Structural and Dimensional Synthesis of Overconstraint Symmetric 3T2R Parallel Robots using Tait-Bryan-Angle Kinematic Constraints

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Abstract The basic parallel robotics principle of defining kinematic constraints as vector loops is transferred from the general 3T3R case to the 3T2R case by applying a nonlinear Tait-Bryan-angle rotation constraint using the intrinsic Z-Y'-X'' convention. This presents an alternative way to the deduction of differential inverse kinematics by the theory of linear transformations. The resulting formulation is used in a permutational combined structural and dimensional synthesis. The modular approach allows to combine databases of serial and parallel robots without manual intervention. The validation shows the reproducibility of existing kinematic structures using the new kinematic formulation. The optimization scheme allows to obtain suitably dimensioned symmetric 3T2R parallel robots for a given task.

Key words: Parallel robot, Parallel manipulator, Overconstraint, 3T2R, Kinematic constraints, Tait-Bryan angles, Euler angles, Dimensional synthesis.

1 Introduction and State of the Art

A special case of parallel robots or manipulators (PMs) are those with five structural degrees of freedom (DoF): three translational and two rotational (3T2R). These PMs provide an interesting kinematic structure e.g. for machining tasks. Symmetric 3T2R PMs with identical limbs were shown to exist rather late [5, 9]. Since then, works on analysis, modeling and synthesis of these types of PMs have increased [16, 7].

The *analysis* of symmetric 3T2R PMs can be performed by several methods. Most prominent is screw theory [5, 10, 13, 7], which can be extended by other algebraic and geometric concepts like Grassmann-Cayley algebra and Grassmann geometry [1] or algebraic geometry (Study parameters and Gröbner bases) [13]. For the related 2T3R PMs Lie groups of displacements were used in [11]. The concept of linear transformations presents an approach of less mathematical complexity and has also been used in the context of general 3T2R PMs [8]; in [16] together with a geometrical analysis. The existing works focus on different aspects like forward kinematics [13], workspace [14], singularity analysis [12, 1] or compliance modeling [3].

The *synthesis* of this type of PM has been performed by screw theory in combination with a constraint method [9] or a virtual chain approach [10]. In [4] screw

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theory is used for synthesis together with a systematic deduction and naming scheme to build up a database of symmetric and asymmetric PMs. In [8] the synthesis is performed by evolutionary morphology and the theory of linear transformations, which was taken up in similar form in [16]. Works on robot synthesis give high focus on the *structural* synthesis of the leg chains and mainly provide rules for the synthesis of complete PMs together with some selected examples. However, the performance of parallel robots is strongly depending on the *dimensioning* of the parameters [15].

To obtain PMs for a given purpose — even for an academic example — the *dimensional* synthesis has to be performed — either manually or automatically. Therefore, the structural synthesis should be *combined* with a dimensional synthesis [6]. The kinematics parameter optimization of a five-DoF PM with complex kinematic structure but few parameters is performed in [18]. Other works on dimensional synthesis like [6] focus on different architectures, but are also transferable to the 3T2R case.

To further the combined synthesis of 3T2R PMs, the paper's contributions are

- an *alternative approach* to the *inverse kinematics* model of symmetric 3T2R PMs with a deduction similar to the theory of linear transformations from [8],
- an optimization scheme suitable for combined structural and dimensional synthesis of 3T2R PMs with *less mathematical complexity* than established methods,
- the reproduction of symmetric 3T2R PMs from literature with the new method,
- an open-source MATLAB toolbox for the kinematics model, the structural and dimensional synthesis toolchain and a serial chain and parallel robot database.

The remainder of the paper is structured as follows. Sect. 2 introduces the kinematic model for symmetric 3T2R parallel robots. The synthesis of these robots is discussed in Sect. 3, followed by results in Sect. 4 and a conclusion in Sect. 5.

2 Inverse Kinematic Model for 3T2R Parallel Robots

The constraints equations for parallel robots are usually defined on or equivalent to the velocity level when using screw theory [10] or the theory of linear transformations [8]. The following section presents an inverse kinematics model based on full kinematic constraints equations, which can be seen as an alternative deduction to the latter method. Approaching the kinematics problem from the nonlinear position level can avoid difficulties regarding the reference frame of angular velocities. A method from [17] for 3T3R parallel robots with functional redundancy for 3T2R tasks is transferred to the case of 3T2R robots without redundancy. It should be kept in mind that the method is dedicated to a numeric evaluation and only parts of the expressions are derived symbolically. An elimination of passive joint coordinates or the use for the forward kinematics problem is not feasible with the proposed approach. This does not present a disadvantage for the combined synthesis or in simulation.

A set of coordinate systems (CS), shown in Fig. 1a, is used to model the parallel robot with m=5 leg chains n=5 platform DoF, extending the model of [2]. The robot base frame $(\mathrm{CS})_0$ is fixed regarding the world frame $(\mathrm{CS})_W$. Each leg chain has a virtual base frame $(\mathrm{CS})_{A_i}$ and a virtual end frame $(\mathrm{CS})_{C_i}$. This allows a modular use of models for serial kinematic leg chains. The cut joint frames at the platform are $(\mathrm{CS})_{B_i}$, corresponding to the leg's $(\mathrm{CS})_{C_i}$. The desired end effector frame $(\mathrm{CS})_D$ is

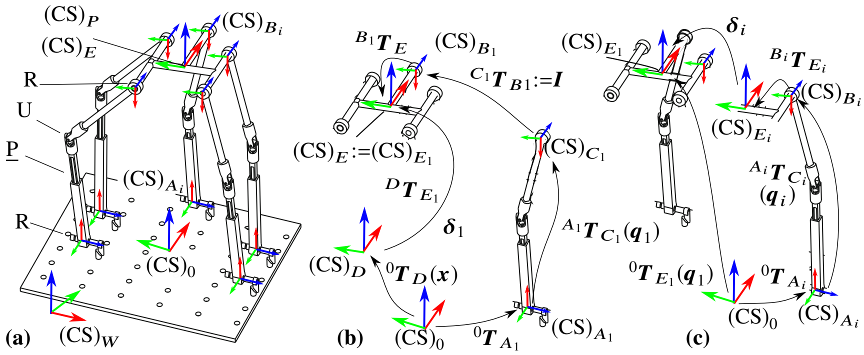

Fig. 1 Sketch of the kinematics model at the example of the modified 5-RP̲UR image of [12]. (**a**) Full model with all coordinate frames, (**b**) constraints for first leg chain and (**c**) other leg chains $i{\neq}1$

used in the derivation of the model and $(\mathrm{CS})_E$ corresponds to the actual frame of the end effector. Geometric parameters are given by the SE(3) matrices ${}^0\boldsymbol{T}_{A_i}$ for the base coupling joint frames, ${}^P\boldsymbol{T}_{B_i}$ for the platform coupling joint frames and ${}^P\boldsymbol{T}_E$ for the (for now neglected) end effector (e.g. tool) frame on the platform.

The translational component of the operational space coordinates $\boldsymbol{x}_\mathrm{t}{=}_{(0)}\boldsymbol{r}_D$ is the vector to the desired end effector frame relative to the base frame. The rotational component marks the pointing direction of the end effector frame with the two angles $\boldsymbol{x}_\mathrm{r}^\mathsf{T}{=}[\varphi_x, \varphi_y]$, as established in literature [16, 14, 1, 18, 7] with $\varphi_z{=}0$. To allow general (tool) frame definitions, using X-Y'-Z'' Tait-Bryan angles for the full orientation of the end effector frame gives the SO(3) rotation matrix ${}^0\boldsymbol{R}_D{=}\boldsymbol{R}_x(\varphi_x)\boldsymbol{R}_y(\varphi_y)\boldsymbol{R}_z(\varphi_z)$. The third angle φ_z=const (depending on frame definitions) corresponds to a rotation around the Z-axis of $(\mathrm{CS})_E$ and — as a dependent variable for 3T2R PMs — is not included in the five-DoF minimal coordinate $\boldsymbol{x}^\mathsf{T} = [\boldsymbol{x}_\mathrm{t}^\mathsf{T}, \boldsymbol{x}_\mathrm{r}^\mathsf{T}]$.

The forward kinematics for a leg chain i with (active and passive) joint coordinates $\boldsymbol{q}_i$ are defined as

$$ {}^0\boldsymbol{T}_{E_i}(\boldsymbol{q}_i) = {}^0\boldsymbol{T}_{A_i}\,{}^{A_i}\boldsymbol{T}_{C_i}(\boldsymbol{q}_i)\,{}^{C_i}\boldsymbol{T}_{B_i}\,{}^{B_i}\boldsymbol{T}_E. \tag{1} $$

The kinematic constraints for the first leg chain $i{=}1$ are expressed as residual regarding $(\mathrm{CS})_E$ from chain 1 and the desired platform frame $(\mathrm{CS})_D$, giving

$$ \boldsymbol{\delta}_{\mathrm{t},i}(\boldsymbol{q}_i, \boldsymbol{x}) = {}_{(0)}\boldsymbol{r}_{D,E}(\boldsymbol{q}_i, \boldsymbol{x}) = -\boldsymbol{x}_\mathrm{t} + {}_{(0)}\boldsymbol{r}_{E_i}(\boldsymbol{q}_i) \in \mathbb{R}^3 \tag{2} $$

for translation, as depicted in Fig. 1b. The rotational part is

$$ \boldsymbol{\delta}_{\mathrm{r},i}(\boldsymbol{q}_i, \boldsymbol{x}) = \begin{bmatrix}\alpha_y & \alpha_x\end{bmatrix}^\mathsf{T} = \boldsymbol{\alpha}_\mathrm{red}\left({}^D\boldsymbol{R}_{E_i}(\boldsymbol{x}, \boldsymbol{q}_i)\right) = \boldsymbol{\alpha}_\mathrm{red}\left({}^0\boldsymbol{R}_D^\mathsf{T}(\boldsymbol{x}_\mathrm{r})\,{}^0\boldsymbol{R}_{E_i}(\boldsymbol{q}_i)\right) \in \mathbb{R}^2. \tag{3} $$

The function $\boldsymbol{\alpha}(\boldsymbol{R})$ computes the intrinsic Z-Y'-X'' Tait-Bryan angles $[\alpha_z, \alpha_y, \alpha_x]$ from the rotation matrix $\boldsymbol{R}$ and $\boldsymbol{\alpha}_\mathrm{red}(\boldsymbol{R}) = [\alpha_y, \alpha_x]^\mathsf{T}$ does the same without the Z angle. Using proper Euler angles like Z-X'-Z'' instead would be impractical due to the singularity for $\boldsymbol{\alpha} = 0$. Therefore, the general term "Euler angles" is omitted. A convention with first rotation around Z should be used for independence of φ_z, [17].

The nonlinear function (3) allows a minimal coordinate representation of the inverse kinematics problem of 3T2R robots (or tasks) and is not dependent on the uncontrollable (or redundant) variable φ_z, as elaborated in more detail in [17].

The rotational constraints of further leg chains are modeled following the first leg chain. See e.g. [8, 2] for similar approaches on velocity level. As sketched in Fig. 1c, this is expressed relative to the first leg's rotation matrix ${}^0\boldsymbol{R}_{E_1}(\boldsymbol{q}_1)$ from (1) as

$$\boldsymbol{\delta}_{\mathrm{r},i}(\boldsymbol{q}_i, \boldsymbol{q}_1) = \boldsymbol{\alpha}\left({}^0\boldsymbol{R}_{E_1}^{\mathsf{T}}(\boldsymbol{q}_1){}^0\boldsymbol{R}_{E_i}(\boldsymbol{q}_i)\right) \in \mathbb{R}^3 \quad \text{for} \quad i = 2, \ldots, m. \tag{4}$$

The translational component for further leg chains is linear and therefore used identically as in (2). The constraints $\boldsymbol{\delta}_i$ for each leg chain and $\boldsymbol{\delta}$ for the full robot are

$$\boldsymbol{\delta}_i^{\mathsf{T}} = \left[\boldsymbol{\delta}_{\mathrm{t},i}^{\mathsf{T}}\ \boldsymbol{\delta}_{\mathrm{r},i}^{\mathsf{T}}\right] \quad \text{and} \quad \boldsymbol{\delta}^{\mathsf{T}} = \left[\boldsymbol{\delta}_1^{\mathsf{T}}\ \boldsymbol{\delta}_2^{\mathsf{T}} \cdots \boldsymbol{\delta}_m^{\mathsf{T}}\right]. \tag{5}$$

A symmetric 3T2R PM has $n_{\boldsymbol{q}}$=25 active and passive joint DoFs and the constraints are $\boldsymbol{\delta}(\boldsymbol{q}, \boldsymbol{x})=\boldsymbol{0} \in \mathbb{R}^{29}$, which descriptively shows the overconstraint of degree four.

The first-order inverse kinematics can be obtained by partial derivatives as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta}(\boldsymbol{q}, \boldsymbol{x}) = \boldsymbol{\delta}_{\partial\boldsymbol{q}}\dot{\boldsymbol{q}} + \boldsymbol{\delta}_{\partial\boldsymbol{x}}\dot{\boldsymbol{x}} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{\delta}_{\partial\boldsymbol{q}} := \frac{\partial}{\partial\boldsymbol{q}}\boldsymbol{\delta} \quad \text{and} \quad \boldsymbol{\delta}_{\partial\boldsymbol{x}} := \frac{\partial}{\partial\boldsymbol{x}}\boldsymbol{\delta}. \tag{6}$$

The gradients in (6) can be obtained in a closed form only depending on known expressions such as the geometric Jacobian of the leg chains and Euler angle transformation matrices [17]. The "direct kinematic matrix" [8] $\boldsymbol{\delta}_{\partial\boldsymbol{q}}$ is rectangular and the linear relation $\dot{\boldsymbol{q}} = \tilde{\boldsymbol{J}}^{-1}\dot{\boldsymbol{x}}$ can be obtained numerically from (6) e.g. using a QR solver. The tilde sign is used to demarcate the Jacobian corresponding to all joint coordinates (including passive and coupling joints). For analysis of the robot the (inverse) Jacobian matrix $\boldsymbol{J}^{-1}$ corresponding to the active joints $\boldsymbol{q}_{\mathrm{a}}$ is needed. The appropriate rows are selected with the matrix $\boldsymbol{P}_{\mathrm{a}}$ giving $\boldsymbol{q}_{\mathrm{a}} = \boldsymbol{P}_{\mathrm{a}}\boldsymbol{q}$ and $\boldsymbol{J}^{-1} = \boldsymbol{P}_{\mathrm{a}}\tilde{\boldsymbol{J}}^{-1}$.

This approach has several properties which makes it favorable to use for a combined structural and dimensional synthesis of symmetric 3T2R PMs:

- The inverse kinematics problem can be solved using the Newton-Raphson algorithm with $\boldsymbol{\delta}(\boldsymbol{q}^{k+1}, \boldsymbol{x})=\boldsymbol{\delta}(\boldsymbol{q}^k, \boldsymbol{x}) + \boldsymbol{\delta}_{\partial\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{x})|_{\boldsymbol{q}^k}(\boldsymbol{q}^{k+1}-\boldsymbol{q}^k)=\boldsymbol{0}$ for a step k.
- Only mathematical concepts in the scope of textbooks like [15] are necessary, avoiding the explicit use of screw theory and Grassmann or Lie algebra.
- A completely numeric and modular implementation allows setting up the kinematics model automatically for creation and use of a database of robot structures.

The model (5) has the property of overconstraint, which shows in a rectangular direct kinematic matrix $\boldsymbol{\delta}_{\partial\boldsymbol{q}}$. This can be avoided by using the reduced orientation residual (3) on all leg chains. This geometric elimination of the overconstraint is only permitted if the kinematic constraints can be met, i.e. $\boldsymbol{\delta}$=$\boldsymbol{0}$. This reduced model is written with letter $\boldsymbol{\psi}$ instead of $\boldsymbol{\delta}$ to easier distinguish the two. The translational part stays unchanged with $\boldsymbol{\psi}_{\mathrm{t},i} = \boldsymbol{\delta}_{\mathrm{t},i}$ and the rotational part is

$$\boldsymbol{\psi}_{\mathrm{r},i}(\boldsymbol{q}_i, \boldsymbol{x}) = \left[\alpha_y\ \alpha_x\right]^{\mathsf{T}} = \boldsymbol{\alpha}_{\mathrm{red}}\left({}^D\boldsymbol{R}_{E_i}(\boldsymbol{x}_{\mathrm{r}}, \boldsymbol{q}_i)\right) \in \mathbb{R}^2 \quad \text{for} \quad i = 1, \ldots, m. \tag{7}$$

The reduced dimension leads to the non-overconstraint $\boldsymbol{\psi} \in \mathbb{R}^{25}$ and $\boldsymbol{\psi}_{\partial\boldsymbol{q}} \in \mathbb{R}^{25\times 25}$.

The Jacobian can be obtained numerically by formulating (6) with $\boldsymbol{\psi}$ instead of $\boldsymbol{\delta}$ using standard solvers for square matrices like the LU solver. For evaluating the mobility of a parallel robot first the full constraints $\boldsymbol{\delta}$ have to be regarded. Otherwise a false positive $\boldsymbol{\psi}=\boldsymbol{0}$ can result while the full constraints are not met, giving an infeasible $\boldsymbol{\delta}\neq\boldsymbol{0}$. The leg chain's coupling joints then perform an inconsistent rotation around the end effector's Z-axis with feasible ${}_{(0)}\boldsymbol{r}_{E_i}$ and last column in ${}^{0}\boldsymbol{R}_{E_i}$.

3 Combined Structural and Dimensional Synthesis of 3T2R PMs

The kinematic model from the previous section is used in a combined structural and dimensional synthesis of symmetric parallel mechanisms (PMs) with 3T2R DoF. In the *combined synthesis* all possible structures are evaluated by an algorithm which first creates a PM database in a structural synthesis (using the dimensional synthesis implementation) and then performs the dimensional synthesis on the database.

The *dimensional synthesis* is performed for an exemplary task by the kinematics simulation of a representative end effector trajectory. A more general, but computationally more expensive solution would include a subsequent workspace analysis. The kinematic parameters are subject to a particle swarm optimization (PSO), as outlined in Fig. 2. The fitness function $\boldsymbol{f}$ for evaluating a set of dimensional parameters $\boldsymbol{p}$ begins with the solution of the first-order inverse kinematics (IK) on position level for reference points in the workspace. If successful, a second-order inverse kinematics is solved for the trajectory using the Jacobian relation. The process is repeated for all the PM's IK configurations, found numerically. Within the fitness function evaluation several *constraints* are checked in a hierarchical manner with decreasing priority, meaning that violation of a constraint leads to the abortion of the evaluation and a penalty corresponding to constraint priority. Some of the constraints (checked in structural "S" or dimensional "D" synthesis) are (in this order)
(S/D) geometric plausibility (e.g. leg chain lengths vs base/platform dimensions),
(S/D) success of the inverse kinematics (first and second order, based on (7), (6)),
(S) validity of full kinematic constraints $\boldsymbol{\delta} = \boldsymbol{0}$ from (5),
(D) self-collisions (with elementary geometry such as spheres and capsules),
(D) installation space (e.g. robot has to be in a cylinder with reasonable radius),
(D) joint angle ranges and velocities (to be in a technically feasible range of values),
(D) singularities of type I (of $\boldsymbol{\delta}_{\partial \boldsymbol{q}}$) and II (of $\boldsymbol{J}$) (by a condition-number threshold).
If all constraints are met, the *optimization objective* is evaluated, as discussed next.

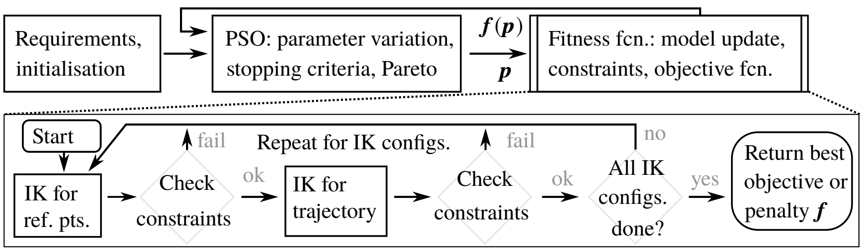

Fig. 2 Overall procedure for the dimensional synthesis of a robot with hierarchical constraints

In the *structural synthesis* mode of the toolchain the *objective* is to obtain the mobility based on the Jacobian $\boldsymbol{J}$. By this framework there is no need of a complex geometric analysis as done by other authors. The underlying assumption is that a numeric evaluation of the PM mobility is possible if (and only if) the kinematic constraints can be met. While many implementation details are different, the general leg chain synthesis approach is similar to [8] and the database approach is similar to [4]. Serial kinematic leg chains are obtained by permutation of Denavit-Hartenberg parameters and successive elimination of isomorphisms which leaves 213 possible serial kinematic leg chains with five joints consisting of revolute (R) or prismatic (P) joints. Substituting some RR-subchains with universal (U) joints further gives 180 variants. The geometric characteristics of possible symmetric parallel robots are obtained from literature. This leads to several possible joint alignments relative to the fixed base (transformation ${}^{0}\boldsymbol{T}_{A_i}$) or the moving platform (transformation ${}^{P}\boldsymbol{T}_{B_i}$). The synthesis is performed by permutation of all implemented possibilities for leg chain, base coupling, platform coupling and actuation with additional filtering for feasibility (e.g. no passive prismatic joints, proximal actuation). Eleven PMs with four joints (classes 5-PRUR, 5-RPUR, 5-RRUR, 5-RUPR, 5-RURR) and 16 PMs with five joints (classes 5-PRRRR, 5-RPRRR, 5-RRPRR, 5-RRRRR) are generated.

4 Exemplary Results of the Combined Synthesis for 3T2R PMs

These structures are validated for an exemplary task with middle position of $[r_{\mathrm{T}x}, r_{\mathrm{T}y}, r_{\mathrm{T}z}]=\boldsymbol{x}_{\mathrm{T,t}}^{\mathsf{T}}=[0, 0, 1500\,\mathrm{mm}]$ in the world frame. The tilting angles $[\varphi_{\mathrm{T}x}, \varphi_{\mathrm{T}y}]=\boldsymbol{x}_{\mathrm{T,r}}^{\mathsf{T}}=[20°, 20°]$ of the tool axis have to be non-zero to avoid singularities. The end effector pose is then changed consecutively by $\pm 300\,\mathrm{mm}$ for each position component of $\boldsymbol{x}_{\mathrm{t}}$ and $\pm 10°$ for the orientation $\boldsymbol{x}_{\mathrm{r}}$. A further required singularity [12] and workspace [14] analysis beyond the specific trajectory is out of this paper's scope.

The PM is floor-mounted and the base position $\boldsymbol{r}_0^{\mathsf{T}}=[r_{0x}, r_{0y}, r_{0z}]$ is *subject to optimization* with $-600\,\mathrm{mm}<r_{0x}, r_{0y}<600\,\mathrm{mm}$ and $0\,\mathrm{mm}<r_{0z}<800\,\mathrm{mm}$. The base and platform diameter are *optimization variables* as well with $1400\,\mathrm{mm}<d_{\mathrm{B}}<3000\,\mathrm{mm}$ and $200\,\mathrm{mm}<d_{\mathrm{P}}<800\,\mathrm{mm}$. Other variables are one scaling parameter, one platform coupling joint alignment angle and one to seven Denavit-Hartenberg parameters [2], representing the link's lengths and alignment angle. The latter *parameter's bounds* are computed from the task dimensions and are not vital for the results.

This leads to nine to 15 *optimization variables* in total, depending on the kinematic structure of the robot. The high number of dimensional parameters facilitates finding a non-singular setting of the robot. *Minimization objectives* in the *multi-objective* PSO are the condition number and the summed lengths of the leg chains as (debatable) indicators for general feasibility. A further elaboration on the optimization details is omitted for the sake of brevity but can be reconstructed from the published source code. The optimization with up to 200 generations and 100 individuals took 7 h to 10 h per robot (with time limit of 10 h) on a state-of-the-art Intel Xeon computing cluster system, running in parallel using a Matlab implementation and mex-compiled functions.

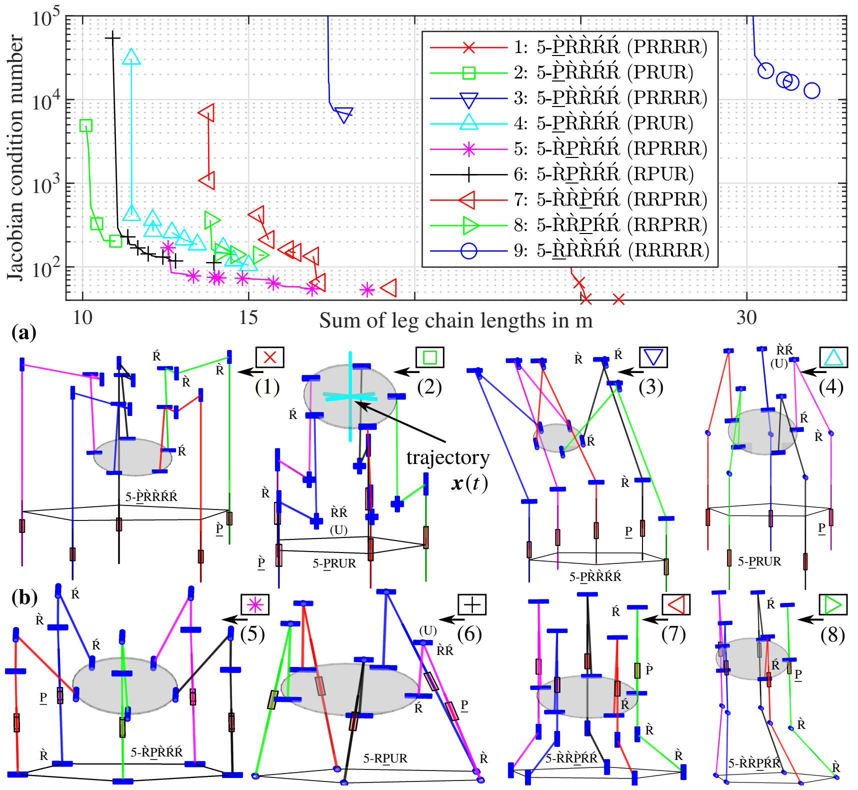

Fig. 3 Results of the synthesis: **(a)** Pareto front, **(b)** robot examples with active prismatic base coupling joint (numbers 1–4), and with revolute base joint and active prismatic joint (no. 5–8)

The results are summarized in a Pareto diagram shown in Fig. 3a. The robots in Fig. 3b are named using the established PM notation [15] together with the kinematic chain notation from [10], where joints with the same accent on R̀ or Ŕ are parallel to each other and underlined R or P are actuated. In addition, P̀R̀ and ṔŔ subchains are used to distinguish the structures as this provides more information than C in [14].

Known structures like the 5-RPUR (no. 6) [13, 1, 7] and 5-PRUR (no. 4) [14] can be reproduced. The 5-R̀PR̀ŔŔ (no. 5) emerges when separating the DoF of the universal joint of no. 6. The same goes for no. 3 emerging from no. 4. Structures like this may not be popular in literature due to the higher number of kinematic parameters, difficulty of analysis and the necessity of the dimensional synthesis beyond manual parameter tuning. Some structures are unconventional like 5-P̀R̀R̀ŔŔ (no. 1 and 2) similar to the CUR chains of the Pentapteron in [14]. Structure no. 7 is disadvantageous for technical realization due to the distal position of the actuation in the chain, despite its acceptable performance. Robots with only revolute joints like the 5-RRUR reported in [3] were not successful due to self collisions with the long limbs required for the large-scale platform motion as can be seen by high values for no. 9 in the Pareto diagram.

5 Conclusion

The presented inverse kinematics model for 3T2R PMs can be regarded as a variation of the theory of linear transformations. Using the model in a combined structural and dimensional synthesis allows to reproduce relevant symmetric 3T2R parallel robots. The practical application of symmetric 3T2R robots will be promoted by the open-source MATLAB tool which generates feasibly-dimensioned robot structures.

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