, \mu_B)$ and Lebesgue's Theorem on dominated convergence we obtain for all $x \in \Omega_\delta$:

$$\begin{aligned} G_{\gamma_a}(x) &= \sum_{\kappa \in \mathcal{Z}_0} \int_H a|e_\kappa|^2 x^\kappa \, d\mu_B \\ &= \int_H a(z) \sum_{\kappa \in \mathcal{Z}_0} \frac{|z^\kappa|^2 x^\kappa}{\lambda^\kappa \kappa!} d\mu_B(z) = \int_H a(z) \exp\left(\sum_{j=1}^{\infty} \frac{|z_j|^2 x_j}{\lambda_j}\right) d\mu_B(z). \end{aligned}$$

Note that the integral on the right hand side extends to a holomorphic function on $\{x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N}) \ : \ \mathrm{Re}(x_j) \leq 0, \ j \in \mathbb{N}\}$. Assume that there is $\delta > 0$ such that

$$\left\{\eta \in \mathbb{C} \ : \ x_\eta \in \Omega_\delta\right\}$$

is an open zero-neighborhood in $\mathbb{C}$. Note that this is the case if, e.g., we choose $\lambda_j := \theta^j$ where $\theta^j \in (0, 1)$. Again we consider the map

$$\mathbb{D} \ni \eta \mapsto G_{\gamma_a}(x_\eta) = \int_H a(z) \exp\left(-\sum_{j=0}^{\infty} \frac{|z_j|^2 \eta^{j+1}}{\lambda_j} \right) d\mu_B(z).$$

The assignment $\eta \mapsto G_{\gamma_a}(x_\eta)$ extends to a real analytic function in $\mathbb{R}_+$. In particular, it has no singularity in $\eta = 1$. We conclude that there is no bounded k-quasi-radial function with $\gamma_a = \gamma$.

To end this subsection, we briefly analyze the maximal ideal space of the algebra $\mathcal{T}_{\text{k-qr}}$. We introduce a corresponding version of the subspaces used in the previous sections. Given $\kappa \in \mathcal{Z}_0$ denote by H_κ the finite dimensional subspace

$$H_\kappa = \text{span}\left\{ e_\alpha \ : \ \left|\alpha_{(j)}\right| = \kappa_j, \quad \forall j \in \mathbb{N} \right\}.$$

Moreover, P_κ will denote the orthogonal projection from $\mathcal{A}_B^2(H)$ onto H_κ. On the other hand, for $d \in \mathbb{Z}_+$ and $j \in \mathbb{N}$ we define the infinite dimensional space $H_d^{(j)}$ by

$$H_d^{(j)} = \overline{\text{span}}\left\{ e_\alpha : \left|\alpha_{(j)}\right| = d \right\},$$

and we denote by $Q_d^{(j)}$ the orthogonal projection from $\mathcal{A}_B^2(H)$ onto $H_d^{(j)}$.

By (5.4) we see that the operators from $\mathcal{T}_{\text{k-qr}}$ leave all subspaces $H_d^{(j)}$ and H_κ invariant. Furthermore, by Theorem 5.5, we conclude that all projections $Q_d^{(j)}$ belong to $\mathcal{T}_{\text{k-qr}}$. Nonetheless, it is interesting to note that the orthogonal projections P_κ, being an infinite product of projections of the form $Q_d^{(j)}$, may not belong to this algebra. A proof of this conjecture however is missing.

Let $M\left(\mathcal{T}_{\text{k-qr}}\right)$ denote the compact space of maximal ideals of $\mathcal{T}_{\text{k-qr}}$. As in the Bergman space case, we can decompose $M\left(\mathcal{T}_{\text{k-qr}}\right)$ by means of these operators. Let Θ be the set of all sequences $\left(\theta_j\right)_{j \in \mathbb{N}}$ with $\theta_j \in \{0, 1\}$ for all $j \in \mathbb{N}$. Given $\theta \in \Theta$ we define the set $J_\theta = \left\{ j : \theta_j = 1 \right\}$ and, in this case, we let $\mathbb{Z}_+^\theta$ denote the set of all sequences of integers κ_θ such that $(\kappa_\theta)_j = 0$ for all $j \notin J_\theta$. Set

$$M_\theta = \left\{ \mu \in M\left(\mathcal{T}_{\text{k-qr}}\right) : \mu(Q_d^{(j)}) = \begin{cases} 0 \text{ for all } d \in \mathcal{Z}_0, & \text{if } \theta_j = 0 \\ 1 \text{ for some } d \in \mathcal{Z}_0, & \text{if } \theta_j = 1 \end{cases} \right\},$$

and

$$M_\theta(\kappa_\theta) = \{ \mu \in M_\theta \colon \mu(Q^{(j)}_{(\kappa_\theta)_j}) = 1 \text{ for all } j \in J_\theta \}.$$

Then, as before, we can write $M\left(\mathcal{T}_{\text{k-qr}}\right)$ as a disjoint union:

$$M\left(\mathcal{T}_{\text{k-qr}}\right) = \bigcup_{\theta \in \Theta} M_\theta = \bigcup_{\theta \in \Theta} \bigcup_{\kappa_\theta \in \Theta} M_\theta(\kappa_\theta).$$

5.2 Commutative Banach Algebras and Gelfand Theory

Finally, we present the corresponding commutative Banach algebras and their Gelfand theory.

Following our previous constructions we consider a sequence of generalized pseudo-homogeneous symbols $\left(\phi_j\right)_{j\in\mathbb{N}}$, where each ϕ_j (formally defined on $\mathbb{B}^{k_j}$) is a function of the form (3.11). That is, our symbols ϕ_j are formally cylindrical functions defined on H depending only on the coordinates $z_{(j)}$, such that the associated function on $\mathbb{C}^{k_j}$ coincides with the extension of a function (3.11) defined on $\mathbb{B}^{k_j}$. For the sake of simplicity, we will denote by ϕ_j all of these functions.

Thus, as one easily sees, we have an infinite dimensional version of the tensor product introduced in Sect. 3.3. That is, for each of the operators T_{ϕ_j} there is a uniquely associated Toeplitz operator $T_{\phi_j}^{(j)}$ acting on the Bergman space $\mathcal{A}_0^2(\mathbb{B}^{k_j})$ such that

$$\langle T_{\phi_j} e_\alpha, e_\beta\rangle_{\mathcal{A}_B^2(H)} = \begin{cases} 0, & \text{if } \alpha_{(j')} \neq \beta_{(j')}, \text{ for some } j' \in \mathbb{N}, \\ \langle T_{\phi_j}^{(j)} e_{\alpha_{(j)}}^{(j)}, e_{\beta_{(j)}}^{(j)}\rangle_{\mathcal{A}_0^2(\mathbb{B}^{k_j})}, & \text{otherwise.} \end{cases}$$

Here $e_{\alpha_{(j)}}^{(j)}$ denotes the canonical basic monomial of $\mathcal{A}_0^2(\mathbb{B}^{k_j})$. Denote by $\mathcal{T}_{\mathrm{ph}}$ the unital Banach algebra generated by the operators $(T_{\phi_j})_{j\in\mathbb{N}}$.

Corollary 5.10 *The sequence of operators $\left(T_{\phi_j}\right)_{j\in\mathbb{N}}$ is a commutative family of operators. In particular, the Banach algebra $\mathcal{T}_{\mathrm{ph}}$ is commutative.*

Proposition 5.11 *The compact set of maximal ideals of $\mathcal{T}_{\mathrm{ph}}$ can be (topologically) identified with the following set*

$$M\left(\mathcal{T}_{\mathrm{ph}}\right) = \prod_{j\in\mathbb{N}} \widehat{\mathrm{sp}\left(T_{\phi_j}\right)}.$$

Proof This follows from standard arguments as in the Bergman space case. □

Denote by $\mathcal{T}_{\text{k-qr,ph}}$ the Banach algebra generated by the algebras $\mathcal{T}_{\text{k-qr}}$ and $\mathcal{T}_{\mathrm{ph}}$. To conclude, we characterize the maximal ideal space of the algebra $\mathcal{T}_{\text{k-qr,ph}}$. As in the previous sections, we have the following natural inclusion:

$$M(\mathcal{T}_{\text{k-qr,ph}}) \subset M(\mathcal{T}_{\text{k-qr}}) \times M(\mathcal{T}_{\mathrm{ph}}) = M(\mathcal{T}_{\text{k-qr}}) \times \prod_{j\in\mathbb{N}} \widehat{\mathrm{sp}\left(T_{\phi_j}\right)}.$$

Although some special care has to be taken in the corresponding calculations, it is not hard to find the points which belong to the maximal ideal space following the same methods as before.

Proposition 5.12 *The maximal ideal space of $\mathcal{T}_{\text{k-qr,ph}}$ coincides with the set*

$$\bigcup_{\theta\in\Theta} \bigcup_{\kappa_\theta\in\mathbb{Z}_+^\theta} M_\theta(\kappa_\theta) \times \prod_{j\in\mathbb{N}} M_{\theta,\kappa_\theta,j},$$

where

$$M_{\theta,\kappa_\theta,j}=\begin{cases}\mathrm{sp}(T^{(j)}_{\phi_j}|_{\widetilde{H}^{(j)}_{(\kappa_\theta)_j}}), & \text{if } j\in J_\theta\\ \widehat{\text{ess-sp}}(T^{(j)}_{\phi_j}), & \text{otherwise.}\end{cases}$$

Furthermore, the Gelfand transform is generated by the following map on the generators of the algebra:

$$\sum_{\rho\in F} D_{\gamma_\rho}T^\rho \longmapsto \sum_{\rho\in F}\gamma_\rho(\mu)\zeta^\rho,\quad (\mu,\zeta)\in M_\theta(\kappa_\theta)\times\prod_{j\in\mathbb{N}} M_{\theta,\kappa_\theta,j}, \tag{5.12}$$

where $F\subset\mathcal{Z}_0$ is a finite subset and $T^\rho := T^{\rho_1}_{\phi_1}\cdots T^{\rho_m}_{\phi_m}$.

Proof This follows from the same arguments as in the Bergman space setting, approximating operators by finite sums of finite products of the generators. □

It seems reasonable to expect that $\mathcal{T}_{\text{k-qr,ph}}$ shares similar properties with the corresponding algebra on the Bergman space (e.g. being not simple but spectral invariant). Yet we leave this for future works, as it requires some detailed analysis.

6 Conclusion and Open Problems

To conclude the present work we compare our results for different function Hilbert spaces. In each section we introduced a commutative Banach algebra $\mathcal{T}_{\text{k-qr,ph}}$ generated by a commutative C^* algebra $\mathcal{T}_{\text{k-qr}}$ and a commutative (not C^*) Banach algebra $\mathcal{T}_{\text{ph}}$. The model cases are the corresponding algebras defined on the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. The other cases are somehow infinite extensions of this: an infinite direct sum for the Hardy space and an infinite tensor product for the Fock space.

The C^* algebras $\mathcal{T}_{\text{k-qr}}$ have already been an interesting object of study in the recent literature. The C^* algebra generated by radial Toeplitz operators on the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ was characterized in [8], whereas the corresponding result for C^* algebras generated by Toeplitz operators with quasi-radial symbols on the Fock space $F^2(\mathbb{C}^n)$ on finitely many variables can be found in [13, 15]. In particular, in both cases it turns out that the closed linear span of the generators coincides with the corresponding C^* algebra.

However, a complete characterization of these algebras for all other function Hilbert spaces and non-radial quasi-radial symbols, presented in this work, remains an open problem (see Problem 1).

As for the algebras $\mathcal{T}_{\text{ph}}$, we have seen that their structure is essentially the same in all three cases, so that the work done on the Bergman space can be reproduced on the Hardy and Fock spaces without much effort. This will probably change when studying symbols of the more general form (3.12), where the corresponding weight parameters have to be carefully analyzed.

In all three cases, after analyzing individually the generating algebras, we introduced the corresponding algebra $\mathcal{T}_{\text{k-qr,ph}}$ and developed its Gelfand theory. In this context, much of the work done on the Bergman space can be reproduced on the other spaces. However, in the case of the Hardy space some careful examination of the set $M(\mathcal{T}_{\text{k-qr}})$ has to be done to show that the points at infinity are separated. We recall that this was not necessary in the Bergman and Fock spaces due to the existence of some particular orthogonal projections inside the algebra.

Finally, in the Bergman space setting, we showed how these results can give structural information (description of the radical and spectral invariance) of the algebra $\mathcal{T}_{\text{k-qr,ph}}$. Such an analysis can probably be performed also on the Hardy and Fock spaces along the same lines and we leave this for future works.

To close this survey, we list some open problems collected along the paper.

(1) Find a characterization of the algebras $\mathcal{T}_{\text{k-qr}}$ (e.g. as uniformly bounded continuous function with respect to some metric) for the remaining cases: (not radial) quasi-radial symbols on the Bergman space, quasi-radial symbols on the Hardy space and quasi-radial symbols on the Fock space in infinitely many variables. Does $\mathcal{T}_{\text{k-qr}}$ coincide with the (linear) closure of Toeplitz operators with k-quasi-radial symbols? Do we have equality in (5.9)? (See Sects. 3.2, 4.2 and 5.1).

(2) Is it possible to determine an asymptotic behaviour for the eigenvalues of the matrices $T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}}$? Under which conditions do we have $|\operatorname{sp}(T_{\phi_j}^{(j)}|_{\widetilde{H}_d^{(j)}})| \to \infty$ as $d \to \infty$? (See Sect. 3.6).

(3) Find a description of the algebra $\operatorname{Rad}(\mathcal{T}_{\text{k-qr,ph}}) \cap \widetilde{\mathcal{T}_{\text{k-qr,ph}}}$ for general symbols of the form (3.11). (See Sect. 3.6).

(4) Develop the Gelfand theory for the corresponding algebras taking more general symbols ϕ_j such as symbols of the form (3.12) or symbols without the continuity assumptions. (See Sect. 3.3).

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