Gromov-Witten- and degeneration invariants: computation and enumerative significance

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Zusammenfassung

Die vorliegende Arbeit ist in drei Kapitel unterteilt. Das erste Kapitel gibt hauptsächlich einen kurzen Überblick über die Konstruktion und Eigenschaften von Gromov-Witten-Invarianten und den sogenannten gravitational descendants, die eine Erweiterung der Gromov-Witten-Invarianten darstellen. Wir beschränken uns dabei auf den Fall des Grundkörpers $\mathbb{C}$ und auf Kurven vom Geschlecht null.

In Kapitel 2 betrachten wir Gromov-Witten-Invarianten von Aufblasungen von Punkten. Aufblasungen sind in gewissem Sinne die einfachsten Fälle von Varietäten, deren Modulräume stabiler Abbildungen zu hohe Dimension haben. Wir untersuchen sowohl die Berechenbarkeit als auch die enumerative Bedeutung der Invarianten. Hierbei ist die enumerative Bedeutung besonders interessant, weil Kurven auf der Aufblasung $\tilde{X}$ von X (als strikte Transformierte) mit solchen auf dem ursprünglichen Raum X zusammenhängen, so daß die Gromov-Witten-Invarianten von $\tilde{X}$ unter Umständen auch auf X interpretiert werden können als Anzahlen von Kurven, die globale Multiplizitätenbedingungen in den aufgeblasenen Punkten erfüllen. Benutzt man exzeptionelle Klassen als Inzidenzbedingungen für die Kurven, so können auch Tangentialbedingungen an Untervarietäten in den aufgeblasenen Punkten behandelt werden.

Was die Berechenbarkeit betrifft, so zeigen wir, daß es zumindest für konvexe Varietäten immer möglich ist, die Gromov-Witten-Invarianten der Aufblasung aus denen der ursprünglichen Varietät zu bestimmen. Hierzu geben wir einen expliziten Algorithmus an, mit dem die Zahlen berechnet werden können. Zur enumerativen Bedeutung zeigen wir, daß die Invarianten die erwartete Deutung haben für Aufblasungen von $\mathbb{P}^r$ in einem Punkt sowie unter gewissen Bedingungen für Aufblasungen von $\mathbb{P}^3$ in bis zu vier Punkten. Andererseits werden wir aber sehen, daß Gromov-Witten-Invarianten von Aufblasungen von $\mathbb{P}^r$ mit $r \geq 4$ in mindestens zwei Punkten fast nie enumerativ sind. Der Fall von Aufblasungen von $\mathbb{P}^2$ wurde bereits von Göttsche und Pandharipande [GP] behandelt, die die Invarianten in diesen Fällen berechnen und bei fast allen ihre enumerative Bedeutung zeigen. Wir geben schließlich noch einige numerische Anwendungen der Gromov-Witten-Invarianten von Aufblasungen an, unter anderem auch als Ausblick im Fall gewisser Aufblasungen entlang von Untervarietäten, wodurch wir wohlbekannte Multisekanten-Formeln erhalten.

In Kapitel 3 betrachten wir Degenerations-Invarianten. Das Hauptresultat ist hier die Erweiterung des Ergebnisses von Vakil [V] auf Degenerationen zu beliebigen Hyperflächen $Q \subset \mathbb{P}^r$ und nicht nur zu Hyperebenen. Im Gegensatz zu [V] müssen hierzu virtuelle Fundamentalklassen auf den betrachteten Modulräumen definiert und benutzt werden, da Q im allgemeinen nicht konvex ist. Wir zeigen, daß die Gleichungen, die man erhält, die Gromov-Witten-Invarianten von $\mathbb{P}^r$ mit denen von Q durch eine Reihe von Degenerations-Invarianten verbindet. Dies beantwortet teilweise die Frage nach dem Zusammenhang zwischen Gromov-Witten- und Degenerations-Invarianten. Als interessantes nicht-triviales Beispiel betrachten wir den Fall einer Quintik $Q \subset \mathbb{P}^4$ und zeigen, wie man die Anzahl rationaler Kurven vom Grad 1 und 2 auf Q aus gewissen Gromov-Witten-Invarianten und gravitational descendants von $\mathbb{P}^4$ berechnen kann. Wir vermuten, daß ähnliche Methoden auch für Kurven höheren Grades und womöglich auch höheren Geschlechts anwendbar sind, da die Arbeit von Vakil gezeigt hat, daß Degenerations-Invarianten im Gegensatz zu Gromov-Witten-Invarianten vergleichsweise gut dazu geeignet sind, Anzahlen von Kurven von höherem Geschlecht zu berechnen.

Schlagworte: enumerative Geometrie, Gromov-Witten-Invarianten, Degenerationsmethoden

Abstract

This thesis is divided into three chapters. The first one mainly gives a short overview of the construction and properties of Gromov-Witten invariants and the so-called gravitational descendants, which are an extension of the Gromov-Witten invariants. Everything will be done over $\mathbb{C}$ and for curves of genus zero.

Chapter 2 deals with Gromov-Witten invariants of blow-ups of points. Blow-ups are in some sense the simplest cases of varieties whose moduli spaces of stable maps have too big dimension. We address the questions both of computation and of enumerative significance of the invariants. Here, the enumerative significance is particularly interesting since (via strict transform) curves on the blow-up $\tilde{X}$ of X are related to curves on the original space X, such that the Gromov-Witten invariants of $\tilde{X}$ should be interpretable on X as numbers of curves satisfying global multiplicity conditions at the blown-up points. Using exceptional classes as incidence conditions for the curves, even tangency conditions to subvarieties at the blown-up points are tractable.

Concerning the computation, we show that at least for convex varieties, it is always possible to compute the Gromov-Witten invariants of the blow-up from those of the original space. This is done by giving an explicit algorithm to calculate the numbers. As for the enumerative significance, we show that the invariants are enumerative on the blow-up of $\mathbb{P}^r$ at one point, and under certain conditions on the blow-up of $\mathbb{P}^3$ at up to four points. On the negative side, we will see that Gromov-Witten invariants on blow-ups of $\mathbb{P}^r$ with $r \geq 4$ in at least two points are almost never enumerative. The case of blow-ups of $\mathbb{P}^2$ has already been considered by Göttsche and Pandharipande [GP] who compute the invariants and prove the enumerative significance of almost all of them in this case. We also give various numerical applications of Gromov-Witten invariants of blow-ups, including as an outlook the case of certain blow-ups along subvarieties, leading to well-known multisecant formulas.

In chapter 3, we consider degeneration invariants. The main result is the extension of the results of Vakil [V] in that we allow degenerations to arbitrary hypersurfaces $Q \subset \mathbb{P}^r$ and not only hyperplanes. In contrast to [V], this requires the use of certain virtual fundamental classes on the moduli spaces, since Q is in general not convex. We will show that the equations we get relate the Gromov-Witten invariants of $\mathbb{P}^r$ to those of Q through a sequence of degeneration invariants, answering in part the question of the connection between Gromov-Witten- and degeneration invariants. As an interesting non-trivial example we take $Q \subset \mathbb{P}^4$ to be a quintic threefold and show how to calculate the numbers of rational curves on Q of degrees 1 and 2 from certain Gromov-Witten invariants and gravitational descendants on $\mathbb{P}^4$. Similar methods are supposed to work for arbitrary degree, and perhaps also for higher genus, as the work of Vakil has shown that degeneration invariants are quite suitable to compute numbers of curves of higher genus, in contrast to Gromov-Witten theory.

Keywords: enumerative geometry, Gromov-Witten invariants, degeneration techniques

Contents

Preface

In the last few years, there has been enormous progress in enumerative geometry inspired by the work of physicists. The starting point was the famous paper by Candelas, de la Ossa, Green, and Parkes [COGP] in which the authors calculated the numbers of rational curves of degree d on a generic quintic threefold, for any d. In fact, these numbers are expected to be finite by a naive dimension count, although it is still unknown whether this really holds for any degree. The methods used by Candelas et al., namely a certain equivalence of string theories called “mirror symmetry”, are however not at all mathematically rigorous, and probably neither from a physicists point of view, since they involve some guesswork motivated merely by the fact that the resulting numbers should be non-negative integers. Mathematicians were able to verify the numbers easily for degrees 1 and 2, but already the case of degree 3 is so complicated to attack with classical methods that Ellingsrud and Strømme [ES] needed some 30 pages to verify only this one number. Similar calculations for higher degree would be increasingly complicated, if not impossible.

Inspired by these facts, the mathematical breakthrough in modern mathematical enumerative geometry has begun about four years ago with the work of Kontsevich and Manin [K], [KM1]. The basic idea is the notion of stable maps, due to Kontsevich, which provides the analogue of Deligne-Mumford stable curves. In the same way as the latter give rise to a proper smooth moduli stack $\bar{M}_{g,n}$ of n-pointed curves of genus g, stable maps can be used to construct a moduli space $\bar{M}_{g,n}(X,\beta)$, where X is a smooth projective variety and $\beta \in A_1(X)$ a homology class. An element of this space is given by a tuple $(C, x_1, \dots, x_n, f)$, where C is a curve of arithmetic genus g with at most nodes as singularities, x_i are distinct smooth marked points on C, $f : C \to X$ is an arbitrary map, and where a certain stability condition is satisfied. The space $\bar{M} := \bar{M}_{g,n}(X,\beta)$ should be viewed as a compactification of the space of those stable maps where the underlying curve C is irreducible, although the latter space is in general not dense in $\bar{M}$. Due to some technical difficulties, the actual construction of $\bar{M}$ has only been given one year later by Behrend and Manin in [BM]. Up to that point, the expected properties of these spaces had been given as axioms [KM1].

Nowadays, the moduli spaces $\bar{M}_{g,n}(X,\beta)$ are the basic objects of study in almost any modern treatment of enumerative geometry. Usual enumerative questions are to count curves of given genus and homology class in a given projective variety X that satisfy certain additional conditions, such as e.g. intersecting given subvarieties of X, being tangent or having higher contact to subvarieties, or having certain types of singularities at some subvarieties. In general, one hopes to express such conditions as suitable cycles on the moduli space $\bar{M}$ and then wants to calculate the intersection product of these cycles for all conditions that one wants to impose. If this intersection is zero-dimensional, one then hopes to be actually able to compute the intersection, and that the degree of this zero-cycle is in fact the answer to the original enumerative problem.

Both these hopes are however far from being fulfilled in general. As for the computation, the moduli spaces of stable maps are in general neither smooth nor even of constant dimension, and their full cohomology groups are extremely complicated. Thus it is almost hopeless to expect to be able to compute any intersection product on them just by computing their full cohomology rings. In general, one cannot even compute the dimensions of their cohomology groups. Hence, to be able to do calculations at all, one has to restrict oneself to certain types of cycles in $\bar{M}$. If one chooses these cycles such that the intersection of two of them is computable and again of the same type, one can at least do calculations in the corresponding subring of the cohomology of $\bar{M}$.

There are several ways how to do this. By far the most effort is nowadays spent on so-called Gromov-Witten theory, which is also a theory inspired by physics. In this theory, first of all one defines a virtual dimension of the moduli space of stable maps which is based on the deformation theory of the elements of this space. If X is not a so-called convex variety, i.e. if there are obstructions to the deformations such that the actual dimension of the moduli space is bigger than the virtual one, one uses the structure of the obstructions to define a so-called virtual fundamental class on $\bar{M}$. This is a cycle in the homology of the moduli space in the virtual dimension. If we have a convex variety, i.e. if there are no obstructions, then the virtual fundamental class will be the usual one. The general theory of these virtual fundamental classes has been introduced in algebraic geometry by the work of Behrend and Fantechi [BF], [B] about two years ago. It is not restricted to the case we have at hand, but the inspiration to give such a construction was certainly given by Gromov-Witten theory. There also exists a symplectic construction of virtual fundamental classes, introduced by Li and Tian [LT1], [LT2], which has recently shown to be the same as the algebro-geometric one [LT3].

One then considers the evaluation maps $ev_i : \bar{M} \to X$ that map $(C, x_1, \dots, x_n, f)$ to $f(x_i)$, pulls back cohomology classes γ_i on X to the moduli space via the various ev_i, and considers the intersection of these pullbacks. If the codimensions of these classes sum up to the virtual dimension of $\bar{M}$, one can evaluate the intersection on the virtual fundamental class to get a number. The numbers obtained that way are the so-called *Gromov-Witten invariants*. They are supposed to represent numbers of curves in X satisfying incidence conditions with generic representatives of the classes γ_i.

The main point of the Gromov-Witten invariants is that, in genus zero, there is actually a way to get relations between them which are often sufficient to compute all of them by a recursive strategy. This is done by considering the morphism $\bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,4} \cong$

$\mathbb{P}^1$ for $n \geq 4$, given by mapping $(C, x_1, \ldots, x_n, f)$ to $(C, x_1, \ldots, x_4)$ and stabilizing. One then looks at a point, i.e. divisor, in $\bar{M}_{0,4}$ corresponding to the curve with two rational components and two of the marked points on each of them. Taking the pullback of this divisor to $\bar{M}$, one obtains a sum of divisors on $\bar{M}$ whose points correspond to certain reducible curves with two components that can be described explicitly. It turns out that, when intersecting such a divisor with pullbacks via the evaluation maps and evaluating the result on the virtual fundamental class, one indeed gets a product of two Gromov-Witten invariants corresponding to the two components. Now, the linear equivalence of two points in $\bar{M}_{0,4}$ as above that differ only by the labeling of the marked points, pulls back to give equations among the Gromov-Witten invariants. In favourable cases, e.g. on $\mathbb{P}^r$, these equations suffice to compute all the invariants. In general, however, this is not the case, e.g. it is impossible to calculate the numbers of rational curves on the quintic threefold mentioned above using these methods.

Recently, there has emerged a different approach to get relations between certain cycles in $\bar{M}$: one again looks at pullbacks of cohomology classes on X via the evaluation maps, but now moves the subvarieties representing these classes to very special positions, e.g. such that they are all contained in a fixed hyperplane. This again causes the stable maps satisfying these incidence conditions to become reducible, making a similar procedure work as above. The types of reducible curves arising here are more complicated, however. These methods, usually called degeneration methods (since one degenerates the incidence subvarieties in X to lie in special positions), have their origin in the work of Caporaso and Harris [CH3]. In this paper, the authors work on $\mathbb{P}^2$ and do not yet use the language of stable maps. The translation of these methods to the spaces of stable maps has been done later by Vakil [V], where they have also been generalized to degenerations to hyperplanes in higher-dimensional projective spaces. The precise connection between the invariants obtained that way, which we call *degeneration invariants*, and the Gromov-Witten invariants, is still unclear in general.

As for the enumerative significance of Gromov-Witten- and degeneration invariants, there are lots of things that may go wrong. Whereas on "nice" spaces such as $\mathbb{P}^r$ it is clear that the invariants actually count the numbers of curves that they are supposed to count, there is no hope to get such a statement on arbitrary varieties X. In the Gromov-Witten case this arises mainly from the virtual fundamental classes, which are in general not interpretable in geometric terms. So whenever the dimension of $\bar{M}$ is too big, such that we have to use virtual fundamental classes and cannot use the ordinary fundamental class of $\bar{M}$, the enumerative significance of the invariants is not at all clear. In the theory of degeneration invariants, no other spaces than $\mathbb{P}^r$ have been considered so far, but of course one has to expect similar problems there.

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Chapter 1

Gromov-Witten invariants and descendants

1.1 Introduction

We start our work by recalling the basic constructions of Gromov-Witten theory and giving various applications to enumerative geometry. Apart from lemma 1.3.3, proposition 1.3.5, and section 1.6, the material in this chapter is not new. Main references are: [ML], [FP], [BM] for section 1.2, [BF], [B] for section 1.3, [ML], [FP], [BM], [KM1], [B] for section 1.4, and [KM2], [G] for section 1.5. Intersection theory on Deligne-Mumford stacks that will be used to construct the Gromov-Witten invariants and descendants has been developed in [Vi].

Throughout our work, we will only consider enumerative problems concerning rational curves. Therefore, whenever we talk of (pre-)stable curves or maps and their moduli spaces in the sequel, it is always assumed tacitly that the curves are of arithmetic genus zero.

Let us first fix some notation that will be used throughout the work. Let X be a complex smooth projective variety of dimension $r = \dim X$. For $0 \le i \le r$, we denote by $\boldsymbol{A_i(X)}$ the algebraic part of $H_{2i}(X)$ modulo torsion and by $\boldsymbol{A^i(X)}$ the algebraic part of $H^{2i}(X)$ modulo torsion. These are finitely generated abelian groups. The classes in $A^i(X)$ will be said to have **codimension** i. By abuse of notation, we will often denote a subvariety of X and its fundamental class in $A_*(X)$ or $A^*(X)$ (via Poincaré duality) by the same symbol if no confusion can result. The intersection product of two elements γ, γ' in $A^*(X)$ (or $A_*(X)$ via Poincaré duality) will be denoted $\gamma \cdot \gamma'$. The class of a point will be denoted pt.

If $X = \mathbb{P}^r$, the hyperplane class will be called $H \in A^1(X)$, and the class of a line will be called $\boldsymbol{H'} \in A_1(X)$.

This chapter is organized as follows. In section 1.2, we recall the construction of the moduli spaces of stable maps and state some of their properties. The question whether these moduli spaces are smooth stacks of the expected dimension and what to do if they are not leads to the definition of virtual fundamental classes in section 1.3. We will then introduce Gromov-Witten invariants in 1.4 and gravitational descendants in 1.5. In section 1.6, we use gravitational descendants to obtain some enumerative results concerning curves with tangency conditions to a hyperplane in $\mathbb{P}^r$ and some virtual numbers of curves satisfying higher order contact conditions, which will be needed in section 3.4.

1.2 Moduli spaces of stable maps

In this section we will recall the construction of the moduli spaces of stable maps, which will be the basic objects of study both for Gromov-Witten theory and degeneration techniques. The main idea of this concept is to find a good compactification of the space of maps from $\mathbb{P}^1$ with n distinct marked points to X. This is done by allowing $\mathbb{P}^1$ to degenerate to certain singular curves, but it is of course crucial to only allow the "right" singular curves to get a well-behaved moduli space. This leads to the definition of stable maps, due to Kontsevich [K].

We start by recalling briefly the definitions of prestable and stable curves and their moduli spaces.

Definition 1.2.1 *An n-pointed **prestable curve** (of genus zero) $(C,x_1,\dots,x_n)$ is a proper, reduced, connected curve C with $h^1(C,\mathcal{O})=0$ and at worst ordinary double points as singularities, together with n distinct smooth points $x_i \in C$. The points x_i will be called the **marked** points of $(C,x_1,\dots,x_n)$. A point on C is called **special** if it is either a singular point of C or one of the x_i. A **morphism** $(C,x_1,\dots,x_n) \to (C',x'_1,\dots,x'_n)$ between n-pointed prestable curves is a morphism $\varphi : C \to C'$ such that $\varphi(x_i) = x'_i$ for all i.*

Note that the condition $h^1(C,\mathcal{O})=0$ means that C is a "tree of smooth rational curves".

Definition 1.2.2 *An n-pointed **stable curve** is an n-pointed prestable curve with finite automorphism group. Equivalently, it is an n-pointed prestable curve $(C,x_1,\dots,x_n)$ such that each irreducible component of C has at least three special points.*

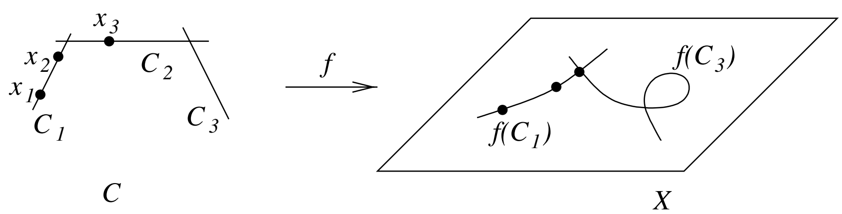
One then defines the notion of a family of stable curves in the usual way: a family of n-pointed stable curves over a base scheme S is a scheme C together with a flat, projective morphism $\pi : C \to S$ and n sections $x_1,\dots,x_n$ of π, such that for each geometric fibre $C_s \to s \in S$ of π, $(C_s,x_1(s),\dots,x_n(s))$ is an n-pointed stable curve. Together with the obvious definition of morphisms between such families, this defines a functor $\boldsymbol{\bar{M}_{0,n}}$ from the category of schemes to the category of sets (the subscript 0 refers to the genus of the curves). It is now a well-known theorem that $\bar{M}_{0,n}$ is actually a smooth proper algebraic Deligne-Mumford stack. There is also a projective scheme which is a coarse moduli space for this stack.

One now adapts this definition to the case of stable maps:

Definition 1.2.3 *An n-pointed* ***prestable map*** *to X is a tuple $\mathcal{C} = (C, x_1, \ldots, x_n, f)$ where $(C, x_1, \ldots, x_n)$ is a prestable curve and $f : C \to X$ is a morphism. We call $f_*[C] \in A_1(X)$ the* ***homology class*** *of $\mathcal{C}$. A* ***morphism*** *$(C, x_1, \ldots, x_n, f) \to (C', x'_1, \ldots, x'_n, f')$ between n-pointed prestable maps to X is a morphism $\varphi : C \to C'$ such that $f' \circ \varphi = f$ and $\varphi(x_i) = x'_i$ for all i.*

Definition 1.2.4 *An n-pointed* ***stable map*** *to X is an n-pointed prestable map to X whose automorphism group is finite. Equivalently, it is an n-pointed prestable map $(C, x_1, \ldots, x_n, f)$ to X such that each irreducible component of C on which f is constant has at least three special points. We will call a stable map $(C, x_1, \ldots, x_n, f)$* ***irreducible*** *if C is irreducible.*

The following picture shows an example of a stable map:



(Here the prestable curve C consists of three components C_1, C_2, C_3, of which the component C_2 gets contracted by f to a point. The map would not be stable without the marked point x_3.)

We will sometimes associate to a stable map a **topology** τ, by which we mean the homeomorphism class of the n-pointed topological space $(C, x_1, \ldots, x_n)$ together with the data of the homology classes $f_*[C_i] \in A_1(X)$ on each irreducible component C_i of C. This definition can be made much more precise and formal using the language of graphs [BM], however then the notation is likely to get very messy, so we will not make use of it.

We now say what a family of stable maps should be. This is exactly what one would expect: a family of n-pointed stable maps to X of homology class $\beta \in A_1(X)$ over a base scheme S is given by the data

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, shift right, "\pi"'] & X \\ S \arrow[u, shift right, bend right, "x_1, \ldots, x_n"'] & \end{tikzcd}$$

where C is a scheme, π is a flat, projective morphism and $x_1,\ldots,x_n$ are sections of π, such that for each geometric fibre $C_s \to s \in S$ of π, $(C_s, x_1(s),\ldots,x_n(s), f|_{C_s})$ is an n-pointed stable map to X of homology class β. Together with the definition of morphisms between such families, this again defines a functor $\bar{\boldsymbol{M}}_{\mathbf{0},\boldsymbol{n}}(\boldsymbol{X},\boldsymbol{\beta})$. The following deep theorem is already a strong indication that we made the right choice about which singular curves to allow.

Theorem 1.2.5 *$\bar{M}_{0,n}(X,\beta)$ is a proper algebraic Deligne-Mumford stack. Moreover, there exists a projective scheme which is a coarse moduli space for this stack.*

Proof See e.g. [BM] theorem 3.14, [ML] part I, [FP] section 1.2. □

Hence, when we talk about $\bar{M}_{0,n}(X,\beta)$ in the sequel we will always mean the corresponding stack. Most of our applications, however, can be done equally well on the coarse moduli space. In general, the moduli space $\bar{M}_{0,n}(X,\beta)$ will neither be smooth, nor irreducible, nor connected — we will meet lots of examples for this throughout our work.

Obviously, we can also consider the substack $M(X,\tau) \subset \bar{M}_{0,n}(X,\beta)$ consisting of all stable maps of topology τ, and the collection of the various $M(X,\tau)$ for fixed β (there are only finitely many of them) forms a decomposition of $\bar{M}_{0,n}(X,\beta)$. The substack of $\bar{M}_{0,n}(X,\beta)$ corresponding to irreducible stable maps will be denoted $\boldsymbol{M}_{\mathbf{0},\boldsymbol{n}}(\boldsymbol{X},\boldsymbol{\beta})$. If $X = \mathbb{P}^r$, one also writes $\bar{M}_{0,n}(\mathbb{P}^r,d)$ instead of $\bar{M}_{0,n}(\mathbb{P}^r,dH')$.

Here are some easy concrete examples for moduli spaces of stable maps.

- $\bar{M}_{0,n}(X,0) = \bar{M}_{0,n} \times X$.
- $\bar{M}_{0,0}(\mathbb{P}^r,1) = G(1,r)$ is the Grassmannian of lines in $\mathbb{P}^r$ (here all stable maps in the moduli space are irreducible).
- $\bar{M}_{0,1}(\mathbb{P}^r,1)$ is the universal line over $G(1,r)$ (again all stable maps in the moduli space are irreducible).

We now list a few standard facts about $\bar{M}_{0,n}(X,\beta)$.

Proposition 1.2.6

(i) There exist evaluation maps $ev_i : \bar{M}_{0,n}(X,\beta) \to X$ sending $(C,x_1,\ldots,x_n,f)$ to $f(x_i)$ $(1 \le i \le n)$.

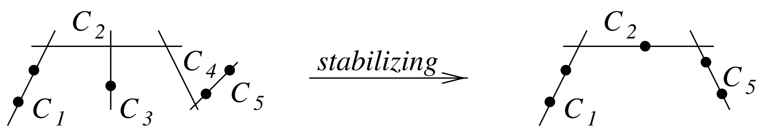
(ii) If $p : X \to Y$ is a morphism between smooth projective varieties, there is an induced map $\phi : \bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,n}(Y,p_*\beta)$ given by composing f with p and stabilizing if necessary.

(iii) If $n \ge 3$ or $\beta \ne 0$ there exist maps $\bar{M}_{0,n+1}(X,\beta) \to \bar{M}_{0,n}(X,\beta)$ given by forgetting x_{n+1} and stabilizing if necessary. These maps identify $\bar{M}_{0,n+1}(X,\beta)$ as the universal curve over $\bar{M}_{0,n}(X,\beta)$.

(iv) *If $n \geq 3$ there exist maps $\bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,n}$ given by forgetting the map f and stabilizing the curve if necessary.*

Proof See [BM]. □

Here, in (ii), (iii), and (iv) by "stabilizing if necessary" we mean that we contract components of C that have become unstable. For example, if we have a stable map in $\bar{M}_{0,n}(X,\beta)$ with underlying prestable curve $(C,x_1,\dots,x_n)$ as in the following picture on the left, and if we apply the map (iv) in the proposition, then the resulting stable curve in $\bar{M}_{0,n}$ will be the one on the right:



Finally, a word of warning: despite the suggestive notation, it is not in general true that $\bar{M}_{0,n}(X,\beta)$ is a compactification of $M_{0,n}(X,\beta)$ in the sense that $M_{0,n}(X,\beta)$ is dense in $\bar{M}_{0,n}(X,\beta)$. In fact, it may even happen that $M_{0,n}(X,\beta)$ is empty but $\bar{M}_{0,n}(X,\beta)$ is not: if we take for example X to be the blow-up of $\mathbb{P}^2$ in one point and $\beta = H+E$ (where H denotes the hyperplane class and E the exceptional divisor), then there are certainly reducible stable maps in X having homology class β but no irreducible ones.

In the next section we will study the question which conditions on X have to be satisfied in order for $\bar{M}_{0,n}(X,\beta)$ to be "well-behaved".

1.3 Virtual fundamental classes

We now try to compute the dimension of the moduli spaces of stable maps $\bar{M}_{0,n}(X,\beta)$. If we do this naively on the level of tangent spaces, we see that a deformation of a stable map $(C,x_1,\dots,x_n,f)$ is comprised of a deformation of the marked curve (this deformation space has dimension $\dim \bar{M}_{0,n} = n-3$) and a deformation of the map f with first order deformation space $H^0(C,f^*T_X)$. If we pretend that the obstruction space $H^1(C,f^*T_X)$ to deforming f vanishes, we would therefore get the result $\chi(C,f^*T_X)+n-3 = -K_X\cdot\beta+\dim X+n-3$ as the expected dimension of $\bar{M}_{0,n}(X,\beta)$. This motivates the following definition.

Definition 1.3.1 *We say that X is* ***convex*** *if $H^1(\mathbb{P}^1,f^*T_X)=0$ for all maps $f:\mathbb{P}^1\to X$. In any case, we call*

$$\mathrm{vdim}\ \bar{M}_{0,n}(X,\beta) := -K_X\cdot\beta+\dim X+n-3$$

the ***virtual*** *or* ***expected dimension*** *of the moduli space $\bar{M}_{0,n}(X,\beta)$.*

Note that this definition of convexity is equivalent to the condition that $H^1(C, f^*T_X) = 0$ for all prestable curves C of genus zero (see [FP] lemma 10).

Proposition 1.3.2 *If X is a convex variety then $\bar{M}_{0,n}(X,\beta)$ is a smooth stack of pure dimension vdim $\bar{M}_{0,n}(X,\beta)$. (Its coarse moduli space is then locally the quotient of a smooth variety by a finite group, and it is actually a fine moduli space away from the elements with non-trivial automorphism group.)*

Proof See e.g. [ML] part I, [FP] section 1.2. □

The most important examples for convex varieties are homogeneous spaces, hence in particular $\mathbb{P}^r$.

If X is convex, the basic idea of Gromov-Witten theory is to compute intersection products of total codimension vdim $\bar{M}_{0,n}(X,\beta)$ on the moduli space and evaluate them on its fundamental class to get some numbers that can then be interpreted geometrically. To be able to do this also in the case when X is not convex, one constructs a "virtual fundamental class"

$$[\bar{M}_{0,n}(X,\beta)]^{virt} \in A_{\mathrm{vdim}\ \bar{M}_{0,n}(X,\beta)}(\bar{M}_{0,n}(X,\beta))$$

that will serve as a replacement for the usual fundamental class. We now describe very briefly the construction of this virtual fundamental class as introduced by K. Behrend and B. Fantechi in [BF], [B].

The first ingredient of the construction of the virtual fundamental class is the **relative intrinsic normal cone** associated to a morphism $p: Y \to Z$, where Y and Z are algebraic stacks and Y is in addition of Deligne-Mumford type. To construct it, one chooses local embeddings (in the étale topology) of Y into a scheme M which is smooth over Z, i.e. we look at commutative diagrams of the form

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "i"'] & M \arrow[d, "j"] \\ Y \arrow[r, "p"] & Z \end{tikzcd}$$

where U and M are affine schemes, $g: U \to M$ is a local immersion, $i: U \to Y$ is étale and $j: M \to Z$ is smooth. One now defines the relative intrinsic normal cone $\mathcal{C}_{Y/Z}$ of $\pi: Y \to Z$ to be the (Artin) stack over Y which is (étale) locally on the various open subsets U of Y given by the stack quotient

$$\mathcal{C}_{Y/Z}|_U := [C_{U/M}/g^*T_{M/Z}]$$

where $C_{U/M}$ denotes the usual normal cone of U in M. (Of course one has to check that these local definitions glue to give a global object on Y.) The dimension of $\mathcal{C}_{Y/Z}$ is always equal to the dimension of Z.

As a simple example, we can look at the case where p itself is smooth on the open subset U of Y. Then we can choose $M = U$ and get the result that $\mathcal{C}_{Y/Z}|_U = BT_{U/Z} := [U/T_{U/Z}]$ is the so-called "classifying stack" of $T_{U/Z}$ over U, the fibre of which over a point $u \in U$ is "a point divided by the relative tangent space $T_{U/Z,u}$". In particular, if $p : Y \to Z$ is smooth everywhere, then $\mathcal{C}_{Y/Z} = BT_{Y/Z}$.

In our case we will apply this construction with $Y = \bar{M}_{0,n}(X,\beta)$ and $Z = \mathcal{M}_{0,n}$, where $\mathcal{M}_{0,n}$ denotes the moduli stack of n-pointed *pre*stable curves of genus zero (this is not a Deligne-Mumford stack since there exist elements in $\mathcal{M}_{0,n}$ with infinite automorphism group). The map p is given by sending $(C, x_1, \dots, x_n, f)$ to $(C, x_1, \dots, x_n)$, i.e. by forgetting the map f, but *without* stabilizing the curve (in contrast to proposition 1.2.6 (iv)). Hence we get a relative intrinsic normal cone $\mathcal{C}_{\bar{M}_{0,n}(X,\beta)/\mathcal{M}_{0,n}}$ over $\bar{M}_{0,n}(X,\beta)$.

The second ingredient of the construction of the virtual fundamental class is a **relative obstruction theory** for $\bar{M}_{0,n}(X,\beta)$ over $\mathcal{M}_{0,n}$. In our case, this is meant to be the two-term complex $R^\bullet\pi_* f^* T_X$ where

$$\begin{array}{ccc} \bar{M}_{0,n+1}(X,\beta) & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\pi} & & \\ \bar{M}_{0,n}(X,\beta) & & \end{array}$$

with $f = ev_{n+1}$ being the evaluation map (see proposition 1.2.6 (i)) and π being the map forgetting the point x_{n+1} and stabilizing (see proposition 1.2.6 (iii)). This complex can be realized in the derived category as a two-term complex of vector bundles $E_0 \to E_1$ (see [B] proposition 5), i.e. in particular we have ker $(E_0 \to E_1) = \pi_* f^* T_X$ and coker $(E_0 \to E_1) = R^1\pi_* f^* T_X$. The construction of the virtual fundamental class will not depend on the choice of this realization of the complex $R^\bullet\pi_* f^* T_X$.

We can now construct the Artin stack $[E_1/E_0]$ over $\bar{M}_{0,n}(X,\beta)$, and there is a natural closed immersion $\mathcal{C}_{\bar{M}_{0,n}(X,\beta)/\mathcal{M}_{0,n}} \hookrightarrow [E_1/E_0]$ (see [BF] theorem 4.5). Now consider the diagram of stacks over $\bar{M}_{0,n}(X,\beta)$

$$\begin{tikzcd} & \mathcal{C}' \arrow[r] \arrow[d, hook] & \mathcal{C}_{\bar{M}_{0,n}(X,\beta)/\mathcal{M}_{0,n}} \arrow[d, hook] \\ \bar{M}_{0,n}(X,\beta) \arrow[r, "0"] & E_1 \arrow[r] & {[E_1/E_0]} \end{tikzcd}$$

where $\mathcal{C}'$ is defined such that the square is cartesian, and where 0 denotes the zero section of the vector bundle stack E_1. We now define the **virtual fundamental class** of $\bar{M}_{0,n}(X,\beta)$ to be

$$[\bar{M}_{0,n}(X,\beta)]^{virt} = 0^![\mathcal{C}'] \in A_*(\bar{M}_{0,n}(X,\beta)).$$

Since the relative dimension of $[E_1/E_0]$ over $\bar{M}_{0,n}(X,\beta)$ is $-(-K_X \cdot \beta + \dim X)$ and the dimension of $\mathcal{C}_{\bar{M}_{0,n}(X,\beta)/\mathcal{M}_{0,n}}$ is $n-3$, the virtual fundamental class is in fact a cycle of dimension

$$n - 3 - K_X \cdot \beta + \dim X = \operatorname{vdim} \bar{M}_{0,n}(X,\beta),$$

as required.

A simple example is the case when there is an open subset $U \subset \bar{M}_{0,n}(X,\beta)$ (again in the étale topology) where there are no obstructions, i.e. $R^1\pi_* f^* T_X = 0$ on U. Then, locally on U, we can take $E_1 = U$ to be the trivial bundle on U and $E_0 = (\pi_* f^* T_X)|_U$. So we must also have $\mathcal{C}' = U$ (note that $\mathcal{C}' \hookrightarrow E_1$ is an inclusion of Deligne-Mumford stacks over U), and taking $0^![\mathcal{C}']$ will of course give us U again — or to be precise, when computing the virtual fundamental class on $\bar{M}_{0,n}(X,\beta)$ we will get the cycle $[\bar{U}]$ plus other cycles with support disjoint from U.

If we finally take into account the semicontinuity of the function $h^1(C, f^*T_X)$ on the moduli space $\bar{M}_{0,n}(X,\beta)$ and observe that $\bar{M}_{0,n}(X,\beta)$ is smooth over $\mathcal{M}_{0,n}$ and hence smooth over $\mathbb{C}$ at all points where there are no obstructions (see [BF] proposition 7.3), we have just proven

Lemma 1.3.3 *Let $(C, x_1, \ldots, x_n, f) \in \bar{M}_{0,n}(X,\beta)$ be a stable map with $h^1(C, f^*T_X) = 0$. Then $(C, x_1, \ldots, x_n, f)$ lies in a unique irreducible component Z of $\bar{M}_{0,n}(X,\beta)$ of dimension vdim $\bar{M}_{0,n}(X,\beta)$, and if R denotes the union of all the other irreducible components, then*

$$[\bar{M}_{0,n}(X,\beta)]^{virt} = [Z] + \textit{some cycle supported on } R.$$

In particular, if X is convex, so that $h^1(C, f^*T_X)$ always vanishes, then the virtual fundamental class coincides with the usual one. This "global version" of lemma 1.3.3 has also been stated in [BF] (proposition 7.3), however we will also need the "local version" from above in the next chapter. If the obstructions do not vanish but form a vector bundle, it follows by the definition of the map $0^!$ that one can compute the virtual fundamental class as follows:

Lemma 1.3.4 *If $E := R^1\pi_* f^* T_X$ is locally free, then*

$$[\bar{M}_{0,n}(X,\beta)]^{virt} = c_{\mathrm{rk}\, E}(E) \cdot [\bar{M}_{0,n}(X,\beta)].$$

Proof See [BF] proposition 7.3. □

We now give another possibility to compute virtual fundamental classes, which will be needed in chapter 3.

Proposition 1.3.5 *Let X be a smooth projective variety and H,Y smooth subvarieties of X intersecting transversally in $Q = H \cap Y$, so that we have a cartesian diagram of inclusions*

$$\begin{tikzcd} Q \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & H \arrow[d, hook, "h"] \\ Y \arrow[r, hook, "g"] & X. \end{tikzcd}$$

Assume that the map $A_1(Q) \to A_1(H) \oplus A_1(Y)$ induced by the inclusions is injective, so that there is a cartesian diagram of inclusions

$$\begin{tikzcd} \bar{M}_{0,n}(Q,\beta) \arrow[r, hook, "i'"] \arrow[d, hook, "j'"'] & \bar{M}_{0,n}(H,i_*\beta) \arrow[d, hook, "h'"] \\ \bar{M}_{0,n}(Y,j_*\beta) \arrow[r, hook, "g'"] & \bar{M}_{0,n}(X,h_*i_*\beta). \end{tikzcd}$$

Then, if X, H, and Y are convex, we have

$$[\bar{M}_{0,n}(Q,\beta)]^{virt} = \bar{M}_{0,n}(H,i_*\beta) \cdot \bar{M}_{0,n}(Y,j_*\beta) \in A_*(\bar{M}_{0,n}(Q,\beta)),$$

where the dot denotes the intersection product in $\bar{M}_{0,n}(X,h_*i_*\beta)$.

Remark 1.3.6 The assumption of the injectivity of the map $A_1(Q) \to A_1(H) \oplus A_1(Y)$ is not essential, it just simplifies the result a little bit. In general, the following proof shows that the intersection product $\bar{M}_{0,n}(H,i_*\beta) \cdot \bar{M}_{0,n}(Y,j_*\beta)$ yields the sum of all virtual fundamental classes $[\bar{M}_{0,n}(Q,\beta')]^{virt}$ with $i_*\beta' = i_*\beta$ and $j_*\beta' = j_*\beta$. Note that there can be only finitely many such β' with $\bar{M}_{0,n}(Q,\beta') \neq \emptyset$.

Remark 1.3.7 The proposition can in particular be used to describe the virtual fundamental class of any smooth hypersurface $Q \subset Y = \mathbb{P}^r$ of degree δ, if one takes $Y = \mathbb{P}^r \to X = \mathbb{P}^N$ with $N = \binom{r+\delta}{\delta} - 1$ to be the degree δ Veronese embedding and $H \subset X$ the hyperplane such that $Q = H \cap Y$.

Proof (of proposition 1.3.5) It is possible to prove this using [BF] proposition 7.5, by showing that the obstruction theories $R^\bullet\pi_{Q*}f_Q^*T_Q$ and $R^\bullet\pi_{H*}f_H^*T_H$ are "compatible" over g' in the sense of [BF]. (Here, as usual, $\pi_Q : \bar{M}_{0,n+1}(Q,\beta) \to \bar{M}_{0,n}(Q,\beta)$ is the universal curve and $f_Q = ev_{n+1} : \bar{M}_{0,n+1}(Q,\beta) \to Q$ the evaluation map, similarly for π_H and f_H.) However, since we did not introduce the notations used there, we will give an alternative proof here.

Recall that $[\bar{M}_{0,n}(Q,\beta)]^{virt}$ was defined by the relative obstruction theory $R^\bullet\pi_{Q*}f_Q^*T_Q$ over $\mathcal{M}_{0,n}$. On the other hand, the intersection product

$$\bar{M}_{0,n}(H,i_*\beta) \cdot \bar{M}_{0,n}(Y,j_*\beta) \in A_*(\bar{M}_{0,n}(Q,\beta))$$

can also be viewed as a virtual fundamental class arising from a relative obstruction theory, namely from the two-term complex $(j'^*T_{\bar{M}_Y} \to i'^*N_{\bar{M}_H/\bar{M}_X})$, where $T_{\bar{M}_Y}$ denotes the relative tangent bundle of $\bar{M}_{0,n}(Y, j_*\beta)$ over $\mathcal{M}_{0,n}$, similarly for $T_{\bar{M}_H}$ and $T_{\bar{M}_Y}$, and $N_{\bar{M}_H/\bar{M}_X} = h'^*T_{\bar{M}_X}/T_{\bar{M}_H}$. This follows e.g. from [BF] section 6, "the basic example". We will now show that these two relative obstruction theories coincide, i.e. that the corresponding two-term complexes are quasi-isomorphic, so that the two virtual fundamental classes agree.

First we look at the complex $(j'^*T_{\bar{M}_Y} \to i'^*N_{\bar{M}_H/\bar{M}_X})$ defining the intersection product. As Y is a convex variety, $\bar{M}_{0,n}(Y, j_*\beta)$ is a smooth stack, and its relative tangent bundle $T_{\bar{M}_Y}$ over $\mathcal{M}_{0,n}$ is given by the degree zero term of its relative obstruction theory, i.e. by $\pi_{Y*}f_Y^*T_Y$ (see [BF] definition 4.4). To compute $j'^*T_{\bar{M}_Y} = j'^*\pi_{Y*}f_Y^*T_Y$, note that we have a commutative diagram

$$\begin{tikzcd} Q \arrow[d, hook, "j"'] & \bar{M}_{0,n+1}(Q,\beta) \arrow[l, "f_Q"'] \arrow[r, "\pi_Q"] \arrow[d, hook, "j''"'] & \bar{M}_{0,n}(Q,\beta) \arrow[d, hook, "j'"] \\ Y & \bar{M}_{0,n+1}(Y, j_*\beta) \arrow[l, "f_Y"'] \arrow[r, "\pi_Y"] & \bar{M}_{0,n}(Y, j_*\beta). \end{tikzcd}$$

As π_Y and π_Q are flat morphisms and the right square is cartesian, it follows by [EGA3] remarques 7.7.9 that

$$\begin{aligned} j'^*T_{\bar{M}_Y} &= j'^*\pi_{Y*}f_Y^*T_Y \\ &= \pi_{Q*}j''^*f_Y^*T_Y \\ &= \pi_{Q*}f_Q^*j^*T_Y. \end{aligned}$$

Moreover, since Y is convex such that for every stable map $(C, x_1, \ldots, x_n, f)$ to Y (and in particular for every such stable map to Q) we have $H^1(C, f^*T_Y) = 0$, it follows that $R^1\pi_{Q*}f_Q^*j^*T_Y = 0$.

Similar calculations apply to H and X instead of Y, so we get by the same reasoning that

$$\begin{aligned} i'^*N_{\bar{M}_H/\bar{M}_X} &= \pi_{Q*}f_Q^*i^*N_{H/X} \\ &= \pi_{Q*}f_Q^*N_{Q/Y} \end{aligned}$$

and that $R^1\pi_{Q*}f_Q^*N_{Q/Y} = 0$.

We have thus shown that the relative obstruction theory $(j'^*T_{\bar{M}_Y} \to i'^*N_{\bar{M}_H/\bar{M}_X})$ used to define the intersection product is given by

$$R^\bullet\pi_{Q*}f_Q^*(j^*T_Y \to N_{Q/Y}).$$

But by the normal sequence of j, the complex (T_Q) is quasi-isomorphic to $(j^*T_Y \to N_{Q/Y})$, so the proposition follows. □

We finish this section by mentioning a non-trivial example of a virtual fundamental class that we will meet again several times throughout our work. We consider multiple covering maps of certain infinitesimally rigid curves on a threefold.

Lemma 1.3.8 *Let X be a smooth projective threefold and $\beta \in A_1(X)$ a homology class with $K_X \cdot \beta = 0$ (hence vdim $\bar{M}_{0,0}(X,\beta) = 0$). Let $L \subset X$ be a smooth, infinitesimally rigid rational curve in X with normal bundle $N_{L/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Assume that for some $d \geq 1$, we have $\beta = d\,[L]$.*

Then the moduli space $\bar{M}_{0,0}(X,\beta)$ contains a connected component $Z \cong \bar{M}_{0,0}(\mathbb{P}^1,d)$ of dimension $2d-2$ corresponding to degree d multiple covering maps $C \to L$, and the virtual fundamental class of ($\bar{M}_{0,0}(X,\beta)$ restricted to) Z is equal to

$$[Z]^{virt} = \int_{\bar{M}_{0,0}(\mathbb{P}^1,d)} c_{2d-2}\left(R^1\pi_* f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))\right) \in A_0(Z) \cong \mathbb{Q}$$

where $\pi : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \bar{M}_{0,0}(\mathbb{P}^1,d)$ is the universal curve and $f : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \mathbb{P}^1$ the evaluation map. (Note that this number need not be an integer since we are working on stacks.)

Proof As the curve L cannot be deformed in X (not even infinitesimally), it is clear that the space of stable maps into L forms a connected component Z of $\bar{M}_{0,0}(X,\beta)$ which is obviously isomorphic to the space of stable maps to $L \cong \mathbb{P}^1$ of degree d, hence $Z = \bar{M}_{0,0}(\mathbb{P}^1,d)$.

Because of the normal sequence

$$0 \to f^*T_L \to f^*N_{L/X} \to f^*T_X \to 0$$

and $h^1(C, f^*T_L) = 0$ we can write

$$R^1\pi_* f^*T_X = R^1\pi_* f^*N_{L/X} = R^1\pi_* f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1)).$$

The statement of the lemma now follows from lemma 1.3.4. □

Note that this integral depends on nothing but d, in particular not on the variety X. We will postpone the actual computation of this number to example 2.8.5, it will turn out to be d^{-3}. In fact, this number has some history. Its most important application is the case where X is a quintic threefold, so that $K_X \cdot \beta = 0$ for all β. All methods to compute the numbers of rational curves of a given degree on X will determine the degree of the zero-cycle $[\bar{M}_{0,0}(X,\beta)]^{virt} \in A_0(\bar{M}_{0,0}(X,\beta))$. The result above tells you that this number counts not only the number of rational curves of class β, but also d-fold covering maps of all rational curves of class β/d. Knowing that these multiple coverings are counted with multiplicity d^{-3}, one can then subtract them from the degree of the zero-cycle $[\bar{M}_{0,0}(X,\beta)]^{virt}$ to get the actual number of rational curves of degree β on X.

When the numbers of rational curves on the quintic threefold had been computed first by physicists [COGP], they just guessed the multiplicity d^{-3} because it was the only one that turned their predictions of the number of rational curves into non-negative integers. Later, Yu. Manin [M] and independently P. Aspinwall and D. Morrison [AM] (using an a priori different definition of the multiplicity) derived this multiplicity rigorously, however their methods are very complicated. In example 2.8.5, we will give a remarkably simple way to compute it as a byproduct of our work on Gromov-Witten invariants of blow-ups.

1.4 Gromov-Witten invariants

We now come to the definition of Gromov-Witten invariants. Let X be a smooth projective r-dimensional variety, $\beta \in A_1(X)$ an effective homology class, and $n \geq 0$. Let $\gamma_1, \ldots, \gamma_n \in A^*(X)$ be classes on X. Then we define the associated **Gromov-Witten invariant** to be the intersection product on $\bar{M}_{0,n}(X,\beta)$

$$\boldsymbol{I^X_\beta(\gamma_1 \otimes \ldots \otimes \gamma_n)} := (ev_1^*\gamma_1 \cdot \ldots \cdot ev_n^*\gamma_n) \cdot [\bar{M}_{0,n}(X,\beta)]^{virt} \in \mathbb{Q}.$$

if $\sum_{i=1}^n \operatorname{codim} \gamma_i = \operatorname{vdim} \bar{M}_{0,n}(X,\beta)$, and zero otherwise.

The idea of this definition is to count (irreducible) stable maps $(C, x_1, \ldots, x_n, f)$ of homology class β with $f(x_i) \in V_i$ for all i, where the V_i are generic subschemes of X representing the classes γ_i. It is however not clear that this interpretation is valid, and indeed in some cases it is not.

Note that, as $\bar{M}_{0,n}(X,\beta)$ is a Deligne-Mumford stack, the Gromov-Witten invariants need not be integers. In many cases, however, they will be non-negative integers, in particular if they have an enumerative meaning as certain numbers of curves.

Now some remarks concerning the notation. We will often drop the superscript X. The Gromov-Witten invariant is by definition multilinear in the γ_i, therefore we use the notation $\gamma_1 \otimes \ldots \otimes \gamma_n$. (It is obviously also symmetric under permutations of the γ_i, but we will not use the notation $\gamma_1 \cdot \ldots \cdot \gamma_n$ because we want to reserve the dot notation for the intersection product of cycles.) Because of the multilinearity, we will often choose a homogeneous basis $\mathcal{B} = \{T_0, \ldots, T_q\}$ of the vector space $A^*(X)$ and only consider invariants where the γ_i are chosen from among this basis. To shorten notation, we will often write $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ or $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ and call $\mathcal{T} \in (A^*(X))^{\otimes n}$ a **collection of classes**. Correspondingly, we write $ev^*\mathcal{T}$ for $ev_1^*\gamma_1 \cdot \ldots \cdot ev_n^*\gamma_n$. If $X = \mathbb{P}^r$, the invariant $I_\beta(\mathcal{T})$ is also denoted by $I_d(\mathcal{T})$, where $\beta = d\,H'$. If the fundamental class of X is in the invariant, we will write this as $I_\beta(X \otimes \ldots)$, as we want to reserve the notation $I_\beta(1)$ for the case where $n = 0$, considering 1 as an element in $(A^*(X))^{\otimes 0}$.

There are now two obvious questions concerning the invariants: firstly how to compute them, and secondly whether they are enumeratively meaningful, i.e. whether they are really equal to the number of curves in X with homology class β satisfying the given incidence conditions.

We will first address the question of computation. The key ingredients of the computation are the following four relations among the invariants:

Proposition 1.4.1 *Properties of Gromov-Witten invariants*

(i) ***(Mapping to a point)*** *If $\beta = 0$, then the invariant is equal to the triple intersection product:*

$$I_0(\gamma_1 \otimes \ldots \otimes \gamma_n) = \begin{cases} \gamma_1 \cdot \gamma_2 \cdot \gamma_3 & \text{if } n = 3 \text{ and } \sum_i \operatorname{codim} \gamma_i = r, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) ***(Fundamental class)*** *If $\beta \neq 0$ and the invariant contains the fundamental class of X, then the invariant is zero:*

$$I_\beta(X \otimes \mathcal{T}) = 0 \quad \text{for all } \mathcal{T} \text{ and all } \beta \neq 0.$$

(iii) ***(Divisor axiom)*** *If $\beta \neq 0$ and $\gamma \in A^1(X)$ is a divisor, then*

$$I_\beta(\gamma \otimes \mathcal{T}) = (\gamma \cdot \beta)\, I_\beta(\mathcal{T}) \quad \text{for all } \mathcal{T}.$$

(iv) ***(Splitting axiom)*** *Choose a homogeneous basis $\mathcal{B} = \{T_0, \dots, T_q\}$ of $A^*(X)$, define $g = (g_{ij})$ to be the intersection matrix*

$$g_{ij} = \begin{cases} T_i \cdot T_j & \text{if codim } T_i + \text{codim } T_j = r, \\ 0 & \text{otherwise,} \end{cases}$$

and let $g^{-1} = (g^{ij})$ be the inverse matrix. Choose $\beta \in A_1(X)$, four classes $\mu_1, \dots, \mu_4 \in A^*(X)$ and a collection $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ of classes such that

$$\sum_{i=1}^{n} \text{codim } \gamma_i + \sum_{i=1}^{4} \text{codim } \mu_i = -K_X \cdot \beta + r + n.$$

Then we have the equation

$$0 = \sum_{\beta_1, \beta_2} \sum_{\mathcal{T}_1, \mathcal{T}_2} \sum_{i,j} g^{ij} \Big(I_{\beta_1}(\mathcal{T}_1 \otimes \mu_1 \otimes \mu_2 \otimes T_i)\, I_{\beta_2}(\mathcal{T}_2 \otimes \mu_3 \otimes \mu_4 \otimes T_j)$$
$$- I_{\beta_1}(\mathcal{T}_1 \otimes \mu_1 \otimes \mu_3 \otimes T_i)\, I_{\beta_2}(\mathcal{T}_2 \otimes \mu_2 \otimes \mu_4 \otimes T_j) \Big).$$

where the sum is taken over

- *all effective classes $\beta_1, \beta_2 \in A_1(X)$ with $\beta_1 + \beta_2 = \beta$,*
- *all $\mathcal{T}_1 = \gamma_{i_1} \otimes \ldots \otimes \gamma_{i_{n_1}}$ and $\mathcal{T}_2 = \gamma_{j_1} \otimes \ldots \otimes \gamma_{j_{n_2}}$ such that $i_1 < \cdots < i_{n_1}$, $j_1 < \cdots < j_{n_2}$, and $\{i_1, \dots, i_{n_1}\} \dot\cup \{j_1, \dots, j_{n_2}\} = \{1, \dots, n\}$ (i.e. "the classes of $\mathcal{T}$ get distributed in all possible ways onto the two factors"),*
- *all $0 \leq i, j \leq q$.*

In the sequel we will call this equation $\boldsymbol{\mathcal{E}_\beta(\mathcal{T}\,;\, \mu_1, \mu_2 \,|\, \mu_3, \mu_4)}$.

Proof See e.g. [ML] part I, [FP] in the case of convex X, or [KM1], [B] for general X. In the convex case, the ideas behind the four properties are as follows:

(i) This follows from the fact that $\bar{M}_{0,n}(X, 0) = \bar{M}_{0,n} \times X$.

(ii) If γ_n is the fundamental class of X in the invariant $I_\beta(\gamma_1 \otimes \ldots \otimes \gamma_n)$, then the intersection product to be computed on $\bar{M}_{0,n}(X, \beta)$ is actually the pull-back of an intersection product on $\bar{M}_{0,n-1}(X, \beta)$, but on $\bar{M}_{0,n-1}(X, \beta)$ it vanishes for dimensional reasons.

(iii) This can be understood geometrically since the condition that an additional marked point is mapped to a divisor γ does not restrict the curve at all, but fixes the marked point to be one of the $\gamma \cdot \beta$ points of intersection of the curve with the divisor.

(iv) One can derive this equation by considering the morphism $\pi : \bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,4} \cong \mathbb{P}^1$ (for $n \geq 4$) forgetting the map and all but the first four marked points (see proposition 1.2.6). One now looks at the inverse image under π of the two points in $\bar{M}_{0,4}$ corresponding to the stable curves

A generic point in this inverse image can be shown to correspond to a reducible curve with two irreducible components, with the marked points x_1,x_2 (resp. x_1,x_3) on the one component and x_3,x_4 (resp. x_2,x_4) on the other, with the other marked points distributed in any way on the two components, and with homology classes β_1,β_2 on the two components such that $\beta_1+\beta_2=\beta$. The linear equivalence of the pullback of the two above points (i.e. divisors) in $\bar{M}_{0,4}$ then yields the desired equation.

□

The relations (i), (ii), and (iii) just tell us that we know all invariants with $\beta = 0$, and that we do not have to consider fundamental classes and divisors in the invariants. The most important (and most complicated) equations are of course those of the splitting axiom, which are sometimes also called the **associativity equations of quantum cohomology** or the **WDVV equations** (the name "splitting *axiom*" for these equations has historical reasons, for they have been written down before the theory of virtual fundamental classes existed). We will use them in the sequel in the form where we split off the summands where β_1 or β_2 are zero: by part (i) of the proposition and the definition of g^{ij} the equation $\mathcal{E}_\beta(\mathcal{T}\,;\mu_1,\mu_2\,|\,\mu_3,\mu_4)$ then becomes

$$\begin{aligned}
0 = {} & I_\beta(\mathcal{T}\otimes\mu_1\otimes\mu_2\otimes\mu_3\cdot\mu_4) + I_\beta(\mathcal{T}\otimes\mu_3\otimes\mu_4\otimes\mu_1\cdot\mu_2) \\
& - I_\beta(\mathcal{T}\otimes\mu_1\otimes\mu_3\otimes\mu_2\cdot\mu_4) - I_\beta(\mathcal{T}\otimes\mu_2\otimes\mu_4\otimes\mu_1\cdot\mu_3) \\
& + \sum_{\beta_1,\beta_2\neq 0}\ \sum_{\mathcal{T}_1,\mathcal{T}_2}\ \sum_{i,j} g^{ij}\Big(I_{\beta_1}(\mathcal{T}_1\otimes\mu_1\otimes\mu_2\otimes T_i)\,I_{\beta_2}(\mathcal{T}_2\otimes\mu_3\otimes\mu_4\otimes T_j) \\
& \qquad - I_{\beta_1}(\mathcal{T}_1\otimes\mu_1\otimes\mu_3\otimes T_i)\,I_{\beta_2}(\mathcal{T}_2\otimes\mu_2\otimes\mu_4\otimes T_j)\Big).
\end{aligned}$$

One can try to use these equations to determine all Gromov-Witten invariants recursively from some hopefully small set of initial numbers that can be calculated by other means. A result in this direction is the following proposition.

Proposition 1.4.2 ***(First Reconstruction Theorem)*** *If $A^*(X)$ is generated as a ring by divisor classes, then there exists an explicit algorithm to compute all Gromov-Witten invariants recursively from those invariants $I_\beta(\mathcal{T})$ where $\mathcal{T}$ contains at most two classes.*

Proof See [KM1] theorem 3.1. The interested reader may perhaps want to look at the proof of lemma 2.2.4 (iii) which is completely analogous and should show how the First Reconstruction Theorem works. □

How strong this statement is depends very much on the variety X. For example, on $X = \mathbb{P}^r$ the only invariant with at most two classes is $I_{H'}(pt \otimes pt)$, which is 1 for geometrical reasons since this invariant simply counts the number of lines through two points. Hence in this case we can calculate all Gromov-Witten invariants. For example, the following table lists some of the famous numbers $N_d = I^{\mathbb{P}^2}_{dH'}(pt^{\otimes(3d-1)})$ of rational curves of degree d in $\mathbb{P}^2$ through $3d-1$ generic points:

d	1	2	3	4	5	6	7
N_d	1	1	12	620	87304	26312976	14616808192

However, if for example X is a quintic threefold in $\mathbb{P}^4$, then the only invariants on X are those corresponding to the numbers of degree d rational curves on X without any further conditions, hence the invariants are $I_{dH'}(1)$, where H' is the class of a line in the quintic and 1 denotes the element $1 \in (A^*(X))^{\otimes 0}$. Therefore, in this case the proposition does not help at all to compute these numbers (and in fact all equations from the splitting axiom are trivial in this case).

Finally we come to the question of enumerative significance of the invariants. The following proposition tells us that in the case of homogeneous varieties the invariants actually count what one would expect.

Proposition 1.4.3 *Let $X = G/P$ be a homogeneous variety, where G is a Lie group and P a parabolic subgroup. Let $\beta \in A_1(X)$ be an effective homology class and let $V_1, \dots, V_n$ be pure-dimensional subvarieties of X with $[V_i] = \gamma_i \in A^*(X)$ such that*

$$\sum_i \mathit{codim}\ \gamma_i = \mathit{vdim}\ \bar{M}_{0,n}(X,\beta).$$

Then, for generic elements $g_i \in G$, the Gromov-Witten invariant $I_\beta(\gamma_1 \otimes \ldots \otimes \gamma_n)$ is equal to the number of irreducible stable maps $(C, x_1, \dots, x_n, f)$ with $f_*[C] = \beta$ and $f(x_i) \in g_i V_i$ for all i, each counted with multiplicity one, and for all these stable maps the morphism $f : C \to X$ is generically injective.

Proof This is basically the Bertini-Kleiman theorem together with the statement that homogeneous varieties are convex so that their virtual fundamental class coincides with the usual one. See e.g. [ML] part I, [FP] lemma 14. □

1.5 Gravitational descendants

We will now extend the definition of Gromov-Witten invariants in that we want to compute intersection numbers on $\bar{M}_{0,n}(X,\beta)$ of a bigger variety of classes than just pullbacks of classes on X via the evaluation maps.

Recall that $\pi : \bar{M}_{0,n+1}(X,\beta) \to \bar{M}_{0,n}(X,\beta)$ can be identified with the universal curve over $\bar{M}_{0,n}(X,\beta)$ by proposition 1.2.6 (iii). So, for any $1 \le i \le n$, we have sections $s_i : \bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,n+1}(X,\beta)$ of π mapping the stable map $\mathcal{C} = (C, x_1, \dots, x_n, f)$ to the point x_i on $\mathcal{C}$. We now define the i-th **cotangent line** to be the line bundle

$$L_i := s_i^* \omega_\pi$$

on $\bar{M}_{0,n}(X,\beta)$, where ω_π denotes the relative dualizing sheaf of π. It can also be characterized by the exact sequence

$$0 \to L_i^\vee \to s_i^* T_{\bar{M}_{0,n+1}(X,\beta)} \xrightarrow{s_i^*(d\pi)} T_{\bar{M}_{0,n}(X,\beta)} \to 0$$

on $\bar{M}_{0,n}(X,\beta)$ (note that $d\pi$ is not of maximal rank everywhere on $\bar{M}_{0,n+1}(X,\beta)$, but it is of maximal rank at all points in $s_i(\bar{M}_{0,n}(X,\beta))$ since the marked points of a stable map are always nonsingular points on the curve). The fibre of L_i over a stable map $(C, x_1, \dots, x_n, f)$ is obviously canonically isomorphic to the cotangent space $T^\vee_{C,x_i}$.

The definition of the so-called **gravitational descendants** is now in complete analogy to the definition of the Gromov-Witten invariants, however we also allow the first Chern classes of the cotangent lines L_i in the intersection product. Hence we define

$$\boldsymbol{I_\beta^X(\gamma_1 c^{k_1} \otimes \ldots \otimes \gamma_n c^{k_n})} := \left(\prod_{i=1}^{n} (ev_i^* \gamma_i \cdot c_1^{k_i}(L_i)) \right) \cdot [\bar{M}_{0,n}(X,\beta)]^{virt} \in \mathbb{Q}$$

where c is to be considered as a formal variable, Π denotes the intersection product, and where $\gamma_i \in A^*(X)$ and $k_i \ge 0$ are such that the dimension condition

$$\sum_{i=1}^{n} (\text{codim } \gamma_i + k_i) = \text{vdim } \bar{M}_{0,n}(X,\beta)$$

is satisfied (otherwise we define the invariant to be zero, as usual). The Gromov-Witten invariants are obviously included here if we set all k_i equal to zero.

In analogy to the properties of the Gromov-Witten invariants in proposition 1.4.1, there are similar rules for the gravitational descendants that allow one to always calculate the descendants from the Gromov-Witten invariants:

Proposition 1.5.1 *For any smooth projective variety X, there is an explicit algorithm to compute all gravitational descendants on X from the Gromov-Witten invariants on X.*

Proof See e.g. [KM2] theorem 1.2, [G] equation (2). □

Hence, in particular, all gravitational descendants are known on $\mathbb{P}^r$.

Concerning the enumerative significance of the gravitational descendants, nothing at all has appeared in the literature so far, and only very few results seem to be known. Indeed, the question of enumerative significance is much more delicate here, especially since there is not even an obvious educated guess one could make about the meaning of the invariants as in the Gromov-Witten case. In general, one will have to combine several Gromov-Witten invariants and descendants to get geometrically interpretable numbers. Nevertheless, gravitational descendants are a useful tool in enumerative geometry. We will see some applications in the next section, and in particular later in section 3.4.

1.6 Curves with higher order contact

In this section we will use gravitational descendants to compute some virtual numbers of rational curves in $X = \mathbb{P}^r$ having contact of given order m to a hyperplane $H \subset X$ in a given subvariety of H (and satisfying additional incidence conditions such that we expect finitely many such curves). We say "virtual" here because in many cases, the numbers that we calculate receive contributions from possibly infinite families of curves with components in H, such that they cannot be interpreted directly in enumerative geometry. We will see how to compute the enumeratively correct numbers in section 3.2. We include the virtual results here mainly because they will be needed in section 3.4, where they give new results in combination with degeneration invariants. As a first application, we give a method to compute enumeratively correct numbers of curves with certain tangency conditions (that could also be obtained by other methods, though).

Let $X = \mathbb{P}^r$ and let $H \subset X$ be a fixed hyperplane. Fix $n \geq 1$ and $d \geq 1$ and consider the moduli space $\bar{M}_{0,n}(X,d)$. We are going to define two subspaces $\bar{M}^{(m)}$, $\bar{M}'^{(m)}$ of $\bar{M}_{0,n}(X,d)$ that can be considered as moduli spaces of stable maps having contact of order m to H at the point x_1 of the curve.

Definition 1.6.1 *For $m \geq 1$, we denote by $\boldsymbol{\bar{M}^{(m)}}$ the closure in $\bar{M}_{0,n}(X,\beta)$ of the space of irreducible stable maps $(C, x_1, \ldots, x_n, f)$ of degree d to X with $f(C) \not\subset H$ such that the divisor f^*H on C contains the point x_1 with multiplicity m.*

To define $\bar{M}'^{(m)}$, we need some preliminary remarks. We set $\boldsymbol{\bar{M}_n} := \bar{M}_{0,n}(X,d)$ and $\boldsymbol{\bar{M}_{n+1}} := \bar{M}_{0,n+1}(X,d)$, and consider the commutative diagram

$$\begin{tikzcd} \bar{M}_{n+1} \arrow[d, shift left, "\pi"] \arrow[dr, "ev_{n+1}"] & \\ \bar{M}_n \arrow[u, shift left, "s_1"] \arrow[r, "ev_1"'] & X \end{tikzcd}$$

where s_1 maps $(C, x_1, \ldots, x_n, f) \in \bar{M}_n$ to the point x_1 on the universal curve $\bar{M}_{n+1}$. Let the equation of H be $h = 0$ for $h \in H^0(\mathcal{O}_X(H))$. Then we have a section

$$ev_{n+1}^* h \in H^0(ev_{n+1}^* \mathcal{O}(H))$$

on $\bar{M}_{n+1}$. Differentiating this section up to order $m-1$ with respect to x_{n+1} yields a section

$$d^{m-1} ev_{n+1}^* h \in H^0(\mathcal{P}^{m-1}_{\bar{M}_{n+1}/\bar{M}_n}(ev_{n+1}^* \mathcal{O}(H))),$$

where $\mathcal{P}^{m-1}_{\bar{M}_{n+1}/\bar{M}_n}$ denotes the functor of relative principal parts of order $m-1$ (or $(m-1)$-jets) and $d^{m-1} = d^{m-1}_{\bar{M}_{n+1}/\bar{M}_n}$ is the derivative up to order $m-1$, see [EGA4] 16.3, 16.7.2.1 for precise definitions. Now we take the pullback of this section via s_1 to obtain a section $s \in H^0(E)$ on $\bar{M}_n$, where

$$\boldsymbol{s} := s_1^* d^{m-1} ev_{n+1}^* h \qquad \text{and} \qquad \boldsymbol{E} := s_1^* \mathcal{P}^{m-1}_{\bar{M}_{n+1}/\bar{M}_n} \otimes ev_1^* \mathcal{O}(H)$$

(where $\mathcal{P}^{m-1}_{\bar{M}_{n+1}/\bar{M}_n} := \mathcal{P}^{m-1}_{\bar{M}_{n+1}/\bar{M}_n}(\mathcal{O})$).

Definition 1.6.2 *With the above notation, we define* $\boldsymbol{\bar{M}'^{(m)}}$ *to be the zero scheme of the section* $s \in H^0(E)$.

The definition of $\bar{M}'^{(m)}$ expresses exactly the condition that the map $h \circ f$ has to vanish up to order $m-1$ at the point x_1. It is obvious from the definitions that $\bar{M}^{(m)} \subset \bar{M}'^{(m)}$. The expected codimension of both spaces in $\bar{M}_{0,n}(X, d)$ is m.

We gave these two definitions because the space $\bar{M}^{(m)}$ is the "enumeratively correct one" in the sense that the curves we want to count are dense in it, whereas the space $\bar{M}'^{(m)}$ is easier to describe since it is the zero locus of a section of a vector bundle that we know well (see next lemma). We are now going to compare the two spaces $\bar{M}^{(m)}$ and $\bar{M}'^{(m)}$. Let $\boldsymbol{R} \subset \bar{M}_{0,n}(X, d)$ be the substack of reducible stable maps and $\boldsymbol{Z} \subset \bar{M}_{0,n}(X, d)$ the substack corresponding to stable maps $(C, x_1, \ldots, x_n, f)$ where x_1 lies on a component C_0 of C with $f(C_0) \subset H$.

Lemma 1.6.3 *E is a vector bundle of rank m on $\bar{M}_{0,n}(X, d)$ with top Chern class $c_m(E) = \prod_{i=0}^{m-1}(i c_1(L_1) + ev_1^* H)$, where L_1 denotes the first cotangent line as in the previous section.*

Proof We use the notations introduced above. As the morphism π is smooth at all points in $s_1(\bar{M}_n)$, there is an exact sequence

$$0 \to L_1^{\otimes i} \to s_1^* \mathcal{P}^i_{\bar{M}_{n+1}/\bar{M}_n} \to s_1^* \mathcal{P}^{i-1}_{\bar{M}_{n+1}/\bar{M}_n} \to 0$$

for all $i > 0$ (see e.g. [EGA4] 16.10.1, 16.7.3), and hence

$$0 \to L_1^{\otimes i} \otimes ev_1^* \mathcal{O}(H) \to s_1^* \mathcal{P}^i_{\bar{M}_{n+1}/\bar{M}_n} \otimes ev_1^* \mathcal{O}(H) \to s_1^* \mathcal{P}^{i-1}_{\bar{M}_{n+1}/\bar{M}_n} \otimes ev_1^* \mathcal{O}(H) \to 0.$$

As $s_1^*\mathcal{P}^0_{\bar{M}_{n+1}/\bar{M}_n} = \mathcal{O}$, it follows by induction that E is a vector bundle of rank m, and that its top Chern class is given by

$$c_m(E) = \prod_{i=0}^{m-1} (i\,c_1(L_1) + ev_1^* H).$$

□

Lemma 1.6.4 *If $\bar{M}'^{(m)} \backslash (R \cup Z)$ is not empty, it is reduced of pure codimension m in $\bar{M}_{0,n}(X,d)$. Moreover, there are local equations defining $\bar{M}'^{(m)}$ away from $R \cup Z$ that form a regular sequence.*

Proof If $m > d$, then $\bar{M}'^{(m)} \backslash (R \cup Z)$ is obviously empty, so we assume from now on that $m \le d$.

For an irreducible stable map $(C, x_1, \dots, x_n, f)$ to $\mathbb{P}^r$ not contained in H, the map f is given by $r+1$ sections $f_0, \dots, f_r \in H^0(\mathbb{P}^1, \mathcal{O}(d))$. Choose coordinates $(u : v)$ on $\mathbb{P}^1$ such that $x_1 \in \mathbb{P}^1$ is the point $(0:1)$, and choose coordinates $(z_0 : \cdots : z_r)$ on $\mathbb{P}^r$ such that the equation of H is $z_0 = 0$. If we then write $f_0(u:v) = \sum_i a_i u^i v^{d-i}$, then the equation $s = 0$ is given by

$$\begin{aligned} f_0(x_1) = \frac{\partial f_0}{\partial u}(x_1) = \cdots = \frac{\partial^{m-1} f_0}{\partial u^{m-1}}(x_1) = 0 \\ \iff a_0 = a_1 = \cdots = a_{m-1} = 0. \end{aligned}$$

From this we see that the m functions defining $\bar{M}'^{(m)}$ locally form a regular sequence and that $\bar{M}'^{(m)}$ is reduced away from $R \cup Z$. □

Lemma 1.6.5

(i) The stack $\bar{M}'^{(m)} \cap (R \backslash Z)$ has codimension at least $m+1$ in $\bar{M}_{0,n}(X,d)$.

(ii) At a point $(C, x_1, \dots, x_n, f) \in \bar{M}'^{(m)} \cap (R \cap Z)$, $\bar{M}'^{(m)} \cap (R \cap Z)$ has codimension at least $d_0 + 2$ in $\bar{M}_{0,n}(X,d)$, where d_0 is the degree of f on the component on which x_1 lies.

Proof We start with (i) and decompose $\bar{M}_{0,n}(X,d)$ into the subspaces $M(X,\tau)$ according to the topology of the curves. So fix a reducible topology τ and consider a stable map $(C, x_1, \dots, x_n, f) \in \bar{M}'^{(m)} \cap M(X,\tau)$ such that x_1 lies on a component C_0 of C that is not mapped into H by f. Consider the connected components $C_1, \dots, C_\ell$ of $\overline{C \backslash C_0}$. Assume that n_i of the marked points are on C_i and that f has degree d_i on C_i for $0 \le i \le \ell$. We also mark the intersection points $C_0 \cap C_i$, which we call $p_1, \dots, p_\ell$ on C_0 and q_i on C_i, such that C_0 becomes a stable map with $n_0 + \ell$ marked points and C_i becomes a stable map with $n_i + 1$ marked points for $i > 0$. Then, by lemma 1.6.4, C_0 varies in a family of dimension

$$\dim \bar{M}_{0,n_0+\ell}(X, d_0) - m = d_0(r+1) + r + n_0 + \ell - m - 3, \tag{1}$$

whereas C_i for $i > 0$ varies in a family of dimension at most

$$\dim \bar{M}_{0,n_i+1}(X,d_i) = d_i(r+1)+r+n_i-2.$$

Moreover, the condition that the points p_i and q_i have to map to the same point in X reduces the dimension of the family by $r = \dim X$ for each pair of points (it is clear, e.g. because of the projective automorphisms, that these are r independent conditions each). So we get the result that the dimension of $\bar{M}'^{(m)} \cap M(X,\tau)$ at our chosen stable map is at most

$$\begin{aligned}\dim \bar{M}_{0,n_0+\ell}(X,d_0) - m - \sum_{i=1}^{\ell} \dim \bar{M}_{0,n_i+1}(X,d_i) - r\ell \\ = d(r+1)+r+n-m-3-\ell \\ = \dim \bar{M}_{0,n}(X,d) - m - \ell.\end{aligned}$$

As $\ell \geq 1$, we have proven (i).

As for (ii), the proof is exactly the same, with the only exception that the curve C_0 is a stable map *contained in H without multiplicity conditions*, so C_0 varies in a family of dimension

$$\dim \bar{M}_{0,n_0+\ell}(H,d_0) = \dim \bar{M}_{0,n_0+\ell}(X,d_0) - d_0 - 1$$

instead of $\bar{M}_{0,n_0+\ell}(X,d_0) - m$ as in (1). This replaces m with d_0+1 in the result. □

We now come to the main result that tells us how to compute numbers of curves satisfying contact conditions, modulo correction terms from certain curves with components in H.

Proposition 1.6.6 *For $m \leq d$, the stack $\bar{M}^{(m)}$ has codimension m in $\bar{M}_{0,n}(X,d)$, and its fundamental class satisfies*

$$[\bar{M}^{(m)}] = \prod_{i=0}^{m-1} (i\,c_1(L_1) + ev_1^* H) + \mu,$$

where μ is a cycle with support in $R \cap Z$, i.e. in the space of those reducible stable maps $(C, x_1, \ldots, x_n, f)$ such that x_1 lies on a component C_0 of C with $f(C_0) \subset H$. If moreover $m = 2$, then μ is a cycle with support in the space of reducible stable maps $(C, x_1, \ldots, x_n, f)$ such that x_1 lies on a component that is contracted *by f.*

Proof By definition, it is clear that (as sets)

$$\bar{M}^{(m)} = \overline{\bar{M}'^{(m)} \backslash (R \cup Z \cup \bar{M}'^{(m+1)})}. \tag{1}$$

First of all, we see that $\bar{M}^{(m)}$ has codimension m in $\bar{M}_{0,n}(X,d)$ by lemma 1.6.4.

Again by lemma 1.6.4, $\bar{M}'^{(m+1)} \backslash (R \cup Z)$ has codimension at least $m+1$ in $\bar{M}_{0,n}(X,d)$, so we can write (1) as

$$\bar{M}^{(m)} = \overline{\bar{M}'^{(m)} \backslash (R \cup Z)}.$$

We now want to take the fundamental class on both sides of the equation. Note that the fundamental class of a section of a vector bundle is given by the top Chern class of the bundle if the functions defining the section locally form a regular sequence (see [F1] proposition 14.1). In our case, this is true away from $R \cup Z$ by lemma 1.6.4, so by lemma 1.6.3 we get

$$[\bar{M}^{(m)}] = \prod_{i=0}^{m-1} (i c_1(L_1) + ev_1^* H) + \mu,$$

where $\mu \in A^m(\bar{M}_{0,n}(X,d))$ is a cycle with support in $\bar{M}'^{(m)} \cap (R \cup Z)$. (This should be viewed as the intersection-theoretic analogue of lemma 1.3.3 and can be proven in the same way. See also [F1] example 14.1.4.)

Note that the codimension of $\bar{M}'^{(m)} \cap (R \backslash Z)$ is at least $m+1$ by lemma 1.6.5 (i). The same is true for $\bar{M}'^{(m)} \cap (Z \backslash R)$, since curves in this stack are contained in H, and

$$\dim \bar{M}_{0,n}(H,d) = \dim \bar{M}_{0,n}(X,d) - d - 1 < \dim \bar{M}_{0,n}(X,d) - m.$$

This means that the support of the cycle $\mu \in A^*(\bar{M}_{0,n}(X,d))$ must actually be contained in $R \cap Z$.

If moreover $m = 2$, then even $\bar{M}'^{(m)} \cap (R \cap Z)$ has codimension at least $m+1 = 3$ whenever f is not of degree zero on the component on which x_1 lies. □

As a first application, we compute the numbers of curves satisfying certain tangency conditions:

Corollary 1.6.7 *Let $X = \mathbb{P}^r$ and $H \subset X$ be a hyperplane. Let $d \geq 2$, $n \geq 1$, and $k \in \{0, \ldots, r-1\}$. Choose an effective class $\gamma_1 \in A^k(X)$ and a collection of effective classes $\mathcal{T} = \gamma_2 \otimes \ldots \otimes \gamma_n$ in $A^{\geq r-k}(X)$ such that $2 + \sum \text{codim } \gamma_i = \dim \bar{M}_{0,n}(X,d)$. Then, for generic subvarieties $V_i \subset X$ with $[V_i] = \gamma_i$, the invariant*

$$I_d(\gamma_1 \cdot H \cdot (H+c) \otimes \mathcal{T})$$

is equal to the number of irreducible stable maps $(C, x_1, \ldots, x_n, f)$ to X of degree d with $f(x_i) \in V_i$ for $i \geq 2$ and $f(x_1) \in V_1 \cap H$ such that $f(C)$ is tangent to H at x_1.

Proof By proposition 1.6.6, the invariant stated in the corollary is equal to the intersection product

$$ev^* \mathcal{T} \cdot ev_1^* \gamma_1 \cdot ([\bar{M}^{(2)}] - \mu) \tag{1}$$

on $\bar{M}_{0,n}(X,d)$, where μ is some cycle supported on the space Z' of reducible stable maps such that x_1 lies on a component that gets contracted by f. As this component must have at least three special points, we can write $Z' = Z_1 \cup Z_2$, where Z_1 is the subspace of Z' where the component on which x_1 lies contains at least one other marked point, and Z_2 is the subspace of Z' where this component contains at least two nodes of the curve.

As the subspace of $\bar{M}_{0,n}(X,d)$ corresponding to curves with at least two nodes has codimension 2, it follows that the subspace corresponding to curves with at least two nodes and x_1 mapped to H has codimension 3. Therefore, the space Z_2 has codimension at least 3, and since μ is a cycle of codimension 2, its support must actually lie in Z_1.

Note that for all curves in the intersection (1), x_1 is mapped to $V_1 \cap H$ and x_i to V_i for $i > 1$. But $V_1 \cap H \cap V_i = \emptyset$ for $i > 0$ by the condition on the codimensions of the cycles γ_i, so we conclude that

$$ev^*\mathcal{T} \cdot ev_1^*\gamma_1 \cdot \mu = 0,$$

such that we can neglect μ in (1).

But now, as by definition the generic element of $\bar{M}^{(2)}$ corresponds to an irreducible stable map with x_1 mapped to H such that $f(C)$ is tangent to H there, and as the subvarieties V_i are chosen generically, the statement of the corollary follows by the Bertini lemma. (For a precise statement, see lemma 2.4.7. The fact that the generic element of $\bar{M}^{(2)}$ has no automorphisms follows in the same way as in lemma 2.4.8, as the space of N-fold coverings in $\bar{M}^{(2)}$ is of dimension at most

$$\begin{aligned}
&\Big(\frac{d}{N}(r+1)+r+n-3-m\Big)+2N-2 \\
&\qquad = (d(r+1)+r+n-3-m)+2N-2+d(r+1)(\frac{1}{N}-1) \\
&\qquad = \dim \bar{M}^{(2)} + (N-1)(2-\frac{d}{N}(r+1)) \\
&\qquad \le \dim \bar{M}^{(2)} + (N-1)(2-1\cdot 3) \\
&\qquad < \dim \bar{M}^{(2)},
\end{aligned}$$

so that these stable maps do not appear in the generic intersection.) $\square$

As a numerical example, the following table lists some numbers of degree d rational curves in $\mathbb{P}^2$ tangent to H (a) at a point in H and (b) somewhere in H, and intersecting in addition $3d-3$ (resp. $3d-2$) generic points:

d	1	2	3	4	5	6
(a) $I_d(pt \cdot (H+c) \otimes pt^{\otimes(3d-3)})$	1	1	10	428	51040	13300176
(b) $I_d(H \cdot (H+c) \otimes pt^{\otimes(3d-2)})$	0	2	36	2184	335792	106976160

As already mentioned, these numbers can also be obtained by the degeneration methods in section 3.2. L. Ernström and G. Kennedy [EK2] have also computed these numbers in $\mathbb{P}^2$, together with those for any number of simultaneous tangency conditions. In section 2.7, we will use Gromov-Witten invariants of blow-ups to compute some numbers of curves tangent to other subvarieties than hyperplanes.

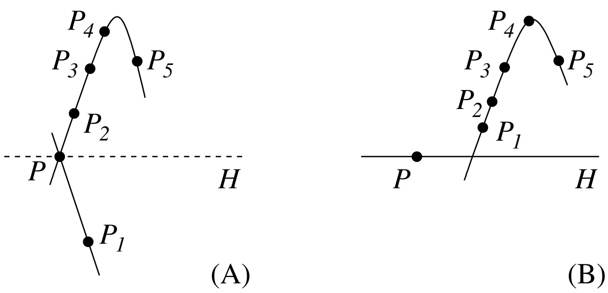
We finish this section with an instructive example showing why the analogue of corollary 1.6.7 gives only "virtual numbers" and not the enumeratively correct ones for higher order contact conditions. Consider the invariant

$$I_3^{\mathbb{P}^2}(pt \cdot (H+c) \cdot (H+2c) \otimes pt^{\otimes 5}) = 14$$

that is supposed to count rational plane cubics intersecting 5 generic points $P_1, \dots, P_5$ and having contact of order 3 to a line $H \subset X = \mathbb{P}^2$ at a point $P \in H$. Degeneration methods tell us that the correct enumerative answer is

$$I_{3,(2,1)}^{H/X}(pt^{\otimes 5} \,|\, H \otimes X) = 7$$

(see section 3.2 for notations and results, and section 3.1 for this particular example). The difference arises from the following two types of curves:



The curves in (A) consist of a line through P and one of the P_i, and a conic through P and the other P_i. The marked point mapped to P lies on a component contracted by f, so $h \circ f$ is identically zero around this point. There are 5 curves of this type, corresponding to the choice of the point P_i lying on the line.

The curves in (B) consist of the line H itself together with a conic through $P_1, \dots, P_5$. As the marked point mapped to P lies on a component that is mapped into H by f, again $h \circ f$ is identically zero around this point. There are 2 curves of this type, according to the fact that the line and the conic can be glued at any of the two intersection points.

This explains the difference of 7 between the two results given above. It should be noted, however, that in general the unwanted stable maps that get counted by the methods of this section will form infinite families, so that it is not always as easy as in the above example to subtract their contribution to get the enumeratively correct result.

Chapter 2

Gromov-Witten invariants of blow-ups

2.1 Introduction

There are at least two motivations to look at Gromov-Witten invariants of blow-ups. Firstly, a blow-up $\tilde{X}$ of a convex variety X provides an easy example for a non-convex variety, in the sense that one has reasonably good control over the stable maps with $h^1(C, f^*T_{\tilde{X}}) \neq 0$ since they all must be such that they intersect the exceptional divisor. Hence this gives a good class of examples where one can study the effects of virtual fundamental classes on Gromov-Witten theory. In fact, so far the Gromov-Witten invariants of no other non-convex variety have been studied in detail, apart from the famous quintic threefold.

Secondly, curves on the blowup $\tilde{X}$ of a variety X are closely related to curves on X. At least for irreducible curves not contained in the exceptional divisor, the strict transform of curves gives a correspondence between curves in $\tilde{X}$ of specified homology class and curves in X intersecting the blown-up variety with a given (global) multiplicity. Hence, being able to calculate Gromov-Witten invariants of blow-ups, one can hope to solve enumerative problems on X involving multiplicity conditions at the blown-up variety.

Apart from the last section of this chapter, we will only be concerned with blow-ups of points, since both the calculation and the question of enumerative significance get very complicated in the case of blow-ups of general subvarieties.

We now introduce some notation which will be used throughout this chapter when dealing with blow-ups. Let X be a smooth r-dimensional convex variety, $r \geq 2$. Fix a homogeneous basis $\mathcal{B} = \{T_0, \ldots, T_q\}$ of $A^*(X)$ of increasing codimension such that $T_0 = [X]$ and $T_q = pt$.

Let $p : \tilde{X} = \tilde{X}(s) \to X$ be the blow-up of X at s generically chosen points $P_1, \ldots, P_s \in X$, and let E_i be the exceptional divisors. If we define $T_{q+1}, \ldots, T_{\tilde{q}}$ with $\tilde{q} = q + s(r-1)$ to be the classes

$$E_i^k \in A^*(\tilde{X}) \quad \text{where } 1 \leq i \leq s, 1 \leq k \leq r-1$$

(in any order), then

$$\tilde{\mathcal{B}} = \{p^*T_1, \dots, p^*T_q, T_{q+1}, \dots, T_{\tilde{q}}\}$$

is a homogeneous basis of $A^*(\tilde{X})$. We call the classes $p^*T_1, \dots, p^*T_q$ **non-exceptional** and $T_{q+1}, \dots, T_{\tilde{q}}$ **exceptional**. A collection of classes $\mathcal{T}$ will be called non-exceptional if all its classes are non-exceptional.

In terms of the basis $\tilde{\mathcal{B}}$, the intersection theory on $\tilde{X}$ is given by

$$\begin{aligned} p^*T_j \cdot p^*T_{j'} &= p^*(T_j \cdot T_{j'}) \\ p^*T_j \cdot E_i^k &= 0 \\ E_i^k \cdot E_{i'}^{k'} &= \delta_{i,i'} E_i^{k+k'} \\ E_i^r &= (-1)^{r-1} pt \end{aligned}$$

for $1 \le j, j' \le q$; $1 \le i, i' \le s$; $1 \le k, k' \le r-1$. If there is no danger of confusion, we will write the classes $p^*T_1, \dots, p^*T_q$ simply as $T_1, \dots, T_q$.

The homology group $A_1(\tilde{X})$ has a canonical decomposition

$$A_1(\tilde{X}) = A_1(X) \oplus \mathbb{Z}E_1' \oplus \cdots \oplus \mathbb{Z}E_s'$$

where $\boldsymbol{E_i'}$ denotes the class of a line in the exceptional divisor $E_i \cong \mathbb{P}^{r-1}$, such that $E_i' = -(-E_i)^{r-1}$ via Poincaré duality. We denote the $s+1$ projections onto the summands of the above decomposition by $\boldsymbol{d : A_1(\tilde{X}) \to A_1(X)}$ and $\boldsymbol{e_1, \dots, e_s : A_1(\tilde{X}) \to \mathbb{Z}}$, and we set $\boldsymbol{e = e_1 + \cdots + e_s}$. If $X = \mathbb{P}^r$, we will identify $A_1(X)$ with $\mathbb{Z}$ in the obvious way and consider d as a function $d : A_1(\tilde{X}) \to \mathbb{Z}$.

For a homology class $\beta \in A_1(\tilde{X})$, we call $d(\beta)$ the **non-exceptional part** and $e(\beta)$ the **exceptional part**. The class β is called a **non-exceptional class** if $e_i(\beta) = 0$ for all i and a **purely exceptional class** if $d(\beta) = 0$ and $e_i(\beta) \neq 0$ for at least one i. For a homology class $\beta \in A_1(X)$, we will denote the corresponding non-exceptional class in $A_1(\tilde{X})$ also by β.

Since the Gromov-Witten invariants are multilinear in the cohomology classes, we will only consider invariants of the form $I_\beta(\mathcal{T})$ where the cohomology classes in $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ are chosen to be in $\tilde{\mathcal{B}}$.

The canonical divisor on $\tilde{X}$ is given by $K_{\tilde{X}} = p^*K_X + (r-1)E$ (see [GH] section 1.4), hence the virtual dimension of the moduli space $\bar{M}_{0,n}(X, \beta)$ is

$$\begin{aligned} \text{vdim } \bar{M}_{0,n}(\tilde{X}, \beta) &= -K_{\tilde{X}} \cdot \beta + n + r - 3 \\ &= \text{vdim } \bar{M}_{0,n}(X, d(\beta)) + (r-1)e(\beta). \end{aligned}$$

This chapter is organized as follows. We first address the question of how one can compute the Gromov-Witten invariants of blow-ups. We state and prove an explicit algorithm how to reconstruct all invariants of $\tilde{X}$ from those of X in section 2.2. In section 2.3, we prove a vanishing theorem saying mainly that if one has a homology class β with $d(\beta) \neq 0$ and $e_i(\beta) > 0$ for some i (such that there are no irreducible curves with this homology class), then a Gromov-Witten invariant $I_\beta(\mathcal{T})$ vanishes if it contains "not too many" exceptional cohomology classes in $\mathcal{T}$. We then come to the enumerative significance of the invariants which will be discussed only on $\tilde{X} = \mathbb{P}^r(s)$. Section 2.4 contains some general remarks, introductory lemmas and counterexamples to enumerative significance. Then we study the cases $\tilde{\mathbb{P}}^r(1)$ in 2.5 and $\tilde{\mathbb{P}}^3(4)$ in 2.6 in detail. In section 2.7 it is shown how Gromov-Witten invariants $I_\beta(\mathcal{T})$ with exceptional cohomology classes in $\mathcal{T}$ can lead to numbers of curves with certain tangency conditions. Finally, we give many numerical examples of Gromov-Witten invariants of blow-ups in 2.8 and finish the chapter with a short outlook on blow-ups of subvarieties in section 2.9.

Independently from our work, L. Göttsche and R. Pandharipande studied blow-ups of $\mathbb{P}^2$. In their paper [GP], they show how to calculate the Gromov-Witten invariants on $\tilde{\mathbb{P}}^2(s)$ and prove enumerative significance for all invariants $I_\beta(\mathcal{T})$ where $e_i(\beta) \in \{-1,-2\}$ for some i or $\mathcal{T}$ contains at least one point class.

2.2 Calculation of the invariants

The aim of this section is to prove the following.

Theorem 2.2.1 *Let X be a convex variety and $\tilde{X}$ the blow-up of X at some points. Then there exists an explicit algorithm to compute the Gromov-Witten invariants of $\tilde{X}$ from those of X.*

The computation is done in three steps. Firstly, we show in lemma 2.2.2 that all invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ with β and $\mathcal{T}$ non-exceptional are actually equal to the corresponding invariants on X. Secondly, in lemma 2.2.4 we compute the invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ with β purely exceptional using a technique similar to the First Reconstruction Theorem of Kontsevich and Manin. Thirdly, we state and prove an algorithm that allows one to compute all Gromov-Witten invariants on $\tilde{X}$ recursively from those obtained in the first two steps.

Lemma 2.2.2 *Let $\mathcal{T} = T_{j_1},\dots,T_{j_n}$ be a collection of non-exceptional classes and let $\beta \in A_1(X)$ be a non-exceptional homology class. Then*

$$I_\beta^{\tilde{X}}(\mathcal{T}) = I_\beta^{X}(\mathcal{T}).$$

*In this case we will say that the invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ is **induced by** X.*

Proof Consider the commutative diagram

$$\begin{tikzcd} \bar{M}_{0,n}(\tilde{X},\beta) \arrow[r, "\phi"] \arrow[d, "ev_i"'] & \bar{M}_{0,n}(X,\beta) \arrow[d, "ev_i"'] \\ \tilde{X} \arrow[r, "p"] & X \end{tikzcd}$$

for $1 \le i \le n$. First we show that $\phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = [\bar{M}_{0,n}(X,\beta)]^{virt}$: since X is convex, $\bar{M}_{0,n}(X,\beta)$ is a smooth stack of the expected dimension $d = \text{vdim } \bar{M}_{0,n}(X,\beta)$. Let $Z_1,\dots,Z_k$ be the connected components of $\bar{M}_{0,n}(X,\beta)$, so that $A_d(\bar{M}_{0,n}(X,\beta)) = \mathbb{Q}[Z_1] \oplus \cdots \oplus \mathbb{Q}[Z_k]$. Since vdim $\bar{M}_{0,n}(\tilde{X},\beta) = d$, we must therefore have

$$\phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = \alpha_1[Z_1] + \cdots + \alpha_k[Z_k]$$

for some $\alpha_i \in \mathbb{Q}$.

To see that all $\alpha_i = 1$, pick a stable map $\mathcal{C}_i \in Z_i$ whose image does not intersect the blown-up points. Then $\phi^{-1}(\mathcal{C}_i)$ consists of exactly one stable map $\tilde{\mathcal{C}_i}$, and the map $\phi : \bar{M}_{0,n}(\tilde{X},\beta) \to \bar{M}_{0,n}(X,\beta)$ is a local isomorphism around the point $\tilde{\mathcal{C}_i}$. Hence $\tilde{\mathcal{C}_i}$ is a smooth point of an irreducible component $\tilde{Z}_i$ of $\bar{M}_{0,n}(\tilde{X},\beta)$. Denote by $\tilde{R}_i$ the union of the other irreducible components of $\bar{M}_{0,n}(\tilde{X},\beta)$. Then, by lemma 1.3.3,

$$[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = [\tilde{Z}_i] + \text{some cycle supported on } \tilde{R}_i.$$

Now, since $\phi : \tilde{Z}_i \to Z_i$ is a local isomorphism around $\tilde{\mathcal{C}_i}$, we have $\phi_*[\tilde{Z}_i] = [Z_i]$. However, the pushforward of a d-cycle supported on $\tilde{R}_i$ will give no contribution to α_i since $\mathcal{C}_i$ and therefore Z_i is not contained in the image of $\tilde{R}_i$ under ϕ. We conclude that all $\alpha_i = 1$ and that therefore

$$\begin{aligned} \phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} &= [Z_1] + \cdots + [Z_k] \\ &= [\bar{M}_{0,n}(X,\beta)] \\ &= [\bar{M}_{0,n}(X,\beta)]^{virt}. \end{aligned}$$

To complete the proof, note that by the projection formula

$$\begin{aligned} I_\beta^{\tilde{X}}(\mathcal{T}) &= (\prod_i ev_i^* p^* T_{j_i}) \cdot [\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i \phi^* ev_i^* T_{j_i}) \cdot [\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i ev_i^* T_{j_i}) \cdot \phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i ev_i^* T_{j_i}) \cdot [\bar{M}_{0,n}(X,\beta)]^{virt} \\ &= I_\beta^X(\mathcal{T}). \end{aligned}$$

□

Remark 2.2.3 This lemma is actually the only point in the proof of theorem 2.2.1 where the convexity of X is needed. Hence, one can formulate the theorem also in the following, more general way:

Let X be a smooth projective variety and $\tilde{X}$ the blow-up of X at some points. There exists an explicit algorithm to compute all Gromov-Witten invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ of $\tilde{X}$ from those where β and $\mathcal{T}$ are non-exceptional.

The proof would be literally the same, just skipping lemma 2.2.2. In fact, it may even be that lemma 2.2.2 also holds for non-convex X, but I do not know how to prove it in this case.

Lemma 2.2.4 *Let $\mathcal{T} = T_{j_1}, \dots, T_{j_n} \in \tilde{\mathcal{B}}$ be a collection of classes and let $\beta \in A_1(\tilde{X})$ be a purely exceptional homology class. Then*

(i) If β is not of the form $d \cdot E_i'$ for $d > 0$ and some $1 \le i \le s$, then $I_\beta^{\tilde{X}}(\mathcal{T}) = 0$. Moreover, the invariant can only be non-zero if all classes in $\mathcal{T}$ are exceptional with support in the exceptional divisor E_i.

(ii) $I_{E_i'}^{\tilde{X}}(E_i^{r-1} \otimes E_i^{r-1}) = 1$ for all $1 \le i \le s$.

(iii) All other invariants with purely exceptional homology class can be computed recursively.

Proof

(i) This follows easily from the fact that a Gromov-Witten invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ is always zero if there is no stable map in $\bar{M}_{0,n}(\tilde{X}, \beta)$ satisfying the conditions given by $\mathcal{T}$.

(ii) Note that $\bar{M}_{0,2}(\tilde{X}, E_i') \cong \bar{M}_{0,2}(\mathbb{P}^{r-1}, 1)$ and that this space is of the expected dimension (which is $2r-2$), hence we do not need virtual fundamental classes to compute this invariant. Choose two curves $Y_1, Y_2 \subset X$ intersecting transversally at the blown-up point P_i, and let $\gamma_1, \gamma_2 \in A^{r-1}(X)$ be their cohomology classes. Let $\tilde{Y}_k$ be the strict transform of Y_k for $k = 1, 2$. Then $\tilde{Y}_1$ and $\tilde{Y}_2$ intersect E_i transversally at different points, so the invariant

$$I_{E_i'}^{\tilde{X}}([\tilde{Y}_1] \otimes [\tilde{Y}_2]) = I_{E_i'}^{\tilde{X}}((\gamma_1 + (-E_i)^{r-1}) \otimes (\gamma_2 + (-E_i)^{r-1}))$$

simply counts the number of lines in E_i through two points in E_i, which is 1. Therefore, by the multilinearity of the Gromov-Witten invariants and by (i) we conclude that

$$\begin{aligned} I_{E_i'}^{\tilde{X}}(E_i^{r-1} \otimes E_i^{r-1}) &= I_{E_i'}^{\tilde{X}}((\gamma_1 + (-E_i)^{r-1}) \otimes (\gamma_2 + (-E_i)^{r-1})) \\ &= 1. \end{aligned}$$

(iii) (This is essentially the First Reconstruction Theorem of Kontsevich and Manin, see proposition 1.4.2.) As in (ii) we assume that $\tilde{X} = \tilde{\mathbb{P}}^r(1)$ and that we want to compute the invariant $I_{dE'}(E^{j_1} \otimes \ldots \otimes E^{j_n})$ for some d and some j_i. Consider the equation $\mathcal{E}_{dE'}(\mathcal{T}\ ;\ E^a, E^b \mid E^c, E)$ for some $\mathcal{T}$ consisting of exceptional classes and for some $2 \le a \le r-1$, $2 \le b \le r-1$, $1 \le c \le r-1$:

$$\begin{aligned}
0 &= I_{dE'}(\mathcal{T} \otimes E^a \otimes E^b \otimes E^c \cdot E) && (1)\\
&+ I_{dE'}(\mathcal{T} \otimes E^c \otimes E \otimes E^a \cdot E^b) && (2)\\
&- I_{dE'}(\mathcal{T} \otimes E^a \otimes E^c \otimes E^b \cdot E) && (3)\\
&- I_{dE'}(\mathcal{T} \otimes E^b \otimes E \otimes E^a \cdot E^c) && (4)\\
&+ (\text{terms with homology classes } d'E' \text{ with } d' < d). && (5)
\end{aligned}$$

We want to compute the invariants by induction on the degree d and on the number of non-divisorial classes in the invariant. Obviously, the terms in (5) have lower degree and those in (2) and (4) have same degree but a smaller number of non-divisorial classes than (1). The degree of (3) is equal to that of (1), and its number of non-divisorial classes is not bigger than that of (1). In any case, we can write

$$\begin{aligned}
I_{dE'}(\mathcal{T} \otimes E^a \otimes E^b \otimes E^{c+1}) &= I_{dE'}(\mathcal{T} \otimes E^a \otimes E^{b+1} \otimes E^c)\\
&\quad + (\text{recursively known terms}).
\end{aligned}$$

Thus if a Gromov-Witten invariant contains at least three non-divisorial classes, we can use this equation repeatedly to express $I_{dE'}(\mathcal{T} \otimes E^a \otimes E^b \otimes E^{c+1})$ in terms of $I_{dE'}(\mathcal{T} \otimes E^a \otimes E^{b+c} \otimes E)$ (and recursively known terms), which again has fewer non-divisorial classes. This makes the induction work and reduces everything to invariants with at most two non-divisorial classes. However, since $\operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, dE') = (r-1)d + r + n - 3$ and each class has codimension at most r, it is easy to check that the only such invariant is the one calculated in (ii).

□

We now come to the main part of the proof of theorem 2.2.1, namely the algorithm to compute all invariants on $\tilde{X}$ from those calculated so far. We will first state the algorithm in such a way that it can be programmed easily on a computer, and afterwards give the proof that it really does the job. Many numbers computed using this algorithm can be found in section 2.8.

From now on, Gromov-Witten invariants will always be on $\tilde{X}$ unless otherwise stated, so we will often write them as $I_\beta(\mathcal{T})$ instead of $I_\beta^{\tilde{X}}(\mathcal{T})$.

Algorithm 2.2.5 *Suppose one wants to calculate an invariant $I_\beta^{\tilde{X}}(\mathcal{T})$. Assume that the invariant is not induced by X and that β is not purely exceptional. We may assume without loss of generality that the sum of the codimensions of the non-exceptional classes* *in $\mathcal{T}$ is at least $r+1$ (hence in particular that there are at least two non-exceptional classes) — otherwise choose a divisor $\rho \in \mathcal{B}$ with $\rho \cdot \beta \neq 0$ (such a ρ exists because β is not purely exceptional) and use $\mathcal{T} \otimes \rho^{\otimes(r+1)}$ instead of $\mathcal{T}$, which gives essentially the same invariant by the divisor axiom.*

We can further assume without loss of generality that $\mathcal{T}$ contains no exceptional divisor class and that the classes $T_{j_1}, \dots, T_{j_n}$ in $\mathcal{T}$ are ordered such that the non-exceptional classes are exactly $T_{j_1}, \dots, T_{j_m}$, where codim $T_{j_1} \geq \cdots \geq$ codim T_{j_m}. In particular, T_{j_1} and T_{j_2} are two non-exceptional classes with maximal codimension in $\mathcal{T}$.

We now distinguish the following three cases.

(A) $n > m$, i.e. $T_{j_n} = E_i^k$ (for some $1 \leq i \leq s$, $2 \leq k \leq r-1$) is an exceptional class. Then use the equation

$$\mathcal{E}_\beta(\mathcal{T}' \,;\, T_{j_1}, T_{j_2} \mid E_i, E_i^{k-1}) \qquad \text{where } \mathcal{T}' = T_{j_3} \otimes \dots \otimes T_{j_{n-1}}.$$

(B) $n = m$ (i.e. there is no exceptional class in $\mathcal{T}$), $T_{j_1} = pt$ and codim $T_{j_2} \geq 2$. Then choose $\mu, \nu \in \mathcal{B}$ such that codim $\mu = 1$, codim $\nu = r-1$, and $\mu \cdot \nu \neq 0$. Since the invariant to be computed is not induced by X, there is an $i \in \{1, \dots, s\}$ such that $E_i \cdot \beta \neq 0$. Use the equation

$$\mathcal{E}_\beta(\mathcal{T}' \,;\, \mu, \nu \mid E_i, T_{j_2}) \qquad \text{where } \mathcal{T}' = T_{j_3} \otimes \dots \otimes T_{j_n}.$$

(C) $n = m$, and it is not true that $T_{j_1} = pt$ and codim $T_{j_2} \geq 2$. Then again there is an $i \in \{1, \dots, s\}$ such that $E_i \cdot \beta \neq 0$. Use the equation

$$\mathcal{E}_{\beta + E_i'}(\mathcal{T}' \,;\, T_{j_1}, T_{j_2} \mid E_i, E_i^{r-1}) \qquad \text{where } \mathcal{T}' = T_{j_3} \otimes \dots \otimes T_{j_n}.$$

Here, "use equation $\mathcal{E}$" means: the Gromov-Witten invariant $I_\beta(\mathcal{T})$ to be calculated appears in $\mathcal{E}$ linearly with non-zero coefficient. Solve this equation for $I_\beta(\mathcal{T})$ and compute recursively with the same rules all other invariants in this equation that are not already known.

Proof (of theorem 2.2.1) Suppose we want to compute an invariant $I_\beta(\mathcal{T})$. If the invariant is induced by X, it is assumed to be known by lemma 2.2.2. If β is purely exceptional, the invariant is known by lemma 2.2.4. In all other cases, use the algorithm 2.2.5 to compute the invariant recursively. We have to show that the equations to be used in fact do contain the desired invariants linearly with non-zero coefficient, and that the recursion stops after a finite number of calculations.

To do this, we will define a partial ordering on pairs $(\beta, \mathcal{T})$ where $\beta \in A_1(\tilde{X})$ is an effective homology class and $\mathcal{T}$ is a collection of cohomology classes. Choose an ordering of the effective homology classes in $A_1(X)$ such that, for $\alpha_1, \alpha_2 \neq 0$ being two such classes, we have $\alpha_1 < \alpha_1 + \alpha_2$ (this is possible since the effective classes in $A_1(X)$ form a semigroup with indecomposable zero). For a collection of classes $\mathcal{T} = T_{j_1} \otimes$

$\ldots \otimes T_{j_n}$, we assume as in the description of the algorithm that the classes are ordered such that the non-exceptional classes are exactly $T_{j_1}, \ldots, T_{j_m}$, where codim $T_{j_1} \geq \cdots \geq$ codim T_{j_m}, and that codim $T_{j_1} + \cdots +$ codim $T_{j_m} \geq r+1$ (by possibly adding non-exceptional divisor classes). Then we define

$$\nu(\mathcal{T}) = \min \{k \,;\, \text{codim } T_{j_1} + \cdots + \text{codim } T_{j_k} \geq r+1\},$$

i.e. "the minimal number of non-exceptional classes in $\mathcal{T}$ whose codimensions sum up to at least $r+1$". With this, we now define the partial ordering on pairs $(\beta, \mathcal{T})$ as follows: say that $(\beta_1, \mathcal{T}_1) < (\beta_2, \mathcal{T}_2)$ if and only if one of the following holds:

- $d(\beta_1) < d(\beta_2)$,
- $d(\beta_1) = d(\beta_2)$ and $\nu(\mathcal{T}_1) < \nu(\mathcal{T}_2)$,
- $d(\beta_1) = d(\beta_2)$, $\nu(\mathcal{T}_1) = \nu(\mathcal{T}_2)$, and $e(\beta_1) < e(\beta_2)$.

Obviously, this defines a partial ordering satisfying the "descending chain condition", i.e. there do not exist infinite chains $(\beta_1, \mathcal{T}_1) > (\beta_2, \mathcal{T}_2) > (\beta_3, \mathcal{T}_3) > \ldots$. This means that, to prove that the recursion stops after finitely many calculations, it suffices to show that the equations in the algorithm compute the desired invariant $I_\beta(\mathcal{T})$ entirely in terms of invariants that are either known by the lemmas 2.2.2 and 2.2.4 or smaller with respect to the above partial ordering. We will do this now for the three cases (A), (B), and (C).

(A) The equation reads

$$\begin{aligned}
0 = {} & I_\beta(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes E_i \cdot E_i^{k-1}) && (1)\\
& + I_\beta(\mathcal{T}' \otimes E_i \otimes E_i^{k-1} \otimes T_{j_1} \cdot T_{j_2}) && (2)\\
& + (\text{no further } I_\beta(\,\cdot\,) I_0(\,\cdot\,)\text{-terms since } E_i \cdot T_{j_1} = E_i^{k-1} \cdot T_{j_2} = 0) && \\
& + (\text{some } I_{\beta - dE_i'}(\,\cdot\,) I_{dE_i'}(\,\cdot\,)\text{-terms}) && (3)\\
& + (\text{some } I_{\beta_1}(\,\cdot\,) I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (4)
\end{aligned}$$

The term (1) is the desired invariant. If the term in (2) is non-zero, it has the same $d(\beta)$ and smaller $\nu(\mathcal{T})$, since the two non-exceptional classes T_{j_1}, T_{j_2} of maximal codimensions codim T_{j_1}, codim T_{j_2} are replaced by one class of codimension codim $T_{j_1} +$ codim T_{j_2}. Hence, the term (2) is smaller with respect to our partial ordering. The terms in (3) have the same d, the same or smaller ν (note that all non-exceptional classes from the original invariant must be in the left invariant $I_{\beta - dE_i'}(\,\cdot\,)$), and smaller e. Finally, the terms in (4) have smaller d. Hence, all terms in (2), (3) and (4) are smaller with respect to our partial ordering.

(B) The equation reads

$$
\begin{aligned}
0 &= I_\beta(\mathcal{T}' \otimes E_i \otimes T_{j_2} \otimes \mu \cdot \nu) && (1)\\
&+ (\text{no further } I_\beta(\,\cdot\,)\, I_0(\,\cdot\,)\text{-terms since } E_i \cdot T_{j_2} = E_i \cdot \mu = T_{j_2} \cdot \nu = 0)\\
&+ (\text{no } I_{\beta - dE_i'}(\,\cdot\,)\, I_{dE_i'}(\,\cdot\,)\text{-terms since } I_{dE_i'}(\,\cdot\,) \text{ would have to contain at least}\\
&\qquad \text{one of the non-exceptional classes } T_{j_2}, \mu, \nu)\\
&+ (\text{some } I_{\beta_1}(\,\cdot\,)\, I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (2)
\end{aligned}
$$

Here, obviously, (1) is the desired invariant and the terms in (2) have smaller d and are therefore smaller with respect to the partial ordering.

(C) The equation reads

$$
\begin{aligned}
0 &= I_{\beta + E_i'}(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes \underbrace{E_i \cdot E_i^{r-1}}_{(-1)^{r-1} pt}) && (1)\\
&+ I_{\beta + E_i'}(\mathcal{T}' \otimes E_i \otimes E_i^{r-1} \otimes T_{j_1} \cdot T_{j_2}) && (2)\\
&+ (\text{no further } I_\beta(\,\cdot\,)\, I_0(\,\cdot\,)\text{-terms})\\
&+ I_\beta(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes E_i)\, \underbrace{I_{E_i'}(E_i \otimes E_i^{r-1} \otimes E_i^{r-1})}_{=-1}\, (-1)^{r-1} && (3)\\
&+ (\text{no further } I_{\beta - dE_i'}(\,\cdot\,)\, I_{dE_i'}(\,\cdot\,)\text{-terms since there are not enough exceptional}\\
&\qquad \text{classes to put into } I_{dE_i'}(\,\cdot\,))\\
&+ (\text{some } I_{\beta_1}(\,\cdot\,)\, I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (4)
\end{aligned}
$$

Here, (3) is the desired invariant. (4) has smaller d, and (2) has the same d and smaller v, as in case (A)-(2). The term (1) has the same d, but is not necessarily smaller with respect to the partial ordering. We distinguish two cases:

(i) If $\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2}$ contains a non-divisorial (non-exceptional) class, then the invariant (1) will be computed in the next step using rule (B), which expresses it entirely in terms of invariants with smaller d.

(ii) If $\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2}$ contains only divisor classes, the invariant (1) will be computed in the next step using (C). This time, (2) vanishes (for $T_{j_1} \cdot T_{j_2} = 0$ since $T_{j_1} = pt$), (4) has smaller d, and (1) will be computed by (B) as in (i) in terms of invariants with smaller d.

Hence, combining (C) with possibly one other application of (B) and/or (C), the desired invariant will again be computed in terms of invariants that are smaller with respect to the partial ordering.

This finishes the proof. □

Corollary 2.2.6 *There exists an explicit algorithm to compute all Gromov-Witten invariants on $\tilde{\mathbb{P}}^r(s)$ for all $r \geq 2$, $s \geq 1$.*

Proof Compute the invariants of $\mathbb{P}^r$ using the First Reconstruction Theorem 1.4.2, and then use theorem 2.2.1. □

2.3 A vanishing theorem

We will now prove a vanishing theorem saying that a Gromov-Witten invariant $I_\beta(\mathcal{T})$ with $d(\beta) \neq 0$ and $e_i(\beta) \geq 0$ for some i vanishes under favourable conditions, mainly if $e_i(\beta) > 0$ and if there are "not too many" exceptional classes in $\mathcal{T}$. The proof of the proposition is quite involved, but as a reward it is also very sharp in the sense that numerical calculations on $\tilde{\mathbb{P}}^r(1)$ have shown that an invariant (with non-vanishing $d(\beta)$ and non-negative $e(\beta)$) is "unlikely to vanish" if the conditions of the proposition are not satisfied. We will then apply the proposition to prove corollary 2.3.2, which is a first hint that Gromov-Witten invariants on blow-ups will lead to enumeratively meaningful numbers.

To state the proposition, we need an auxiliary definition. For $T \in \tilde{\mathcal{B}}$ and $1 \leq i \leq s$ we define

$$\boldsymbol{w_i(T)} = \begin{cases} m-1 & \text{if } T = E_i^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ is a collection of classes, we set $w_i(\mathcal{T}) = w_i(T_{j_1}) + \cdots + w_i(T_{j_n})$.

Proposition 2.3.1 *Let β and $\mathcal{T}$ be such that for some $1 \leq i_0 \leq s$ the following three conditions hold:*

(i) $d(\beta) \neq 0$,

(ii) $w_{i_0}(\mathcal{T}) > 0$ *or* $e_{i_0}(\beta) > 0$,

(iii) $w_{i_0}(\mathcal{T}) < (e_{i_0}(\beta) + 1)(r-1)$.

Then $I_\beta(\mathcal{T}) = 0$.

Proof The proof will be given inductively following the lines of the algorithm 2.2.5. For invariants induced by X or invariants with purely exceptional homology class, the proposition does not say anything, so all we have to do is to go through the three equations (A) to (C) and show that the statement of the proposition is correct for the invariant to be determined if it is correct for all the others.

For the proof of the proposition, we will refer to the classes T_i and T_j in the splitting axiom (see proposition 1.4.1 (iv))

$$0 = \sum g^{ij} \Big(I(\ldots \otimes T_i) I(\ldots \otimes T_j) \Big)$$

as the **additional classes** of a certain summand in the equation.

Assume that we are calculating an invariant $I_\beta(\mathcal{T})$ and that a term $I_{\beta_1}(\mathcal{T}_1) I_{\beta_2}(\mathcal{T}_2)$ occurs in the corresponding equation (A), (B), or (C) such that $(\beta, \mathcal{T})$ satisfies the conditions of the proposition, but neither $(\beta_1, \mathcal{T}_1)$ nor $(\beta_2, \mathcal{T}_2)$ does. We will show that this assumption leads to a contradiction.

We first distinguish the two cases $w_{i_0}(\mathcal{T}) > 0$ and $e_{i_0}(\beta) > 0$ according to $(\beta, \mathcal{T})$ satisfying (ii).

- $w_{i_0}(\mathcal{T}) > 0$. This means that we have an exceptional non-divisorial class in the invariant and hence that we are in case (A) of the algorithm. Moreover, we can assume that we use case (A) of the algorithm with $i = i_0$. Since the term in (A)-(2) in the proof of theorem 2.2.1 satisfies the conditions of the proposition if the desired invariant (A)-(1) does, we only need to consider the terms (A)-(3) and (A)-(4).

  From (A)-(1) we know that

  $$w_i(\mathcal{T}) = w_i(\mathcal{T}') + w_i(E_i^k) = w_i(\mathcal{T}') + k - 1,$$

  whereas in all other terms $I_{\beta_1}(\mathcal{T}_1) I_{\beta_2}(\mathcal{T}_2)$ we have

  $$w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) = w_i(\mathcal{T}') + w_i(E_i^{k-1}) + \varepsilon(r-2) = w_i(\mathcal{T}') + k - 2 + \varepsilon(r-2), \tag{1}$$

  where $\varepsilon = 1$ if the additional classes happen to be classes in the exceptional divisor E_i, and $\varepsilon = 0$ otherwise. Combining both equations, we get

  $$w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) = w_i(\mathcal{T}) - 1 + \varepsilon(r-2). \qquad (*)$$

  Now we again distinguish two cases.

  (a) $(\beta_1, \mathcal{T}_1)$ and $(\beta_2, \mathcal{T}_2)$ satisfy (ii). If $(\beta_1, \mathcal{T}_1)$ does not satisfy (i), then β_1 is a purely exceptional class, so all classes in $\mathcal{T}_1$ must be exceptional, i.e.

  $$\begin{aligned} w_i(\mathcal{T}_1) = \operatorname{vdim} \bar{M}_{0,0}(\tilde{X}, \beta_1) &= e_i(\beta_1)(r-1) + r - 3 \\ &= (e_i(\beta_1)+1)(r-1) - 2. \end{aligned}$$

  So we have the two possibilities

  $$\begin{aligned} (\beta_1, \mathcal{T}_1) \text{ does not satisfy (i)} &\Rightarrow w_i(\mathcal{T}_1) \geq (e_i(\beta_1)+1)(r-1) - 2, \\ (\beta_1, \mathcal{T}_1) \text{ does not satisfy (iii)} &\Rightarrow w_i(\mathcal{T}_1) \geq (e_i(\beta_1)+1)(r-1). \end{aligned}$$

The same is true for $(\beta_2, \mathcal{T}_2)$. However, since β is not purely exceptional, it is not possible that both $(\beta_1, \mathcal{T}_1)$ and $(\beta_2, \mathcal{T}_2)$ do not satisfy (i). We conclude that

$$\begin{aligned} w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) &\geq (e_i(\beta_1) + 1 + e_i(\beta_2) + 1)(r-1) - 2 \\ &= (e_i(\beta) + 2)(r-1) - 2 \\ &> w_i(\mathcal{T}) + r - 3 \qquad \text{since } (\beta, \mathcal{T}) \text{ satisfies (iii).} \end{aligned}$$

This is a contradiction to (1).

(b) $(\beta_1, \mathcal{T}_1)$ does not satisfy (ii), i.e. $w_i(\mathcal{T}_1) = e_i(\beta_1) = 0$. Since $w_i(\mathcal{T}_1) = 0$, $\mathcal{T}_1$ does not contain exceptional classes E_i^k for $k > 1$. Since $e_i(\beta_1) = 0$, $\mathcal{T}_1$ also does not contain E_i (otherwise $I_{\beta_1}(\mathcal{T}_1) = 0$ by the divisor axiom). Hence $\mathcal{T}_1$ does not contain E_i^k for any k, and in particular we conclude that $\varepsilon = 0$ in (1):

$$\begin{aligned} w_i(\mathcal{T}_2) = w_i(\mathcal{T}) - 1 < w_i(\mathcal{T}) & \\ &< (e_i(\beta) + 1)(r-1) \\ &= (e_i(\beta_2) + 1)(r-1). \end{aligned}$$

Therefore $(\beta_2, \mathcal{T}_2)$ satisfies (iii). It also satisfies (ii), since otherwise we would have $e_i(\beta_1) = e_i(\beta_2) = 0$ and hence get zero by the divisor axiom from the class E_i in (A). Hence, $(\beta_2, \mathcal{T}_2)$ cannot satisfy (i), i.e. we must be looking at the invariants (A)-(3). However, the invariant $I_{d' E_i'}(\,\cdot\,)$ appearing there can never be non-zero if the additional classes are non-exceptional. We reach a contradiction.

- $e_{i_0}(\beta) > 0$ and $w_{i_0}(\mathcal{T}) = 0$. Then we can be in any of the cases (A) to (C) of the algorithm. Note that $e_{i_0}(\beta_1) + e_{i_0}(\beta_2)$ is equal to $e_{i_0}(\beta)$ or $e_{i_0}(\beta) + 1$ (the latter case appearing exactly if we are in case (C) and $i = i_0$). In any case, it follows that

$$e_{i_0}(\beta_1) + e_{i_0}(\beta_2) \geq e_{i_0}(\beta) \geq 1,$$

hence we can assume without loss of generality that $e_{i_0}(\beta_1) \geq 1$. In particular, $(\beta_1, \mathcal{T}_1)$ satisfies (ii). We are going to show that it also satisfies (i) and (iii), which is then a contradiction to our assumptions.

The case that $(\beta_1, \mathcal{T}_1)$ does not satisfy (i), i.e. that $d(\beta_1) = 0$, could only occur in (A)-(3) and for $\beta_1 = d E_i'$. Since

$$1 \leq e_{i_0}(\beta_1) = e_{i_0}(d E_i') = d\, \delta_{i, i_0}$$

we must have $i = i_0$. But this means that we have a class $E_i^k = E_{i_0}^k$ in $\mathcal{T}$ which is a contradiction to $w_{i_0}(\mathcal{T}) = 0$. Hence $(\beta_1, \mathcal{T}_1)$ must satisfy (i).

As for (iii), we compute $w_{i_0}(\mathcal{T}_1)$. There are no exceptional classes $E_{i_0}^2, \dots, E_{i_0}^{r-1}$ in $\mathcal{T}'$ since $w_{i_0}(\mathcal{T}) = 0$. Hence the only such classes in $\mathcal{T}_1$ can come from

- the additional classes,
- the four special classes used in the equation (A), (B), or (C).

Both can contribute at most $r-2$ to $w_{i_0}(\mathcal{T}_1)$, hence

$$w_{i_0}(\mathcal{T}_1) \leq 2r-4 < 2(r-1) \leq (e_{i_0}(\beta_1)+1)(r-1).$$

Therefore $(\beta_1, \mathcal{T}_1)$ also satisfies (iii), arriving at the contradiction we were looking for.

$\square$

As a corollary we can now prove a relation between the Gromov-Witten invariants of $\tilde{X}$ that one would expect from geometry. Namely, if we want to express the condition that curves of homology class β pass through a generic point in X, we expect to be able to do this in two different ways: either we add the class of a point to $\mathcal{T}$, or we blow up the point and count curves with homology class $\beta - E'$. The following corollary states that these two methods will always give the same result, no matter whether the invariants are actually enumeratively meaningful or not.

Corollary 2.3.2 *Let $(\beta, \mathcal{T})$ be such that, for some $1 \leq i \leq s$, we have $e_i(\beta) = w_i(\mathcal{T}) = 0$ and $d(\beta) \neq 0$. Then*

$$I_{\beta - E'_i}(\mathcal{T}) = I_\beta(\mathcal{T} \otimes pt).$$

Proof Consider the equation $\mathcal{E}_\beta(\mathcal{T}\,;\,\lambda, \lambda \,|\, E_i, E_i^{r-1})$ for an arbitrary divisor $\lambda \in \mathcal{B}$ with $\lambda \cdot \beta \neq 0$:

$$\begin{aligned}
0 = {} & I_\beta(\mathcal{T} \otimes \lambda \otimes \lambda \otimes E_i \cdot E_i^{r-1}) && (1)\\
& + (\text{no further } I_\beta(\,\cdot\,) I_0(\,\cdot\,)\text{-terms}) \\
& + I_{\beta - E'_i}(\mathcal{T} \otimes \lambda \otimes \lambda \otimes E_i)\, \underbrace{I_{E'_i}(E_i \otimes E_i^{r-1} \otimes E_i^{r-1})}_{=-1}\, (-1)^{r-1} && (2)\\
& + (\text{no further } I_{\beta - d E'_i}(\,\cdot\,) I_{d E'_i}(\,\cdot\,)\text{-terms since there are not enough exceptional} \\
& \qquad \text{classes to put into } I_{d E'_i}(\,\cdot\,)) \\
& + (\text{some } I_{\beta_1}(\,\cdot\,) I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (3)
\end{aligned}$$

Using proposition 2.3.1, we will show for any term $I_{\beta_1}(\mathcal{T}_1) I_{\beta_2}(\mathcal{T}_2)$ in (3) that it vanishes. Since $e_i(\beta_1) + e_i(\beta_2) = e_i(\beta) = 0$, we have without loss of generality one of the following cases:

- $\underline{e_i(\beta_1) = e_i(\beta_2) = 0.}$ Then $I_{\beta_1}(\mathcal{T}_1) I_{\beta_2}(\mathcal{T}_2) = 0$ by the divisor axiom because of the class E_i in the equation.

- $e_i(\beta_1) > 0$. Then we show that $(\beta_1, \mathcal{T}_1)$ satisfies conditions (i) to (iii) of the proposition and hence vanishes. (i) and (ii) are obvious. As for (iii), the only classes contributing to $w_i(\mathcal{T}_1)$ can come from
  - the additional classes,
  - the special class E_i^{r-1} used in the equation.

  Both can contribute at most $r-2$ to $w_i(\mathcal{T}_1)$, hence
  $$w_i(\mathcal{T}_1) \le 2r-4 < 2(r-1) \le (e_i(\beta_1)+1)(r-1).$$
  Therefore $(\beta_1, \mathcal{T}_1)$ also satisfies (iii).

Now that we know that all terms in (3) vanish, the above equation becomes
$$I_\beta(\mathcal{T} \otimes \lambda \otimes \lambda \otimes E_i \cdot E_i^{r-1}) = I_{\beta - E_i'}(\mathcal{T} \otimes \lambda \otimes \lambda \otimes E_i)\,(-1)^{r-1}.$$
Since $E_i \cdot E_i^{r-1} = (-1)^{r-1} pt$ and $E_i \cdot (\beta - E_i') = 1$, the corollary follows. $\square$

2.4 Enumerative significance — general remarks

After having computed all Gromov-Witten invariants on blow-ups of projective space (see corollary 2.2.6), we now come to the question of enumerative significance of the invariants. For most of the time, we will be concerned with invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ where $\mathcal{T}$ is non-exceptional, leading to numbers of curves on X intersecting the blown-up points with prescribed multiplicities. Only in section 2.7 we will consider some invariants containing exceptional classes in $\mathcal{T}$, leading to numbers of curves on X with certain tangency conditions.

For the rest of the chapter, we will only work with $\tilde{X} = \tilde{\mathbb{P}}^r(s)$. We start by giving a precise definition of an enumeratively significant invariant.

Definition 2.4.1 *Let $\beta \in A_1(\tilde{X})$ a homology class with $d(\beta) \neq 0$ and $e_i(\beta) \le 0$, and let $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ be a collection of non-exceptional effective classes $\gamma_i \in A^{\ge 1}(X)$ such that $\sum_i \operatorname{codim} \gamma_i = \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta)$.*

Then we call the Gromov-Witten invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ ***enumerative*** *if, for generic subschemes $V_i \subset \tilde{X}$ with $[V_i] = \gamma_i$, it is equal to the number of irreducible stable maps $(C, x_1, \ldots, x_n, f)$ with f being generically injective, $f_*[C] = \beta$, and $f(x_i) \in V_i$ for all i (where each such stable map is counted with multiplicity one).*

Note that irreducible stable maps $(C, x_1, \ldots, x_n, f)$ on $\tilde{X}$ of homology class β with f generically injective correspond bijectively to irreducible curves in $\tilde{X}$ of homology class β, and hence via strict transform to irreducible curves in X of homology class $d(\beta)$ intersecting the blown-up points P_i with global multiplicities $-e_i(\beta)$. Hence it is clear that we can also give the following interpretation of enumerative invariants:

Lemma 2.4.2 *If $I_\beta(\mathcal{T})$ is enumerative, then for generic subschemes $V_i \subset \tilde{X}$ with $[V_i] = \gamma_i$, it is equal to the number of irreducible rational curves $C \subset X$ of homology class $d(\beta)$ intersecting all V_i, and in addition passing through each P_i with global multiplicity $-e_i(\beta)$. Every such curve is counted with multiplicity $\sharp(C \cap V_1) \cdot \ldots \cdot \sharp(C \cap V_n)$.*

In general, one would then expect these curves to have $-e_i$ smooth local branches at every point P_i.

We will now give an overview of the results about enumerative significance of Gromov-Witten invariants on $\tilde{\mathbb{P}}^r(s)$. Assume that $d(\beta) \neq 0$, $e_i(\beta) \leq 0$, and that $\mathcal{T}$ is a collection of non-exceptional effective classes.

(i) If $s = 1$ then $I_\beta(\mathcal{T})$ is enumerative. This will be shown in theorem 2.5.3.

(ii) If $r = 2$ then $I_\beta(\mathcal{T})$ is enumerative if $e_i(\beta) \in \{-1, -2\}$ for some i or $\mathcal{T}$ contains at least one point class. This has been proven by L. Göttsche and R. Pandharipande in [GP].

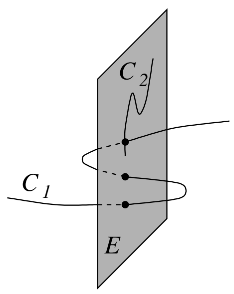
(iii) If $r = 3$, $s \leq 4$, and $\mathcal{T}$ contains only point classes, then $I_\beta(\mathcal{T})$ is enumerative if and only if β is not equal to $dH' - dE'_i - dE'_j$ for some $d \geq 2$ and $i \neq j$ with $1 \leq i, j \leq s$. We will prove this in theorem 2.6.4.

(iv) If $r = 3$ and $\mathcal{T}$ contains not only point classes, then $I_\beta(\mathcal{T})$ is in general not enumerative.

(v) If $r \geq 4$ and $s \geq 2$ then $I_\beta(\mathcal{T})$ is "almost never" enumerative.

We start our study of enumerative significance by showing the origin of potential problems with enumerative significance, thereby giving counterexamples to enumerative significance in the cases (iv) and (v) above.

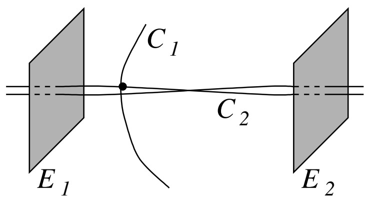
The most obvious problem is that a stable map $(C, x_1, \ldots, x_n, f)$ may be reducible, with some of the components mapped to the exceptional divisor. The part of the moduli space corresponding to such curves will in general have too big dimension. For example, consider the case $\tilde{X} = \tilde{\mathbb{P}}^3(1)$, $\beta = 4H'$. Stable maps in $M_{0,0}(\tilde{X}, \beta)$ will not intersect the exceptional divisor at all, hence $M_{0,0}(\tilde{X}, \beta)$ has the expected dimension. However, consider reducible curves $C = C_1 \cup C_2$ where f is of homology class $4H' - 3E'$ on C_1 and of homology class $3E'$ on C_2. These can be depicted as follows:



The space of such curves C_1 is (at least) of dimension vdim $\bar{M}_{0,0}(\tilde{X}, 4H' - 3E') = 4 \cdot 4 - 3 \cdot 2 = 10$, the space of curves C_2 of homology class $3E'$ through a given point (namely one of the points of intersection of C_1 with E) is of dimension $3 \cdot 3 - 1 - 1 = 7$ (note that $E \cong \mathbb{P}^2$). Hence the part of the moduli space $\bar{M}_{0,0}(\tilde{X}, \beta)$ corresponding to those curves has dimension (at least) 17, but we have vdim $\bar{M}_{0,0}(\tilde{X}, \beta) = 4 \cdot 4 = 16$. Note that this is in agreement with the fact that these curves certainly cannot be deformed into smooth quartics not intersecting the exceptional divisor, hence they are not contained in the closure of $M_{0,0}(\tilde{X}, \beta)$ in $\bar{M}_{0,0}(\tilde{X}, \beta)$.

However, this will cause no problems when computing Gromov-Witten invariants, since, intuitively speaking, the curve C_2 cannot satisfy any incidence conditions with generic non-exceptional varieties. So if we try to impose vdim $\bar{M}_{0,0}(\tilde{X}, \beta) = 16$ non-exceptional conditions on these curves, we will get zero, since the curve C_1 can satisfy at most 10 of the conditions and C_2 can satisfy none at all. For a mathematically more precise statement of this fact, see proposition 2.5.2 (i) which is the important step in the proof of enumerative significance in the case of only one blow-up.

When we consider more than one blow-up, things get more complicated, since then for example multiple coverings of the lines joining the blown-up points will cause problems. As an example, consider $\tilde{X} = \tilde{\mathbb{P}}^r(2)$, $\beta = (d+q)H' - qE'_1 - qE'_2$ for some $r \geq 2$, $d \geq 1$, $q \geq 2$, and look at reducible stable maps as above with C_1 of homology class dH' and C_2 of homology class $qH' - qE'_1 - qE'_2$, being a q-fold covering of the strict transform of the line between P_1 and P_2:



We have just learned that C_2 for itself will make no problems, since no generic (non-divisorial) non-exceptional incidence conditions can be satisfied on this component. However, it may well happen that the dimension of the moduli space of curves C_1 meeting the line through P_1 and P_2 (i.e. vdim $\bar{M}_{0,0}(\tilde{X}, dH') - (r-2)$) is *bigger* than that of both components together:

$$\begin{aligned}
\text{vdim } \bar{M}_{0,0}(\tilde{X}, dH') - (r-2) &= (r+1)d + r - 3 - (r-2), \\
\text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) &= (r+1)d + (1-q)(r-3), \\
\Rightarrow \text{vdim } \bar{M}_{0,0}(\tilde{X}, dH') - (r-2) - \text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) &= \underline{(q-1)(r-3) - 1}.
\end{aligned}$$

If this last number is non-negative, we will obviously get non-wanted contributions to our Gromov-Witten invariants from these reducible curves, since all vdim $\bar{M}_{0,0}(X, \beta)$ conditions that we impose on the curve can be satisfied on C_1. This will always happen if $r \geq 4$, showing that in this case there is no chance of getting enumerative invariants.

The reader who wants to convince himself of this fact numerically can find some obviously non-enumerative invariants of this kind in example 2.8.4. For $r = 3$, we will see that multiple coverings of lines joining blown-up points only make problems if they form the only component of an irreducible curve, see theorem 2.6.4 and example 2.8.3. In fact, in the case where $\beta = dH' - dE_1' - dE_2'$, such that we "count" d-fold coverings of lines, we get other important invariants, see example 2.8.5.

Since the case of $\tilde{\mathbb{P}}^4(s)$ for $s \geq 2$ will not lead to enumerative invariants and the case of $\tilde{\mathbb{P}}^2(s)$ has been studied almost exhaustively in [GP], it only remains to look at blow-ups of $\mathbb{P}^3$. We will look at the case $\tilde{X} = \tilde{\mathbb{P}}^3(4)$ in detail in section 2.6 (which then includes, of course, also the cases $\tilde{X} = \tilde{\mathbb{P}}^3(s)$ with $s < 4$). Here, in analogy to the situation discussed above, one gets problems with too big dimensions for reducible curves as above, where C_2 is now a curve contained in a plane spanned by three of the blown-up points. These problems arise in particular because in this case it is no longer true that C_2 can satisfy no incidence conditions. To be more precise, C_2 can satisfy incidence conditions with generic curves, but *not* with generic points in $\tilde{\mathbb{P}}^3(4)$. This is the reason why we have to make the assumption that all cohomology classes in the invariant are point classes (see theorem 2.6.4). If we do not assume this, we can again easily get non-enumerative invariants, e.g. $I^{\tilde{\mathbb{P}}^3(4)}_{4H'-2E_1'-2E_2'-2E_3'}((H^2)^{\otimes 4}) = -1$, to mention the easiest one.

In the remainder of this section, we will prove some statements about irreducible curves in blow-ups that will be needed for both cases $\tilde{\mathbb{P}}^r(1)$ and $\tilde{\mathbb{P}}^3(4)$. We start by computing $h^1(\mathbb{P}^1, f^*T_{\tilde{X}})$ in the next two lemmas.

Lemma 2.4.3 *Let $p : \tilde{X} \to X$ be the blow-up of a smooth variety at some points $P_1, \dots, P_s$ and let $E = E_1 \cup \dots \cup E_s$ be the exceptional divisor. Let C be a smooth curve and $f : C \to \tilde{X}$ a map such that $f(C) \not\subset E$. Then there is an injective morphism of sheaves on $\tilde{X}$*

$$f^* p^* T_X(-f^*E) \to f^* T_{\tilde{X}}$$

which is an isomorphism away from $f^{-1}(E)$.

Proof Since $E = \{P_1, \dots, P_s\} \times_X \tilde{X}$, we have $i^*\Omega_{\tilde{X}/X} = \Omega_{E/\{P_1,\dots,P_s\}} = \Omega_E$ where $i : E \to \tilde{X}$ is the inclusion. As $\Omega_{\tilde{X}/X}$ has support on E, this can be rewritten as $i_*\Omega_E = \Omega_{\tilde{X}/X}$. Hence, there is an exact sequence of sheaves on $\tilde{X}$

$$0 \to p^*\Omega_X \to \Omega_{\tilde{X}} \to i_*\Omega_E \to 0.$$

Dualizing, we get

$$0 \to T_{\tilde{X}} \to p^*T_X \to \mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_X) \to 0.$$

By duality (see [H] theorem III 6.7), we have

$$\mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_X) = i_*\mathcal{E}xt^1(\Omega_E, N_{E/\tilde{X}}) = i_*T_E(-1)$$

where $\mathcal{O}(-1) := \mathcal{O}_{E_1}(-1) \otimes \ldots \otimes \mathcal{O}_{E_s}(-1)$. Therefore we get a morphism $p^* T_X \to i_* T_E(-1)$ which we can restrict to E to get a morphism $p^* T_X|_E \to i_* T_E(-1)$ fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & p^* T_X(-E) \arrow[r] & p^* T_X \arrow[r] \arrow[d, equal] & p^* T_X|_E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\tilde{X}} \arrow[r] & p^* T_X \arrow[r] & i_* T_E(-1) \arrow[r] & 0.
\end{tikzcd}$$

From this we can deduce the existence of an injective map $p^* T_X(-E) \to T_{\tilde{X}}$ which is clearly an isomorphism away from E. Applying the functor f^* we get the desired morphism $f^* p^* T_X(-f^* E) \to f^* T_{\tilde{X}}$. Since the image of f is not contained in E, this morphism is also injective and an isomorphism away from $f^{-1}(E)$. $\square$

Lemma 2.4.4 *Let $C = \mathbb{P}^1$, $\tilde{X} = \mathbb{P}^r(s)$, $f : C \to \tilde{X}$ a morphism, $\beta = f_*[C] \in A_1(\tilde{X})$, and $\varepsilon \in \{0, 1\}$.*

(i) If $f(C) \not\subset E$ or f is a constant map then $h^1(C, f^* T_{\tilde{X}}(-\varepsilon)) = 0$ whenever $d(\beta) + e(\beta) \geq 0$. (Here, $f^* T_{\tilde{X}}(-\varepsilon)$ is to be interpreted as $f^* T_{\tilde{X}} \otimes \mathcal{O}_C(-\varepsilon)$.) In particular, this always holds for $s = 1$ (since then $d(\beta) + e(\beta) = \deg\ f^*(H - E)$ and $f^*(H - E)$ is an effective divisor on C).

(ii) If $f(C) \subset E$ and the map $f : C \to E \cong \mathbb{P}^{r-1}$ has degree $e > 0$ then

$$h^1(C, f^* T_{\tilde{X}}(-\varepsilon)) = e + \varepsilon - 1.$$

Proof

(i) If f is a constant map then the assertion is trivial, so assume that $f(C) \not\subset E$ and set $d = \deg\ f^* H$, $e = -\deg\ f^* E$. By lemma 2.4.3 we have an exact sequence

$$0 \to f^* p^* T_X(e) \to f^* T_{\tilde{X}} \to Q \to 0$$

with some sheaf Q on C with zero-dimensional support. Hence to prove the lemma it suffices to show that $h^1(C, f^* p^* T_X(e - \varepsilon)) = 0$. But this follows from the Euler sequence on $\mathbb{P}^r$ pulled back to C and twisted by $\mathcal{O}_C(e - \varepsilon)$:

$$0 \to \mathcal{O}_C(e - \varepsilon) \to (r + 1)\, \mathcal{O}_C(d + e - \varepsilon) \to f^* p^* T_X(e - \varepsilon) \to 0$$

since $d + e - \varepsilon \geq -1$ by assumption.

(ii) We consider the normal sequence

$$0 \to T_E \to i^* T_{\tilde{X}} \to N_{E/\tilde{X}} \to 0.$$

As $N_{E/\tilde{X}} = \mathcal{O}_E(-1)$, pulling back to C and twisting by $\mathcal{O}_C(-\varepsilon)$ yields

$$0 \to f^* T_E(-\varepsilon) \to f^* T_{\tilde{X}}(-\varepsilon) \to \mathcal{O}_C(-e - \varepsilon) \to 0. \tag{1}$$

In complete analogy to (i), it follows by the Euler sequence of $E \cong \mathbb{P}^{r-1}$

$$0 \to \mathcal{O}_C(-\varepsilon) \to r\,\mathcal{O}_C(e-\varepsilon) \to f^*T_E(-\varepsilon) \to 0$$

that $h^1(C, f^*T_E(-\varepsilon)) = 0$. Hence we deduce from (1) that

$$h^1(f^*T_{\tilde{X}}(-\varepsilon)) = h^1(C, \mathcal{O}_C(-e-\varepsilon)) = e+\varepsilon-1.$$

□

We now come to the Bertini lemma 2.4.7 which is our main tool to prove the transversality of the intersection products in the Gromov-Witten invariants.

Lemma 2.4.5 *Let M be a scheme of finite type and $f: M \to \mathbb{P}^r$ a morphism. Then, for a generic hyperplane $H \subset \mathbb{P}^r$, we have:*

(i) $f^{-1}(H)$ is (empty or) of pure codimension 1 in M.

(ii) If M is smooth then the divisor $f^{-1}(H)$ is a smooth subscheme of M counted with multiplicity one.

Proof See e.g. [J] corollary 6.11. □

Lemma 2.4.6 *Let M be a scheme of finite type, X a smooth, connected, projective scheme, and $f: M \to X$ a morphism. Let L be a base point free linear system on X. Then, for generic $D \in L$, we have:*

(i) $f^{-1}(D)$ is (empty or) purely 1–codimensional.

(ii) If M is smooth then the divisor $f^{-1}(D)$ is a smooth subscheme of M counted with multiplicity one.

Proof The base point free linear system L on X gives rise to a morphism $s: X \to \mathbb{P}^m$ where $m = \dim L$. Composing with f yields a morphism $M \to \mathbb{P}^m$, and the divisors $D \in L$ correspond to the inverse images under s of the hyperplanes in $\mathbb{P}^m$. Hence, the statement follows from lemma 2.4.5, applied to the map $M \to \mathbb{P}^m$. □

Lemma 2.4.7 *Let M be a Deligne-Mumford stack of finite type, X a smooth, connected, projective scheme and $f_i: M \to X$ morphisms for $i = 1, \dots, n$. Let $\gamma_i \in A^{c_i}(X)$ be cycles of codimensions $c_i \geq 1$ on X that can be written as intersection products of divisors on X*

$$\gamma_i = [D'_{i,1}] \cdot \dots \cdot [D'_{i,c_i}] \qquad (i = 1, \dots, n)$$

such that the complete linear systems $|D'_{i,j}|$ are base point free (this always applies, for example, for effective cycles in the case $X = \mathbb{P}^r$). Let $c = c_1 + \dots + c_n$. Then, for generic $D_{i,j} \in |D'_{i,j}|$, we have:

(i) *$V_i := D_{i,1} \cap \cdots \cap D_{i,c_i}$ is smooth of pure codimension c_i in X, and the intersection is transverse. In particular, $[V_i] = \gamma_i$.*

(ii) *$V := f_1^{-1}(V_1) \cap \cdots \cap f_n^{-1}(V_n)$ is of pure codimension c in M. In particular, if dim $M < c$ then $V = \emptyset$.*

(iii) *If dim $M = c$ and M contains a dense, open, smooth substack U such that each geometric point of U has no non-trivial automorphisms then V consists of exactly $(f_1^*\gamma_1 \cdot \ldots \cdot f_n^*\gamma_n)[X]$ points of M which lie in U and are counted with multiplicity one.*

Proof

(i) follows immediately by recursive application of lemma 2.4.5 to the scheme X.

(ii) If M is a scheme, then the statement follows by recursive application of lemma 2.4.6. If M is a Deligne-Mumford stack, then it has an étale cover $S \to M$ by a scheme S, so (ii) holds for the composed maps $S \to M \to X$. But since the map $S \to M$ is étale, the statement is also true for the maps $M \to X$.

(iii) A Deligne-Mumford stack U whose generic geometric point has no non-trivial automorphisms always has a dense open substack U' which is a scheme (see e.g. [Vi]. To be more precise, U is a functor and hence an algebraic space ([DM] ex. 4.9), but an algebraic space always contains a dense open subset U' which is a scheme ([Kn] p. 25)). Since U' is dense in M and therefore $M\backslash U'$ has smaller dimension, applying (ii) to the restrictions $f_i|_{M\backslash U'} : M\backslash U' \to X$ gives that V is contained in the smooth scheme U', hence it suffices to consider the restrictions $f_i|_{U'} : U' \to X$. But since U' is a smooth scheme, we can apply lemma 2.4.6 (ii) recursively and get the desired result.

□

As we needed for lemma 2.4.7 (iii) that the generic element of M has no non-trivial automorphisms, we now give a criterion under which circumstances this is satisfied for our moduli spaces of stable maps.

Lemma 2.4.8 *Let $\tilde{X} = \mathbb{P}^r(s)$ and $\beta \in A_1(\tilde{X})$ with $d(\beta) > 0$ and $d(\beta) + e(\beta) \geq 0$. Assume that β is not of the form $d\,H' - d\,E_i'$ for $1 \leq i \leq s$ and $d \geq 2$. Then, if $M_{0,n}(\tilde{X},\beta)$ is not empty, it is a smooth stack of the expected dimension, and if $\mathcal{C} = (C, x_1, \ldots, x_n, f)$ is a generic element of $M_{0,n}(\tilde{X},\beta)$ then $\mathcal{C}$ has no automorphisms and f is generically injective.*

Proof Set $d = d(\beta)$ and $e = e(\beta)$. We can assume that $e \leq 0$ since otherwise $M_{0,n}(\tilde{X},\beta)$ is empty.

It follows from lemma 2.4.4 (i) that $M_{0,n}(\tilde{X},\beta)$ is a smooth stack of the expected dimension. Note that an irreducible stable map can only have automorphisms if it is a multiple covering map onto its image. Therefore it suffices if we compute, for all $N \geq 2$, the dimension of the subspace $Z_N \subset M_{0,n}(\tilde{X},\beta)$ consisting of N-fold coverings and show that it is smaller than the dimension of $M_{0,n}(\tilde{X},\beta)$.

So assume that $N \geq 2$ and that $Z_N \neq \emptyset$, so that $\beta = N\beta'$ for some $\beta' \in A_1(\tilde{X})$. We set $d' = d(\beta')$ and $e' = e(\beta')$. Since $d' + e' \geq 0$, we can apply lemma 2.4.4 (i) to see that the space of stable maps of homology class β' is of the expected dimension $(r+1)d' + (r-1)e' + r + n - 3$. The dimension of Z_N is exactly bigger by $2N-2$ because of the moduli of the covering. Hence we have

$$\begin{aligned}
\dim Z_N &= (r+1)d' + (r-1)e' + r + n - 3 + 2N - 2 \\
&= (r+1)d + (r-1)e + r + n - 3 + ((r+1)d' + (r-1)e')(1-N) + 2N - 2 \\
&= \dim M_{0,n}(\tilde{X},\beta) + ((r+1)d' + (r-1)e' - 2)(1-N).
\end{aligned}$$

Therefore, to prove the lemma, it suffices to show that $(r+1)d' + (r-1)e' > 2$. We distinguish two cases:

- If $e' = 0$, then
$$(r+1)d' + (r-1)e' = (r+1)d' \geq (2+1)\cdot 1 = 3 > 2.$$

- If $e' \leq -1$, then
$$(r+1)d' + (r-1)e' = (r+1)(d'+e') - 2e' \geq -2e' \geq 2,$$
but if we had equality, this would mean $d' + e' = 0$ and $e' = -1$, hence $\beta' = H' - E'_i$ for some i and therefore $\beta = NH' - NE'_i$, which is the case we excluded in the lemma.

This finishes the proof. $\square$

2.5 Enumerative significance — the case $\tilde{\mathbb{P}}^r(1)$

In this section we will prove that all invariants $I_\beta(\mathcal{T})$ on $\tilde{X} = \tilde{\mathbb{P}}^r(1)$ are enumerative. We start with the computation of $h^1(C, f^*T_{\tilde{X}})$ for arbitrary stable maps. To state the result, we need the following definition: for any prestable map $(C, x_1, \dots, x_n, f)$ to $\tilde{X}$ we define $\eta(C,f)$ to be "the sum of the exceptional degrees of all irreducible components of C which are mapped into E", i.e.

$$\boldsymbol{\eta(C,f)} := \sum_{C'} \{\, e \mid C' \text{ is an irreducible component of } C \text{ such that } f_*[C'] = eE' \,\}.$$

Obviously, $\eta(C,f)$ only depends on the topology τ of the prestable map in the sense of section 1.2, so we will write $\eta(\tau) = \eta(C,f)$.

Lemma 2.5.1 *Let C be a prestable curve, $\tilde{X} = \tilde{\mathbb{P}}^r(1)$, and $f : C \to \tilde{X}$ a morphism. Then $h^1(C, f^*T_{\tilde{X}}) \leq \eta(C, f)$, with strict inequality holding if $\eta(C, f) > 0$.*

Proof The proof is by induction on the number of irreducible components of C. If C itself is irreducible, the statement follows immediately from lemma 2.4.4 for $\varepsilon = 0$.

Now let C be reducible, so assume $C = C_0 \cup C'$ where $C' \cong \mathbb{P}^1$, $C_0 \cap C' = \{Q\}$, and where C_0 is a prestable curve for which the induction hypothesis holds. If $\eta(C, f) > 0$, we can arrange this such that $\eta(C_0, f_0) > 0$.

Consider the exact sequences

$$0 \to f^*T_{\tilde{X}} \to f_0^*T_{\tilde{X}} \oplus f'^*T_{\tilde{X}} \xrightarrow{\varphi} f_Q^*T_{\tilde{X}} \to 0$$
$$0 \to f'^*T_{\tilde{X}}(-Q) \to f'^*T_{\tilde{X}} \xrightarrow{\psi} f_Q^*T_{\tilde{X}} \to 0$$

where f_0, f', and f_Q denote the restrictions of f to C_0, C', and Q, respectively.

From these sequences we deduce that

$$\dim \operatorname{coker} H^0(\varphi) = h^1(C, f^*T_{\tilde{X}}) - h^1(C_0, f_0^*T_{\tilde{X}}) - h^1(C', f'^*T_{\tilde{X}})$$
$$\dim \operatorname{coker} H^0(\psi) = h^1(C', f'^*T_{\tilde{X}}(-Q)) - h^1(C', f'^*T_{\tilde{X}}).$$

Since we certainly have $\dim \operatorname{coker} H^0(\varphi) \leq \dim \operatorname{coker} H^0(\psi)$, we can combine these equations into the single inequality

$$h^1(C, f^*T_{\tilde{X}}) \leq h^1(C_0, f_0^*T_{\tilde{X}}) + h^1(C', f'^*T_{\tilde{X}}(-Q)).$$

Now, by the induction hypothesis on f_0, we have $h^1(C_0, f_0^*T_{\tilde{X}}) \leq \eta(C_0, f_0)$ with strict inequality holding if $\eta(C_0, f_0) > 0$. On the other hand, we get $h^1(C', f'^*T_{\tilde{X}}(-Q)) \leq \eta(C', f')$ by lemma 2.4.4 for $\varepsilon = 1$. As $\eta(C, f) = \eta(C_0, f_0) + \eta(C', f')$, the proposition follows by induction. □

We now come to the central proposition already alluded to in section 2.4: given a part $M(\tilde{X}, \tau)$ of the moduli space $\bar{M}_{0,n}(\tilde{X}, \beta)$ corresponding to the topology τ (see section 1.2), we consider the map

$$\phi : M(\tilde{X}, \tau) \hookrightarrow \bar{M}_{0,n}(\tilde{X}, \beta) \to \bar{M}_{0,n}(X, d(\beta))$$

given by mapping $(C, x_1, \dots, x_n, f)$ to $(C, x_1, \dots, x_n, p \circ f)$ and stabilizing if necessary (ϕ exists by the functoriality of the moduli spaces of stable maps, see proposition 1.2.6 (ii)). We show that, although $M(\tilde{X}, \tau)$ may have too big dimension, the image $\phi(M(\tilde{X}, \tau))$ has not. Part (ii) of the proposition, which is of similar type, will be needed later in section 2.7.

Proposition 2.5.2 *Let $\tilde{X} = \tilde{\mathbb{P}}^r(1)$ and $\beta \in A_1(\tilde{X})$ with $d(\beta) > 0$. Let $\phi : \bar{M}_{0,n}(\tilde{X}, \beta) \to \bar{M}_{0,n}(X, d(\beta))$ be the morphism as above, and let τ be a topology of stable maps of homology class β (so that $M(\tilde{X}, \tau) \subset \bar{M}_{0,n}(\tilde{X}, \beta)$). Then we have* *(i)* *dim* $\phi(M(\tilde{X},\tau)) \leq$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta)$. *Moreover, strict inequality holds if and only if* τ *is a topology corresponding to reducible curves.*

(ii) *At least one of the following holds:*

(a) *dim* $\phi(M(\tilde{X},\tau)) \leq$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta) - r$,

(b) *dim* $M(\tilde{X},\tau) \leq$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta) - 2$,

(c) *dim* $M(\tilde{X},\tau) \leq$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta) - 1$ *and* $\eta(\tau) = 0$,

(d) *dim* $M(\tilde{X},\tau) =$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta)$ *and* τ *is the topology corresponding to irreducible curves,*

(e) *dim* $M(\tilde{X},\tau) =$ *vdim* $\bar{M}_{0,n}(\tilde{X},\beta) - 1$ *and* τ *is a topology corresponding to reducible curves having exactly two irreducible components, one with homology class* $\beta - E'$ *and the other with homology class* E'.

Proof We start by defining some numerical invariants of the topology τ that will be needed in the proof.

- Let $\boldsymbol{S}$ be the number of nodes of a curve with topology τ. We divide this number into $S = S_{EE} + S_{XX} + S_{XE}$, where $\boldsymbol{S_{EE}}$ (resp. $\boldsymbol{S_{XX}}$, $\boldsymbol{S_{XE}}$) denotes the number of nodes joining two exceptional components of C (resp. two non-exceptional components, or one exceptional with one non-exceptional component). Here and in the following we call an irreducible component of C exceptional if it is mapped by f into the exceptional divisor and it is not contracted by f, and non-exceptional otherwise.

- Let $\boldsymbol{P}$ be the (minimal) number of additional marked points which are necessary to stabilize C. We divide the number P into $P = P_E + P_X$, where $\boldsymbol{P_E}$ (resp. $\boldsymbol{P_X}$) is the number of marked points that have to be added on exceptional components (resp. non-exceptional components) of C to stabilize C.

Now we give an estimate for the dimension of $M(\tilde{X},\tau)$. The tangent space $T_{M(\tilde{X},\tau),\mathcal{C}}$ at a point $\mathcal{C} = (C, x_1, \ldots, x_n, f) \in M(\tilde{X},\tau)$ is given by the hypercohomology group (see [K] section 1.3.2)

$$T_{M(\tilde{X},\tau),\mathcal{C}} = \mathbb{H}^1(T'_C \to f^* T_{\tilde{X}})$$

where $T'_C = T_C(-x_1 - \cdots - x_n)$ and where we put the sheaves T'_C and $f^* T_{\tilde{X}}$ in degrees 0 and 1, respectively. This means that there is an exact sequence

$$0 \to H^0(C, T'_C) \to H^0(C, f^* T_{\tilde{X}}) \to T_{M(\tilde{X},\tau),\mathcal{C}} \to H^1(C, T'_C) \qquad (1)$$

(note that the first map is injective because $\mathcal{C}$ is a stable map). By lemma 2.5.1, we have

$$\dim H^0(C, f^* T_{\tilde{X}}) \leq \chi(C, f^* T_{\tilde{X}}) + \eta(C, f). \qquad (2)$$

Moreover, by Riemann-Roch we get $\chi(C,T'_C) = S+3-n$. It follows that

$$\begin{aligned}\dim\, T_{M(\tilde{X},\tau),\mathcal{C}} &\leq \chi(C,f^*T_{\tilde{X}}) + \eta(C,f) + n - S - 3 \\ &= \text{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) + \eta(C,f) - S,\end{aligned}$$

and therefore

$$\dim\, M(\tilde{X},\tau) \leq \text{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) + \eta(\tau) - S.$$

If $\eta(\tau) - S < 0$, then statement (i) is obviously satisfied. Moreover, if $\eta(\tau) = 0$ then we also have (ii)-(c), and if $\eta(\tau) > 0$ then we have strict inequality also in (2) and therefore (ii)-(b). Therefore we can assume from now on that $\underline{\eta(\tau) - S \geq 0}$. If $\eta(\tau) = 0$, then we must also have $S = 0$, which means that the curve is irreducible. But then (i) and (ii)-(d) are satisfied. So we can also assume in the sequel that $\underline{\eta(\tau) > 0}$. It follows then from lemma 2.5.1 that we have strict inequality in (2), hence

$$\dim\, T_{M(\tilde{X},\tau),\mathcal{C}} \leq \text{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) + \eta(C,f) - S - 1. \tag{3}$$

We now give an estimate of the dimension of the image $\phi(M(\tilde{X},\tau))$. As we always work over the ground field $\mathbb{C}$, we can do this on the level of tangent spaces, i.e. we have

$$\dim\, \phi(M(\tilde{X},\tau)) \leq \max{}_{\mathcal{C}\in M(\tilde{X},\tau)} \dim\, (d\phi)(T_{M(\tilde{X},\tau),\mathcal{C}}).$$

Hence our goal is to find as many vectors in ker $d\phi$ as possible. We do this by finding elements in the kernel of the composite map (see (1))

$$H^0(C,f^*T_{\tilde{X}})/H^0(C,T'_C) \hookrightarrow T_{M(\tilde{X},\tau),\mathcal{C}} \to T_{\bar{M}_{0,n}(X,d(\beta)),\phi(\mathcal{C})}.$$

Let C_0 be a maximal connected subscheme of C consisting only of exceptional components of C. Let f_0 be the restriction of f to C_0 and let $Q_1,\dots,Q_a$ be the nodes of C which join C_0 with the rest of C (they are of type S_{XE}). Now every section of $f_0^*T_E(-Q_1-\cdots-Q_a)$ can be extended by zero to a section of $f^*T_{\tilde{X}}$ which is mapped to zero by $d\phi$ since these deformations of the map take place entirely within the exceptional divisor. As $E\cong\mathbb{P}^{r-1}$ is a convex variety, we have

$$h^0(C_0,f_0^*T_E) = \chi(C_0,f_0^*T_E) = r-1+r\eta(C_0,f_0)$$

and therefore we can estimate the dimension of the space of deformations that we have just found:

$$h^0(C_0,f_0^*T_E(-Q_1-\cdots-Q_a)) \geq r-1+r\eta(C_0,f_0)-(r-1)\,a.$$

(The right hand side of this inequality may well be negative, but nevertheless the statement is correct also in this case, of course.)

We will now add up these numbers for all possible C_0, say there are $\boldsymbol{B}$ of them. The sum of the $\eta(C_0, f_0)$ will then give $\eta(C, f)$, and the sum of the a will give S_{XE}. Note that there is a P_E-dimensional space of infinitesimal automorphisms of C, i.e. a subspace of $H^0(C, T'_C)$, included in the deformations that we have just found, and that these are exactly the trivial elements in the kernel of $d\phi$. Therefore we have

$$\begin{aligned}
\dim \ker d\phi &\geq B(r-1) + r\eta(C, f) - (r-1)S_{XE} - P_E \\
&= (r-2)(\underbrace{B}_{\geq 1} + \underbrace{\eta(C, f) - S_{XE}}_{\geq 0}) + B + 2\eta(C, f) - S_{XE} - P_E \\
&\qquad (B \geq 1 \text{ since } \eta(C, f) > 0 \\
&\qquad\ \text{and } \eta(C, f) - S_{XE} \geq 0 \text{ since } \eta(C, f) - S \geq 0) \\
&\geq (r-2) + B + 2\eta(C, f) - S_{XE} - P_E.
\end{aligned}$$

Combining this with (3), we get the estimate

$$\begin{aligned}
\dim \phi(M(\tilde{X}, \tau)) &\leq \dim T_{M(\tilde{X},\tau),\mathcal{C}} - \dim \ker d\phi \\
&\leq \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) - r + 1 - (S_{XX} + S_{EE} + B + \eta(\tau) - P_E).
\end{aligned}$$

To prove the proposition, it remains to look at the term in brackets. First we will show that

$$P_E \leq S_{XX} + S_{EE} + B + \eta(\tau). \tag{4}$$

Look at P_E, i.e. the exceptional components of C where marked points have to be added to stabilize C. We have to distinguish three cases:

(A) Components on which two points have to be added, and whose (only) node is of type S_{EE}: those give a contribution of 2 to P_E, but they also give at least 1 to $\eta(\tau)$ and to S_{EE} (and every node of type S_{EE} belongs to at most one such component).

(B) Components on which two points have to be added, and whose (only) node is of type S_{XE}: those give a contribution of 2 to P_E, but they also give at least 1 to $\eta(\tau)$ and to B (since such a component alone is one of the C_0 considered above).

(C) Components on which only one point has to be added: those give a contribution of 1 to P_E, but they also give at least 1 to $\eta(\tau)$.

This shows (4), finishing the proof of (i). As for (ii), (a) is satisfied if we have strict inequality in (4), so we assume from now on that this is not the case and determine necessary conditions for equality by looking at the proof of (4) above. First of all, we see that every maximal connected subscheme of C consisting only of exceptional components contributes 1 to B, but this gets accounted for only in case (B) above, so if we want to have equality, every such maximal connected subscheme must actually be an irreducible component of type (B), which in addition gives a contribution of *exactly*

2 to P_E and *exactly* 1 to $\eta(\tau)$. So all exceptional components of the curve must actually be lines with no marked points, connected at exactly one point to a non-exceptional component of the curve. Moreover, for equality we must also have $S_{XX} = 0$, since these nodes have not been considered above at all.

Hence, in summary, we must have one non-exceptional irreducible component C_0 of homology class $\beta - qE'$, and q exceptional components of homology class E' with no marked points, each connected at exactly one point to C_0. But it is easy to compute the dimension of $\phi(M(X,\tau))$ for these topologies: the map ϕ simply forgets the q exceptional components, so

$$\begin{aligned} \dim\, \phi(M(\tilde{X},\tau)) &= \dim\, M_{0,n}(\tilde{X},\beta - qE') \\ &= \mathrm{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta - qE') \qquad \text{by (i)} \\ &= \mathrm{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) - q(r-1). \end{aligned}$$

Hence we see that (ii)-(a) is satisfied for $q > 1$ and (ii)-(e) for $q = 1$.

This completes the proof. □

We now combine our results to prove the enumerative significance of the Gromov-Witten invariants of $\tilde{\mathbb{P}}^r(1)$. Some examples of these numbers can be found in 2.8.1 and 2.8.2.

Theorem 2.5.3 *Let $\tilde{X} = \tilde{\mathbb{P}}^r(1)$, $\beta = dH' + eE' \in A_1(\tilde{X})$ an effective homology class with $d > 0$ and $e \le 0$, and $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ a collection of non-exceptional effective classes such that $\sum_i$ codim γ_i = vdim $\bar{M}_{0,n}(\tilde{X},\beta)$. Then $I_\beta(\mathcal{T})$ is enumerative.*

Proof The proof goes along the same lines as that of lemma 2.2.2. For irreducible stable maps $(C,x_1,\ldots,x_n,f)$ we have $h^1(C,f^*T_{\tilde{X}}) = 0$ by lemma 2.4.4 (i). Therefore, if $Z \subset \bar{M}_{0,n}(\tilde{X},\beta)$ denotes the closure of $M_{0,n}(\tilde{X},\beta)$, then lemma 1.3.3 tells us that

$$[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = [Z] + \alpha$$

where α is a cycle of dimension vdim $\bar{M}_{0,n}(\tilde{X},\beta)$ supported on $\bar{M}_{0,n}(\tilde{X},\beta)\backslash M_{0,n}(\tilde{X},\beta)$. But if $\phi : \bar{M}_{0,n}(\tilde{X},dH'+eE') \to \bar{M}_{0,n}(X,dH')$ denotes the morphism induced by the map $p : \tilde{X} \to X$, we must have $\phi_*\alpha = 0$ by proposition 2.5.2 (i). So, considering the commutative diagram

$$\begin{tikzcd} \bar{M}_{0,n}(\tilde{X},\beta) \arrow[r, "\phi"] \arrow[d, "ev_i"'] & \bar{M}_{0,n}(X,dH') \arrow[d, "ev_i"'] \\ \tilde{X} \arrow[r, "p"] & X \end{tikzcd}$$

for $1 \le i \le n$, it follows by the projection formula that

$$\begin{aligned} I_\beta^{\tilde{X}}(\mathcal{T}) &= (\prod_i ev_i^* p^* \gamma_i) \cdot [\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} \\ &= (\prod_i ev_i^* \gamma_i) \cdot \phi_* [\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} \\ &= (\prod_i ev_i^* \gamma_i) \cdot \phi_* [Z]. \\ &= (\prod_i ev_i^* p^* \gamma_i) \cdot [Z]. \end{aligned}$$

Hence we are evaluating an intersection product on the stack Z.

Unless $d+e=0$ and $d \ge 2$, the theorem now follows from the Bertini lemma 2.4.7 (iii) in combination with lemma 2.4.8 saying that the generic element of Z has no automorphisms and corresponds to a generically injective stable map. However, if $d+e=0$ and $d \ge 2$, then the image of every stable map in $M_{0,n}(\tilde{X}, dH' - dE')$ is a line through the blown-up point. These curves can obviously only satisfy as many incidence conditions as the curves in $M_{0,n}(\tilde{X}, H' - E')$. But vdim $\bar{M}_{0,n}(\tilde{X}, dH' - dE') >$ vdim $\bar{M}_{0,n}(\tilde{X}, H' - E')$, so the Gromov-Witten invariant will be zero, which is also the enumeratively correct number. □

2.6 Enumerative significance — the case $\tilde{\mathbb{P}}^3(4)$

In this section, we discuss the enumerative significance of the Gromov-Witten invariants on $\tilde{X} = \tilde{\mathbb{P}}^3(4)$. First we fix some notation. As the four points to blow up on $X = \mathbb{P}^3$ we choose $P_1 = (1:0:0:0)$, $P_2 = (0:1:0:0)$, $P_3 = (0:0:1:0)$, and $P_4 = (0:0:0:1)$. For $1 \le i < j \le 4$, we denote by $L_{ij} \subset \tilde{X}$ the strict transform of the line $\overline{P_i P_j}$. The L_{ij} are disjoint from each other, and we set $\mathcal{L} = \bigcup_{i<j} L_{ij}$. For $1 \le i \le 4$, we let H_i be the strict transform of the hyperplane in X spanned by the three points P_j with $j \ne i$, and we set $\mathcal{H} = \bigcup_i H_i$. As usual, E_i denotes the exceptional divisor over P_i. We set $\mathcal{E} = \bigcup_i E_i$.

Let $\beta \in A_1(\tilde{X})$ be an effective homology class with $d(\beta) > 0$. The first thing to do is to look at *irreducible* curves of homology class β and to see whether their moduli space $M_{0,0}(\tilde{X}, \beta)$ is smooth and of the expected dimension, which in this case is

$$\text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) = 4d(\beta) + 2e(\beta).$$

In the case of one blow-up in section 2.5, this followed easily from lemma 2.4.4 (i) since there we always have $d(\beta) + e(\beta) \ge 0$. However, for multiple blow-ups, this is not necessarily the case. Our way to solve this problem is to use a certain Cremona map to transform curves with $d(\beta) + e(\beta) \le 0$ into others with $d(\beta) + e(\beta) \ge 0$, so that lemma 2.4.4 can be applied again. Before we can describe this map, we need a definition.

Definition 2.6.1 *Let $(C,f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4),\beta)$ be an irreducible stable map such that $f(C) \not\subset \mathcal{L}$. Then we set $\lambda_{ij}(C,f)$ to be the "multiplicity of f along L_{ij}", defined as follows: if $\varphi_1 : \tilde{Y} \to \tilde{\mathbb{P}}^3(4)$ is the blow-up of $\tilde{\mathbb{P}}^3(4)$ along $\mathcal{L}$ with exceptional divisors F_{ij} over L_{ij}, then there is a well-defined map $\varphi_1^{-1} \circ f : C \to \tilde{Y}$, and we define*

$$\boldsymbol{\lambda_{ij}(C,f)} := F_{ij} \cdot (\varphi_1^{-1} \circ f)_*[C] \geq 0.$$

Finally, we define $\boldsymbol{\vec{\lambda}(C,f)}$ to be the vector consisting of all $\lambda_{ij}(C,f)$, and set

$$\boldsymbol{\lambda(C,f)} = \sum_{i<j} \lambda_{ij}(C,f).$$

We can now describe the Cremona map announced above.

Lemma 2.6.2 *There exists a birational map $\varphi : \tilde{\mathbb{P}}^3(4) \dashrightarrow \tilde{\mathbb{P}}^3(4)$ which is an isomorphism outside $\mathcal{L}$ with the following property:*

If $(C,f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4),\beta)$ is an irreducible stable map such that $f(C) \not\subset \mathcal{L}$, so that the transformed stable map $(C,\varphi \circ f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4),\beta')$ exists, then the homology class β' of the transformed stable map satisfies

$$\begin{aligned} d(\beta') &= 3d(\beta) + 2e(\beta) - \lambda(C,f), \\ e(\beta') &= -4d(\beta) - 3e(\beta) + 2\lambda(C,f). \end{aligned}$$

Hence, in particular, we have

- $4d(\beta') + 2e(\beta') = 4d(\beta) + 2e(\beta)$,
- *if* $d(\beta) + e(\beta) \leq 0$, *then* $d(\beta') + e(\beta') \geq 0$.

Proof The birational map $\varphi : \tilde{\mathbb{P}}^3(4) \dashrightarrow \tilde{\mathbb{P}}^3(4)$ we want to consider is most easily described in the language of toric geometry (see e.g. [F2]). Let Δ' in $\mathbb{R}^3$ be the complete simplicial fan with one-dimensional cones $\{\langle v_i \rangle \mid 1 \leq i \leq 4\}$, where

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1),\ v_4 = (-1,-1,-1),$$

corresponding to the toric variety $X_{\Delta'} = \mathbb{P}^3$. Let Δ be the blow-up of Δ' at the four torus-invariant points as described in [F2] section 2.4, so that the toric variety X_Δ associated to Δ is $\tilde{\mathbb{P}}^3(4)$. The fan Δ can be described explicitly as follows: it is the complete fan with one-dimensional cones

$$\{\pm\langle v_i \rangle \mid 1 \leq i \leq 4\}$$

and two-dimensional cones

$$\{\langle v_i, -v_j \rangle \mid 1 \leq i,j \leq 4; i \neq j\} \cup \{\langle v_i, v_j \rangle\ ;\ 1 \leq i < j \leq 4\}.$$

The Picard group of X_Δ is generated by the divisors corresponding to the one-dimensional cones, we will denote the divisor corresponding to the cone $\langle v_i \rangle$ by H_i and the divisor corresponding to the cone $-\langle v_i \rangle$ by E_i. This coincides with the definition of H_i and E_i given above, and these divisors satisfy the three relations

$$\begin{aligned} H := {} & H_1 + E_2 + E_3 + E_4 \\ = {} & H_2 + E_1 + E_3 + E_4 \\ = {} & H_3 + E_1 + E_2 + E_4 \\ = {} & H_4 + E_1 + E_2 + E_3 \end{aligned} \tag{1}$$

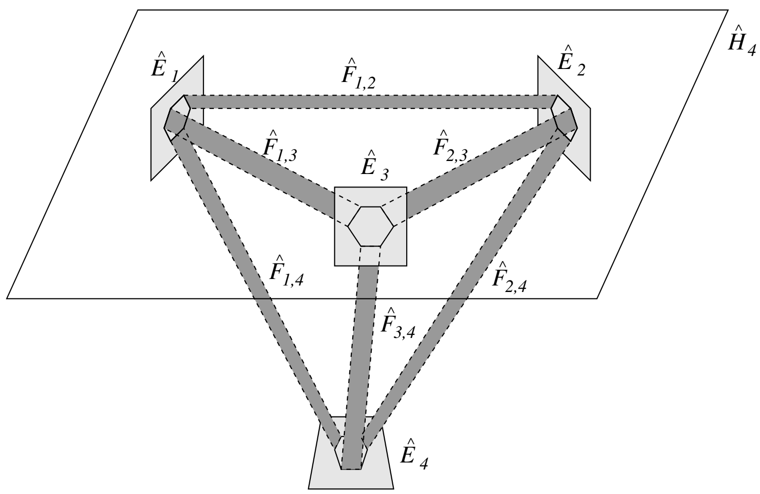
where H denotes the pullback of the hyperplane class under the map $p : \tilde{\mathbb{P}}^3(4) \to \mathbb{P}^3$.

Now denote by $-\Delta$ the fan obtained by mirroring Δ at the origin in $\mathbb{R}^3$. Then, of course, we also have $X_{-\Delta} \cong \tilde{\mathbb{P}}^3(4)$. The map φ we want to consider is now the obvious rational map $\varphi : X_\Delta \dashrightarrow X_{-\Delta}$ which is the identity on the torus $(\mathbb{C}^*)^3$ contained in both X_Δ and $X_{-\Delta}$. Note that the one-dimensional cones of Δ and $-\Delta$ are the same, so that φ is an isomorphism away from a subvariety of $\tilde{\mathbb{P}}^3(4)$ of codimension 2.

In more geometric terms, we can describe φ as the so-called "flip" of the 6 lines $\mathcal{L}$, i.e. one blows up these lines (that have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in $\tilde{\mathbb{P}}^3(4)$) to get a variety $\tilde{Y}$ with the 6 exceptional divisors $\hat{F}_{ij} \cong \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to L_{ij}, and then blows down the F_{ij} again with the roles of base and fibre reversed in $\mathbb{P}^1 \times \mathbb{P}^1$. One can write these two steps as in the following diagram:

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ X_\Delta \cong \tilde{\mathbb{P}}^3(4) \arrow[rr, dashed, "\varphi"] & & X_{-\Delta} \cong \tilde{\mathbb{P}}^3(4). \end{tikzcd}$$

The variety $\tilde{Y}$ can be depicted as follows:



Here, we denoted the strict transforms of H_i and E_i under φ_1 by $\hat{H}_i$ and $\hat{E}_i$, respectively. These are all isomorphic to $\tilde{\mathbb{P}}^2(3)$. The divisors $\hat{H}_1$, $\hat{H}_2$, and $\hat{H}_3$ have not been drawn to keep the picture simple.

We now look more closely at the divisors in $\tilde{Y}$. Obviously, we have

$$\begin{aligned}\varphi_1^* H_1 &= \hat{H}_1 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34},\\ \varphi_1^* E_1 &= \hat{E}_1,\end{aligned}$$

and similarly for H_i and E_i with $i = 2,3,4$. The Picard group of $\tilde{Y}$ is the free abelian group generated by the $\hat{H}_i$, $\hat{E}_i$, and $\hat{F}_{ij}$, modulo the three relations induced by (1)

$$\begin{aligned}\hat{H} := \varphi_1^* H &= \hat{H}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}\\ &= \hat{H}_2 + \hat{E}_1 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{13} + \hat{F}_{14} + \hat{F}_{34}\\ &= \hat{H}_3 + \hat{E}_1 + \hat{E}_2 + \hat{E}_4 + \hat{F}_{12} + \hat{F}_{14} + \hat{F}_{24}\\ &= \hat{H}_4 + \hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{F}_{12} + \hat{F}_{13} + \hat{F}_{23}.\end{aligned} \tag{2}$$

If we now have a stable map in (C,f) in $\tilde{Y}$, we also get stable maps (C_i, f_i) in $\tilde{\mathbb{P}}^3(4)$ by composing f with φ_i for $i = 1,2$. We will now compute the homology classes of these two stable maps.

The homology class of (C_1, f_1) is $\beta = d H' + \sum_i e_i E_i'$ where

$$\begin{aligned}d &= H \cdot \varphi_{1*} f_*[C]\\ &= \hat{H} \cdot f_*[C]\\ &= (\hat{H}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}) \cdot f_*[C],\\ e_i &= -E_i \cdot \varphi_{1*} f_*[C]\\ &= -\hat{E}_i \cdot f_*[C].\end{aligned}$$

The homology class of (C_2, f_2) is obtained by reversing the roles of $\hat{H}_i$ and $\hat{E}_i$ and substituting $\hat{F}_{12} \leftrightarrow \hat{F}_{34}$, $\hat{F}_{13} \leftrightarrow \hat{F}_{24}$, and $\hat{F}_{14} \leftrightarrow \hat{F}_{23}$, so it is $\beta' = d' H' + \sum_i e_i' E_i'$ where

$$\begin{aligned}d' &= (\hat{E}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \hat{F}_{14} + \hat{F}_{13} + \hat{F}_{12}) \cdot f_*[C]\\ &= (3\hat{H}_1 - 2\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 - \hat{F}_{12} - \hat{F}_{13} - \hat{F}_{14} + 2\hat{F}_{23} + 2\hat{F}_{24} + 2\hat{F}_{34}) \cdot f_*[C]\\ &\qquad \text{(by substituting } \hat{H}_2,\ \hat{H}_3, \text{ and } \hat{H}_4 \text{ from (2))}\\ &= 3d + 2(e_1 + e_2 + e_3 + e_4) - \underbrace{\Big(\sum_{i<j} F_{ij}\Big) \cdot f_*[C]}_{=\lambda(C_1,f_1)=\lambda(C_2,f_2)=:\lambda},\\ e_1' &= -\hat{H}_1 \cdot f_*[C]\\ &= -d - e_2 - e_3 - e_4 + (\hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}) \cdot f_*[C],\end{aligned}$$

and similarly for e_2, e_3, and e_4. Defining $e = \sum_i e_i$ and $e' = \sum_i e_i'$, we arrive at the equations

$$\begin{aligned}d' &= 3d + 2e - \lambda,\\ e' &= -4d - 3e + 2\lambda.\end{aligned}$$

In particular, we see that $4d'+2e' = 4d+2e$ and that, if $d+e \leq 0$, then

$$d'+e' = -d-e+\lambda \geq \lambda \geq 0.$$

□

We now use this map to prove some properties of irreducible stable maps in $\tilde{X} = \tilde{\mathbb{P}}^3(4)$. As already mentioned in section 2.4, apart from the case where $M_{0,n}(\tilde{X},\beta)$ is smooth of the expected dimension (case (iii) below), we have to consider the cases where the curves are multiple coverings of one of the L_{ij} (case (i)) and where they are contained in one of the H_i (such that they cannot satisfy any incidence conditions with generic points in $\tilde{X}$, see case (ii)). One of the most important statements of the next lemma is the final conclusion that, although the dimension of the moduli space may be too big, the curves can never satisfy more incidence conditions (with points) as one would expect from the virtual dimension of the moduli space.

Lemma 2.6.3 *Let $\beta \in A_1(\tilde{X})$ be a homology class such that $M_{0,0}(\tilde{X},\beta) \neq \emptyset$. Set*

$$n := \frac{1}{2}\,\mathrm{vdim}\,\bar{M}_{0,0}(\tilde{X},\beta) = 2d(\beta)+e(\beta).$$

Then at least one of the following statements holds:

(i) $n = 0$ and $\beta = dH' - dE'_i - dE'_j$ for some $d > 0$, $1 \leq i < j \leq 4$. All curves in $M_{0,0}(\tilde{X},\beta)$ are contained in L_{ij}.

(ii) $n > 0$, and for generic points $Q_1,\dots,Q_n \in \tilde{X}$, we have

$$ev_1^{-1}(Q_1) \cap \dots \cap ev_n^{-1}(Q_n) = \emptyset$$

in $M_{0,n}(\tilde{X},\beta)$.

(iii) $n > 0$, $\dim M_{0,0}(\tilde{X},\beta) = \mathrm{vdim}\,\bar{M}_{0,0}(\tilde{X},\beta)$, and for a generic element $\mathcal{C} = (C,f) \in M_{0,0}(\tilde{X},\beta)$, f is generically injective, $\mathcal{C}$ has no automorphisms, and $f(C)$ intersects neither $\mathcal{L}$ (which is a disjoint union of 6 smooth rational curves) nor $\mathcal{H} \cap \mathcal{E}$ (which is a union of 12 smooth rational curves).

In particular, it is impossible that $n < 0$, and in any case we have

$$ev_1^{-1}(Q_1) \cap \dots \cap ev_{n'}^{-1}(Q_{n'}) = \emptyset$$

in $M_{0,n'}(\tilde{X},\beta)$ for generic points $Q_1,\dots,Q_{n'} \in \tilde{X}$ if $n' > n$.

Proof Let $(C,f) \in M_{0,0}(\tilde{X},\beta)$ be a stable map, $d = d(\beta)$, $e_i = e_i(\beta)$, $e = \sum_i e_i$, and assume that $\beta \neq 0$ (since otherwise $M_{0,0}(\tilde{X},\beta) = \emptyset$).

If $d = 0$, then $n = e(\beta) > 0$ and $f(C)$ is contained in an exceptional divisor. Then it is clear that for a generic point in $\tilde{X}$, no curve in $M_{0,0}(\tilde{X},\beta)$ meets this point. Therefore, (ii) is satisfied.

Now assume $d > 0$, then we must have $e_i \leq 0$ for all i. The curve $f(C)$ cannot be contained at the same time in three of the H_i, since their intersection is empty. This means that there are at least two of the H_i, say H_1 and H_2, in which $f(C)$ is not contained. It follows that

$$d+e_2+e_3+e_4 = \deg\, f^*H_1 \geq 0 \quad \text{and} \quad d+e_1+e_3+e_4 = \deg\, f^*H_2 \geq 0.$$

Since $e_4 \leq 0$ and $e_3 \leq 0$, this also means that $d+e_2+e_3 \geq 0$ and $d+e_1+e_4 \geq 0$, and therefore $n = 2d+e \geq 0$: the virtual dimension of the moduli space cannot be negative. Moreover, if $n = 0$ then we must have equality everywhere, which means

$$e_1 = -d,\ e_2 = -d,\ e_3 = 0,\ e_4 = 0.$$

Hence we are in case (i), and it is clear that all these curves are d-fold coverings of L_{12}.

It remains to consider the case when $n > 0$. We distinguish four cases.

Case 1: $\beta = dH' - dE_i'$ for $d > 1$ and some $1 \leq i \leq 4$. Then the curves in $M_{0,0}(\tilde{X},\beta)$ must obviously be d-fold coverings of a line through the exceptional divisor E_i. Those cannot pass through two generic points, however $n = 2d-d = d \geq 2$, hence (ii) is satisfied.

We assume therefore from now on that β is not of this form.

Case 2: $d+e \geq 0$. We show that (iii) is satisfied.

- $\dim M_{0,0}(\tilde{X},\beta) = \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta)$: This follows because $h^1(C, f^*T_{\tilde{X}}) = 0$ by lemma 2.4.4 (i).

- the generic element of $M_{0,0}(\tilde{X},\beta)$ has no automorphisms and corresponds to a generically injective map: This follows from lemma 2.4.8.

- the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect $\mathcal{L}$ and $\mathcal{H}\cap\mathcal{E}$: Let L be one of the 18 smooth rational curves in $\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E})$, we will show that the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect L. Assume that (C,f) is a stable map in $\tilde{X}$ such that there is a point $x \in C$ with $f(x) = Q \in L$. Consider $\mathcal{C} = (C,x,f)$ as an element of $M = M_{0,1}(\tilde{X},\beta)$. The tangent space to M at the point $\mathcal{C}$ is (see [K] section 1.3.2)

  $$T_{M,\mathcal{C}} = H^0(C, f^*T_{\tilde{X}})/H^0(C, T_C(-x)).$$

  If $Z \subset M$ denotes the substack of those stable maps with $f(x) \in L$, then the tangent space to Z at $\mathcal{C}$ is

  $$T_{Z,\mathcal{C}} = \{s \in T_{M,\mathcal{C}}\ ;\ s(x) \in f^*T_{L,Q}\}.$$

  However, by lemma 2.4.4 (i) for $\varepsilon = 1$ we see that

  $$h^0(C, f^*T_{\tilde{X}}(-x)) = h^0(C, f^*T_{\tilde{X}}) - 3,$$

i.e. that the map $H^0(C, f^*T_{\tilde{X}}) \to f^*T_{\tilde{X},Q}$, $s \mapsto s(x)$ is surjective. Therefore the tangent space to Z at $\mathcal{C}$ has smaller dimension than that to M. Since M is smooth at $\mathcal{C}$, it follows that Z has smaller dimension than M at $\mathcal{C}$, proving the statement that the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect L.

Case 3: $d+e<0$ and $e_i=0$ for some i. Without loss of generality assume that $e_4=0$. Since then $0>d+e=\deg f^*(H-E_1-E_2-E_3)=\deg f^*H_4$, we conclude that $f(C)$ must be contained in H_4. Hence (ii) is satisfied.

Case 4: $d+e<0$ and all $e_i \neq 0$. We show that (iii) is satisfied using the Cremona map of lemma 2.6.2. We use in the following proof the notations of this lemma. Certainly no curve in $M_{0,0}(\tilde{X},\beta)$ is contained in $\mathcal{L}$. So if we decompose $M_{0,0}(\tilde{X},\beta)$ into parts $M_{\vec{\lambda}}$ according to the value of $\vec{\lambda}(C)$ then φ gives injective morphisms from $M_{\vec{\lambda}}$ to $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$ with $\beta_{\vec{\lambda}}$ calculated in the proof of lemma 2.6.2. In particular we have $d(\beta_{\vec{\lambda}})+e(\beta_{\vec{\lambda}}) \geq 0$, so that we can apply the results of case 2 to $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$. We therefore have

$$\begin{aligned} \dim M_{\vec{\lambda}} &\leq \dim M_{0,0}(\tilde{X},\beta_{\vec{\lambda}}) \\ &= \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta_{\vec{\lambda}}) \quad \text{by case 2} \\ &= 4d(\beta_{\vec{\lambda}})+2e(\beta_{\vec{\lambda}}) \\ &= 4d(\beta)+2e(\beta) \quad \text{by lemma 2.6.2} \\ &= \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta). \end{aligned} \tag{1}$$

If $\vec{\lambda} \neq 0$, i.e. if all curves in $M_{\vec{\lambda}}$ intersect $\mathcal{L}$, then the transformed curves in $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$ also have to intersect $\mathcal{L}$. But the generic curve in $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$ does not intersect $\mathcal{L}$ by the results of case 2, so it follows that we must have strict inequality in (1). Since the dimension of $\bar{M}_{0,0}(\tilde{X},\beta)$ cannot be smaller than its virtual dimension, this means that $M_{\vec{\lambda}}$ is nowhere dense in $M_{0,0}(\tilde{X},\beta)$ for $\vec{\lambda} \neq \vec{0}$. In other words, $M_{\vec{0}}$ is dense in $M_{0,0}(\tilde{X},\beta)$, so it obviously suffices to prove (iii) for $M_{\vec{0}}$.

But this is now easy: it follows from the above calculation that the dimension of $M_{\vec{0}}$ is equal to the virtual dimension of $\bar{M}_{0,0}(\tilde{X},\beta)$. The other statements of (iii) about the generic curves in the moduli space are obviously preserved by the Cremona map φ, so they follow from the fact that the space $M_{0,0}(\tilde{X},\beta_{\vec{0}})$ has these properties.

This completes the proof that we always have one of the cases (i) to (iii). The statement that $n \geq 0$ has already been proven, and the fact that

$$ev_1^{-1}(Q_1) \cap \cdots \cap ev_{n'}^{-1}(Q_{n'}) = \emptyset$$

in $M_{0,n'}(\tilde{X},\beta)$ for generic points $Q_1,\dots,Q_{n'} \in \tilde{X}$ if $n'>n$ follows easily in all cases: for (i) because the image of all curves in the moduli space is contained in an L_{ij}, for (ii) it is trivial, and for (iii) it follows from the Bertini lemma 2.4.7 (ii). □

To prove enumerative significance for the Gromov-Witten invariants on $\tilde{\mathbb{P}}^3(4)$, we now finally have to consider reducible stable maps. Some numerical examples can be found in 2.8.3.

Theorem 2.6.4 *Let $\tilde{X} = \tilde{\mathbb{P}}^3(4)$ and $\beta \in A_1(\tilde{X})$ an effective homology class which is not of the form $dH' - dE'_i - dE'_j$ for some $d \geq 2$ and $i \neq j$. Let $\mathcal{T} = pt^{\otimes n}$, where $n = 2d(\beta) + e(\beta)$. Then $I_\beta(\mathcal{T})$ is enumerative.*

Proof Let $Q_1, \dots, Q_n$ be generic points in $\tilde{X}$. First we want to show that all points in the intersection

$$I := ev_1^{-1}(Q_1) \cap \cdots \cap ev_n^{-1}(Q_n) \tag{1}$$

on $\bar{M}_{0,n}(\tilde{X}, \beta)$ correspond to irreducible stable maps. To do this, we decompose the moduli space $\bar{M}_{0,n}(\tilde{X}, \beta)$ into the spaces $M_\tau := M(\tilde{X}, \tau)$ according to the topology of the curves and show that $I \cap M_\tau$ is empty for each τ corresponding to reducible curves.

So assume that τ is a topology corresponding to stable maps (C, f) whose irreducible components *that are not contracted by* f are $C_1, \dots, C_a$. For $1 \leq i \leq a$, let $\beta_i \neq 0$ be the homology class of f on C_i and let n_i be the number of markings on the component C_i.

By a **maximal contracted subscheme** we will mean a maximal connected subscheme of C consisting only of components of C that are contracted by f. A maximal contracted subscheme will be called **marked** if it contains at least one of the marked points. For each $1 \leq i \leq a$, we define ρ_i to be the number of marked maximal contracted subschemes of C that have non-empty intersection with C_i.

We can assume that each maximal contracted subscheme hast at most one marked point, since otherwise the intersection (1) will certainly be empty. This means that each maximal contracted subscheme must have at least two points of intersection with the other components of the curve, since otherwise the prestable map $(C, x_1, \dots, x_n, f)$ would not be stable. We conclude that each marked point that lies in a contracted component (there are $(n - \sum_i n_i)$ of them) must be counted in at least two of the ρ_i:

$$\sum_i \rho_i \geq 2(n - \sum_i n_i). \tag{2}$$

Now there is a morphism

$$\Phi : M_\tau \to M_{0,n_1+\rho_1}(\tilde{X}, \beta_1) \times \cdots \times M_{0,n_a+\rho_a}(\tilde{X}, \beta_a) \tag{3}$$

mapping a stable map $\mathcal{C}$ to its non-contracted components, where on each such component we take as marked points the n_i marked points of $\mathcal{C}$ lying on this component together with the intersection points of the component with the maximal contracted subschemes. We denote by $\Phi_i : M_\tau \to M_{0,n_i+\rho_i}(\tilde{X}, \beta_i)$ the composition of Φ with the projections onto the factors of the right hand side of (3).

We now consider again the intersection I in (1) and show that $\Phi(I \cap M_\tau)$ is empty for all topologies τ but the trivial one, hence showing that $I \cap M_\tau$ is empty. Note that in $\Phi_i(I \cap M_\tau)$ the image point of each of the $n_i + \rho_i$ marked points is fixed to be a certain

Q_j. But we have seen in lemma 2.6.3 that, if $\Phi_i(I\cap M_\tau)\subset M_{0,n_i+\rho_i}(\tilde X,\beta_i)$ is non-empty, this requires $n_i+\rho_i$ to be at most $2d(\beta_i)+e(\beta_i)$. Therefore we get

$$n \le 2n-\sum_i n_i \overset{(2)}{\le} \sum(n_i+\rho_i) \le \sum_i (2d(\beta_i)+e(\beta_i))$$
$$= 2d(\beta)+e(\beta) = \frac{1}{2}\,\mathrm{vdim}\,\bar M_{0,0}(\tilde X,\beta) = n.$$

Hence we must have equality everywhere, which means first of all that $\sum_i n_i = n$ and therefore $\rho_i = 0$ for all i. Moreover, it follows that the number n_i of marked points with prescribed image in $\Phi_i(I\cap M_\tau)$ is equal to $2d(\beta_i)+e(\beta_i)$ for all i, showing that there can be no component of C of type (ii) according to the classification of lemma 2.6.3 (to be precise, that for all i, C is mapped under Φ_i to a moduli space which is not of type (ii)). If there are only components of type (i), then we have the case that $\beta = dH-dE'_i-dE'_j$ for some $d>2$ and $i\ne j$ (note that there cannot be two components of type (i) with different (i,j) since the L_{ij} do not intersect). As we excluded this case in the theorem, we conclude that there must be at least one component of C of type (iii). We are going to show that there is in fact only one component which must then necessarily be of type (iii).

We first exclude components of type (i). Note that on each component C_i of type (iii) we impose n_i generic point conditions. Since $\dim M_{0,n_i}(\tilde X,\beta_i) = 3n_i$, this means by the Bertini lemma 2.4.7 (ii) that $\Phi_i(I\cap M_\tau)\subset M_{0,n_i}(\tilde X,\beta_i)$ is zero-dimensional (if not empty). Moreover, if we let $Z_i\subset M_{0,n_i}(\tilde X,\beta_i)$ be the substack of curves intersecting $\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E})$, then $\dim Z_i < 3n_i$ by lemma 2.6.3, and hence again by Bertini, $\Phi_i(I\cap M_\tau)$ will not intersect Z_i, i.e. the curves in $\Phi_i(I\cap M_\tau)$ do not intersect $\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E})$. This is true for any component of type (iii). Hence, if there were also a component of type (i) which is contained in an L_{ij}, the curve would not be connected, which is impossible. Therefore we can only have components of type (iii).

Assume now that we have at least two components of type (iii). We will again show that these components do not intersect, leading to a contradiction. We define

$$V_1 := \bigcup_{(C,x_1,\dots,x_{n_1},f)\in\Phi_1(I\cap M_\tau)} f(C)\subset \tilde X,$$

$$V_2 := \bigcup_{i=2}^{a}\ \bigcup_{(C,x_1,\dots,x_{n_i},f)\in\Phi_i(I\cap M_\tau)} f(C)\subset \tilde X.$$

We already remarked that $\Phi_i(I\cap M_\tau)$ is zero-dimensional for all i and corresponds to curves none of which intersects $\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E})$, hence V_1 and V_2 are one-dimensional subvarieties of $\tilde X\backslash(\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E}))$. We now define

$$\mathcal{M} := \{\mathrm{diag}\,(v_0,v_1,v_2,v_3)\mid v_i\in\mathbb{C}^*\}/\mathbb{C}^*\subset \mathrm{PGL}\,(3)$$

to be the space of all invertible projective diagonal matrices. Obviously the elements of $\mathcal{M}$ can be considered as automorphisms of $\tilde{\mathbb{P}}^3(4)$ with our choice of the blown-up points. We now consider the map

$$\begin{aligned} \Psi : V_1 \times \mathcal{M} &\to \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E})) \\ (Q,\mu) &\mapsto \mu(Q) \end{aligned}$$

and determine the dimension of its fibres. Fix a point $Q' \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$.

- If $Q' \notin \mathcal{H} \cup \mathcal{E}$, then for any $Q \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$ there is at most one $\mu \in \mathcal{M}$ such that $\mu(Q) = Q'$ (in fact, there is exactly one such μ if $Q \notin \mathcal{H} \cup \mathcal{E}$ and no such μ otherwise). Therefore the fibre $\Psi^{-1}(Q')$ is one-dimensional (in fact, isomorphic to $V_1 \backslash (\mathcal{H} \cup \mathcal{E})$).

- If $Q' \in H_i$ for some i, then any $Q \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$ that can be transformed into Q' by an element of $\mathcal{M}$ must also lie in H_i. In this case, we then have a $\mathbb{C}^*$-family of elements of $\mathcal{M}$ mapping Q to Q'. Since V_1 meets H_i only in finitely many points (otherwise we would be in case (ii) of lemma 2.6.3), the fibre $\Psi^{-1}(Q')$ is again (at most) one-dimensional.

- If $Q' \in E_i$ for some i, we again get at most one-dimensional fibres by exactly the same reasoning as for the H_i.

We have thus shown that all fibres of Ψ are at most one-dimensional. Hence $\Psi^{-1}(V_2)$ can be at most two-dimensional. But this means that there must be a $\mu \in \mathcal{M}$ such that $V_1 \times \{\mu\} \cap \Psi^{-1}(V_2) = \emptyset$, or in other words such that $\mu(V_1) \cap V_2 = \emptyset$. So if we now transform the prescribed images $Q_i \in \tilde{X}$ of those marked points lying on the component C_1 by μ, this will transform V_1 to $\mu(V_1)$, with the result that the component C_1 does not intersect the others. This would lead to curves that are not connected, which is a contradiction.

So we finally see that only the trivial topology τ corresponding to irreducible curves can contribute to I, and moreover that these irreducible curves are of type (iii) according to lemma 2.6.3. Hence if we let $Z \subset \bar{M}_{0,n}(\beta)$ be the closure of the substack corresponding to irreducible curves and R be the union of the other irreducible components, then by lemma 1.3.3 we can write

$$[\bar{M}_{0,n}(\beta)]^{virt} = [Z] + \text{some cycle supported on } R.$$

But as we have just shown, the intersection I to be considered is disjoint from R, so we can drop this additional cycle and evaluate the intersection on Z. Then it follows from the Bertini lemma 2.4.7 (iii) that the invariant $I_\beta(\mathcal{T})$ is enumerative, since the generic element of Z has no automorphisms, as shown in lemma 2.6.3. □

2.7 Tangency conditions via blow-ups

In this section we will show how to count curves in $X = \mathbb{P}^r$ of given homology class β that intersect a fixed point $P \in X$ with tangent direction in a specified linear subspace of $T_{X,P}$. One would expect that this can be done on the blow-up $\tilde{X}$ of X at P, since the condition that a curve in X has tangent direction in a specified linear subspace of $T_{X,P}$ of codimension k (where $1 \leq k \leq r-1$) translates into the statement that the strict transform of the curve intersects the exceptional divisor E in a specified k-codimensional projective subspace of $E \cong \mathbb{P}^{r-1}$. As such a k-codimensional projective subspace of E has class $-(-E)^{k+1}$, we would expect that the answer to our problem is

$$I^{\tilde{X}}_{\beta-E'}(\mathcal{T} \otimes -(-E)^{k+1})$$

where $\mathcal{T}$ denotes as usual the other incidence conditions that the curves should satisfy.

We will show in theorem 2.7.1 that this is in fact the case as long as $k \neq r-1$. However, if $k = r-1$, so that we want to have a fixed tangent direction at P, things get more complicated. This can be seen as follows: consider the invariant $I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2})$ on X, about which we know that it counts the number of curves on X through the classes in $\mathcal{T}$ and through two generic points P and P' in X. We now want to see what happens if P' and P approach each other and finally coincide. Basically, if P' approaches P, there are two possibilities: either the two points x and x' on the curve that are mapped to P and P' also approach each other (left picture), or they do not (right picture):

In the limit $P' \to P$, the curves on the left become curves through P tangent to the limit of the lines $\overline{PP'}$, and those on the right simply become curves intersecting P with global multiplicity two. But the latter we have already counted in theorem 2.5.3. So we expect in this case

$$\begin{aligned} I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2}) &= \text{(curves through } \mathcal{T} \text{ and through } P \text{ with specified tangent)} \\ &\quad + 2 I^{\tilde{X}}_{\beta-2E'}(\mathcal{T}) \end{aligned}$$

where the factor two arises because in the right picture, the points x and x' on the curve can be interchanged in the limit where $P = P'$ and $x \neq x'$. This should motivate the results of the following theorem. Some numerical examples can be found in 2.8.6.

Theorem 2.7.1 *Let $X = \mathbb{P}^r$ and let $0 \neq \beta \in A_1(X)$ be an effective homology class. Let $P \in X$ be a point, $k \in \{1,\dots,r-1\}$ and W a generic projective subspace of $\mathbb{P}(T_{X,P})$ of codimension k. Let $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ be a collection of effective classes in X such that $\sum_i$ codim γ_i = vdim $\bar{M}_{0,n}(X,\beta) - r + 1 - k$.*

Then, for generic subschemes $V_i \subset X$ with $[V_i] = \gamma_i$, the number of irreducible stable maps $(C, x_1, \dots, x_{n+1}, f)$ satisfying the conditions

- *f generically injective,*
- *$f_*[C] = \beta$,*
- *$f(x_i) \in V_i$ for all i,*
- *$f(x_{n+1}) = P$,*
- *the tangent direction of f at x_{n+1} lies in W (i.e. if $\tilde{f} : C \to \tilde{X}$ is the strict transform, then $\tilde{f}(x_{n+1}) \in W \subset \mathbb{P}(T_{X,P}) \cong E$),*

is equal to

$$\begin{aligned} I^{\tilde{X}}_{\beta - E'}(\mathcal{T} \otimes -(-E)^{k+1}) \qquad & \text{if } k < r-1, \\ I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2}) - 2 I^{\tilde{X}}_{\beta - 2E'}(\mathcal{T}) \qquad & \text{if } k = r-1, \end{aligned}$$

where each such curve is counted with multiplicity one.

Proof Consider the Gromov-Witten invariant $I^{\tilde{X}}_{\beta - E'}(\mathcal{T} \otimes -(-E)^{k+1})$. We will show that this invariant counts what we want, apart from a correction term in the case $k = r-1$.

As usual, we decompose the moduli space $\bar{M}_{0,n+1}(\tilde{X}, \beta - E')$ according to the topology of the curves

$$\bar{M}_{0,n+1}(\tilde{X}, \beta - E') = \bigcup_\tau M(\tilde{X}, \tau)$$

and determine which parts $M(\tilde{X}, \tau)$ give rise to contributions to the intersection

$$ev_1^{-1}(V_1) \cap \dots \cap ev_n^{-1}(V_n) \cap ev_{n+1}^{-1}(W) \tag{1}$$

on $\bar{M}_{0,n+1}(\tilde{X}, \beta - E')$ (note that $[W] = -(-E)^{k+1}$ on $\tilde{X}$).

We use proposition 2.5.2 (ii) and distinguish the five cases of this proposition. Assume that $M(\tilde{X}, \tau)$ satisfies (a). Set $I := ev_1^{-1}(V_1) \cap \dots \cap ev_n^{-1}(V_n)$ on $\bar{M}_{0,n+1}(X, \beta)$. By the Bertini lemma 2.4.7 (ii), this intersection is of codimension

$$\begin{aligned} \sum_i \text{codim } V_i &= \text{vdim } \bar{M}_{0,n}(\tilde{X}, \beta) - r + 1 - k \\ &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X}, \beta - E') - k - 1 \\ &\geq \dim \phi(M(\tilde{X}, \tau)) + r - k - 1 \qquad \text{(by (a))} \\ &\geq \dim \phi(M(\tilde{X}, \tau)), \qquad \text{(since } k \leq r-1) \end{aligned}$$

where $\phi : M(\tilde{X},\tau) \hookrightarrow \bar{M}_{0,n+1}(\tilde{X},\beta - E') \to \bar{M}_{0,n+1}(X,\beta)$ is the morphism given by proposition 1.2.6 (ii). Hence, by Bertini again, $\phi^{-1}(I)$ will be a finite set of points. But since the point x_{n+1} of the curves in $\phi^{-1}(I)$ is not restricted at all, it is actually impossible that $\phi^{-1}(I)$ is finite unless it is empty. So we see that we get no contribution to the intersection (1) from $M(\tilde{X},\tau)$.

Before we look at the cases (b) to (e) of proposition 2.5.2 (ii), we set $Z = ev_{n+1}^{-1}(E) \subset \bar{M}_{0,n+1}(\tilde{X},\beta - E')$ and decompose Z analogously to $\bar{M}_{0,n+1}(\tilde{X},\beta - E')$ as $Z = \bigcup_\tau Z(\tau)$. Then we obviously have

$$\dim Z(\tau) = \begin{cases} \dim M(\tilde{X},\tau) - 1 & \text{if } x_{n+1} \text{ is on a non-exceptional component of the curve,} \\ \dim M(\tilde{X},\tau) & \text{if } x_{n+1} \text{ is on an exceptional component of the curve.} \end{cases} \tag{2}$$

There are evaluation maps $ev_i : Z(\tau) \to \tilde{X}$ for $1 \le i \le n$ and $\widetilde{ev}_{n+1} : Z(\tau) \to E \cong \mathbb{P}^{r-1}$, and the intersection (1) now becomes the intersection

$$ev_1^{-1}(V_1) \cap \cdots \cap ev_n^{-1}(V_n) \cap \widetilde{ev}_{n+1}^{-1}(W), \tag{3}$$

on $Z(\tau)$, where $V_i \subset \tilde{X}$ and $W \subset \mathbb{P}^{r-1}$ are chosen generically.

We now continue to look at the cases (b) to (e) of proposition 2.5.2 (ii). If $M(\tilde{X},\tau)$ satisfies (b), then the intersection (3) will be empty by Bertini, since

$$\begin{aligned} \sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n}(X,\beta) - r + 1 \\ &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &\ge \dim M(\tilde{X},\tau) + 1 \qquad \text{(by (b))} \\ &\ge \dim Z(\tau) + 1. \qquad \text{(by (2))} \end{aligned}$$

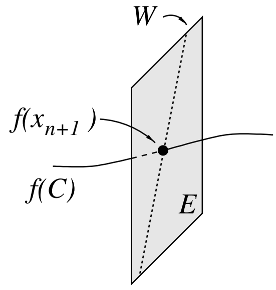
Similarly, this follows for (c): because of $\eta(\tau) = 0$ we have no exceptional component, hence we must have the first possibility in (2), i.e.

$$\begin{aligned} \sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &\ge \dim M(\tilde{X},\tau) \qquad \text{(by (c))} \\ &\ge \dim Z(\tau) + 1. \qquad \text{(by (2))} \end{aligned}$$

Hence we are only left with the cases (d) and (e). In case (d) we must have the first possibility in (2) since the curve is irreducible, hence

$$\begin{aligned} \sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &= \dim M(\tilde{X},\tau) - 1 \qquad \text{(by (d))} \\ &= \dim Z(\tau). \qquad \text{(by (2))} \end{aligned}$$

The intersection (3) is transverse and finite by Bertini. Moreover, the dimension of $M(\tilde{X},\tau)$ coincides with vdim $\bar{M}_{0,n+1}(\tilde{X},\beta-E')$, and there are no obstructions on $\bar{M}(\tilde{X},\tau)$ by lemma 2.4.4 (i). Hence, using lemma 1.3.3 in the same way as we did in the proof of theorem 2.5.3, we see that we get a contribution to the Gromov-Witten invariant $I^{\tilde{X}}_{\beta-E'}(\mathcal{T}\otimes -(-E)^{k+1})$ from exactly the curves we wanted. One can depict these curves as follows:



Note that, by corollary 2.3.2, in the case $k=r-1$ we have

$$I^{\tilde{X}}_{\beta-E'}(\mathcal{T}\otimes -(-E)^{r}) = I^{\tilde{X}}_{\beta-E'}(\mathcal{T}\otimes pt) = I^{X}_{\beta}(\mathcal{T}\otimes pt^{\otimes 2}).$$

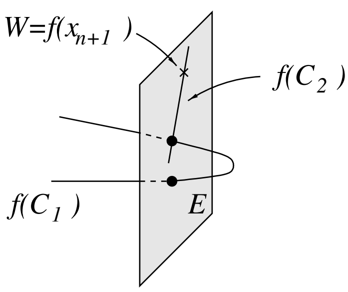
It remains to look at case (e). There we have

$$\begin{aligned}\sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta-E') - 1 \\ &= \dim M(\tilde{X},\tau) \qquad \text{(by (e))} \\ &\geq \dim Z(\tau). \qquad \text{(by (2))}\end{aligned}$$

Note that again there are no obstructions on $\bar{M}(\tilde{X},\tau)$ by lemma 2.5.1.

Hence, to get a non-zero contribution from (e) to the intersection (3), we must have equality in the last line, which fixes the component where x_{n+1} lies. We thus have reducible curves with exactly two components, one component C_1 with marked points $x_1,\dots,x_n$ and homology class $\beta-2E'$, and the other component C_2 with marked point x_{n+1} and homology class E'. Moreover, the intersection (3) must be transverse and finite by Bertini. But this is only possible if $k=r-1$, since the only conditions on the exceptional line C_2 are that it has to intersect C_1 and that x_{n+1} maps to W, and this cannot fix C_2 uniquely unless W is a point, i.e. $k=r-1$. This finishes the proof of the theorem in the case $k<r-1$.

In the case $k=r-1$, we have just shown that the curves in the intersection (3) look as follows:



Here, one has to show that the generic curve of homology class $\beta - 2E'$ intersects the exceptional divisor twice, and not only once with multiplicity two. But this is easy to see: irreducible curves of homology class $\beta - 2E'$ intersecting the exceptional divisor once with multiplicity two correspond via strict transform to curves of homology class β in $\mathbb{P}^r$ having a cusp at P. For maps $f : \mathbb{P}^1 \to X = \mathbb{P}^r$ it is however easy to see that the requirement that a specified point $x \in \mathbb{P}^1$ is mapped to P and that $df(x) = 0$ imposes $2r$ independent conditions, so the space of irreducible stable maps of homology class β with a cusp at P has dimension

$$\dim M_{0,1}(X, \beta) - 2r = \dim M_{0,0}(\tilde{X}, \beta - 2E') - 1,$$

so the generic curve in $\tilde{X}$ of homology class $\beta - 2E'$ does indeed intersect the exceptional divisor twice and looks as in the picture above.

Therefore, to get the correct enumerative answer, we have to subtract the contribution from this case (e). But this is easily done, since we now know that this contribution is twice the number of curves of homology class $\beta - 2E'$ satisfying the conditions $\mathcal{T}$ (the factor two arises since the component C_2 can be attached to both points of intersection of the component $f(C_1)$ with E). By theorem 2.5.3, we know that this number is $I^{\tilde{X}}_{\beta-2E'}(\mathcal{T})$. This finishes the proof also in the case $k = r - 1$. □

These results should be compared with corollary 1.6.7 where we already computed other numbers of curves with tangency conditions.

One can of course ask whether the analogue of theorem 2.7.1 is true also for several tangency conditions at different points. As imaginable from our work in this chapter, the answer in general is no, and the problems arising here are essentially the same as those discussed in the previous sections when considering multiple blow-ups.

However, as (most) invariants on $\tilde{\mathbb{P}}^2(s)$ are enumerative by [GP], one can expect an analogue of theorem 2.7.1 in this case. Indeed, numerical calculations show that this seems to be true: if one calculates with these methods what should be the number of rational curves in $\mathbb{P}^2$ tangent to c general lines at c fixed points, and intersecting additional a general points, one obtains exactly the numbers $N(a, 0, c)$ of Ernström and Kennedy [EK2] that have been computed by completely different methods and shown to be enumeratively correct.

2.8 Numerical examples

Example 2.8.1 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^2(1)$

According to theorem 2.5.3, the Gromov-Witten invariants $I^{\tilde{\mathbb{P}}^2(1)}_{dH'+eE'}(pt^{\otimes(3d+e-1)})$ for $d > 0$ are equal to the numbers of degree d plane rational curves meeting $3d + e - 1$ generic points in the plane, and in addition passing through a fixed point in $\mathbb{P}^2$ with global multiplicity $-e$. All these curves are counted with multiplicity one. Some of the invariants are listed in the following table.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$e=0$	1	1	12	620	87304	26312976	14616808192
$e=-1$	1	1	12	620	87304	26312976	14616808192
$e=-2$	0	0	1	96	18132	6506400	4059366000
$e=-3$	–	0	0	1	640	401172	347987200
$e=-4$	–	0	0	0	1	3840	7492040
$e=-5$	–	0	0	0	0	1	21504
$e=-6$	–	–	0	0	0	0	1

The equality of the first two lines follows from the geometric meaning of the invariants (see theorem 2.5.3) as well as from corollary 2.3.2. In [GP], L. Göttsche and R. Pandharipande also compute the numbers given here, together with those for blow-ups of $\mathbb{P}^2$ in any number of points, and they prove the enumerative significance of all these numbers if the prescribed multiplicity in at least one of the blown-up points is one or two. The numbers for $e=-2$ have been computed earlier by different methods in [P2].

The fact that $I^{\tilde{\mathbb{P}}^2(1)}_{dH'-(d-1)E'}(pt^{\otimes 2d}) = 1$ can also be understood geometrically: a curve C of degree d in $\mathbb{P}^2$ passing with multiplicity $d-1$ through a point P has genus

$$\frac{1}{2}(d-1)(d-2) - \frac{1}{2}(d-1)(d-2) = 0,$$

i.e. it is always a rational curve. Hence the space of degree d rational curves with a $(d-1)$-fold point in P is simply a linear system of the expected dimension, showing that the corresponding Gromov-Witten invariant must be 1.

Example 2.8.2 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^3(1)$

As in the previous example, the Gromov-Witten invariants $I^{\tilde{\mathbb{P}}^3(1)}_{dH'+eE'}(pt^{\otimes(2d+e)})$ for $d > 0$ are equal to the numbers of degree d rational curves in $\mathbb{P}^3$ meeting $2d+e$ generic points, and in addition passing through a fixed point in $\mathbb{P}^3$ with global multiplicity $-e$.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$e=0$	1	0	1	4	105	2576	122129	7397760
$e=-1$	1	0	1	4	105	2576	122129	7397760
$e=-2$	0	0	0	0	12	384	23892	1666128
$e=-3$	–	0	0	0	0	0	620	72528
$e=-4$	–	0	0	0	0	0	0	0

Example 2.8.3 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^3(2)$

By theorem 2.6.4, the numbers $I^{\tilde{\mathbb{P}}^3(2)}_{dH'+e_1E'_1+e_2E'_2}(pt^{\otimes(2d+e_1+e_2)})$ for $d > 0$ are enumerative unless $d > 2$, $e_1 = -d$, $e_2 = -d$ (for those cases, see proposition 2.8.5). This means that they are equal to the numbers of degree d rational curves in $\mathbb{P}^3$ meeting $2d+e_1+e_2$ generic points in $\mathbb{P}^3$, and in addition passing through two fixed points with global multiplicities $-e_1$ and $-e_2$, respectively.

(e_1,e_2)	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$(-2,-2)$	$1/8$	0	0	1	48	4374	360416	39100431
$(-3,-2)$	–	0	0	0	0	96	14040	2346168
$(-3,-3)$	–	$1/27$	0	0	0	1	384	119134
$(-4,-2)$	–	0	0	0	0	0	0	18132
$(-4,-3)$	–	–	0	0	0	0	0	640
$(-4,-4)$	–	–	$1/64$	0	0	0	0	1

The numbers with one of the $e_i=-1$ can be obtained from corollary 2.3.2 and example 2.8.2.

Example 2.8.4 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^4(2)$

The invariants $I^{\tilde{\mathbb{P}}^4(2)}_{dH'+e_1E_1'+e_2E_2'}(\,\cdot\,)$ for $d>0$ are enumerative if only one of the blown-up points is involved (i.e. if one of the e_i is zero) or if one of the e_i is equal to -1 (by corollary 2.3.2). It has already been mentioned that in almost all other cases, the invariants are not enumerative. As examples, we list in the following table some invariants $I^{\tilde{\mathbb{P}}^4(2)}_{dH'+e_1E_1'+e_2E_2'}(\mathcal{T})$ where $\mathcal{T}=pt^{\otimes a}\otimes(H^2)^{\otimes b}$ with $a\geq 0$, $0\leq b\leq 2$ being the unique numbers such that $5d+3e_1+3e_2+1=3a+b$.

(e_1,e_2)	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$(-1,-1)$	1	0	1	161	270	831	1351863
$(-2,-1)$	0	0	0	9	16	105	233040
$(-2,-2)$	–	$1/4$	0	$5/4$	$9/4$	$29/2$	$154683/4$
$(-3,-1)$	–	0	0	0	0	0	2625
$(-3,-2)$	–	0	0	0	$3/4$	1	$2533/2$
$(-3,-3)$	–	–	$1/27$	$13/108$	$-1/12$	$-1/54$	$32471/108$
$(-4,-1)$	–	0	0	0	0	0	0
$(-4,-2)$	–	–	0	0	0	0	16

Example 2.8.5 *Non-enumerative invariants on* $\tilde{\mathbb{P}}^3(4)$

We have seen in theorem 2.6.4 that the only non-enumerative invariants on $\tilde{\mathbb{P}}^3(4)$ are those of the form $I_{dH'-dE_1'-dE_2'}(1)$ for $d\geq 2$. We will now compute these invariants.

Let $\tilde{X}=\tilde{\mathbb{P}}^3(2)$. Let L be the strict transform of the line joining the two blown-up points, its normal bundle in $\tilde{X}$ is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. If we let $\beta=dH'-dE_1'-dE_2'$ for some $d\geq 2$, then we also have $K_{\tilde{X}}\cdot\beta=0$, so that we can apply lemma 1.3.8 to see that the Gromov-Witten invariant $I^{\tilde{\mathbb{P}}^3(2)}_{dH'-dE_1'-dE_2'}(1)$ is equal to the integral

$$\int_{\bar{M}_{0,0}(\mathbb{P}^1,d)} c_{2d-2}\left(R^1\pi_*f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))\right)$$

where $\pi : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \bar{M}_{0,0}(\mathbb{P}^1,d)$ is the universal curve and $f : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \mathbb{P}^1$ the evaluation map. The importance of this invariant has already been discussed in detail in the end of section 1.3.

To actually compute the invariant, we use the equation $\mathcal{E}_{\beta+E_1'}(1\,;\,H,H \mid E_1,E_1^2)$. The only possibilities how the homology class $\beta+E_1' = d\,H' - (d-1)\,E_1' - d\,E_2'$ can split up into two effective classes are

$$\beta_1 = d_1\,H' - d_1\,E_1' - d_1\,E_2', \qquad \beta_2 = d_2\,H' - (d_2-1)\,E_1' - d_2\,E_2'$$

for $d_1+d_2 = d$ and $d_1,d_2 \geq 0$. First we look at the invariants with homology class β_2 and claim that they all vanish for $d_2 \geq 2$. The virtual dimension of $\bar{M}_{0,0}(\tilde{X},\beta_2)$ is 2, so we have to impose two conditions on the curves we are counting. It is easy to see that all stable maps with homology class β_2 are reducible, such that one component maps to a line in the exceptional divisor $E_1 \cong \mathbb{P}^2$, and all the others into L. This means that no such curve can intersect the strict transform of a general line in $\tilde{\mathbb{P}}^3(2)$ or of a general line through P_2, and hence $I_{\beta_2}(\mathcal{T})$ vanishes whenever $\mathcal{T}$ contains one of the classes H^2, E_2^2, and pt. But also no such curve can intersect *two* strict transforms of general lines in $\tilde{\mathbb{P}}^3(2)$ through P_1, so we also have $I_{\beta_2}((H^2-E_1^2)^{\otimes 2}) = 0$. Hence, by the multilinearity of the Gromov-Witten invariants it follows that all invariants with homology class β_2 vanish for $d_2 \geq 2$.

The equation $\mathcal{E}_{\beta+E_1'}(1\,;\,H,H \mid E_1,E_1^2)$ reduces therefore to the simple statement

$$\begin{aligned} 0 = {} & I_{dH'-dE_1'-dE_2'}(H\otimes H\otimes E_1)\,\underbrace{I_{E_1'}(E_1\otimes E_1^2\otimes E_1^2)}_{=-1} \\ & - I_{(d-1)H'-(d-1)E_1'-(d-1)E_2'}(H\otimes E_1\otimes E_1)\,I_{H'-E_2'}(H\otimes E_1^2\otimes E_1^2). \end{aligned}$$

The invariant $I_{H'-E_2'}(H\otimes E_1^2\otimes E_1^2)$ is easily computed to be -1, e.g. using the algorithm 2.2.5. Hence, by the divisor axiom we get

$$d^3\,I_{dH'-dE_1'-dE_2'}(1) = (d-1)^3\,I_{(d-1)H'-(d-1)E_1'-(d-1)E_2'}(1).$$

Together with $I_{H'-E_1'-E_2'}(1) = 1$ (which follows for example from corollary 2.3.2), we see that

$$I_{dH'-dE_1'-dE_2'}(1) = d^{-3}.$$

It should be noted that our additional considerations above to prove the vanishing of Gromov-Witten invariants of homology class $d_2\,H' - (d_2-1)\,E_1' - d_2\,E_2'$ for $d_2 > 0$ would not have been necessary to compute the desired invariants, they just made the calculation easier. According to theorem 2.2.1, we could of course also use the algorithm 2.2.5 without further thinking, and everything would take care of itself.

Example 2.8.6 *Curves with tangency conditions*

The following table shows some of the numbers

$$N_{r,k,d,\mathcal{T}} = \begin{cases} I^{\tilde{\mathbb{P}}^r(1)}_{dH'-E'}(\mathcal{T} \otimes -(-E)^{k+1}) & \text{if } k < r-1 \\ I^{\mathbb{P}^r}_{dH'}(\mathcal{T} \otimes pt^{\otimes 2}) - 2 I^{\tilde{\mathbb{P}}^r(1)}_{dH'-2E'}(\mathcal{T}) & \text{if } k = r-1 \end{cases}$$

which are according to theorem 2.7.1 equal to the numbers of curves in $\mathbb{P}^r$ of degree d through generic subspaces of $\mathbb{P}^r$ according to $\mathcal{T}$, and intersecting a fixed point $P \in \mathbb{P}^r$ with tangent direction contained in a given linear subspace of $T_{\mathbb{P}^r,P}$ of codimension k.

(r,k)	$\mathcal{T}$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$(2,1)$	$pt^{\otimes(3d-3)}$	1	10	428	51040	13300176	6498076192
$(3,1)$	$pt^{\otimes(2d-2)} \otimes H^2$	1	3	28	485	14376	639695
$(3,2)$	$pt^{\otimes(2d-2)}$	0	1	4	81	1808	74345

The numbers in the first row have already been computed by L. Ernström and G. Kennedy [EK2] by different methods.

2.9 Blow-ups of subvarieties

In the last section of this chapter we will discuss two examples of blow-ups of $\mathbb{P}^r$ along higher-dimensional subvarieties, leading to well-known classical results about multisecants of space curves and abelian surfaces in $\mathbb{P}^4$, respectively.

Example 2.9.1 *Blow-ups of curves in* $\mathbb{P}^3$

Let $X = \mathbb{P}^3$ and $Y \subset X$ be a smooth curve of degree d and genus g. Let $\tilde{X}$ be the blow-up of X along Y. We are going to compute the Gromov-Witten invariants

$$q := I^{\tilde{X}}_{H'-4E'}(1) \quad \text{and} \quad t := I^{\tilde{X}}_{H'-3E'}(H^2)$$

where E' is the class of a fibre over a point in Y. Irreducible curves of homology class $H' + eE'$ for $e < 0$ obviously correspond to lines in Y intersecting the curve Y with multiplicity $-e$, i.e. to $(-e)$-secants of Y. Hence, we expect t to be the number of 3-secants of Y intersecting a fixed line and q to be the number of 4-secants of Y. It is however not at all clear that this interpretation is valid, and indeed in some cases it is not, since there are e.g. space curves with infinitely many 4-secants. We will be able to see this already from the result since the numbers t and q can well be negative.

Nevertheless, t and q can be regarded to be the “virtual” number of 3-secants through a line and 4-secants, respectively. These (virtual) numbers have already been computed classically — the computation goes back to Cayley (1863). Some more recent work on this topic has been done by Le Barz [L]. We will see that the numbers we obtain by Gromov-Witten theory are the same, although it is not clear that, in the case where there are infinitely many such multisecants, the classical and the Gromov-Witten definition of the “virtual number” agree.

Of course, the algorithms we developed so far do not tell us how to compute the numbers, so we will sketch here a possible way to calculate them.

Step 1: Intersection ring. (This can be computed easily using the methods of [F1].) The ring structure of $A^*(\tilde{X})$ is determined by $A^1(\tilde{X}) = \langle H, E\rangle$ and $A^2(\tilde{X}) = \langle H^2, F\rangle$ (where E is the exceptional divisor and F is the Poincaré dual of the homology class E' introduced above) and the following non-zero intersection products involving at least one exceptional class:

$$\begin{aligned} E \cdot E &= (4d+2g-2)F - dH^2, \\ E \cdot H &= dF, \\ E \cdot F &= -pt. \end{aligned}$$

Step 2: Invariants with homology class $\beta = eE'$, $e > 0$. Since these curves have to be contained in the exceptional divisor, the invariants $I_{eE'}(\mathcal{T})$ are certainly zero if $\mathcal{T}$ contains a non-exceptional class. By the divisor axiom, the only independent classes to compute are therefore $I_{eE'}(F^{\otimes e})$. The curves that are counted there must be e-fold coverings of a fibre over a point in Y, so this invariant is zero for $e \geq 2$ since we then require the curve to lie in two different fibres. Finally, the geometric statement that $I_{E'}(H^2 - F) = 1$ (we count curves that are a fibre over a point in Y, and the condition $H^2 - F$ fixes the point) means that $I_{E'}(F) = -1$.

Step 3: Invariants with homology class $\beta = H'$. For geometric reasons, the invariant $I_{H'}(\mathcal{T})$ is zero if $\mathcal{T}$ contains an exceptional class and coincides with the corresponding one on $\mathbb{P}^3$ otherwise, i.e.

$$I_{H'}((H^2)^{\otimes 4}) = 2, \quad I_{H'}((H^2)^{\otimes 2} \otimes pt) = 1, \quad I_{H'}(pt^{\otimes 2}) = 1.$$

Step 4: Invariants with homology class $\beta = H' + eE'$, $e < 0$. The main equation that we use is $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}\,;\, H, H \mid E, E)$ for $e < 0$. Assume that $\mathcal{T}$ contains no divisor classes. Let α be the number of classes F in $\mathcal{T}$ and assume further that $\alpha + e \neq 0$. Then the equation reads after some ordering of the terms

$$\begin{aligned} I_{H'+eE'}(\mathcal{T}) = \frac{1}{\alpha+e} \Big(&(2g-2+(6+2e)d) I_{H'+(e+1)E'}(\mathcal{T} \otimes F) \\ &+ ((e+1)^2 - d) I_{H'+(e+1)E'}(\mathcal{T} \otimes H^2) \Big). \end{aligned}$$

We now list the results in the order they can be computed recursively (and state the equations used to compute the invariant in the cases where $\alpha + e = 0$ such that the above equation is not applicable).

$$
\begin{aligned}
I_{H'-E'}((H^2)^{\otimes 3}) &= 2d, \\
I_{H'-E'}(H^2 \otimes pt) &= d, \\
I_{H'-E'}(\mathcal{T} \otimes F^{\otimes 2}) &= 0 \qquad \text{for any } \mathcal{T}, \\
I_{H'-E'}(F \otimes H^2 \otimes H^2) &= 1 \qquad \text{using } \mathcal{E}_{H'}(H^2 \otimes H^2 \,;\, H,H \mid E,F), \\
I_{H'-E'}(F \otimes pt) &= 1 \qquad \text{using } \mathcal{E}_{H'}(pt \,;\, H,H \mid E,F), \\
I_{H'-2E'}(H^2 \otimes H^2) &= d(d-2)+1-g, \\
I_{H'-2E'}(pt) &= \frac{d(d-3)}{2}+1-g, \\
I_{H'-2E'}(F \otimes H^2) &= d-1, \\
I_{H'-2E'}(F \otimes F) &= 1 \qquad \text{using } \mathcal{E}_{H'-E'}(F \,;\, H,H \mid E,F), \\
I_{H'-3E'}(H^2) &= \boxed{t = \frac{(d-1)(d-2)(d-3)}{3} - g(d-2),} \\
I_{H'-3E'}(F) &= \frac{(d-1)(d-4)}{2}+1-g, \\
I_{H'-4E'}(1) &= \boxed{q = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{g}{2}(d^2-7d+13-g).}
\end{aligned}
$$

The numbers t and q coincide with the classical ones stated in [L].

Example 2.9.2 *Blow-up of an abelian surface in* $\mathbb{P}^4$

In analogy to example 2.9.1 we will now blow up an abelian surface Y of degree 10 in $X = \mathbb{P}^4$. The invariant $I_{H'-6E'}(1)$, where E' again denotes the fibre over a point in Y, is expected to be the number of 6-secants of the abelian variety, which is known to be 25. One can show that this is indeed the case. Since the calculation is very similar to the one in 2.9.1, we will sketch only very briefly the steps to obtain the result.

Step 1: Intersection ring. Assume that Y is generic such that $A^1(Y)$ is one-dimensional. Let $\alpha \in A^1(Y)$ be a hyperplane section of Y. Define $\gamma = j_* g^* \alpha$, where $j : E \to \tilde{X}$ is the inclusion and $g : E \to Y$ the projection. Let F be the Poincaré dual of E' introduced above. Then $A^*(\tilde{X})$ is determined by

$$A^1(\tilde{X}) = \langle H, E \rangle,\ A^2(\tilde{X}) = \langle H^2, \gamma \rangle,\ A^3(\tilde{X}) = \langle H^3, F \rangle$$

and the following non-zero intersection products involving at least one of the exceptional classes:

$$
\begin{aligned}
E \cdot E &= 5\gamma - 10H^2, \\
E \cdot H &= \gamma, \\
E \cdot \gamma &= 50F - 10H^3, \\
E \cdot H^2 &= 10F, \\
E \cdot F &= -pt, \\
\gamma \cdot \gamma &= -10pt, \\
\gamma \cdot H &= 10F.
\end{aligned}
$$

Step 2: Initial data for the recursion. The invariants with homology class H' again coincide with those on $\mathbb{P}^4$ or are zero if they contain an exceptional cohomology class. Invariants with homology class eE' are zero for $e \geq 2$, and the relevant invariants for $e = 1$ are $I_{E'}(F) = -1$ and $I_{E'}(\gamma \otimes \gamma) = 10$.

Step 3: Recursion relations. To determine an invariant $I_{H'+eE'}(\mathcal{T})$ for $e < 0$, use the following equations:

- If $\mathcal{T}$ contains a class F, use equation $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}' \,;\, H,H \mid E,F)$, where $\mathcal{T}'$ is defined by $\mathcal{T} = \mathcal{T}' \otimes F$.
- If $\mathcal{T}$ contains a class γ, use equation $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}' \,;\, H,H \mid \gamma,E)$, where $\mathcal{T}'$ is defined by $\mathcal{T} = \mathcal{T}' \otimes \gamma$.
- If $\mathcal{T}$ contains no exceptional class, use $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T} \,;\, H,H \mid E,E)$.

Using these equations, one can determine the invariants recursively for decreasing values of e and finally obtain $I_{H'-6E'}(1) = 25$.

It should be remarked that this calculation can be done for any surface in $\mathbb{P}^4$. The computations can then still be done in the same way, however they get of course much more complicated since they will involve the numerical invariants of the surface.

Chapter 3

Degeneration invariants

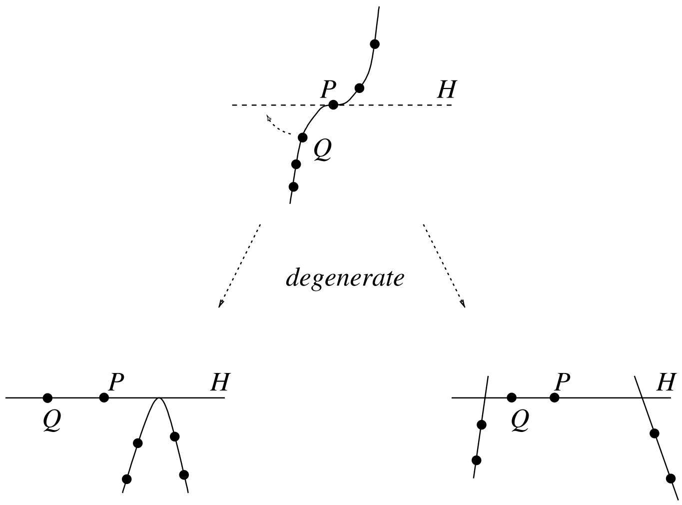
3.1 Introduction

In this chapter, we will study a different method to compute enumerative results on rational curves in $X = \mathbb{P}^r$, which goes back to L. Caporaso and J. Harris [CH3] and has been studied extensively in a recent paper by R. Vakil [V]. To state the idea, recall that for Gromov-Witten invariants we studied intersections of the form $ev_1^{-1}(V_1) \cap \cdots \cap ev_n^{-1}(V_n)$ on the moduli space $\bar{M}_{0,n}(X,\beta)$ of stable maps $(C, x_1, \ldots, x_n, f)$, where $ev_i : (C, x_1, \ldots, x_n, f) \mapsto f(x_i)$ are the evaluation maps, and $V_i \subset X$ are subvarieties of X. Usually, one can then prove or at least expect statements of the form that for generically chosen V_i, the above intersection is transverse and of the expected dimension, such that it can be interpreted in enumerative geometry and calculated as an intersection product on $\bar{M}_{0,n}(X,\beta)$ (see e.g. proposition 1.4.3).

The idea of degeneration methods is now to fix a hyperplane H in X and let one V_i after the other degenerate to lie in H, such that they are not generic any more. The result is that in each degeneration step, some of the curves in the intersection $ev_1^{-1}(V_1) \cap \cdots \cap ev_n^{-1}(V_n)$ will become reducible, with irreducible components in H. If one can count these (note that they are built up of components with smaller degree than the original curves, so that inductive computation methods are supposed to work), this gives a method to compute the original invariant, since the intersection product on the moduli space is of course not affected by the degeneration.

It turns out that, to describe the curves appearing in the degenerations, one also has to consider moduli spaces of curves having intersections with H with prescribed multiplicities, e.g. curves tangent to H at a certain point. This is a big difference between the Gromov-Witten program and the degeneration techniques: the recursive equations we obtain in this chapter involve a by far bigger set of invariants.

To see in a simple example how a degeneration works, suppose we want to count rational plane cubics through 5 generic points in the plane and having contact of order 3 to a line H in a specific point $P \in H$ (see upper picture on the next page). One can show that there are finitely many such curves (see lemma 3.2.2). We now let one of the 5 generic marked points, which we called Q, degenerate to lie in H. The methods of this chapter will show that this makes the curves degenerate in two possible ways: some of the curves will become the union of H with a conic tangent to H somewhere, where the conic intersects the remaining four generic marked points (bottom left). One can show that there are 2 such conics (e.g. by applying degeneration again), and that each of them counts with multiplicity 2 (in the language of corollary 3.2.12, we have $m_1 = 2$), so the contribution from these curves is 4. The other possible degeneration is that the curves split up into three lines, one of which equal to H. The four remaining generic marked points are distributed on the two lines not in H (bottom right). As there are $\frac{1}{2}\binom{4}{2} = 3$ ways of doing this, these curves give a contribution of 3, so that we conclude that the answer to our original problem in the upper picture is $4+3=7$. (This number has already been mentioned in the end of section 1.6.)



From this example we can see another point in which the calculations will get more complicated than in the Gromov-Witten case: we have to consider reducible curves consisting of more than two components (in contrast to proposition 1.4.1 (iv)).

But degeneration techniques have also advantages over the computation of Gromov-Witten invariants by means of proposition 1.4.1, apart from the obvious one that one can count curves with multiplicity conditions to a hyperplane, if one is interested in them. One of the big advantages is that degeneration methods seem to be better suited to count curves of higher genus, see e.g. [V], where all numbers of elliptic curves in $\mathbb{P}^r$ satisfying generic incidence conditions are computed. This has not been achieved so far with Gromov-Witten methods. Another important advantage is that, since the reducible curves appearing in the degeneration have components contained in H, the equations arising from degeneration techniques will relate curves in the ambient space X to curves in the hyperplane H, whereas the equations 1.4.1 for Gromov-Witten invariants do not combine invariants on different varieties. In fact, we will see in section 3.3, when we generalize the degeneration methods to the case of an arbitrary hypersurface in $\mathbb{P}^r$, that also in this case the equations relate Gromov-Witten invariants on $\mathbb{P}^r$ to Gromov-Witten invariants on the hypersurface, even if the hypersurface is not convex.

This chapter is organized as follows. In section 3.2, we give a short introduction to the results of R. Vakil [V] on degenerations to hyperplanes in $\mathbb{P}^r$. We will then generalize the construction to the case of hypersurfaces in section 3.3, involving the definition of virtual fundamental classes on the moduli spaces (which was not necessary for hyperplanes). Certain degeneration invariants for lines, i.e. numbers of lines with contact of given order to the hypersurface, are calculated in section 3.4 using gravitational descendants. Finally, to give a non-trivial explicit application, we show in section 3.5 how to compute the numbers of lines and conics on the quintic threefold using degenerations of curves in $\mathbb{P}^4$ to the quintic — a calculation which is not possible using just the properties 1.4.1 of Gromov-Witten invariants on the quintic. We hope that this method will generalize to arbitrary degree. The calculation also indicates how the considerably complicated equations among the degeneration invariants may be organized in terms of generating functions and differential operators.

3.2 Degeneration to a hyperplane in $\mathbb{P}^r$

We will start our study of degeneration invariants by stating the important constructions and results of R. Vakil [V] on degenerations to hyperplanes in $\mathbb{P}^r$. A slightly different notation will be used to make the connection with Gromov-Witten invariants more obvious. As in the Gromov-Witten case, the first thing to do is to define the moduli spaces.

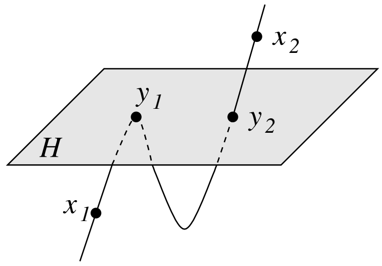
Definition 3.2.1 *Let $X = \mathbb{P}^r$ with $r \geq 2$, $d > 0$, and $n \geq 0$. Fix a hyperplane $H \subset X$. Let $s > 1$ and $\alpha = (\alpha_1, \dots, \alpha_s)$ be an s-tuple of positive integers such that $\sum_i \alpha_i = d$.*

Then we define the moduli space $\boldsymbol{\bar{M}_{0,n,s}(H/X,d\,|\,\alpha)}$ to be the closure in $\bar{M}_{0,n+s}(X,d)$ of the space of irreducible stable maps $(C, x_1, \dots, x_n, y_1, \dots, y_s, f)$ of degree d to X with $f(C) \not\subset H$ such that the divisor f^*H on $C \cong \mathbb{P}^1$ is equal *(not just linearly equivalent) to $\sum_i \alpha_i y_i$.*

To simplify notation, we will often drop the H in the notation of the moduli space and write $\boldsymbol{\bar{M}_{0,n,s}(X,d\,|\,\alpha)}$. If we want to indicate the names of the marked points, we will also write $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ as $\bar{M}_{0,I,J}(X,d\,|\,\alpha)$ if $I = \{x_1, \dots, x_n\}$ and $J = \{y_1, \dots, y_s\}$ with the marked points named as above. The same convention will be applied for the moduli spaces of stable maps $\bar{M}_{0,n+s}(X,d)$ which we then write as $\bar{M}_{0,I\cup J}(X,d)$.

As usual, the evaluation maps will be denoted $ev_{x_i} : \bar{M}_{0,n,s}(X,d\,|\,\alpha) \to X$ and $ev_{y_i} : \bar{M}_{0,n,s}(X,d\,|\,\alpha) \to H$ (note that the evaluation maps ev_{y_i} map to H and not to X, as we always have $f(y_i) \in H$ by definition).

Hence, the generic curve in $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ consists of irreducible curves, with exactly s points of intersection $y_1, \dots, y_s$ with H, and with prescribed local multiplicities $\alpha_1, \dots, \alpha_s$ to H at these points. As an example, the following picture shows (the image of) a generic stable map in $\bar{M}_{0,2,2}(X,3\,|\,(2,1))$:



It is, however, not at all obvious how the "boundary curves" in $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ look like, i.e. those that do not satisfy $f(C) \not\subset H$ and $f^*H = \sum_i \alpha_i y_i$. We will come to this question later.

Lemma 3.2.2 *The dimension of $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ is the expected one, namely*

$$\begin{aligned} \dim\, \bar{M}_{0,n,s}(X,d\,|\,\alpha) &= \dim\, \bar{M}_{0,n+s}(X,d) - \sum_i \alpha_i \\ &= (r+1)d + r + n + s - 3 - d. \end{aligned}$$

Proof See [V] proposition 2.11. □

We now come to the definition of the degeneration invariants, which is completely analogous to the definition of the Gromov-Witten invariants.

Definition 3.2.3 *With notations as above, let $\gamma_1,\dots,\gamma_n \in A^*(X)$ and $\mu_1,\dots,\mu_s \in A^*(H)$ be classes on X and H, respectively. Then we define the associated* ***degeneration invariant*** *to be the intersection product on $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$*

$$\begin{aligned} &\boldsymbol{I^{H/X}_{d,\alpha}(\gamma_1 \otimes \dots \otimes \gamma_n \,|\, \mu_1 \otimes \dots \otimes \mu_s)} \\ &\qquad := (ev^*_{x_1}\gamma_1 \cdot \dots \cdot ev^*_{x_n}\gamma_n \cdot ev^*_{y_1}\mu_1 \cdot \dots \cdot ev^*_{y_s}\mu_s) \cdot [\bar{M}_{0,n,s}(X,d\,|\,\alpha)] \in \mathbb{Q} \end{aligned}$$

if the dimension condition $\sum_i \operatorname{codim}\gamma_i + \sum_i \operatorname{codim}\mu_i = \dim\, \bar{M}_{0,n,s}(X,d\,|\,\alpha)$ is satisfied, and zero otherwise. As in the previous section, we will often abbreviate $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ and $\mathcal{D} = \mu_1 \otimes \dots \otimes \mu_s$ and write the invariant as $I^{H/X}_{d,\alpha}(\mathcal{T}\,|\,\mathcal{D})$ or simply $I_{d,\alpha}(\mathcal{T}\,|\,\mathcal{D})$. Also, to shorten notation, we sometimes write $ev^*\mathcal{T}$ for $ev^*_{x_1}\gamma_1 \cdot \dots \cdot ev^*_{x_n}\gamma_n$. In addition, if $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ is as above and $\gamma \in A^*(X)$, we define

$$\boldsymbol{\gamma \cdot_i \mathcal{T}} := \gamma_1 \otimes \dots \otimes \gamma_{i-1} \otimes \gamma \cdot \gamma_i \otimes \gamma_{i+1} \otimes \dots \otimes \gamma_n$$

for $1 \le i \le n$. Analogous notations will be used for $\mathcal{D}$.

Remark 3.2.4 As we defined the moduli space $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ such that the space of irreducible stable maps $(C, x_1,\dots,x_n, y_1,\dots,y_s, f)$ with $f(C) \not\subset H$ and $f^*H = \sum_i \alpha_i y_i$ is dense in it, and since we know that it is of the expected dimension, it is clear by the Bertini lemma 2.4.7 (iii) that the invariants $I^{H/X}_{d,\alpha}(\mathcal{T}\,|\,\mathcal{D})$ have an enumerative meaning: if V_i and W_j are generic subvarieties of X and H, respectively, with $[V_i] = \gamma_i$ and

$[W_j] = \mu_j$, then the invariant counts (with multiplicity one) irreducible stable maps $(C, x_1, \dots, x_n, y_1, \dots, y_s, f)$ of degree d with $f(x_i) \in V_i$ for $1 \le i \le n$ and $f(y_j) \in W_j$ for $1 \le j \le s$, such that f has multiplicity α_j along H at the point y_j. This also explains the name "degeneration invariant", since, in contrast to the Gromov-Witten case, the subvarieties that the curve has to meet are no longer completely generic, but s of the subvarieties, namely W_j for $1 \le j \le s$, have been degenerated to lie in H.

As in the Gromov-Witten case, an interpretation of degeneration invariants as numbers of *curves* in X (instead of *maps* to X) is also possible.

Example 3.2.5 From the geometric meaning of the invariants it follows that we can recover the Gromov-Witten invariants of X from the degeneration invariants of H in X if we set $\alpha = (1, \dots, 1)$ and $\mathcal{D} = X^{\otimes d}$: we then just require d points of transverse intersection with H at arbitrary points of H, which is the generic case for a degree d curve in X. The only difference is that in the degeneration invariant, the d points of intersection have been marked $y_1, \dots, y_d$, which gives a factor of $d!$ corresponding to the permutation of these points. Hence we have

$$I_d^X(\mathcal{T}) = \frac{1}{d!} I_{d,(1,\dots,1)}^{H/X}(\mathcal{T} \mid H^{\otimes d})$$

(where the H in the invariant denotes the fundamental class).

Example 3.2.6 The number of degree d rational curves in $\mathbb{P}^r$ tangent to H at a specified point $P \in H$ and intersecting additional classes $\mathcal{T}$ such that the dimension condition is satisfied, has been calculated in theorem 2.7.1 to be $I_{dH'-E'}^{\tilde{X}(1)}(\mathcal{T} \otimes -E^2)$. In terms of degeneration invariants, it is also given by $\frac{1}{(d-2)!} I_{d,(2,1,\dots,1)}^{H/X}(\mathcal{T} \mid pt \otimes H^{\otimes(d-2)})$, where the factor $\frac{1}{(d-2)!}$ again corresponds to the permutations of the $d-2$ unconstrained points of intersection of the curve with H. See also corollary 1.6.7 for an alternative description of these numbers in terms of gravitational descendants.

We now come to the key idea in the theory of degeneration invariants, which will also lead to a possibility to compute all of them. In the definition of the invariants, assume that $n \ge 1$ and look at a partial intersection

$$ev_{x_1}^* \hat{H} \cdot [\bar{M}_{0,n,s}(X, d \mid \alpha)] \in A_*(\bar{M}_{0,n,s}(X, d \mid \alpha)),$$

where $\hat{H}$ is a generic hyperplane in X. The generic point in $ev_{x_1}^{-1}(\hat{H})$ then corresponds to an irreducible curve not contained in H, and by intersecting with further pullbacks of classes via the evaluation maps, we obviously get a degeneration invariant. But now we let $\hat{H}$ degenerate to our distinguished hyperplane H. This does of course not affect the above cycle in $A_*(\bar{M}_{0,n,s}(X, d \mid \alpha))$, but the geometric appearance of $ev_{x_1}^{-1}(H)$ is drastically different: in general it will contain many components corresponding to reducible curves which are partly contained in H. In [V] theorem 2.13 the divisor $ev_{x_1}^* H$ has been computed explicitly, it consists of spaces $D_\ell(\vec{d}, \mathcal{I}, \mathcal{J})$ in $\bar{M}_{0,n,s}(X, d \mid \alpha)$ which we now describe.

Definition 3.2.7 *Fix a moduli space $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ as above with $n \geq 1$.*

Let $\ell \geq 0$ be a non-negative integer. Let $\vec{d} = (d_0,\dots,d_\ell)$ be $\ell+1$ integers with $\sum_{i=0}^{\ell} d_i = d$, $d_0 \geq 0$, and $d_i > 0$ for $i > 0$. Let $\mathcal{I} = (I_0,\dots,I_\ell)$ be a decomposition of the set $\{x_1,\dots,x_n\}$ such that $x_1 \in I_0$, and let $\mathcal{J} = (J_0,\dots,J_\ell)$ be a decomposition of the set $\{y_1,\dots,y_s\}$.

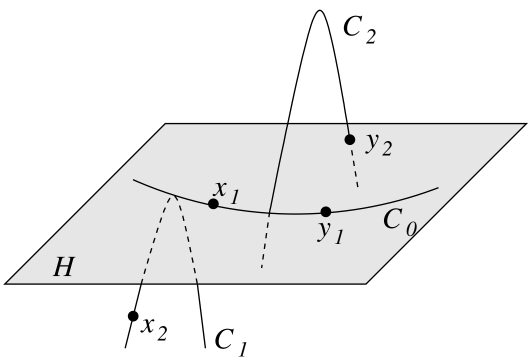
For all $1 \leq i \leq \ell$, define $\boldsymbol{m_i} := d_i - \sum_{k\in J_i} \alpha_k$ and assume that $m_i > 0$.

Then we define $\boldsymbol{D_\ell^{H/X}(\vec{d},\mathcal{I},\mathcal{J}) = D_\ell(\vec{d},\mathcal{I},\mathcal{J})}$ to be the closure in $\bar{M}_{0,n+s}(X,d)$ of the space of stable maps $(C,x_1,\dots,x_n,y_1,\dots,y_s,f)$ satisfying:

- *C has exactly $\ell+1$ irreducible components $C_0,\dots,C_\ell$,*
- *$C_0 \cap C_i \neq \emptyset$ for all $i > 0$, i.e. the curve C consists of a component C_0 with ℓ attached components C_i,*
- *$f(C_0) \subset H$, $f(C_i) \not\subset H$ for $i > 0$,*
- *f has degree d_i on C_i for $0 \leq i \leq \ell$,*
- *for each $0 \leq i \leq \ell$, the marked points of C_i are exactly $I_i \cup J_i$,*
- *for each $1 \leq i \leq \ell$, we have $(f|_{C_i})^* H = m_i(C_0 \cap C_i) + \sum_{k\in J_i} \alpha_k y_k$.*

We call two such spaces $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$, $D_\ell(\vec{d}',\mathcal{I}',\mathcal{J}')$ (and also the corresponding triples $(\vec{d},\mathcal{I},\mathcal{J})$ and $(\vec{d}',\mathcal{I}',\mathcal{J}')$) ***equivalent*** *if they differ just by relabeling of the components $C_1,\dots,C_\ell$.*

Thus, a generic element in $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ may look as follows:



(Here, we have $\ell = 2$, $\mathcal{I} = \{\{x_1\},\{x_2\},\emptyset\}$, $\mathcal{J} = \{\{y_1\},\emptyset,\{y_2\}\}$, $m_1 = 2$, $m_2 = 1$, $\alpha_2 = 1$. The value α_1 cannot be seen in the picture since no multiplicity condition has to be satisfied at points y_i in C_0.)

With this definition, we can now describe the main result of R. Vakil (on curves of genus zero):

Theorem 3.2.8 *Every $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ is a divisor in $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$, and we have*

$$ev_{x_1}^* H = \sum_{\ell \geq 0} \sum_{\vec{d},\mathcal{I},\mathcal{J}} m_1 \cdot \ldots \cdot m_\ell D_\ell(\vec{d},\mathcal{I},\mathcal{J}),$$

where the sum is taken over all equivalence classes $(\vec{d},\mathcal{I},\mathcal{J})$ as in definition 3.2.7, and where m_i is defined as above.

Proof See [V] proposition 2.11, theorem 2.16, theorem 2.19. □

To get some information on degeneration invariants from this equation, we will intersect it later with pullbacks of classes via the evaluation maps such that the total intersection becomes zero-dimensional. Then we obviously get a degeneration invariant on the left hand side of the equation. We claim that on the right hand side, we get a sum over products of a Gromov-Witten invariant and various degeneration invariants. To see this, we write the spaces $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ in a different way that makes it more obvious how the curves in these spaces are built up of $\ell+1$ components.

Lemma 3.2.9 *For fixed $\vec{d},\mathcal{I},\mathcal{J}$ let*

$$\boldsymbol{\Pi_X} := \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\ldots,p_\ell\}}(X,d_0) \times \prod_{i=1}^{\ell} \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(X,d_i).$$

Let $ev_{p_i}, ev_{q_i} : \Pi_X \to X$ be the evaluation maps. Define

$$\boldsymbol{Z_X} := \bigcap_{i=1}^{\ell} \left((ev_{p_i} \times ev_{q_i})^{-1}(\Delta_X) \right) \subset \Pi_X$$

where $\Delta_X \subset X \times X$ is the diagonal. Then there is an inclusion $Z_X \hookrightarrow \bar{M}_{0,n+s}(X,d)$, and Z_X is smooth of dimension $\bar{M}_{0,n+s}(X,d) - \ell$. Moreover, we have

$$Z_X = \prod_{i=1}^{\ell} (ev_{p_i} \times ev_{q_i})^*(\Delta_X)$$

as cycles on Π_X.

Proof See [BM] chapter 7 property III, and proposition 7.4 for smoothness and the statement on the dimension. The idea is of course that Z_X is the closure of the space of curves in $\bar{M}_{0,n+s}(X,d)$ consisting of $\ell+1$ components with the specified topology, degrees, and marked points. The factors of Π_X describe the $\ell+1$ components, the points p_i and q_i mark the gluing points $C_0 \cap C_i$ of the components, and the gluing itself is accomplished by the pullback of the diagonal Δ_X via $ev_{p_i} \times ev_{q_i}$. □

It is obvious that our spaces $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ are contained in Z_X, which we will regard from now on either as substack of Π_X or of $\bar{M}_{0,n+s}(X,\beta)$. On can describe the spaces $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ explicitly as follows:

Lemma 3.2.10 *Let Π_X be as in lemma 3.2.9. Let*

$$\Gamma := \bar{M}_{0,I_0\cup J_0\cup\{p_1,\dots,p_\ell\}}(H,d_0)\times\prod_{i=1}^{\ell}\bar{M}_{0,I_i,J_i\cup\{q_i\}}(X,d_i\,|\,(\alpha_{J_i},m_i))\subset\Pi_X,$$

where $(\boldsymbol{\alpha_{J_i}},\boldsymbol{m_i})$ is denotes the sequence $(\alpha_{k_1},\dots,\alpha_{k_{s_i}},m_i)$ with $J_i=\{\alpha_{k_1},\dots,\alpha_{k_{s_i}}\}$. Let $ev_{p_i},ev_{q_i}:\Gamma\to H$ be the evaluation maps and $\Delta_H\subset H\times H$ the diagonal. Then the cycle $D_\ell(\vec{d},\mathcal{I},\mathcal{J})$ on Π_X is given by

$$D_\ell(\vec{d},\mathcal{I},\mathcal{J})=\Gamma\cdot\prod_{i=1}^{\ell}(ev_{p_i}\times ev_{q_i})^*(\Delta_H).$$

Remark 3.2.11 In this and the following section, we will meet various evaluation maps to various spaces A. If we want to indicate the target space of the evaluation maps, we will write ev_A,x for the evaluation at the point x to the space A. However, by abuse of notation we will often drop this subscript if the target space is clear from the context, in particular if we have constructions of the form $(ev\times ev)^*(\Delta_A)$ which we will meet frequently in the sequel.

Proof (of lemma 3.2.10) This is the content of [V] section 2.5.1 translated into our language. The space Γ restricts the curves such that the component C_0 lies in H and such that the other components have the right multiplicities along H. The evaluation maps again perform the gluing of the components C_i with C_0. □

We will meet many of these gluing arguments in the next section.

As we have already mentioned, we now get relations between the degeneration invariants by intersecting equation 3.2.8 with pullbacks via the evaluation maps. Since we will do this again in more detail in the next section, we just state the result. We write it in a form that it generalizes easily to the case of hypersurfaces in the next section. Many numerical examples can be found in [V].

Corollary 3.2.12 *Let $X=\mathbb{P}^r$ with $r\geq 2$, $d>0$, and $n\geq 0$. Fix a hyperplane $H\subset X$ and denote the inclusion by $i:H\hookrightarrow X$. Let $s>1$ and $\alpha=(\alpha_1,\dots,\alpha_s)$ be an s-tuple of positive integers such that $\sum_i\alpha_i=d$. Let $\gamma\in A^*(X)$, and let $\mathcal{T}=\gamma_1\otimes\ldots\otimes\gamma_n$ and $\mathcal{D}=\mu_1\otimes\ldots\otimes\mu_s$ be collections of classes on X and H, respectively, such that $\sum codim\ \gamma_i+\sum codim\ \mu_i+codim\ \gamma+1=dim\ \bar{M}_{0,n+1,s}(X,d\,|\,\alpha)$. Then we have*

$$\begin{aligned} I^{H/X}_{d,\alpha}(\gamma\cdot H\otimes\mathcal{T}\,|\,\mathcal{D}) &= \sum_{k=1}^{s}\alpha_k I^{H/X}_{d,\alpha}(\mathcal{T}\,|\,(i^*\gamma)\cdot_k\mathcal{D}) && (1)\\ &+\sum_{\ell\geq 0}\sum_{\vec{d},\mathcal{I},\mathcal{J}}\sum_{i_k,j_k} g^{i_1j_1}\cdots g^{i_\ell j_\ell} I^H_{d_0}(i^*\gamma\otimes i^*\mathcal{T}_0\otimes\mathcal{D}_0\otimes T_{i_1}\otimes\ldots\otimes T_{i_\ell}) && (2)\\ &\cdot\prod_{k=1}^{\ell}\Big(m_k I^{H/X}_{d_k,(\alpha_{J_k},m_k)}(\mathcal{T}_k\,|\,\mathcal{D}_k\otimes T_{j_k})\Big) && (3)\end{aligned}$$

where $\mathcal{T}_k$ denotes the classes γ_i with $i\in I_k$, $\mathcal{D}_k$ the classes μ_i with $i\in J_k$. As in proposition 1.4.1 (iv), we have chosen a basis $\{T_0,\dots,T_q\}$ (as a vector space) of $A^*(H)$

and denote by g^{ij} *the inverse intersection matrix on* H *with respect to this basis. The sum is taken over all equivalence classes* $(\vec{d}, \mathcal{I}, \mathcal{J})$ *where* $\vec{d} = (d_0, \dots, d_\ell)$ *with* $\sum_i d_i = d$ *and* $d(i) > 0$ *for* $i \geq 0$, *where* $\mathcal{I} = (I_0, \dots, I_\ell)$ *is a partition of* $\{x_1, \dots, x_n\}$, *and* $\mathcal{J} = (J_0, \dots, J_\ell)$ *a partition of* $\{y_1, \dots, y_s\}$. *Here, the numbers* m_k *are defined to be* $d_k - \sum_{i \in J_k} \alpha_i$ *for* $k > 0$, *and it is assumed that the sum is taken only over those* $\vec{d}$ *and* $\mathcal{J}$ *such that all* m_k *are positive.*

Proof See [V] theorem 2.20. □

This corollary suffices to compute all degeneration invariants of H/X recursively: suppose we want to compute an invariant $I_{d,\alpha}(\mathcal{T}' \mid \mathcal{D})$. It can be shown that, by the dimension condition, there must be at least one class γ' in $\mathcal{T}'$ which is not the fundamental class of X such that it can be written as $\gamma' = \gamma \cdot H$. Then apply corollary 3.2.12, which expresses $I_{d,\alpha}(\mathcal{T}' \mid \mathcal{D}) = I_{d,\alpha}(\gamma \cdot H \otimes \mathcal{T} \mid \mathcal{D})$ entirely in terms of invariants (1) with fewer classes in $\mathcal{T}$, invariants (2) on a variety of lower dimension, and invariants (3) of smaller degree d. This finally reduces everything to the number of lines through two points in $\mathbb{P}^1$, which is 1.

It should be noted that, in contrast to the equations among the Gromov-Witten invariants, corollary 3.2.12 relates invariants on different varieties, namely invariants on X to invariants on H. Therefore it is interesting to generalize the theory of degeneration invariants to the case where H becomes a hypersurface of any degree so that one can relate invariants on $\mathbb{P}^r$ to invariants on hypersurfaces. This will be done in the next section.

3.3 Degeneration to a hypersurface in $\mathbb{P}^r$

We will now carry over the theory developed in the previous section to the case of a smooth hypersurface Q of arbitrary degree δ in $Y = \mathbb{P}^r$. The main problem here is that the various moduli spaces need not have the expected dimension — although a definition of moduli spaces $\bar{M}_{0,n,s}(Q/Y, d \mid \alpha)$ and $D_\ell^{Q/Y}(\vec{d}, \mathcal{I}, \mathcal{J})$ could be written down in the same way, the statements of lemma 3.2.2 and proposition 3.2.8 would in general not be true. The main reason for this is that the curves in $D_\ell(\vec{d}, \mathcal{I}, \mathcal{J})$ contain a component in the hypersurface Q, and if Q is not convex such that the spaces of curves in Q have too big dimension, then $D_\ell(\vec{d}, \mathcal{I}, \mathcal{J})$ will have too big dimension as well.

Our idea to solve this problem is as follows. Assume that we have a hypersurface Q of degree δ in $Y = \mathbb{P}^r$. By the degree δ Veronese embedding, we consider Y as a subvariety of $X = \mathbb{P}^N$ (where $N = \binom{r+\delta}{\delta} - 1$), such that $Q = X \cap H$ for a hyperplane H:

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d, hook] & H \arrow[d, hook] \\ Y \arrow[r, hook] & X. \end{tikzcd}$$

There is an induced cartesian diagram of moduli spaces

$$\begin{tikzcd} \bar{M}_{0,n}(Q,d) \arrow[r, hook] \arrow[d, hook] & \bar{M}_{0,n}(H,\delta d) \arrow[d, hook] \\ \bar{M}_{0,n}(Y,d) \arrow[r, hook] & \bar{M}_{0,n}(X,\delta d). \end{tikzcd}$$

where $\bar{M}_{0,n}(Q,d)$ is defined to be the union of all $\bar{M}_{0,n}(Q,\beta)$ such that $Q \cdot i_* \beta = \delta d$ in Y. (If dim $Q \geq 3$, then by the Lefschetz theorem $A_1(Q)$ is one-dimensional, such that there is only one such space $\bar{M}_{0,n}(Q,\beta)$.) We fix this setup for the rest of the section.

This description of Q as an intersection of two projective spaces Y and H gives us the following possibility to define moduli spaces $\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)$ and a virtual fundamental class on them:

Definition 3.3.1 *Let $d > 0$, $n \geq 0$, $s > 1$ and $\alpha = (\alpha_1,\ldots,\alpha_s)$ be an s-tuple of positive integers such that $\sum_i \alpha_i = \delta d$. Then we define the moduli space* $\boldsymbol{\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)}$ *by the cartesian diagram*

$$\begin{tikzcd} \bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha) \arrow[r, hook] \arrow[d, hook] & \bar{M}_{0,n,s}(H/X,\delta d\,|\,\alpha) \arrow[d, hook] \\ \bar{M}_{0,n+s}(Y,d) \arrow[r, hook] & \bar{M}_{0,n+s}(X,\delta d), \end{tikzcd}$$

i.e. as the intersection $\bar{M}_{0,n,s}(H/X,\delta d\,|\,\alpha) \cap \bar{M}_{0,n+s}(Y,d)$ in $\bar{M}_{0,n+s}(X,\delta d)$. (Here and in the rest of this section, when we write $\cap$ for the intersection of moduli spaces, we always mean the fibre product as above.) We define the ***virtual fundamental class*** *on $\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)$ to be the corresponding intersection product*

$$[\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)]^{virt} := \bar{M}_{0,n,s}(H/X,\delta d\,|\,\alpha) \cdot \bar{M}_{0,n+s}(Y,d) \in A_*(\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)).$$

We use analogous simplifications of the notation as in definition 3.2.1, in particular we will often write $\bar{M}_{0,n,s}(Y,d\,|\,\alpha)$ instead of $\bar{M}_{0,n,s}(Q/Y,d\,|\,\alpha)$.

This definition gives us the analogue of the moduli spaces $\bar{M}_{0,n,s}(X,d\,|\,\alpha)$ in section 3.2: the space $\bar{M}_{0,n+s}(Y,d)$ expresses that the curves have to lie in Y, and the space $\bar{M}_{0,n,s}(X,\delta d\,|\,\alpha)$ expresses that these curves in Y must have the local multiplicities α_i to H and hence to $Q = Y \cap H$. The expected dimension of the space of curves satisfying these conditions is

$$\begin{aligned} \textbf{vdim } \boldsymbol{\bar{M}_{0,n,s}(Y,d\,|\,\alpha)} :&= \dim \bar{M}_{0,n+s}(Y,d) - \sum_i \alpha_i \\ &= (r+1)d + r + n + s - 3 - \delta d, \end{aligned}$$

which is also the dimension of the cycle $\bar{M}_{0,n+s}(Y,d) \cdot \bar{M}_{0,n,s}(X,\delta d\,|\,\alpha)$, as is easy to check using lemma 3.2.2. Hence the virtual fundamental class of $\bar{M}_{0,n,s}(Y,d\,|\,\alpha)$ is really a cycle of dimension vdim $\bar{M}_{0,n,s}(Y,d\,|\,\alpha)$.

The definition of the degeneration invariants of Q/Y is now obvious:

Definition 3.3.2 *With notations as above, let $\gamma_1, \dots, \gamma_n \in A^*(Y)$ and $\mu_1, \dots, \mu_s \in A^*(Q)$ be classes on Y and Q, respectively. Then we define the associated* ***degeneration invariant*** *on Q/Y to be the intersection product on $\bar{M}_{0,n,s}(Y, d \,|\, \alpha)$*

$$\boldsymbol{I^{Q/Y}_{d,\alpha}(\gamma_1 \otimes \dots \otimes \gamma_n \,|\, \mu_1 \otimes \dots \otimes \mu_s)}$$
$$:= (ev^*_{x_1}\gamma_1 \cdot \ldots \cdot ev^*_{x_n}\gamma_n \cdot ev^*_{y_1}\mu_1 \cdot \ldots \cdot ev^*_{y_s}\mu_s) \cdot [\bar{M}_{0,n,s}(Y, d \,|\, \alpha)]^{virt} \in \mathbb{Q}$$

if the dimension condition $\sum_i \mathrm{codim}\, \gamma_i + \sum_i \mathrm{codim}\, \mu_i = \mathrm{vdim}\, \bar{M}_{0,n,s}(Y, d \,|\, \alpha)$ is satisfied, and zero otherwise. Note that the evaluation maps ev_{x_i} map to Y, whereas ev_{y_i} map to Q. We will apply the simplifications of the notation of definition 3.2.3.

According to the definition, the degeneration invariants of Q/Y have an *expected* enumerative meaning, which is literally the same as in 3.2.4, replacing H and X by Q and Y. In general, it is however not clear that this interpretation is valid, and in many cases it will not be. But by the same argument as in example 3.2.5, it is at least still true that one can recover the Gromov-Witten invariants of Y from the degeneration invariants of Q/Y if we set $\alpha = (1, \dots, 1)$:

$$I^Y_d(\mathcal{T}) = \frac{1}{(\delta d)!} I^{Q/Y}_{d,(1,\dots,1)}(\mathcal{T} \,|\, Q^{\otimes \delta d}).$$

Remark 3.3.3 Our definition of the moduli spaces $\bar{M}_{0,n,s}(Y, d \,|\, \alpha)$ and their virtual fundamental classes is a little bit unsatisfactory. We have learned from Gromov-Witten theory that a better way would probably be to define the moduli spaces without an embedding of Y into some auxiliary space, and to define their virtual fundamental classes using a suitable obstruction theory on the moduli space. This would allow us to extend the definition to hypersurfaces in arbitrary smooth projective varieties. In view of chapter 2, an interesting example we have in mind is of course the case of the exceptional divisor in a blow-up. The invariants on the blow-up $\tilde{X}$ are then supposed to count curves on $\tilde{X}$ with given local multiplicities to the exceptional divisor, hence via strict transform they should count curves on X with given local multiplicities to the blown-up point, e.g. multiplicity 2 would correspond to curves with a cusp there. Indeed, some very few numerical calculations have shown that this seems to be possible. We hope to be able to work out a theory of these generalized degeneration invariants in the future.

Returning to our original situation of a hypersurface Q in $Y = \mathbb{P}^r$, we now want to compute the degeneration invariants of Q/Y. To do this, we will show in the remaining part of this section that corollary 3.2.12 carries over almost without change. Consider the equation of theorem 3.2.8, with d replaced by δd, and intersect it with $\bar{M}_{0,n+s}(Y, d)$ in $\bar{M}_{0,n+s}(X, \delta d)$:

$$ev^*_{X,x_1} H \cdot \bar{M}_{0,n,s}(X, \delta d \,|\, \alpha) \cdot \bar{M}_{0,n+s}(Y, d)$$
$$= \sum_{\ell \geq 0} \sum_{\vec{\delta}, \mathcal{I}, \mathcal{J}} m_1 \cdot \ldots \cdot m_\ell D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cdot \bar{M}_{0,n+s}(Y, d)$$

where now $\vec{\delta} = (\delta_0, \dots, \delta_\ell)$ is a vector such that $\sum \delta_i = \delta d$, and $\boldsymbol{m_i} := \delta d - \sum_{k \in J_i} \alpha_k$. Obviously, when we intersect this equation with further pullbacks via evaluation maps, we will get a degeneration invariant of Q/Y on the left hand side of the equation. To analyze the right hand side, we now study the intersection $D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d)$. The main point of the following proposition is that everything splits up into $\ell+1$ factors corresponding to the components of the curves in $D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J})$.

Proposition 3.3.4 *The intersection $D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cap \bar{M}_{0,n+s}(Y,d)$ on $\bar{M}_{0,n+s}(X, \delta d)$ can only be non-zero if $\vec{\delta} = (\delta_0, \dots, \delta_\ell)$ is of the form $(\delta d_0, \dots, \delta d_\ell)$ for some integers d_i. In this case, it is isomorphic to a subspace of $\Gamma_0 \times \cdots \times \Gamma_\ell$, where*

$$\begin{aligned}
\Gamma_0 &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(H, \delta d_0) \cap \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(Y, d_0) \\
&\qquad \subset \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(X, \delta d_0), \\
\Gamma_i &= \bar{M}_{0,I_i,J_i \cup \{q_i\}}(X, \delta d_i \,|\, (\alpha_{J_i}, m_i)) \cap \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(Y, d_i) \\
&\qquad \subset \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(X, \delta d_i) \qquad \text{for } i > 0,
\end{aligned}$$

and where, as before, (α_{J_i}, m_i) is meant to be the sequence $(\alpha_{k_1}, \dots, \alpha_{k_{s_i}}, m_i)$, where $J_i = \{\alpha_{k_1}, \dots, \alpha_{k_{s_i}}\}$.

Moreover, the intersection product $D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d)$ in $\bar{M}_{0,n+s}(X, \delta d)$, viewed as a cycle on $\Gamma_0 \times \cdots \times \Gamma_\ell$ via the above inclusion, is equal to

$$D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d) = \left(\prod_{i=1}^{\ell} (ev_{p_i} \times ev_{q_i})^* (\Delta_Q) \right) \cdot \Gamma_0' \times \cdots \times \Gamma_\ell',$$

where ev_{p_i}, ev_{q_i} are the evaluation maps to Q, Δ_Q is the diagonal in $Q \times Q$, and where we denote by Γ_i' the intersection products corresponding to Γ_i

$$\begin{aligned}
\Gamma_0' &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(H, \delta d_0) \cdot \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(Y, d_0) \in A_*(\Gamma_0), \\
\Gamma_i' &= \bar{M}_{0,I_i,J_i \cup \{q_i\}}(X, \delta d_i \,|\, (\alpha_{J_i}, m_i)) \cdot \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(Y, d_i) \in A_*(\Gamma_i) \qquad \text{for } i > 0.
\end{aligned}$$

Proof First of all, it is clear that $\vec{\delta}$ must be of the form $(\delta d_0, \dots, \delta d_\ell)$, since every component of the curves in $D_\ell(\vec{\delta}, \mathcal{I}, \mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d)$ is contained in Y, hence its homology class in X must be a multiple of δ.

We start with the definition of the various moduli spaces that we will need in the proof. First of all, we abbreviate

$$\begin{aligned}
\boldsymbol{\bar{M}_0^X} &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(X, \delta d_0), \\
\boldsymbol{\bar{M}_i^X} &= \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(X, \delta d_i) \qquad \text{for } i > 0, \\
\boldsymbol{\bar{M}_0^Y} &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(Y, d_0), \\
\boldsymbol{\bar{M}_i^Y} &= \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(Y, d_i) \qquad \text{for } i > 0, \\
\boldsymbol{\bar{M}_0^H} &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\dots,p_\ell\}}(H, \delta d_0), \\
\boldsymbol{\bar{M}_i'} &= ev_{q_i}^{-1}(H) \subset \bar{M}_i^X \qquad \text{for } i > 0, \\
\boldsymbol{\bar{M}_i^\alpha} &= \bar{M}_{0,I_i,J_i \cup \{q_i\}}(X, \delta d_i \,|\, (\alpha_{J_i}, m_i)) \qquad \text{for } i > 0.
\end{aligned}$$

As in the previous section, these are the moduli spaces of the components of the curves in $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})$, with the points where they will be glued later marked p_i and q_i. The space $\bar{M}'_i$ corresponds to curves in X with marked point q_i on H, the meaning of the other spaces is obvious. With these notations, define the following moduli spaces, each of which describes the $\ell+1$ components, only with different additional conditions:

$$\boldsymbol{\Pi_X} = \bar{M}_0^X \times \cdots \times \bar{M}_\ell^X$$

("all components in X, not connected"),

$$\boldsymbol{Z_X} = \bigcap_{i=1}^{\ell} \big((ev_{p_i} \times ev_{q_i})^{-1}(\Delta_X)\big) \subset \Pi_X$$

("all components in X, connected"),

$$\boldsymbol{\Pi_Y} = \bar{M}_0^Y \times \cdots \times \bar{M}_\ell^Y$$

("all components in Y, not connected"),

$$\boldsymbol{Z_Y} = \bigcap_{i=1}^{\ell} \big((ev_{p_i} \times ev_{q_i})^{-1}(\Delta_Y)\big) \subset \Pi_Y$$

("all components in Y, connected"),

$$\boldsymbol{\tilde{\Pi}_X} = \bar{M}_0^H \times \bar{M}_1^X \times \cdots \times \bar{M}_\ell^X$$

("component C_0 in H, others in X, not connected"),

$$\boldsymbol{\Pi'_X} = \bar{M}_0^H \times \bar{M}'_1 \times \cdots \times \bar{M}'_\ell$$

("component C_0 in H, others in X with marked point q_i mapped to H, not connected"),

$$\boldsymbol{Z'_X} = Z_X \cap \tilde{\Pi}_X$$

("component C_0 in H, connected").

The moduli spaces Π_X and Z_X have already been introduced in lemma 3.2.9, the others are completely analogous. The proof will now contain various gluing arguments how spaces corresponding to "non-connected" curves (denoted by the letter Π above) can be made into others corresponding to "connected" curves (denoted by the letter Z above) by pullbacks of diagonals via various evaluation maps. This is done in complete analogy to lemmas 3.2.9 and 3.2.10 and can be proven in the same way, as we will always do it in the case where the ambient spaces are projective spaces.

We now start with the proof of the proposition, which consists of several steps.

Step 1. There is a cartesian diagram

$$\begin{tikzcd}
 & Z_Y \arrow[r, hook] \arrow[d, hook] & \bar{M}_{0,n+s}(Y,d) \arrow[d, hook] \\
D_\ell(\vec{\delta},\mathcal{I},\mathcal{J}) \arrow[r, hook] & Z_X \arrow[r, hook, "i"] & \bar{M}_{0,n+s}(X,\delta d)
\end{tikzcd}$$

with $i^![\bar{M}_{0,n+s}(Y,d)] = Z_Y$, so that the intersection $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})\cdot\bar{M}_{0,n+s}(Y,d)$ to be calculated is equal to the intersection product

$$D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})\cdot Z_Y$$

on Z_X. As $Z_X\subset\Pi_X$, this enables us to work entirely on Π_X and its various subspaces. Hence, unless otherwise stated, all intersection products from now on will be assumed to be on Π_X.

Step 2. Consider the diagram

$$\begin{tikzcd}
 & & Z_Y \arrow[d, hook] \\
D_\ell(\vec{\delta},\mathcal{I},\mathcal{J}) \arrow[r, hook] & Z'_X \arrow[r, hook] & Z_X .
\end{tikzcd}$$

This allows us to break up the intersection product $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})\cdot Z_Y$ on Z_X into two steps: first we will intersect Z'_X with Z_Y in Z_X, and then intersect the result (which will be a cycle in a space contained in Z'_X) with $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})$ in Z'_X.

Step 3. We compute the intersection of Z'_X and Z_Y in Z_X. By definition, we have $Z'_X = Z_X\cap\tilde{\Pi}_X$ and $Z_Y=\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Y)\cdot\Pi_Y$, so the intersection of Z'_X and Z_Y in Z_X is equal to

$$\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Y)\cdot\Pi_Y\cdot\tilde{\Pi}_X.$$

We claim that this is equal to

$$\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Q)\cdot\Pi_Y\cdot\Pi'_X.$$

To see this, consider the following commutative diagrams:

$$\begin{tikzcd}
(Q\times Q)^\ell \arrow[r, hook] & (Y\times Y)^\ell & \\
S \arrow[u] \arrow[r, hook] \arrow[d, hook] & R \arrow[u] \arrow[r, hook] \arrow[d, hook] & \Pi_Y \arrow[d, hook] \\
\Pi'_X \arrow[r, hook] \arrow[d] & \tilde{\Pi}_X \arrow[r, hook, "i"] \arrow[d] & \Pi_X \\
(H\times H)^\ell \arrow[r, hook, "k"] & (X\times X)^\ell &
\end{tikzcd}
\qquad
\begin{tikzcd}
(Q\times Q)^\ell \arrow[rrr] \arrow[ddd] & & & (Y\times Y)^\ell \arrow[ddd] \\
 & \Delta_Q^\ell \arrow[ul, "g"'] \arrow[r] \arrow[d] & \Delta_Y^\ell \arrow[ur, "j"] \arrow[d] & \\
 & \Delta_H^\ell \arrow[r] \arrow[dl] & \Delta_X^\ell \arrow[dr] & \\
(H\times H)^\ell \arrow[rrr, "k"] & & & (X\times X)^\ell
\end{tikzcd}$$

where the exponent ℓ denotes the ℓ-fold cartesian product. The spaces R and S are defined such that the middle two squares in the left diagram are cartesian. Note that the lower left square is also cartesian by the definition of the spaces Π'_X and $\tilde{\Pi}_X$, as well as all squares in the right diagram. The vertical maps to $(Q\times Q)^\ell$, $(Y\times Y)^\ell$, $(H\times H)^\ell$, and $(X\times X)^\ell$ are the evaluation maps at the points p_i and q_i.

Now, the intersection product $\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Y)\cdot\Pi_Y\cdot\tilde{\Pi}_X$ is by definition equal to

$$j^! i^! \Pi_Y \in A_*(R\times_{(Y\times Y)^\ell}\Delta_Y^\ell)$$

which is the same as

$$k^! j^! i^! \Pi_Y \in A_*(S\times_{(Q\times Q)^\ell}\Delta_Q^\ell)$$

because $S\times_{(Q\times Q)^\ell}\Delta_Q^\ell$ is isomorphic to $R\times_{(Y\times Y)^\ell}\Delta_Y^\ell$ and $k^!$ is the identity under this identification: to see this, note that

$$S\times_{(Q\times Q)^\ell}\Delta_Q^\ell=\left(R\times_{(Y\times Y)^\ell}\Delta_Y^\ell\right)\times_{(X\times X)^\ell}(H\times H)^\ell$$

by construction of the diagram. But the right hand side of this equation is actually equal to $R\times_{(Y\times Y)^\ell}\Delta_Y^\ell$ since all elements in this space map to $(H\times H)^\ell$ under the evaluation map to $(X\times X)^\ell$, because the elements in R correspond to connected curves with component C_0 in H.

On the other hand, the intersection product $\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Q)\cdot\Pi_Y\cdot\Pi'_X$ is by definition equal to

$$g^! k^! i^! \Pi_Y \in A_*(S\times_{(Q\times Q)^\ell}\Delta_Q^\ell).$$

Hence, as $g^!=j^!$ (see [F1] theorem 6.2), the statement that

$$\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Y)\cdot\Pi_Y\cdot\tilde{\Pi}_X=\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Q)\cdot\Pi_Y\cdot\Pi'_X$$

follows from the commutativity $k^! j^! = j^! k^!$ (see [F1] section 6.4). To summarize, we have shown that the intersection product of Z'_X and Z_Y in Z_X is equal to

$$\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Q)\cdot\Pi_Y\cdot\Pi'_X.$$

Step 4. We describe $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})$ as subspace of Z'_X, in analogy to lemma 3.2.10: if we denote by $p_i: Z'_X\to\bar{M}'_i$ the projections for $i>0$, then we can write $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})$ as

$$\prod_{i=1}^{\ell} p_i^*\bar{M}_i^\alpha\cdot Z'_X.$$

(From the geometric construction, it is clear that this is true on the level of sets. The proof that it is also an equality of cycles follows in the same way as in lemma 3.2.10, as this just depends on H and X, but not on Q and Y.)

Step 5. Inserting the results of steps 3 and 4 into step 2, we get the result that the intersection product $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J})\cdot Z_Y$ on Z_X is equal to

$$\prod(ev_{p_i}\times ev_{q_i})^*(\Delta_Q)\cdot\prod_{i=1}^{\ell} p_i^*\bar{M}_i^\alpha\cdot\Pi_Y\cdot\Pi'_X.$$

(where we denote the projections $\Pi'_X \to \bar{M}'_i$ also by p_i, by abuse of notation). But note that

$$\prod_{i=1}^{\ell} p_i^* \bar{M}_i^{\alpha} \cdot \Pi'_X = \bar{M}_0^H \times \bar{M}_1^{\alpha} \times \cdots \times \bar{M}_{\ell}^{\alpha},$$

which follows directly from the definitions. So we finally see that the desired intersection is equal to

$$\begin{aligned}
&\prod (ev_{p_i} \times ev_{q_i})^*(\Delta_Q) \cdot (\bar{M}_0^H \times \bar{M}_1^{\alpha} \times \cdots \times \bar{M}_{\ell}^{\alpha}) \cdot \Pi_Y \\
&\quad = \prod (ev_{p_i} \times ev_{q_i})^*(\Delta_Q) \cdot \big((\bar{M}_0^H \cdot \bar{M}_0^Y) \times (\bar{M}_1^{\alpha} \cdot \bar{M}_1^Y) \times \cdots \times (\bar{M}_{\ell}^{\alpha} \cdot \bar{M}_{\ell}^Y) \big) \\
&\quad = \prod (ev_{p_i} \times ev_{q_i})^*(\Delta_Q) \cdot (\Gamma'_0 \times \cdots \times \Gamma'_{\ell}).
\end{aligned}$$

This finishes the proof. □

This proposition now tells us that we get recursion formulas for the invariants of Q/Y which already look very similar to those of H/X in corollary 3.2.12:

Corollary 3.3.5 *Let $Y = \mathbb{P}^r$ with $r \geq 2$, $d > 0$, and $n \geq 0$. Fix a smooth hypersurface $Q \subset Y$ of degree δ and denote the inclusion by $i : Q \hookrightarrow Y$. Let $s > 1$ and $\alpha = (\alpha_1, \dots, \alpha_s)$ be an s-tuple of positive integers such that $\sum_i \alpha_i = \delta d$. Let $\gamma \in A^*(Y)$, and let $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ and $\mathcal{D} = \mu_1 \otimes \ldots \otimes \mu_s$ be collections of classes on Y and Q, respectively, such that $\sum$ codim $\gamma_i + \sum$ codim μ_i + codim $\gamma + 1$ = vdim $\bar{M}_{0,n+1,s}(Y,d\,|\,\alpha)$. Then we have*

$$I_{d,\alpha}^{Q/Y}(\gamma \cdot Q \otimes \mathcal{T} \,|\, \mathcal{D}) = \sum_{\ell \geq 0} \sum_{\vec{d},\mathcal{I},\mathcal{J}} \sum_{i_k,j_k} g^{i_1 j_1} \cdots g^{i_\ell j_\ell} I_{d_0}^{Q}(i^*\gamma \otimes i^*\mathcal{T}_0 \otimes \mathcal{D}_0 \otimes T_{i_1} \otimes \ldots \otimes T_{i_\ell}) \cdot \prod_{k=1}^{\ell} \Big(m_k I_{d_k,(\alpha_{J_k},m_k)}^{Q/Y}(\mathcal{T}_k \,|\, \mathcal{D}_k \otimes T_{j_k}) \Big)$$

where $\mathcal{T}_k$ denotes the classes γ_i with $i \in I_k$, $\mathcal{D}_k$ the classes μ_i with $i \in J_k$. As in proposition 1.4.1 (iv), we have chosen a basis $\{T_0, \dots, T_q\}$ (as a vector space) of $A^*(Q)$ and denote by g^{ij} the inverse intersection matrix on Q with respect to this basis. The sum is taken over all equivalence classes $(\vec{d},\mathcal{I},\mathcal{J})$ where $\vec{d} = (d_0, \dots, d_\ell)$ with $\sum_i d_i = d$, $d(0) \geq 0$ and $d(i) > 0$ for $i \geq 0$, where $\mathcal{I} = (I_0, \dots, I_\ell)$ is a partition of $\{x_1, \dots, x_n\}$, and $\mathcal{J} = (J_0, \dots, J_\ell)$ a partition of $\{y_1, \dots, y_s\}$. Here, the numbers m_k are defined to be $\delta d_k - \sum_{i \in J_k} \alpha_i$ for $k > 0$, and it is assumed that the sum is taken only over those $\vec{d}$ and $\mathcal{J}$ such that all m_k are positive.

Remark 3.3.6 The Gromov-Witten invariant $I_{d_0}^{Q}(\,\cdot\,)$ in the corollary is to be interpreted as the corresponding intersection product on the space $\bar{M}(Q,d_0)$ as introduced in the beginning of this section. This means that the invariant is equal to the sum of all $I_{\beta_0}^{0}(\,\cdot\,)$ such that $i_*\beta_0$ is d_0 times the class of a line, where $i : Q \hookrightarrow Y$ is the inclusion.

Remark 3.3.7 The main difference in the equations of corollaries 3.2.12 and 3.3.5 is that in 3.3.5, we still sum over all $\vec{d}$ with $d(0) \geq 0$, whereas in 3.2.12, we require $d(0) > 0$. We will see in the next corollary that this difference is exactly the line (1) in 3.2.12.

Proof (of corollary 3.3.5) This follows almost immediately from proposition 3.3.4. As indicated before this proposition, we consider the equation of theorem 3.2.8, with d replaced by δd, and intersect it with $\bar{M}_{0,n+s}(Y,d)$ in $\bar{M}_{0,n+s}(X,\delta d)$ to obtain

$$\begin{aligned} &ev_{X,x_1}^* H \cdot \bar{M}_{0,n,s}(X,\delta d \,|\, \alpha) \cdot \bar{M}_{0,n+s}(Y,d) \\ &\qquad = \sum_{\ell \geq 0} \sum_{\vec{\delta},\mathcal{I},\mathcal{J}} m_1 \cdot \ldots \cdot m_\ell D_\ell(\vec{\delta},\mathcal{I},\mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d). \end{aligned}$$

Intersecting with $ev^*\mathcal{T}$ and $ev^*\mathcal{D}$ for $\mathcal{T}$ and $\mathcal{D}$ as in the corollary yields on the left hand side of this equation the degeneration invariant $I_d^{Q/Y}(\gamma \cdot Q \otimes \mathcal{T} \,|\, \mathcal{D})$ (note that $ev_{X,x_1}^* H = ev_{Y,x_1}^* Q$). To evaluate the right hand side, insert the result of proposition 3.3.4 for the intersection product $D_\ell(\vec{\delta},\mathcal{I},\mathcal{J}) \cdot \bar{M}_{0,n+s}(Y,d)$ to see that $\vec{\delta} = (\delta d_0, \ldots, \delta d_\ell)$ and that this intersection is equal to

$$\left(\prod_{i=1}^{\ell} (ev_{p_i} \times ev_{q_i})^* (\Delta_Q) \right) \cdot \Gamma'_0 \times \cdots \times \Gamma'_\ell \tag{2}$$

where

$$\begin{aligned} \Gamma'_0 &= \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\ldots,p_\ell\}}(H,\delta d_0) \cdot \bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\ldots,p_\ell\}}(Y,d_0) \\ \Gamma'_i &= \bar{M}_{0,I_i,J_i \cup \{q_i\}}(X,\delta d_i \,|\, (\alpha_{J_i},m_i)) \cdot \bar{M}_{0,I_i \cup J_i \cup \{q_i\}}(Y,d_i) \qquad \text{for } i > 0. \end{aligned}$$

By proposition 1.3.5 and remark 1.3.6, we see that Γ'_0 is just equal to the virtual fundamental class $[\bar{M}_{0,I_0 \cup J_0 \cup \{p_1,\ldots,p_\ell\}}(Q,d)]^{virt}$. By definition, Γ'_i is equal to the virtual fundamental class of $\bar{M}_{0,I_i,J_i \cup \{q_i\}}(Y,d_i \,|\, (\alpha_{J_i},m_i))$. Hence, noting that $\Delta_Q = \sum_{ij} g^{ij} T_i \times T_j$, we get the desired result. □

As indicated, to obtain the final form of the recursion relations, which is then almost the same as in the previous section, we now consider the terms in corollary 3.3.5 with $d_0 = 0$ separately.

Corollary 3.3.8 *With the notations as in corollary 3.3.5, we have*

$$\begin{aligned} I_{d,\alpha}^{Q/Y}(\gamma \cdot Q \otimes \mathcal{T} \,|\, \mathcal{D}) &= \sum_{k=1}^{s} \alpha_k I_{d,\alpha}^{Q/Y}(\mathcal{T} \,|\, (i^*\gamma) \cdot_k \mathcal{D}) && (1) \\ &\quad + \sum_{\ell \geq 0} \sum_{\vec{d},\mathcal{I},\mathcal{J}} \sum_{i_k,j_k} g^{i_1 j_1} \cdots g^{i_\ell j_\ell} I_{d_0}^{Q}(i^*\gamma \otimes i^*\mathcal{T}_0 \otimes \mathcal{D}_0 \otimes T_{i_1} \otimes \ldots \otimes T_{i_\ell}) && (2) \\ &\quad \cdot \prod_{k=1}^{\ell} \left(m_k I_{d_k,(\alpha_{J_k},m_k)}^{Q/Y}(\mathcal{T}_k \,|\, \mathcal{D}_k \otimes T_{j_k}) \right) && (3) \end{aligned}$$

where the second sum is as in corollary 3.3.5, but only over those $\vec{d}$ such that $d_0 > 0$.

Proof Look at the summands in corollary 3.3.5 where $d_0 = 0$. They contain a factor

$$I_0^Q(i^*\gamma \otimes i^*\mathcal{T}_0 \otimes \mathcal{D}_0 \otimes T_{i_1} \otimes \ldots \otimes T_{i_\ell})$$

which is a Gromov-Witten invariant with homology class zero. We know by proposition 1.4.1 (i) that there can only be three classes in such an invariant. But there are also at least three classes in this invariant:

- $i^*\gamma$,
- T_{i_1}, because we must have $\ell \geq 1$, since otherwise we would have $d = 0$,
- one of the classes in $\mathcal{D}_0$: note that we have $m_i = \delta d_i - \sum_{k \in J_i} \alpha_k > 0$, so by summing these inequalities up we get $\delta d - \delta d_0 - \sum_{k \notin J_0} \alpha_k > 0$. But as $d_0 = 0$ and $\sum_{k=1}^{s} \alpha_k = \delta d$, we conclude that $J_0 \neq \emptyset$, so that we have at least one class μ_j in $\mathcal{D}_0$.

This means that there can be no further classes in the Gromov-Witten invariant than those, i.e. we must have $\ell = 1$, $I_0 = \emptyset$ and $J_0 = \{j\}$ for some $1 \leq j \leq s$, and the invariant becomes

$$I_0^Q(i^*\gamma \otimes \mu_j \otimes T_{i_1}).$$

This means that the corresponding summand in corollary 3.3.5 is

$$\sum_{i,j} g^{ij} m_1 I_0^Q(i^*\gamma \otimes \mu_j \otimes T_i)\, I_{d,(\alpha',m_1)}^{Q/Y}(\mathcal{T} \,|\, \mathcal{D}' \otimes T_j)$$

where $\mathcal{D}'$ contains all classes of $\mathcal{D}$ except μ_j, and α' contains all numbers of α except α_j. By definition of the inverse intersection matrix, this summand becomes

$$m_1 I_{d,(\alpha',m_1)}^{Q/Y}(\mathcal{T} \,|\, \mathcal{D}' \otimes (i^*\gamma) \cdot \mu_j) = \alpha_j I_{d,\alpha}^{Q/Y}(\mathcal{T} \,|\, (i^*\gamma) \cdot_j \mathcal{D})$$

(note that $m_1 = \delta d - \sum_{k \in J_1} \alpha_k = \sum_{k \in J_0} \alpha_k = \alpha_j$).

Inserting this into the result of corollary 3.3.5, we get the stated equation. □

We thus got indeed equations relating Gromov-Witten invariants on Y (they are included in the degeneration invariants) to Gromov-Witten invariants on Q. However, although the equations that we get are the same as in the case of a hyperplane, they do not suffice to calculate all degeneration invariants in the case of a general hypersurface. The reason is that there are invariants $I_{d,\alpha}(\mathcal{T} \,|\, \mathcal{D})$ where $\mathcal{T}$ contains no classes at all, so that the equation cannot be applied directly to compute these invariants. To say the same thing geometrically, this means that for hypersurfaces Q, even after degenerating all incidence conditions into Q, there are still irreducible curves not contained in Q satisfying these conditions. We will see examples for this in 3.4.3 and 3.4.4.

Therefore, to get concrete relations between the Gromov-Witten invariants of Y and Q, we need to calculate the other degeneration invariants appearing in the equations by different methods. A possibility how to do this for $d = 1$ in some cases will be given in the next section.

3.4 Degeneration invariants and descendants

We will now give an explicit formula for some degeneration invariants of degree $d = 1$, some of which could not be obtained by the equations in corollary 3.3.8. The main application of these results is that they allow us in some cases to get explicit relations between the Gromov-Witten invariants of Y and those of Q up to degree 2 involving no unknown numbers any more. This will be applied explicitly in section 3.5 for the quintic threefold.

The computation will be done using gravitational descendants, using the results of section 1.6. In fact, most of the work has already been done in that section.

Proposition 3.4.1 *Let $Y = \mathbb{P}^r$ and $Q \subset Y$ be a smooth hypersurface of degree δ. Let $n \geq 0$ and $m \in \{1,\dots,\delta\}$. Let $\gamma \in A^{\geq r-2}(Y)$ and $\gamma_1,\dots,\gamma_n \in A^{\geq 2}(Y)$ be classes in Y and set $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$. Assume that the dimension condition*

$$\begin{aligned} \textit{codim } \gamma + \sum_i \textit{codim } \gamma_i &= \textit{vdim } \bar{M}_{0,n,\delta-m+1}(Y,1\,|\,(m,1,\dots,1)) \\ &= \textit{dim } \bar{M}_{0,n+1}(Y,1) - m \\ &= 2r+n-m-1 \end{aligned}$$

is satisfied, and that there are only finitely many lines in Q. Then the degeneration invariant

$$I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T}\,|\,i^*\gamma \otimes Q^{\otimes(\delta-m)})$$

is equal to the gravitational descendant

$$(\delta-m)!\, I^Y_1(\gamma \cdot \prod_{i=0}^{m-1}(Q+ic) \otimes \mathcal{T}).$$

Proof By definition, the degeneration invariant in the proposition is equal to the intersection product

$$(ev^*_{x_1}\gamma_1 \cdot \dots \cdot ev^*_{x_n}\gamma_n \cdot ev^*_{Q,y_1} i^*\gamma) \cdot [\bar{M}_{0,n,\delta-m+1}(Y,1\,|\,(m,1,\dots,1))]^{virt}.$$

Inserting the definition of the virtual fundamental class, this becomes the intersection product

$$(ev^*\mathcal{T} \cdot ev^*_{Y,y_1}\gamma) \cdot \bar{M}_{0,n,\delta-m+1}(X,\delta\,|\,(m,1,\dots,1)) \cdot \bar{M}_{0,n+\delta-m+1}(Y,1) \qquad (1)$$

on $\bar{M}_{0,n+\delta-m+1}(X,\delta)$, where H and X are as in the previous section. Denote by

$$p : \bar{M}_{0,n+\delta-m+1}(X,\delta) \to \bar{M}_{0,n+1}(X,\delta)$$

the projection map that forgets all points y_i except y_1. Pushing the zero-cycle (1) forward via p yields

$$(ev^*\mathcal{T} \cdot ev^*_{Y,y_1}\gamma) \cdot p_*(\bar{M}_{0,n,\delta-m+1}(X,\delta\,|\,(m,1,\dots,1)) \cdot \bar{M}_{0,n+\delta-m+1}(Y,1))$$
$$= (ev^*\mathcal{T} \cdot ev^*_{Y,y_1}\gamma) \cdot \bar{M}_{0,n+1}(Y,1) \cdot p_*\bar{M}_{0,n,\delta-m+1}(X,\delta\,|\,(m,1,\dots,1)).$$

Recall that in 1.6.1 we defined a space $\bar{M}^{(m)}$ which is, in our case, the closure in $\bar{M}_{0,n+1}(X,\delta)$ of the space of irreducible stable maps $(C,x_1,\dots,x_n,f)$ of degree δ to X with $f(C) \not\subset H$ such that the divisor f^*H on C contains the point x_1 with multiplicity m. Therefore, by definition of the moduli space $\bar{M}_{0,n,\delta-m+1}(X,\delta\,|\,(m,1,\dots,1))$ it follows that

$$p_*\bar{M}_{0,n,\delta-m+1}(X,\delta\,|\,(m,1,\dots,1)) = (\delta-m)!\,\bar{M}^{(m)},$$

where the factor $(\delta-m)!$ arises from the permutations of the $\delta-m$ forgotten marked points. Thus, the degeneration invariant stated in the proposition is equal to

$$(\delta-m)!\,(ev^*\mathcal{T} \cdot ev^*_{Y,y_1}\gamma) \cdot \bar{M}_{0,n+1}(Y,1) \cdot \bar{M}^{(m)}. \tag{2}$$

But we have already calculated the class of $\bar{M}^{(m)}$: by proposition 1.6.6 it is given by

$$\prod_{i=0}^{m-1}(i\,c_1(L_{y_1}) + ev^*_{X,y_1}H) + \mu,$$

where L_{y_1} denotes the cotangent line of the point y_1 (see section 1.5), and where μ is some cycle with support on the space of reducible stable maps $(C,x_1,\dots,x_n,y_1,f)$ such that y_1 lies on a component C_0 of C with $f(C_0) \subset H$. But note that there can be no such curves in the intersection (2) since

- if f has degree 0 on C_0, then by stability there must be at least one other marked point x_i in C_0. Hence this point maps to Q and must satisfy the incidence conditions $\gamma \in A^{\geq r-2}(Y)$ and $\gamma_i \in A^{\geq 2}(Y)$, which is impossible,
- if f has degree 1 on C_0, then the curve is mapped to a line contained in Q. But by assumption there are only finitely many lines in Q, so no marked point on such a line can satisfy a generic incidence condition in $A^{\geq 2}(Y)$. This would mean that there are actually no marked points x_i, but then the curve could not be reducible by stability.

In any case, we can drop the cycle μ, so (2) becomes the intersection product on $\bar{M}_{0,n+1}(Y,1)$

$$(\delta-m)!\,(ev^*\mathcal{T} \cdot ev^*_{Y,y_1}\gamma) \cdot \prod_{i=0}^{m-1}(i\,c_1(L_{y_1}) + ev^*_{Y,y_1}Q),$$

which is by definition the gravitational descendant stated in the proposition. □

Note that by proposition 1.5.1, this allows us to compute the degeneration invariants explicitly in the cases where proposition 3.4.1 is applicable. We finish this section by giving three such examples.

Example 3.4.2 Let $Q \subset Y = \mathbb{P}^2$ be a smooth curve of degree δ. We want to count the tangents to Q through a fixed point $P \in \mathbb{P}^2$.

The "classical solution" is to consider the projection from P onto a line $L \subset \mathbb{P}^2$. Then, in the generic situation, the tangents to Q through P correspond to the ramification points of the projection $Q \to L$. Hence the desired number is given by the Hurwitz formula

$$2g(Q) - 2 - \delta(2g(L) - 2) = \delta^2 - \delta.$$

The alternative solution using degeneration invariants and proposition 3.4.1 is

$$\begin{aligned}
\frac{1}{(\delta-2)!} I^{Q/Y}_{1,(2,1,\dots,1)}(pt \,|\, Q^{\otimes(\delta-1)}) &= I^Y_1(Q \cdot (Q+c) \otimes pt) \\
&= \delta^2 I^Y_1(H^2 \otimes pt) + \delta I^Y_1(H \cdot c \otimes pt) \\
&= \delta^2 \cdot 1 + \delta \cdot (-1) \\
&= \delta^2 - \delta
\end{aligned}$$

(where we divided by $(\delta-2)!$ in the beginning to remove the permutations of the marked points $y_2, \dots, y_{\delta-1}$).

Example 3.4.3 Again let $Q \subset Y = \mathbb{P}^2$ be a smooth curve of degree δ. We want to compute the number of inflection points of Q, i.e. the number of lines in $\mathbb{P}^2$ having contact of order at least 3 with the curve. In analogy to the previous example, our answer will be

$$\begin{aligned}
\frac{1}{(\delta-3)!} I^{Q/Y}_{1,(3,1,\dots,1)}(1 \,|\, Q^{\otimes(\delta-2)}) &= I^Y_1(Q \cdot (Q+c) \cdot (Q+2c)) \\
&= 3\delta^2 I^Y_1(pt \cdot c) + 2\delta I^Y_1(H \cdot c^2) \\
&= 3\delta^2 \cdot 1 + 2\delta \cdot (-3) \\
&= 3\delta^2 - 6\delta.
\end{aligned}$$

This coincides with the well-known classical result.

Example 3.4.4 As the most important example, we will now compute some degree 1 degeneration invariants of a quintic threefold $Q \subset Y = \mathbb{P}^4$ that will be needed in the next section. The input data, namely the corresponding Gromov-Witten invariants and gravitational descendants in $\mathbb{P}^4$, are well-known to be the following:

$\mathcal{T}$	$I^Y_1(\mathcal{T})$	$\mathcal{T}$	$I^Y_1(\mathcal{T})$	$\mathcal{T}$	$I^Y_1(\mathcal{T})$
$(H^2)^{\otimes 6}$	5	$pt \cdot c \otimes (H^2)^{\otimes 2}$	1	$pt \cdot c^2 \otimes H^2$	1
$(H^2)^{\otimes 4} \otimes H^3$	3	$pt \cdot c \otimes H^3$	1	$H^3 \cdot c^2 \otimes (H^2)^{\otimes 2}$	-1
$(H^2)^{\otimes 2} \otimes (H^3)^{\otimes 2}$	2	$H^3 \cdot c \otimes (H^2)^{\otimes 3}$	1	$H^3 \cdot c^2 \otimes H^3$	-2
$(H^3)^{\otimes 3}$	1	$H^3 \cdot c \otimes H^3 \otimes H^2$	0	$pt \cdot c^3$	1
$pt \otimes (H^2)^{\otimes 3}$	1	$H^3 \cdot c \otimes pt$	-1	$H^3 \cdot c^3 \otimes H^2$	-3
$pt \otimes H^3 \otimes H^2$	1			$H^3 \cdot c^4$	-5
$pt^{\otimes 2}$	1				

Using proposition 3.4.1, we can now calculate the following degeneration invariants

$$I(m,\mathcal{T},\gamma) := \frac{1}{(5-m)!}\, I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T}\,|\,i^*\gamma\otimes Q^{\otimes(5-m)}) :$$

$\gamma=H^2$			$\gamma=H^3$		
m	$\mathcal{T}$	$I(m,\mathcal{T},\gamma)$	m	$\mathcal{T}$	$I(m,\mathcal{T},\gamma)$
2	$(H^2)^{\otimes 3}$	6	2	$(H^2)^{\otimes 2}$	1
2	$H^2\otimes H^3$	5	2	H^3	1
2	pt	4	3	H^2	2
3	$(H^2)^{\otimes 2}$	13	4	1	6
3	H^3	11			
4	H^2	37			
5	1	130			

The numbers with $m=1$ are not stated as they are trivially equal to the corresponding Gromov-Witten invariants.

3.5 Lines and conics on the quintic threefold

In this last section, we will give a non-trivial example for the techniques developed in this chapter: we compute the number of lines and conics on a quintic threefold $Q\subset Y=\mathbb{P}^4$ using degeneration invariants of Q/Y. The input to the calculation will be the Gromov-Witten invariants up to degree 2 in $\mathbb{P}^4$ and the degree 1 degeneration invariants of example 3.4.4. The fact that we know the latter only for degree 1 is the only reason why the method of computing the numbers of rational curves on Q by means of degeneration invariants only works up to degree 2 so far. These numbers of lines and conics on the quintic threefold are of course well-known [Ka], however we hope to generalize our methods to higher degree in the future, and, to be optimistic, also to higher genus, as degeneration methods seem to be quite suitable to compute numbers of curves of higher genus (see [CH3] and [V]). For rational curves on Q, there exist by now mathematically rigorous methods [K], [Gi] to verify the physicists' numbers [COGP]. The numbers of elliptic curves on Q, however, as conjectured by physicists [BCOV], have not been verified mathematically so far.

As the recursion relations of corollary 3.3.8 are quite complicated, we will organize the invariants and equations in terms of generating functions and differential operators, which makes the result easier to state. As this requires the tensor products in collections of classes $\gamma_1\otimes\dots\otimes\gamma_n$ to become multiplication in a polynomial ring, we have to change the notation in this section:

Definition 3.5.1 *Consider polynomial rings in formal variables* $\boldsymbol{V} := \mathbb{Q}[H_2, H_3, H_4]$ *and* $\boldsymbol{W} := \mathbb{Q}[H_2, H_3, H_4, M_3, M_4, C]$*. We give* V *and* W *a grading by*

$$\deg H_2 = 1,\ \deg H_3 = 2,\ \deg H_4 = 3,\ \deg M_3 = 2,\ \deg M_4 = 3,\ \deg C = 1.$$

Next, we define algebra homomorphisms by

$$\begin{aligned} \boldsymbol{i_Y} : &V \to \bigotimes A^*(Y) \\ &H_2 \mapsto H^2,\ H_3 \mapsto H^3,\ H_4 \mapsto H^4, \\ \boldsymbol{i_Q} : &V \to \bigotimes A^*(Q) \\ &H_2 \mapsto \frac{1}{5} i^* H,\ H_3 \mapsto \frac{1}{5} i^* H^2,\ H_4 \mapsto \frac{1}{5} i^* H^3, \end{aligned}$$

where $i : Q \hookrightarrow Y$ *is the inclusion. With these notations, define a functional* $\boldsymbol{\Phi} : V \times W \to \mathbb{Q}$*, the "functional of Gromov-Witten- and degeneration invariants on* Y*", on pairs of monomials as follows:*

If p, q *are monomials in* V *with* $\deg(pq) = 5d+1$ *for* $d \in \{1, 2\}$*, set* j *to be* $5d$ *minus the number of factors in* q*, and define*

$$\Phi(p,q) := \begin{cases} \frac{1}{j!} I^{Q/Y}_{d,(1,\ldots,1)}(i_Y(p) \,|\, Q^{\otimes j} \otimes i_Q(q)) & \text{if } j \geq 0, \\ I^Q_d(i_Q(q)) & \text{if } j = -1. \end{cases}$$

(Note that the only possibility for j *to be negative is* $j = -1$*,* $p = 1$*, and* $q = H_2^{5d+1}$*.) If* $m \geq 1$ *and* p, q *are monomials in* V *with* $\deg(pqM_3C^{m-1}) = 6$*, set* j *to be* $5 - m$ *minus the number of factors in* q *(which is always non-negative), and define*

$$\Phi(p, qM_3C^{m-1}) := \frac{1}{j!} I^{Q/Y}_{1,(m,1,\ldots,1)}(i_Y(p) \,|\, i_Q(H_3) \otimes Q^{\otimes j} \otimes i_Q(q)).$$

Analogously, if $m \geq 1$ *and* p, q *are monomials in* V *with* $\deg(pqM_4C^{m-1}) = 6$*, set* j *to be* $5 - m$ *minus the number of factors in* q *(which is again always non-negative), and define*

$$\Phi(p, qM_4C^{m-1}) := \frac{1}{j!} I^{Q/Y}_{1,(m,1,\ldots,1)}(i_Y(p) \,|\, i_Q(H_4) \otimes Q^{\otimes j} \otimes i_Q(q)).$$

We set $\Phi(p,q)$ *to zero on all other pairs of monomials and extend* Φ *linearly to* $V \times W$*. For* $p \in V$*, we abbreviate*

$$\begin{aligned} \Phi(p) &:= \Phi(p, 1), \\ \Phi(pM_3C^{m-1}) &:= \Phi(p, M_3C^{m-1}), \\ \Phi(pM_4C^{m-1}) &:= \Phi(p, M_4C^{m-1}). \end{aligned}$$

Remark 3.5.2 As the Gromov-Witten invariants are included in the degeneration invariants by means of the equation

$$I_d^Y(\mathcal{T}) = \frac{1}{(5d)!} I_{d,(1,\dots,1)}^{Q/Y}(\mathcal{T} \,|\, Q^{\otimes 5d}),$$

the numbers $\Phi(p) = \Phi(p,1)$ for $p \in V$ are just the Gromov-Witten invariants of Y (of degrees 1 and 2) of Y. The invariants $\Phi(pM_3C^{m-1})$ and $\Phi(pM_4C^{m-1})$ are the degree 1 degeneration invariants that we know by example 3.4.4. Hence, the functional $\Phi(\,\cdot\,)$ (with only one entry) can be assumed to be completely known. On the other hand, the numbers that we want to calculate, namely the degree 1 and 2 Gromov-Witten invariants on Q, are

$$\begin{aligned} I_1^Q(1) &= 5^6\,\Phi(1,H_2^6) = \boldsymbol{n_1}, \\ I_2^Q(1) &= (\frac{5}{2})^{11}\,\Phi(1,H_2^{11}) = \boldsymbol{n_2} + \frac{1}{8}n_1. \end{aligned}$$

Here, n_d denotes the number of rational curves of degree d on Q. The fact that the lines on Q also contribute to $I_2^Q(1)$ with a factor of $\frac{1}{8}$ has already been discussed in the end of section 1.3, in particular in lemma 1.3.8.

Our goal is now to translate the equations of corollary 3.3.8 into our new language. This will be done in the following lemma.

Lemma 3.5.3 *For all $p \in V$, $q \in W$ we have*

$$\begin{aligned} \Phi(5pH_2,q) &= \Phi\left(p,(5H_2 + H_3\partial_{H_2} + H_4\partial_{H_3} + M_4\partial_{M_3}(1+C\partial_C))\,q\right) \\ &\qquad + n_1\,\Phi\left(p, \sum_{m\ge 1} mM_3C^{m-1}\frac{\partial_{H_2}^{m+5}}{5^{m+5}(m+5)!}\,q\right), \\ \Phi(5pH_3,q) &= \Phi\left(p,(5H_3 + H_4\partial_{H_2})\,q\right), \\ \Phi(5pH_4,q) &= \Phi\left(p,5H_4\,q\right). \end{aligned}$$

Proof We give the proof for the first equation, the others are completely analogous (and easier). We may assume that p and q are monomials and that the dimension condition $\deg(pq) = 5d$ is satisfied, since otherwise there is nothing to show. Assume for a moment that $q \in V$. As in the definition above, let j be $5d$ minus the number of factors in q. Let $\mathcal{T} = i_Y(p)$, $\mathcal{D} = Q^{\otimes j} \otimes i_Q(q)$, and $\gamma = H \in A^*(Y)$. Then, the left hand side of the recursion equation in corollary 3.3.8 (for $\alpha = (1,\dots,1)$) is by definition equal to

$$I_{d,\alpha}^{Q/Y}(\gamma \cdot Q \otimes \mathcal{T} \,|\, \mathcal{D}) = j!\,\Phi(5pH_2,q).$$

We now consider the right hand side of the equation 3.3.8 and start with the terms (1)

$$\sum_k I_{d,(1,\dots,1)}^{Q/Y}(\mathcal{T} \,|\, (i^*\gamma)\cdot_k \mathcal{D}).$$

If $\mathcal{D} = \mu_1 \otimes \ldots \otimes \mu_s$, then (1) this is a sum over s terms, where in each sum one of the μ_i is replaced according to the rule

$$Q \mapsto i^*\gamma \cdot Q = i^*H, \tag{A}$$

$$i^*H \mapsto i^*\gamma \cdot H = i^*H^2, \tag{B}$$

$$i^*H^2 \mapsto i^*\gamma \cdot H^2 = i^*H^3, \tag{C}$$

$$i^*H^3 \mapsto i^*\gamma \cdot H^3 = i^*H^4 = 0. \tag{D}$$

Translating this into statements about the monomial q, this means that we have e.g. in (C) to replace H_3 with H_4, and the coefficient of this monomial will be the number of factors H_3 in q. This is accomplished by the differential operator $H_4\partial_{H_3}$. Similarly, the contributions from (B) are counted by $H_3\partial_{H_2}$. The terms (D) give no contribution at all. Finally, in (A), we have to distinguish two cases:

- If $j > 0$, then for (A) we have to multiply q with $5H_2$ (note that $i_Q(5H_2) = i^*H$), and the coefficient of this term should be equal to the number of classes Q in $\mathcal{D}$, namely j. However, since the definition of $\Phi(p,q)$ involves a factor of $\frac{1}{j!}$ and multiplying q by $5H_2$ decreases j by one, this automatically produces the desired factor of j. Hence, the contributions from (A) are described by $5H_2q$, i.e. counted by $j!\Phi(5p, H_2q)$.

- If $j = 0$, then there are no terms in (A) at all. Note that $j = 0$ is only possible if $p = 1$ and $q = H_2^{5d}$. The result $5H_2q$ that we got for $j > 0$ therefore yields $\Phi(1, 5H_2 \cdot H_2^{5d}) = \Phi(1, 5H_2^{5d+1})$, which has by definition a different meaning, namely $5I_d^Q(i_Q(q))$. If we look at the equation 3.3.8, we see that this is exactly the term (2) for $\ell = 0$. Hence, in this case $5H_2q$ describes (A) as well as the terms in (2) with $\ell = 0$.

But note that terms in (2) with $\ell = 0$ can only occur only if $j = 0$, i.e. if $p = 1$ and q is a power of H_2, because otherwise there would be non-divisorial classes in the invariant $I_d^Q(\,\cdot\,)$, which is impossible because vdim $\bar{M}_{0,0}(Q,d) = 0$. In summary, we have shown that in any case, the sum of the terms in (1) and those in (2) with $\ell = 0$ is given by

$$j!\Phi(p, (5H_2 + H_3\partial_{H_2} + H_4\partial_{H_3})\,q),$$

still under the assumption that $q \in V$. If now $q \in W \backslash V$, such that it contains a factor M_iC^{m-1} for $i \in \{3,4\}$, the replacement above has to be done in exactly the same way, with the only difference that the equations 3.2.12 give a multiplicity of m. As $(1 + C\partial_C)M_iC^{m-1} = mM_iC^{m-1}$, it follows that for general $p \in V$ and $q \in W$, the sum of the terms in (1) and those in (2) with $\ell = 0$ is given by

$$j!\,\Phi\left(p, (5H_2 + H_3\partial_{H_2} + H_4\partial_{H_3} + M_4\partial_{M_3}(1 + C\partial_C))\,q\right).$$

Now we consider the terms in (2) and (3) of corollary 3.3.8 in the remaining case $\ell = 1$. These can only occur for $d = 2$, and then for fixed $\mathcal{I}$ and $\mathcal{J}$ they give the contribution

$$\sum_{ik} m\, g^{ik}\, I_1^Q(i^*\gamma \otimes i^*\mathcal{T}_0 \otimes \mathcal{D}_0 \otimes T_i)\, I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T}_1 \,|\, T_k \otimes \mathcal{D}_1). \tag{E}$$

As already mentioned above, there can be no non-divisorial classes among $i^*\gamma \otimes i^*\mathcal{T}_0 \otimes \mathcal{D}_0 \otimes T_i$, so it follows that $\mathcal{T}_0 = 1$ (and hence $\mathcal{T}_1 = \mathcal{T}$), and that $T_i = i^*H$, where we have chosen the basis $i^*\{Y, H, H^2, H^3\}$ of $A^*(Q)$. This means that $T_k = i^*H_2$ and $g^{ik} = \frac{1}{5}$ (with no sum any more). Moreover, all classes in $\mathcal{D}_0$ must be of the form $i_Q(H_2) = \frac{1}{5}\, i^*H$.

Denote by $q_0, q_1 \in V$ the monomials with $q_0 q_1 = q$ corresponding to the decomposition of $\mathcal{D}$ in $\mathcal{D}_0$ and $\mathcal{D}_1$. Then the dimension condition for $I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T} \,|\, i^*H^2 \otimes \mathcal{D}_1)$ is $\deg\,(pq_1) = 5 - m$. As $\deg\,(pq) = 10$, it follows that $\deg\, q_0 = 10 - (5 - m) = 5 + m$. As we have just seen that $q_0 \in \mathbb{Q}[H_2]$, it follows that $q_0 = H_2^{m+5}$, such that (E) evaluates to

$$\begin{aligned} & m\frac{1}{5} I_1^Q(i^*\gamma \otimes (\frac{1}{5} i^*H)^{\otimes (m+5)} \otimes i^*H)\, I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T} \,|\, i^*H^2 \otimes \mathcal{D}_1) \\ &\quad = \frac{1}{5^{m+6}}\, m n_1\, I^{Q/Y}_{1,(m,1,\dots,1)}(\mathcal{T} \,|\, i^*H^2 \otimes \mathcal{D}_1). \end{aligned}$$

Moreover, note that $q_1 = q/q_0 = q/(H_2)^{m+5}$, hence $\mathcal{D}_1$ is obtained from $\mathcal{D}$ by deleting $m+5$ classes $\frac{1}{5} i^*H$ corresponding to $i_Y(H_2)$. The choice of which classes to delete gives a combinatorial factor of $\binom{N}{m+5}$, where N denotes the exponent of H_2 in q. Putting all this together, we see that we can write the contribution (E) as

$$\frac{j!}{5^{m+5}}\, m n_1\, \Phi\left(p, M_3 C^{m-1} \frac{\partial_{H_2}^{m+5}}{(m+5)!}\, q\right).$$

Summing this over all possible multiplicities $m \geq 1$ (in fact we only need $1 \leq m \leq 5$ for dimensional reasons) now yields the desired result. □

We now want to write these equations in a from such that it becomes easy to apply them recursively. To do this, we make the following definition:

Definition 3.5.4 *Define the differential operator $f : W \to W$ by*

$$\boldsymbol{f} := \frac{1}{10} H_3 \partial_{H_2}^2 + \frac{1}{5} H_4 \partial_{H_2} \partial_{H_3} - \frac{1}{300} H_4 \partial_{H_2}^3 + \frac{1}{5} M_4 \partial_{M_3} \partial_{H_2} (1 + C\partial_C),$$

and set $\boldsymbol{F} := \exp\,(f) : W \to W$.

Note that F is an invertible operator that preserves degrees and maps V to V. The operator has been designed to make the following lemma work:

Lemma 3.5.5 *For all $p \in V$, $q \in W$ we have*

$$\begin{aligned}
\Phi(pH_2, Fq) &= \Phi(p, FH_2 q) + n_1 \Phi\left(p, \sum_{m \geq 1} mM_3 C^{m-1} \frac{\partial_{H_2}^{m+5}}{5^{m+6}(m+5)!} Fq\right), \\
\Phi(pH_3, Fq) &= \Phi(p, FH_3 q), \\
\Phi(pH_4, Fq) &= \Phi(p, FH_4 q).
\end{aligned}$$

Proof Firstly, we claim that $F = \exp(f_1) \exp(f_2)$ where

$$f_1 = \frac{1}{10} H_3 \partial_{H_2}^2 \quad \text{and} \quad f_2 = \frac{1}{5} H_4 \partial_{H_2} \partial_{H_3} + \frac{1}{150} H_4 \partial_{H_2}^3 + \frac{1}{5} M_4 \partial_{M_3} \partial_{H_2} (1 + C\partial_C).$$

Indeed, this can be verified immediately using the Baker-Campbell-Hausdorff formula stating that $\exp(A) \exp(B) = \exp(A + B + \frac{1}{2}[A, B])$ if $[A, [A, B]] = [B, [A, B]] = 0$.

Now the proof of the lemma is be done by direct computation, using lemma 3.5.3 and the fact that $[A, B^n] = n[A, B]B^{n-1}$ and $[A, \exp(B)] = [A, B] \exp(B)$ if $[B, [A, B]] = 0$. We do it explicitly for the first equation, the other two are proven in the same way. Consider the product $FH_2 q$ and push H_2 through to the left:

$$\begin{aligned}
FH_2 q &= \exp(f_1) \exp(f_2) H_2 q \\
&= \exp(f_1)(H_2 + \frac{1}{5} H_4 \partial_{H_3} + \frac{1}{50} H_4 \partial_{H_2}^2 + \frac{1}{5} M_4 \partial_{M_3} (1 + C\partial_C)) \exp(f_2) q \\
&= (H_2 + \frac{1}{5} H_3 \partial_{H_2} + \frac{1}{5} H_4 \partial_{H_3} + \frac{1}{5} M_4 \partial_{M_3} (1 + C\partial_C)) \exp(f_1) \exp(f_2) q \\
&= (H_2 + \frac{1}{5} H_3 \partial_{H_2} + \frac{1}{5} H_4 \partial_{H_3} + \frac{1}{5} M_4 \partial_{M_3} (1 + C\partial_C)) Fq.
\end{aligned}$$

But now, writing down the first equation of lemma 3.5.3 for Fq instead of Q and inserting the above equation yields the desired result. □

These equations of lemma 3.5.5 are much nicer than those of lemma 3.5.3, since it is easy to apply them recursively:

Corollary 3.5.6 *For any monomials $p, q \in W$ with deg $(pq) = 6$, where at most one of them is not in V, we have the equation $\Phi(pq) = \Phi(p, Fq)$.*

Proof As we consider invariants of degree 1, the additional term $n_1 \Phi(\,\cdot\,)$ in the upper equation of lemma 3.5.5 vanishes. This means that $\Phi(p' H_i, Fq') = \Phi(p', FH_i q')$ for $i = 2, 3, 4$, $p' \in V$, and $q' \in W$. Applying this equation recursively obviously yields

$$\Phi(p'r, Fq') = \Phi(p', Fq'r) \tag{1}$$

for $r \in V$. So, if $q \in V$, we can set $p' = p$, $q' = 1$, and $r = q$ to prove the statement of the corollary. If $q \notin V$ but $p \in V$, then q is of the form $q = M_i C^{m-1} \tilde{q}$, and we conclude that

$$\begin{aligned}
\Phi(p,Fq) &= \Phi(p,F\tilde{q}M_iC^{m-1}) \\
&= \Phi(p\tilde{q},FM_iC^{m-1}) \qquad \text{by (1)} \\
&= \Phi(p\tilde{q},M_iC^{m-1}) \qquad \text{by definition of } F \\
&= \Phi(p\tilde{q}M_iC^{m-1}) \qquad \text{by definition 3.5.1} \\
&= \Phi(pq).
\end{aligned}$$

□

This result already enables us to compute the number of lines on Q:

Corollary 3.5.7 *The number of lines on the quintic threefold is* $n_1 = \Phi(F^{-1}(5H_2)^6) = 2875$.

Proof By corollary 3.5.6,

$$n_1 = \Phi(1,(5H_2)^6) = \Phi(F^{-1}(5H_2)^6,1) = \Phi(F^{-1}(5H_2)^6).$$

But the invariant $\Phi(F^{-1}(5H_2)^6)$ is known (see remark 3.5.2) since it is made up entirely of Gromov-Witten invariants of $\mathbb{P}^4$. The explicit computation using a Maple program is given at the end of this section. □

We now come to the case of conics. The main difference is of course that we have to cope with the additional term in lemma 3.5.5 that corresponds to reducible curves. This can be done using the following operator:

Definition 3.5.8 *Define the differential operator* $K : W \to W$ *by*

$$\boldsymbol{K} := n_1 \sum_{m=1}^{5} \left(mM_3C^{m-1} \frac{\partial_{H_2}^{m+6}}{5^{m+6}(m+6)!} - m^2(m+6)M_4C^{m-1} \frac{\partial_{H_2}^{m+7}}{5^{m+7}(m+7)!} \right).$$

The following lemma should be viewed as an analogue of corollary 3.5.6.

Lemma 3.5.9 *For any* $p \in V$ *with* $\deg\, p = 11$, *we have* $\Phi(p) = \Phi(1,Fp) + \Phi(Kp)$.

Proof We can assume that p is a monomial and write it as $p = H_2^k p'$ with $p' \in \mathbb{Q}[H_3,H_4]$. Then, applying lemma 3.5.5 recursively, we get

$$\begin{aligned}
\Phi(p) &= \Phi(H_2^k, Fp') \\
&= \Phi(H_2^{k-1}, FH_2p') + n_1\, \Phi\left(H_2^{k-1}, \sum_{m\geq 1} mM_3C^{m-1} \frac{\partial_{H_2}^{m+5}}{5^{m+6}(m+5)!} Fp' \right) \\
&= \dots \\
&= \Phi(1,Fp) + n_1 \sum_{i=0}^{k-1} \Phi\left(H_2^{k-1-i}, \sum_{m\geq 1} mM_3C^{m-1} \frac{\partial_{H_2}^{m+5}}{5^{m+6}(m+5)!} FH_2^i p' \right).
\end{aligned}$$

We want to push the operator F through to the left in the second entry of Φ. To do this, we compute by a standard calculation the commutator for $q \in V$

$$[M_3 C^{m-1} \partial_{H_2}^{m+5}, F] q = -\frac{1}{5} m F M_4 C^{m-1} \partial_{H_2} q.$$

Inserting this into the above calculation gives

$$\begin{aligned}
\Phi(p) - \Phi(1, Fp) &= n_1 \sum_{i,m} \Phi\left(H_2^{k-1-i}, m F C^{m-1} (M_3 - \frac{m}{5} M_4 \partial_{H_2}) \frac{\partial_{H_2}^{m+5}}{5^{m+6}(m+5)!} H_2^i p' \right) \\
&= n_1 \sum_{i,m} \Phi\left(H_2^{k-1-i} C^{m-1} (m M_3 - \frac{m^2}{5} M_4 \partial_{H_2}) \frac{\partial_{H_2}^{m+5}}{5^{m+6}(m+5)!} H_2^i p' \right)
\end{aligned}$$

by corollary 3.5.6. We can now perform the sum over i explicitly, since for $a \in \{m+5, m+6\}$

$$\begin{aligned}
\sum_{i=0}^{k-1} H_2^{k-i-1} \frac{\partial_{H_2}^a}{a!} H_2^i &= \sum_{i=0}^{k-1} H_2^{k-i-1} H_2^{i-a} \binom{i}{a} \\
&= H_2^{k-a-1} \sum_{i=0}^{k-1} \binom{i}{a} \\
&= H_2^{k-a-1} \binom{k}{a+1}.
\end{aligned}$$

Continuing the calculation from above, this gives

$$\begin{aligned}
\Phi(p) - \Phi(1, Fp) &= n_1 \sum_m \Phi\Bigg(\frac{m}{5^{m+6}} \binom{k}{m+6} M_3 C^{m-1} H_2^{k-m-6} p' \\
&\qquad - \frac{m^2(m+6)}{5^{m+7}} \binom{k}{m+7} M_4 C^{m-1} H_2^{k-m-7} p' \Bigg) \\
&= n_1 \sum_m \Phi\left(m M_3 C^{m-1} \frac{\partial_{H_2}^{m+6}}{5^{m+6}(m+6)!} p - m^2 (m+6) M_4 C^{m-1} \frac{\partial_{H_2}^{m+7}}{5^{m+7}(m+7)!} p \right) \\
&= \Phi(Kp).
\end{aligned}$$

□

Corollary 3.5.10 *The number of conics on the quintic threefold is*

$$n_2 = \frac{1}{2^{11}} \Phi((1-K) F^{-1} (5H_2)^{11}) - \frac{n_1}{8} = 609250.$$

Proof Apply lemma 3.5.9 to $p = F^{-1}(5H_2)^{11}$ to get

$$\begin{aligned}
I_2^Q(1) &= \frac{1}{2^{11}} \Phi(1,(5H_2)^{11}) \\
&= \frac{1}{2^{11}} \Phi(1, F\,F^{-1}(5H_2)^{11}) \\
&= \frac{1}{2^{11}} (\Phi(F^{-1}(5H_2)^{11}) - \Phi(KF^{-1}(5H_2)^{11})) \\
&= \frac{1}{2^{11}} \Phi((1-K)F^{-1}(5H_2)^{11}).
\end{aligned}$$

As $n_2 = I_2^Q(1) - \frac{n_1}{8}$, the result follows. $\square$

To finish, we give a short Maple program that can be used to do the explicit calculations of the numbers of lines and conics, i.e. of the numbers

$$\begin{aligned}
n_1 &= \Phi(F^{-1}(5H_2)^6) = 2875 \\
\text{and} \quad n_2 &= \frac{1}{2^{11}} \Phi((1-K)F^{-1}(5H_2)^{11}) - \frac{n_1}{8} = 609250.
\end{aligned}$$

```
# A Maple program to compute the numbers of lines and conics on a
# quintic threefold Q using degeneration invariants of Q/P^4

# These are the necessary Gromov-Witten- and degeneration invariants
# of P^4:

# d=1 Gromov-Witten invariants:
inv [           H2^6] := 5:
inv [      H3  *H2^4] := 3:
inv [      H3^2*H2^2] := 2:
inv [      H3^3     ] := 1:
inv [H4        *H2^3] := 1:
inv [H4   *H3  *H2  ] := 1:
inv [H4^2           ] := 1:

# d=1 degeneration invariants:
inv [M3              *H2^4]:= 3:
inv [M3         *H3  *H2^2]:= 2:
inv [M3         *H3^2     ]:= 1:
inv [M3    *H4       *H2  ]:= 1:
inv [M4              *H2^3]:= 1:
inv [M4         *H3  *H2  ]:= 1:
inv [M4    *H4            ]:= 1:
inv [M3*C            *H2^3]:= 6:
inv [M3*C       *H3  *H2  ]:= 5:
inv [M3*C  *H4            ]:= 4:
inv [M4*C            *H2^2]:= 1:
inv [M4*C       *H3       ]:= 1:
inv [M3*C^2          *H2^2]:= 13:
inv [M3*C^2     *H3       ]:= 11:
inv [M4*C^2          *H2  ]:= 2:
inv [M3*C^3          *H2  ]:= 37:
```

```
inv [M4*C^3               ]:= 6:
inv [M3*C^4               ]:= 130:

# d=2 Gromov-Witten invariants:
inv [          H2^11] := 6620:
inv [     H3  *H2^9 ] := 1734:
inv [     H3^2*H2^7 ] := 473:
inv [     H3^3*H2^5 ] := 132:
inv [     H3^4*H2^3 ] := 36:
inv [     H3^5*H2   ] := 10:
inv [H4       *H2^8 ] := 219:
inv [H4  *H3  *H2^6 ] := 67:
inv [H4  *H3^2*H2^4 ] := 21:
inv [H4  *H3^3*H2^2 ] := 6:
inv [H4  *H3^4      ] := 2:
inv [H4^2     *H2^5 ] := 11:
inv [H4^2*H3  *H2^3 ] := 4:
inv [H4^2*H3^2*H2   ] := 1:
inv [H4^3     *H2^2 ] := 1:
inv [H4^3*H3        ] := 0:


# The differential operator f...

f := a -> H3/10*diff(diff(a,H2),H2) + H4/5*diff(diff(a,H2),H3)
            - H4/300*diff(diff(diff(a,H2),H2),H2)
            + (1+C*diff(a,C))*M4/5*diff(diff(a,H2),M3):

# ... and F^(-1) ...

FI := proc (p) local x,s,i;
  x := p: s := 0: i := 0:
  while (x <> 0) do s := s+x: i := i+1: x := -f(x)/i: od:
  expand (s):
end:

# ... and K.

K := proc (p) local s,m,dfac;

  dfac := proc (p,n) local i;
    p:
    for i from 1 to n do diff(",H2)/5/i: od:
    expand ("):
  end:

  s := 0:
  for m from 1 to 5 do
    s := s + M3*C^(m-1)*m*dfac(p,m+6)
          - M4*C^(m-1)*m^2*(m+6)*dfac(p,m+7):
  od:
  expand (s):
end:

# The functional Phi:
```

```
Phi := proc (p) local i,co,var;
  co := [coeffs (expand (p),{H2,H3,H4,M3,M4,C},'var')]:
  var := [var]:
  sum (co[i]*inv[var[i]],i=1..nops(var)):
end:

# Now calculate the invariants:

# lines: (result 2875)
N1 := Phi ( FI ((5*H2)^6) );

# conics: (result 609250)
N2 := (Phi ( FI ((5*H2)^11) - N1 * K (FI((5*H2)^11)) ) - 2^8 * N1)
      / 2^11;
```

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