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Stable Equilibria to Parabolic Systems in Unbounded Domains

Joachim ESCHER† *and Zhaoyang YIN*‡

†*Institute for Applied Mathematics, University of Hannover*
D-30167 Hannover, Germany
E-mail: escher@ifam.uni-hannover.de

‡*Department of Mathematics, Zhongshan University, 510275 Guangzhou*
E-mail: mcsyzy@zsu.edu.cn

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Abstract

We investigate weakly coupled semilinear parabolic systems in unbounded domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ with polynomial nonlinearities. Three sufficient conditions are presented to ensure the stability of the zero solution with respect to non-negative H^2-perturbations.

1 Introduction

In this paper, we study the following parabolic system

$$\begin{cases} u_t - \Delta u = f(u,v), & \text{in} \quad (0,\infty)\times\Omega, \\ v_t - \Delta v = g(u,v), & \text{in} \quad (0,\infty)\times\Omega, \\ u(t,x) = v(t,x) = 0, & \text{on} \quad [0,\infty)\times\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an unbounded domain in $\mathbb{R}^2$ or in $\mathbb{R}^3$ with a sufficiently smooth boundary and where f and g are polynomials in u, v such that $f(u,v) = g(u,v) = 0$.

System (1.1) is a widely used mathematical model for many chemical, physical, biological, or ecological phenomena. A simple situation arising in population dynamics will be discussed as Example 1 in Section 4. For details on physical and chemical models involving more general reaction-diffusion systems we refer to [4, 22, 25].

Many papers are devoted to the study of system (1.1) either in bounded domains or as a Cauchy problem in the whole of $\mathbb{R}^n$ (see [3, 5, 11, 12, 14, 16, 17, 18, 19, 20, 22, 25, 27] and the citations therein). They mainly discuss existence and uniqueness of local solutions, positivity of solutions, global existence, and blow-up behavior of solutions.

On unbounded domains with an unbounded inradius $\rho(\Omega) := \sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)$, Poincaré's inequality does not hold, cf. [27, Theorem 2.1]. Hence the spectrum of the linear part of (1.1) contains in general 0 as a cluster point so that in these situations the principle of linearized stability is not applicable. In fact, to the best of our knowledge, it seems

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that the stability properties of the trivial solution to system (1.1) in the case of general unbounded domains have so far not yet been investigated.

The purpose of this paper is to present several results ensuring the stability of the zero solution to the system (1.1). Our methods do not only rely on maximum principles and comparison principles for parabolic systems as presented in [13, 19] but also on recent abstract stability results for equilibria of parabolic evolution equations, cf. [8, 9, 10]. These methods permit us to obtain several new results on the stability of equilibria of the system (1.1) in unbounded domains under quite general assumptions on the nonlinearities.

Notation. Throughout this paper we assume that Ω is an unbounded domain in $\mathbb{R}^2$ or in $\mathbb{R}^3$. If the boundary $\partial\Omega$ of Ω is not empty it assumed to be uniformly C^4-regular, see [1] page 67 or [6] page 28 for a precise definition. Let f and g be polynomials of the form

$$f(u,v)=\sum_{3\le j+k\le d_f} a_{jk}u^j v^k, \tag{1.2}$$

$$g(u,v)=\sum_{3\le j+k\le d_g} b_{jk}u^j v^k, \tag{1.3}$$

where d_f, $d_g \ge 3$ denote the degrees and a_{jk}, $b_{jk} \in \mathbb{R}$ for $(j,k) \in \mathbb{N}^2$ with $3 \le j+k \le \max\{d_f, d_g\}$ the coefficients of f and g, respectively.

In the following we identify $L^2(\Omega;\mathbb{R}^2)$ with $L^2(\Omega)\times L^2(\Omega)$. Similarly we denote by $H^m(\Omega;\mathbb{R}^2) \cong H^m(\Omega)\times H^m(\Omega)$ and $H_0^m(\Omega;\mathbb{R}^2) \cong H_0^m(\Omega)\times H_0^m(\Omega)$ the usual Sobolev spaces based on $L^2(\Omega;\mathbb{R}^2)$ so that $H^0(\Omega;\mathbb{R}^2)=L^2(\Omega;\mathbb{R}^2)$. The scalar product $(\cdot,\cdot)_m$ in $H^m(\Omega;\mathbb{R}^2)$ is given by

$$(w,z)_m := \sum_{|\mu|\le m}(D^\mu w, D^\mu z)_0 \quad \text{with} \quad (w,z)_0 := \int_\Omega (w,z)_{\mathbb{R}^2}\,dx.$$

As usual let $C^0(\overline{\Omega};\mathbb{R}^2)$ denote the Banach space of all bounded and uniformly continuous vector functions $w=(u,v):\overline{\Omega}\to\mathbb{R}^2$ with the norm

$$\| w \|_{C^0} := \sup_{x\in\Omega} | w(x) | = \sup_{x\in\Omega}(| u(x) | + | v(x) |).$$

Let also $C^m(\overline{\Omega};\mathbb{R}^2)$ with $m\in\mathbb{N}$ denote the Banach space of all $w\in C^0(\overline{\Omega};\mathbb{R}^2)$ having derivatives up to order m all belonging to $C^0(\overline{\Omega};\mathbb{R}^2)$. The norm in $C^m(\overline{\Omega};\mathbb{R}^2)$ is given by $\| w \|_{C^m} = \sum_{|\mu|\le m} \| D^\mu w(x) \|_{C^0}$. Moreover, given $\alpha\in(0,1)$ and $m\in\mathbb{N}$, let $C^{m+\alpha}(\overline{\Omega};\mathbb{R}^2)$ denote the Banach space of all $w\in C^m(\overline{\Omega};\mathbb{R}^2)$ such that $D^\mu w$ with $| \mu | = m$ are uniformly α-Hölder continuous on $\overline{\Omega}$. The norm in $C^{m+\alpha}(\overline{\Omega};\mathbb{R}^2)$ is given by

$$\| u \|_{C^{m+\alpha}} := \| w \|_{C^m} + \sum_{|\mu|\le m}\Big(\sup_{x\ne y}\frac{| D^\mu w(x) - D^\mu w(y) |}{| x-y |^\alpha}\Big).$$

For brevity we write $\| \cdot \|_{H^m}$ for the Hilbert norm in $H^m(\Omega;\mathbb{R}^2)$ induced by $(\cdot,\cdot)_m$. Given two Banach spaces X and Y, let $L(X,Y)$ denote the Banach space of all bounded linear operators from X to Y with the usual operator norm $\| \cdot \|_{L(X,Y)}$. For convenience we let $L(X) := L(X,X)$ and $\| \cdot \|_{L(X)} := \| \cdot \|_{L(X,Y)}$ if $X=Y$. We write $X \hookrightarrow Y$ if X is continuously injected in Y, i.e. $X\subset Y$ and $id_X \in L(X,Y)$. The Laplace operator in the

distributional sense is denoted by $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$. Finally, given (ξ_1, ξ_2) and $(\eta_1, \eta_2) \in \mathbb{R}^2$, we write $(\xi_1, \xi_2) \leq (\eta_1, \eta_2)$ if $\xi_1 \leq \eta_1$ and $\xi_2 \leq \eta_2$.

The remaining part of this paper is organized as follows. In Section 2 we introduce the functional analytic frame and state the main result. Section 3 is devoted to the proof of the main result. We present some applications of our main result in the last section.

2 The analytic frame and main result

In this section we firstly introduce the functional analytic frame in which we treat system (1.1). Then we present main result of the paper.

Our starting point is the following unbounded operator A_0 in $E_0 = L^2(\Omega; \mathbb{R}^2)$ defined by

$$dom(A_0) := H^2(\Omega; \mathbb{R}^2) \cap H_0^1(\Omega; \mathbb{R}^2), \quad A_0 w = (\Delta u, \Delta v), \quad w = (u, v) \in dom(A_0). \tag{2.1}$$

Then A_0 is a non-positive selfadjoint operator on E_0, cf. [6]. However, since $(f(u, v), g(u, v))$ is not a mapping from E_0 into itself, the space E_0 is not suited to deal with the system (1.1). In order to overcome this difficulty, we follow the method in [10] to take as underlying space the Hilbert space $E_1 = H^2(\Omega; \mathbb{R}^2) \cap H_0^1(\Omega; \mathbb{R}^2)$ endowed with the scalar product

$$(w, z)_{E_1} = ((A_0 - 1)w, (A_0 - 1)z)_{E_0}. \tag{2.2}$$

By the open mapping theorem we obtain that the norm on E_1 induced by the above scalar product is equivalent to the norm $\| \cdot \|_{H^2}$, i.e. there are positive constants C_0 and C_1 such that

$$C_0 \parallel w \parallel_{H^2} \leq \parallel (A_0 - 1)w \parallel_{L^2} \leq C_1 \parallel w \parallel_{H^2}, \quad w \in E_1. \tag{2.3}$$

It follows that the restriction of A_0 to E_1 induces an unbounded operator A_1 in E_1 according to

$$dom(A_1) := dom(A_0^2), \quad A_1 w := A_0 w = (\Delta u, \Delta v), \quad w = (u, v) \in dom(A_1). \tag{2.4}$$

As was shown in Section 2.2 in [26], we have

$$dom(A_1) = \{w = (u, v) \in H^4(\Omega; \mathbb{R}^2) \cap H_0^1(\Omega; \mathbb{R}^2)\,;\, (\Delta u, \Delta v) \in H_0^1(\Omega; \mathbb{R}^2)\}. \tag{2.5}$$

Similarly the graph norm of A_1 is an equivalent norm on $dom(A_1)$, i.e.

$$C_2 \parallel w \parallel_{H^4} \leq \parallel (A_1 - 1)w \parallel_{H^2} \leq C_3 \parallel w \parallel_{H^4}, \quad w \in dom(A_1), \tag{2.6}$$

where C_2 and C_3 are positive constants.

It also follows that A_1 is a non-positive selfadjoint operator in E_1. Hence both A_0 and A_1 generate analytic C_0-semigroups $\{W_0(t),\, t \geq 0\}$ on E_0 and $\{W_1(t),\, t \geq 0\}$ on E_1, respectively. Moreover, representing W_0 and W_1 by means of the spectral resolution of A_0 and A_1, respectively, it is not too difficult to see that

$$W_1(t)z = W_0(t)z \quad \text{for} \quad t \geq 0,\, z \in E_1. \tag{2.7}$$

Note that $H^2(\Omega) \cap H_0^1(\Omega)$ is a Banach algebra and that $(f(0,0), g(0,0)) = (0,0)$. Hence the mapping $E_1 \longrightarrow E_1$, $(u,v) \mapsto (f(u,v), g(u,v))$ is analytic cf. Lemma 3 in [26]. This allows us to regard the system (1.1) as the following abstract evolution equation in E_1:

$$w_t = A_1 w + F(w), \quad w(0) = w_0 = (u_0, v_0) \in E_1, \tag{2.8}$$

where $F(w) := (f(u,v), g(u,v))$. Then standard theorems for abstract evolution equations, see Chapter 6 in [23], yield the existence of a unique strong solution to the above equation in E_1. In fact, the abstract Cauchy problem in E_1 provides the basic frame in which we prove our stability result.

However, in order to derive suitable a priori estimates for solutions to (2.8), we also need to formulate system (1.1) in a different functional analytic setting. This was introduced earlier in [20, 21, 26]. Given $\alpha \in (0,1)$, we define the spaces

$$\hat{C}^{m+\alpha}(\overline{\Omega}; \mathbb{R}^2) := \begin{cases} \{w \in C^{m+\alpha}(\overline{\Omega}; \mathbb{R}^2);\ w = 0 \text{ on } \partial\Omega\} & \text{if } m = 0, 1, \\ \{w \in C^{m+\alpha}(\overline{\Omega}; \mathbb{R}^2);\ w = (\Delta u, \Delta v) = 0 \text{ on } \partial\Omega\} & \text{if } m = 2, 3. \end{cases} \tag{2.9}$$

Obviously $\hat{C}^{m+\alpha}(\overline{\Omega}; \mathbb{R}^2) \subset C^{m+\alpha}(\overline{\Omega}; \mathbb{R}^2)$ is a Banach space. Let

$$\begin{pmatrix} \Delta_{2+\alpha} & 0 \\ 0 & \Delta_{2+\alpha} \end{pmatrix} :\ \hat{C}^{2+\alpha}(\overline{\Omega}; \mathbb{R}^2) \subset \hat{C}^0(\overline{\Omega}; \mathbb{R}^2) \longrightarrow \hat{C}^0(\overline{\Omega}; \mathbb{R}^2),$$

$w \mapsto (\Delta u, \Delta v)$ be the Laplacian on $\hat{C}^0(\overline{\Omega}; \mathbb{R}^2)$ restricted to $\hat{C}^{2+\alpha}(\overline{\Omega}; \mathbb{R}^2)$. It follows that the operator $\begin{pmatrix} \Delta_{2+\alpha} & 0 \\ 0 & \Delta_{2+\alpha} \end{pmatrix}$ is closable and its closure A_C generates a holomorphic semigroup $\{W_C(t);\ t \geq 0\}$ on $\hat{C}^0(\overline{\Omega}; \mathbb{R}^2)$, see Theorem 2.4 in [21]. Moreover the domain of A_C can be characterized as

$$dom(A_C) = \{w \in \cap_{p \geq 1} W^2_{p,loc}(\Omega; \mathbb{R}^2);\ w,\ \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} w\ \in \hat{C}^0(\overline{\Omega}; \mathbb{R}^2)\},$$

see Lemma 4.2 in [9]. Since $(f,g) : \hat{C}^0(\overline{\Omega}; \mathbb{R}^2) \to \hat{C}^0(\overline{\Omega}; \mathbb{R}^2)$ is smooth, we can regard the system (1.1) as the following evolution equation in $\hat{C}^0(\overline{\Omega}; \mathbb{R}^2)$:

$$w_t = A_C w + F(w), \quad w(0) = w_0 = (u_0, v_0). \tag{2.10}$$

Then well-known results for abstract evolution equation, see Chapter 6 in [23], ensure the existence of a unique strong solution z of (2.10) on the maximal interval of existence $[0, t_C^+)$. Moreover, a standard continuation argument yields the following result:

$$t_C^+ < \infty \quad \text{implies} \quad \overline{\lim_{t \uparrow t_C^+}} \parallel z(t) \parallel_{C^0} = \infty. \tag{2.11}$$

By Sobolev's embedding theorem we have

$$dom(A_0) \hookrightarrow \hat{C}^{\alpha}(\overline{\Omega}; \mathbb{R}^2) \quad \text{and} \quad dom(A_1) \hookrightarrow \hat{C}^{2+\alpha}(\overline{\Omega}; \mathbb{R}^2), \tag{2.12}$$

provided $\alpha \in (0, \frac{1}{2})$, cf. [1]. Hence it follows that $dom(A_1) \subset dom(A_C)$. Therefore, given $w_0 \in dom(A_1)$, we can solve (2.8) and (2.10) with the same initial data w_0. Our next result clarifies the relation of the corresponding solutions of (2.8) and (2.10), respectively. For simplicity we write in the following $X := \hat{C}^0(\Omega, \mathbb{R}^2)$.

Lemma 2.1. *Given $w_0 \in dom(A_1)$, let $w \in C^1([0,t_1^+),E_1)$ be the strong solution of (2.8) on the maximal interval of existence $[0,t_1^+)$. Moreover, let $z \in C^1([0,t_C^+),X)$ be the strong solution of (2.10) on the maximal interval of existence $[0,t_C^+)$. Then $t_1^+ = t_C^+$ and $w(t) = z(t)$ for $t \in [0,t_1^+)$.*

Proof: The proof of Lemma 2.1 is similar to that of Theorem 1 in [9] and so we omit it here. □

In the following, given $w_0 \in dom(A_1)$, we denote the maximal existence time of (2.8) (or of (2.10)) by $t^+(w_0)$. Moreover, we say that the equilibrium $w = (u,v) = 0$ of (2.8) is *positively Ljapunov stable* if it is Ljapunov stable under non-negative perturbations in $H^2(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2)$, i.e. if there is a $\tau > 0$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: Given $w_0 \in H^2(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2)$ with $\| w_0 \|_{H^2} \leq \delta$ and $w_0 \geq 0$, the solution w of (2.8) with initial data $w(0) = w_0$ exists globally and satisfies $\| w \|_{H^2} \leq \varepsilon$ for all $t \geq \tau$.

We now present the main result of this paper:

Theorem 2.2. *Let f and g be polynomials of the form (1.2) and (1.3) and assume that one of the following conditions holds true:*

(i) Given $u \in \mathbb{R}_+$, $f(u,v)$ is increasing in v on $\mathbb{R}_+$, given $v \in \mathbb{R}_+$, $g(u,v)$ is increasing in u on $\mathbb{R}_+$, and there exist $\xi_1 > 0$, $\xi_2 > 0$ such that

$$f(\xi_1,\xi_2) = g(\xi_1,\xi_2) = 0.$$

(ii) Given $u \in \mathbb{R}_+$, $f(u,v)$ is decreasing in v on $\mathbb{R}_+$, given $v \in \mathbb{R}_+$, $g(u,v)$ is increasing in u on $\mathbb{R}_+$, and there exist $\xi_1 > 0$, $\xi_2 > 0$ such that

$$f(\xi_1,0) = f(0,\xi_2) = g(\xi_1,\xi_2) = 0.$$

(iii) Given $u \in \mathbb{R}_+$, $f(u,v)$ is decreasing in v on $\mathbb{R}_+$, given $v \in \mathbb{R}_+$, $g(u,v)$ is decreasing in u on $\mathbb{R}_+$, and there exist $\xi_1 > 0$, $\xi_2 > 0$ such that

$$f(\xi_1,0) = f(0,\xi_2) = g(\xi_1,0) = g(0,\xi_2) = 0.$$

Then the trivial solution $w = (0,0)$ of (1.1) is positively Ljapunov stable.

Remarks: (a) If $f(u,v)$ is independent of v and $g(u,v)$ is independent of u (this means that the system (1.1) is decoupled), then the result of Theorem 2.2 covers recent results of Theorem 3 presented in [9] and in Theorem 2.2 in [10]. Moreover Theorem 2.1 in [10] shows that the zero solution of (1.1) is in general unstable if the degree of the polynomials f and g is two. If f and g contain linear terms then the stability of the trivial solution is decidable by means of spectral methods.
(b) We do not know whether or not the assumptions on the quasi-monotonicity of f and g in Theorem 2.2 can be relaxed.
(c) The assumptions on the existence of positive roots of the polynomials f and g in Theorem 2.2 may be relaxed, using an approach similar to the one devised in [7]. We will investigate this in a forthcoming paper.
(d) The result of Theorem 2.2 is also true for more general uniformly strongly elliptic

operators in divergence form, $div(a(x)\nabla u)$ and $div(b(x)\nabla v)$, with bounded smooth coefficients instead of Δu and Δv in (1.1), respectively.
(e) Finally, we mention that Theorem 2.2 can easily be generalized to systems of N equations with quasi-monotonic polynomials.

3 Proof of the main result

In this section we prove Theorem 2.2. To do so we firstly introduce the following abstract stability result.

Theorem 3.1. *Given $w_0 \in E_1$, let $t^+(w_0)$ be the maximal existence time of the corresponding solution w to (2.8) with the initial data w_0. Assume that:*
(i) There exists a $\delta_0 > 0$ such that $t^+(w_0) = \infty$, if $\| w_0 \|_{E_1} \leq \delta_0$.
(ii) There exists a $k_0 > 0$ such that

$$\| w(t+2) \|_{E_1} \leq k_0 \| w(t) \|_{E_0}, \quad \forall t \geq 0.$$

(iii) There exists $\varepsilon_0 > 0$ such that

$$(\tilde{w},\ F(\tilde{w}) - A_0\tilde{w})_0 \leq 0, \quad \text{if } \| \tilde{w} \|_{E_1} \leq \varepsilon_0.$$

Then $(0,0)$ is E_1-Ljapunov stable.

Proof: Let $I_k = [2k, 2k+2]$, $k \in \mathbb{N}$. Given $\varepsilon \in (0, \varepsilon_0)$, choose $\delta > 0$ such that $k_0\delta \leq \varepsilon$. By (i) it suffices to show that there is a $\delta_0 \in (0, \varepsilon)$ such that for every $k \in \mathbb{N}$

$$(S_k) \qquad \| w(t) \|_{E_1} \leq \varepsilon \quad \text{and} \quad \| w(t) \|_{E_0} \leq \delta, \quad \forall t \in I_k,$$

provided $\| w_0 \|_{E_1} \leq \delta_0$.

We prove the above assertion by induction.
Note that $F(0,0) = (0,0)$. Hence the continuous dependence of the corresponding solution to (2.8) with respect to the initial data implies that (S_0) is true.

Assume that (S_k) is true. We prove that (S_{k+1}) holds true as well.
Pick $t \in I_{k+1}$. Then $s := t - 2 \in I_k$ and, by (ii) and the hypothesis (S_k), we have

$$\| w(t) \|_{E_1} = \| w(s+2) \|_{E_1} \leq k_0 \| w(s) \|_{E_0} \leq k_0\delta \leq \varepsilon.$$

Furthermore, using (2.8) we get, in view of (iii), the fact that $\varepsilon < \varepsilon_0$, and the hypothesis (S_k):

$$\begin{aligned} \frac{d}{dt} \| w(t) \|_{E_0}^2 &= 2(w(t),\ w'(t))_{E_0} \\ &= 2(w(t),\ F(w(t)) - A_0(w(t)))_{E_0} \leq 0. \end{aligned}$$

Thus $\| w(t) \|_{E_0}$ is non-increasing on $I_{k+1} = [2k+2, 2k+4]$, and we obtain

$$\| w(t) \|_{E_0} \leq \| w(2k+2) \|_{E_0} \leq \delta.$$

This shows that (S_{k+1}) is true and completes the proof of Theorem 3.1. □

In order to prove Theorem 2.2, in view of Theorem 3.1, we have to verify the three assumptions (i)-(iii) of Theorem 3.1. We firstly need the following important comparison principle for the system (1.1).

Lemma 3.2. *[19] Let $\Omega_T = [0,T) \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$ and suppose that there is a 6-tuple $U = \{u, v, \underline{u}, \underline{v}, \overline{u}, \overline{v}\}$ of functions on Ω_T such that $U, U_t, U_{x_j x_k}$, $1 \le j,k \le n$ are bounded and continuous on $\overline{\Omega}_T$ and such that*

$$(\underline{u}(0,x), \underline{v}(0,x)) \le (u(0,x), v(0,x)) \le (\overline{u}(0,x), \overline{v}(0,x)), \quad \forall x \in \Omega,$$
$$(\underline{u}(t,x), \underline{v}(t,x)) \le (\overline{u}(t,x), \overline{v}(t,x)), \qquad \forall (t,x) \in \Omega_T.$$

Moreover let

$$\underline{\alpha} := \inf_{(t,x)\in\Omega_T} \underline{u}(t,x), \qquad \overline{\alpha} := \sup_{(t,x)\in\Omega_T} \overline{u}(t,x),$$
$$\underline{\beta} := \inf_{(t,x)\in\Omega_T} \underline{v}(t,x), \qquad \overline{\beta} := \sup_{(t,x)\in\Omega_T} \overline{v}(t,x)$$

and assume that one of the following conditions holds true:
(i) $\tilde{f}(u,v)$ is increasing in v on $(\underline{\beta}, \overline{\beta})$ for all $u \in (\underline{\alpha}, \overline{\alpha})$, $\tilde{g}(u,v)$ is increasing in u on $(\underline{\alpha}, \overline{\alpha})$ for all $v \in (\underline{\beta}, \overline{\beta})$, and

$$\underline{u}_t - \Delta\underline{u} - \tilde{f}(\underline{u}, \underline{v}) \le u_t - \Delta u - \tilde{f}(u,v) \le \overline{u}_t - \Delta\overline{u} - \tilde{f}(\overline{u}, \overline{v}),$$
$$\underline{v}_t - \Delta\underline{v} - \tilde{g}(\underline{u}, \underline{v}) \le v_t - \Delta v - \tilde{g}(u,v) \le \overline{v}_t - \Delta\overline{v} - \tilde{g}(\overline{u}, \overline{v})$$

for all $(t,x) \in \Omega_T$.
(ii) $\tilde{f}(u,v)$ is decreasing in v on $(\underline{\beta}, \overline{\beta})$ for all $u \in (\underline{\alpha}, \overline{\alpha})$, $\tilde{g}(u,v)$ is increasing in u on $(\underline{\alpha}, \overline{\alpha})$ for all $v \in (\underline{\beta}, \overline{\beta})$, and

$$\underline{u}_t - \Delta\underline{u} - \tilde{f}(\underline{u}, \overline{v}) \le u_t - \Delta u - \tilde{f}(u,v) \le \overline{u}_t - \Delta\overline{u} - \tilde{f}(\overline{u}, \underline{v}),$$
$$\underline{v}_t - \Delta\underline{v} - \tilde{g}(\underline{u}, \underline{v}) \le v_t - \Delta v - \tilde{g}(u,v) \le \overline{v}_t - \Delta\overline{v} - \tilde{g}(\overline{u}, \overline{v})$$

for all $(t,x) \in \Omega_T$.
(iii) $\tilde{f}(u,v)$ is decreasing in v on $(\underline{\beta}, \overline{\beta})$ for all $u \in (\underline{\alpha}, \overline{\alpha})$, $g(u,v)$ is decreasing in u on $(\underline{\alpha}, \overline{\alpha})$ for all $v \in (\underline{\beta}, \overline{\beta})$, and

$$\underline{u}_t - \Delta\underline{u} - \tilde{f}(\underline{u}, \overline{v}) \le u_t - \Delta u - \tilde{f}(u,v) \le \overline{u}_t - \Delta\overline{u} - \tilde{f}(\overline{u}, \underline{v}),$$
$$\underline{v}_t - \Delta\underline{v} - \tilde{g}(\overline{u}, \underline{v}) \le v_t - \Delta v - \tilde{g}(u,v) \le \overline{v}_t - \Delta\overline{v} - \tilde{g}(\underline{u}, \overline{v})$$

for all $(t,x) \in \Omega_T$.
Then $(\underline{u}(t,x), \underline{v}(t,x)) \le (u(t,x), v(t,x)) \le (\overline{u}(t,x), \overline{v}(t,x))$ in Ω_T.

For the proof of Lemma 3.2 we refer to the main results in [19]. □

Remark: Note that Lemma 3.2 is a slightly modified version of Theorem 4.5, Theorem 5.3, and Theorem 6.3 in [19]. However, scrutinizing the proofs of Theorem 2.9, Theorem 3.5, and Theorem 4.1 in [19], we find that the assertion of Lemma 3.2 holds true.

Lemma 3.3. *Assume that the conditions of Theorem 2.2 are satisfied. Let $w(t)$ be the strong solution to (2.8) with the initial data w_0. Then there exist $\delta_0 > 0$ and $M \ge 0$ such that $t^+(w_0) = \infty$ and $\sup_{t\ge 0} \| w(t) \|_X \le M$, provided $\| w_0 \|_{E_1} \le \delta_0$ and $w_0 \ge (0,0)$.*

Proof: Note that $E_1 \hookrightarrow C^0(\overline{\Omega};\mathbb{R}^2)$ by Sobolev's embedding theorem. Hence there is a positive constant C such that

$$\| \tilde{w} \|_{C^0} \leq C \| \tilde{w} \|_{E_1} \quad \text{for all} \quad \tilde{w} \in E_1.$$

Set $\delta_0 := \frac{1}{C}\min\{\xi_1,\xi_2\}$ and pick $w_0 \in dom(A_1)$ with $\| w_0 \|_{E_1} < \delta_0$ and $w_0 \geq (0,0)$. Then $(0,0) \leq w(0) \leq (\xi_1,\xi_2)$ and Lemma 3.2 and the boundary conditions in (1.1) imply that

$$(0,0) \leq w(t,x) \leq (\xi_1,\xi_2) \quad \text{for all} \quad (t,x) \in [0,t^+(w_0)) \times \overline{\Omega}.$$

Invoking (2.11) and Lemma 2.1 we get the assertion. □

We now prove that the assumption (ii) of Theorem 3.1 is true, provided the solution to (2.8) is bounded a priori in X.

Lemma 3.4. *Assume that w is a global strong solution to (2.8) and that*

$$\sup_{t\geq 0} \| w(t) \|_X \leq M, \tag{3.1}$$

for some $M > 0$. Then there exists a $k_0 > 0$ such that

$$\| w(t+2) \|_{E_1} \leq k_0 \| w(t) \|_{E_0}, \quad \forall t \geq 0.$$

Proof: Note that f and g are polynomials satisfying (1.2) and (1.3), respectively. Thus there are polynomials, q_1, q_2, q_3, and q_4, such that

$$F(\tilde{w}) = (f(\tilde{u},\tilde{v}), g(\tilde{u},\tilde{v})) = \begin{pmatrix} q_1(\tilde{u}) & q_2(\tilde{u},\tilde{v}) \\ q_3(\tilde{u},\tilde{v}) & q_4(\tilde{v}) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

for all $\tilde{w} = (\tilde{u},\tilde{v}) \in E_1$. Let $w = (u,v) \in C^1(\mathbb{R}_+, E_1)$ be a global solution to (2.8) and set

$$Q(t) := \begin{pmatrix} q_1(u(t)) & q_2(u(t),v(t)) \\ q_3(u(t),v(t)) & q_4(v(t)) \end{pmatrix}, \quad t \in \mathbb{R}_+. \tag{3.2}$$

Using the fact that pointwise multiplication maps $L^\infty(\Omega) \times L^2(\Omega)$ bilinearly into $L^2(\Omega)$, it is not difficult to see that (3.1) implies that $Q \in C^1(\mathbb{R}_+; L(E_0))$ and that there is a $M_0 > 0$ such that

$$\| Q(t) \|_{L(E_0)} \leq M_0 \quad \text{for} \quad t \geq 0. \tag{3.3}$$

Clearly the Fréchet derivative of $F \in C^\infty(X,X)$ is bounded on bounded subsets of X. Therefore assumption (3.1) and Proposition 4.1 in [10] imply that there is $C > 0$ such that

$$\| \frac{d}{dt} w(t) \|_X \leq C \quad \text{for} \quad t \geq 0. \tag{3.4}$$

Note further that

$$\frac{d}{dt}Q(t) = \begin{pmatrix} \partial q_1(u(t))u'(t) & \Big(\partial_u q_2(u(t),v(t))u'(t) \\ & +\partial_v q_2(u(t),v(t))v'(t)\Big) \\ \Big(\partial_u q_3(u(t),v(t))u'(t) & \partial q_4(v(t))v'(t) \\ +\partial_v q_3(u(t),v(t))v'(t)\Big) & \end{pmatrix}$$

for $t \geq 0$. Thus we conclude from (3.1) and (3.4) that there is a $M_1 > 0$ such that

$$\left\| \frac{d}{dt} Q(t) \right\|_{L(E_0)} \leq M_1 \quad \text{for} \quad t \geq 0. \tag{3.5}$$

Combining this with (3.3), we find

$$\| Q(t) \|_{L(E_0)} + \| \frac{d}{dt} Q(t) \|_{L(E_0)} \leq C(M) \quad \text{for} \quad t \geq 0. \tag{3.6}$$

Since w is the unique strong to (2.8) with the initial data w_0 and $F(w(t)) = Q(t)w(t)$ for $t \geq 0$, it follows that w is also a mild solution to

$$\frac{dz}{dt} = (A_0 + Q(t))z, \quad t \in \mathbb{R}_+. \tag{3.7}$$

Let $\{U(t,\tau);\ 0 \leq \tau \leq t \leq s \leq \tau + 2\}$ be the evolution operator generated by the family $\{A_0 + Q(t);\ t \in [\tau, \tau + 2]\}$. Then (3.6) and Theorem II.5.1.1 in [2] yield

$$\max_{t \in [\tau, \tau+2]} \| U(\tau, t) \|_{L(E_0)} \leq C,$$

where the constant $C > 0$ is independent of τ. Since $w(t) = U(\tau, t)w(\tau)$ for $t \in [\tau, \tau + 2]$, it follows that

$$\| w(t) \|_{E_0} \leq C \| w(\tau) \|_{E_0}, \quad t \in [\tau, \tau + 2]. \tag{3.8}$$

Moreover, (3.6) and the estimate (A.8) in [9] imply that there exists a constant $C(M_0) > 0$ such that

$$\| A_0 w(\tau + 2) \|_{E_0} \leq C(M_0) \max_{t \in [\tau, \tau+2]} \| w(t) \|_{E_0} \leq C. \tag{3.9}$$

By (3.8), (3.9) and (2.3) we obtain the statement of the Lemma. □

We now prove that the assumption (iii) in Theorem 3.1 is true.

Lemma 3.5. *There exists $\varepsilon_0 > 0$ such that*

$$(w,\ F(w) - A_0 w)_0 \leq 0, \quad \textit{provided } \| w \|_{E_1} \leq \varepsilon_0.$$

Proof: By (2.8) we have

$$\begin{aligned} &(w,\ F(w) - A_0 w)_0 \\ &= (\Delta u, u)_0 + (\Delta v, v)_0 + (f(u,v), u)_0 + (g(u,v), v)_0 \\ &= - \| \nabla u \|_{L^2}^2 - \| \nabla v \|_{L^2}^2 + (f(u,v), u)_0 + (g(u,v), v)_0. \end{aligned} \tag{3.10}$$

Using Young's inequality in the form

$$ab \leq \frac{j}{j+k} a^{\frac{j+k}{j}} + \frac{k}{j+k} b^{\frac{j+k}{k}}, \quad a,\ b \geq 0, \quad j,\ k > 0, \tag{3.11}$$

we conclude from (1.2) that

$$| u\, f(u,v) | \leq d_0 \mid u \mid^4 + d_1 \mid u \mid^5 + d_2 \mid u \mid^6 + \cdots + e_0 \mid v \mid^4 + e_1 \mid v \mid^5 + e_2 \mid v \mid^6 \cdots, \quad (3.12)$$

where $d_i \geq 0$ and $e_i \geq 0$, $i = 0, 1, \cdots$, are constants which depend only on the coefficients a_{jk} of f. Similarly we have

$$| v\, g(u,v) | \leq l_0 \mid u \mid^4 + l_1 \mid u \mid^5 + l_2 \mid u \mid^6 + \cdots + m_0 \mid v \mid^4 + m_1 \mid v \mid^5 + m_2 \mid v \mid^6 + \cdots, \quad (3.13)$$

where $l_i \geq 0$ and $m_i \geq 0$, $i = 0, 1, \cdots$, are constants which depend only on coefficients b_{jk} of g.

(i) We firstly consider the case $n = 3$. By the Nirenberg-Gagliardo inequality in [15] we have

$$\begin{aligned} &\int_\Omega \mid u \mid^4 dx \leq C \parallel \nabla u \parallel_{L^2}^3 \cdot \parallel u \parallel_{L^2}, \\ &\int_\Omega \mid v \mid^4 dx \leq C \parallel \nabla v \parallel_{L^2}^3 \cdot \parallel v \parallel_{L^2}, \end{aligned} \quad (3.14)$$

where $C > 0$ is a constant. Choose now $\varepsilon_0 > 0$ such that

$$\begin{aligned} &\parallel u \parallel_{C^0} \leq \min\{\xi_1, \xi_2, 1\}, \quad \text{provided} \quad \parallel u \parallel_{H^2} \leq \varepsilon_0, \\ &\parallel v \parallel_{C^0} \leq \min\{\xi_1, \xi_2, 1\}, \quad \text{provided} \quad \parallel v \parallel_{H^2} \leq \varepsilon_0. \end{aligned} \quad (3.15)$$

Hence there exists a constant $K > 0$, depending only on the coefficients of f and g, such that for $\parallel w \parallel_{H^2} \leq \varepsilon_0$ we have that

$$\begin{aligned} &\int_\Omega \mid uf(u,v) \mid dx \leq K\Big(\int_\Omega \mid u \mid^4 dx + \int_\Omega \mid v \mid^4 dx\Big), \\ &\int_\Omega \mid vg(u,v) \mid dx \leq K\Big(\int_\Omega \mid u \mid^4 dx + \int_\Omega \mid v \mid^4 dx\Big). \end{aligned} \quad (3.16)$$

Thus by (3.10), (3.14) and (3.16) we obtain

$$\begin{aligned} (w,\ F(w) - A_0 w)_0 \leq &- \parallel \nabla u \parallel_{L^2}^2 (1 - 2K \parallel \nabla u \parallel_{L^2} \parallel u \parallel_{L^2}) \\ &- \parallel \nabla v \parallel_{L^2}^2 (1 - 2K \parallel \nabla v \parallel_{L^2} \parallel v \parallel_{L^2}). \end{aligned} \quad (3.17)$$

Shrinking $\varepsilon_0 > 0$, we have

$$2K \parallel \nabla u \parallel_{L^2} \parallel u \parallel_{L^2} < 1 \quad \text{and} \quad 2K \parallel \nabla v \parallel_{L^2} \parallel v \parallel_{L^2} < 1, \quad (3.18)$$

provided $\parallel w \parallel_{H^2} \leq \varepsilon_0$. Combining (3.17) and (3.18) we get the assertion for the case $n = 3$.

(ii) Assume now that $n = 2$. By the Nirenberg-Gagliardo inequality in [15] we have

$$\begin{aligned} &\int_\Omega \mid u \mid^4 dx \leq C \parallel \nabla u \parallel_{L^2}^2 \cdot \parallel u \parallel_{L^2}^2, \\ &\int_\Omega \mid v \mid^4 dx \leq C \parallel \nabla v \parallel_{L^2}^2 \cdot \parallel v \parallel_{L^2}^2, \end{aligned} \quad (3.19)$$

where $C > 0$ is a constant. Following arguments similar to those in (i) we get

$$\begin{aligned}(w,\ F(w) - A_0 w)_0 \leq &- \parallel \nabla u \parallel_{L^2}^2 (1 - 2K \parallel u \parallel_{L^2}^2) \\ &- \parallel \nabla v \parallel_{L^2}^2 (1 - 2K \parallel v \parallel_{L^2}^2).\end{aligned} \tag{3.20}$$

Again, by shrinking $\varepsilon_0 > 0$, we may assume that

$$2K \parallel u \parallel_{L^2}^2 < 1 \quad \text{and} \quad 2K \parallel v \parallel_{L^2}^2 < 1, \tag{3.21}$$

for $w = (u, v) \in E_1$ with $\parallel w \parallel_{H^2} \leq \varepsilon_0$. It remains to combine (3.20) and (3.21) to complete the proof. □

Proof of Theorem 2.2: The assertions of Theorem 2.2 follow immediately by combining Theorem 3.1, Lemma 3.3, Lemma 3.4, and Lemma 3.5. □

4 Applications

In this section we apply our main results to three concrete examples and show that their trivial solutions $(0, 0)$ are positively Ljapunov stable.

Throughout this section we assume that Ω is an unbounded domain in $\mathbb{R}^2$ or in $\mathbb{R}^3$ with a uniformly C^4-regular boundary.

Example 1 Consider the following weakly coupled reaction-diffusion system

$$\begin{cases} u_t - \Delta u = -u^p + v^r, & \text{in} \quad (0, \infty) \times \Omega, \\ v_t - \Delta v = -v^q + u^s, & \text{in} \quad (0, \infty) \times \Omega, \\ u(t, x) = v(t, x) = 0, & \text{on} \quad [\,0, \infty) \times \partial\Omega, \end{cases} \tag{4.1}$$

with p, q, r, $s \geq 3$. System (4.1) provides a simple model to describe e.g. the cooperative interaction of two diffusing biological species. Here it is assumed that each species finds its subsistance from the activity of the other one by the reaction terms v^r and u^s, respectively, and disappears by a destruction mechanism, represented by the absorbtion terms $-u^p$ and $-v^q$, respectively.
Applying Theorem 2.2 with $(\xi_1, \xi_2) = (1, 1)$, $f(u, v) = -u^p + v^r$ and $g(u, v) = -v^q + u^s$, we get that the trivial solution $w = (0, 0)$ to (4.1) is positively Ljapunov stable. □

Example 2 Consider the following weakly coupled reaction-diffusion system

$$\begin{cases} u_t - \Delta u = u^p - u^r - u^m v^l, & \text{in} \quad (0, \infty) \times \Omega, \\ v_t - \Delta v = -v^q + u^s, & \text{in} \quad (0, \infty) \times \Omega, \\ u(t, x) = v(t, x) = 0, & \text{on} \quad [\,0, \infty) \times \partial\Omega, \end{cases} \tag{4.2}$$

where p, q, r, $s \geq 3$ and m, $l \geq 1$ with $m + l \geq 3$. Applying Theorem 2.2 with $(\xi_1, \xi_2) = (1, 1)$, $f(u, v) = u^p - u^r - u^m v^l$ and $g(u, v) = -v^q + u^s$, we get that the trivial solution $w = (0, 0)$ to (4.2) is positively Ljapunov stable. □

Example 3 Consider the following weakly coupled reaction-diffusion system

$$\begin{cases} u_t - \Delta u = u^p - u^r - u^m v^l, & \text{in} \quad (0,\infty)\times\Omega, \\ v_t - \Delta v = v^q - v^s - v^n u^k, & \text{in} \quad (0,\infty)\times\Omega, \\ u(t,x) = v(t,x) = 0, & \text{on} \quad [\,0,\infty)\times\partial\Omega, \end{cases} \tag{4.3}$$

where p, q, r, $s \geq 3$ and m, l, n, $k \geq 1$ with $m + l \geq 3$ and $n + k \geq 3$. Applying Theorem 2.2 with $(\xi_1, \xi_2) = (1, 1)$, $f(u, v) = u^p - u^r - u^m v^l$ and $g(u, v) = v^q - v^s - v^n u^k$, we get that the trivial solution $w = (0, 0)$ to (4.3) is positively Ljapunov stable. □

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