

A mixed finite element method for solving coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term

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This paper is concerned with the numerical approximation of the solution of the coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term using a mixed finite element method. The Raviart-Thomas mixed finite element method is one of the most prominent techniques to discretize the second-order wave equations; therefore, we apply this scheme for space discretization. Furthermore, an L^2 -in-space error estimate is presented for this mixed finite element approximation. Finally, the efficiency of the method is verified by a numerical example.

KEYWORDS

convergence, nonlinear wave equation, Raviart-Thomas mixed finite element, semi-discretization, 65N15 error bounds for boundary value problems involving PDEs, 65N22 Numerical solution of discretized equations for boundary value problems involving PDEs

MSC CLASSIFICATION

65M15; 65M22; 65M60; 65N30

1 | INTRODUCTION

Mixed finite element methods are among the most efficient techniques to solve second-order differential equations; therefore, this method has become an interesting research area during recent years. In particular, in Pani and Yuan,¹ a mixed finite element Galerkin method is used for a strongly damped wave equation. Liu et al.² presented an H^1 -Galerkin mixed finite element method for a class of second-order Schrödinger equation. Liu et al.³ considered a new numerical scheme based on the H^1 -Galerkin mixed finite element method for a class of second-order pseudo-hyperbolic equations. Liu et al.⁴ presented two splitting mixed finite element schemes for the pseudo-hyperbolic equation where a mixed finite element method applied for approximating the solution of nearly incompressible elasticity and Stokes equations.

Wang and Ye⁵ introduced a novel mixed finite element for the second-order elliptic equation formulated as a system of two first-order linear equations. In Monk et al.,⁶ a hybridized mixed method was presented to discretize the Helmholtz

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problem using Raviart-Thomas finite elements. Discretization of the shallow-water equations using the Raviart-Thomas finite element method was presented in Rostand and Le Roux.⁷ Ainsworth⁸ obtained an a posteriori error estimator for the lowest order Raviart-Thomas mixed finite element which provided the computable upper bounds on the error in the flux variable regardless of jumps in the material coefficients across interfaces. The numerical approximation of the displacement form of the acoustic wave equation using mixed finite elements was explained in Jenkins.⁹

In Glowinski and Lapin,¹⁰ the space-time discretization using a combination of mixed finite element method (Raviart-Thomas) and a finite difference scheme was applied for the time discretization of the wave equation. A splitting positive definite mixed finite element method was used for the second-order viscous elasticity wave equation in Liu et al.¹¹ In Parvizi and Eslahchi,¹² an extended Raviart-Thomas mixed finite element method was applied to approximate the solution of damped Boussinesq equation. In Gao et al.,¹³ the numerical solution of a viscoelasticity wave equation using mixed finite element approximations is studied. Jenkins et al.¹⁴ presented a priori error estimation for the mixed finite element displacement formulations of the acoustic wave equation. In He et al.,¹⁵ mixed finite element methods, explicit and implicit in time, for a fourth-order wave equation were studied. Liu and Li¹⁶ applied Galerkin mixed finite element methods to approximate the solution of a class of second-order pseudo-hyperbolic equations. In Bécache et al.,¹⁷ a new family of quadrangular (two-dimensional) or cubic (three-dimensional) mixed finite elements for the approximation of elastic wave equation was used. A new approach for stabilization of low-order velocity-pressure pairs for the incompressible Stokes equations using mixed finite element was presented in Bochev et al.¹⁸ For several model problems, the general inf-sup condition for the mixed finite element was reviewed in Bathe.¹⁹ In Shi et al.,²⁰ a new stabilized mixed finite element method for the Poisson equation was presented. For more details, we refer to the following paper and corresponding equations, Cahn-Hilliard equation,^{21,22} phase-field fracture,²³ drift-diffusion-Poisson system,²⁴ and damped Boussinesq equation.²⁵

A crucial step in the analysis of the mixed finite element schemes is to study the error analysis of the discretized solutions obtained from the mixed finite element method. But first, we need to verify conditions such as consistency, ellipticity, and the so-called inf-sup (or LBB) condition.²⁶ A main advantage of using the mixed finite element method is the freedom to choose suitable approximation spaces for different types of equations. Robey²⁷ shows that in the mixed framework, it is not always necessary to meet the LBB condition, and instead, the idea of a weak LBB condition is introduced in this paper.

This paper is concerned with the Raviart-Thomas mixed finite element method for the following coupled Kirchhoff-type wave equation with nonlinear boundary damping and memory term^{28,29}:

$$U_{tt}(x, t) - (1 + \|\nabla U\|_{\Omega}^2 + \|\nabla V\|_{\Omega}^2) \Delta U(x, t) - \Delta U_t(x, t) = f(x, t) \quad x \in \Omega, \quad t \in [0, T], \tag{1a}$$

$$V_{tt}(x, t) - (1 + \|\nabla U\|_{\Omega}^2 + \|\nabla V\|_{\Omega}^2) \Delta V(x, t) - \Delta V_t(x, t) = f(x, t) \quad x \in \Omega, \quad t \in [0, T], \tag{1b}$$

$$U = V = \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \Sigma_1, \tag{1c}$$

$$(1 + \|\nabla U\|_{\Omega}^2 + \|\nabla V\|_{\Omega}^2) \frac{\partial U}{\partial \nu} + \frac{\partial U_t}{\partial \nu} + U + U_t + g(t)|U_t|^p U_t = g * |U|^\gamma U \quad \text{on } \Sigma_0, \tag{1d}$$

$$(1 + \|\nabla U\|_{\Omega}^2 + \|\nabla V\|_{\Omega}^2) \frac{\partial V}{\partial \nu} + \frac{\partial V_t}{\partial \nu} + V + V_t + g(t)|V_t|^p V_t = g * |V|^\gamma V \quad \text{on } \Sigma_0, \tag{1e}$$

$$U(x, 0) = U_0(x), \quad U_t(x, 0) = U_1(x) \quad x \in \Omega, \tag{1f}$$

$$V(x, 0) = V_0(x), \quad V_t(x, 0) = V_1(x) \quad x \in \Omega, \tag{1g}$$

where U and V represent the transverse displacements, Ω is a polygon domain of \mathbb{R}^2 with a boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and Γ_0 and Γ_1 have positive measures, and $\Sigma_1 := \Gamma_1 \times [0, T]$ and $\Gamma_0 := \times [0, T]$. Here, (1a) has its origin in the mathematical description of small amplitude vibrations of an elastic string.²⁹ We define $\frac{\partial U}{\partial \nu} := \nabla U \cdot \nu$ where ν is the unit outer normal vector pointing toward Ω and

$$g * u(t) := \int_0^t g(t-r)u(r)dr, \quad \|\nabla U\|_{\Omega}^2 := \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial U}{\partial x_i}(x) \right|^2 dx,$$

and $0 \leq g \leq \alpha_1$ (a constant), $\gamma > 0, p \geq \gamma$. We have the following assumptions on the kernel g of the memory term

- $\frac{\partial g}{\partial t} \in L^\infty(0, \infty) \cap L^1(0, \infty)$,
- $g(t) \geq 0t \geq t_0$, where t_0 is a constant,
- $g(t) \leq g'(t) \leq -m_2g(t) \forall t \in [0, t_0]$,
- $g(0) = 0 \quad |g'(t)| \leq m_2g(t) \forall t \in [0, t_0]$,

for some $m_0, m_1, m_2 > 0, m_1 > 2(\gamma + 2)$, and $1 - \int_0^\infty g(r)dr$. The function f satisfies the mentioned conditions of Ferreira et al.,³⁰ which guaranties its meaningfulness. Finally, the existence and the uniqueness of the system Equation 1 can be found in Ferreira et al.³⁰ A fully discretization analysis is done by Parvizi et al.³¹

Let us define $W := \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_1\}$ and assume U_0, U_1, V_0 , and V_1 belong to $H^{3/2}(\Omega) \cap W$ should satisfy the following assumptions:

$$(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2) \Delta U_0 + \Delta U_1 = (1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2) \Delta V_0 + \Delta V_1 \quad \text{on } \Omega, \tag{2a}$$

$$U_0 = V_0 = \frac{\partial U_0}{\partial \nu} = \frac{\partial V_0}{\partial \nu} = 0 \quad \text{on } \Omega, \tag{2b}$$

$$(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2) \frac{\partial U_0}{\partial \nu} + \frac{\partial U_1}{\partial \nu} + U_0 + U_1 + g(t)|U_1|^p U_1 = 0 \quad \text{on } \Gamma_0, \tag{2c}$$

$$(1 + \|\nabla U_0\|_\Omega^2 + \|\nabla V_0\|_\Omega^2) \frac{\partial V_0}{\partial \nu} + \frac{\partial V_1}{\partial \nu} + V_0 + V_1 + g(t)|V_1|^p V_1 = 0 \quad \text{on } \Gamma_0. \tag{2d}$$

Now, we review the history and the background of (1). The first Kirchhoff equation was of the form

$$\rho h \frac{\partial^2 U}{\partial t^2} = \left(\rho_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \right) \frac{\partial^2 U}{\partial x^2} \quad x \in [0, L], \quad t \geq 0, \tag{3}$$

where this equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Equation 3 is called the wave equation of the Kirchhoff type because Kirchhoff was the first one who introduced this equation in the study of oscillation of stretched strings and plates.³² The equation

$$\rho h \frac{\partial^2 U}{\partial t^2} + C \frac{\partial U}{\partial t} + \left(\rho_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \right) \frac{\partial^2 U}{\partial x^2} = q(x, t) \quad x \in [0, L], \quad t \geq 0$$

is a mathematical Kirchhoff model of nonlinear transverse vibration, neglecting the displacements along the string's axis and averaging tension N over its length L . In this equation, ρ is the density of the string material, E is the Young modulus of the latter, C is the viscous damping parameter, ρ_0 is the initial string tension value, and $q(x, t)$ is the transverse load intensity.³³ Here, we consider the following equation³⁴:

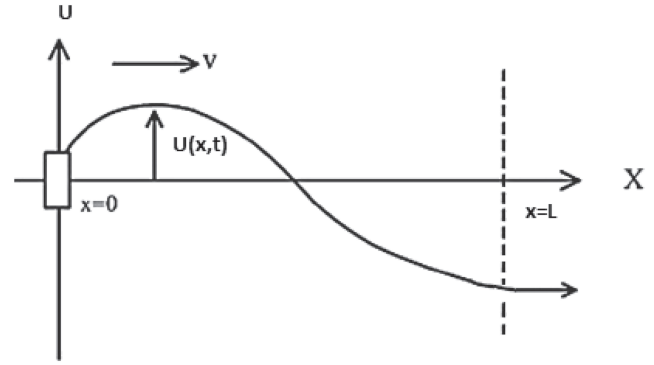
$$\begin{cases} U_{tt} - \varphi (\|\nabla U\|_\Omega^2) \Delta U - a \Delta U_t = b|U|^{\beta-2}U & \text{in } \Omega \times (0, \infty), \\ U(x, t) = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \varphi (\|\nabla U\|_\Omega^2) \frac{\partial U}{\partial \nu} + a \frac{\partial U}{\partial \nu} = g(U_t) & \text{on } \Gamma_0 \times (0, \infty), \\ U(x, 0) = U_1, \quad U_t(x, 0) = U_2 & \text{in } \Omega. \end{cases} \tag{4}$$

In the special case of (4), the dynamics of the moving string in Figure 1 can be described by

$$\rho h \frac{\partial^2 U}{\partial t^2} - a \frac{\partial^3 U}{\partial^2 x \partial t} = \left(\rho_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \right) \frac{\partial^2 U}{\partial x^2} + f \quad x \in [0, L], \quad t \geq 0, \tag{5}$$

where $U = U(x, t)$ is the lateral displacement at the space coordinate x and the time t . Additionally, E, ρ, h, ρ_0, a , and f are the Young modulus of the latter, the mass density, the cross-section of the area, the length, the initial axial tension, and the

FIGURE 1 Schematic diagram of the axially moving Kirchhoff string



resistance modulus, respectively. Physically, Equation 4 occurs in the study of vibrations of damped flexible space structures in a bounded domain in \mathbb{R}^n . The term $a \Delta u_t$ is the internal material damping of Kelvin-Voigt type of the structure. For a freely vibrating fixed-fixed string with two polarized displacements U_1 and U_2 , the Kirchhoff-Carrier equations are described by^{35,36}

$$\begin{cases} \rho h \frac{\partial^2 U_1}{\partial t^2} - (\rho_0 + N) \frac{\partial^2 U_1}{\partial x^2} = 0 & x \in [0, L] & t \geq 0, \\ \rho h \frac{\partial^2 U_2}{\partial t^2} - (\rho_0 + N) \frac{\partial^2 U_2}{\partial x^2} = 0 & x \in [0, L] & t \geq 0, \end{cases} \quad (6)$$

where N is the axial tension created by the large amplitude motions and the coupling with the transverse motion and defined by

$$N = \frac{Eh}{2L} \int_0^L \left(\left(\frac{\partial U_1}{\partial x} \right)^2 + \left(\frac{\partial U_2}{\partial x} \right)^2 \right) dx. \quad (7)$$

In these equations, ρ is density, E is Young's modulus, h is the cross section, ρ_0 is the tension, and L is the length.

Equation 1 is a nonlinear PDE with a gradient term. In order to obtain the convergence rate for the approximation technique, the efficient strategy is converting the equation into a system of two nonlinear first-order equations (using an auxiliary variable). Then, the Raviart-Thomas mixed finite element can be employed to solve the system.

The paper is organized as follows. In Section 2, the necessary definitions and lemma that will be used in the next sections are introduced. In Section 3, the semi-discrete Raviart-Thomas mixed finite element method³⁷ for solving (1) is presented in detail, and the main theorem of this paper is mentioned. In Section 4, we provide a proof for the main result mentioned in Section 3. In Section 5, we demonstrate the accuracy of theoretical results using a numerical example.

2 | PRELIMINARY AND NOTATIONS

This section is devoted to the necessary definitions and some preliminaries. We also mention a lemma regarding the regularity of the solutions of Equation 1 and introduce two required lemmas that will be used later in our analysis.

For the Hilbert spaces $H^k(\Omega)$ and $(H^k(\Omega))^2$, $k \geq 1$, the standard inner product is denoted by $(\cdot, \cdot)_{k,\Omega}$, and the associated norm is denoted by $\|\cdot\|_{k,\Omega}$. For $k = 0$, $H^k(\Omega)$ and $(H^k(\Omega))^2$ coincide with $L^2(\Omega)$ and $(L^2(\Omega))^2$, respectively, and we use $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$ to denote the corresponding inner product and norm. In order to define a weak formulation of the problem, we introduce the following spaces

$$\begin{aligned} H_0^k(\Omega) &:= \{u \in H^k(\Omega) : u|_{\Gamma_1} = 0\}, \\ L^\infty([0, T], H_0^k(\Omega)) &:= \{u : u(\cdot, t) \in H_0^k(\Omega) \text{ and } u(x, \cdot) \in L^\infty([0, T])\}, \\ L^\infty([0, T], L^2(\Omega)) &:= \{u : u(\cdot, t) \in L^2(\Omega) \text{ and } u(x, \cdot) \in L^\infty([0, T])\}, \\ L^\infty([0, T], H^k(\Omega)) &:= \{u : u(\cdot, t) \in H^k(\Omega) \text{ and } u(x, \cdot) \in L^\infty([0, T])\}, \\ \mathbf{H}(\text{div}, \Omega) &:= \{\mathbf{q} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{q} \in L^2(\Omega)\}, \end{aligned}$$

and we equip these spaces with the norms $\|\cdot\|_{k,\Omega,0}$, $\|\cdot\|_{\infty,k,\Omega,0}$, $\|\cdot\|_{\infty,0,\Omega}$, and $\|\cdot\|_{\infty,k,\Omega}$, respectively. Furthermore, $\mathbf{H}(\text{div}, \Omega)$ is endowed with the following norm:

$$\|\mathbf{q}\|_{\mathbf{H}(\text{div},\Omega)} = (\|\mathbf{q}\|^2 + \|\nabla \cdot \mathbf{q}\|^2)^{1/2}. \quad (8)$$

Lemma 1. *Under the mentioned assumptions for Equation 1, these equations have unique solutions $U, V : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that $U, V \in L^\infty([0, T], H_0^1(\Omega))$, $U_t, V_t \in L^\infty([0, T], H_0^1(\Omega))$ and $U_{tt}, V_{tt} \in L^\infty([0, T], L^2(\Omega))$.*

Proof. See Bae.²⁸

Lemma 2 (The Gronwall lemma).³⁸ *Let $\varphi \in C([0, t_1])$, $\varphi \in C^1((0, t_1))$, and there exists a constant $\alpha \in \mathbb{R}$ and a continuous function g_1 such that for $t \in (0, t_1)$, φ satisfies one of the following inequalities:*

$$\frac{d}{dt}\varphi(t) \leq \alpha\varphi(t) + g_1(t), \quad (9)$$

$$\varphi(t) \leq \varphi(0) + \int_0^t [\alpha\varphi(s) + g_1(s)] ds; \quad (10)$$

then, φ satisfies the following estimate

$$\varphi(t) \leq e^{\alpha t}\varphi(0) + \int_0^t g_1(s)e^{\alpha(t-s)} ds.$$

Lemma 2.3. (Houston et al³⁹ and Liu & Barrett⁴⁰). *Let $f \in C([0, T] \times \bar{\Omega})$ satisfies the following estimate:*

$$m_f(x-s) \leq f(t,x)x - f(t,s)s \leq M_f(x-s), \quad x \geq 0, s \geq 0, t \geq 0,$$

where m_f and M_f are positive constants. Then, there exist the constants $C_1 \geq C_2 > 0$, such that for vectors $u, v \in \mathbb{R}$ and $t \in [0, T]$, we have

$$\begin{aligned} |f(t, |v|)v - f(t, |u|)u| &\leq C_1 |v - u|, \\ C_2 |v - u|^2 &\leq (f(t, |v|)v - f(t, |u|)u) \cdot (u - v). \end{aligned}$$

3 | MIXED FINITE ELEMENT METHOD (SPATIAL DISCRETIZATION)

In this section, we present a mixed variational formulation for Equation 1 and describe the Raviart-Thomas mixed finite element scheme associated with this variational formulation. Furthermore, we mention the main theorem of the paper which is related to an a priori estimate of the error of the semi-discrete scheme.

3.1 | The weak formulation

This subsection is devoted to introduce the variational formulation of Equation 1. We denote $X := L^2(\Omega)$ and $\mathbf{M} := \mathbf{H}(\text{div}, \Omega)$; then, the mixed formulation is obtained by splitting each of (1a) and (1b) into two equations where $\mathbf{W} := \nabla U \in \mathbf{M}$ and $\mathbf{Z} := \nabla V \in \mathbf{M}$. Thus, Equation 1 can be written in the following form:

$$U_{tt}(x, t) - (1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) \nabla \cdot \mathbf{W}(x, t) - \Delta U_t(x, t) = f(x, t) \quad x \in \Omega, t \in [0, T], \quad (11a)$$

$$\mathbf{W} = \nabla Ux \in \Omega, t \in [0, T], \quad (11b)$$

$$V_{tt}(x, t) - (1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) \nabla \cdot \mathbf{Z}(x, t) - \Delta V_t(x, t) = f(x, t)x \in \Omega t \in [0, T], \quad (11c)$$

$$\mathbf{Z} = \nabla Vx \in \Omega t \in [0, T], \quad (11d)$$

$$U = V = \mathbf{W} \cdot \nu = \mathbf{Z} \cdot \nu = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \quad (11e)$$

$$(1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) \mathbf{W} \cdot \nu + \frac{\partial U_t}{\partial \nu} + U + U_t + g(t)|U_t|^p U_t = g * |U|^\gamma U \quad \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \quad (11f)$$

$$(1 + \|\mathbf{W}\|_{\Omega}^2 + \|\mathbf{Z}\|_{\Omega}^2) \mathbf{Z} \cdot \mathbf{v} + \frac{\partial V_t}{\partial \mathbf{v}} + V + V_t + g(t)|V_t|^p V_t = g * |V|^{\gamma} V \text{ on } \Sigma_0 = \Gamma_0 \times [0, T], \quad (11g)$$

$$U(x, 0) = U_0(x), \quad U_t(x, 0) = U_1(x) x \in \Omega, \quad (11h)$$

$$V(x, 0) = V_0(x), \quad V_t(x, 0) = V_1(x) x \in \Omega. \quad (11i)$$

Then, the weak formulation associated with (11) gives rise to the following variational formulation of (1): find $(U, V, \mathbf{W}, \mathbf{Z}) \in X \times X \times \mathbf{M} \times \mathbf{M}$ such that

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} U, \varphi \right)_{\Omega} + (1 + \|\mathbf{W}\|_{\Omega}^2 + \|\mathbf{Z}\|_{\Omega}^2) (\mathbf{W}, \nabla \varphi)_{\Omega} + \left(\nabla \frac{\partial}{\partial t} U, \nabla \varphi \right)_{\Omega} + (U, \varphi)_{\Gamma_0} \\ + \left(\frac{\partial}{\partial t} U, \varphi \right)_{\Gamma_0} + g(t) \left(\left| \frac{\partial}{\partial t} U \right|^p \frac{\partial}{\partial t} U, \varphi \right)_{\Gamma_0} = (g * |U|^{\gamma} U, \varphi)_{\Gamma_0} + (f, \varphi)_{\Omega} \quad \varphi \in X, \end{aligned} \quad (12a)$$

$$(\mathbf{W}, \boldsymbol{\psi})_{\Omega} = (\nabla U, \boldsymbol{\psi})_{\Omega} \boldsymbol{\psi} \in \mathbf{M}, \quad (12b)$$

$$\left(\frac{d^2}{dt^2} V, \varphi \right)_{\Omega} + (1 + \|\mathbf{W}\|_{\Omega}^2 + \|\mathbf{Z}\|_{\Omega}^2) (\mathbf{Z}, \nabla \varphi)_{\Omega} + \left(\nabla \frac{\partial}{\partial t} V, \nabla \varphi \right)_{\Omega} + (V, \varphi)_{\Gamma_0} \quad (12c)$$

$$+ \left(\frac{\partial}{\partial t} V, \varphi \right)_{\Gamma_0} + g(t) \left(\left| \frac{\partial}{\partial t} V \right|^p \frac{d}{dt} V, \varphi \right)_{\Gamma_0} = (g * |V|^{\gamma} V, \varphi)_{\Gamma_0} + (f, \varphi)_{\Omega} \varphi \in X, \quad (12d)$$

$$(\mathbf{Z}, \boldsymbol{\psi})_{\Omega} = (\nabla \mathbf{W}, \boldsymbol{\psi})_{\Omega} \boldsymbol{\psi} \in \mathbf{M}, \quad (12e)$$

$$(U(0), \varphi)_{\Omega} = (U_0, \varphi)_{\Omega}, \quad \left(\frac{\partial U}{\partial t} \Big|_{t=0}, \varphi \right)_{\Omega} = (U_1, \varphi)_{\Omega} \varphi \in X, \quad (12f)$$

$$(V(0), \varphi)_{\Omega} = (V_0, \varphi)_{\Omega}, \quad \left(\frac{\partial V}{\partial t} \Big|_{t=0}, \varphi \right)_{\Omega} = (V_1, \varphi)_{\Omega} \varphi \in X. \quad (12g)$$

3.2 | Mixed finite element discretization

Let $\mathcal{T}_h = \{T_1, \dots, T_N\}$ be a quasi-uniform triangulation of Ω with the mesh width $h := \max_{T_j \in \mathcal{T}_h} \text{diam}(T_j)$. Also, the mesh \mathcal{T}_h assume to be regular in the Ciarlet sense. Furthermore, we assume the elements are γ -shape regular in the sense that we have $\text{diam}(T_j) \leq \gamma |T_j|^{1/3}$ for all $T_j \in \mathcal{T}_h$. For $T \in \mathcal{T}_h$, let $P_k(T)$ denote the space of polynomials of degree $\leq k$ over T . For $k \geq 0$ and $T \in \mathcal{T}_h$, we define

$$RT_k(T) := \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + p_3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad : \quad p_i \in P_k(T), \quad i = 1, 2, 3 \right\}.$$

Then, we define the following spaces:

$$\begin{aligned} \mathbf{M}_h &:= \{ \mathbf{v}_h \in \mathbf{M} \quad : \quad \mathbf{v}_h|_T \in RT_k(T), \quad \forall T \in \mathcal{T}_h \}, \\ X_h &:= \{ v_h \in X \quad : \quad v_h|_T \in P_k(T), \quad \forall T \in \mathcal{T}_h \}. \end{aligned}$$

Finally, based on the variational formulation (12) and the mentioned finite element spaces, we define the Raviart-Thomas mixed finite element method for (1):

find $(u_h, v_h, \mathbf{w}_h, \mathbf{z}_h) \in X_h \times X_h \times \mathbf{M}_h \times \mathbf{M}_h$ such that

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} u_h, \varphi \right)_{\Omega} + (1 + \|\mathbf{w}_h\|_{\Omega}^2 + \|\mathbf{z}_h\|_{\Omega}^2) (\mathbf{w}_h, \nabla \varphi)_{\Omega} + \left(\nabla \frac{\partial}{\partial t} u_h, \nabla \varphi \right)_{\Omega} + (u_h, \varphi)_{\Gamma_0} \\ + \left(\frac{\partial}{\partial t} u_h, \varphi \right)_{\Gamma_0} + g(t) \left(\left| \frac{\partial}{\partial t} u_h \right|^p \frac{\partial}{\partial t} u_h, \varphi \right)_{\Gamma_0} = (g * |u_h|^{\gamma} u_h, \varphi)_{\Gamma_0} + (f, \varphi)_{\Omega} \quad \varphi \in X_h, \end{aligned} \quad (13a)$$

$$(\mathbf{w}_h, \boldsymbol{\psi})_{\Omega} = (\nabla u_h, \boldsymbol{\psi})_{\Omega} \boldsymbol{\psi} \in \mathbf{M}_h, \quad (13b)$$

$$\begin{aligned} & \left(\frac{d^2}{dt^2} v_h, \varphi \right)_\Omega + (1 + \|\mathbf{w}_h\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) (\mathbf{z}_h, \nabla \varphi)_\Omega + \left(\nabla \frac{\partial}{\partial t} v_h, \nabla \varphi \right)_\Omega + (v_h, \varphi)_{\Gamma_0} \\ & + \left(\frac{\partial}{\partial t} v_h, \varphi \right)_{\Gamma_0} + g(t) \left(\left| \frac{\partial}{\partial t} v_h \right|^p \frac{d}{dt} v_h, \varphi \right)_{\Gamma_0} = (g * |v_h|^\gamma v_h, \varphi)_{\Gamma_0} + (f, \varphi)_\Omega \quad \varphi \in X_h, \end{aligned} \quad (13c)$$

$$(\mathbf{z}_h, \boldsymbol{\psi})_\Omega = (\nabla \mathbf{w}_h, \boldsymbol{\psi})_\Omega \boldsymbol{\psi} \in \mathbf{M}_h, \quad (13d)$$

$$(u_h(0), \varphi)_\Omega = (U_0, \varphi)_\Omega, \quad \left(\frac{\partial u_h}{\partial t} \Big|_{t=0}, \varphi \right)_\Omega = (U_1, \varphi)_\Omega \varphi \in X_h, \quad (13e)$$

$$(v_h(0), \varphi)_\Omega = (V_0, \varphi)_\Omega, \quad \left(\frac{\partial v_h}{\partial t} \Big|_{t=0}, \varphi \right)_\Omega = (V_1, \varphi)_\Omega \varphi \in X_h. \quad (13f)$$

Considering suitable regularity assumptions on the exact solutions, the following theorem provides the theoretical rate of convergence for the semi-discretized Raviart-Thomas method (13).

Theorem 3.1. *Let $U, V \in L^\infty([0, T], H_0^l(\Omega))$, and $\mathbf{W}, \mathbf{Z} \in (L^\infty([0, T], H^l(\Omega)))^2$, $1 \leq l \leq k + 1$, be the unique solutions of the continuous formulation (12) and $u_h, v_h \in X_h$ and $\mathbf{w}_h, \mathbf{z}_h \in \mathbf{M}_h$ be the unique solutions of the discrete formulation (13). Then, the following error estimation holds:*

$$\begin{aligned} & \|U(t) - u_h(t)\|_\Omega^2 + \|V(t) - v_h(t)\|_\Omega^2 + \|\mathbf{W}(t) - \mathbf{w}_h(t)\|_\Omega^2 + \|\mathbf{Z}(t) - \mathbf{z}_h(t)\|_\Omega^2 \\ & \leq Ch^{2l-1} \left(\|U\|_{\infty, l, \Omega}^2 + \|V\|_{\infty, l, \Omega}^2 \right) + Ch^{2l} \left(\|\mathbf{W}\|_{\infty, l, \Omega}^2 + \|\mathbf{Z}\|_{\infty, l, \Omega}^2 \right. \\ & \quad \left. + \|\mathbf{W}\|_{\infty, l, \Omega}^4 + \|\mathbf{Z}\|_{\infty, l, \Omega}^4 \right), \quad \forall t \in [0, T], \quad 1 \leq l \leq k + 1, \end{aligned}$$

where c is a constant depending only on Ω and \mathcal{T}_h .

The mentioned theorem will be proved in Section 4. Before we start to prove the theorem, some necessary lemmas should be presented and proved.

4 | PROOF OF THE MAIN RESULTS

The main purpose of this section is to prove Theorem 3.1, which is mentioned in Section 3. Hence, for the sake of simplicity and preventing the complexity in the proof, we separate this section into three subsections and prove the required lemma and theorems. Then, we employ these results to prove the main theorem.

Lemma 4.1. *There exist projection operators $\Pi_h \times R_h : X \times \mathbf{M} \rightarrow X_h \times \mathbf{M}_h$ such that^{1,41}*

1. *The operator $\Pi_h : X \rightarrow X_h$ is a L^2 -projection, i.e.,*

$$(\nabla \cdot \mathbf{w}_h, u - \Pi_h u)_\Omega = 0 \quad \forall \mathbf{w}_h \in \mathbf{M}_h, \quad u \in X, \quad (14)$$

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq ch^l \|u\|_{l, \Omega} \quad 0 \leq l \leq k + 1 \quad u \in H^l(\Omega) \cap X. \quad (15)$$

2. *The operator $R_h : \mathbf{M} \rightarrow \mathbf{M}_h$ with $\nabla \cdot R_h \mathbf{w} = \Pi_h \nabla \cdot \mathbf{w}$ satisfies*

$$(\nabla \cdot (\mathbf{w} - R_h \mathbf{w}), \varphi)_\Omega = 0 \quad \forall \varphi \in X_h \quad \mathbf{w} \in \mathbf{M}, \quad (16)$$

$$\|\mathbf{w} - R_h \mathbf{w}\|_\Omega \leq ch^l \|\mathbf{w}\|_{l, \Omega} \quad 0 \leq l \leq k + 1 \quad \mathbf{w} \in \mathbf{M} \cap (H^l(\Omega))^2, \quad (17)$$

$$\|\nabla \cdot (\mathbf{w} - R_h \mathbf{w})\|_\Omega \leq ch^l \|\nabla \cdot \mathbf{w}\|_{l, \Omega} \quad 0 \leq l \leq k + 1 \quad \mathbf{w} \in \mathbf{M} \cap (H^{l+1}(\Omega))^2, \quad (18)$$

where c is a constant depending only on Ω and \mathcal{T}_h .

Let us define the error terms of U, V, \mathbf{W} , and \mathbf{Z} as

$$\begin{aligned} e_{1,h} &:= u_h - \Pi_h U & e_{2,h} &:= v_h - \Pi_h V, \\ e_{3,h} &:= \mathbf{w}_h - R_h \mathbf{W} & e_{4,h} &:= \mathbf{z}_h - R_h \mathbf{Z}. \end{aligned}$$

Subtracting (12a–12d) from (13a–13d) gives us

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} e_{1,h}, \varphi\right)_\Omega + (1 + \|\mathbf{w}_h\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) (\mathbf{w}_h, \nabla \varphi)_\Omega - (1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) (\mathbf{W}, \nabla \varphi)_\Omega \\ &+ \left(\nabla \frac{\partial}{\partial t} e_{1,h}, \nabla \varphi\right)_\Omega + (e_{1,h}, \varphi)_{\Gamma_0} + \left(\frac{\partial}{\partial t} e_{1,h}, \varphi\right)_{\Gamma_0} + g(t) \left(\left|\frac{\partial}{\partial t} u_h\right|^p \frac{\partial}{\partial t} u_h, \varphi\right)_{\Gamma_0} \\ &- g(t) \left(\left|\frac{\partial}{\partial t} U\right|^p \frac{\partial}{\partial t} U, \varphi\right)_{\Gamma_0} = (g * |u_h|^\gamma u_h, \varphi)_{\Gamma_0} - (g * |U|^\gamma U, \varphi)_{\Gamma_0} \\ &+ \left(\frac{\partial^2}{\partial t^2} (U - \Pi_h U), \varphi\right)_\Omega + \left(\nabla \frac{\partial}{\partial t} (U - \Pi_h U), \nabla \varphi\right)_\Omega \\ &+ (U - \Pi_h U, \varphi)_{\Gamma_0} + \left(\frac{\partial}{\partial t} (U - \Pi_h U), \varphi\right)_{\Gamma_0}, \quad \forall \varphi \in X_h, \end{aligned} \tag{19}$$

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} e_{2,h}, \varphi\right)_\Omega + (1 + \|\mathbf{w}_h\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) (\mathbf{z}_h, \nabla \varphi)_\Omega - (1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) (\mathbf{Z}, \nabla \varphi)_\Omega \\ &+ \left(\nabla \frac{\partial}{\partial t} e_{2,h}, \nabla \varphi\right)_\Omega + (e_{2,h}, \varphi)_{\Gamma_0} + \left(\frac{\partial}{\partial t} e_{2,h}, \varphi\right)_{\Gamma_0} + g(t) \left(\left|\frac{\partial}{\partial t} v_h\right|^p \frac{\partial}{\partial t} v_h, \varphi\right)_{\Gamma_0} \\ &- g(t) \left(\left|\frac{\partial}{\partial t} V\right|^p \frac{\partial}{\partial t} V, \varphi\right)_{\Gamma_0} = (g * |v_h|^\gamma v_h, \varphi)_{\Gamma_0} - (g * |V|^\gamma V, \varphi)_{\Gamma_0} \\ &+ \left(\frac{\partial^2}{\partial t^2} (V - \Pi_h V), \varphi\right)_\Omega + \left(\nabla \frac{\partial}{\partial t} (V - \Pi_h V), \nabla \varphi\right)_\Omega \\ &+ (V - \Pi_h V, \varphi)_{\Gamma_0} + \left(\frac{\partial}{\partial t} (V - \Pi_h V), \varphi\right)_{\Gamma_0}, \quad \forall \varphi \in X_h, \end{aligned} \tag{20}$$

$$(e_{3,h}, \boldsymbol{\psi})_\Omega = (\nabla e_{1,h}, \boldsymbol{\psi})_\Omega + ((\Pi_h U - U), \nabla \cdot \boldsymbol{\psi})_\Omega + (\mathbf{W} - R_h \mathbf{W}, \boldsymbol{\psi})_\Omega, \quad \forall \boldsymbol{\psi} \in \mathbf{M}_h, \tag{21}$$

$$(e_{4,h}, \boldsymbol{\psi})_\Omega = (\nabla e_{2,h}, \boldsymbol{\psi})_\Omega + ((\Pi_h V - V), \nabla \cdot \boldsymbol{\psi})_\Omega + (\mathbf{Z} - R_h \mathbf{Z}, \boldsymbol{\psi})_\Omega, \quad \forall \boldsymbol{\psi} \in \mathbf{M}_h. \tag{22}$$

Plugging $\varphi = \frac{\partial}{\partial t} e_{1,h}$ into (19) and $\varphi = \frac{\partial}{\partial t} e_{2,h}$ into (20) as well as adding these equations together yields

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} e_{1,h}, \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\frac{\partial^2}{\partial t^2} e_{2,h}, \frac{\partial}{\partial t} e_{2,h}\right)_\Omega + \left(\nabla \frac{\partial}{\partial t} e_{1,h}, \nabla \frac{\partial}{\partial t} e_{1,h}\right)_\Omega + \left(\nabla \frac{\partial}{\partial t} e_{2,h}, \nabla \frac{\partial}{\partial t} e_{2,h}\right)_\Omega \\ &+ \left(e_{1,h}, \frac{\partial}{\partial t} e_{1,h}\right)_{\Gamma_0} + \left(e_{2,h}, \frac{\partial}{\partial t} e_{2,h}\right)_{\Gamma_0} + \left(\frac{\partial}{\partial t} e_{1,h}, \frac{\partial}{\partial t} e_{1,h}\right)_{\Gamma_0} + \left(\frac{\partial}{\partial t} e_{2,h}, \frac{\partial}{\partial t} e_{2,h}\right)_{\Gamma_0} \\ &+ L_1 = L_2 + L_3 + L_4 + L_5 + L_6, \end{aligned} \tag{23}$$

where

$$\begin{aligned}
L_1 &:= (1 + \|\mathbf{w}_h\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) \left(\mathbf{w}_h + \mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - (1 + \|\mathbf{W}\|_\Omega^2 + \|\mathbf{Z}\|_\Omega^2) \left(\mathbf{W} + \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega, \\
L_2 &:= g(t) \left(\left(\left| \frac{\partial}{\partial t} u_h \right|^p \frac{\partial}{\partial t} u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} U \right|^p \frac{\partial}{\partial t} U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) + g(t) \left(\left(\left| \frac{\partial}{\partial t} v_h \right|^p \frac{\partial}{\partial t} v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right. \\
&\quad \left. - \left(\left| \frac{\partial}{\partial t} V \right|^p \frac{\partial}{\partial t} V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right), \\
L_3 &:= \left(g * |u_h|^\gamma u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} + \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \\
&\quad - \left(g * |U|^\gamma U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} + \left(g * |v_h|^\gamma v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \\
&\quad + \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(g * |V|^\gamma V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0}, \\
L_4 &:= \left(\frac{\partial^2}{\partial t^2} (U - \Pi_h U), \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \left(\frac{\partial^2}{\partial t^2} (V - \Pi_h V), \frac{\partial}{\partial t} e_{2,h} \right)_\Omega, \\
L_5 &:= \left(\nabla \frac{\partial}{\partial t} (U - \Pi_h U), \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \left(\nabla \frac{\partial}{\partial t} (V - \Pi_h V), \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega, \\
L_6 &:= \left(U - \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} + \left(V - \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} + \left(\frac{\partial}{\partial t} (U - \Pi_h U), \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \\
&\quad + \left(\frac{\partial}{\partial t} (V - \Pi_h V), \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0}.
\end{aligned}$$

From now on, we consider the following regularity assumptions.

Assumptions 4.1. We assume $U(t), V(t) \in H^l(\Omega)$ and $\mathbf{W}(t), \mathbf{Z}(t) \in (H^l(\Omega))^2$, $1 \leq l \leq k + 1$, are the exact solutions of (11) and $u_h(t), v_h(t) \in X_h$ and $\mathbf{w}_h, \mathbf{z}_h \in \mathbf{M}_h$ are the discretized solutions of (11) obtained from (13). Additionally, we assume $U_0, U_1, V_0, V_1 \in H^l(\Omega)$, $1 \leq l \leq k + 1$.

In order to prove Theorem 3.1, the challenging part is to deal with L_1 and L_2 . Therefore, the following subsections are devoted to obtain upper bounds for L_1 and L_2 .

4.1 | An upper bound for L_1

Lemma 4.2. Using the definition of $e_{i,h}$, $i = 1, \dots, 4$, we get the following estimates:

$$\frac{d}{dt} \|e_{3,h}\|_\Omega^2 = \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h \mathbf{W}), e_{3,h} \right)_\Omega,$$

and

$$\frac{d}{dt} \|e_{4,h}\|_\Omega^2 = \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h \mathbf{Z}), e_{4,h} \right)_\Omega.$$

Proof. Taking derivative with respect to t from (21) and (22), and putting $\boldsymbol{\psi} = e_{3,h}$ and $\boldsymbol{\psi} = e_{4,h}$ into these equations as test functions, respectively, yield

$$\frac{d}{dt} \|e_{3,h}\|_\Omega^2 = \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h \mathbf{W}), e_{3,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\Pi_h U - U), \nabla \cdot e_{3,h} \right)_\Omega,$$

and

$$\frac{d}{dt} \|e_{4,h}\|_\Omega^2 = \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h \mathbf{Z}), e_{4,h} \right)_\Omega + \left(\frac{\partial}{\partial t} (\Pi_h V - V), \nabla \cdot e_{4,h} \right)_\Omega.$$

Considering the orthogonality estimate from Equation 14, we have

$$\left(\frac{\partial}{\partial t} (\Pi_h U - U), \nabla \cdot e_{3,h} \right)_\Omega = 0,$$

$$\left(\frac{\partial}{\partial t} (\Pi_h V - V), \nabla \cdot e_{4,h} \right)_\Omega = 0.$$

Then, the combination of (26)–(29) gives us the desired result.

Lemma 4.3. Let U , V , \mathbf{W} , and \mathbf{Z} satisfy the regularity assumptions of Assumption 4.1. Also, let us define

$$L_{1,0} := \|\mathbf{w}_h\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|\mathbf{z}_h\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h \mathbf{Z}\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega;$$

then, we have

$$L_{1,0} \geq \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_\Omega^4 + \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_\Omega^4 + \mathbf{M}_0,$$

where

$$\begin{aligned} 3 |\mathbf{M}_0| &\leq (c_1 h^{2l} + c_2) \left(\|e_{3,h}\|_\Omega^2 + \|e_{3,h}\|_\Omega^4 \right) + (c_3 h^{2l} + c_4) \left(\|e_{4,h}\|_\Omega^2 + \|e_{4,h}\|_\Omega^4 \right) \\ &\quad + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 + c_5 (h^{2l} + h^{6l}), \end{aligned} \quad (31)$$

where c_i , $i = 1, \dots, 4$ are constants depending only on Ω and \mathcal{T}_h .

Proof. First, we should note that $L_{1,0}$ can be written in the following form:

$$\begin{aligned} L_{1,0} &= \|\mathbf{w}_h\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|\mathbf{w}_h\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad - \|\mathbf{w}_h\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|R_h \mathbf{W}\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad + \|\mathbf{z}_h\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h \mathbf{Z}\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \|\mathbf{z}_h\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\ &\quad - \|\mathbf{z}_h\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \|R_h \mathbf{Z}\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h \mathbf{Z}\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega. \end{aligned} \quad (32)$$

Then, rearranging the terms of (32) gives rise to

$$\begin{aligned} L_{1,0} &= (\|\mathbf{w}_h\|_\Omega^2 + \|R_h \mathbf{W}\|_\Omega^2) \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad + (\|\mathbf{w}_h\|_\Omega^2 - \|R_h \mathbf{W}\|_\Omega^2) \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + (\|\mathbf{z}_h\|_\Omega^2 + \|R_h \mathbf{Z}\|_\Omega^2) \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\ &\quad - \|R_h \mathbf{Z}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + (\|\mathbf{z}_h\|_\Omega^2 - \|R_h \mathbf{Z}\|_\Omega^2) \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega. \end{aligned} \quad (33)$$

Employing Lemma 4.2 and using the following estimates

$$\|e_{3,h}\|_\Omega^2 \frac{d}{dt} \|e_{3,h}\|_\Omega^2 = \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_\Omega^4, \quad \|e_{4,h}\|_\Omega^2 \frac{d}{dt} \|e_{4,h}\|_\Omega^2 = \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_\Omega^4,$$

result in the following inequality:

$$\begin{aligned} L_{1,0} &\geq \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_\Omega^4 - \|R_h \mathbf{W}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + (\|\mathbf{w}_h\|_\Omega^2 - \|R_h \mathbf{W}\|_\Omega^2) \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad + \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_\Omega^4 - \|R_h \mathbf{Z}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + (\|\mathbf{z}_h\|_\Omega^2 - \|R_h \mathbf{Z}\|_\Omega^2) \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\ &\quad + (\|R_h \mathbf{W}\|_\Omega^2 + \|\mathbf{w}_h\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h \mathbf{W}), e_{3,h} \right)_\Omega + (\|R_h \mathbf{Z}\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h \mathbf{Z}), e_{4,h} \right)_\Omega. \end{aligned} \quad (34)$$

In order to simplify (34), we rewrite it as

$$L_{1,0} \geq \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_\Omega^4 + \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_\Omega^4 + \mathbf{M}_0, \quad (35)$$

where

$$\begin{aligned}
3\mathbf{M}_0 := & (\|\mathbf{W}\|_\Omega^2 - \|R_h\mathbf{W}\|_\Omega^2) \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + (\|\mathbf{w}_h\|_\Omega^2 - \|R_h\mathbf{W}\|_\Omega^2) \left(R_h\mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\
& + (\|\mathbf{w}_h\|_\Omega^2 - \|R_h\mathbf{W}\|_\Omega^2) \left(\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|\mathbf{W}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\
& + (\|\mathbf{Z}\|_\Omega^2 - \|R_h\mathbf{Z}\|_\Omega^2) \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + (\|\mathbf{z}_h\|_\Omega^2 - \|R_h\mathbf{Z}\|_\Omega^2) \left(R_h\mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\
& + (\|\mathbf{z}_h\|_\Omega^2 - \|R_h\mathbf{Z}\|_\Omega^2) \left(\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|\mathbf{Z}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\
& + (\|R_h\mathbf{W}\|_\Omega^2 + \|\mathbf{w}_h\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h\mathbf{W}), e_{3,h} \right)_\Omega + (\|R_h\mathbf{Z}\|_\Omega^2 + \|\mathbf{z}_h\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h\mathbf{Z}), e_{4,h} \right)_\Omega \\
& + (\|R_h\mathbf{W}\|_\Omega^2 + 2\|\mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h\mathbf{W}), e_{3,h} \right)_\Omega - (\|R_h\mathbf{W}\|_\Omega^2 + 2\|\mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h\mathbf{W}), e_{3,h} \right)_\Omega \\
& + (\|R_h\mathbf{Z}\|_\Omega^2 + 2\|\mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h\mathbf{Z}), e_{4,h} \right)_\Omega - (\|R_h\mathbf{Z}\|_\Omega^2 + 2\|\mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h\mathbf{Z}), e_{4,h} \right)_\Omega. \tag{36}
\end{aligned}$$

Finally, applying the Cauchy-Schwarz and Young inequalities and approximate properties of Π_h and R_h from Lemma 4.1 yields

$$\begin{aligned}
|\mathbf{M}_0| \leq & c_1 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \left(\|e_{3,h}\|_\Omega^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 \right) + c_2 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|e_{3,h}\|_\Omega^4 + 8 \|\mathbf{W}\|_\Omega^2 \|e_{3,h}\|_\Omega^4 + 8 \|\mathbf{W}\|_\Omega^4 \|e_{3,h}\|_\Omega^2 \\
& + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + c_3 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \left(\|e_{4,h}\|_\Omega^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \right) + c_4 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|e_{4,h}\|_\Omega^4 + 8 \|\mathbf{Z}\|_\Omega^4 \|e_{4,h}\|_\Omega^2 \\
& + 8 \|\mathbf{Z}\|_\Omega^2 \|e_{4,h}\|_\Omega^4 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 + c_5 h^{2l} \|e_{3,h}\|_\Omega^4 \left\| \frac{\partial}{\partial t} \mathbf{W} \right\|_{l,\Omega}^2 + \frac{1}{4} \|e_{3,h}\|_\Omega^4 + 2c_6 h^{6l} \|\mathbf{W}\|_{l,\Omega}^2 \left\| \frac{\partial}{\partial t} \mathbf{W} \right\|_{l,\Omega}^2 \\
& + c_7 h^{2l} \|e_{3,h}\|_\Omega^2 \|\mathbf{W}\|_{l,\Omega}^2 + c_8 h^{2l} \|e_{4,h}\|_\Omega^4 \left\| \frac{\partial}{\partial t} \mathbf{Z} \right\|_{l,\Omega}^2 + \frac{1}{4} \|e_{4,h}\|_\Omega^4 + 2c_9 h^{6l} \|\mathbf{Z}\|_{l,\Omega}^2 \left\| \frac{\partial}{\partial t} \mathbf{Z} \right\|_{l,\Omega}^2 \\
& + c_{10} h^{2l} \|e_{4,h}\|_\Omega^2 \|\mathbf{Z}\|_{l,\Omega}^2, \tag{37}
\end{aligned}$$

which gives us the desired results. \square

Lemma 4.4. Let us denote $L_{1,2}$ by

$$\begin{aligned}
L_{1,2} := & \|\mathbf{z}_h\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h\mathbf{Z}\|_\Omega^2 \left(R_h\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\
& + \|\mathbf{w}_h\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h\mathbf{W}\|_\Omega^2 \left(R_h\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega;
\end{aligned}$$

then, under Assumption 4.1, $L_{1,2}$ can be written in the following form:

$$L_{1,2} = \frac{d}{dt} \left(\|e_{3,h}\|_\Omega^2 \|e_{4,h}\|_\Omega^2 \right) + L_{1,2,0} + L_{1,2,1}, \tag{38}$$

where $L_{1,2,0}$ and $L_{1,2,1}$ satisfy the following properties:

$$|L_{1,2,0}| \leq c_1 h^{2l} \|e_{4,h}\|_\Omega^4 + c_2 \|e_{4,h}\|_\Omega^2 + c_3 h^{4l} \|e_{3,h}\|_\Omega^2 + c_4 \|e_{3,h}\|_\Omega^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 \tag{39}$$

and

$$\begin{aligned}
|L_{1,2,1}| \leq & (c_1 h^{2l} + c_2) \left(\|e_{3,h}\|_\Omega^4 + \|e_{3,h}\|_\Omega^2 \right) + (c_3 h^{2l} + c_4) \left(\|e_{4,h}\|_\Omega^2 + \|e_{4,h}\|_\Omega^4 \right) \\
& + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 + c_5 h^{6l}. \tag{40}
\end{aligned}$$

Proof. Using the definition of $L_{1,2}$ as well as adding and subtracting some terms, one arrives at

$$\begin{aligned} L_{1,2} &= \|\mathbf{z}_h\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|R_h \mathbf{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|\mathbf{z}_h\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad - \|R_h \mathbf{Z}\|_\Omega^2 \left(\mathbf{w}_h, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega + \|\mathbf{w}_h\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \|R_h \mathbf{W}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\ &\quad + \|\mathbf{w}_h\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(\mathbf{z}_h, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + \|R_h \mathbf{Z}\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad + \|R_h \mathbf{W}\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|R_h \mathbf{Z}\|_\Omega^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|R_h \mathbf{W}\|_\Omega^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega. \end{aligned}$$

Substituting (26) and (27) from Lemma 4.2 into the above formula gives us

$$\begin{aligned} 3L_{1,2} &= (\|\mathbf{z}_h\|_\Omega^2 + \|R_h \mathbf{Z}\|_\Omega^2) \frac{d}{dt} \|e_{3,h}\|_\Omega^2 + (\|\mathbf{z}_h\|_\Omega^2 - \|R_h \mathbf{Z}\|_\Omega^2) \left((R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h})_\Omega \right. \\ &\quad \left. + (\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h})_\Omega \right) - (\|R_h \mathbf{Z}\|_\Omega^2 - \|\mathbf{Z}\|_\Omega^2) \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|\mathbf{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega \\ &\quad + (\|\mathbf{w}_h\|_\Omega^2 + \|R_h \mathbf{W}\|_\Omega^2) \frac{d}{dt} \|e_{4,h}\|_\Omega^2 + (\|\mathbf{w}_h\|_\Omega^2 - \|R_h \mathbf{W}\|_\Omega^2) \left((R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h})_\Omega \right. \\ &\quad \left. - (\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h})_\Omega \right) - (\|R_h \mathbf{W}\|_\Omega^2 - \|\mathbf{W}\|_\Omega^2) \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega - \|\mathbf{W}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega \\ &\quad + (\|\mathbf{z}_h\|_\Omega^2 + \|R_h \mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (R_h \mathbf{W} - \mathbf{W}), e_{3,h} \right)_\Omega + (\|\mathbf{w}_h\|_\Omega^2 + \|R_h \mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (R_h \mathbf{Z} - \mathbf{Z}), e_{4,h} \right)_\Omega \\ &\geq \|e_{4,h}\|_\Omega^2 \frac{d}{dt} \|e_{3,h}\|_\Omega^2 + \|e_{3,h}\|_\Omega^2 \frac{d}{dt} \|e_{4,h}\|_\Omega^2 + L_{1,2,0} + L_{1,2,1} \\ &= \frac{d}{dt} \left(\|e_{3,h}\|_\Omega^2 \|e_{4,h}\|_\Omega^2 \right) + L_{1,2,0} + L_{1,2,1}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} 3L_{1,2,0} &:= (\|\mathbf{z}_h\|_\Omega^2 - \|R_h \mathbf{Z}\|_\Omega^2) \left((R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h})_\Omega + (\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h})_\Omega \right) \\ &\quad - (\|R_h \mathbf{Z}\|_\Omega^2 - \|\mathbf{Z}\|_\Omega^2) \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega - \|\mathbf{Z}\|_\Omega^2 \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_\Omega, \end{aligned} \tag{42}$$

and

$$\begin{aligned} L_{1,2,1} &:= (\|\mathbf{w}_h\|_\Omega^2 - \|R_h \mathbf{W}\|_\Omega^2) \left((R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h})_\Omega + (\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h})_\Omega \right) \\ &\quad - (\|R_h \mathbf{W}\|_\Omega^2 - \|\mathbf{W}\|_\Omega^2) \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega + (\|\mathbf{z}_h\|_\Omega^2 + \|R_h \mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (R_h \mathbf{W} - \mathbf{W}), e_{3,h} \right)_\Omega \\ &\quad + (\|\mathbf{w}_h\|_\Omega^2 + \|R_h \mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (R_h \mathbf{Z} - \mathbf{Z}), e_{4,h} \right)_\Omega + (\|R_h \mathbf{W}\|_\Omega^2 + 2\|\mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h \mathbf{W}), e_{3,h} \right)_\Omega \\ &\quad - (\|R_h \mathbf{W}\|_\Omega^2 + 2\|\mathbf{W}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{W} - R_h \mathbf{W}), e_{3,h} \right)_\Omega + (\|R_h \mathbf{Z}\|_\Omega^2 + 2\|\mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h \mathbf{Z}), e_{4,h} \right)_\Omega \\ &\quad - (\|R_h \mathbf{Z}\|_\Omega^2 + 2\|\mathbf{Z}\|_\Omega^2) \left(\frac{\partial}{\partial t} (\mathbf{Z} - R_h \mathbf{Z}), e_{4,h} \right)_\Omega \\ &\quad - \|\mathbf{W}\|_\Omega^2 \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_\Omega. \end{aligned} \tag{43}$$

Employing the Cauchy-Schwarz and Young inequalities and approximate property of R_h from Lemma 4.1, we conclude

$$\begin{aligned} |L_{1,2,0}| &\leq c_1 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|e_{4,h}\|_\Omega^4 + 16 \|\mathbf{W}\|_\Omega^2 \|e_{4,h}\|_\Omega^2 + c_2 h^{4l} \|\mathbf{Z}\|_{l,\Omega}^4 \|e_{3,h}\|_\Omega^2 \\ &\quad + 16 \|\mathbf{Z}\|_\Omega^4 \|e_{3,h}\|_\Omega^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2, \end{aligned} \tag{44}$$

and

$$\begin{aligned}
\|L_{1,2,1}\|_{\Omega}^2 &\leq c_1 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|e_{3,h}\|_{\Omega}^4 + 16 \|\mathbf{Z}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^4 + c_3 h^{4l} \|\mathbf{W}\|_{l,\Omega}^4 \|e_{4,h}\|_{\Omega}^2 + 16 \|\mathbf{Z}\|_{\Omega}^4 \|e_{4,h}\|_{\Omega}^2 \\
&\quad + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 + c_3 h^{2l} \|e_{3,h}\|_{\Omega}^4 \left\| \frac{\partial}{\partial t} \mathbf{W} \right\|_{l,\Omega}^2 + \frac{1}{4} \|e_{3,h}\|_{\Omega}^4 + c_4 h^{6l} \|\mathbf{W}\|_{l,\Omega}^2 \left\| \frac{\partial}{\partial t} \mathbf{W} \right\|_{l,\Omega}^2 \\
&\quad + c_5 h^{2l} \|e_{3,h}\|_{\Omega}^2 \|\mathbf{W}\|_{l,\Omega}^2 + c_6 h^{2l} \|e_{4,h}\|_{\Omega}^4 \left\| \frac{\partial}{\partial t} \mathbf{Z} \right\|_{l,\Omega}^2 + \frac{1}{4} \|e_{4,h}\|_{\Omega}^4 + c_7 h^{6l} \|\mathbf{Z}\|_{l,\Omega}^2 \left\| \frac{\partial}{\partial t} \mathbf{Z} \right\|_{l,\Omega}^2 \\
&\quad + c_8 h^{2l} \|e_{4,h}\|_{\Omega}^2 \|\mathbf{Z}\|_{l,\Omega}^2.
\end{aligned} \tag{45}$$

The combination of (41), (44), and (45) completes the proof. \square

Theorem 4.1. Let $\mathbf{W}(t), \mathbf{Z}(t) \in (H^l(\Omega))^2$, $1 \leq l \leq k + 1$; then, L_1 satisfies the following inequality:

$$L_1 \geq \frac{d}{dt} \|e_{3,h}\|_{\Omega}^2 + \frac{d}{dt} \|e_{4,h}\|_{\Omega}^2 + \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_{\Omega}^4 + \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_{\Omega}^4 + \frac{d}{dt} \left(\|e_{3,h}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^2 \right) \& + L_{1,1} + L_{1,2,0} + L_{1,2,1} + L_{1,3} + \mathbf{M}_0,$$

where

$$L_{1,1} := \|R_h \mathbf{W}\|_{\Omega}^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} - \|\mathbf{W}\|_{\Omega}^2 \left(\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} \& + \|R_h \mathbf{W}\|_{\Omega}^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} - \|\mathbf{W}\|_{\Omega}^2 \left(\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega},$$

$$L_{1,3} := \|R_h \mathbf{Z}\|_{\Omega}^2 \left(R_h \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} - \|\mathbf{Z}\|_{\Omega}^2 \left(\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} \& + \|R_h \mathbf{Z}\|_{\Omega}^2 \left(R_h \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} - \|\mathbf{Z}\|_{\Omega}^2 \left(\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega},$$

and $L_{1,2,0}$ and $L_{1,2,1}$ are defined in (42) and (43), respectively, and \mathbf{M}_0 can be found in (36). Furthermore, $L_{1,1}$ and $L_{1,3}$ satisfy the following estimates:

$$|L_{1,1}| \leq c_1 h^{2l} + c_2 h^{4l} + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2, \tag{46}$$

and

$$|L_{1,3}| \leq c_1 h^{2l} + c_2 h^{4l} + c_3 h^{6l} + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2. \tag{47}$$

Proof. Rearranging L_1 gives us

$$L_1 = \left(e_{3,h}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + \left(e_{4,h}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} + L_{1,0} + L_{1,1} + L_{1,2} + L_{1,3} + L_{1,4}, \tag{48}$$

where $L_{1,0}$ and $L_{1,2}$ are defined in Lemmas 4.3 and 4.4, respectively, and we denote $L_{1,4}$ by

$$L_{1,4} := \left(R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + \left(R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega}.$$

Considering (48) and Lemmas 4.2–4.4, L_1 satisfies the following estimate:

$$\begin{aligned}
L_1 &\geq \frac{d}{dt} \|e_{3,h}\|_{\Omega}^2 + \frac{d}{dt} \|e_{4,h}\|_{\Omega}^2 + \frac{1}{2} \frac{d}{dt} \|e_{3,h}\|_{\Omega}^4 + \frac{1}{2} \frac{d}{dt} \|e_{4,h}\|_{\Omega}^4 + \frac{d}{dt} \left(\|e_{3,h}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^2 \right) \\
&\quad + L_{1,1} + L_{1,2,0} + L_{1,2,1} + L_{1,3} + \mathbf{M}_0.
\end{aligned} \tag{49}$$

Adding and subtracting some terms allow us to rewrite $L_{1,1}$ as

$$\begin{aligned} L_{1,1} &= (\|R_h \mathbf{W}\|_{\Omega}^2 - \|\mathbf{W}\|_{\Omega}^2) \left(R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + \|\mathbf{W}\|_{\Omega}^2 \left(R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} \\ &\quad + (\|R_h \mathbf{W}\|_{\Omega}^2 - \|\mathbf{W}\|_{\Omega}^2) \left(\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + (\|R_h \mathbf{W}\|_{\Omega}^2 - \|\mathbf{W}\|_{\Omega}^2) \left(R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} \\ &\quad + \|\mathbf{W}\|_{\Omega}^2 \left(R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} + (\|R_h \mathbf{W}\|_{\Omega}^2 - \|\mathbf{W}\|_{\Omega}^2) \left(\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega}. \end{aligned}$$

Applying the Cauchy-Schwarz and Young inequalities and Equation 18, we conclude the following estimate for $L_{1,1}$:

$$\begin{aligned} 3 |L_{1,1}| &\leq \|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 \left(\|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 + \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 \right) + 4 \|\mathbf{W}\|_{\Omega}^4 \|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 \\ &\quad + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 \left(\|\mathbf{W}\|_{\Omega}^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 \right) \\ &\quad + \|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 \left(\|R_h \mathbf{Z} - \mathbf{Z}\|_{\Omega}^2 + \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 \right) + 4 \|\mathbf{W}\|_{\Omega}^4 \|R_h \mathbf{Z} - \mathbf{Z}\|_{\Omega}^2 \\ &\quad + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 + \|R_h \mathbf{W} - \mathbf{W}\|_{\Omega}^2 \left(\|\mathbf{Z}\|_{\Omega}^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 \right) \\ &\leq c_1 h^{4l} \|\mathbf{W}\|_{l,\Omega}^4 + c_1 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|\mathbf{W}\|_{\Omega}^4 + c_1 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|\mathbf{W}\|_{\Omega}^2 + c_2 h^{2l} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 \\ &\quad + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + c_1 h^{4l} \|\mathbf{W}\|_{l,\Omega}^2 \|\mathbf{Z}\|_{l,\Omega}^2 + c_1 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|\mathbf{W}\|_{\Omega}^4 + c_1 h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|\mathbf{Z}\|_{\Omega}^4 \\ &\quad + c_2 h^{2l} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2. \end{aligned} \quad (50)$$

Adding and subtracting some terms allow us to rewrite $L_{1,3}$ in the following form:

$$\begin{aligned} L_{1,3} &= (\|R_h \mathbf{Z}\|_{\Omega}^2 - \|\mathbf{Z}\|_{\Omega}^2) \left(R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + (\|R_h \mathbf{Z}\|_{\Omega}^2 - \|\mathbf{Z}\|_{\Omega}^2) \left(\mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} \\ &\quad + \|\mathbf{Z}\|_{\Omega}^2 \left(R_h \mathbf{W} - \mathbf{W}, \nabla \frac{\partial}{\partial t} e_{1,h} \right)_{\Omega} + (\|R_h \mathbf{Z}\|_{\Omega}^2 - \|\mathbf{Z}\|_{\Omega}^2) \left(R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} \\ &\quad + \|\mathbf{Z}\|_{\Omega}^2 \left(R_h \mathbf{Z} - \mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega} + (\|R_h \mathbf{Z}\|_{\Omega}^2 - \|\mathbf{Z}\|_{\Omega}^2) \left(\mathbf{Z}, \nabla \frac{\partial}{\partial t} e_{2,h} \right)_{\Omega}. \end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities and the approximate property of R_h from Lemma 4.1, the following inequality can be deduced:

$$\begin{aligned} |L_{1,3}| &\leq c_1 h^{6l} \|\mathbf{Z}\|_{l,\Omega}^4 \|\mathbf{W}\|_{l,\Omega}^2 + c_2 h^{4l} \|\mathbf{Z}\|_{l,\Omega}^4 \|\mathbf{W}\|_{\Omega}^2 + c_3 h^{4l} \|\mathbf{Z}\|_{\Omega}^4 \|\mathbf{W}\|_{l,\Omega}^2 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 \\ &\quad + c_4 h^{6l} \|\mathbf{Z}\|_{l,\Omega}^6 + c_5 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|\mathbf{Z}\|_{\Omega}^2 + c_6 h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|\mathbf{Z}\|_{\Omega}^4 + \frac{1}{16} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2. \end{aligned} \quad (51)$$

Finally, the combination of (49), (50), and (51) completes the proof. \square

4.2 | An upper bound for L_2

Theorem 4.2. *Let the conditions of Assumption 4.1 are satisfied; then following estimate for L_2 holds*

$$L_2 \geq c \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) + L_{2,2}, \quad (52)$$

where

$$|L_{2,2}| \leq c_1 h^{2l} \left\| \frac{\partial}{\partial t} U \right\|_{2,\Omega}^2 + \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + c_2 h^{2l} \left\| \frac{\partial}{\partial t} V \right\|_{l,\Omega}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2. \quad (53)$$

Proof. It is easy to see that L_2 can be rewritten in the following form:

$$\begin{aligned} L_2 = & g(t) \left(\left(\left| \frac{\partial}{\partial t} u_h \right|^p \frac{\partial}{\partial t} u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h U \right|^p \frac{\partial}{\partial t} \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) \\ & + g(t) \left(\left(\left| \frac{\partial}{\partial t} \Pi_h U \right|^p \frac{\partial}{\partial t} \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} U \right|^p \frac{\partial}{\partial t} U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) \\ & + g(t) \left(\left(\left| \frac{\partial}{\partial t} v_h \right|^p \frac{\partial}{\partial t} v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h V \right|^p \frac{\partial}{\partial t} \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right) \\ & + g(t) \left(\left(\left| \frac{\partial}{\partial t} \Pi_h V \right|^p \frac{\partial}{\partial t} \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} V \right|^p \frac{\partial}{\partial t} V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right). \end{aligned}$$

Let us denote $L_{2,1}$ and $L_{2,2}$ as

$$\begin{aligned} L_{2,1} := & g(t) \left(\left(\left| \frac{\partial}{\partial t} u_h \right|^p \frac{\partial}{\partial t} u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h U \right|^p \frac{\partial}{\partial t} \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) \\ & + g(t) \left(\left(\left| \frac{\partial}{\partial t} v_h \right|^p \frac{\partial}{\partial t} v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h V \right|^p \frac{\partial}{\partial t} \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right), \\ L_{2,2} := & g(t) \left(\left(\left| \frac{\partial}{\partial t} \Pi_h U \right|^p \frac{\partial}{\partial t} \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} U \right|^p \frac{\partial}{\partial t} U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) \\ & + g(t) \left(\left(\left| \frac{\partial}{\partial t} \Pi_h V \right|^p \frac{\partial}{\partial t} \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} V \right|^p \frac{\partial}{\partial t} V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right). \end{aligned}$$

Therefore, L_2 can be written as

$$L_2 = L_{2,1} + L_{2,2}. \quad (54)$$

It follows from Lemma 2.3 that

$$\left(\left(\left| \frac{\partial}{\partial t} u_h \right|^p \frac{\partial}{\partial t} u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h U \right|^p \frac{\partial}{\partial t} \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \right) \geq c \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2, \quad (55)$$

and

$$\left(\left(\left| \frac{\partial}{\partial t} v_h \right|^p \frac{\partial}{\partial t} v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(\left| \frac{\partial}{\partial t} \Pi_h V \right|^p \frac{\partial}{\partial t} \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} \right) \geq c \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2. \quad (56)$$

From (55) and (56), we get the following estimate for L_2 :

$$L_2 \geq cg(t) \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) + L_{2,2}. \quad (57)$$

Finally, the Trace theorem and Cauchy-Schwarz inequality and Equation 15 imply that

$$\begin{aligned}
 |L_{2,2}| &\leq g(t) \left(c_9 \left\| \frac{\partial}{\partial t} (\Pi_h U - U) \right\|_{\Gamma_0}^2 + c_2 \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 \right) + g(t) \left(c_9 \left\| \frac{\partial}{\partial t} (\Pi_h V - V) \right\|_{\Gamma_0}^2 + c_2 \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) \\
 &\leq c_{10} h^{2l-1} \left\| \frac{\partial}{\partial t} U \right\|_{l,\Omega}^2 + \frac{g(t)}{8} \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + c_{10} h^{2l-1} \left\| \frac{\partial}{\partial t} V \right\|_{l,\Omega}^2 + \frac{g(t)}{8} \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2.
 \end{aligned} \tag{58}$$

4.3 | Proof of Theorem 3.1

Proof. Let us denote $L_{3,1}$ and $L_{3,2}$ by

$$\begin{aligned}
 L_{3,1} &:= \left(g * |u_h|^\gamma u_h, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \\
 &\quad + \left(g * |v_h|^\gamma v_h, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0},
 \end{aligned} \tag{59}$$

and

$$\begin{aligned}
 L_{3,2} &:= \left(g * |\Pi_h U|^\gamma \Pi_h U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} - \left(g * |U|^\gamma U, \frac{\partial}{\partial t} e_{1,h} \right)_{\Gamma_0} \\
 &\quad + \left(g * |\Pi_h V|^\gamma \Pi_h V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0} - \left(g * |V|^\gamma V, \frac{\partial}{\partial t} e_{2,h} \right)_{\Gamma_0}.
 \end{aligned} \tag{60}$$

Then, considering the definition of $L_{3,1}$ and $L_{3,2}$ yields

$$L_3 = L_{3,1} + L_{3,2}. \tag{61}$$

By applying the Cauchy-Schwarz and Young inequalities, we can see

$$\begin{aligned}
 |L_{3,1}| &= \left| \int_0^t g(t-r) \left((|u_h|^\gamma u_h - |\Pi_h U|^\gamma \Pi_h U)(r), \frac{\partial}{\partial t} e_{1,h}(t) \right)_{\Gamma_0} dr \right| \\
 &\quad + \left| \int_0^t g(t-r) \left((|v_h|^\gamma v_h - |\Pi_h V|^\gamma \Pi_h V)(r), \frac{\partial}{\partial t} e_{2,h}(t) \right)_{\Gamma_0} dr \right| \\
 &\leq \int_0^t |g(t-r)| \left(\|g\|_\infty \|e_{1,h}(r)\|_{\Gamma_0}^2 + \frac{1}{4\|g\|_\infty} \left\| \frac{\partial}{\partial t} e_{1,h}(t) \right\|_{\Gamma_0}^2 \right) dr \\
 &\quad + \int_0^t |g(t-r)| \left(\|g\|_\infty \|e_{2,h}(r)\|_{\Gamma_0}^2 + \frac{1}{4\|g\|_\infty} \left\| \frac{\partial}{\partial t} e_{2,h}(t) \right\|_{\Gamma_0}^2 \right) dr \\
 &\lesssim \int_0^t \left((\|g\|_\infty)^2 \|e_{1,h}(r)\|_{\Gamma_0}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{1,h}(t) \right\|_{\Gamma_0}^2 \right) dr \\
 &\quad + \int_0^t \left((\|g\|_\infty)^2 \|e_{2,h}(r)\|_{\Gamma_0}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{2,h}(t) \right\|_{\Gamma_0}^2 \right) dr.
 \end{aligned} \tag{62}$$

Using the Cauchy-Schwarz and Young inequalities, Trace theorem, and Lemma 2.3, we get

$$\begin{aligned}
 |L_{3,2}| &\lesssim \int_0^t \left((\|g\|_\infty)^2 \|(\Pi_h U - U)(r)\|_{\Gamma_0}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 \right) dr \\
 &\quad + \int_0^t \left((\|g\|_\infty)^2 \|(\Pi_h V - V)(r)\|_\Omega^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) dr \\
 &\lesssim \int_0^t \left(c_{11} h^{2l-1} \|U(r)\|_{l,\Omega}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 \right) dr \\
 &\quad + \int_0^t \left(c_{11} h^{2l-1} \|V(r)\|_{l,\Omega}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) dr, \tag{63}
 \end{aligned}$$

and

$$\begin{aligned}
 |L_4| &\leq \left\| \frac{\partial^2}{\partial t^2} (\Pi_h U - U) \right\|_\Omega^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \left\| \frac{\partial^2}{\partial t^2} (\Pi_h V - V) \right\|_\Omega^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \\
 &\leq c_{14} h^{2l} \left\| \frac{\partial^2}{\partial t^2} U \right\|_{k,\Omega}^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + c_{15} h^{2l} \left\| \frac{\partial^2}{\partial t^2} V \right\|_{k,\Omega}^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2. \tag{64}
 \end{aligned}$$

Employing Cauchy-Schwarz and Young inequalities and Lemma 2.3 leads to

$$\begin{aligned}
 |L_5| &\leq \left\| \nabla \frac{\partial}{\partial t} (\Pi_h U - U) \right\|_\Omega^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \left\| \nabla \frac{\partial}{\partial t} (\Pi_h V - V) \right\|_\Omega^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \\
 &\leq c_{16} h^{2l} \left\| \nabla \frac{\partial}{\partial t} U \right\|_{l,\Omega}^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + c_{17} h^{2l} \left\| \nabla \frac{\partial}{\partial t} V \right\|_{l,\Omega}^2 + \frac{1}{4} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2, \tag{65}
 \end{aligned}$$

and

$$\begin{aligned}
 |L_6| &\leq \|U - \Pi_h U\|_\Omega^2 + \left\| \frac{\partial}{\partial t} (U - \Pi_h U) \right\|_\Omega^2 + \frac{1}{2} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \|V - \Pi_h V\|_\Omega^2 + \left\| \frac{\partial}{\partial t} (V - \Pi_h V) \right\|_\Omega^2 + \frac{1}{2} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \\
 &\lesssim h^{2l} \|U\|_{l,\Omega}^2 + h^{2l} \left\| \frac{\partial}{\partial t} U \right\|_{l,\Omega}^2 + \frac{1}{2} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + h^{2l} \|V\|_{l,\Omega}^2 + h^{2l} \left\| \frac{\partial}{\partial t} V \right\|_{l,\Omega}^2 + \frac{1}{2} \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2. \tag{66}
 \end{aligned}$$

Applying (23) and Lemmas 4.1 and 4.2 gives rise to

$$\begin{aligned}
 &3 \frac{1}{2} \frac{d}{dt} \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \right) + \left(\left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_\Omega^2 + \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_\Omega^2 \right) + \frac{1}{2} \frac{d}{dt} \left(\|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 \right) \\
 &\quad + (1 + g(t)) \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) + \frac{1}{2} \frac{d}{dt} \left(\|e_{3,h}\|_\Omega^2 \|e_{4,h}\|_\Omega^2 \right) \\
 &\quad + \frac{1}{2} \frac{d}{dt} \left(\|e_{3,h}\|_\Omega^4 + \|e_{4,h}\|_\Omega^4 \right) + \|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 \frac{d}{dt} \left(\|e_{3,h}\|_\Omega^2 \|e_{4,h}\|_\Omega^2 \right) \\
 &\quad \leq \|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 + |L_{1,1}| + |L_{1,2,0}| + |L_{1,2,1}| + |L_{1,3}| \\
 &\quad + |\mathbf{M}_0| + |L_{2,2}| + |L_{3,1}| + |L_{3,2}| + |L_4| + |L_5| + |L_6|. \tag{67}
 \end{aligned}$$

Employing Lemmas 4.2–4.4 and Theorems 4.1 and 4.2 as well as Equations 62–67 results in

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 \right) + \frac{7}{8} \left(\left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \left\| \nabla \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 \right) + \frac{1}{2} \frac{d}{dt} \left(\|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 \right) \\
 & + (1 + g(t)) \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) + \frac{1}{2} \frac{d}{dt} \left(\|e_{3,h}\|_{\Omega}^2 + \|e_{4,h}\|_{\Omega}^2 \right) \\
 & + \frac{1}{2} \frac{d}{dt} \left(\|e_{3,h}\|_{\Omega}^4 + \|e_{4,h}\|_{\Omega}^4 \right) + \|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 + \frac{d}{dt} \left(\|e_{3,h}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^2 \right) \\
 & \lesssim \|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 + h^{2l} \left\| \nabla \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + h^{2l} \|\mathbf{W}\|_{l,\Omega}^2 \|e_{4,h}\|_{\Omega}^4 \\
 & + \|\mathbf{W}\|_{l,\Omega}^2 \|e_{3,h}\|_{\Omega}^2 + h^{4l} \|\mathbf{Z}\|_{l,\Omega}^4 \|e_{3,h}\|_{\Omega}^2 + \|\mathbf{Z}\|_{\Omega}^4 \|e_{3,h}\|_{\Omega}^2 \\
 & + h^{2l} \|\mathbf{Z}\|_{l,\Omega}^2 \|e_{4,h}\|_{\Omega}^4 + \|\mathbf{Z}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^4 + h^{4l} \|\mathbf{Z}\|_{l,\Omega}^4 \|e_{4,h}\|_{\Omega}^2 \\
 & + \|\mathbf{Z}\|_{l,\Omega}^4 \|e_{4,h}\|_{\Omega}^2 + \left(\int_0^t \|e_{1,h}(r)\|_{\Gamma_0}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 \right) dr \\
 & + \int_0^t \left(\|e_{2,h}(r)\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) dr + h^{2l-1} \left(\|U\|_{\infty,l,\Omega}^2 + \|V\|_{\infty,l,\Omega}^2 \right) \\
 & + h^{2l} \left(\|\mathbf{W}\|_{\infty,l,\Omega}^2 + \|\mathbf{Z}\|_{\infty,l,\Omega}^2 + \|\mathbf{W}\|_{\infty,l,\Omega}^4 + \|\mathbf{Z}\|_{\infty,l,\Omega}^4 \right). \tag{68}
 \end{aligned}$$

Now, from Lemma 2, we get

$$\begin{aligned}
 & 3 \frac{1}{2} \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Omega}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Omega}^2 \right) + \frac{3}{2} \left(\|e_{1,h}\|_{\Gamma_0}^2 + \|e_{2,h}\|_{\Gamma_0}^2 \right) + (1 + g(t)) \left(\left\| \frac{\partial}{\partial t} e_{1,h} \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h} \right\|_{\Gamma_0}^2 \right) \\
 & + \frac{1}{2} \left(\|e_{3,h}\|_{\Omega}^2 + \|e_{4,h}\|_{\Omega}^2 \right) + \frac{1}{2} \left(\|e_{3,h}\|_{\Omega}^4 + \|e_{4,h}\|_{\Omega}^4 \right) + \left(\|e_{3,h}\|_{\Omega}^2 \|e_{4,h}\|_{\Omega}^2 \right) \\
 & \lesssim \left(\left\| \frac{\partial}{\partial t} e_{1,h}(0) \right\|_{\Omega}^2 + \left\| \frac{\partial}{\partial t} e_{2,h}(0) \right\|_{\Omega}^2 \right) + \left(\|e_{1,h}(0)\|_{\Gamma_0}^2 + \|e_{2,h}(0)\|_{\Gamma_0}^2 \right) \\
 & + \left(\left\| \frac{\partial}{\partial t} e_{1,h}(0) \right\|_{\Gamma_0}^2 + \left\| \frac{\partial}{\partial t} e_{2,h}(0) \right\|_{\Gamma_0}^2 \right) + \left(\|e_{3,h}(0)\|_{\Omega}^4 + \|e_{4,h}(0)\|_{\Omega}^4 \right) \\
 & + \left(\|e_{3,h}(0)\|_{\Omega}^2 \|e_{4,h}(0)\|_{\Omega}^2 \right) + h^{2l-1} \left(\|U\|_{\infty,l,\Omega}^2 + \|V\|_{\infty,l,\Omega}^2 \right) \\
 & + h^{2l} \left(\|\mathbf{W}\|_{\infty,l,\Omega}^2 + \|\mathbf{Z}\|_{\infty,l,\Omega}^2 + \|\mathbf{W}\|_{\infty,l,\Omega}^4 + \|\mathbf{Z}\|_{\infty,l,\Omega}^4 \right). \tag{69}
 \end{aligned}$$

Assume X_h and \mathbf{M}_h are $(N_1 + 1)$ - and $(N_2 + 1)$ - dimensional spaces, respectively. Also, we assume $\mathcal{A} = \{\varphi_j\}_{j=1}^{N_1}$ and $\mathcal{B} = \{\psi_j\}_{j=1}^{N_2}$ are basis for X_h and \mathbf{M}_h , respectively; therefore, from (13), we have

$$\begin{aligned}
 (u_h|_{t=0}, \varphi_j)_{\Omega} &= (U_0, \varphi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_1, & (\mathbf{w}_h|_{t=0}, \psi_j)_{\Omega} &= (\nabla U_0, \psi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_2, \\
 \left(\frac{\partial}{\partial t} u_h \Big|_{t=0}, \varphi_j \right)_{\Omega} &= (U_1, \varphi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_1, & (v_h|_{t=0}, \varphi_j)_{\Omega} &= (V_0, \varphi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_1, \\
 (\mathbf{z}_h|_{t=0}, \psi_j)_{\Omega} &= (\nabla V_0, \psi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_2 & \left(\frac{\partial}{\partial t} v_h \Big|_{t=0}, \varphi_j \right)_{\Omega} &= (V_1, \varphi_j)_{\Omega} \quad j = 0, 1, 2, \dots, N_1.
 \end{aligned}$$

Using the above equations and applying (15)–(18), we find that

$$\|e_{1,h}(0)\|_{\Omega}^2 \leq \|U_0 - \Pi_h U_0\|_{\Omega}^2 \leq c_{29} h^{2l} \|U_0\|_{l,\Omega}^2, \tag{70}$$

$$\|e_{3,h}(0)\|_{\Omega}^2 \leq \|\nabla U_0 - R_h \nabla U_0\|_{\Omega}^2 \leq c_{30} h^{2l} \|\nabla U_0\|_{l,\Omega}^2, \tag{71}$$

$$\left\| \frac{\partial}{\partial t} e_{1,h}(0) \right\|_{\Omega}^2 \leq c_{31} h^{2l} \|U_1\|_{L^2(\Omega)}^2, \quad (72)$$

$$\|e_{2,h}(0)\|_{\Omega}^2 \leq \|V_0 - \Pi_h V_0\|_{\Omega}^2 \leq c_{32} h^{2l} \|V_0\|_{L^2(\Omega)}^2, \quad (73)$$

$$\|e_{4,h}(0)\|_{\Omega}^2 \leq \|\nabla V_0 - R_h \nabla V_0\|_{\Omega}^2 \leq c_{33} h^{2l} \|\nabla V_0\|_{L^2(\Omega)}^2, \quad (74)$$

$$\left\| \frac{\partial}{\partial t} e_{2,h}(0) \right\|_{\Omega}^2 \leq c_{36} h^{2l} \|V_1\|_{L^2(\Omega)}^2. \quad (75)$$

Also, it is easy to see that

$$\|e_{1,h}\|_{\Omega}^2 - \|e_{1,h}(0)\|_{\Omega}^2 \leq \left\| \int_0^t \frac{\partial}{\partial t} e_{1,h} \, dt \right\|_{\Omega}^2 \leq \int_0^t \left\| \frac{\partial}{\partial t} e_{1,h} \, dt \right\|_{\Omega}^2, \quad (76)$$

$$\|e_{2,h}\|_{\Omega}^2 - \|e_{2,h}(0)\|_{\Omega}^2 \leq \left\| \int_0^t \frac{\partial}{\partial t} e_{2,h} \, dt \right\|_{\Omega}^2 \leq \int_0^t \left\| \frac{\partial}{\partial t} e_{2,h} \, dt \right\|_{\Omega}^2. \quad (77)$$

Finally, substituting (70)–(75) into (69) and using (76) and (77) completes the proof. \square

5 | NUMERICAL EXAMPLE

In this section, we validate the theoretical results obtained in Theorem 3.1 by a test problem. The computational geometry ($\Omega = [0, 1] \times [0, 1]$) including the boundaries is shown in Figure 2. In this section, we use the following spaces:

$$\begin{aligned} \mathbf{M}_h &:= \{\mathbf{v}_h \in \mathbf{M} & : & \quad \mathbf{v}_h|_T \in RT_1(T), \forall T \in \mathcal{T}_h\}, \\ X_h &:= \{v_h \in X & : & \quad v_h|_T \in P_1(T), \forall T \in \mathcal{T}_h\}. \end{aligned}$$

The initial conditions for the model problem (1) are given as

$$\begin{aligned} U(x, y, 0) &= (\sin(y) + y^2(y-1) - y \sin(y)) \times (\sin(x) + x^2(x-1) - x \sin(x)), \\ U_t(x, y, 0) &= -(\sin(y) + y^2(y-1) - y \sin(y)) \times (\sin(x) + x^2(x-1) - x \sin(x)), \\ V(x, y, 0) &= (\sin(y) + y^2(y-1) - y \sin(y)) \times (\sin(x) + x^2(x-1) - x \sin(x)), \\ V_t(x, y, 0) &= -(\sin(y) + y^2(y-1) - y \sin(y)) \times (\sin(x) + x^2(x-1) - x \sin(x)). \end{aligned}$$

The right-hand side of the Equations 1a and 1b is given by

$$\begin{aligned} f(x, y, t) &= \exp(-t) \left((\sin(x) + x^2(x-1) - x \sin(x)) (2 \cos(y) - 6y + \sin(y) - y \sin(y) + 2) + \right. \\ &\quad \left. \exp(-t) (\sin(y) + y^2(y-1) - y \sin(y)) (2 \cos(x) - 6x + \sin(x) - x \sin(x) + 2) \right) \times \\ &\quad (0.0046 \exp(-2t) + 1) + \exp(-t) \left((\sin(x) + x^2(x-1) - x \sin(x)) \times \right. \\ &\quad \left. (\sin(y) + y^2(y-1) - y \sin(y)) (\sin(x) + x^2(x-1) - x \sin(x)) \times \right. \\ &\quad \left. (2 \cos(y) - 6y + \sin(y) - y \sin(y) + 2) - (\sin(y) + y^2(y-1) - y \sin(y)) \times \right. \\ &\quad \left. (2 \cos(x) - 6x + \sin(x) - x \sin(x) + 2) \right). \end{aligned}$$

In this example, the boundary Γ consists of Γ_0 and Γ_1 (see Figure 1) such that the homogenous Neumann and Dirichlet boundary conditions are defined on Γ_1 . For the boundary conditions (1d) and (1e) defined on Γ_0 , we set $g(t) := 0.1 \exp(-t)$, $p := 3.5$, $\gamma := 2.5$. Finally, the exact solutions of Equation 1 considering the mentioned boundary and initial conditions are given by

$$U(x, y, t) = V(x, y, t) = \exp(-t) (\sin(y) + y^2(y-1) - y \sin(y)) \times (\sin(x) + x^2(x-1) - x \sin(x)).$$

FIGURE 2 A simple computational domain including its respective boundaries Γ_0 and Γ_1 [Colour figure can be viewed at wileyonlinelibrary.com]

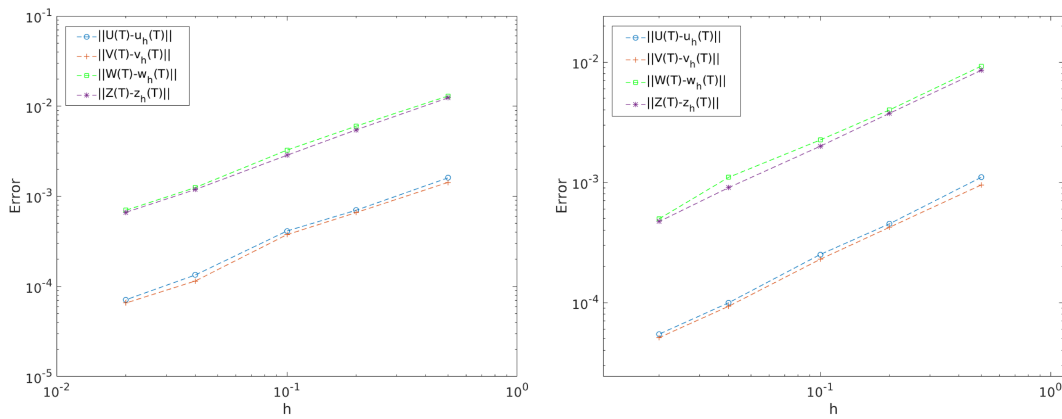
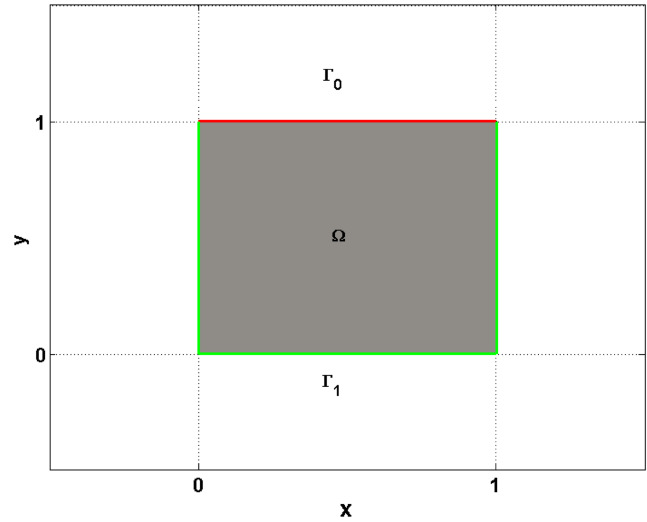


FIGURE 3 The approximation error in L^2 norm for $\Delta t = 0.005$ (left) and $\Delta t = 0.003$ (right) at $T = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

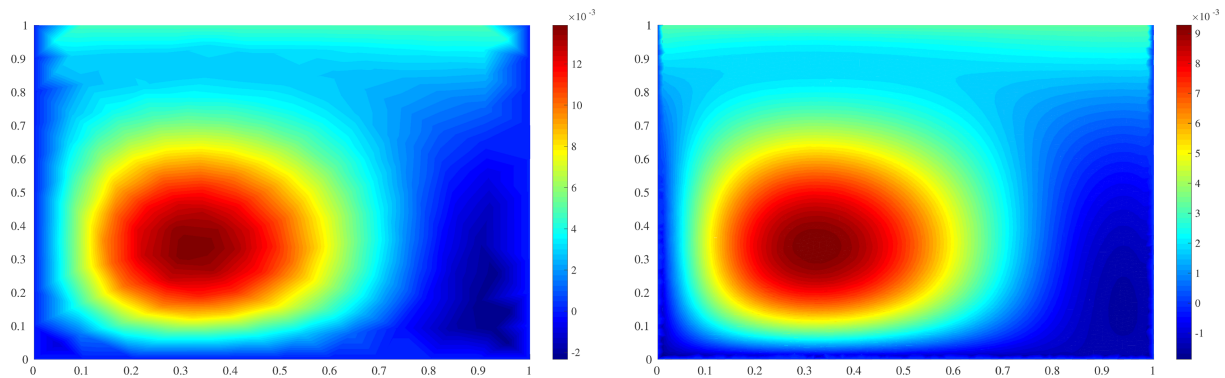


FIGURE 4 The approximated error of $\|U(T) - u_h(T)\|_\Omega$ for $h = 0.1$ (left) and $\|V(T) - v_h(T)\|_\Omega$ for $h = 0.01$ (right) with $\Delta t = 0.005$ at $T = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

In order to discretize in the space, we use the mixed variational formulation given in (13). The time discretization is done by an implicit finite difference (Crank-Nicolson method) scheme. Furthermore, to treat the nonlinearity, a Newton scheme is used. The numerical results for two different time step sizes ($\Delta t = 0.005$ and $\Delta t = 0.003$) are presented in Figure 3. Moreover, for the final time step, the discretization errors ($U(T) - u_h(T)$ and $V(T) - v_h(T)$) with respect to the exact solutions for two specific mesh sizes are illustrated in Figure 4.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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