Irreducible tensor products for symmetric and alternating groups in small characteristics

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Abstract

In characteristic 0 irreducible tensor products of representations of symmetric and alternating groups have been described by Zisser and by Bessenrodt and Kleshchev. In positive characteristic a classification conjecture for such products for symmetric groups S_n was formulated by Gow and Kleshchev. Parts of the conjecture were proved shortly after in papers of Bessenrodt and Kleshchev and of Graham and James. However many cases in characteristic 2 for n even were still open. For alternating groups in characteristics $p \geq 7$ irreducible tensor products have been described in a paper of Bessenrodt and Kleshchev, though not for $p \leq 5$.

In the submitted papers I consider the still open cases, completing the classification of irreducible tensor products of representations of symmetric and alternating groups up to a certain class of tensor products for alternating groups in characteristic 2.

In Charakteristik 0 wurden die einfachen Tensorprodukte von Darstellungen der symmetrischen und alternierenden Gruppen von Zisser und von Bessenrodt und Kleshchev beschrieben. In positiver Charakteristik formulierten Gow und Kleshchev eine Klassifizierungsvermutung für solche Produkte der symmetrischen Gruppen S_n. Teile der Vermutung konnten kurz danach von Bessenrodt und Kleshchev und von Graham und James bewiesen werden, trotzdem blieben vielen Fälle in Charakteristik 2 für gerade n noch offen. Für alternierende Gruppen in Charakteristik $p \geq 7$ wurden die einfachen Tensorprodukte in einem Artikel von Bessenrodt und Kleshchev beschrieben, für $p \leq 5$ jedoch nicht.

In den eingereichten Artikeln betrachte ich die noch offenen Fälle. Dabei kann ich die Klassifizierung von einfachen Tensorprodukten von Darstellungen der symmetrischen und alternierenden Gruppen abschließen, mit der einzigen Ausnahme einer Klasse von Tensorprodukten für alternierenden Gruppen in Charakteristik 2.

Keywords: symmetric groups, alternating groups, tensor products
Schlagworte: symmetrische Gruppen, alternierende Gruppen, Tensorprodukte

Introduction

Let G be a group and F be a field. A FG-representation is an homomorphism $f : G \to \mathrm{GL}(V)$, where V is a F-vector space. Given a FG-representation $f : G \to \mathrm{GL}(V)$ we often simply refer to V as a FG-representation (or FG-module), with G-action given by $gv = f(g)v$ for $g \in G$ and $v \in V$. The dimension of a FG-representation V is just the dimension of V as a F-vector space. The notions and results presented below on representation theory are standard and well known, see for example [Se].

The simplest example of representation is the trivial representation $\mathbf{1}_G$, that is the FG-representation with $V \cong F$ and G-action $gv = v$ for each $g \in G$ and $v \in V$. A further basic example of representation is the regular representation, the FG-representation where V has F-basis $\{e_g|g \in G\}$ and G-action given by $ge_h = e_{gh}$ for every $g, h \in G$.

In the following we will always assume that G is a finite group, F is an algebraically closed field and that representations have finite dimension. Although these assumptions could be somehow weakened, this setting is enough for the purpose of this introduction and we will always work within this setting in the submitted papers [M_1, M_2, M_3].

Given a FG-representation V and a subspace $V' \subseteq V$ we say that V' is a subrepresentation (or submodule) of V if V' is stable under the action of G. If there exists a proper G-stable subspace $0 \subsetneq V' \subsetneq V$ then V is called reducible. If no such proper G stable subspace exists and $V \neq 0$ then V is called irreducible. Any 1-dimensional FG-representation is irreducible, but in general there exist irreducible FG-representations which are not 1-dimensional. Clearly if V and W are isomorphic representations, that is there exists an F-vector space isomorphism $\varphi : V \to W$ such that $g\varphi(v) = \varphi(gv)$ for every $g \in G$ and $v \in V$, then V is irreducible if and only if W is irreducible. Thus irreducible FG-representations are usually only considered up to isomorphism.

Note that if V is any irreducible FG-representation and $0 \neq v \in V$ is any non-zero vector of V, then V is the F-span of the vectors gv for $g \in G$. Since by assumption G is a finite group we then have that V is finite dimensional. So the assumption of V being finite dimensional does not limit the study of irreducible representations of finite groups.

Let now V and W be two FG-representations. The tensor product space $V \otimes W$ can be viewed as a FG-representation using the G-action

$$g(v \otimes w) = (gv) \otimes (gw)$$

for each $g \in G$, $v \in V$, $w \in W$ and extending the action linearly. Note that $V \otimes W \cong W \otimes V$ and that if $V' \cong V$ and $W' \cong W$ are two further FG-representations then $V' \otimes W' \cong V \otimes W$.

If at least one of V or W has dimension 0, then so does the tensor product $V \otimes W$. So let us now assume that this is not the case. If V is reducible and $0 \subsetneq V' \subsetneq V$ is a proper submodule of V then $0 \subsetneq V' \otimes W \subsetneq V \otimes W$ is a proper submodule of $V \otimes W$. So in this case $V \otimes W$ is also reducible. Similarly $V \otimes W$ is reducible if W is reducible. Thus if the tensor product

$V \otimes W$ is irreducible then both V and W have to be irreducible, but this assumption is not enough to imply that $V \otimes W$ is irreducible.

One class of tensor products which are always irreducible are tensor products of a 1-dimensional representation with an irreducible representation: let V be a 1-dimensional FG-representation and W be an irreducible FG-representation. Since V is 1-dimensional there exists an homomorphism $f : G \to F^\times$ such that $gv = f(g)v$ for each $g \in G$ and $v \in V$. Let $f^* : G \to F^\times$ be given by $f^*(g) = f(g)^{-1}$ for each $g \in G$. Since $F^\times$ is abelian, f^* defines a 1-dimensional FG-representation V^*. For fixed non-zero vectors $v \in V$ and $v^* \in V^*$ we then have that for each $g \in G$ and $w \in W$

$$g(v \otimes v^* \otimes w) = (gv) \otimes (gv^*) \otimes (gw) = f(g) \cdot f^*(g)(v \otimes v^* \otimes (gw)) = v \otimes v^* \otimes (gw).$$

Thus W and $V \otimes V^* \otimes W$ are isomorphic representations (through the vector space map $w \to v \otimes v^* \otimes w$). Since $V \otimes V^* \otimes W \cong W$ is irreducible, it follows that $V \otimes W$ is also irreducible. As this class of tensor products is always irreducible, we refer to such tensor products as trivial irreducible tensor products. On the other hand irreducible tensor products $V \otimes W$ where neither V nor W is 1-dimensional are called non-trivial irreducible tensor products.

In order to describe techniques that help studying irreducible tensor products we first have to define some representations and state some basic results.

The construction of the module V^* introduced above for 1-dimensional representations, can be generalised also to module of dimension larger than 1 as follows. Let $f : G \to \mathrm{GL}(V)$ be a FG-representation. Let $V^* \cong V$ as vector space. Upon choosing bases for V and V^* and representing f though matrices, define a FG-representation $f^* : G \to \mathrm{GL}(V^*)$ by $f^*(g) = (f(g)^{-1})^t$ (with M^t being the transpose matrix of M). The FG-representation f^* is called dual representation of f (and V^* with the corresponding G-action is called dual representation of V).

Given two FG-representations $f_V : G \to \mathrm{GL}(V)$ and $f_W : G \to \mathrm{GL}(W)$ it is possible to define a FG-representation $f : G \to \mathrm{GL}(\mathrm{Hom}_F(V, W))$ on the homomorphism space $\mathrm{Hom}_F(V, W)$ by $f(g)\varphi := f_W(g) \circ \varphi \circ f_V(g)^{-1}$ for $g \in G$ and $\varphi \in \mathrm{Hom}_F(V, W)$. We have $V^* \otimes W \cong \mathrm{Hom}_F(V, W)$. In particular $V^* \cong V^* \otimes \mathbf{1}_G \cong \mathrm{Hom}_F(V, \mathbf{1}_G)$.

Consider now the space of G-homomorphisms $\mathrm{Hom}_G(V, W)$, that is homomorphisms $\varphi \in \mathrm{Hom}_F(V, W)$ for which $f_W(g)\varphi(v) = \varphi(f_V(g)v)$ for each $g \in G$ and $v \in V$. Then $\mathrm{Hom}_G(V, W)$ is exactly the set of homomorphisms $\varphi \in \mathrm{Hom}_F(V, W)$ on which G acts trivially, that is $f(g)\varphi = \varphi$ for all $g \in G$ (with f the above defined FG-representation on $\mathrm{Hom}_F(V, W)$). For V an irreducible FG-representation Schur's Lemma states that $\mathrm{End}_G(V)$ is 1-dimensional as F-vector space.

Let us now go back to the tensor product of two representations V and W. If $V \otimes W$ is irreducible then by the previous paragraph $\mathrm{End}_G(V \otimes W)$ is 1-dimensional. Further

$$\mathrm{End}_G(V \otimes W) \cong \mathrm{Hom}_G(V^* \otimes V, W^* \otimes W) = \mathrm{Hom}_G(\mathrm{End}_F(V), \mathrm{End}_F(W)).$$

Thus if $V \otimes W$ is irreducible then

$$\dim \mathrm{Hom}_G(\mathrm{End}_F(V), \mathrm{End}_F(W)) = 1. \quad (1)$$

This observation suggests us to study the FG-representations $\mathrm{End}_F(V)$ and $\mathrm{End}_F(W)$ separately in order to decide if the tensor product $V \otimes W$ might be irreducible. In particular we can use this idea to obtain reduction results on which $V \otimes W$ might be irreducible and then study such tensor products more in details (for example using knowledge on at least one of the two representations V or W).

One way to study the endomorphism modules $\mathrm{End}_F(V)$ is to study restrictions $V{\downarrow}_H^G$ of V to subgroups H of G and the corresponding permutation modules $\mathbf{1}{\uparrow}_H^G$. Given any FG-representation A and any subgroup $H \leq G$, the restriction $A{\downarrow}_H^G$ is simply the FH-representation A obtained by restricting the G-action to a H-action. Given a FH-representation B, again with $H \leq G$, it is possible to define the induced FG-representation as follows. Let K be a set of representatives of G/H and define $B{\uparrow}_H^G := \oplus_{k \in K} kB$ as a F-vector space. The $B{\uparrow}_H^G$ becomes a FG-representation with the G-action given by $g(kb) = \overline{k}((\overline{k}^{-1} gk)b)$, with $\overline{k} \in K$ the representative of the coset gkH. If $B = \mathbf{1}_H$ then the H-action is trivial, so given any non-zero vector $x \in \mathbf{1}_H$, elements of G permute the vectors $\{kx | k \in K\}$ which build a basis of $\mathbf{1}{\uparrow}_H^G := \mathbf{1}_H{\uparrow}_H^G$, explaining why such representations are called permutation representations.

By Frobenius reciprocity

$$\mathrm{Hom}_G(A, B{\uparrow}_H^G) \cong \mathrm{Hom}_H(A{\downarrow}_H^G, B) \quad \text{and} \quad \mathrm{Hom}_G(B{\uparrow}_H^G, A) \cong \mathrm{Hom}_H(B, A{\downarrow}_H^G)$$

for every subgroup $H \leq G$, FG-representation A and FH-representation B. In particular

$$\begin{aligned} \mathrm{Hom}_G(\mathbf{1}{\uparrow}_H^G, \mathrm{End}_F(A)) &\cong \mathrm{Hom}_G(\mathbf{1}{\uparrow}_H^G, A^* \otimes A) \\ &\cong \mathrm{Hom}_H(\mathbf{1}_H, A^*{\downarrow}_H^G \otimes A{\downarrow}_H^G) \\ &\cong \mathrm{End}_H(A{\downarrow}_H^G). \end{aligned}$$

Thus studying restrictions $A{\downarrow}_H^G$ and the dimension of the corresponding H-endomorphism space as well as the submodule structure of the permutation modules $\mathbf{1}{\uparrow}_H^G$ can bring informations on certain submodules of $\mathrm{End}_F(A)$. Since the modules $\mathrm{End}_F(A)$ are self-dual, this also gives informations on some quotients of $\mathrm{End}_F(A)$.

In order to obtain reduction results on possible irreducible tensor products, the idea is thus to show that in most cases (1) does not hold by studying homomorphisms

$$\mathrm{End}_F(A) \to \mathbf{1}{\uparrow}_H^G \to \mathrm{End}_F(B)$$

for different subgroups $H \leq G$.

We will now show more in details how this is applied to the case where G is a symmetric or alternating group. Before doing this we introduce notation for irreducible representations of these two classes of groups.

For the symmetric group S_n it is well known that irreducible representations are labelled by partitions of n in characteristic 0 and by p-regular partitions of n (that is partitions of n which have no part repeated p or more times) in positive characteristic p. We refer the reader to [J] for more informations on the representation theory of symmetric groups.

Let $\mathscr{P}(n) = \mathscr{P}_0(n)$ be the set of partitions of n and, for p positive, $\mathscr{P}_p(n)$ be the set of p-regular partition of n. For each partition $\lambda \in \mathscr{P}_p(n)$ let D^λ be the corresponding irreducible S_n-representation (which is well defined up to isomorphism). When considering irreducible representations of alternating groups it is well known (see [Ben, BO, FK]) that either $D^\lambda{\downarrow}_{\mathsf{A}_n}^{\mathsf{S}_n} \cong E^\lambda$ is irreducible or that $D^\lambda{\downarrow}_{\mathsf{A}_n}^{\mathsf{S}_n} \cong E^\lambda_+ \oplus E^\lambda_-$ is the direct sum of two non-isomorphic irreducible representations. Further each irreducible representation of A_n is isomorphic to E^λ or $E^\lambda_\pm$ for some partition $\lambda \in \mathscr{P}_p(n)$.

In order to classify irreducible tensor products $D^\lambda \otimes D^\mu$ for symmetric groups the first step, using the idea above is to study the endomorphism modules $\mathrm{End}_F(D^\lambda)$ and $\mathrm{End}_F(D^\mu)$ separately using certain permutation modules. It turns out that this approach works using permutation modules induced from large Young subgroups S_α. Here if $\alpha = (\alpha_1, \ldots, \alpha_h) \in \mathscr{P}(n)$ then $\mathsf{S}_\alpha \subseteq \mathsf{S}_n$ is the Young subgroup

$$\mathsf{S}_{\{1,\ldots,\alpha_1\}} \times \mathsf{S}_{\{\alpha_1+1,\ldots,\alpha_1+\alpha_2\}} \times \cdots \times \mathsf{S}_{\{n-\alpha_h+1,\ldots,n\}} \cong \mathsf{S}_{\alpha_1} \times \mathsf{S}_{\alpha_2} \times \cdots \times \mathsf{S}_{\alpha_h}.$$

By large Young subgroup we mean that $n - \alpha_1 \leq k$ for some constant k. For example if $p = 0$ then this approach works taking $h = \mathsf{S}_{n-1}$ and $\mathsf{S}_{n-2,2}$, that is studying the restrictions $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n}$ and $D^\lambda{\downarrow}_{\mathsf{S}_{n-2,2}}^{\mathsf{S}_n}$ for any partition λ as well as the permutation modules $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n}$ and $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-2,2}}^{\mathsf{S}_n}$. The structure of both restrictions of irreducible representations and permutation modules induced from Young subgroups is well understood in characteristic 0, allowing to tell that

$$\dim \mathrm{Hom}_{\mathsf{S}_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) \geq 2$$

if neither D^λ nor D^μ is 1-dimensional, so that $D^\lambda \otimes D^\mu$ is not irreducible in this case. This approach had been used by Zisser in [Z] to prove this result.

Consider now $p > 0$. If $p \neq 2$ or if n is odd and $p = 2$ then the structure of the permutation modules $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n}$ and $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-2,2}}^{\mathsf{S}_n}$ is still easy enough. Thus this same idea allowed Bessenrodt and Kleshchev in [BeK$_2$] to prove that also in these cases if $D^\lambda \otimes D^\mu$ is irreducible then either D^λ or D^μ needs to be 1-dimensional.

If n is even and $p = 2$ however there exist some non-trivial irreducible tensor products $D^\lambda \otimes D^\mu$ (at least for $n \equiv 2 \pmod 4$). It had been conjectured by Gow and Kleshchev in [GK] that non-trivial irreducible tensor products for symmetric groups can only occur for $n \equiv 2 \pmod 4$ and that in this case $D^\lambda \otimes D^\mu$ is a non-trivial irreducible tensor product if and only if, up to exchange, $\lambda = (n/2+1, n/2-1)$ and $\mu = (n-2a-1, 2a+1)$ with $0 \leq a < n/2$. Such tensor products have been proved to be indeed irreducible by Graham and James in [GJ]. Due to the aforementioned paper of Bessenrodt and Kleshchev it thus remained open to prove that no further non-trivial irreducible tensor products existed (some reductions also in this case had been obtained by Bessenrodt and Kleshchev). The main reason why this case had remained open was that structure of the modules $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n}$ and $\mathbf{1}{\uparrow}_{\mathsf{S}_{n-2,2}}^{\mathsf{S}_n}$ becomes more complicated when n is even and $p = 2$. Using the submodule structure of these two permutation modules as well as that of the permutation module $\mathbf{1}\uparrow_{\mathsf{S}_{n-2}}^{\mathsf{S}_n}$, the proof of the conjecture of Gow and Kleshchev was completed in the first submitted paper [M$_1$]. This required a careful analysis of the restrictions of D^λ to the subgroups S_{n-2} and $\mathsf{S}_{n-2,2}$ and their submodule structure.

Consider now alternating groups. Here three kind of tensor products have to be considered. The first kind of products are those of the form $E^\lambda \otimes E^\mu$. In this case if $E^\lambda \otimes E^\mu$ is irreducible then $D^\lambda \otimes D^\mu$ is also irreducible. Since whenever $D^\lambda \otimes D^\mu$ is irreducible the partition $\nu \in \mathscr{P}_p(n)$ with $D^\lambda \otimes D^\mu \cong D^\nu$ is known and since it is also known when $D^\nu\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}$ splits, these tensor products are easily checked for irreducibility (and actually none of them turns out to be irreducible if neither E^ν nor E^μ is 1-dimensional).

The other two classes of tensor products that have to be considered are those of the forms $E^\lambda \otimes E^\mu_\pm$ and $E^\lambda_\pm \otimes E^\mu_\pm$. A problem when studying the endomorphism rings $\mathrm{End}_F(E^\pi_\pm)$ of splitting modules is that the modules $E^\pi_\pm$ are less understood than the modules D^π (at least in positive characteristic). One way to avoid this problem is to study the modules

$$\mathrm{Hom}_F(E^\pi_\pm, E^\pi_+ \oplus E^\pi_-) \quad \text{and} \quad \mathrm{Hom}_F(E^\pi_+ \oplus E^\pi_-, E^\pi_\pm)$$

instead using that $E^\pi_\pm\uparrow_{\mathsf{A}_n}^{\mathsf{S}_n} \cong D^\pi$. Given any $F\mathsf{S}_n$-representation M,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}_n}(M\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}, \mathrm{Hom}_F(E^\pi_\pm, E^\pi_+ \oplus E^\pi_-)) &\cong \mathrm{Hom}_{\mathsf{A}_n}(M\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}, (E^\pi_\pm)^* \otimes (D^\pi\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}))\\ &\cong \mathrm{Hom}_{\mathsf{S}_n}(M, ((E^\pi_\pm)^* \otimes (D^\pi\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}))\uparrow_{\mathsf{A}_n}^{\mathsf{S}_n})\\ &\cong \mathrm{Hom}_{\mathsf{S}_n}(M, (E^\pi_\pm\uparrow_{\mathsf{A}_n}^{\mathsf{S}_n})^* \otimes D^\pi)\\ &\cong \mathrm{Hom}_{\mathsf{S}_n}(M, (D^\pi)^* \otimes D^\pi)\\ &\cong \mathrm{Hom}_{\mathsf{S}_n}(M, \mathrm{End}_F(D^\pi))\end{aligned}$$

and similarly

$$\mathrm{Hom}_{\mathsf{A}_n}(M\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n}, \mathrm{Hom}_F(E^\pi_+ \oplus E^\pi_-, E^\pi_\pm)) \cong \mathrm{Hom}_{\mathsf{S}_n}(M, \mathrm{End}_F(D^\pi)).$$

So knowing information on the submodule structure of the $F\mathsf{S}_n$-module $\mathrm{End}_F(D^\pi)$ can lead to knowing information on the submodule structures of the $F\mathsf{A}_n$-modules $\mathrm{Hom}_F(E^\pi_\pm, E^\pi_+ \oplus E^\pi_-)$ and $\mathrm{Hom}_F(E^\pi_+ \oplus E^\pi_-, E^\pi_\pm)$. We also obtain corresponding informations on quotients of these modules, since the irreducible representations of the symmetric groups are self dual and so

$$E^\pi_+ \oplus E^\pi_- \cong D^\pi\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n} \cong (D^\pi\downarrow_{\mathsf{A}_n}^{\mathsf{S}_n})^* \cong (E^\pi_+)^* \oplus (E^\pi_-)^*,$$

thus for some $\varepsilon \in \{\pm\}$

$$\mathrm{Hom}_F(E^\pi_\pm, E^\pi_+ \oplus E^\pi_-) \cong \mathrm{Hom}_F(E^\pi_+ \oplus E^\pi_-, E^\pi_{\varepsilon\pm}).$$

Assume now that $E^\lambda \otimes E^\mu_\pm \cong E$ is irreducible. Since E^μ_+ and E^μ_- have the same dimension we have that either $E^\lambda \otimes E^\mu_\mp$ is also isomorphic to E or that E is not isomorphic to any quotient of $E^\lambda \otimes E^\mu_\mp$. So by Schur's lemma

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathsf{A}_n}(\mathrm{End}_F(E^\lambda), \mathrm{Hom}_F(E^\mu_\varepsilon, E^\mu_\pm))\\ &= \dim \mathrm{Hom}_{\mathsf{A}_n}(E^\lambda \otimes E^\mu_\varepsilon, E^\lambda \otimes E^\mu_\pm) \leq 1\end{aligned}$$

for $\varepsilon \in \{\pm\}$ and then

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathsf{S}_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu))\\ &= \dim \mathrm{Hom}_{\mathsf{A}_n}(\mathrm{End}_F(E^\lambda), \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\mu_\pm)) \leq 2.\end{aligned}$$

Similarly if $E^\lambda_\pm \otimes E^\mu_\pm$ is irreducible then

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathsf{A}_n}(\mathrm{Hom}_F(E^\lambda_\pm, E^\lambda_\delta), \mathrm{Hom}_F(E^\mu_\varepsilon, E^\mu_\pm))\\ &= \dim \mathrm{Hom}_{\mathsf{A}_n}(E^\lambda_\delta \otimes E^\mu_\varepsilon, E^\lambda_\pm \otimes E^\mu_\pm) \leq 1\end{aligned}$$

for each choice of $\delta, \varepsilon \in \{\pm\}$ and then

$$\dim \mathrm{Hom}_{\mathsf{A}_n}(\mathrm{Hom}_F(E^\lambda_\pm, E^\lambda_+ \oplus E^\lambda_-), \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\mu_\pm)) \leq 4.$$

Thus we obtain some generalisations of (1). In many cases these dimension bounds are shown not to hold through study of $\mathrm{End}_F(D^\lambda)$ and $\mathrm{End}_F(D^\mu)$ separately. However since the bounds are now larger we require more permutation modules to obtain enough informations on $\mathrm{End}_F(D^\lambda)$ and $\mathrm{End}_F(D^\mu)$. Cases where these bounds do not allow to exclude irreducibility are checked more in details (using the structure of at least one of the two modules appearing in the product).

In characteristic 0 more informations are known on the splitting modules $E^\lambda_\pm$, so that this case is easier to cover and had already been covered implicitly by Zisser in [Z] and explicitly by Bessenrodt and Kleshchev in [BeK$_1$]. In particular they showed that the only non-trivial irreducible tensor products for alternating groups in characteristic 0 are exactly those of the form $E^{(n-1,1)} \otimes E^{(a^a)}_\pm$, for $n = a^2$. It is easy to see that, in characteristic 0, an irreducible module D^λ of a symmetric group S_n restricts irreducibly to S_{n-1} but splits upon restriction to A_n if and only if $n = a^2$ and $\lambda = (a^a)$ for some a. This gives a different description of the non-trivial irreducible tensor products for alternating groups in characteristic 0.

In positive characteristic $p > 5$ the structure of the modules $\mathbf{1}\uparrow^{\mathsf{S}_n}_{\mathsf{S}_{n-k,k}}$ for $k \leq 5$ allowed Bessenrodt and Kleshchev in [BeK$_3$] to completely characterise non-trivial irreducible tensor products. The given classifications in these cases nicely extends the above description from characteristic 0: up to certain congruences modulo p on n and the number of parts of λ, the non-trivial irreducible tensor products of alternating groups are exactly those of the form $E^{(n-1,1)} \otimes E^\lambda_\pm$ where $D^\lambda\downarrow^{\mathsf{S}_n}_{\mathsf{S}_{n-1}}$ is irreducible. The problem with characteristics 2, 3 and 5 is that the structure of the modules $\mathbf{1}\uparrow^{\mathsf{S}_n}_{\mathsf{S}_{n-k,k}}$ for $k \leq 5$ can in these cases be more complicated.

Using also permutation modules corresponding to other Young subgroups as well as a more detailed analysis of the restriction of the modules D^λ to the corresponding Young subgroups the classification of non-trivial irreducible tensor products in characteristics 3 and 5 was completed in the second and third submitted papers [M$_2$, M$_3$] to completely characterise non-trivial irreducible tensor products in characteristics 3 and 5. In characteristic 2, non-trivial irreducible tensor products which do not involve basic spin modules (that is irreducible representations which are not labelled by the partition

$(\lceil (n+1)/2\rceil, \lfloor (n-1)/2\rfloor))$ have been completely described in [M_2]. In characteristics 3 and 5 the classification extends that from larger characteristic, with only one further non-trivial irreducible tensor product in characteristic 3. In characteristic 2 the tensor products corresponding to the above characterisation are shown to be irreducible and it is shown that any other possible non-trivial tensor product must involve a basic spin modules as one of its factors.

Irreducible tensor products for alternating groups in characteristic 2 with basic spin modules are considered in a paper currently in writing [M_5].

I will now present some connection to other results.

Irreducible tensor products for covering groups of symmetric and alternating groups. Let $\widetilde{\mathsf{S}}_n$ and $\widetilde{\mathsf{A}}_n$ be double covers of S_n and A_n respectively. The irreducible representations of $\widetilde{\mathsf{S}}_n$ (resp. $\widetilde{\mathsf{A}}_n$) which are not irreducible representations of S_n (resp. A_n) are called irreducible spin representations. When considering irreducible tensor products $V \otimes W$ of $\widetilde{\mathsf{S}}_n$ or $\widetilde{\mathsf{A}}_n$ there are three cases to be considered: neither V nor W is a spin representation, V is not a spin representation but W is or both V and W are spin representations. In the first case $V \otimes W$ is irreducible for $\widetilde{\mathsf{S}}_n$ (resp. $\widetilde{\mathsf{A}}_n$) if and only if it is irreducible for S_n (resp. A_n), so these tensor products do not require extra consideration. The other two cases can be considered using arguments similar to those described above for symmetric and alternating groups. For double covers of symmetric groups in characteristic 0 this has been done by Bessenrodt [Bes] and Bessenrodt and Kleshchev [BeK_4]. In positive characteristic $p \geq 5$ some reduction results had been obtained by Kleshchev and Tiep in [KT]. In [M_4] the characterisation of non-trivial irreducible tensor products for double covers of symmetric and alternating groups in arbitrary odd characteristic was completed. Since there are no spin representations in characteristic 2, this case does not have to be considered for these groups.

Characteristic 0. In characteristic 0 results are often easier since in this case any representation of a finite group is always isomorphic to a direct sum of irreducible representations. So in this case knowing the character of a representation is equivalent to knowing the representation itself (up to isomorphism). Further in this case more informations on irreducible representations are known, for example for symmetric groups dimensions and characters of irreducible representations are known in characteristic 0 but they are in general not known in positive characteristic. So more questions can often be answered in this setting.

For symmetric groups, known results on Kronecker coefficients (that is the multiplicity of D^ν as a composition factor of $D^\lambda \otimes D^\mu$) allow to describe tensor products with certain properties, like tensor products which have only few homogeneous components [BeK_1] or multiplicity free tensor products [BB]. A different question connected to Kronecker coefficients is the Saxl conjecture, which studies the existence of partitions λ such that any irreducible representation D^ν appears with positive multiplicity in the tensor square $D^\lambda \otimes D^\lambda$.

For covering groups, multiplicities of composition factors of tensor products of an irreducible representation with a basic spin module are known, see [St, Theorem 9.3]. Further (almost) homogeneous tensor products have been characterised in [Bes, BeK_4].

A different kind of coefficients considers multiplicities of compositions factors of representations obtained by inducing an irreducible representation of a Young subgroup. Since Young subgroups are direct products of symmetric groups, this means considering induced outer tensor products. Multiplicities of composition factors in this case are given by the Littlewood-Richardson coefficients, see [J, 16.4]. The Littlewood-Richardson coefficients also gives multiplicities of composition factors of tensor products of representations of general linear groups, see [J, 26.13] and the remark just after.

For spin irreducible representations coefficients for inducing from the double covers $\widetilde{\mathsf{S}}_\lambda \subseteq \widetilde{\mathsf{S}}_n$ of the Young subgroups S_λ are also known. Here outer tensor products are replaced by reduced Clifford products (since $\widetilde{\mathsf{S}}_\lambda$ is not a direct product of smaller covering groups). Formulas for the multiplicities of composition factors of induced reduced Clifford products are given in [St, Theorems 8.1, 8.3].

Irreducible restrictions. Let V a representation of G and $H \leq G$ be a subgroup. If the restriction $V{\downarrow}_H$ is irreducible then

$$\dim \mathrm{Hom}_G(\mathbf{1}{\uparrow}_H^G, \mathrm{End}_F(V)) = \dim \mathrm{End}_H(V{\downarrow}_H) = 1.$$

This suggests that when studying irreducible restrictions of representations of a given group to subgroups reduction results can be obtained by studying the permutation module $\mathbf{1}{\uparrow}_H^G$ and the endomorphism module $\mathrm{End}_F(V)$ separately. Similar to the tensor product case one can try to show that $\mathrm{Hom}_G(\mathbf{1}{\uparrow}_H^G, \mathrm{End}_F(V))$ is not 1-dimensional by constructing homomorphisms which factor through some other permutation module $\mathbf{1}{\uparrow}_K^G$, that is homomorphisms

$$\mathbf{1}{\uparrow}_H^G \to \mathbf{1}{\uparrow}_K^G \to \mathrm{End}_F(V)$$

for different subgroups $K \leq G$.

For symmetric groups, taking $K = \mathsf{S}_{n-1}$, $\mathsf{S}_{n-2,2}$ or $\mathsf{S}_{n-3,3}$ this approach often allows to obtain reductions to transitive or 2- or 3-homogeneous subgroups. Once these reductions are obtained, the remaining cases are then considered more in details (using informations on the representations or subgroups) to complete the classification. For alternating groups, when considering restrictions of splitting modules, slight modifications of this idea are used, similarly to what was done for tensor products.

Since the structure of the endomorphism rings $\mathrm{End}_F(D^\lambda)$ is studied using the structure of permutation modules induced from Young subgroups when considering either irreducible restrictions or irreducible tensor products, there are multiple references between papers studying these two problems.

In characteristic 0 irreducible restrictions for symmetric and alternating groups are studied by Saxl in [Sa]. The cases for $p \geq 5$ are considered in papers of Brundan and Kleshchev [BrK] and Kleshchev and Sheth [KS_1, KS_2]. Maximal intransitive cases for symmetric groups were covered by Jantzen and Seitz [JS], Kleshchev [K] and Phillips [P], but for the other cases in characteristics 2 and 3 only some reduction results were available, due to Kleshchev and Sheth [KS_1] as well as Kleshchev, Sin and Tiep [KST]. In [KMT_1, KMT_2, KMT_3] the classification of irreducible restrictions of representations of symmetric and alternating groups was essentially completed. Again due to the often more complicated structure of the permutation modules $\mathbf{1}\uparrow_{\mathsf{S}_{n-2,2}}^{\mathsf{S}_n}$ and $\mathbf{1}\uparrow_{\mathsf{S}_{n-3,3}}^{\mathsf{S}_n}$ in characteristics 2 and 3, these cases required a much more careful analysis of the modules involved.

For covering groups of symmetric and alternating groups classifications of irreducible restrictions were obtained by Kleidman and Wales [KW] in characteristic 0 and Kleshchev and Tiep [KT] in positive characteristic.

Aschbacher-Scott classification of maximal subgroups of finite classical groups. Aschbacher and Scott showed in [AS] that understanding the conjugacy classes of maximal subgroups of almost simple groups is one of two problems which is needed in order to be able to completely describe maximal subgroups of arbitrary finite groups. For G a symmetric or alternating group, maximal subgroups of G have been described in [LPS]. For sporadic groups informations on maximal groups can be found in the ATLAS [Atl]. For G a finite classical group, Aschbacher defined in [A] certain classes of subgroups $\mathscr{C}_i(G)$ for $1 \leq i \leq 8$ and $\mathscr{S}(G)$ consisting of possible maximal subgroups of G. Since not all subgroups in the classes $\mathscr{C}_i(G)$ and $\mathscr{S}(G)$ are maximal, the subgroups appearing in these classes have to be further studied to determine which of them are indeed maximal. In dimension at most 12 the question of maximality of these subgroups has been answered in [BHR]. For dimension at least 13, it has been studied in [KL] which subgroups of the classes $\mathscr{C}_i(G)$ are maximal, but maximality of subgroups of the class $\mathscr{S}(G)$ is still open.

If $H \in \mathscr{S}(G)$ is not maximal then there exists $H < L < G$ with $L \in \mathscr{C}_i(G)$ for some $i \in \{2, 4, 6, 7\}$ or $L \in \mathscr{S}(G)$ (see [Ma]). One of the conditions on $H \in \mathscr{S}(G)$ is that H acts absolutely irreducibly on the natural module of G.

If L is also in class $\mathscr{S}(G)$ then in particular there exists an irreducible representation D of L such that $D\downarrow_H^L$ is irreducible.

If L is in class $\mathscr{C}_4(G)$ and V is the natural representation of G then $V\downarrow_L^G \cong V_1 \otimes V_2$ with neither V_1 nor V_2 of dimension 1. In particular upon further restriction to H one can see that H has a non-trivial irreducible tensor product.

Since groups H in class $\mathscr{S}(G)$ are almost quasi-simple, that is $S \trianglelefteq H/Z(H) \leq \mathrm{Aut}(S)$ for S a non-abelian simple groups, the study of irreducible restrictions of representations of almost quasi-simple groups to almost quasi-simple subgroups and of irreducible tensor products of almost quasi-simple groups has an application to the Aschbacher-Scott classification. Note that although the existence of an irreducible restriction (resp. non-trivial irreducible tensor product) is necessary for the existence of a maximal subgroup subgroup L with $H < L < G$ and $L \in \mathscr{S}(G)$ (resp. $L \in \mathscr{C}_4(G)$), this is not a sufficient condition.

In particular the above described work on irreducible tensor products and irreducible restrictions of representations of symmetric and alternating groups and their covering groups is needed for this classification.

More details on this can be found in [Ma]. In this survey paper references can also be found for the irreducible tensor products and irreducible restrictions problems for finite groups of Lie type. Further informations about containments $H < L < G$ with $L \in \mathscr{C}_i(G)$ with $i \in \{2, 6, 7\}$ can also be found as well as additional informations on the structure of subgroups contained in class $\mathscr{S}(G)$ and resulting further restrictions on possible subgroups containments $H < L < G$.

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