Spectral analysis of classes of subriemannian manifolds

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Abstract

In this thesis we study classes of subriemannian manifolds and their sublaplacians.

We determine the spectrum of the intrinsic sublaplacian on pseudo H-type nilmanifolds. Moreover, we construct an arbitrary number of isospectral but pairwise non-homeomorphic pseudo H-type nilmanifolds with respect to the sublaplacian.

Furthermore, with the help of the representation theory of step 2 nilpotent Lie groups, we show that the spectrum of the intrinsic sublaplacian on nilmanifolds whose covering space is a step 2 Carnot group can be completely expressed in terms of the underlying metric Lie algebra data. In the case of H-type nilmanifolds whose covering space has odd dimensional center, we prove a Poisson summation formula that relates the spectrum of the intrinsic sublaplacian to the lengths of closed subriemannian geodesics.

Based on the nilpotent approximation of subriemannian manifolds, we compare two subriemannian structures of step 2 and rank 4 on the Euclidean sphere $\mathbb{S}^7$ with regards to various geometric and analytical properties. We show that both subriemannian structures are neither locally isometric nor isospectral with respect to the subaplacian.

Finally, we study the heat kernel associated to the intrinsic sublaplacian on a quaternionic contact manifold considered as a subriemannian manifold. More precisely, we explicitly compute the first two coefficients appearing in the small time asymptotic expansion of the heat kernel on the diagonal. We show that the second coefficient coincides with the quaternionic contact scalar curvature up to a (universal) constant multiple.

Keywords: subriemannian geometry, sublaplacian, sub-elliptic heat kernel, asymptotics.

Zusammenfassung

In dieser Arbeit untersuchen wir bestimmte Klassen subriemannscher Mannigfaltigkeiten und die zugehörigen sub-Laplace Operatoren.

Wir bestimmen das Spektrum des intrinsischen sub-Laplace Operators auf pseudo H-Typ Nilmannigfaltigkeiten. Außerdem, konstruieren wir zu jeder beliebigen Zahl $m \in \mathbb{N}$, eine Familie von m pseudo H-Typ Nilmannigfaltigkeiten, die isospektral und paarweise nicht homöomorph bezüglich des sub-Laplace Operators sind.

Mithilfe der Darstellungstheorie 2-stufig nilpotenter Lie Gruppen zeigen wir, dass das Spektrum des intrinsischen sub-Laplace Operators auf Nilmannigfaltigkeiten, deren Überlagerungsraum eine 2-stufige Carnot Gruppe ist, vollständig durch Daten der zugehörigen, metrischen Lie Algebra beschrieben werden kann. Im Falle einer Nilmannigfaltigkeit, deren Überlagerungsraum eine H-Typ Gruppe ist, zeigen wir eine Poisson Summationsformel, die dieses Spektrum und Längen der geschlossenen, subriemannaschen geodätis-

chen Kuren verknüpft.

Basierend auf der nilpotenten Approximation subriemannscher Mannigfaltigkeiten, vergleichen wir zwei subriemannsche Strukturen von Rang 4 und Stufe 2 auf der euklidischen Sphäre $\mathbb{S}^7$ hinsichtlich verschiedener geometrischer und analytischer Eigenschaften. Wir zeigen, dass beide Strukturen weder isospektral noch (lokal) isometrisch bezüglich des intrinsischen sub-Laplace Operators sind.

Schließlich betrachten wir den Wärmeleitungskern des intrinsischen sub-Laplace Operators auf quaternionischen Kontaktmannigfaltigkeiten. Wir berechnen die ersten Koeffizienten c_0 und c_1 in der asymptotischen Entwicklung des Wärmeleitungskerns auf der Diagonale für kleine Zeiten. Wir zeigen, dass der zweite Koeffizient c_1 mit der quaternionischen skalarkrümmung bis auf ein konstantes (universelles) Vielfaches übereinstimmt.

Schlagworte: subriemannsche Geometrie, sub-Laplace Operator, sub-elliptischer Wärmeleitungskern, asymptotische Analysis.

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Contents

Chapter 1

Introduction

Subriemannian geometry can be seen as a generalization of Riemannian geometry under non-holonomic constraints. Non-holonomic constraints are given by non-integrable distributions, i.e. taking the Lie bracket of two vector fields in such a distribution may give rise to a vector field pointing outside the distribution. Moreover, subriemannian geometry is associated to the study of hypoelliptic operators similar to the interplay between Riemannian geometry and the theory of elliptic operators (see [26, 27, 34, 51, 102]). There is also a relation between (optimal) control theory of non-holonomic systems and subriemannian geometry. In this framework and under some additional assumptions, the (optimal) control problem can be reduced to the problem of finding (length mimimizing) horizontal curves with respect to the subriemannian structure. Concrete applications of subriemannian geometry appears e.g. in robotic engineering (see [35]) and in the field of neurobiology (see [42, 43, 103]) where the functional architecture of the primary visual cortex V_1 has been modeled using subriemannian geometry for image completion.

A subriemannian manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a smooth manifold M equipped with a smoothly varying inner product $\langle\cdot,\cdot\rangle$ on a subbundle (distribution) $\mathcal{H}$ of the tangent bundle TM, where $\mathcal{H}$ is assumed to be bracket generating, i.e. sections of $\mathcal{H}$ together with all brackets generate TM as a module over the functions on M. The so-called *Carnot-Carathéodory* distance between two points $p, q \in M$ is defined as the infimum of the lengths of horizontal curves (i.e. curves on the manifold that are tangent to the distribution $\mathcal{H}$) joining these points. One of the main results in subriemannian geometry is the famous *Chow-Rashevsky Theorem* (see [40, 99]) which asserts that assuming the bracket generating condition, each pair of points can be joined by a horizontal curve. The bracket generating condition is also called the *Hörmander condition*. In fact, Hörmander showed in his famous work [61] that this condition is sufficient for the hypoellipticity (or even subellipticity) of certain second order differential operators defined by a sum of squares of vector fields plus a first order term. Therefore, subriemannian geometry can be used to study and understand some hypoelliptic operators. Conversely, hypoelliptic operators can be associated to subriemannian manifolds in order to analyze and determine invariants of the underlying geometric structure.

A classical example showing this link is the Heisenberg group. Beals, Gaveau and Greiner (see [27]) studied the geometry of the Heisenberg group to compute the fundamental solution of the heat operator associated to the Heisenberg sublaplacian. Their work shows the strong link between properties of fundamental solutions of certain hypoelliptic operators and the related subriemannian geometry (e.g. subriemannian distance, cut locus, Hausdorff dimension). In the general setting, the interplay between the analysis of second order hypoelliptic operators and subriemannian geometry was intensively studied and various results are known (see [9, 14, 26, 27, 34, 51, 100, 102, 110]).

In subriemannian geometry, there are various second order differential operators that can be related to the geometric structure. These are called sublaplacians. Let $X_1, \cdots, X_m$ be a local orthonormal frame of the rank m distribution $\mathcal{H}$. Then a sublaplacian Δ_{sub} can be defined as the sum of squares of these vector fields

$$\Delta_{sub} = -\sum_{i=1}^{m} X_i^2$$

wherein we think of the X_i's as first-order differential operators. However, the drawback of this definition is that this sublaplacian depends in general on the chosen orthonormal frame. Another attempt to define a sublaplacian is to consider some smooth volume form μ on M and to set

$$\Delta_{sub} = -\mathrm{div}_\mu \circ \mathrm{grad}_{\mathcal{H}}, \tag{1.1}$$

where div_μ denotes the divergence operator with respect to μ and $\mathrm{grad}_{\mathcal{H}}$ is the so-called *horizontal gradient* (see [11]). This approach requires the choice of a canonical smooth volume form. Here, we mean by canonical volume a smooth volume form which only depends on the subriemannian structure and not on a particular choice of orthonormal frame or local coordinates. A partial solution to the latter problem is given by Montgomery [93] when we limit ourselves to a sub-class of subriemannian manifolds, called *equiregular*. These are subriemannian manifolds such that the dimension of the following sequence of linear subspaces (growth vector) of the space T_qM, (for each given point $q \in M$):

$$\mathcal{H}_q^1 := \mathcal{H}_q \text{ and } \mathcal{H}_q^{i+1} := \mathcal{H}_q^i + [\mathcal{H}_q^i, \mathcal{H}_q]$$

does not depend on the point q for every $i \geq 1$. On such a manifold there always is a canonical smooth volume called *Popp's volume*. A sublaplacian defined by formula (1.1) and using the Popp's volume μ is called the *intrinsic sublaplacian* [4, 11]. Another advantage of the Popp's volume is its invariance under subriemannian isometries. In particular, this imply that the intrinsic sublaplacian commutes with subriemannian isometries. Therefore, the Popp's volume (resp. intrinsic sublaplacian) of an equiregular subriemannian manifold can be seen as the equivalent of the Riemannian volume (resp. Laplace-Beltrami operator) of a Riemannian manifold (see [4, 11]).

As mentioned above, one can associate to every equiregular subriemannian manifold a natural fundamental object, namely the intrinsic sublaplacian. A classical problem is to define and classify subriemannian structures defined in a specific way on a given manifold up to local subriemannian isometries, or to compare them based on their geometric and topological properties (see [14, 20, 85, 93]). Another classical problem is to extract topological and geometric information from the (spectral) analysis of this operator. If M is compact, then it is known that the spectrum of this operator is discrete and consists of eigenvalues with finite multiplicity [61]. One of the results in this framework is a Weyl law for the spectral counting function $N(\lambda)$ established by many authors using different methods (see [87, 91, 105, 110]):

$$N(\lambda) = c(M)\lambda^{Q/2} + o(\lambda^{Q/2}) \text{ for } \lambda \to \infty.$$

Here $c(M)$ is a constant depending only on M and Q is the Hausdorff dimension of the metric space (M, d) associated to the subriemannian structure on M. In particular, this implies that the Hausdorff dimension can be extracted from the spectrum of the intrinsic sublaplacian. It is interesting to compare the above asymptotic with the classical Weyl law in the Riemannian case where the same formula holds true with the topological dimension instead of the Hausdorff dimension. Extending a well-known problem in Riemannian geometry, one may ask whether the geometry of a subriemannian manifold can be recovered from the spectrum of the intrinsic sublaplacian. From an analytical point of view one may also study distributions associated to operators related to the intrinsic sublaplacian like the heat kernel K_t, the wave kernel W_t or the Poisson distribution $tr(e^{it\sqrt{\Delta_{sub}}})$. A natural question is: how many geometric informations are encoded in these distributions.

From the work of R.B. Melrose in [88] it can be deduced in the case of 3-dimensional contact subriemannian manifolds (with some additional assumtions) that the singular support of the Poisson distribution is contained in $\{0\} \cup L(M)$ where $L(M)$ denotes the set of lengths of closed geodesics in the subriemannian manifold M. A similar result has been shown for 4-dimensional quasi-contact subriemannian manifolds in [104]. The wave kernel W_t has been studied in [60] on the Heisenberg group. Therein, it has been shown that $W_t(x, y)$ is a real analytic function of all its arguments whenever t is different from lengths of geodesics joining x and y. To our knowledge, the interplay of analysis and subriemannian geometry based on the wave kernel and the Poisson distribution have been considered only in these special cases of subriemannian manifolds.

In the general setting of subriemannian manifolds, one of the most intensively studied objects is the heat kernel induced by the (intrinsic) sublaplacian. On the diagonal, the heat kernel K_t admits an asymptotic expansion for small times as follows (see [29, 46, 64, 110]):

$$K_t(q, q) = \frac{1}{t^{Q/2}} \left(c_0(q) + c_1(q)t + \cdots + c_N(q)t^N + O(t^{N+1}) \right).$$

Here Q denotes the Hausdorff dimension of the metric space (M, d) where d is the subriemannian distance (Carnot-Carathéodory distance) on M. The coefficients $c_0, c_1, \cdots$ are called *heat invariants* of the (equiregular) subriemannian manifold M.

A challenging problem in this framework is to give a geometric meaning to the heat invariants. In the recent work [110], it was shown that the heat invariants can be computed with the help of some convolution integrals involving the heat kernel of the nilpotentization of the structure. However, if we consider a subriemannian manifold together with a canonical linear connection adapted to the subriemannian structure, it is not clear if the heat invariants can be interpreted geometrically in terms of tensors associated to this connection. In case of a subriemannian structure induced by a specific structure (e.g. H-type manifolds, contact manifolds), it is not clear if we can identify the heat invariants with geometric invariants of the underlying structure. To our best knowledge, such results exist only for specific classes of subriemannian manifolds, which we recall in the following. The first result in this framework has been established for the class of strictly pseudoconvex CR manifolds endowed with a Levi-metric [28]. There, the authors showed by developing an appropriate pseudodifferential calculus that the second heat invariant c_1 can be interpreted as the scalar curvature of the Tanaka-Webster connection. Furthermore, the Popp's volume of some small balls (balls defined by using some geodesic curves of the Tanaka-Webster connection) admit an asymptotic expansion where the scalar curvature appears in the second coefficient. Similar results were found in [7, 8]. Therein, using the nilpotent approximation together with a special construction of privileged coordinates, it was shown in the case of a 3-dimensional contact subriemannian manifold, that the second coefficient c_1 can be interpreted as the scalar curvature defined in this class of manifolds. Furthermore, the Popp's volume of small subriemannian balls has a two-term asymptotic where the first (resp. second) coefficient is exactly the volume of the unit ball in the Heisenberg group (resp. the scalar curvature). Therefore, based on the sign of the scalar curvature one can compare Popp's volume of small subriemannian balls in 3-dimensional subriemannian manifolds to the volume of the unit ball in the Heisenberg group, which is the local model of such subriemannian manifolds.

In this thesis, we consider some of the classical problems cited above for specific subriemannian structures on the following manifolds:

1. Compact quotient manifolds of the form $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is a certain step 2 nilpotent Lie group called *pseudo H-type* Lie group and Γ is a lattice of $\mathbb{G}$.

2. The Euclidean sphere $\mathbb{S}^7$.

3. Quaternionic contact manifolds.

The structure of the thesis is the following:

In Chapter 2 we introduce the basic definitions and some classical results in subriemannian geometry. Moreover, we recall briefly the concept of the nilpotent approximation and cite one of its applications.

Chapter 3 is identical with our paper [22], which is done jointly with W. Bauer, K. Furutani and C. Iwasaki. We consider an isospectrality problem in the subriemannian setting introduced by Mark Kac in his paper [70]. Here we construct isospectral but non-homeomorphic subriemannian manifolds extending known examples and methods in the Riemannian case [57, 58, 62, 63, 86]. With such examples one should be able to determine topological properties that are not encoded in the spectrum.

More precisely, in this Chapter we consider nilmanifolds $\Gamma\backslash\mathbb{G}$ of step 2 where $\mathbb{G}$ is a simply connected nilpotent Lie group called *pseudo H-type* [41]. Such groups are generalizations of the well known *Heisenberg type groups* (shortly H-type groups) introduced by A. Kaplan in [71]. These groups and their Lie algebras are constructed from Clifford algebras $C\ell_{r,s}$ of signature (r, s) and their *(admissible) modules* (see [41, 53, 54, 55]). Since the existence of lattices Γ in pseudo H-type Lie groups has been proved (see [55]), we can consider the compact manifold $\Gamma\backslash\mathbb{G}$ where Γ is a standard (integral) lattice in the pseudo H-type group $\mathbb{G}$. In the following we denote by $\mathbb{G}_{r,s}(V)$ the pseudo H-type Lie group constructed from the Clifford algebra $C\ell_{r,s}$ with an admissible module V and by $\Gamma_{r,s}(V)$ the standard lattice of $\mathbb{G}_{r,s}(V)$. Also we set

$$N_{r,s}(V) := \Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V).$$

Then a subriemannian structure on $N_{r,s}(V)$ can be defined by descending the natural left-invariant subriemannian structure on the covering space $\mathbb{G}_{r,s}(V)$. We first give an explicit heat trace formula for the sublaplacian. It turns out that this formula on $N_{r,s}(V)$ remains unchanged by permuting the role of r and s if the dimension of the corresponding Clifford admissible modules remains invariant. Therefore, under the latter condition the constructed pairs of nilmanifolds are isospectral with respect to the sublaplacian. On the other hand, from a topological point of view, a homeomorphism of two nilmanifolds induces an isomorphism of their fundamental groups and hence, by applying a general fact from [98] their associated pseudo H-type Lie algebras will be isomorphic. Based on the recent classification of pseudo H-type Lie algebras in [53, 54], we were able to construct pairs of isospectral, but non-homeomorphic nilmanifolds with respect to the sublaplacian. Note that the considered subriemannian structure has a natural extension to a Riemannian one and we may as well consider the corresponding problem from the Riemannian point of view. As it turns out, the constructed examples are also isospectral, non-homeomorphic manifolds in the Riemannian setting.

If we deal with arbitrary admissible modules instead of modules of minimal dimension and by choosing the space dimension suitably high, the method described above allows us to construct any given number of pairwise isospectral, non-homeomorphic nilmanifolds of dimension ≥ 11. To our knowledge, these are the first examples of this kind.

Finally, we give a negative answer to the (weak) spectral invariance of the topological dimension of a nilmanifold in the subriemannian setting. More precisely, we construct two nilmanifolds of different dimensions such that the short time heat trace asymptotics of the corresponding sublaplacians coincide up to a term which vanishes at infinite order as time tends to zero. However, the problem of spectral invariance of the topological dimension of nilmanifolds (in the subriemannian setting) remains unsolved. In some specific cases (for $s = 0$), this problem can be reduced to the injectivity of a function involving the multiple Hurwitz zeta function (see the last part of Chapter 2). We note that this situation cannot occur in the framework of Riemannian geometry, since the topological dimension of a manifold can be extracted from the heat trace asymptotics for small times.

Chapter 4 is based on the paper [76]. We consider the classical problem of extracting geometric information from the spectrum of the intrinsic sublaplacian on nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is an H-type group. It is known that the classical Poisson summation formula on a torus $\Gamma\backslash\mathbb{R}^n$

$$\sum_{X\in\Gamma^*} e^{-4\pi^2\|X\|^2 t} = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{R}^n)}{(4\pi t)^{n/2}} \sum_{x\in\Gamma} e^{-\frac{\|x\|^2}{4t}} \tag{1.2}$$

shows that the spectrum of the Laplacian determines the length spectrum of the Riemannian manifold $\Gamma\backslash\mathbb{R}^n$ and vice versa. Generalizations (in a specific sense) of the above formula to compact Riemannian manifolds can be found in [111, 48, 96]. However, in the subriemannian setting, the relation between the spectrum of the intrinsic sublaplacian and the length spectrum of the underlying structure is far from being fully understood. Here, to our knowledge, only specific classes of subriemannian manifolds have been considered, as mentioned above. In this Chapter, we mainly study the relation between the spectrum of the sublaplacian and the length spectrum, i.e. the set of lengths of closed subriemannian geodesics, on nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is an H-type group. For this, we first compute the spectrum of the intrinsic sublaplacian on compact nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is a step 2 Carnot group and Γ is assumed to be a lattice of $\mathbb{G}$. The key idea here is that the Popp measure on a nilpotent Lie group $\mathbb{G}$ is also right-invariant and hence, the Popp measure on $\Gamma\backslash\mathbb{G}$ is $\mathbb{G}$-invariant. Therefore the Popp measure and the quotient measure coincide up to a constant multiple. In particular, this implies that the intrinsic sublaplacian on $\Gamma\backslash\mathbb{G}$ can be written as a sum of squares of vector fields. Then, the computation of the spectrum is similar to the one in the Riemannian setting (see [95]). Here we show that the spectrum of the intrinsic sublaplacian can be completely expressed in terms of the underlying metric Lie algebra data.

Every lattice Γ in a H-type Lie group $\mathbb{G}$ induces a horizontal lattice Γ_1 and a vertical lattice Γ_2 (see Chapter 4). We show that the subriemannian length spectrum of H-type nilmanifolds can be explicitly expressed using these lattices and the metric on the Lie algebra level. By combining these results and using the classical Poisson summation formula, we present a similar formula for compact nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is an H-type group with odd dimensional center. Here, the calculations involve the Fourier transformation of the standard measure on the Euclidean unit sphere $\mathbb{S}^{d-1}$, where d denotes the dimension of the center of $\mathbb{G}$. By the assumption d is odd, this integral can be explicitly computed. However, when d is even this transformation can be expressed with the help of modified Bessel functions and the method used in the proof of the Poisson summation formula does not work. Here, new methods must be developed to study the remaining case. Finally, we conclude that the spectrum of the sublaplacian determines the length spectrum of $\Gamma\backslash\mathbb{G}$ whenever $\mathbb{G}$ has odd dimensional center.

Chapter 5 is identical with our paper [23], which is done jointly with W. Bauer. We study and compare two subriemannian structures defined on the Euclidean sphere $\mathbb{S}^7$ with regards to a various amount of their topological, geometric and spectral properties. But before presenting our results, we recall typical methods of defining a subriemannian structure on a given Euclidean sphere $\mathbb{S}^N$ which depend on the dimension N as follows:

1. If $N = 3$, we can use the Lie group structure on $\mathbb{S}^3$ (s. [37]).
2. If N is odd, we can use a contact structure on the sphere $\mathbb{S}^N$ (s. [84, 85]).
3. If $N = 2n + 1$, we can use the principle bundle structure of the complex Hopf fibration $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ (s. [84]).
4. If $N = 4n + 3$, we can use the principle bundle structure of the quaternionic Hopf fibration $\mathbb{S}^{4n+3} \to \mathbb{HP}^n$ (s. [84]).
5. If $N = 3, 7, 15$, we can use a suitable number of some (canonical) vector fields on the sphere to define a trivializable distribution (s. [1]).

In the special case of the sphere with lowest dimension $\mathbb{S}^3$, it turns out that all these structures are isometric as subriemannian manifolds (see [85]).

In the following we briefly recall the construction of trivial distributions on spheres. Consider a family of $(N + 1) \times (N + 1)$ skew-symmetric real matrices $A_1, \cdots, A_k$ such that

$$A_iA_j + A_jA_i = -2\delta_{ij} \quad \textit{for} \quad i, j = 1, \cdots, k.$$

Then a collection of k linear vector fields on $\mathbb{S}^N$ that are orthonormal at each point of the sphere can be defined in global coordinates of $\mathbb{R}^{N+1}$ by:

$$X(A_l) := \sum_{i,j=1}^{N+1} (A_l)_{ij} x_j \frac{\partial}{\partial x_i}, \qquad (l = 1, \cdots, k).$$

As pointed out in [21] the rank k distribution

$$\mathcal{H} := \mathrm{Span}\{X(A_l) : l = 1, \cdots, k\} \subset T\mathbb{S}^N$$

is of step two and trivial as a vector bundle by definition. Moreover, this distribution is bracket generating only for the following particular choices of N and k:

1. $N = 3$ and $k = 2$.
2. $N = 7$ and $k = 4, 5, 6$.
3. $N = 15$ and $k = 8$.

In this Chapter, we consider the subriemannian structure on $\mathbb{S}^7$ induced by a trivial distribution of rank 4 as above and compare it with the rank 4 distribution induced by the quaternionic Hopf fibration [15, 84]

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4.$$

From a geometric point of view, the above trivializable subriemannian structures on $\mathbb{S}^7$ have been studied in [25]. Therein, the authors analyzed the corresponding geodesic flow and constructed a family of normal subriemannian geodesics (i.e. locally length minimizing curves defined from the geodesic equations). However, the problem of constructing all normal subriemannian geodesics seems to be unsolved.

Since a distribution constructed from canonical vector fields is trivial by definition, it is natural to ask whether the quaternionic distribution is also trivial (see [85]). Due to some topological obstruction, it turns out that this is not the case. In fact, using results in topological K-theory in [81] we show that the horizontal distribution induced by the quaternionic Hopf fibration on $\mathbb{S}^7$ does not admit a nowhere vanishing and globally defined vector field. In particular, this distribution is not trivial. As a consequence, this implies that the corresponding subriemannian structures on $\mathbb{S}^7$ are not isometric. To compare these structures locally, we need to extract further invariants and to compare them. A useful tool we used is the nilpotent approximation. Here we compare the so-called tangent groups i.e. local approximations of the considered subriemannian structures on $\mathbb{S}^7$. We find out that the tangent groups of the trivializable subriemannian structure may change from point to point. Then by comparing some local invariants of the considered structures, we obtain as a consequence that the considered subriemannian structures on $\mathbb{S}^7$ are not locally isometric. Furthermore, the isometry group of the trivializable subriemannian structure does not act transitively on $\mathbb{S}^7$.

Moreover, the trivializable distribution is of elliptic type inside an open dense subset. In contrast, the quaternionic distribution is everywhere elliptic on $\mathbb{S}^7$, which implies that the corresponding symmetry group is finite dimensional (see [93]). Furthermore, we give an example of a 3-dimensional family of subriemannian isometries with respect to the trivializable subriemannian structure.

Also we study the isospectrality of these subriemannian manifolds. In fact, by applying recent results in [110] combined with an explicit form of the subelliptic heat kernel on step two nilpotent Lie groups in [26, 36, 52] we compute the first heat invariants associated to the intrinsic sublaplacians. By comparing the first heat invariant corresponding to these structures, we show that the considered subriemannian structures are not isospectral with respect to the intrinsic sublaplacians.

Whereas an explicit formula for the heat kernel of the intrinsic sublaplacian induced from the quaternionic contact structure has been found in [15], the situation for the trivializable subriemannian structure is completely different. Here, the subriemannian isometry group does not act transitively on $\mathbb{S}^7$. Therefore, the heat kernel must be calculated at every point of $\mathbb{S}^7$ and an explicit formula is yet to be known. Also, it is interesting to compare the latter structure with the rang 5 trivializable subriemannian structure on $\mathbb{S}^7$ where its subriemannian isometry group acts transitively on $\mathbb{S}^7$ and an explicit integral formula for the corresponding heat kernel was found in [24].

Finally, we consider the sublaplacian associated to the trivializable subriemannian structure endowed with the standard Riemannian volume form. Here we compute a part of its spectrum, which is exactly the spectrum of the intrinsic sublaplacian corresponding to the quaternionic subriemannian structure. However, it is not clear if these spectra coincide.

In Chapter 6 which is based on the paper [75], we study the first heat invariant associated to the sublaplacian on a class of subriemannian manifolds called *quaternionic contact* (shortly qc). These were introduced by Biquard in [33] as a manifold of the conformal boundary at infinity of quaternionic Kähler manifolds. A qc manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is a $(4n+3)$-dimensional connected manifold endowed with a codimension 3 distribution $\mathcal{H}$ and a fiberwise inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. Locally, the distribution $\mathcal{H}$ is given as the kernel of three contact forms η_1, η_2, η_3 and there are almost complex structures I_1, I_2, I_3 on $\mathcal{H}$ satisfying the quaternionic commutation relations

$$(I_i)^2 = I_1 I_2 I_3 = -\mathrm{Id} \text{ for } i = 1, 2, 3.$$

Moreover, the following compatibility conditions hold

$$2\langle I_i \cdot, \cdot \rangle = d\eta_i(\cdot, \cdot) \text{ on } \mathcal{H} \text{ for } i = 1, 2, 3.$$

Biquard in [33] gave an infinite family of examples of qc structures defined as the conformal boundary of the quaternionic Kähler manifolds constructed in [78]. Conversely, every real analytic qc structure on a manifold M can be seen as the conformal infinity of a quaternionic Kähler metric defined in a neighborhood of M (s. [33]).

A classical example of a qc manifold is given by the quaternionic Hopf fibration

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n.$$

Here the distribution $\mathcal{H}_q$ at $q \in \mathbb{S}^{4n+3}$ is defined as the orthogonal complement of the vertical space (tangent space to the fiber through q) with respect to the standard Riemannian metric on $\mathbb{S}^{4n+3}$. This example is included in the sub-class of 3-*Sasakian manifolds* which are also qc manifolds.

On a qc manifold, there exists a linear connection adapted to the qc structure, the so-called Biquard connection [33] . This connection is canonical in the sence that it is unique under some conditions. Also, it plays a similar role to the Levi-Civita connection in Riemannian geometry or the Tanaka-Webster connection on CR manifolds [109, 113]. Furthermore, qc manifolds can be seen as a quaternionic analogue of CR manifolds. Therefore, it is natural to ask which properties can be extended to this class of subriemnnian manifolds.

In this Chapter, we consider the classical problem of finding a relation between the second heat invariant and geometric invariants of the underlying qc structure. First, we compute the Popp measure on such a qc manifold and determine the intrinsic sublaplacian. The inner product on the horizontal distribution of a qc manifold extends canonically to a Riemannian metric. With this in mind, we show that the Popp's volume of a qc manifold coincides with the associated Riemannian volume up to a constant multiple.

For the geometric interpretation of the second heat invariant, we use two crucial tools: the so-called *qc normal coordinates* constructed in [73] and the recent results from [110]. In these coordinates the anisotropic asymptotic expansion of a special horizontal frame can be computed in terms of tensors induced by the Biquard connection. Then using symmetries of the heat kernel of the sublaplacian on the quaternionic Heisenberg group (which is the local model of a qc manifold) and identities of some tensors induced by the Biquard connection we prove that the second heat invariant coincides with the scalar curvature induced by the Biquard connection up to a multiple (universal) constant. This universal constant can be calculated by considering a special qc manifold.

The results presented in this thesis are extracted from the following articles:

(A1) W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi, *Spectral theory of a class of nilmanifolds attached to Clifford modules*, Mathematische Zeitschrift (2021), 297(1), 557-583.

(A2) A. Laaroussi, *A Poisson formula for subriemannian H-type nilmanifolds*, in preparation.

(A3) W. Bauer, A. Laaroussi, *Trivializable and quaternionic subriemannian structure on $\mathbb{S}^7$ and subelliptic heat kernel*, submitted.

(A4) A. Laaroussi, *Heat kernel asymptotics for quaternionic contact manifolds*, arXiv:2103.00892 (2021).

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Chapter 2

Preliminaries

We start recalling basic definitions in subriemannian geometry [3, 93, 106, 107].

A subriemannian manifold is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ where

(a) M is a connected orientable smooth manifold of dimension $n \geq 3$.

(b) $\mathcal{H}$ is a smooth distribution of constant rank $m < n$ which is bracket generating, i.e. if we set for $j \geq 1$

$$\mathcal{H}^1 := \mathcal{H} \text{ and } \mathcal{H}^{j+1} := \mathcal{H}^j + [\mathcal{H}, \mathcal{H}^j],$$

then for all $q \in M$ there is $N(q) \in \mathbb{N}$ such that $\mathcal{H}_q^{N(q)} = T_qM$.

(c) $\langle\cdot,\cdot\rangle$ is a fiber inner product on $\mathcal{H}$, i.e.

$$\langle\cdot,\cdot\rangle_q : \mathcal{H}_q \times \mathcal{H}_q \longrightarrow \mathbb{R}$$

is an inner product for all $q \in M$ and it smoothly varies with $q \in M$.

We say that a subriemannian manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is equiregular, if for all $j \geq 1$ the dimension of $\mathcal{H}_q^j$ does not depend on the point $q \in M$. Furthermore, M is said to be of step r if r is the smallest integer such that $\mathcal{H}^r = TM$.
In this thesis we only consider equiregular subriemannian manifolds of step 2. Therefore, in what follows, we recall the required concepts only for this class of manifolds.
A local frame $\{X_1, \cdots, X_m, X_{m+1}, \cdots, X_n\}$ is called *adapted*, if the vector fields $X_1, \cdots, X_m$ form a local orthonormal frame of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.
To every equiregular subriemannian manifold M of step 2, a family of graded step 2 nilpotent Lie algebras

$$\mathfrak{g}M := \mathcal{H} \oplus \left(\mathcal{H}^2/\mathcal{H}\right)$$

is associated with Lie brackets induced by the Lie brackets of vector fields on M, [93]. Note that the defined Lie brackets respect the above grading, i.e.

$$[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}^2/\mathcal{H} \quad \textit{and} \quad [\mathcal{H}, \mathcal{H}^2/\mathcal{H}] = [\mathcal{H}^2/\mathcal{H}, \mathcal{H}^2/\mathcal{H}] = 0.$$

We call $\mathfrak{g}M(q) = \mathfrak{g}M_q$ the *tangent algebra* of $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ at $q \in M$, [93].

Let $(M_i, \mathcal{H}_i, \langle\cdot,\cdot\rangle_i)$ for $i = 1, 2$ be subriemannian manifolds. A (local) diffeomorphism $\phi : M_1 \longrightarrow M_2$ is a (local) subriemannian isometry if its differential ϕ_* fulfills:

1. $\phi_*(\mathcal{H}_{1,q}) = \mathcal{H}_{2,\phi(q)}$ for all $q \in M_1$.
2. $\langle\phi_*(X), \phi_*(X)\rangle_{2,\phi(q)} = \langle X, Y\rangle_{1,q}$ for all $q \in M_1$ and $X, Y \in \mathcal{H}_{1,q}$.

We denote by $\mathcal{I}(M)$ the group of all subriemannian isometries of M (i.e. isometries from M to M). By the assumption M is equiregular, the isometry group $\mathcal{I}(M)$ admits a structure of finite-dimensional Lie group (s. [38]). In the literature, another (much larger) group has been considered, the so-called *symmetry group.* It consists in diffeomorphisms of M which preserve its distribution, i.e. condition (1) above holds. However, it is not clear if it admits a structure of Lie group (s. [93]).

In subriemannian geometry, curves that are admissible in a certain sense are of crucial interest. To be precise, an absolutely continuous curve $\gamma : [0, 1] \longrightarrow M$ is said to be horizontal, if $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$ almost everywhere. One of the main results in subriemannian geometry is the *Chow-Rashevskii* theorem about the path-connectedness (by means of horizontal curves) which follows under the bracket generating condition on $\mathcal{H}$:

Theorem 2.0.1 ([40, 99])**.** *Let M be a connected manifold with a bracket generating distribution $\mathcal{H}$. Then the set of points that can be connected to a given point $q \in M$ by a horizontal curve coincides with M.*

By defining the length of a horizontal curve $\gamma : [0, 1] \longrightarrow M$ to be

$$l(\gamma) := \int_0^1 \sqrt{\langle\dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}}dt,$$

the subriemannian structure on M induces the so-called *Carnot-Carathéodory distance d* on M defined by

$$d(p, q) := \inf l(\gamma),$$

where the infimum is taken over all horizontal curves joining p and q.
A natural problem in subriemannian geometry is to find a length minimizing horizontal curve between a given point $p \in M$ and q sufficiently close to p (i.e. a horizontal curve that realizes the distance between p and q). A (partial) solution to this problem is given by considering the subriemannian Hamiltonian system which we recall next. Consider a trivializing neighborhood U_p around $p \in M$ for the subbundle $\mathcal{H}$ and a local orthonormal basis $X_1, \cdots, X_m$ of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then the associated subriemannian Hamiltonian $H : T^*U_p \longrightarrow \mathbb{R}$ is defined by

$$H(q, \lambda) := \frac{1}{2}\sum_{j=1}^{m} \lambda\left(X_j(q)\right)^2 \text{ for } (q, \lambda) \in T^*U_p.$$

A normal subriemannian geodesic is defined as the projection to $U_p \subset M$ of a solution $\Gamma(t)$ to the following Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H}{\partial q},$$

with the property $H(\Gamma(t)) \neq 0$. Here, (q, λ) are the coordinates in the cotangent bundle. Then any normal subriemannian geodesic is locally a length minimizer [80, 93]. In strong contrast to the Riemannian case, length minimizing horizontal curves are in general not necessary normal subriemannian geodesics. These curves are known as *abnormal subriemannian geodesics*. If in addition, we assume that the distribution $\mathcal{H}$ has the *strong bracket generating condition*, i.e. $TM = \mathcal{H} + [X, \mathcal{H}]$ for every nonzero section X of $\mathcal{H}$ (fat subriemannian structures), then abnormal geodesics do not exist [106, 107].

On a subriemannian manifold the definition of a sublaplacian requires the data of a smooth measure μ on M. We denote by div_μ the divergence operator associated with μ, defined by

$$\mathcal{L}_X \mu = \mathrm{div}_\mu(X)\mu$$

for every smooth vector field X on M. Here $\mathcal{L}_X$ denotes the Lie derivative along the vector field X. Then we can associate to μ a sublaplacian Δ^μ_{sub} defined as the second order differential operator

$$\Delta^\mu_{sub} f := -\mathrm{div}_\mu \left(\nabla_{\mathcal{H}} f \right) \text{ for } f \in C^\infty(M).$$

Here $\nabla_{\mathcal{H}} f$ denotes the horizontal gradient of f with respect to the horizontal metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ defined as the unique horizontal vector field with the property

$$X(f) = \langle \nabla_{\mathcal{H}} f, X \rangle$$

for every smooth horizontal vector field on M. One of the central results in the analysis of subriemannian structures is the famous Hörmander theorem:

Theorem 2.0.2 ([61])**.** *Let $X_0, X_1, \cdots, X_m$ be smooth vector fields defined on some open set $U \subseteq \mathbb{R}^n$ and let $c \in C^\infty(U)$. We assume that among the vector fields*

$$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, X_{j_s}]]]$$

where $j_s = 0, 1, \cdots, m$, there exist n which are linearly independent at any given point in U. Then the operator

$$P := \sum_{i=1}^{m} X_i^2 + X_0 + c$$

is hypoelliptic.

Now, if we take a local orthonormal frame $X_1, \cdots, X_m$ of the distribution $\mathcal{H}$, the sublaplacian Δ^{μ}_{sub} can be expressed locally in the form (see [11])

$$\Delta^{\mu}_{sub} = -\sum_{i=1}^{m} \left(X_i^2 + \mathrm{div}_{\mu}(X_i) X_i \right).$$

Hence, by the bracket generating condition on $\mathcal{H}$, Hörmander's theorem implies that the sublaplacian Δ^{μ}_{sub} is hypoelliptic.

Since by assumption the subriemannian manifold M is equiregular, there is a canonical choice of smooth measure on M, namely the Popp measure. In this case, the sublaplacian defined from the Popp measure is called the intrinsic sublaplacian.

Note that the sublaplacian is positive and if the manifold M endowed with the subriemannian distance is complete then Δ^{μ}_{sub} is essentially selfadjoint on compactly supported smooth functions and has a unique selfadjoint extension on $L^2(M, \mu)$ (see [106]). Therefore the heat semigroup $\left(e^{-t\Delta^{\mu}_{sub}}\right)_{t>0}$ is a well-defined one-parameter family of bounded operators on $L^2(M, \mu)$. In the following, we denote by $K_t(\cdot, \cdot)$ the heat kernel of the operator $e^{-t\Delta^{\mu}_{sub}}$ which is smooth due to the hypoellipticity of $\frac{d}{dt} + \Delta^{\mu}_{sub}$.

Moreover, if M is compact, the subellipticity of Δ^{μ}_{sub} imply that its unique selfadjoint extension has compact resolvent. Therefore, its spectrum is discrete and consists of eigenvalues with finite multiplicity

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \cdots \to \infty.$$

We recall the following formula for the small time asymptotic expansion of the heat kernel on the diagonal:

Theorem 2.0.3 ([29, 110])**.** *For all $q \in M$ and $N \in \mathbb{N}$, it holds:*

$$K_t(q, q) = \frac{1}{t^{Q/2}} \left(c_0(q) + c_1(q) t + \cdots + c_N(q) t^N + o(t^N) \right) \quad \text{as } t \to 0.$$

Moreover, when assuming equiregularity of the subriemannian manifold, the functions c_i are smooth in a neighborhood of q. Here Q denotes the Hausdorff dimension of the metric space (M, d) where d denotes the subriemannian distance (Carnot Carathéodory distance) on M.

Theorem 2.0.3 was first proven by G. Ben Arous in [29] and then treated again by Y. C. de Verdière et al. in [110]. The advances in [110] consists in the interpretation of the heat invariants (i.e. the coefficients $c_0, c_1, \cdots$) in terms of the so-called *nilpotent approximation* of the underlying subriemannian structure which we will recall briefly.

In the setting of general subriemannian manifolds, a powerful method in the analysis of a sublaplacian is given by the so-called nilpotent approximation. The idea consists in an approximation of the subriemannian manifold at a given point by a nilpotent Lie group endowed with a left-invariant subriemannian structure. In the following, we briefly recall the relevant concepts. For more details we refer to [3, 6, 39, 110].

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a step two equiregular subriemannian manifold of dimension n and denote by

$$\{X_1, \cdots, X_m, X_{m+1}, \ldots, X_n\}$$

a local adapted frame at $q \in M$.
Given a smooth function f, the nonholonomic order of f at q is defined by

$$\mathrm{ord}_q(f) := \min\{s \in \mathbb{N} : \exists i_1, \cdots, i_s \in \{1, \cdots, m\} \text{ such that } (X_{i_1} \cdots X_{i_s} f)(q) \neq 0\}.$$

Assume now that M is an equiregular subriemannian manifold of step r and consider the increasing sequence of vector subspaces:

$$\{0\} =: \mathcal{H}_q^0 \subset \mathcal{H}_q^1 \subset \cdots \subset \mathcal{H}_q^r = T_qM$$

together with a basis $v_1, \cdots, v_n$ adapted to this flag.

Consider the following weights for $1 \leq j \leq n$:

$$w_j(q) := s \text{ if } v_j \in \mathcal{H}_q^s \text{ but } v_j \notin \mathcal{H}_q^{s-1}.$$

In the following we call a system of coordinates $x_1, \cdots, x_n$ *linearly adapted* (to the flag) at q, if it is centered at q and

$$dx_i(\mathcal{H}_q^{w_i(q)}) \neq 0 \text{ and } dx_i(\mathcal{H}_q^{w_i(q)-1}) = 0.$$

In subriemannian geometry, coordinates $x_1, \cdots, x_n$ linearly adapted at q such that every coordinate x_j has maximal order play a crucial role. Such a system of coordinates is called a system of *privileged coordinates* at q. More precisely, a system of privileged coordinates $x_1, \cdots, x_n$ at q is a system of coordinates linearly adapted at q such that

$$\mathrm{ord}_q(x_i) = w_i(q) \text{ for } i = 1, \cdots, n.$$

An example of privileged coordinates at $q \in M$ is given by the so-called *canonical coordinates of the first kind* defined as the inverse of the local diffeomorphism:

$$(x_1, \cdots, x_n) \longmapsto \exp(x_1X_1 + \cdots + x_nX_n)(q).$$

Here $X_1, \cdots, X_m, X_{m+1}, \cdots, X_n$ is an adapted local frame at q.

In a system of privileged coordinates at q we consider the natural dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined for $\lambda > 0$ by

$$\delta_\lambda(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) := (\lambda x_1, \cdots, \lambda x_m, \lambda^2 x_{m+1}, \cdots, \lambda^2 x_n).$$

Using the infinitesimal generator P of the action δ_λ on $\mathbb{R}^n$:

$$P := \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i} + 2 \sum_{i=m+1}^{n} x_i \frac{\partial}{\partial x_i},$$

it is possible to define the notion of a homogeneous tensor field φ with respect to the above dilations by the condition:

$$\mathcal{L}_P(\varphi) = l\varphi, \tag{2.1}$$

where $\mathcal{L}_P$ denotes the Lie derivative with respect to P and l is called the degree of homogeneity of φ. In the following, we denote by $\varphi^{(l)}$ the homogeneous part of degree l of a given tensor field φ.

Every smooth vector field X on $\mathbb{R}^n$ has a formal expansion near $q = (0, \cdots, 0)$ of the form:

$$X \simeq X^{(-2)} + X^{(-1)} + \cdots,$$

where $X^{(l)}$ is a polynomial vector field of degree l, i.e. homogeneous of degree l with respect to the dilations δ_λ.

One of the most important facts about privileged coordinates is that every horizontal vector field X_i, for $i =, \cdots, m$ has an expansion near 0 of the form

$$X_i \simeq X_i^{(-1)} + X_i^{(0)} + \cdots, \tag{2.2}$$

i.e. the first homogeneous term appearing in this expansion has order $-1 = -\mathrm{ord}_q(x_i)$. In the literature, some authors use this property to define privileged coordinates. We refer to the work [39] for a comparison of the different definitions of privileged coordinates.

Note that the vector fields $X_1^{(-1)}, \cdots, X_m^{(-1)}$ on $\mathbb{R}^n$ generate a graded step two nilpotent Lie algebra $\mathfrak{g}(q)$. The system of vector fields $\{X_1^{-1}, \cdots, X_m^{(-1)}\}$ is called the *canonical nilpotent homogeneous approximation* of the system $\{X_1, \cdots, X_m\}$ at q. Note also that the tangent algebra $\mathfrak{g}M(q)$ and the Lie algebra $\mathfrak{g}(q)$ are isomorphic as Lie algebras (s. [39]). Let us denote by $(\mathbb{G}(q), *)$ the corresponding step two nilpotent Lie group defined as follows. As a manifold we take $\mathbb{G}(q) = \mathfrak{g}(q)$, while the group law is given by

$$\xi_1 * \xi_2 := \xi_1 + \xi_2 + \frac{1}{2}[\xi_1, \xi_2] \quad \textit{for} \quad \xi_1, \xi_2 \in \mathbb{G}(q).$$

In the following, we fix some privileged coordinates $x_1, \cdots, x_n$ defined near q on a chart ψ.

Definition 2.0.4. Given a smooth measure μ on M, its *nilpotentization* at q is a measure $\widetilde{\mu}_q$ on $\mathbb{G}(q)$ defined in the chart ψ by

$$\widetilde{\mu}_q := \lim_{\epsilon \to 0} \frac{1}{\epsilon^Q} \delta_\epsilon^* \mu.$$

Here the convergence is understood in the weak-$*$-topology of $C_c(M)'$ and Q denotes the Hausdorff dimension of the equiregular SR manifold M (for more details see [110]). Due to the equiregularity assumption of the SR manifold M, the measure $\widetilde{\mu}_q$ is in fact a left-invariant measure on $\mathbb{G}(q)$. Since $\mathbb{G}(q)$ is nilpotent and hence unimodular, the measure $\widetilde{\mu}^q$ is a Haar measure on $\mathbb{G}(q)$ (see [110]).

Now we recall the interpretation of the heat invariants c_0 and c_1 with the help of the nilpotentization of the subriemannian manifold M. For a horizontal vector field X, we define its ϵ-dilation X^ϵ (in privileged coordinates) by

$$X^\epsilon := \epsilon \delta_\epsilon^*(X).$$

If the sublaplacian Δ_{sub}^μ has the form

$$\Delta_{sub}^\mu = -\sum_{i=1}^m X_i^2 + Y,$$

where Y is a horizontal vector field, then its ϵ-dilation $\Delta_{sub}^{\mu,\epsilon}$ is defined by the formula

$$\Delta_{sub}^{\mu,\epsilon} := -\sum_{i=1}^m (X_i^\epsilon)^2 + \epsilon Y^\epsilon. \tag{2.3}$$

Note that the second term in (2.3) has an ϵ-coefficient in order to ensure that $\Delta_{sub}^{\mu,\epsilon}$ converges (in C^∞ topology) to the sublaplacian

$$\widetilde{\Delta}_{sub} := \sum_{i=1}^m \left(X_i^{(-1)}\right)^2$$

as $\epsilon \to 0$. Furthermore, the heat kernels K_t^ϵ and K_t of the operators $\Delta_{sub}^{\mu,\epsilon}$ and Δ_{sub}^μ are related by the formula

$$K_t^\epsilon(q', q'') = \epsilon^Q K_{\epsilon^2 t}(\delta_\epsilon(q'), \delta_\epsilon(q'')) + O(\epsilon^\infty), \tag{2.4}$$

where Q denotes the Hausdorff dimension of M (s. [110, 7]).
We consider the anisotropic expansion of the horizontal vector fields X_i^ϵ and Y^ϵ as $\epsilon \to 0$:

$$\begin{aligned} X_i^\epsilon &= X_i^{(-1)} + \epsilon X_i^{(0)} + \epsilon^2 X_i^{(1)} + \cdots \\ Y^\epsilon &= Y^{(-1)} + \epsilon Y^{(0)} + \epsilon^2 Y^{(1)} + \cdots \end{aligned}$$

Then the sublaplacian $\Delta_{sub}^{\mu,\epsilon}$ has an expansion as $\epsilon \to 0$ of the form

$$\Delta_{sub}^{\mu,\epsilon} = -\left(\widetilde{\Delta}_{sub} + \epsilon \mathcal{P}_1 + \epsilon^2 \mathcal{P}_2 + \cdots\right) \tag{2.5}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are second-order differential operators given by

$$\begin{aligned}\mathcal{P}_1 &:= \sum_{i=1}^m \left(X_i^{(-1)} X_i^{(0)} + X_i^{(0)} X_i^{(-1)} \right) + Y^{(-1)} \\ \mathcal{P}_2 &:= \sum_{i=1}^m X_i^{(-1)} X_i^{(1)} + \sum_{i=1}^m X_i^{(1)} X_i^{(-1)} + (X_i^{(0)})^2 + Y^{(0)}.\end{aligned}$$

By using Duhamel formula and setting $t = 1, \epsilon = \sqrt{s}$ in (2.4), we obtain the following expressions for the heat invariants $c_0(q)$ and $c_1(q)$ (s. [110]):

$$c_0(q) = K_s^{\mathbb{G}(q)}(0,0), \tag{2.6}$$

where $K_s^{\mathbb{G}(q)}$ denotes the heat kernel of the intrinsic sublaplacian

$$\widetilde{\Delta}_{sub} = \sum_{i=1}^m \left(X_i^{(-1)} \right)^2$$

on $\mathbb{G}(q)$ with respect to the Haar measure $\widetilde{\mu}_q$ at time s. The second heat invariant $c_1(q)$ is given as a convolution integral by the formula

$$c_1(q) = \int_0^1 \int_{\mathbb{G}(q)} K_{1-s}^{\mathbb{G}(q)}(0,\xi) \mathcal{P}_2 \left(K_s^{\mathbb{G}(q)}(\xi, 0) \right) d\xi ds, \tag{2.7}$$

where $\mathcal{P}_2$ acts on the first variable.

Chapter 3

Spectral theory of a class of nilmanifolds attached to Clifford modules

This Chapter is taken from the article $A1$. In order to detect isospectral (subriemannian) nilmanifolds we first need to determine the spectrum of the sublaplacian $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ on a step 2 nilmanifold $M = \Gamma\backslash\mathbb{G}$. Based on an explicit expression of the heat kernel for Δ_{sub} on the covering group $\mathbb{G}$ in [27, 36, 52] a formula for the heat trace of $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ descended from $\mathbb{G}$ to $M = \Gamma\backslash\mathbb{G}$ was obtained in [17]. In case of a pseudo H-type group $\mathbb{G}$ this trace formula simplifies further and in principle can be used to explicitly calculate the spectrum of Δ_{sub} on M. However, we need not to perform the full calculation. In order to identify isospectral manifolds it is sufficient to compare the corresponding trace formulas. In a second step we need to classify non-homeomorphic nilmanifolds $\Gamma_1\backslash\mathbb{G}_1$ and $\Gamma_2\backslash\mathbb{G}_2$ of the same dimension. First, we reduce this task to a classification of pseudo H-type Lie algebras up to isomorphisms (cf. Corollary 3.6.2). Then we apply the recent classification results in [54, 53].

This chapter is organized as follows: In Section 3.1 we introduce the sublaplacian on a general step 2 nilpotent Lie group $\mathbb{G}$ and we recall an explicit integral expression of its heat kernel known as *Beals-Gaveau-Greiner formula*, cf. [26, 27, 36]. Assuming the existence of a lattice Γ in $\mathbb{G}$ we decompose the sublaplacian $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ on the compact nilmanifold $M = \Gamma\backslash\mathbb{G}$ into an infinite sum of elliptic operators acting on line bundles in Section 3.2. Via this method we obtain a decomposition of the heat trace of $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ into the heat traces of its component elliptic operators, cf. [17], and we present a trace formula for the sublaplacian on M. In Section 3.3 we recall the notion of pseudo H-type Lie algebras and groups following [41, 54, 55]. We discuss the existence and some basic properties of integral lattices for such groups. These will play a role in our construction in Section 3.6. In Section 3.4 we study the eigenvalues of a matrix-valued function which encode the structure constants of the pseudo H-type Lie algebra. These data are essential in the calculation of the heat kernel of the sublaplacian in Section 3.1 and the trace formula in Section 3.2. Based on the trace formula we give a criterion for isospectrality of two pseudo H-type nilmanifolds in Section 3.5 (Theorem 3.5.3). The last sections contain our main results. We use the classification of pseudo H-type Lie algebras in [53, 54] to construct finite families of isospectral, non-homeomorphic pseudo H-type nilmanifolds. Finally, we present two nilmanifolds of different dimensions such that the short time heat trace expansions of the corresponding sublaplacians coincide up to a term vanishing to infinite order as time tends to zero.

3.1 Heat kernel on two step nilpotent Lie groups

We recall the integral form of the heat kernel for a sublaplacian on simply connected two step nilpotent Lie groups given in [26, 27], see also [36, 52].

3.1.1 Sublaplacian on two step nilpotent groups:

Let $\mathbb{G}$ be a simply connected two step nilpotent Lie group with Lie algebra $\mathfrak{g}$. We assume that

$$\big[\mathfrak{g}, \mathfrak{g}\big] = \textit{center of } \mathfrak{g} \tag{3.1}$$

and we fix a basis $\{X_i,\, Z_k \,:\, i = 1, \cdots, N,\ \ k = 1, \cdots, d\}$ of $\mathfrak{g}$ such that $\{Z_k\}_{k=1}^{d}$ and $\{X_i\}_{i=1}^{N}$ span the center $[\mathfrak{g}, \mathfrak{g}]$ and its complement, respectively. Moreover, we assume that $\mathfrak{g}$ is equipped with an inner product with respect to which $\{X_i,\, Z_k\}$ becomes an orthonormal basis. Hence the Lie algebra $\mathfrak{g}$ is decomposed into an orthogonal sum

$$\mathfrak{g} = \operatorname{span}\big\{X_1, \cdots, X_N\big\} \oplus_{\perp} \big[\mathfrak{g}, \mathfrak{g}\big] \cong \mathbb{R}^N \oplus_{\perp} \mathbb{R}^d,$$

i.e. $\mathbb{G}$ is a step 2 *Carnot group*. The expansion of Lie brackets

$$[X_i, X_j] = \sum_{k=1}^{d} c_{i\,j}^{k} Z_k, \tag{3.2}$$

defines the structure constants $c_{i\,j}^{k} = -c_{j\,i}^{k}$. Given $Z = \sum_{k=1}^{d} z_k Z_k \in [\mathfrak{g}, \mathfrak{g}]$ we denote by $\Omega(Z)$ the skew-symmetric matrix

$$\Omega(Z) = \frac{1}{2} \sum_{k=1}^{d} z_k \left(c_{i\,j}^{k}\right)_{i,j} \in \mathbb{R}(N) =: \textit{algebra of } N \times N \textit{ real matrices.} \tag{3.3}$$

Remark 3.1.1. *Throughout this Chapter we identify the group $\mathbb{G}$ with $\mathbb{R}^N \times \mathbb{R}^d$ via the group exponential coordinates, i.e.*

$$\mathbb{G} \ni g = \sum_{i=1}^{N} x_i X_i + \sum_{k=1}^{d} z_k Z_k \longleftrightarrow (x_1, \cdots, x_N, z_1, \cdots, z_d) \in \mathbb{R}^N \times \mathbb{R}^d \cong \mathfrak{g}.$$

Then the exponential map $\exp : \mathfrak{g} \xrightarrow{\approx} \mathbb{G}$ *is the identity. Via the Baker-Campbell Hausdorff formula and this identification we can express the group product* $*$ *on* $\mathbb{G} \cong \mathfrak{g}$ *in the form*

$$g * h = g + h + \frac{1}{2}[g, h].$$

More explicitly and with respect to the above coordinates one has:

$$\begin{aligned} g * h &= (x_1, \cdots, x_N, z_1, \cdots, z_d) * (x_1', \cdots, x_N', z_1', \cdots, z_d') \\ &= \Big(x_1 + x_1', \cdots, x_N + x_N', z_1 + z_1' + \frac{1}{2}\sum_{i,j} c_{i\,j}^1 x_i x_j', \cdots, z_d + z_d' + \frac{1}{2}\sum_{i,j} c_{i\,j}^d x_i x_j'\Big). \end{aligned}$$

Let $\tilde{X}_i$ denote the left-invariant vector field on $\mathbb{G}$ corresponding to $X_i \in \mathfrak{g}$ and consider the intrinsic *sublaplacian*

$$\Delta_{\text{sub}}^{\mathbb{G}} = -\sum_{i=1}^{N} \tilde{X}_i^2. \tag{3.4}$$

Based on (3.1) the operator $\Delta_{\text{sub}}^{\mathbb{G}}$ is known to be sub-elliptic [61] and essentially selfadjoint in $L_2(\mathbb{G})$ with respect to the Haar measure and considered on compactly supported smooth functions $C_0^\infty(\mathbb{G})$, cf. [106, 107].

3.1.2 Beals-Gaveau-Greiner formula:

Next we recall the integral expression of the kernel function $K_t(g, h) \in C^\infty(\mathbb{R}_+ \times \mathbb{G} \times \mathbb{G})$ of the heat operator

$$e^{-t\,\Delta_{\text{sub}}^{\mathbb{G}}}, \tag{3.5}$$

where $\mathbb{G}$ is a general step 2 nilpotent Lie group as above. The existence of a smooth kernel has been shown in [106, 107] and since the sublaplacian $\Delta_{\text{sub}}^{\mathbb{G}}$ is a left-invariant operator it follows that K is a convolution kernel, i.e.

$$K_t^{\mathbb{G}}(g, h) = k^{\mathbb{G}}(t,\, g^{-1} \, * \, h)$$

with a smooth function $k^{\mathbb{G}} \in C^\infty(\mathbb{R}_+ \times \mathbb{G})$.

In [26, 27, 36, 52] an integral expression of $k^{\mathbb{G}}$ is given explicitly. Below we will calculate the spectrum of the sublaplacian on a class of nilmanifolds by using this expression. Recall that in the integrand of $k^{\mathbb{G}}$ two functions (*action and volume function*) appear. The integration is taken over a space which can be interpreted as the *characteristic variety* of the sublaplacian. Here we will neither present the details of this structure nor a proof of the next theorem.

Theorem 3.1.2 (Beals-Gaveau-Greiner formula, [26, 36])**.** *The integral kernel of the heat operator (3.5) has the form:*

$$K_t^{\mathbb{G}}(g, h) = k^{\mathbb{G}}(t, g^{-1} \, * \, h) = \frac{1}{(4\pi t)^{N/2+d}} \int_{\mathbb{R}^d} e^{-\frac{f(\tau,\, g^{-1} * h)}{2t}}\, W(\tau)\, d\tau,$$

where the functions $f = f(\tau, g) \in C^\infty(\mathbb{R}^d \times \mathbb{G})$ *and* $W(\tau) \in C^\infty(\mathbb{R}^d)$ *are given as follows: put* $g = (x, z) \in \mathbb{R}^N \times \mathbb{R}^d$*, then*

$$f(\tau, g) = f(\tau, x, z) = \sqrt{-1}\langle \tau,\, z\rangle + \frac{1}{2}\Big\langle \Omega(\sqrt{-1}\tau)\coth\big(\Omega(\sqrt{-1}\tau)\big)\cdot x, x\Big\rangle,$$

$$W(\tau) = \left\{\det \frac{\Omega(\sqrt{-1}\tau)}{\sinh \Omega(\sqrt{-1}\tau)}\right\}^{1/2},$$

where $\langle z, z'\rangle = \sum_{k=1}^{d} z_k z'_k$ *denotes the Euclidean inner product on* $\mathbb{R}^d$*.*

Remark 3.1.3. *Later on we will use the notation* $\langle \bullet,\, \bullet\rangle_{r,s}$ *for a non-degenerate indefinite scalar product with the signature* (r, s) *such that* $\langle \cdot,\, \cdot\rangle = \langle \bullet,\, \bullet\rangle_{d,0}$*.*

We call $f = f(\tau, x, z)$ and $W(\tau)d\tau$ the *complex action function* and the *volume form*, respectively. Recall that f is constructed by the *complex Hamilton-Jacobi method*, and the volume function $W(\tau)$ is sometimes referred to as *van Vleck determinant*. It is the Jacobian of the correspondence between the space of initial conditions and boundary conditions when we solve the Hamilton equation associated to the symbol of the sublaplacian. The solution can be interpreted as the bi-characteristic flow in the subriemannian setting. We recall that the volume function satisfies a *transportation equation*.

3.2 Lattices and decomposition of a sublaplacian

Based on Theorem 3.1.2 we describe the heat kernel of the sublaplacian descended to the quotient space $\Gamma\backslash\mathbb{G}$ (left coset space) by a lattice Γ. Such a space is called a (compact) *step* 2 *nilmanifold*. In the following we assume that there exists a *lattice* (cocompact discrete subgroup) in $\mathbb{G}$. We recall *Malćev's Theorem*:

Theorem 3.2.1 (Malćev, [83, 98])**.** *A nilpotent Lie group* G *possesses a lattice* Γ*, i.e.* $\Gamma\backslash\mathbb{G}$ *is compact, if and only if there exists a basis* $\{Y_i\}$ *in its Lie algebra* $\mathfrak{g}$ *such that the structure constants* $\{\alpha_{ij}^k\}$ *defined by*

$$[Y_i,\, Y_j] = \sum_k \alpha_{ij}^k Y_k$$

are all rational numbers.

3.2.1 Torus bundle and a family of elliptic operators

We recall a heat trace formula which previously has been obtained in [17, Theorem 4.2]. Our analysis is essential based on this formula and in order to keep the Section self-contained we now repeat the main steps of the calculation.

Let Γ be a lattice in a simply connected step 2 nilpotent Lie group $\mathbb{G} \cong \mathbb{R}^N \times \mathbb{R}^d$. The quotient space $\Gamma\backslash\mathbb{G}$ can be equipped with a subriemannian structure naturally inherited from that of $\mathbb{G}$. Its intrinsic sublaplacian, which we now denote by $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$, is the operator descended from the sublaplacian $\Delta_{\text{sub}}^{\mathbb{G}}$ on $\mathbb{G}$ (see Chapter 4).

For an element $g \in \mathbb{G}$ we will denote by $[g] \in \Gamma\backslash\mathbb{G}$ the corresponding class in the quotient space. Then, the heat kernel

$$K_t^{\Gamma\backslash\mathbb{G}}([g],[h]) \in C^\infty\Big(\mathbb{R}_+ \times \Gamma\backslash\mathbb{G} \times \Gamma\backslash\mathbb{G}\Big)$$

of the sublaplacian $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ on the nilmanifold $\Gamma\backslash\mathbb{G}$ is given by

$$\begin{aligned} K_t^{\Gamma\backslash\mathbb{G}}([g],[h]) &= \sum_{\gamma\in\Gamma} K_t^{\mathbb{G}}\big(\gamma * g, h\big) \\ &= \sum_{\gamma\in\Gamma} k^{\mathbb{G}}\big(t, g^{-1} * \gamma * h\big) \in C^\infty\Big(\mathbb{R}_+ \times \Gamma\backslash\mathbb{G} \times \Gamma\backslash\mathbb{G}\Big). \end{aligned} \tag{3.6}$$

Assuming the existence of a lattice Γ in $\mathbb{G}$ we can decompose the sublaplacian into a family of differential operators acting on invariant subspaces according to a torus bundle structure of $\Gamma\backslash\mathbb{G}$. Next, we present some details and give the heat kernel expression for each component elliptic operator.

Let $\mathbb{A} \cong \mathbb{R}^d$ be the center of the group $\mathbb{G}$ where as before the identification is done with respect to the fixed orthonormal basis $\{Z_k\}$ of $[\mathfrak{g},\mathfrak{g}]$. We obtain a principal bundle with the structure group $\mathbb{A}/(\Gamma\cap\mathbb{A}) \cong \mathbb{T}^{\dim\mathbb{A}} = \mathbb{T}^d$:

$$\Gamma\backslash\mathbb{G} \longrightarrow (\Gamma/\Gamma\cap\mathbb{A})\backslash(\mathbb{G}/\mathbb{A}) \cong (\Gamma * \mathbb{A})\backslash\mathbb{G}.$$

Note that the base space $(\Gamma/\Gamma\cap\mathbb{A})\backslash(\mathbb{G}/\mathbb{A}) \cong (\Gamma * \mathbb{A})\backslash\mathbb{G}$ is also a torus of dimension $\dim\mathbb{G} - \dim\mathbb{A} = N + d - d = N$. Since $\mathbb{A}$ is abelian, the subgroup $\Gamma * \mathbb{A}$ coincides with $\Gamma + \mathbb{A}$, i.e. with the sum in the Lie algebra.

Let $\mathbf{n}$ be an element in the "dual lattice" $[\Gamma\cap\mathbb{A}]^*$ of $\Gamma\cap\mathbb{A}$, that is, $\mathbf{n}$ is a linear functional on $\mathbb{A}$ with the property that

$$\mathbf{n}(\gamma) \in \mathbb{Z} \textit{ for all } \gamma \in \Gamma\cap\mathbb{A}.$$

We may express $\mathbf{n}$ in the form $\mathbf{n} = \sum_{k=1}^{d} n_k Z_k$ with integer coefficients $n_k \in \mathbb{Z}$ such that

$$\mathbf{n}(\gamma) = \langle \mathbf{n}, \gamma\rangle = \sum n_k \langle Z_k, \gamma\rangle \in \mathbb{Z} \textit{ for all } \gamma \in \Gamma\cap\mathbb{A}.$$

Then, the function space $C^\infty(\Gamma\backslash\mathbb{G})$ is decomposed via a Fourier series expansion:

$$C^\infty(\Gamma\backslash\mathbb{G}) \ni {}^\forall f \ ; \ f(g) = \sum_{\mathbf{n}\in[\Gamma\cap\mathbb{A}]^*} \int_{\mathbb{T}^d} f(g*\lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda,$$

where $\chi_{\mathbf{n}} : \mathbb{T}^d \cong \mathbb{A}/(\Gamma \cap \mathbb{A}) \to U(1)$ with $\chi_{\mathbf{n}}(\lambda) = e^{2\pi\sqrt{-1}\langle \mathbf{n},\lambda\rangle}$ is a unitary character corresponding to a dual element $\mathbf{n} \in [\Gamma \cap \mathbb{A}]^*$. So, we decompose

$$C^\infty(\Gamma\backslash\mathbb{G}) = \sum_{\mathbf{n}\in[\Gamma\cap\mathbb{A}]^*} \mathcal{F}^{(\mathbf{n})},$$

where

$$\mathcal{F}^{(\mathbf{n})} = \left\{ \int_{\mathbb{T}^d} f(g * \lambda)\, \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \mid f \in C^\infty(\Gamma\backslash\mathbb{G}) \right\}.$$

The subspace $\mathcal{F}^{(\mathbf{n})}$ can be seen as a space of smooth sections of a line bundle $E^{(\mathbf{n})}$ on the base space $(\Gamma + \mathbb{A})\backslash\mathbb{G} \cong (\Gamma/\Gamma \cap \mathbb{A})\backslash(\mathbb{G}/\mathbb{A})$ associated to the character $\chi_{\mathbf{n}}$. The sublaplacian leaves invariant each subspace $\mathcal{F}^{(\mathbf{n})}$ and therefore it can be interpreted as a differential operator $\mathcal{D}^{(\mathbf{n})}$ acting on the line bundle $E^{(\mathbf{n})}$. Since the subbundle spanned by the (left)-invariant vector fields $\{\tilde{X}_i \mid i = 1, \cdots, N\}$ defines a connection, i.e., its linear span is equivariant and transversal to the structure group action by $\mathbb{A}/(\Gamma \cap \mathbb{A})$, each operator $\mathcal{D}^{(\mathbf{n})}$ is elliptic. Hence the sublaplacian $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ can be seen as an infinite sum of elliptic operators on the torus $\Gamma * \mathbb{A}\backslash\mathbb{G}$

As a consequence we obtain a decomposition of the operator trace:

$$\operatorname{tr}\left(e^{-\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}}\right) = \sum_{\mathbf{n}\in[\Gamma\cap\mathbb{A}]^*} \operatorname{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right). \tag{3.7}$$

Recall that $\{Z_k : k = 1, \cdots, d\}$ denotes an orthonormal basis of the center $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. As before we write $\tilde{Z}_k, k = 1, \cdots, d$ for the corresponding left-invariant vector fields on the group $\mathbb{G}$. We equip $\mathbb{G}$ with a left-invariant Riemannian metric defined by assuming that the frame $[\tilde{X}_1, \cdots, \tilde{X}_N, \tilde{Z}_1, \cdots, \tilde{Z}_d]$ is orthonormal at any point of $\mathbb{G}$. Then the corresponding Laplacian has the form

$$\Delta^{\mathbb{G}} = \Delta_{\text{sub}}^{\mathbb{G}} - \sum_{k=1}^{d} \tilde{Z}_k^2. \tag{3.8}$$

The action of the difference $\Delta^{\mathbb{G}} - \Delta_{\text{sub}}^{\mathbb{G}}$ on the subspace $\mathcal{F}^{(\mathbf{n})}$ for each dual element $\mathbf{n} \in [\Gamma \cap \mathbb{A}]^*$ is given as follows:

Proposition 3.2.2. *Let $f \in \mathcal{F}^{(\mathbf{n})}$, then*

$$(\Delta^{\mathbb{G}} - \Delta_{\text{sub}}^{\mathbb{G}})f = -\sum_{k=1}^{d} \tilde{Z}_k^2(f) = 4\pi^2 \sum_{k=1}^{d} {n_k}^2 \cdot f.$$

3.2.2 Heat trace of the component operators

Next we give an expression of the heat trace of each operator $\mathcal{D}^{(\mathbf{n})}$. Recall that the heat kernel $K^{\Gamma\backslash\mathbb{G}}$ of $\Delta_{\text{sub}}^{\Gamma\backslash\mathbb{G}}$ is given by (3.6). Let $\mathcal{F}_\Gamma$ and $\mathcal{F}_{\Gamma\cap\mathbb{A}}$ be a fundamental domain for the lattice Γ in $\mathbb{G}$ and $\Gamma \cap \mathbb{A}$ in the Euclidean space $\mathbb{A}$, respectively. Then the integral

$$k_{\mathcal{D}^{(\mathbf{n})}}\big(t, [g], [h]\big) = \int_{\mathcal{F}_{\Gamma\cap\mathbb{A}}} K_t^{\Gamma\backslash\mathbb{G}}\big([g], [h] * \lambda\big)\overline{\chi_{\mathbf{n}}(\lambda)}\, d\lambda$$

is the kernel function for the heat operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$, that is it satisfies

$$k_{\mathcal{D}^{(\mathbf{n})}}(t, [g] * \theta, [h]) = k_{\mathcal{D}^{(\mathbf{n})}}(t, [g * \theta], [h]) = \overline{\chi_{\mathbf{n}}(\theta)} k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h]),$$
$$k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h] * \theta) = k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h * \theta]) = \chi_{\mathbf{n}}(\theta) k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h]),$$

where $\theta \in \mathbb{A}$. Let $\mathbb{M} = \{\mu_i\}$ be a set of complete representatives of the coset space $\Gamma/(\Gamma \cap \mathbb{A})$, then the trace of the heat operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$ is given as follows:

Proposition 3.2.3 (see [17])**.** *For each* $\mathbf{n}$ *in the dual lattice* $[\Gamma \cap \mathbb{A}]^*$ *and with the heat kernel* $K^{\mathbb{G}}$ *of the sublaplacian on* $\mathbb{G}$*:*

$$\mathrm{Vol}\big(\mathbb{A}/(\Gamma \cap \mathbb{A})\big) \cdot tr\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \int_{\mathcal{F}_\Gamma} \left(\sum_{\gamma \in \Gamma} \int_{\mathcal{F}_{\Gamma \cap \mathbb{A}}} K_t^{\mathbb{G}}(g,\, \gamma * g * \lambda) \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \right) dg$$
$$= \int_{\mathcal{F}_\Gamma} \left(\sum_{\mu \in \mathbb{M}} \sum_{\nu \in \Gamma \cap \mathbb{A}} \int_{\mathcal{F}_{\Gamma \cap \mathbb{A}}} k^{\mathbb{G}}\big(t, g^{-1} * \mu * g * \nu * \lambda\big) \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \right) dg$$
$$= \int_{\mathcal{F}_\Gamma} \left(\sum_{\mu \in \mathbb{M}} \int_{\mathbb{R}^d} k^{\mathbb{G}}\big(t, g^{-1} * \mu * g * \lambda\big) \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \right) dg$$
$$= \sum_{\mu \in \mathbb{M}} \int_{\mathcal{F}_\Gamma} \int_{\mathbb{R}^d} k^{\mathbb{G}}\big(t, g^{-1} * \mu * g * \lambda\big) \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \, dg.$$

Applying Theorem 3.1.2 we can give a more concrete expression of the formula in Proposition 3.2.3. For this purpose and for the sake of simplicity, we assume that the structure constants $c_{i\,j}^k$ in (3.2) are of the form

$$c_{i\,j}^k = \frac{2q_{i\,j}^k}{p_0}$$

with a common positive integer $p_0 \geq 1$ and integers $q_{i\,j}^k$. Then we fix a lattice Γ

$$\Gamma := \left\{ \sum_{1 \leq i \leq N} m_i X_i + \sum_{1 \leq k \leq d} \frac{\ell_k}{p_0} Z_k \;\Big|\; m_i,\, \ell_k \in \mathbb{Z} \right\},$$

and we choose the set $\mathbb{M} = \Big\{ \mu = \sum_{1 \leq i \leq N} m_i X_i \mid m_i \in \mathbb{Z} \Big\}$ of representatives of the quotient group $(\Gamma \cap \mathbb{A}) \backslash \Gamma = \Gamma/(\Gamma \cap \mathbb{A})$. For each fixed

$$\mathbf{n} = p_0 \sum_{k=1}^d n_k Z_k \in [\Gamma \cap \mathbb{A}]^*,$$

where $n_k \in \mathbb{Z}$ we have

$$\begin{aligned}
&\mathrm{Vol}\big(\mathbb{A}/(\Gamma\cap\mathbb{A})\big)\cdot \mathrm{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right)\\
&=\frac{1}{(4\pi t)^{N/2+d}}\int_{\mathcal{F}_\Gamma}\sum_{\mu\in\mathbb{M}}\int_{\mathbb{A}}\int_{\mathbb{R}^d} e^{-\sqrt{-1}\frac{\langle[\mu,x]+\lambda,\tau\rangle}{2t}}\cdot\varphi_t(\tau,\mu)d\tau\,\overline{\chi_{\mathbf{n}}(\lambda)}\,d\lambda\,dx\,dz=\ (*),
\end{aligned}$$

where the function $\varphi_t(\tau,\mu)$ in the integrand is given by:

$$\varphi_t(\tau,\mu)=\exp\Big\{-\frac{1}{4t}\big\langle\Omega(\sqrt{-1}\tau)\coth\Omega(\sqrt{-1}\tau)\cdot\mu,\mu\big\rangle\Big\}W(\tau).$$

In the following we write $\hat{\varphi}_t(\tau,\mu)$ for the Fourier transform of φ_t with respect to the τ-variable. Then

$$\begin{aligned}
(*)&=\frac{1}{(2t)^{N/2+d}\cdot(2\pi)^{(N+d)/2}}\int_{\mathcal{F}_\Gamma}\sum_{\mu\in\mathbb{M}}\int_{\mathbb{A}}\hat{\varphi}_t\left(\frac{[\mu,x]+\lambda}{2t},\mu\right)\cdot e^{-2\pi\sqrt{-1}\langle\mathbf{n},\lambda\rangle}\,d\lambda\,dx\,dz\\
&=\frac{1}{(2t)^{N/2+d}\cdot(2\pi)^{(N+d)/2}}\int_{\mathcal{F}_\Gamma}\sum_{\mu\in\mathbb{M}}\int_{\mathbb{A}}\hat{\varphi}_t\left(u,\mu\right)\cdot e^{-2\pi\sqrt{-1}\langle\mathbf{n},\,2tu+[x,\mu]\rangle}(2t)^d\,du\,dx\,dz,\\
&=\frac{1}{(4\pi t)^{N/2}}\cdot {p_0}^d\cdot\sum_{\mu\in\mathbb{M}}\varphi_t(-4\pi t\mathbf{n},\mu)\cdot\int_{\underbrace{[0,1]\times\cdots\times[0,1]}_{N\text{ times}}} e^{-2\pi\sqrt{-1}\langle\mathbf{n},\,[x,\mu]\rangle}\,dx.
\end{aligned}$$

With a suitable set of linear independent vectors $a_1(\mathbf{n}),\,\cdots,a_{b(\mathbf{n})}(\mathbf{n})$ in Γ the solution space $\mathbb{M}(\mathbf{n})=\{\mu\in\mathbb{M}\mid\Omega(\mathbf{n})(\mu)=0\ \}$ can be written as

$$\mathbb{M}(\mathbf{n})=\Big\{\ \mu=\sum_{i=1}^{b(\mathbf{n})}m_i\,a_i(\mathbf{n}),\ \Big|\ m_i\in\mathbb{Z}\Big\}.$$

Here $b(\mathbf{n})\le N$ and $b(\mathbf{n})=N$ if and only if $\mathbf{n}=0$. Hence

Theorem 3.2.4. *For each* $\mathbf{n}$ *in the dual lattice* $[\Gamma\cap\mathbb{A}]^*$ *and with the above notation:*

$$\begin{aligned}
tr\left(e^{-t\,\mathcal{D}^{(\mathbf{n})}}\right)&=\frac{1}{(4\pi t)^{N/2}}\sum_{\mu\in\mathbb{M}(\mathbf{n})}e^{-\frac{\left\langle\Omega(4\pi t\sqrt{-1}\mathbf{n})\,\coth\Omega(4\pi\sqrt{-1}t\mathbf{n})\mu,\,\mu\right\rangle}{4t}}\sqrt{\det\frac{\Omega(4\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(4\pi\sqrt{-1}t\mathbf{n})}}\\
&=\frac{1}{(4\pi t)^{N/2}}\sum_{\mu\in\mathbb{M}(\mathbf{n})}e^{-\frac{<\mu,\mu>}{4t}}\sqrt{\det\frac{\Omega(4\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(4\pi\sqrt{-1}t\mathbf{n})}}.
\end{aligned}\tag{3.9}$$

In particular, it holds:

$$tr\left(e^{-t\Delta_{\mathrm{sub}}^{\Gamma\backslash\mathbb{G}}}\right)=\frac{1}{(4\pi t)^{N/2}}\sum_{\mathbf{n}\in[\Gamma\cap\mathbb{A}]^*}\sum_{\mu\in\mathbb{M}(\mathbf{n})}e^{-\frac{<\mu,\mu>}{4t}}\sqrt{\det\frac{\Omega(4\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(4\pi\sqrt{-1}t\mathbf{n})}}.$$

Proof. It suffices to show the second equation in (3.9). Note that the defining equation $\Omega(\mathbf{n})(\mu) = 0$ for $\mu \in \mathbb{M}(\mathbf{n})$ implies:

$$\big\langle \Omega(2\pi t\sqrt{-1}\mathbf{n}) \coth \Omega(2\pi\sqrt{-1}t\mathbf{n})(\mu),\, \mu\big\rangle = \langle \mu,\, \mu\rangle = \sum m_i m_j \big\langle a_i(\mathbf{n}), a_j(\mathbf{n})\big\rangle.$$

The last statement follows from (3.7) and (3.9). □

Corollary 3.2.5. *For $\mathbf{n}, -\mathbf{n} \in [\Gamma \cap \mathbb{A}]^*$ the traces $tr\left(e^{-t\,\mathcal{D}(\mathbf{n})}\right)$ and $tr\left(e^{-t\,\mathcal{D}(-\mathbf{n})}\right)$ coincide.*

3.3 Pseudo H-type algebras and groups

For the rest of the Chapter we consider a specific subclass of step 2 nilpotent Lie groups, the so called *pseudo H-type groups*. These are generalizations of Heisenberg type groups in [71, 72] and have been first introduced in [41]. An extensive analysis of the structure and classification of pseudo H-type groups and their algebras can be found in the recent papers [53, 54, 55]. For completeness we recall the relevant definitions:

We write $\mathbb{R}^{r,s}$ for the Euclidean space $\mathbb{R}^{r+s}$ equipped with the non-degenerate scalar product

$$\langle x,\, y\rangle_{r,s} := \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Consider the quadratic form $q_{r,s}(x) = \langle x, x\rangle_{r,s}$ and let $C\ell_{r,s}$ denote the Clifford algebra generated by $(\mathbb{R}^{r,s}, q_{r,s})$. We call a $C\ell_{r,s}$-module $\mathcal{V}$ *admissible*, if there is a non-degenerate bilinear form (= scalar product) $\langle \bullet, \bullet\rangle_{\mathcal{V}}$ on $\mathcal{V}$ satisfying the following conditions:

(a) There is a *Clifford module action* $J : C\ell_{r,s} \times \mathcal{V} \to \mathcal{V} : (Z, X) \mapsto J_Z X$, i.e.

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z'\rangle_{r,s} I \quad \textit{for all} \quad Z, Z' \in \mathbb{R}^{r,s}. \tag{3.10}$$

(b) For all $Z \in \mathbb{R}^{r,s}$ the map J_Z is skew-symmetric on $\mathcal{V}$ with respect to $\langle \bullet, \bullet\rangle_{\mathcal{V}}$, i.e.

$$\big\langle J_Z X, Y\big\rangle_{\mathcal{V}} + \big\langle X, J_Z Y\big\rangle_{\mathcal{V}} = 0 \quad \textit{for all} \quad X, Y \in \mathcal{V}. \tag{3.11}$$

Moreover, from (a) and (b) one concludes:

$$\big\langle J_Z X, J_Z Y\big\rangle_{\mathcal{V}} = \langle Z, Z\rangle_{r,s} \langle X, Y\rangle_{\mathcal{V}} \quad \textit{where} \quad X, Y \in \mathcal{V} \ \text{ and } Z \in \mathbb{R}^{r,s}. \tag{3.12}$$

We write $\{J,\ \mathcal{V},\ \langle \bullet,\, \bullet\rangle_{\mathcal{V}}\}$ for an admissible module of the Clifford algebra $C\ell_{r,s}$ with the module action $J = J_Z$ and the scalar product $\langle \bullet,\, \bullet\rangle_{\mathcal{V}}$.

Remark 3.3.1. The existence of an admissible $C\ell_{r,s}$-module $\mathcal{V}$ has been shown in [41]. If $s \neq 0$ then an admissible module $\mathcal{V}$ needs not to be irreducible. More precisely, five cases are possible which all are present in the classification. If $C\ell_{r,s}$ has, up to equivalence, only one irreducible representation $(J, \mathcal{V})$, then either $\mathcal{V}$ or the sum $\mathcal{V} \oplus \mathcal{V}$ is admissible. In the case where $C\ell_{r,s}$ has two non-equivalent irreducible representations $(J^{(i)}, \mathcal{V}_i)$, $i = 1, 2$, then either $\mathcal{V}_i$ for $i = 1, 2$ both are admissible, or only $\mathcal{V}_1 \oplus \mathcal{V}_2$ is admissible, or $\mathcal{V}_1 \oplus \mathcal{V}_1$ and $\mathcal{V}_2 \oplus \mathcal{V}_2$ simultaneously are admissible. These cases are complementary to each other (cf. [41, 53, 54, 55]).

In the case $s = 0$ the situation is simpler. Every irreducible module $\mathcal{V}$ is admissible with respect to an inner product (i.e. $\langle \bullet, \bullet \rangle_{\mathcal{V}}$ is positive definite). Originally such cases have been defined and studied by A. Kaplan in [71].

In the following, we call a vector $X \in \mathcal{V}$ *positive* (resp. *negative*) if the scalar product $\langle X, X \rangle_{\mathcal{V}}$ is *positive* (resp. *negative*) and *null vector* if $\langle X, X \rangle_{\mathcal{V}} = 0$. A similar notation is used for vectors $Z \in \mathbb{R}^{r,s}$. If $s > 0$, then an admissible module $\mathcal{V}$ with scalar product $\langle \bullet, \bullet \rangle_{\mathcal{V}}$ has positive and negative subspaces of the same dimension N with respect to the above scalar product $\langle \bullet, \bullet \rangle_{\mathcal{V}}$, cf. [41]. In particular, $\dim \mathcal{V} = 2N$ is even.

Moreover, $\mathcal{V}$ decomposes into the orthogonal sum of *minimal dimensional admissible modules*. In fact, since the scalar product restricted to such an invariant subspace is non-degenerate, the orthogonal complement is also an admissible module.

Definition 3.3.2. Let $\{J, \mathcal{V}, \langle \bullet, \bullet \rangle_{\mathcal{V}}\}$ be an admissible $C\ell_{r,s}$-module.

(1) The step 2 nilpotent Lie algebra $\mathcal{V} \oplus_{\perp} \mathbb{R}^{r,s}$ with center $\mathbb{R}^{r,s}$ and Lie brackets defined via the relation

$$\big\langle J_Z(X), Y \big\rangle_{\mathcal{V}} = \big\langle Z, [X, Y] \big\rangle_{r,s}, \quad Z \in \mathbb{R}^{r,s}, \text{ and } X, Y \in \mathcal{V}, \tag{3.13}$$

will be denoted by $\mathfrak{g}_{r,s}(\mathcal{V})$. We write $\mathbb{G}_{r,s}(\mathcal{V})$ for the corresponding simply connected Lie group and call it a *pseudo H-type group*, cf. [41, 55].

(2) If $\mathcal{V}$ is of minimal dimension among all admissible modules, then we call $\mathcal{V}$ *minimal admissible* and we shortly write $\mathfrak{g}_{r,s} := \mathfrak{g}_{r,s}(\mathcal{V})$ and $\mathbb{G}_{r,s} := \mathbb{G}_{r,s}(\mathcal{V})$.

Remark 3.3.3. Note that minimal admissible modules are cyclic and the nilpotent Lie algebra $\mathfrak{g}_{r,s}$ is unique up to isomorphisms, even if the Clifford algebra $C\ell_{r,s}$ admits two non-equivalent irreducible modules (cf. [55]). Furthermore, by definition of Lie brackets (3.13), the left-invariant distribution on $\mathbb{G}_{r,s}(\mathcal{V})$ induced by the admissible module $\mathcal{V}$ has the bracket generating property. Moreover, it is fat if and only if $r = 0$ or $s = 0$.

We fix an orthonormal basis $\{Z_k\}_{k=1}^{r+s}$ in $\mathbb{R}^{r,s}$, i.e. we assume that:

$$\langle Z_i, Z_i \rangle_{r,s} = 1 \ (i = 1, \cdots, r), \ \langle Z_{r+j}, Z_{r+j} \rangle_{r,s} = -1 \ (j = 1, \cdots, s), \text{ and}$$
$$\langle Z_i, Z_j \rangle_{r,s} = 0 \ (i \neq j).$$

Let $\{J, \ \mathcal{V}, \ \langle \bullet, \bullet \rangle_{\mathcal{V}}\}$ be an admissible $C\ell_{r,s}$-module.

Theorem 3.3.4 (cf. [45, 55])**.** *Assume that $s > 0$. Then there exists an orthonormal basis $\{X_i\}_{i=1}^{2N}$ of $\mathcal{V}$ such that*

(1) $\langle X_i, X_i\rangle_{\mathcal{V}} = 1$ $(i = 1 \cdots, N)$, $\langle X_i, X_i\rangle_{\mathcal{V}} = -1$ $(i = N+1, \cdots, 2N)$ *and* $\langle X_i, X_j\rangle_{\mathcal{V}} = 0$ *for* $i \neq j$,

(2) *For each k, the operator J_{Z_k} maps X_i to some X_j or $-X_j$ with $j \neq i$.*

Definition 3.3.5. We call a basis $\{X_i, Z_j\}$ satisfying the properties in Theorem 3.3.4 an *integral basis* of the algebra $\mathfrak{g}_{r,s}(\mathcal{V})$.

Remark 3.3.6. An interesting problem, which we will postpone to a future work, consists in a classification of *integral bases* up to isomorphisms. Consider an orthonormal basis $\{Z_k\}_{k=1}^{r+s}$ of $\mathbb{R}^{r,s}$ in the above sense. If $\mathcal{V}$ is a minimal admissible $\mathrm{C}\ell_{r,s}$-module, then we can define a finite subgroup G in $\mathrm{GL}(\mathcal{V})$ generated by $\{J_{Z_k} \,:\, k = 1, \dots, r+s\}$. Consider the commutative subgroup:

$$\mathbb{S} := \big\{A \in \mathrm{G} \,:\, A^2 = \mathrm{Id},\ A = J_{Z_{i_1}} \dots J_{Z_{i_r}} > 0, A \neq -\mathrm{Id}\big\} \subset \mathrm{G}.$$

By "$A > 0$" we mean that A maps positive (resp. negative) vectors in $\mathcal{V}$ to positive (resp. negative) vectors. Such groups are partially ordered with respect to the inclusion and we assume that $\mathbb{S}$ is a maximal element. Further, we assume that $X \in \mathcal{V}$ is a common eigenvector of elements in $\mathbb{S}$. Necessarily, X is not a null vector, i.e. $\langle X, X\rangle_{\mathcal{V}} \neq 0$. Consider

$$\big\{\pm X_i\big\} = \big\{A(X) \,:\, A \in \mathrm{G}\big\}.$$

We conjecture that a suitable choice of the common eigenvector X leads to an integral basis $\{X_i, Z_k\}$ of the pseudo H-type Lie algebra $\mathfrak{g}_{r,s}(\mathcal{V})$.

Conversely, let $\{X_i, Z_k\}$ be an integral basis and put $\pm\mathcal{B} := \{\pm X_\ell \,:\, \ell = 1, \dots, 2N = \dim \mathcal{V}\}$. Then each J_{Z_i} defines a bijective map

$$J_{Z_i} : \pm\mathcal{B} \to \pm\mathcal{B}$$

and elements in the group G act on $\pm\mathcal{B}$. We obtain a subgroup $\mathbb{S}$ as above from this basis by defining

$$\big\{A \in \mathrm{G} \,:\, A(X_1) = X_1\big\} =: \mathbb{S} \subset \mathrm{G}.$$

We conjecture that every maximal subgroup $\mathbb{S}$ defines an integral basis.

From now on we assume that $\{X_i, Z_k\}$ is an integral basis of $\mathfrak{g}_{r,s}(\mathcal{V})$.

Corollary 3.3.7. *If there exists $i \in \{1, \cdots, 2N\}$ such that $J_{Z_k}(X_i) = \pm J_{Z_\ell}(X_i)$, then $k = \ell$. Hence any basis vector X_i is mapped to some X_j or $-X_j$ by at most one operator J_{Z_k}.*

Proof. If $k \leq r$ then J_{Z_k} maps positive to positive and negative to negative elements. Similarly, if $k > r$, then J_{Z_k} maps positive to negative and negative to positive elements. Therefore, under the above assumption only the cases $k, \ell \leq r$ or $k, \ell > r$ are possible.

Let us assume $k \neq \ell$ such that $\pm X_i = J_{Z_k} J_{Z_\ell}(X_i)$. By the previous remark we have

$$J_{Z_k} J_{Z_\ell} \circ J_{Z_k} J_{Z_\ell} = -J_{Z_k} \circ J_{Z_\ell}^2 \circ J_{Z_k} = -\langle Z_k, Z_k \rangle_{r,s} \langle Z_\ell, Z_\ell \rangle_{r,s} = -I.$$

This equation contradicts the existence of the eigenvalue 1 or -1 of $J_{Z_k} J_{Z_\ell}$. □

Corollary 3.3.8. *If we put* $[X_i,\, X_j] = \sum c_{i\,j}^k Z_k$, *then* $c_{i\,j}^k$ *can be non-zero for at most one* k. *If* $c_{i\,j}^k$ *is non-zero then it equals* ± 1.

Proof. The statement follows from Corollary 3.3.7 and

$$\langle J_{Z_\ell} X_i,\, X_j \rangle_{\mathcal{V}} = \langle Z_\ell,\, [X_i,\, X_j] \rangle_{r,s} = \begin{cases} c_{i\,j}^\ell & \text{if } \ell \leq r \\ -c_{i\,j}^\ell & \text{if } \ell > r. \end{cases}$$

□

Definition 3.3.9. From an integral basis $\{X_i, Z_k\}$ of $\mathfrak{g}_{r,s}(\mathcal{V})$ we define a *lattice* in the pseudo H-type group $\mathbb{G}_{r,s}(\mathcal{V})$ by

$$\Gamma_{r,s}(\mathcal{V}) := \left\{ \sum_{m_i \in \mathbb{Z}} m_i X_i + \frac{1}{2} \sum_{k_j \in \mathbb{Z}} k_j Z_j \right\}.$$

In the following we call $\Gamma_{r,s}(\mathcal{V})$ a *standard integral lattice* in $\mathfrak{g}_{r,s}(\mathcal{V})$. If $\mathfrak{g}_{r,s}$ is constructed from a minimal admissible module $\mathcal{V}$ (cf. Definition 3.3.2), then we write $\Gamma_{r,s} := \Gamma_{r,s}(\mathcal{V})$.

Remark 3.3.10. A *standard integral lattice* is not unique. A complete classification will be subject of another work. For particular cases the construction of $\Gamma_{r,s}$ is found in [55].

In the following two sections we consider the sublaplacian

$$\Delta_{\text{sub}}^{\mathbb{G}_{r,s}(\mathcal{V})} = -\sum_{i=1}^{2N} \tilde{X}_i^2 \tag{3.14}$$

on $\mathbb{G}_{r,s}(\mathcal{V})$, where $\{X_i \,:\, i = 1, \cdots, 2N\}$ is the basis of the module $\mathcal{V}$ in the definition of the standard integral lattice $\Gamma_{r,s}(\mathcal{V})$. We determine the heat trace of the sublaplacian

$$\Delta_{\text{sub}}^{\Gamma_{r,s}(\mathcal{V}) \backslash \mathbb{G}_{r,s}(\mathcal{V})} \tag{3.15}$$

descended from (3.14) to the nilmanifold $\Gamma_{r,s}(\mathcal{V}) \backslash \mathbb{G}_{r,s}(\mathcal{V})$. Based on the sub-ellipticity of (3.15) it is known that the spectrum of the sublaplacian on a compact quotient only consists of eigenvalues with finite multiplicities. In principle our trace formula in Theorem 3.2.4 can be used to obtain the spectrum of (3.15). However, we will not calculate the eigenvalues and multiplicities explicitly since a comparison of heat traces is sufficient to decide isospectrality.

3.4 The structure constants of pseudo H-type groups

In the case of pseudo H-type groups we calculate the characteristic polynomial of the matrix $\Omega(Z)$ in (3.3) in the case where $s > 0$ in Definition 3.3.2 of the pseudo H-type group $\mathbb{G}_{r,s}(\mathcal{V})$. Recall that this matrix is an essential ingredient for the integral expression of the heat kernel in Theorem 3.1.2.

Throughout this section we assume that $s > 0$ so that we can use the integral basis in Theorem 3.3.4. Let us start by decomposing the Clifford module $\mathcal{V} = \mathcal{V}_+ \oplus_\perp \mathcal{V}_-$ into a positive and a negative subspace $\mathcal{V}_+ := \mathrm{Span}\{X_i : i = 1, \cdots, N\}$ and $\mathcal{V}_- = \mathrm{Span}\{X_{N+j} : j = 1, \cdots, N\}$, where $\{X_i : i = 1, \cdots, 2N\}$ is part of a standard integral basis $\{X_i, Z_j\}$ of $\mathfrak{g}_{r,s}(\mathcal{V})$. Then the Clifford module action

$$J_Z(X_i) = \sum_j c_{ij}(Z) X_j$$

with

$$Z = \sum_{i=1}^{r} \mu_i Z_i + \sum_{j=1}^{s} \nu_j Z_{r+j} \cong (\mu, \nu)^T \in \mathbb{R}^{r,s}$$

can be written in form of a matrix with respect to the basis $\{X_i\}$ of $\mathcal{V}$:

$$J_Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{V}_+ \\ \oplus_\perp \\ \mathcal{V}_- \end{matrix} \longrightarrow \begin{matrix} \mathcal{V}_+ \\ \oplus_\perp \\ \mathcal{V}_- \end{matrix} .$$

By (3.12) the map J_Z leaves $\mathcal{V}_\pm$ invariant whenever Z is positive in $\mathbb{R}^{r,s}$. If $Z \in \mathbb{R}^{r,s}$ is negative then J_Z maps $\mathcal{V}_+$ to $\mathcal{V}_-$ and vice versa. It follows that the component matrices A, B, C, D are of the forms $A = A(\mu)$, $B = B(\nu)$, $C = C(\nu)$ and $D = D(\mu)$. Due to the admissibility condition (3.11) of the Clifford action on the module $\mathcal{V}$ we have

$$A^T(\mu) = -A(\mu), \quad B^T(\nu) = C(\nu) \quad \textit{and} \quad D^T(\mu) = -D(\mu).$$

Here $A^T(\mu)$ denotes the transposed matrix of $A(\mu) \in \mathbb{R}(N)$. Moreover, the identity $J_Z^2 = -\langle Z, Z\rangle_{r,s}$ yields additional relations between the component matrices A, B, C and D which are collected in the next lemma.

Lemma 3.4.1. *With the notion* $\|\mu\|^2 := \sum_{i=1}^r \mu_i^2$ *and* $\|\nu\|^2 := \sum_{j=1}^s \nu_j^2$ *we have:*

(a) $A(\mu)^2 + B(\nu)C(\nu) = -\langle Z, Z\rangle_{r,s} = -(\|\mu\|^2 - \|\nu\|^2)$,

(b) $A(\mu)B(\nu) + B(\nu)D(\mu) = 0$ *and* $C(\nu)A(\mu) + D(\mu)C(\nu) = 0$,

(c) $C(\nu)B(\nu) + D(\mu)^2 = -\langle Z, Z\rangle_{r,s}$.

In particular, it follows $A(\mu)^2 = -\|\mu\|^2$ *and* $D(\mu)^2 = -\|\mu\|^2$.

Let L be a linear map on $\mathcal{V} \cong \mathbb{R}^{2N}$. The same notation is used for its matrix representation with respect to the basis $\{X_i\}$. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{R}^{2N}$ and fix $X, Y \in \mathcal{V}$. We calculate the matrix representation of the transpose L^* of L with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. Consider the matrix

$$\tau := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in \mathbb{R}(2N) \quad \textit{where} \quad I = \text{identity} \in \mathbb{R}(N).$$

Then we have

$$\big\langle LX, Y \big\rangle_{\mathcal{V}} = \big\langle LX, \tau Y \big\rangle = \big\langle X, L^T \tau Y \big\rangle = \big\langle X, \tau L^T \tau Y \big\rangle_{\mathcal{V}},$$

which implies that $L^* = \tau L^T \tau$. A direct calculation shows that the skew-symmetric matrix $\Omega(Z)$ in (3.3) is related to the above matrix representation of J_Z as follows:

$$\Omega(Z) = \tau J_Z^T = \begin{pmatrix} -A(\mu) & B(\nu) \\ -B^T(\nu) & D(\mu) \end{pmatrix}. \tag{3.16}$$

Remark 3.4.2. If $r = 0$, then $A(\mu) = D(\mu) = 0$.

In order to determine the eigenvalues of the matrix $\Omega(\sqrt{-1}Z)$ we employ the relation

$$\underbrace{\begin{pmatrix} -A + \lambda & B \\ -C & D + \lambda \end{pmatrix}}_{=\Omega(Z)+\lambda} \begin{pmatrix} I & -(-A+\lambda)^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} -A + \lambda & 0 \\ -B^T & B^T(-A+\lambda)^{-1}B + \lambda + D \end{pmatrix}.$$

Hence we have

$$\det\big(\Omega(Z) + \lambda\big) = \det\big(- A + \lambda\big) \ \det\Big(B^T(-A+\lambda)^{-1}B + \lambda + D\Big).$$

According to Lemma 3.4.1 one has $B^T B = \|\nu\|^2 = BB^T$ and $B^T AB + \|\nu\|^2 D = 0$, showing that $B^T\big(- A + \lambda\big)B = \|\nu\|^2(\lambda + D)$. Together with the skew-symmetry of $\Omega(Z)$:

$$\begin{aligned}
&\det\big(\Omega(Z) + \lambda\big) = \det\big(\Omega(Z) - \lambda\big) \\
&= \det\big(- A + \lambda\big) \ \det\Big(B^T(-A+\lambda)^{-1}B + \frac{1}{\|\nu\|^2}\, B^T\,(-A+\lambda)\,B\Big) \\
&= \det\big(- A + \lambda\big) \ \det\left(B^T\Big[(-A+\lambda)^{-1} + \frac{1}{\|\nu\|^2}(-A+\lambda)\Big]B\right) \\
&= \det\Big(\|\nu\|^2 - \|\mu\|^2 + \lambda^2 - 2\lambda A\Big).
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\det\big(\Omega(Z) + \lambda\big)^2 = \det\big(\Omega(Z) + \lambda\big) \ \det\big(\Omega(Z) - \lambda\big) \\
&= \det\Big(\big(\|\nu\|^2 - \|\mu\|^2 + \lambda^2\big)^2 + 4\lambda^2\|\mu\|^2\Big).
\end{aligned}$$

Proposition 3.4.3. *With $s > 0$ and $z \in \mathbb{R}^{r,s}$ we have:*

$$\begin{aligned}\det(\Omega(Z)+\lambda)^2 =& \Big(\big(\lambda^2 + \|\mu\|^2 + \|\nu\|^2\big)^2 - 4\|\mu\|^2\|\nu\|^2 \Big)^N \\ =& \Big[\big(\lambda^2 + (\|\mu\| + \|\nu\|)^2\big)\big(\lambda^2 + (\|\mu\| - \|\nu\|)^2\big) \Big]^N.\end{aligned}$$

By replacing Z with $\sqrt{-1}Z$, μ with $\sqrt{-1}\mu$ and ν with $\sqrt{-1}\nu$ we have:

Corollary 3.4.4. *The eigenvalues λ of the matrix $\Omega(\sqrt{-1}Z)$ are $\lambda = \pm(\|\mu\| \pm \|\nu\|)$.*

If $Z \neq 0$ then the matrix $\Omega(Z)$ has the eigenvalue zero only when $\|\mu\| = \|\nu\|$. In this case the matrices $B(\nu)$ and $D(\mu)$ are non-singular so that the dimension of the solution space $\Omega(Z) \cdot X = 0$ is N (= half the dimension of $\mathcal{V}$).

Proposition 3.4.5. *Assume that $Z \neq 0$ and $\|\mu\| = \|\nu\|$. The kernel of $\Omega(Z)$ is given by*

$$Ker\ \Omega(Z) = \left\{ \begin{pmatrix} B(\nu)x \\ -D(\mu)x \end{pmatrix} \;\middle|\; x = (x_1, \dots, x_N)^T \in \mathbb{R}^N \right\}.$$

Finally we determine the dimension of the eigenspaces corresponding to the above eigenvalues $\pm(\|\mu\| \pm \|\nu\|)$.

Proposition 3.4.6. *Let $Z \neq 0$. Then the dimensions of the eigenspaces E_λ of $\Omega(\sqrt{-1}Z)$ with respect to the eigenvalue $\lambda = \pm(\|\mu\| \pm \|\nu\|)$ are given as follows:*

(i) If neither μ nor ν is zero, then $\dim E_\lambda = N/2$ and therefore $\frac{\dim \mathcal{V}}{2} = N$ is even.

(ii) If $\mu = 0$ and $\nu \neq 0$, then $\dim E_{\|\nu\|} = N = \dim E_{-\|\nu\|}$.

(iii) If $\mu \neq 0$ and $\nu = 0$, then $\dim E_{\|\mu\|} = N = \dim E_{-\|\mu\|}$.

Proof. (i): Let $(x,y)^T \in \mathbb{R}^N \times \mathbb{R}^N \cong \mathbb{R}^{2N}$ be an eigenvector of the matrix $\Omega(\sqrt{-1}Z)$ with respect to the eigenvalue $\lambda = \|\mu\| + \|\nu\|$, where $\mu, \nu \neq 0$, then

$$\Omega(\sqrt{-1}Z) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -A(\sqrt{-1}\mu) & B(\sqrt{-1}\nu) \\ -C(\sqrt{-1}\nu) & D(\sqrt{-1}\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\|\mu\| + \|\nu\|) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.17}$$

Multiplying the first equation by $A(\sqrt{-1}\mu)$ and the second equation by $B(\sqrt{-1}\nu)$ gives

$$\begin{aligned} A(\sqrt{-1}\mu)\Big(- A(\sqrt{-1}\mu)\, x + B(\sqrt{-1}\nu)\, y \Big) &= (\|\mu\| + \|\nu\|) A(\sqrt{-1}\mu)\, x \\ B(\sqrt{-1}\nu)\Big(- C(\sqrt{-1}\nu)\, x + D(\sqrt{-1}\mu)\, y \Big) &= (\|\mu\| + \|\nu\|) B(\sqrt{-1}\nu)\, y. \end{aligned}$$

On the left hand side we use Lemma 3.4.1 and deduce the following two equations

$$-\|\mu\|^2\, x + A(\sqrt{-1}\mu) B(\sqrt{-1}\nu)\, y = (\|\mu\| + \|\nu\|) A(\sqrt{-1}\mu)\, x$$

$$\|\nu\|^2\, x + B(\sqrt{-1}\nu)D(\sqrt{-1}\mu)\, y = (\|\mu\| + \|\nu\|)B(\sqrt{-1}\nu)\, y.$$

Adding these identities and using Lemma 3.4.1, (ii) gives

$$\big(\|\nu\| - \|\mu\|\big)\, x = A(\sqrt{-1}\mu)\, x + B(\sqrt{-1}\nu)\, y.$$

Together with the Equation (3.17) we find that $B(\sqrt{-1}\nu)y = \|\nu\|x$. This shows that

$$A(\sqrt{-1}\mu)x = -\|\mu\|x.$$

Since $B(\nu)$ is non-singular for $\nu \neq 0$ the vector y is uniquely determined by x. Conversely, the eigenvector x of the matrix $A(\sqrt{-1}\mu)$ with the eigenvalue $-\|\mu\| \neq 0$, determines the eigenvector of the matrix $\Omega(\sqrt{-1}Z)$ by putting $y = \|\nu\|B(\sqrt{-1}\nu)^{-1}x$.

Since $A(\mu)$ is skew-symmetric for any real vector μ and $A(\mu)^2 = -\|\mu\|^2$, the dimension of the eigenspace of the matrix $A(\sqrt{-1}\mu)$ with respect to the eigenvalue $\|\mu\|$ is half the size of the matrix A, i.e. it equals $\frac{N}{2}$. The remaining eigenvalues can be treated similarly and therefore (i) follows.

(ii): Under the assumption of (ii) and by applying the relations in Lemma 3.4.1 we have $A = D = 0$ and $B(\nu)C(\nu) = \|\nu\|^2 = C(\nu)B(\nu)$. Let $(x, y)^T \in \mathbb{R}^N \times \mathbb{R}^N \cong \mathbb{R}^{2N}$ be an eigenvector of $\Omega(\sqrt{-1}Z)$ with eigenvalue $\lambda = \|\nu\|$. Then the equation

$$\Omega(\sqrt{-1}Z)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & B(\sqrt{-1}\nu) \\ \|\nu\|^2 B^{-1}(\sqrt{-1}\nu) & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \|\nu\|\begin{pmatrix} x \\ y \end{pmatrix}$$

is equivalent to $B(\sqrt{-1}\nu)y = \|\nu\|x$, which can be uniquely solved for any given $x \in \mathbb{R}^N$. The case $\lambda = -\|\nu\|$ is treated in the same way and (ii) follows.

(iii): If $\mu \neq 0$ and $\nu = 0$, then $C = B = 0$ and with $\lambda = \|\mu\|$ we have the equation

$$\Omega(\sqrt{-1}Z)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -A(\sqrt{-1}\mu) & 0 \\ 0 & D(\sqrt{-1}\mu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \|\mu\|\begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\Omega(\sqrt{-1}Z)$ is Hermitian and therefore can be diagonalized. From the expression of det $(\Omega(Z) + \lambda)^2$, we deduce that the eigenspaces have dimension N. The case $\lambda = -\|\mu\|$ can be treated similarly. □

Corollary 3.4.7. *Let $s > 0$, then the characteristic polynomial of $\Omega(\sqrt{-1}Z)$ is given by:*

$$\det\Big(\Omega(\sqrt{-1}Z) + \lambda\Big) = \big(\lambda^2 - (\|\mu\| + \|\nu\|)^2\big)^{N/2}\big(\lambda^2 - (\|\mu\| - \|\nu\|)^2\big)^{N/2}.$$

Remark 3.4.8. In the case $s = 0$ we can find an admissible module $\mathcal{V}$ with respect to an inner product (= positive definite scalar product) and so we obtain:

$$\Omega(Z) = -J_Z \quad \textit{and} \quad \Omega(Z)^2 = J_Z^2 = -\|Z\|^2 I.$$

Therefore, the statement of Corollary 3.4.7 remains valid even in this case:

$$\det\ \big(\Omega(\sqrt{-1}Z) + \lambda\big) = \big(\lambda^2 - \|Z\|^2\big)^N.$$

3.5 Spectrum of the sublaplacian on pseudo H-type nilmanifolds

Let $\{J,\ \mathcal{V},\ \langle\bullet,\bullet\rangle_{\mathcal{V}}\}$ be an admissible module of the Clifford algebra $C\ell_{r,s}$. Based on the results of the previous sections we derive an explicit expression for the heat trace of the sublaplacian $\Delta_{\text{sub}}^{\Gamma_{r,s}(\mathcal{V})\backslash\mathbb{G}_{r,s}(\mathcal{V})}$ on the nilmanifolds $\Gamma_{r,s}(\mathcal{V})\backslash\mathbb{G}_{r,s}(\mathcal{V})$ for $r>0$ and $s>0$. In fact, with a view to the decomposition (3.7), it suffices to calculate the heat trace of each component operator $\mathcal{D}^{(\mathbf{n})}$ with respect to the element $\mathbf{n}$ in the dual $[\mathbb{A}\cap\Gamma_{r,s}(\mathcal{V})]^*$ of the lattice $\mathbb{A}\cap\Gamma_{r,s}(\mathcal{V})$.

Recall that $\mathbb{A}\cong\mathbb{R}^{r,s}$ denotes the center of the group $\mathbb{G}_{r,s}(\mathcal{V})$. However, our notation will not indicate the dependence on the parameter (r,s).

3.5.1 Determination of the spectrum

An element $\mathbf{n}$ in the dual lattice $\big[\Gamma_{r,s}\cap\mathbb{A}\big]^*$ can be expressed as

$$\mathbf{n}=2\left(\sum_{i=1}^{r}m_iZ_i+\sum_{j=1}^{s}n_jZ_{r+j}\right)\quad \textit{where}\quad (m,n)\in\mathbb{Z}^{r+s}.$$

We also use the notation $\mathbf{n}=2(\mu+\nu)$ with $\mu+\nu=(m_1,\cdots,m_r,n_1\cdots,n_s)\in\mathbb{Z}^{r+s}$. Now Theorem 3.2.4 implies:

(1) If $\mathbf{n}=0$, then the trace of the operator $e^{-t\mathcal{D}^{(0)}}$ is given by

$$\operatorname{tr}\Big(e^{-t\mathcal{D}^{(0)}}\Big)=\frac{1}{(4\pi\,t)^N}\sum_{\ell\in\mathbb{Z}^{2N}}e^{-\frac{\|\ell\|^2}{4t}}. \tag{3.18}$$

(2) Assume that $\mathbf{n}\in\big[\Gamma_{r,s}\cap\mathbb{A}\big]^*$ with

$$\sum_{i=1}^{r}{m_i}^2=\sum_{j=1}^{s}{n_j}^2,$$

and let $d_0>0$ be the greatest common divisor of $(\mu,\,\nu)=(m_1,\cdots,m_r,n_1,\cdots,n_s)$. Define integers m_i' and n_i' through the equations $m_i={m_i}'d_0$ and $n_j={n_j}'d_0$. According to Proposition 3.4.5 the solution space $\mathbb{M}(\mathbf{n})=\big\{\ell\in\mathbb{Z}^{2N}\mid\Omega(\mathbf{n})\,\ell=0\big\}$ is given by:

$$\mathbb{M}(\mathbf{n})=\left\{\begin{pmatrix}B(\nu')\ell\\-D(\mu')\ell\end{pmatrix}\ \Big|\ \ell=(\ell_1,\cdots,\ell_N)^T\in\mathbb{Z}^N\right\},$$

where $(\mu',\nu')=({m_1}',\cdots,{m_r}',{n_1}',\cdots,{n_s}')$. Hence

$$\operatorname{tr}\Big(e^{-t\mathcal{D}^{(\mathbf{n})}}\Big)=\frac{1}{(2\pi\,t)^{N/2}}\sum_{\ell\in\mathbb{Z}^{2N}}e^{-\frac{\|\mu\|^2\,\|\ell\|^2}{2{d_0}^2\,t}}\left(\frac{\|\mu\|}{\sinh\big(4\pi\,t\|\mu\|\big)}\right)^{N/2}. \tag{3.19}$$

(3) For $\mathbf{n} = 2(\mu + \nu)$ with $\|\mu\| \neq \|\nu\|$ the matrix $\Omega(\mathbf{n})$ is non-singular. In this case the solution space $\mathbb{M}(\mathbf{n}) = \{0\}$ is trivial and

$$\operatorname{tr}\left(e^{-t\mathcal{D}(\mathbf{n})}\right) = \left(\frac{\|\mu\|^2 - \|\nu\|^2}{\sinh\{2\pi\, t(\|\mu\| + \|\nu\|)\}\sinh\{2\pi\, t(\|\mu\| - \|\nu\|)\}}\right)^{N/2.}. \tag{3.20}$$

Remark 3.5.1. If $s = 0$, then the matrix $\Omega(\mathbf{n})$ is always non-singular for $\mathbf{n} \neq 0$. In this case we find:

$$\operatorname{tr}\left(e^{-t\mathcal{D}(\mathbf{n})}\right) = \left(\frac{\|\mathbf{n}\|}{\sinh 2\pi t\|\mathbf{n}\|}\right)^N.$$

Furthermore, from the heat trace formula (3.7) we conclude that the eigenvalues of the sublaplacian on $\Gamma_{r,0}(V)\backslash\mathbb{G}_{r,0}(V)$ are given by

- $\lambda_l = 4\pi^2\|l\|^2$ for $l \in \mathbb{Z}^{2N}$.
- $\beta_{n,m} = 4\pi\|n\|(2m+N)$ for $n \in \mathbb{Z}^r\backslash\{0\}$ and $m \in \mathbb{N}$ with multiplicity $2^N\|n\|^N\binom{m+N-1}{N-1}$.

Since $\lambda_l^2 \in \pi^4\mathbb{Z}$ and $\beta_{n,m}^2 \in \pi^2\mathbb{Z}$, we can distinguish between these different numbers, i.e. knowing the eigenvalues, we can extract the dimension of the admissible module $\mathcal{V}$. We conclude that if two manifolds $\Gamma_{r,0}(\mathcal{V})\backslash\mathbb{G}_{r,0}(\mathcal{V})$ and $\Gamma_{r',0}(\mathcal{V}')\backslash\mathbb{G}_{r',0}(\mathcal{V}')$ are isospectral with respect to the sublaplacian, then they have the same dimension.

Based on the dependence of the above heat traces on the parameters $\|\mu\|$, $\|\nu\|$ and N we conclude:

Corollary 3.5.2. *$\Gamma_{r,s}\backslash\mathbb{G}_{r,s}$ and $\Gamma_{s,r}\backslash\mathbb{G}_{s,r}$ are isospectral with respect to the Laplacian and the sublaplacian, if the dimensions of their admissible modules coincide.*

The pseudo H-type algebra $\mathfrak{g}_{r,s}$ does not depend on the chosen minimal admissible module, cf. [55]. Moreover, in the above determination we do not explicitly use the assumption that the admissible module is minimal. This implies:

Theorem 3.5.3. *If $\mathcal{V}$ is a sum of k minimal admissible modules, then the heat trace in each of the cases, (3.18), (3.19) and (3.20) above is the k-th power of the corresponding heat trace for the manifold $\Gamma_{r,s}\backslash\mathbb{G}_{r,s}$. Let $\mathcal{U}$ be an admissible module of $C\ell_{s,r}$ with $\dim\mathcal{V} = \dim\mathcal{U}$, then the two nilmanifolds $\Gamma_{r,s}(\mathcal{V})\backslash\mathbb{G}_{r,s}(\mathcal{V})$ and $\Gamma_{s,r}(\mathcal{U})\backslash\mathbb{G}_{s,r}(\mathcal{U})$ are isospectral.*

3.6 Isospectral, but non-homeomorphic nilmanifolds

By applying Theorem 3.5.3 and the classification of pseudo H-type Lie algebras in [53, 54] we detect finite families of *isospectral* but *mutually non-homeomorphic* pseudo H-type nilmanifolds. If the module $\mathcal{V}$ in the construction of the pseudo H-type Lie algebra is minimal admissible, then pairs of non-isomorphic pseudo H-type Lie algebras $\mathfrak{g}_{r,s} \ncong \mathfrak{g}_{s,r}$ of the same dimension $\dim\mathfrak{g}_{r,s} = \dim\mathfrak{g}_{s,r}$ are known (see [53, 54] or Table 3.6.1). By choosing integral lattices $\Gamma_{r,s}$ and $\Gamma_{s,r}$ in the corresponding Lie groups $\mathbb{G}_{r,s}$ and $\mathbb{G}_{s,r}$, respectively, we first detect pairs

$$M_1 := \Gamma_{r,s} \backslash \mathbb{G}_{r,s} \qquad \textit{and} \qquad M_2 := \Gamma_{s,r} \backslash \mathbb{G}_{s,r} \tag{3.21}$$

of isospectral, non-homeomorphic manifolds. The minimal dimension of such examples arise for $(r, s) = (1, 3)$ in which case $\dim M_1 = \dim M_2 = 12$. By dropping the minimality condition on the module we can produce many more examples. More generally, for any $k \in \mathbb{N}$ we can find a family $M_1, \cdots, M_k$ of nilmanifolds such that for $i \neq j \in \{1, \cdots, k\}$:

$$M_i \sim_{\text{isosp}} M_j \qquad \textit{and} \qquad M_i \nsim_{\text{homeo}} M_j. \tag{3.22}$$

Here $\sim_{\text{isosp}}$ means *isospectral* with respect to the sublaplacian and $\nsim_{\text{homeo}}$ indicates that two manifolds are *non-homeomorphic*. First, we explain the method of detecting non-homeomorphic nilmanifolds.

If there is a homeomorphism between the nilmanifolds $N_{r,s} := \Gamma_{r,s}(\mathcal{V}) \backslash \mathbb{G}_{r,s}(\mathcal{V})$ and $N_{s,r} = \Gamma_{s,r}(\mathcal{U}) \backslash \mathbb{G}_{s,r}(\mathcal{U})$, then their fundamental groups

$$\pi_1 \left(\Gamma_{r,s}(\mathcal{V}) \backslash \mathbb{G}_{r,s}(\mathcal{V}) \right) \cong \Gamma_{r,s}(\mathcal{V}) \quad \text{and} \quad \pi_1 \left(\Gamma_{s,r}(\mathcal{U}) \backslash \mathbb{G}_{s,r}(\mathcal{U}) \right) \cong \Gamma_{s,r}(\mathcal{U})$$

are isomorphic. We can apply the following general fact from [98]:

Proposition 3.6.1. *Any isomorphism between lattices in simply connected nilpotent Lie groups can be extended to an isomorphism between the whole groups.*

From these observations we conclude:

Corollary 3.6.2. *If the nilmanifolds $N_{r,s}$ and $N_{s,r}$ are homeomorphic, then $\mathfrak{g}_{r,s}(\mathcal{V})$ and $\mathfrak{g}_{s,r}(\mathcal{U})$ have to be isomorphic as Lie algebras.*

3.6.1 Pairs of non-homeomorphic, isospectral nilmanifolds via minimal admissible modules

To obtain pairs $(M_1 = N_{r,s}, M_2 = N_{s,r})$ of nilmanifolds with (3.22) we determine pairs (r, s) such that $\dim N_{r,s} = \dim N_{s,r}$, but $N_1 \ncong N_2$. The classification of pseudo H-type algebras in TABLE 3.1 constructed from minimal admissible modules was obtained in [53, 54]. This table gives us only information about the cases $0 \leq r, s \leq 8$. For the remaining cases we use the following periodicity:

Lemma 3.6.3. *For $\mu, \nu = 0$ mod 8 or $\mu = \nu, \mu = 0$ mod 4 we have*

$$N_{r,s} \cong N_{s,r} \textit{ iff } N_{r+\mu,s+\nu} \cong N_{s+\nu,r+\mu}.$$

From Corollary 3.6.2 and TABLE 3.1 we see that for $(r, s) \in \{(3, 1), (3, 2), (3, 7)\}$ both nilmanifolds $N_{r,s}, N_{s,r}$ have the same dimension but they are non-homeomorphic.

Hence we obtain the following result:

Table 3.1: *Classification of pseudo H-type Lie algebras defined via minimal admissible modules.* **Notation**: If $\mathcal{V}^{r,s}_{\min}$ denotes a minimal admissible $C\ell_{r,s}$-module, then the letter 'd' = *double* (or 'h'=*half*, respectively) at the position (r,s) means that $\dim \mathcal{V}^{r,s}_{\min} = 2 \dim \mathcal{V}^{s,r}_{\min}$ (or $\dim \mathcal{V}^{r,s}_{\min} = 1/2 \dim \mathcal{V}^{s,r}_{\min}$, respectively). The symbol '$\cong$' indicates that the Lie algebras $\mathfrak{g}_{r,s}$ and $\mathfrak{g}_{s,r}$ are isomorphic while '$\ncong$' means that they are non-isomorphic.

8	$\cong$	$\cong$	$\cong$	h					
7	d	d	d	$\ncong$					
6	d	$\cong$	$\cong$	h	$\cong$				
5	d	$\cong$	$\cong$	h	$\cong$				
4	$\cong$	h	h	h		$\cong$	$\cong$		
3	d	$\ncong$	$\ncong$		d	d	d	$\ncong$	d
2	$\cong$	h		$\ncong$	d	$\cong$	$\cong$	h	$\cong$
1	$\cong$		d	$\ncong$	d	$\cong$	$\cong$	h	$\cong$
0		$\cong$	$\cong$	h	$\cong$	h	h	h	$\cong$
s/r	0	1	2	3	4	5	6	7	8

Corollary 3.6.4. *The following pairs of nilmanifolds are isospectral and non-homeomorphic:*

1. $(N_{r,s}, N_{s,r})$ *for* $r \equiv 3 \bmod 8$ *and* $s \equiv 1, 2, 7 \bmod 8$.
2. $(N_{r+4k,s+4k}, N_{s+4k,r+4k})$ *for* $(r,s) \in \{(3,1),(3,2),(3,7)\}$ *and* $k \in \mathbb{N}_0$.

3.6.2 Finite families of non-homeomorphic, isospectral nilmanifolds

In case the module $\mathcal{V}$ in the construction of the Lie algebra $\mathfrak{g}_{r,s}(\mathcal{V})$ is not minimal admissible we can use the classification result in [53, Theorem 4.1.2 and Theorem 4.1.3] to determine families $\{M_1, \cdots, M_k\}$ of a given lenght $k \in \mathbb{N}$ of isospectral, mutually non-homeomorphic nilmanifolds, i.e. (3.22) holds. First, we fix the pair (r,s) and study the Lie algebra $\mathfrak{g}_{r,s}(\mathcal{U})$, constructed from different admissible modules. In the general case the classification of isomorphic pseudo H-type Lie algebras is more subtle and to state the result we need to introduce some notation from [53]. Note that for any given minimal admissible module $\{J, \mathcal{V}, \langle \bullet, \bullet \rangle_{\mathcal{V}}\}$ also the module $\{J, \mathcal{V}, -\langle \bullet, \bullet \rangle_{\mathcal{V}}\}$ is minimal admissible. The upper index in the notation $\mathcal{V}^{r,s;\pm}_{\min;\pm}$ indicate that the scalar product of the two minimal admissible modules $\mathcal{V}^{r,s;+}_{\min;\pm}$ and $\mathcal{V}^{r,s;-}_{\min;\pm}$ differ by a sign.

Theorem 3.6.5 (K. Furutani, I. Markina, [53])**.** *Let* $r \equiv 3 \bmod 4$, $s \equiv 1, 2, 3 \bmod 4$ *and* $\mathcal{U}, \widetilde{\mathcal{U}}$ *be admissible modules decomposed into the direct sums:*

$$\mathcal{U} = \left(\bigoplus^{p^+} \mathcal{V}^{r,s;+}_{\min} \right) \bigoplus \left(\bigoplus^{p^-} \mathcal{V}^{r,s;-}_{\min} \right),$$

$$\widetilde{\mathcal{U}} = \left(\bigoplus^{\widetilde{p}^+} \mathcal{V}^{r,s;+}_{\min} \right) \bigoplus \left(\bigoplus^{\widetilde{p}^-} \mathcal{V}^{r,s;-}_{\min} \right).$$

Then the Lie algebras $\mathfrak{g}_{r,s}(\mathcal{U}), \mathfrak{g}_{r,s}(\widetilde{\mathcal{U}})$ are isomorphic, if and only if:

$$\left[p^+ = \widetilde{p}^+ \quad \text{and} \quad p^- = \widetilde{p}^- \right] \quad \text{or} \quad \left[p^+ = \widetilde{p}^- \quad \text{and} \quad p^- = \widetilde{p}^+ \right].$$

Let $R \in \mathbb{N}$ be fixed. If we consider admissible modules of the form

$$\mathcal{U}(p,q) := \left(\bigoplus^{p} \mathcal{V}^{r,s;+}_{\min} \right) \bigoplus \left(\bigoplus^{q} \mathcal{V}^{r,s;-}_{\min} \right), \quad p+q = R,$$

then we obtain non-isomorphic Lie algebras $\mathfrak{g}_{r,s}(\mathcal{U}(p_1,q_1))$ and $\mathfrak{g}_{r,s}(\mathcal{U}(p_2,q_2))$ of the same dimension $R \cdot \dim \mathcal{V}^{r,s;+}_{min}$ if simultaneously $(p_1,q_1) \neq (p_2,q_2)$ and $(p_1,q_1) \neq (q_2,p_2)$.

We fix an integer k and determine all pairs of integers (p_i,q_i) with the properties:

(a) $p_i \leq q_i$ for all i.

(b) $p_i + q_i = k$ for all i.

(c) $(p_i,q_i) \neq (p_j,q_j)$ and $(p_i,q_i) \neq (q_j,p_j)$ for $i \neq j$.

With such pairs we define:

$$\mathcal{U}_i := \left(\bigoplus^{p_i} \mathcal{V}^{r,s;+}_{\min} \right) \bigoplus \left(\bigoplus^{q_i} \mathcal{V}^{r,s;-}_{\min} \right).$$

From Theorem 3.6.5 and the above remark we conclude that the Lie algebras $\mathfrak{g}_{r,s}(\mathcal{U}_i)$ and $\mathfrak{g}_{r,s}(\mathcal{U}_j)$ are mutually non-isomorphic for $r \equiv 3 \bmod 4$, $s \equiv 1,2,3 \bmod 4$. In order to present a concrete family of nilmanifolds with the required properties we choose $r = 3$ and $s = 1$ such that $\dim \mathcal{V}^{3,1}_{\min} = 8$. Let k be even and choose $m \in \mathbb{N}$ such that $k = 2m$. The pairs (p_i,q_i) with the properties (a) - (c) are of the form $\{(i, k-i) \mid 0 \leq i \leq m\}$ and we conclude:

Corollary 3.6.6. *The following $m+1$ nilmanifolds*

$$\left(\Gamma_{3,1}(\mathcal{U}_i) \backslash \mathbb{G}_{3,1}(\mathcal{U}_i) \right)_{0 \leq i \leq m}$$

are isospectral, but mutually non-homeomorphic with respect to the (sub)-Laplacian.

Remark 3.6.7. For any given integer $m \in \mathbb{N}$ and by using the above method, we can construct $m+1$ nilmanifolds of the common dimension $4+16m$ which are isospectral but mutually non-homeomorphic. In particular, one obtains a pair of such manifolds of the (minimal) dimension $4+16 \times 1 = 20$. Note that via the first method (i.e. Corollary 3.6.4) we can find a pair of such nilmanifolds of dimension 12.

To minimize the dimension of the constructed family of nilmanifolds we should use a third method which is based on [53, Theorem 4.2], which treat the case $r \equiv 3 \bmod 4$ and $s \equiv 0 \bmod 4$. In this situation there are two non-equivalent irreducible modules and we use the lower index $\pm$ in the notation below to distinguish the minimal admissible modules corresponding to each irreducible modules (or to each sum of irreducible modules, cf. Remark 3.3.1).

Theorem 3.6.8 (K. Furutani, I. Markina, [53])**.** *For $r \equiv 3 \bmod 4$ and $s \equiv 0 \bmod 4$, let $\mathcal{U}, \widetilde{\mathcal{U}}$ be admissible modules decomposed into the direct sums:*

$$\mathcal{U} = \left(\bigoplus^{p_+^+} \mathcal{V}^{r,s;+}_{\min,+}\right) \bigoplus \left(\bigoplus^{p_+^-} \mathcal{V}^{r,s;-}_{\min,+}\right) \bigoplus \left(\bigoplus^{p_-^+} \mathcal{V}^{r,s;+}_{\min,-}\right) \bigoplus \left(\bigoplus^{p_-^-} \mathcal{V}^{r,s;-}_{\min,-}\right),$$

$$\widetilde{\mathcal{U}} = \left(\bigoplus^{\widetilde{p}_+^+} \mathcal{V}^{r,s;+}_{\min,+}\right) \bigoplus \left(\bigoplus^{\widetilde{p}_+^-} \mathcal{V}^{r,s;-}_{\min,+}\right) \bigoplus \left(\bigoplus^{\widetilde{p}_-^+} \mathcal{V}^{r,s;+}_{\min,-}\right) \bigoplus \left(\bigoplus^{\widetilde{p}_-^-} \mathcal{V}^{r,s;-}_{\min,-}\right).$$

Then the Lie algebras $\mathfrak{g}_{r,s}(\mathcal{U})$ and $\mathfrak{g}_{r,s}(\widetilde{\mathcal{U}})$ are isomorphic, if and only if one of the following conditions are fulfilled:

$$\left[p_+^+ + p_-^- = \widetilde{p}_+^+ + \widetilde{p}_-^- \text{ and } p_+^- + p_-^+ = \widetilde{p}_+^- + \widetilde{p}_-^+\right],$$

or

$$\left[p_+^+ + p_-^- = \widetilde{p}_+^- + \widetilde{p}_-^+ \text{ and } p_+^- + p_-^+ = \widetilde{p}_+^+ + \widetilde{p}_-^-\right].$$

To simplify the construction we choose $p_-^+ = p_-^- = 0$, $\widetilde{p}_-^+ = \widetilde{p}_-^- = 0$. The condition in Theorem 3.6.8 take the form:

$$\left[p_+^+ = \widetilde{p}_+^+ \text{ and } p_+^- = \widetilde{p}_+^-\right] \quad \text{or} \quad \left[p_+^+ = \widetilde{p}_+^- \text{ and } p_+^- = \widetilde{p}_+^+\right].$$

Next, we choose $r = 3$, $s = 0$ and fix $m \in \mathbb{N}$. Then we consider the following family of $m+1$ admissible modules:

$$\mathcal{V}_i := \left(\bigoplus^{i} \mathcal{V}^{3,0;+}_{\min;+}\right) \bigoplus \left(\bigoplus^{2m-i} \mathcal{V}^{3,0;-}_{\min;+}\right), \qquad (0 \leq i \leq m).$$

We obtain a family of $m+1$ mutually non-homeomorphic, isospectral nilmanifolds of common dimension $3 + 8m$.

Corollary 3.6.9. *For $0 \leq i \leq m$, the $m+1$ nilmanifolds*

$$(\Gamma_{3,0}(\mathcal{V}_i) \backslash \mathbb{G}_{3,0}(\mathcal{V}_i))_{0 \leq i \leq m}$$

are isospectral but mutually non-homeomorphic.

Remark 3.6.10. 1. By choosing $m = 1$ we obtain a pair of nilmanifolds both having dimension $3 + 8 = 11$. This dimension is minimal among the previous examples.

2. By definition, the subriemannian structure on $\Gamma_{r,s}(\mathcal{V})\backslash\mathbb{G}_{r,s}(\mathcal{V})$ is defined such that the canonical projection $\mathbb{G}_{r,s}(\mathcal{V}) \longrightarrow \Gamma_{r,s}\backslash\mathbb{G}_{r,s}(\mathcal{V})$ is a local subriemannian isometry. Since the considered subriemannian structure on $\mathbb{G}_{r,s}(\mathcal{V})$ is left-invariant, the tangent algebra at an arbitrary point of $\Gamma_{r,s}(\mathcal{V})\backslash\mathbb{G}_{r,s}(\mathcal{V})$ is isomorphic to $\mathfrak{g}_{r,s}(\mathcal{V})$. This implies that the constructed families of isospectral but mutually non-homeomorphic nilmanifolds have also the property, that their tangent algebras are not isomorphic. In particular, it follows that they can not be locally isometric as subriemannian manifolds (see Lemma 5.4.2). This means that even, if the tangent algebras at different points of a subriemannian manifold are of the same nature (i.e. isomorphic), we can not read the isomorphism class of these Lie algebras from the spectrum of the associated intrinsic sublaplacian.

Finally, we present the dimensions of minimal admissible modules for some basic cases in form of a table. These data are taken from [53, 54] which we refer to for more details and notations. The remaining cases can be obtained by $(4,4)$, $(8,0)$ and $(0,8)$-periodicities with respect to the signature (r,s), respectively. In particular, the table indicates the cases in which two non-equivalent minimal admissible modules exist. However, it is known that pseudo H-type algebras constructed from two non-equivalent minimal admissible modules are isomorphic.

Table 3.2: Dimensions of minimal admissible modules

8	$16^{\pm}$								
7	16^N	32^N	$\mathbf{64}^N$	$64^{\pm}$					
6	16^N	$16^N_{\times 2}$	32^N	$32^{\pm}$					
5	$\mathbf{16}^N$	16^N	16^N	$16^{\pm}$					
4	$8^{\pm}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}_{\times 2}$	$16^{\pm}$				
3	$\mathbf{8}^N$	8^N	8^N	$8^{\pm}$	16^N	32^N	$\mathbf{64}^N$	$64^{\pm}$	
2	$\mathbf{4}^N$	$\mathbf{4}^N_{\times 2}$	$\mathbf{8}^N$	$8^{\pm}$	16^N	$16^N_{\times 2}$	32^N	$32^{\pm}$	
1	$\mathbf{2}^N$	$\mathbf{4}^N$	$\mathbf{8}^N$	$8^{\pm}$	$\mathbf{16}^N$	16^N	16^N	$16^{\pm}$	
0	$1^{\pm}$	$2^{\pm}$	$4^{\pm}$	$4^{\pm}_{\times 2}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}_{\times 2}$	$16^{\pm}$
s/r	0	1	2	3	4	5	6	7	8

regular text = irreducible, **bold text** = double of irreducible,
$*_{\times 2}$ = two non-equivalent minimal dimensional admissible modules

3.7 Subriemannian structure and heat trace expansion

To every pseudo H-type nilmanifold $N_{r,s} = \Gamma_{r,s}\backslash\mathbb{G}_{r,s}$ with $r+s>1$ and based on [16, Theorem 3.3] we construct a Heisenberg manifold $H = \Gamma\backslash\mathbb{H}_{2n+1}$ such that the short time heat trace asymptotic expansions corresponding to the sublaplacians on $N_{r,s}$ and H, respectively, coincide up to a term vanishing to infinite order. Moreover, in our construction the manifolds $N_{r,s}$ and H have different dimensions. Recall that in the case of a Riemannian structure on M the heat trace expansion corresponding to the Laplacian encodes the dimension of M and therefore such examples do not exist in the framework of Riemannian geometry.

Let $d = r+s > 1$, and with our previous notation consider the nilmanifold

$$N_{r,s} = \Gamma_{r,s}\backslash\mathbb{G}_{r,s} \quad \textit{with} \quad \dim N_{r,s} = 2N + d.$$

We write $\mathbb{H}_{2n+1} = \mathbb{G}_{1,0}(\mathbb{R}^{2n})$ for the $(2n+1)$-dimensional Heisenberg group and with $\alpha > 0$ we define a lattice $\Gamma_\alpha \subset \mathbb{H}_{2n+1}$ of the form:

$$\Gamma_\alpha = \left\{ \sqrt{\alpha}\sum m_i X_i + \frac{\alpha}{2} kZ : m_i, k \in \mathbb{Z} \right\}.$$

Here $\{X_i, Z : i = 1, \cdots, 2n\}$ denotes a basis of the Lie algebra of $\mathbb{H}_{2n+1}$ with the non-trivial bracket relations

$$\big[X_i, X_{n+i}\big] = Z, \qquad (i = 1, \cdots, n).$$

The corresponding one-parameter family of Heisenberg manifolds will be denoted by:

$$H_\alpha = \Gamma_\alpha\backslash\mathbb{H}_{2n+1} \quad \textit{where} \quad \alpha > 0.$$

We recall the form of the short time heat trace asymptotic expansion of the sublaplacian on a general compact step 2 nilmanifold $\Gamma\backslash\mathbb{G}$ in [16, Theorem 3.3]:

Theorem 3.7.1 (W. Bauer, K. Furutani, C. Iwasaki [16])**.** *Let $M = \Gamma\backslash\mathbb{G}$ be a step 2 compact nilmanifold of dimension $2N+d$. Then*

$$tr\left(e^{-t\Delta_{\text{sub}}^M}\right) = \frac{c_M}{t^{N+d}} + O(t^\infty) \quad \textit{as} \quad t \to 0.$$

The constant $c_M > 0$ explicitly is given by

$$c_M = \frac{\mathrm{Vol}(M)}{(2\pi)^{N+d}} \int_{\mathbb{R}^d} W(\tau)d\tau. \tag{3.23}$$

If we apply the above theorem to H_α, we obtain:

$$c_{H_\alpha} = \frac{\mathrm{Vol}(H_\alpha)}{(2\pi)^{n+1}} \int_{\mathbb{R}} W(\tau)d\tau.$$

Here $W(\tau)d\tau$ is simply the volume form in Theorem 3.1.2 and associated to the Heisenberg group with the above structure constants. Note that

$$\mathrm{Vol}(H_\alpha) = \alpha^{n+1}\mathrm{Vol}(H_1).$$

If we choose $n \in \mathbb{N}$ with $n+1 = N+d$ and $\alpha > 0$ such that $c_{H_\alpha} = c_M$, then we obtain a pair of compact, subriemannian manifolds with the properties:

1. $tr\left(e^{-t\Delta^M_{\mathrm{sub}}}\right) - tr\left(e^{-t\Delta^{H_\alpha}_{\mathrm{sub}}}\right) = O(t^\infty)$ as $t \to 0$

2. $2N+d = \dim\, M \neq \dim\, H_\alpha = 2N+2d-1$ (since $d > 1$).

Remark 3.7.2. From the heat trace expansion for small times we can read the Hausdorff dimension $2(N+d)$ of the nilmanifold $M = \Gamma\backslash\mathbb{G}$ in Theorem 3.7.1 considered as a metric space with respect to the Carnot-Carathéodory distance. However, the last example indicates that we cannot read the Euclidean dimension of M from the coefficient (3.23) of the heat trace expansion. However, in Remark 3.7 we have pointed out that in some cases this dimension can be obtained from the full spectrum of the sublaplacian. In the case of the nilmanifolds M in this paper and which are constructed from a standard lattice we have $\mathrm{vol}(M) = 1$ and therefore, with the notation in Theorem 3.7.1:

$$c_M = C_M(N,d) := \frac{1}{(2\pi)^{N+d}} \int_{\mathbb{R}^d} W(\tau)d\tau. \tag{3.24}$$

We list a few problems concerning the geometric information contained in the spectrum of the sublaplacian on a nilmanifold:

Problems:

(a) Let $\mathbb{G}_{r,s}(\mathcal{V})$ be a pseudo H-type group with standard lattice $\Gamma_{r,s}(\mathcal{V})$ as explained in Section 3.3. Can we determine the numbers $2N = \dim \mathcal{V}$ and $d = \dim \mathbb{G}_{r,s}(\mathcal{V}) - 2N$ from the coefficient (3.24)?

Consider the case $s = 0$. Then integration with respect to polar coordinates shows:

$$\int_{\mathbb{R}^d} W(\tau)d\tau = \int_{\mathbb{R}^d} \frac{|\tau|^N}{(\sinh|\tau|)^N} d\tau = 2V_d \int_0^\infty \frac{r^{N+d-1}e^{-Nr}}{(1-e^{-2r})^N} dr = (*),$$

where $V_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ denotes the volume of the $(d-1)$-dimensional unit sphere. Now we use the following power series expansion for $|x| < 1$:

$$\frac{1}{(1-x)^N} = \sum_{\alpha \in \mathbb{N}_0^N} x^{|\alpha|}.$$

A change of variables in the integral is applied to obtain:

$$(*) = 2V_d \sum_{\alpha \in \mathbb{N}_0^N} \int_0^\infty r^{N+d-1} e^{-(N+2|\alpha|)r} dr$$

$$
\begin{aligned}
&= 2V_d \sum_{\alpha \in \mathbb{N}_0^N} \frac{1}{(N+2|\alpha|)^{N+d}} \int_0^\infty r^{N+d-1} e^{-r} dr \\
&= \frac{V_d \Gamma(N+d)}{2^{N+d-1}} \sum_{\alpha \in \mathbb{N}_0^N} \frac{1}{\left(\frac{N}{2}+|\alpha|\right)^{N+d}} \\
&= \frac{V_d \Gamma(N+d)}{2^{N+d-1}} \zeta_N\big(N+d, \frac{N}{2}\big).
\end{aligned}
$$

The infinite sum is called multiple Hurwitz zeta function and previously has been studied in the literature:

$$
\zeta_N\big(N+d, \frac{N}{2}\big) := \sum_{\alpha \in \mathbb{N}_0^N} \frac{1}{\left(\frac{N}{2}+|\alpha|\right)^{N+d}}.
$$

Hence:

$$
c_M(N,d)(2\pi)^{N+d} \frac{2^{N+d-2}}{\Gamma(N+d)} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \zeta_N\Big(N+d, \frac{N}{2}\Big).
$$

The left hand side can be calculated from the spectral data (more precisely, from the heat trace expansion in Theorem 3.7.1). Hence the problem reduces to the question, whether for each $k \in \mathbb{N}$ the assignment:

$$
N_k := \Big\{(N,d) \in \mathbb{N}^2 \;:\; N+d = k\Big\} \ni (N,d) \mapsto \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \zeta_N\Big(k, \frac{N}{2}\Big)
$$

is injective.

(b) Consider two isospectral compact nilmanifolds $M_j = \Gamma_j \backslash G_j$ where $j = 1, 2$. Assume that both are equipped with a left-invariant subriemannian structure as described in this paper. Is it true that $\dim G_1 = \dim G_2$ (see Remark 3.5.1)?

(c) The distribution of eigenvalues for classes of hypoelliptic operators with double characteristics on compact manifolds and under additional conditions is well-studied (e.g. see the work by A. Menikoff and J. Sjöstrand [90, 91]). Moreover, in the case of "sum-of-squares operators" satisfying *Hörmander's bracket generating condition* the asymptotic of the heat kernel at small times was found by G. Ben Arous, R. Léandre (see [29, 30]) in a form which encodes geometric data of an induced sub-Riemannian structure (such as the Carnot Carathéodory metric). However, not much seems to be known on the precise growth order or coefficient of the second term in the expansion of the eigenvalue counting function for the sublaplacian on compact nilmanifolds. Based on a classification of lattices and the explicit spectral data such question in the case of Heisenberg manifolds has been discussed in [105]. In generalizing R. Strichartz's result one may study the following problem:

Let $M := \Gamma \backslash G$ denote a compact nilmanifold (e.g. modeled over a pseudo H-type Lie group). Determine the growth order or even the coefficient of the second term in the eigenvalue counting function for the corresponding sublaplacian.

Chapter 4

A Poisson formula for subriemannian H-type nilmanifolds

This chapter is organized as follows. In Section 4.1 by using representation theory of step 2 nilpotent Lie groups, we compute the spectrum of the intrinsic sublaplacian on step 2 nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is a step 2 Carnot group and Γ a lattice. Then we calculate the subriemannian length spectrum of $\Gamma\backslash\mathbb{G}$ with $\mathbb{G}$ an H-type group in Section 4.2. In Section 4.3 we prove a Poisson summation formula relating these spectra on $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is an H-type group with odd dimensional center.

4.1 Spectrum of step 2 nilmanifolds

In this section we compute the spectrum of the intrinsic sublaplacian induced by a natural subriemannian structure on nilmanifolds $\Gamma\backslash\mathbb{G}$, where $\mathbb{G}$ is a step 2 Carnot group and Γ is a lattice of $\mathbb{G}$. This can be done by considering the regular representation of $\mathbb{G}$ in $L^2(\Gamma\backslash\mathbb{G})$ and using representation theory of step 2 nilpotent Lie groups. In comparison with the method in the last Chapter, where the main idea was to use an associated torus bundle structure, the advances are that we need neither further assumptions on the group $\mathbb{G}$ nor a formula for the heat kernel of the sublaplacian on the covering group $\mathbb{G}$. Note that the method given in this section for the computation of the spectrum is an adaptation to the subriemannian case of the standard one in [95]. Also, we refer to the work [4] where a similar method has been used for the explicit computation of the heat kernel on some Lie groups.

Let $\mathbb{G}$ be a Carnot group of step 2, i.e. $\mathbb{G}$ is a connected, simply connected nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ is stratified and nilpotent of step 2:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \text{ and } [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \;\; [\mathfrak{g}_i, \mathfrak{g}_2] = 0 \text{ for } i = 1, 2.$$

Furthermore, we assume the existence of an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$.

In the following we denote by $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ the exponential map and by log its inverse. Note that in the above case the exponential map exp is a diffeomorphism.
A straightforward calculation shows that $(\mathfrak{g}_1, \langle\cdot,\cdot\rangle)$ induces a left-invariant subriemannian structure $(\mathbb{G}, \mathcal{H}, \langle\cdot,\cdot\rangle)$ where

$$\mathcal{H}_x := (L_x)_*(\mathfrak{g}_1) \text{ for all } x \in \mathbb{G}.$$

Here L_x, for $x \in \mathbb{G}$, denotes the left-translation by x. Furthermore, if we denote by $\{X_j : j = 1, \cdots, m\}$ an orthonormal basis of $\mathfrak{g}_1$, the intrinsic sublaplacian on $\mathbb{G}$ can be expressed as a sum of squares (see [4]):

$$\Delta_{\text{sub}}^{\mathbb{G}} = -\sum_{j=1}^{m} \widetilde{X}_j^2,$$

where $\widetilde{X}_j$ denotes the left-invariant vector field on $\mathbb{G}$ associated to $X_j \in \mathfrak{g}$. Here the sublaplacian is defined with respect to the Popp measure, which coincides with the Haar measure up to a constant multiple, since the considered subriemannian structure is left-invariant.

Let Γ be a lattice in $\mathbb{G}$ and consider the compact quotient manifold $\Gamma\backslash\mathbb{G}$. Then we define a subriemannian structure on $\Gamma\backslash\mathbb{G}$ such that the canonical projection

$$p : \mathbb{G} \longrightarrow \Gamma\backslash\mathbb{G}$$

is a local subriemannian isometry (this is possible because the subriemannian structure on $\mathbb{G}$ is left-invariant). Since $\mathbb{G}$ is nilpotent and hence unimodular, the Popp measure on $\mathbb{G}$ is also right-invariant. Therefore, the Popp measure on $\Gamma\backslash\mathbb{G}$ is $\mathbb{G}$-invariant and hence, coincides with the quotient measure up to a constant multiple. The intrinsic sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$ on $C^\infty(\Gamma\backslash\mathbb{G})$ is also given by a sum of squares:

$$\Delta_{sub}^{\Gamma\backslash\mathbb{G}} = -\sum_{j=1}^{m} p_*\left(\widetilde{X}_j\right)^2.$$

Let us consider the regular representation ρ of $\mathbb{G}$ on $L^2(\Gamma\backslash\mathbb{G})$ defined by

$$\rho(x)f([y]) = f([y * x]) \text{ for } x \in \mathbb{G}, [y] \in \Gamma\backslash\mathbb{G} \text{ and } f \in L^2(\Gamma\backslash\mathbb{G}).$$

Note that since every nilpotent Lie group is unimodular, the regular representation is well defined and unitary. An important fact in the analysis of the sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$ is that the horizontal vector fields $p_*(\widetilde{X}_j)$ can be rewritten in the form

$$p_*(\widetilde{X}_j)(f) = \frac{d}{dt}|_{t=0}\rho(\exp(tX_j))f =: \rho_*(X_j)(f) \text{ for } f \in C^\infty(\Gamma\backslash\mathbb{G}),$$

i.e. the differential operator $p_*(\widetilde{X}_j)$ can be identified with the infinitesimal generator $\rho_*(X_j)$ of the one parameter group of operators $(\rho(\exp tX_j))_t$ on $L^2(\Gamma\backslash\mathbb{G})$.

Therefore, the sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$ can be expressed in the form

$$\Delta_{sub}^{\Gamma\backslash\mathbb{G}} = -\sum_{j=1}^{m} \rho_*(X_j)^2. \tag{4.1}$$

Since $\Gamma\backslash\mathbb{G}$ is a compact manifold, the representation ρ decomposes into a discrete direct sum of irreducible unitary representations π_λ with finite multiplicities:

$$\rho \simeq \sum_{\lambda\in\hat{\mathbb{G}}} m_\lambda \pi_\lambda, \tag{4.2}$$

where the sum runs over the unitary dual $\hat{\mathbb{G}}$ of $\mathbb{G}$ (i.e. the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$) and m_λ is the finite multiplicity of $\pi_\lambda \in \hat{\mathbb{G}}$. Hence, to compute the spectrum of $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$, we consider its restriction to every representation space $\mathcal{V}_\lambda$ associated to a representation π_λ which is by (4.1), unitary equivalent to

$$\Delta_{sub}^{\pi_\lambda} := -\sum_{j=1}^{m} (\pi_\lambda)_*(X_j)^2.$$

For this, we recall first the description of the unitary dual $\hat{\mathbb{G}}$ and second, which representation π_λ appears in (4.2), i.e. $m_\lambda > 0$ (see [44, 95, 97]).

4.1.1 Unitary dual $\hat{\mathbb{G}}$

We recall briefly the construction of the irreducible unitary representations of $\mathbb{G}$. We follow the presentations in [74, 95].

Let $\mathbf{n} \in \mathfrak{g}^*$. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called subordinate to $\mathbf{n}$ if

$$\mathbf{n}([X, Y]) = 0 \text{ for all } X, Y \in \mathfrak{h}.$$

The subalgebra $\mathfrak{h}$ is called maximal subordinate to $\mathbf{n}$, if it is subordinate to $\mathbf{n}$ and of maximal dimension among all subordinate subalgebras to $\mathbf{n}$. Every subalgebra $\mathfrak{h}$ which is subordinate to $\mathbf{n}$ induces a character $\chi_{\mathbf{n}}$ on $\exp\mathfrak{h}$ defined by the formula

$$\chi_{\mathbf{n}}(\exp X) := e^{2\pi i\mathbf{n}(X)} \text{ for } X \in \mathfrak{h}.$$

Now, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ subordinate to $\mathbf{n} \in \mathfrak{g}^*$ and $Y_1, \cdots, Y_k$ be elements of $\mathfrak{g}$ such that

1. $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}Y_1 \oplus \cdots \oplus \mathbb{R}Y_k$;

2. $\mathfrak{h}_j := \mathfrak{h} \oplus \mathbb{R}Y_1 \oplus \cdots \oplus \mathbb{R}Y_j$ is a subalgebra of $\mathfrak{g}$ for every $j = 1, \cdots, k$.

Then the map

$$\begin{aligned}\exp(\mathfrak{h}) \times \mathbb{R}^k &\longrightarrow \mathbb{G} \\ (h,t) &\longmapsto h * \prod_{j=1}^{k} \exp(t_i Y_i)\end{aligned}$$

where $t = (t_1, \cdots, t_k)$, defines a diffeomorphism [97]. Furthermore, this diffeomorphism induces another diffeomorphism between $(\exp \mathfrak{h})\backslash\mathbb{G}$ and $\mathbb{R}^k$. Let us denote by $\mathcal{F}_0$ the space of continuous functions on $\mathbb{G}$ with the property:

$$f(h * x) = \chi_{\mathbf{n}}(h) f(x) \text{ for } h \in \exp \mathfrak{h} \text{ and } x \in \mathbb{G}.$$

The space $\mathcal{F}_0$ can be identified (with help of the above diffeomorphism) with $C(\mathbb{R}^k)$ by the map $\alpha : \mathcal{F}_0 \longrightarrow C(\mathbb{R}^k)$ defined by

$$\alpha(f)(t) := f\left(\prod_{j=1}^{k} \exp(t_j Y_j)\right), \text{ for } t = (t_1, \cdots, t_k).$$

Let $\mathcal{F}_1$ denotes the subspace of $\mathcal{F}_0$ such that $\int_{\mathbb{R}^k} |\alpha(f)(t)|^2 dt < \infty$ and define $\mathcal{F}$ to be the completion of $\mathcal{F}_1$ with respect to the L^2-norm ($\mathcal{F}$ can be identified with $L^2(\mathbb{R}^k)$). Then we obtain a representation $\pi(\mathbf{n}, \exp \mathfrak{h})$ induced by $(\mathbf{n}, \exp \mathfrak{h})$ and defined by the formula

$$(\pi(\mathbf{n}, \exp \mathfrak{h})(x) f)(y) := f(y * x),$$

for $x, y \in \mathbb{G}$ and $f \in \mathcal{F}$. The following crucial result can be found in [74]:

Theorem 4.1.1. *With the above notations, it holds:*

1. $\pi(\mathbf{n}, \exp \mathfrak{h})$ *is a unitary representation. Furthermore, this representation is irreducible if and only if* $\mathfrak{h}$ *is maximal subordinate to* $\mathbf{n}$. *In this case, up to unitary equivalence, this representation depends only on* $\mathbf{n}$ *and not on the choice of the maximal subordinate subalgebra* $\mathfrak{h}$. *We set* $\pi_{\mathbf{n}} := \pi(\mathbf{n}, \mathfrak{h})$.

2. *Every unitary irreducible representation of* $\mathbb{G}$ *is unitary equivalent to some* $\pi_{\mathbf{n}}$ *with* $\mathbf{n} \in \mathfrak{g}^*$.

3. *For* $\mathbf{n}, \mathbf{m} \in \mathfrak{g}^*$, *the representations* $\pi_{\mathbf{n}}$ *and* $\pi_{\mathbf{m}}$ *are unitary equivalent if and only if there exists* $x \in \mathbb{G}$ *such that* $\mathbf{m} = \mathbf{n} \circ (I_x)_*$. *Here* I_x *denotes the following map on* $\mathbb{G}$, $I_x(y) := x * y * x^{-1}$.

By the above theorem, the unitary dual $\hat{\mathbb{G}}$ can be identified with $\mathfrak{g}^*/\sim$, where $\sim$ is the following equivalence relation on $\mathfrak{g}^*$:

$$\mathbf{n} \sim \mathbf{m} \text{ if and only if there exists } x \in \mathbb{G} \text{ with } \mathbf{m} = \mathbf{n} \circ (I_x)_*.$$

4.1.2 Representations appearing in the regular representation

Let $\mathbf{n} \in \mathfrak{g}^*$ and let us denote by $B_{\mathbf{n}}$ the bilinear form defined on $\mathfrak{g}$ by the formula

$$B_{\mathbf{n}}(X, Y) = \mathbf{n}([X, Y]) \text{ for } X, Y \in \mathfrak{g}.$$

Let $\mathfrak{g}_{\mathbf{n}}$ denote the kernel of $B_{\mathbf{n}}$, i.e. $\mathfrak{g}_{\mathbf{n}} = \{X \in \mathfrak{g} : B_{\mathbf{n}}(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$. Note that by definition of the Carnot group $\mathbb{G}$, it holds $\mathfrak{g}_2 \subseteq \mathfrak{g}_{\mathbf{n}} \subseteq \mathfrak{g}$. Using the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$, we associate to the bilinear form $B_{\mathbf{n}}$ a skew-symmetric operator $u_{\mathbf{n}}$ on $\mathfrak{g}_1$ defined by the formula

$$B_{\mathbf{n}}(X, Y) = \langle u_{\mathbf{n}}(X), Y \rangle \text{ for } X, Y \in \mathfrak{g}_1.$$

Since $u_{\mathbf{n}}$ is skew-symmetric, its non-zero eigenvalues are of the form $\pm i d_j$ for $j = 1, \cdots, k$ with

$$k = \frac{\dim \mathfrak{g}_1 - \dim(\mathfrak{g}_{\mathbf{n}} \cap \mathfrak{g}_1)}{2} \quad \text{and} \quad 0 < d_1 \leq \cdots \leq d_k.$$

Note that the bilinear form $B_{\mathbf{n}}$ induces a non-degenerate bilinear form $\widetilde{B}_{\mathbf{n}}$ on $\mathfrak{g}/\mathfrak{g}_{\mathbf{n}}$ and that the image $p_{\mathbf{n}}(\log \Gamma)$, where $p_{\mathbf{n}}$ is the projection from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{g}_{\mathbf{n}}$, is a lattice which we denote by $\mathcal{L}_{\mathbf{n}}$. We set

$$l_{\mathbf{n}} := \sqrt{\det(\widetilde{B}_{\mathbf{n}})},$$

where the determinant is defined with respect to an arbitrary basis of the lattice $\mathcal{L}_{\mathbf{n}}$. The following result was proved in [95]:

Proposition 4.1.2. *For $\mathbf{n} \in \mathfrak{g}^*$, the induced unitary irreducible representation $\pi_{\mathbf{n}}$ appears in the regular representation ρ of $\mathbb{G}$ in $L^2(\Gamma\backslash\mathbb{G})$ if and only if $\mathbf{n}(\log \Gamma \cap \mathfrak{g}_{\mathbf{n}}) \subset \mathbb{Z}$. Furthermore, the multiplicity $m_{\mathbf{n}}$ of $\pi_{\mathbf{n}}$ equals 1 if $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}$ and $m_{\mathbf{n}} = l_{\mathbf{n}}$ if $\mathfrak{g}_{\mathbf{n}} \neq \mathfrak{g}$.*

Hence, if we define

$$\mathcal{A}(\Gamma) := \{\mathbf{n} \in \mathfrak{g}^*/\sim : \mathbf{n}(\log \Gamma \cap \mathfrak{g}_{\mathbf{n}}) \subset \mathbb{Z}\},$$

then we can write

$$\rho = \sum_{\mathbf{n} \in \mathcal{A}(\Gamma)} m_{\mathbf{n}} \pi_{\mathbf{n}}$$

with $m_{\mathbf{n}} > 0$ for all $\mathbf{n} \in \mathcal{A}(\Gamma)$.

4.1.3 The spectrum of the intrinsic sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$

The computation of the spectrum of the sublaplacian is done like in the Riemannian case [95]. The idea is to explicitly realize the representations $\pi_{\mathbf{n}}$ and then to compute the action of the operators $(\pi_{\mathbf{n}})_*(X_i)$ for some special orthonormal basis $X_1, \cdots, X_m$ of $\mathfrak{g}_1$. Let $\mathbf{n} \in \mathcal{A}(\Gamma)$ and $\pi_{\mathbf{n}}$ be the induced representation with representation space $\mathcal{V}_{\mathbf{n}}$. We distinguish the following cases:

1. If $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}$, then $\mathfrak{g}$ is the unique maximal subordinate subalgebra to $\mathbf{n}$. Furthermore, the representation space $\mathcal{V}_{\mathbf{n}}$ is 1-dimensional and the representation $\pi_{\mathbf{n}}$ is given by

$$\pi_{\mathbf{n}}(\exp X) = e^{2\pi i \mathbf{n}(X)} \text{ for } X \in \mathfrak{g}.$$

Hence, the operator $\Delta_{sub}^{\pi_{\mathbf{n}}}$ is the multiplication operator by $4\pi^2 \sum_{j=1}^{m} \mathbf{n}(X_j)^2$. Therefore, the spectrum of $\Delta_{sub}^{\pi_{\mathbf{n}}}$ is $\sigma_{\mathbf{n}} = \{4\pi^2 \sum_{j=1}^{m} \mathbf{n}(X_j)^2\}$.

2. If $\mathfrak{g}_{\mathbf{n}} \neq \mathfrak{g}$ and $\mathfrak{g}_{\mathbf{n}} \cap \mathfrak{g}_1 \neq 0$, or equivalently $\mathfrak{g}_2 \subsetneq \mathfrak{g}_{\mathbf{n}} \subsetneq \mathfrak{g}$, we construct a basis $\{U_1, \cdots, U_k, V_1, \cdots, V_k, W_1, \cdots, W_l\}$ of $\mathfrak{g}$ such that:

   - $\{W_1, \cdots, W_d\}$ is a basis of $\mathfrak{g}_2$ and $\{W_{d+1}, \cdots, W_l\}$ is an orthonormal basis of $\mathfrak{g}_{\mathbf{n}} \cap \mathfrak{g}_1$.
   - $\{U_1, \cdots, U_k, V_1, \cdots, V_k, W_{d+1}, \cdots, W_l\}$ is an orthonormal basis of $\mathfrak{g}_1$.
   - For $i, j = 1, \cdots, k$ it holds:

$$\langle U_i, u_{\mathbf{n}}(U_j)\rangle = \langle V_i, u_{\mathbf{n}}(V_j)\rangle = \langle U_i, u_{\mathbf{n}}(V_j)\rangle - \delta_{ij} d_j = 0.$$

In this case, a maximal subordinate subalgebra to $\mathbf{n}$ is given by (see [95])

$$\mathfrak{h}_{\mathbf{n}} := \mathfrak{g}_{\mathbf{n}} \oplus \mathbb{R}V_1 \oplus \cdots \oplus \mathbb{R}V_k.$$

With the special basis constructed above, the operator $\Delta_{sub}^{\pi_{\mathbf{n}}}$ can be expressed in the form

$$\Delta_{sub}^{\pi_{\mathbf{n}}} = -\sum_{j=1}^{k} \left((\pi_{\mathbf{n}})_*(U_j)^2 + (\pi_{\mathbf{n}})_*(V_j)^2 \right) - \sum_{j=d+1}^{l} (\pi_{\mathbf{n}})_*(W_j)^2.$$

Now, we use the above basis to explicitly construct the representation $\pi_{\mathbf{n}}$ with representation space $\mathcal{V}_{\mathbf{n}} \simeq L^2(\mathbb{R}^k)$ like in Subsection 4.1.1. The proof of the following identities is similar to the ones in [95]: For $f \in L^2(\mathbb{R}^k)$ and $t = (t_1, \cdots, t_k)$ it holds

   - $(\pi_{\mathbf{n}})_*(U_j) f(t) = \frac{\partial f}{\partial t_j}(t)$,
   - $(\pi_{\mathbf{n}})_*(V_j) f(t) = 2\pi i(\mathbf{n}(V_j) + d_j t_j) f(t)$,
   - $(\pi_{\mathbf{n}})_*(W_r) f = 2\pi i \mathbf{n}(W_r) f$.

for $j = 1, \cdots, k$ and $r = 1, \cdots, l$. Therefore, the operator $\Delta_{sub}^{\pi_{\mathbf{n}}}$ can be expressed in the form

$$\Delta_{sub}^{\pi_{\mathbf{n}}} = -\sum_{j=1}^{k} \frac{\partial^2}{\partial t_j^2} + 4\pi^2 \left(\sum_{j=1}^{k} (\mathbf{n}(V_j) + d_j t_j)^2 + \sum_{j=d+1}^{l} \mathbf{n}(W_j)^2 \right).$$

Note that this operator is unitary equivalent to the *harmonic oscillator Hamiltonian*

$$-\sum_{j=1}^{k} \frac{\partial^2}{\partial t_j^2} + 4\pi^2 \left(\sum_{j=1}^{k} d_j^2 t_j^2 + \sum_{j=d+1}^{l} \mathbf{n}(W_j)^2 \right).$$

Hence, the spectrum of $\Delta_{sub}^{\pi_{\mathbf{n}}}$ is

$$\sigma_{\mathbf{n}} = \{4\pi^2 \sum_{j=d+1}^{l} \mathbf{n}(W_j)^2 + 2\pi \sum_{j=1}^{k} (2p_j + 1)d_j : p = (p_1, \cdots, p_k) \in \mathbb{N}^k\}.$$

3. If $\mathfrak{g}_{\mathbf{n}} \neq \mathfrak{g}$ and $\mathfrak{g}_{\mathbf{n}} \cap \mathfrak{g}_1 = 0$ or equivalently $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}_2$, the same arguments as in the second case shows that the spectrum of $\Delta_{sub}^{\pi_{\mathbf{n}}}$ is

$$2\pi \sum_{j=1}^{k} (2p_j + 1)d_j \text{ for } p = (p_1, \cdots, p_k) \in \mathbb{N}^k$$

with $2k = m$.

Theorem 4.1.3. *Let $\mathbb{G}$ be a step 2 Carnot group with left invariant metric $\langle \cdot, \cdot \rangle$ and Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We assume that there exists a lattice Γ in $\mathbb{G}$. Then the spectrum of the intrinsic sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$ associated to the subriemannian manifold $(\Gamma\backslash\mathbb{G}, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is given as follows:*

$$\sigma(\Delta_{sub}^{\Gamma\backslash\mathbb{G}}) = \bigcup_{\mathbf{n} \in \mathcal{A}(\Gamma)} \sigma_{\mathbf{n}}$$

where

1. $\sigma_{\mathbf{n}} = \{4\pi^2 \sum_{i=1}^{m} \mathbf{n}(X_i)^2\}$ *if* $\mathbf{n} \in \mathcal{A}(\Gamma)$ *with* $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}$. *Here* $X_1, \cdots, X_m$ *is an orthonormal basis of* $\mathfrak{g}_1$.
2. $\sigma_{\mathbf{n}} = \{4\pi^2 \sum_{j=d+1}^{l} \mathbf{n}(W_j)^2 + 2\pi \sum_{j=1}^{k} d_j(2p_j + 1)$ *for* $p \in \mathbb{N}^k\}$ *if* $\mathbf{n} \in \mathcal{A}(\Gamma)$ *with* $\mathfrak{g}_2 \subsetneq \mathfrak{g}_{\mathbf{n}} \subsetneq \mathfrak{g}$. *Here* $\pm i d_j$, $j = 1 \cdots, k$ *denote the non-zero eigenvalues of the operator* $u_{\mathbf{n}}$ *and* $W_{d+1}, \cdots, W_l$ *is an orthonormal basis of* $\mathfrak{g}_{\mathbf{n}} \cap \mathfrak{g}_1$.
3. $\sigma_{\mathbf{n}} = \{2\pi \sum_{j=1}^{k} d_j(2p_j + 1)$ *for* $p = (p_1, \cdots, p_k) \in \mathbb{N}^k\}$ *if* $\mathbf{n} \in \mathcal{A}(\Gamma)$ *with* $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}_2$. *Here* $\pm i d_j$, $j = 1 \cdots, k$ *denote the non-zero eigenvalues of the operator* $u_{\mathbf{n}}$.

Furthermore, the multiplicity of an eigenvalue λ is given by $\sum_{\mathbf{n}} m_{\mathbf{n}}$ where the sum runs over all $\mathbf{n} \in \mathcal{A}(\Gamma)$ such that $\lambda \in \sigma_{\mathbf{n}}$.

Remark 4.1.4. For $X \in \mathfrak{g}$ it holds $(I_{\exp X})_* = \mathrm{Id} + [X, \cdot]$. It follows that

$$\mathbf{m} \sim \mathbf{n} \Longleftrightarrow \mathbf{m} = \mathbf{n} + \mathbf{n}([X, \cdot]) \text{ for some } X \in \mathfrak{g}.$$

Therefore, we have $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}_{\mathbf{m}}$, $u_{\mathbf{n}} = u_{\mathbf{m}}$ and $l_{\mathbf{n}} = l_{\mathbf{m}}$, i.e. $\sigma_{\mathbf{n}}$ depends only on the equivalence class $\mathbf{n}$.

For the rest of this chapter, we consider a specific subclass of all step 2 Carnot groups, the so-called H-type Lie groups [71]. We recall that an H-type Lie group $\mathbb{G}$ is a pseudo H-type Lie group with $s = 0$, i.e. $\mathbb{G}$ is of the form $\mathbb{G}_{d,0}(\mathcal{V})$ with an admissible Clifford module $\mathcal{V}$ (see Definition 3.3.2). Also $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \mathcal{V} \simeq \mathbb{R}^{2N}$ and $\mathfrak{g}_2 = \mathbb{R}^d$ and the Lie brackets are given by the formula

$$\langle Z, [X, Y] \rangle = \langle J_Z X, Y \rangle$$

for $X, Y \in \mathfrak{g}_1$ and $Z \in \mathfrak{g}_2$. Here J_Z denotes the map introduced in Chapter 3, Section 3.3.

Let $\mathbb{G}$ be an H-type group and Γ be a lattice in $\mathbb{G}$. Then $\Gamma_1 := p(\log \Gamma)$ (resp. $\Gamma_2 := \log \Gamma \cap \mathfrak{g}_2$) where $p : \mathfrak{g} \longrightarrow \mathfrak{g}_1$ denotes the orthogonal projection, is also a lattice in $\mathfrak{g}_1$ (resp. in $\mathfrak{g}_2$), which we call the induced horizontal lattice (resp. vertical lattice). By $\mathbb{T}_1 := \Gamma_1 \backslash \mathfrak{g}_1$ and $\mathbb{T}_2 := \Gamma_2 \backslash \mathfrak{g}_2$ we denote the associated tori.

Next, we characterize explicitly the elements of $\mathcal{A}(\Gamma)$ for a lattice Γ in the H-type group $\mathbb{G}$. Let $\mathbf{n} \in \mathfrak{g}^*$ and let $X \in \mathfrak{g}$ be the corresponding element throughout the identification $\mathfrak{g}^* \cong \mathfrak{g}$ via the inner product $\langle \cdot, \cdot \rangle$. Then the operator $u_{\mathbf{n}}$ defined in Subsection 4.1.2 coincides with the map J_Z, where $Z \in \mathfrak{g}_2$ denotes the orthogonal projection of X onto $\mathfrak{g}_2$. Furthermore, we have two possibilities for $\mathfrak{g}_{\mathbf{n}}$:

1. If $\mathbf{n}(\mathfrak{g}_2) = \{0\}$, then $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}$ and the orbit of $\mathbf{n}$ in $\mathfrak{g}^*/\sim$ contains only $\mathbf{n}$.

2. If $\mathbf{n}(\mathfrak{g}_2) \neq \{0\}$, then $\mathfrak{g}_{\mathbf{n}} = \mathfrak{g}_2$ and $u_{\mathbf{n}}$ is invertible with eigenvalues $\pm i\|\mathbf{n}\|$. Furthermore, we can choose an element $\mathbf{m}$ in the orbit of $\mathbf{n}$ with the property:

$$\mathbf{m} = \mathbf{n} \text{ on } \mathfrak{g}_2 \text{ and } \mathbf{m} = 0 \text{ on } \mathfrak{g}_1.$$

Hence, the orbit space $\mathcal{A}(\Gamma)$ can be decomposed as follows:

$$\mathcal{A}(\Gamma) = \Gamma_1^* \cup \Gamma_2^* \backslash \{0\},$$

where Γ_1^* (resp. Γ_2^*) denotes the dual lattice of Γ_1 (resp. Γ_2). Furthermore, since the eigenvalues of $u_{\mathbf{n}}$ are $\pm i\|\mathbf{n}\|$, a straightforward calculation shows that for $\mathbf{n} \in \Gamma_2^*$

$$m_{\mathbf{n}} = \|\mathbf{n}\|^N \mathrm{vol}(\mathbb{T}_1).$$

As corollary of Theorem 5.6.5 we obtain:

Corollary 4.1.5. *The spectrum of the intrinsic sublaplacian on $\Gamma \backslash \mathbb{G}$ is given by*

$$\sigma(\Delta_{sub}^{\Gamma \backslash \mathbb{G}}) = \bigcup_{\mathbf{n} \in \mathcal{A}(\Gamma)} \sigma_{\mathbf{n}}$$

with

1. $\sigma_{\mathbf{n}} = \{4\pi^2 \|\mathbf{n}\|^2\}$ *for* $\mathbf{n} \in \Gamma_1^*$.

2. $\sigma_{\mathbf{n}} = \{2\pi \|\mathbf{n}\| \sum_{j=1}^N (2p_j + 1) : p = (p_1, \cdots, p_N) \in \mathbb{N}^N\}$ *for* $\mathbf{n} \in \Gamma_2^* \backslash \{0\}$.

The multiplicity of an eigenvalue λ *is given by* $\sum_{\mathbf{n}} m_{\mathbf{n}}$ *where the sum runs over all* $\mathbf{n} \in \mathcal{A}(\Gamma)$ *such that* $\lambda \in \sigma_{\mathbf{n}}$.

Remark 4.1.6.
1. If Γ is the standard integral lattice defined in section 3.3 of Chapter 3, we find the same result as in Remark 3.5.1.
2. In [50], the spectrum of the sublaplacian was computed on nilmanifolds $\Gamma\backslash\mathbb{G}$ where $\mathbb{G}$ is the Heisenberg group. However, the expression of the spectrum given therein differs from our expression due to different normalizations in the definition of the Heisenberg group and of the considered sublaplacian.

4.2 Subriemannian length spectrum

In this section we study the subriemannian geodesics on the nilmanifolds $\Gamma\backslash\mathbb{G}$ in order to compute the subriemannian length spectrum. Since the canonical projection

$$\pi : \mathbb{G} \longrightarrow \Gamma\backslash\mathbb{G}$$

is a local subriemannian isometry and a covering map, we only need to study the normal geodesics $\gamma(t)$ in the H-type group $\mathbb{G}$ which are lifts of closed subriemannian geodesics of $\Gamma\backslash\mathbb{G}$, i.e. there is an element $\phi \in \Gamma$ and $\omega \in \mathbb{R}$ with the property:

$$\phi * \gamma(t) = \gamma(t+\omega) \text{ for all } t \in \mathbb{R}.$$

The projection of such a geodesic to the base space $\Gamma\backslash\mathbb{G}$ will be a closed geodesic that belongs to the free homotopy class of closed curves in $\Gamma\backslash\mathbb{G}$ determined by ϕ. Note also that elements of Γ that are conjugate in Γ determine the same free homotopy class in $\Gamma\backslash\mathbb{G}$. As we will see below, every non-trivial free homotopy class is represented by a closed subriemannian geodesic like in the Riemannian case. This result remains also true in the general framework of compact, connected subriemannian manifolds [79].

Definition 4.2.1. The subriemannian length spectrum of $\Gamma\backslash\mathbb{G}$ is defined as the set of lengths of closed subriemannian geodesics. Furthermore, we denote it by $L(\Gamma\backslash\mathbb{G})$.

Since the considered subriemannian structure on $\mathbb{G}$ is left-invariant, it is sufficient to consider geodesics starting at the identity $e = (0,0) \in \mathbb{R}^{2N} \times \mathbb{R}^d \cong \mathbb{G}$. Furthermore, we use the coordinates on $T_e^*\mathbb{G}$ which are dual to the exponential coordinates, i.e. for a covector $\lambda \in T_e^*\mathbb{G}$ we write

$$\lambda = \sum_{i=1}^{2N} u_i dx_i + \sum_{j=1}^{d} v_i dz_i \text{ with } (u,v) \in \mathbb{R}^{2N} \times \mathbb{R}^d.$$

Also we set

$$U := \sum_{i=1}^{2N} u_i X_i \text{ and } V := \sum_{k=1}^{d} v_k Z_k.$$

The following expression of geodesics on H-type groups can be found in [12, 26]:

Lemma 4.2.2 ([12, 26])**.** *Let $\lambda = (u, v) \in T_e^*\mathbb{G}$. Then the subriemannian geodesic in $\mathbb{G}$ starting at the identity with initial covector λ is $\alpha(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathfrak{g}_1$ and $Z(t) \in \mathfrak{g}_2$ are given for $t \in \mathbb{R}$ by*

$$X(t) = \left(\frac{\sin t\|V\|}{\|V\|} I + \frac{\cos(t\|V\|) - 1}{\|V\|} J_V\right) U,$$
$$Z(t) = \|U\|^2 \left(\frac{t\|V\| - \sin t\|V\|}{2\|V\|^2}\right) \frac{V}{\|V\|}.$$

Here we assumed that $V \neq 0$. Otherwise, we have $X(t) = tU$ and $Z(t) = 0$ for all t.

The following theorem is the main result of this section:

Theorem 4.2.3. *Let $\phi = \exp(X + Z) \in \Gamma$.*

1. *If $X \neq 0$, then every closed geodesic in the free homotopy class determined by ϕ has length $\|X\|$.*
2. *If $X = 0$, then the lengths of closed geodesics in the free homotopy class determined by ϕ are given by*
$$\sqrt{4k\pi\|Z\|} \text{ for } k = 1, 2, \cdots$$

Proof. Let $\gamma(t)$ be a unit speed subriemannian geodesic starting at $g = \gamma(0) \in \mathbb{G}$ and lying in the free homotopy class determined by ϕ, i.e. there is some $\omega \in \mathbb{R}$ such that

$$\phi * \gamma(t) = \gamma(t + \omega) \text{ for all } t \in \mathbb{R}.$$

Then $\alpha(t) := L_{g^{-1}}(\gamma(t))$ is a unit speed subriemannian geodesic starting at the identity with

$$\psi * \alpha(t) = \alpha(t + \omega) \text{ for all } t, \tag{4.3}$$

where $\psi := g^{-1} * \phi * g = \exp(X + Z')$ for some $Z' \in \mathfrak{g}_2$. We set $\alpha(t) = \exp(X(t) + Z(t))$. Then, equation (4.3) is equivalent to

$$X(t) + X = X(t + \omega) \text{ and } Z(t) + Z' + \frac{1}{2}[X, X(t)] = Z(t + \omega) \text{ for all } t. \tag{4.4}$$

1. We assume $X \neq 0$. Let us also assume that the geodesic $\alpha(t)$ has $\lambda = (u, v)$ as initial covector with $v \neq 0$. Then, differentiating the first equation of (4.4) and using Lemma 4.2.2 yields

$$(\cos t\|V\|I - \sin t\|V\|J_V)\, U = (\cos(t + \omega)\|V\|I - \sin(t + \omega)\|V\|J_V)\, U,$$

 for all t. Therefore, there is some $k \in \mathbb{N}$ such that $\omega\|V\| = 2k\pi$. Inserting this is the first equation of (4.4) imply $X = 0$, which is a contradiction. Hence v must be zero and so $X(t) = tU$ and $Z(t) = 0$. Therefore, equation (4.4) imply $\omega = \|X\|$.
 Conversely, let us consider the unit speed geodesic $\alpha(t)$ starting at the identity with

initial covector $\lambda = (u, 0)$ such that $U = \frac{X}{\|X\|}$. Then a straightforward calculation shows that $\phi * \alpha(t) = \alpha(t + \|X\|)$ for all t. Therefore, the geodesic $\alpha(t)$ projects to a geodesic in $\Gamma\backslash\mathbb{G}$ with period $\|X\|$ which lies in the free homotopy class determined by ϕ.

2. Now, we assume that $X = 0$. Then, similar arguments like in the first case show that if the geodesic $\alpha(t)$ has $\lambda = (u, v)$ as initial covector, then $v \neq 0$ and hence, from the first equation of (4.4) it follows that there is some $k \in \mathbb{N}$ such that $\omega\|V\| = 2k\pi$. Now, let us consider the second equation of (4.4). Since $X = 0$, it follows that ϕ lies in the center of the group and so $\psi = \phi$ and we can write

$$Z(t) + Z = Z(t + \omega) \text{ for all } t.$$

By Lemma 4.2.2 and using the fact that $\omega\|V\| = 2k\pi$, we can write

$$\begin{aligned} Z(t+\omega) &= \left(\frac{(t+\omega)\|V\| - \sin(t+\omega)\|V\|}{2\|V\|^2} \right) \frac{V}{\|V\|} \\ &= Z(t) + \frac{\omega}{2\|V\|^2} V. \end{aligned}$$

Therefore, V and Z are related by the relation $Z = \frac{\omega}{2\|V\|^2} V$ and hence

$$\omega^2 = 2\omega\|V\|\|Z\| = 4k\pi\|Z\|,$$

i.e. ω can be expressed in the form $\sqrt{4k\pi\|Z\|}$.
Conversely, let $k = 1, 2, \cdots$ and $\omega_k^2 = 4k\pi\|Z\|$. let us consider the unit speed geodesic $\alpha_k(t)$ starting at the identity with initial covector $\lambda_k = (u, v_k)$ such that $V_k = \frac{2k\pi}{\omega_k\|Z\|} Z$ and $\|U\| = 1$ is arbitrary. Then a straightforward calculation shows that $\omega_k\|V_k\| = 2k\pi$ and hence, the geodesic $\alpha_k(t)$ fulfils the equation (4.4) with ω_k. This means that the geodesic $\alpha_k(t)$ projects to a geodesic in $\Gamma\backslash\mathbb{G}$ with period $\sqrt{4k\pi\|Z\|}$ which lies in the free homotopy class determined by ϕ.

□

Remark 4.2.4. From Theorem 4.2.3, we conclude that in every free homotopy class determined by $\phi = \exp Z$ with $Z \in \Gamma_2\backslash\{0\}$, there are infinitely many closed geodesics with lengths given by $\sqrt{4k\pi\|Z\|}$ for $k = 1, 2, \cdots$. The situation is very different if we consider the nilmanifold $\Gamma\backslash\mathbb{G}$ with the natural left-invariant Riemannain metric. In this case, in every free homotopy class there are only a finite number of lengths corresponding to closed geodesics lying in this class (see [49]).

The length spectrum $L(\Gamma\backslash\mathbb{G})$ can be completely expressed using the lattices Γ_1 and Γ_2 introduced in Section 4.1.3 and the metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$:

Corollary 4.2.5. *Let $\mathbb{G}$ be an H-type Lie group and Γ be a lattice in $\mathbb{G}$. Then the subriemannian length spectrum of the nilmanifold $\Gamma\backslash\mathbb{G}$ can be expressed in the form*

$$L(\Gamma\backslash\mathbb{G}) = \{\|X\| : \ X \in \Gamma_1\} \cup \{\sqrt{4k\pi\|Z\|} : \ k \in \mathbb{N}, Z \in \Gamma_2\}.$$

4.3 Heat trace and length spectrum

In this section we establish a relation between the spectrum of the sublaplacian $\Delta_{sub}^{\Gamma\backslash\mathbb{G}}$ and the subriemannian length spectrum $L(\Gamma\backslash\mathbb{G})$ computed in the previous sections.
In the following $\mathbb{G}$ is assumed to be an H-type group and Γ denotes an arbitrary lattice in $\mathbb{G}$. Next, we prove a useful formula for the heat trace

$$\mathrm{tr}(e^{-t\Delta_{sub}^{\Gamma\backslash\mathbb{G}}}) = \sum_{\lambda\in\sigma(\Delta_{sub})} e^{-\lambda t}, \text{ for } t>0.$$

Lemma 4.3.1. *For $t>0$, it holds:*

$$tr(e^{-t\Delta_{sub}^{\Gamma\backslash\mathbb{G}}}) = \sum_{\mathbf{n}\in\Gamma_1^*} e^{-4\pi^2\|\mathbf{n}\|^2 t} + \frac{vol(\mathbb{T}_1)}{(4\pi t)^N}\sum_{\mathbf{n}\in\Gamma_2^*\backslash\{0\}} f_t(\mathbf{n}),$$

where $f_t(\lambda) = \left(\frac{2\pi t\|\lambda\|}{\sinh 2\pi t\|\lambda\|}\right)^N$ for $\lambda\in\mathbb{R}^d$.

Proof. Using the formula

$$\frac{1}{\sinh 2\pi t\|\lambda\|} = 2\sum_{p=0}^{\infty} e^{-2\pi t\|\lambda\|(2p+1)}$$

we write

$$\begin{aligned}\frac{\mathrm{vol}(\mathbb{T}_1)}{(4\pi t)^N}\sum_{\mathbf{n}\in\Gamma_2^*\backslash\{0\}} f_t(\mathbf{n}) &= \frac{\mathrm{vol}(\mathbb{T}_1)}{(4\pi t)^N}\sum_{\mathbf{n}\in\Gamma_2^*\backslash\{0\}}\left(\frac{2\pi t\|\mathbf{n}\|}{\sinh 2\pi t\|\mathbf{n}\|}\right)^N\\ &= \sum_{\mathbf{n}\in\Gamma_2^*\backslash\{0\}}\sum_{p\in\mathbb{N}_0^N} \mathrm{vol}(\mathbb{T}_1)\|\mathbf{n}\|^N e^{-2\pi t\|\mathbf{n}\|\sum_{j=1}^N(2p_j+1)}.\end{aligned}$$

Now, the result follows by using Corollary 4.1.5. □

Applying the Poisson summation formula on the tori $\mathbb{T}_1$ and $\mathbb{T}_2$, we obtain

$$\sum_{\mathbf{n}\in\Gamma_1^*} e^{-4\pi^2\|\mathbf{n}\|^2 t} = \frac{\mathrm{vol}(\mathbb{T}_1)}{(4\pi t)^N} + \frac{\mathrm{vol}(\mathbb{T}_1)}{(4\pi t)^N}\sum_{X\in\Gamma_1\backslash\{0\}} e^{-\frac{\|X\|^2}{4t}}$$

and

$$\sum_{\mathbf{n}\in\Gamma_2^*} f_t(\mathbf{n}) = \mathrm{vol}(\mathbb{T}_2)\sum_{Z\in\Gamma_2}\hat{f}_t(Z),$$

where $\hat{f}_t$ denotes the Fourier transformation of the function f_t. Therefore, we can write

$$\mathrm{tr}(e^{-t\Delta_{sub}^{\Gamma\backslash\mathbb{G}}}) = \frac{\mathrm{vol}(\mathbb{T}_1)}{(4\pi t)^N}\sum_{X\in\Gamma_1\backslash\{0\}} e^{-\frac{\|X\|^2}{4t}} + \frac{\mathrm{vol}(\mathbb{T}_1)\mathrm{vol}(\mathbb{T}_2)}{(4\pi t)^N}\sum_{Z\in\Gamma_2}\hat{f}_t(Z). \tag{4.5}$$

Note that lengths of closed geodesics lying in free homotopy classes corresponding to elements $\phi = \exp(X+Z) \in \Gamma$ with $X \neq 0$, appear in the first summation of (4.5). As we will see below, the second summation of (4.5) contains informations about lengths of closed geodesics lying in free homotopy classes corresponding to elements $\phi = \exp(X+Z) \in \Gamma$ with $X = 0$.

Next, we evaluate the Fourier transformation of f_t using the fact that it is a radial function. For this, we assume first that $d > 1$:

$$\begin{aligned}\hat{f}_t(Z) &= \int_{\mathbb{R}^d} e^{-2\pi i\langle Z,y\rangle}\left(\frac{2\pi t\|y\|}{\sinh 2\pi t\|y\|}\right)^N dy\\ &=\frac{1}{(2\pi t)^d}\int_{\mathbb{R}^d} e^{-\frac{i}{t}\langle Z,\xi\rangle}\left(\frac{\|\xi\|}{\sinh\|\xi\|}\right)^N d\xi\\ &=\frac{1}{(2\pi t)^d}\int_0^\infty\left(\int_{\|\eta\|=1} e^{-\frac{ir}{t}\langle Z,\eta\rangle}d\sigma(\eta)\right)\left(\frac{r}{\sinh r}\right)^N r^{d-1}dr\\ &=\frac{1}{(2\pi t)^d}\int_0^\infty\left(S_{d-2}\int_0^\pi e^{-\frac{ir}{t}\|Z\|\cos\theta}\sin(\theta)^{d-2}d\theta\right)\left(\frac{r}{\sinh r}\right)^N r^{d-1}dr.\end{aligned}$$

Here we used spherical coordinates and S_{d-2} denotes the volume of the $(d-2)$-dimensional Euclidean sphere $\mathbb{S}^{d-2}$. If $d = 1$, then the Fourier transformation of f_t simplify to

$$\hat{f}_t(Z) = \frac{1}{(2\pi t)^d}\int_{\mathbb{R}} e^{-\frac{i}{t}\langle Z,\xi\rangle}\left(\frac{\xi}{\sinh\xi}\right)^N d\xi.$$

In the following we only treat the case when $d > 1$ is odd. The case $d = 1$ can be achieved in a similar way.

For $\alpha \in \mathbb{R}\backslash\{0\}$, we consider the following integral

$$I(\alpha) := \int_0^\pi e^{-i\alpha\cos\theta}\sin(\theta)^{d-2}d\theta$$

Lemma 4.3.2. *We assume that* $d \geq 3$ *is odd. Then, there is a polynomial* $P(\cdot,\cdot)$ *of degree* $d-2$ *(resp.* $d-3$*) in the first (resp. second) variable such that:*

$$I(\alpha) = \left[e^{-i\alpha u}P\left(\frac{1}{\alpha},u\right)\right]_{-1}^{1}.$$

Furthermore, it holds:

$$P(-\alpha,u) = -P(\alpha,-u) \;\; \textit{for all } \alpha, u.$$

Proof. We make the ansatz $\cos\theta = u$ to obtain

$$I(\alpha) = \int_{-1}^{1} e^{-i\alpha u}(1-u^2)^{\frac{d-3}{2}}du.$$

Now, we define the polynomial P by the equation:

$$(1-u^2)^{\frac{d-3}{2}} = \frac{\partial P}{\partial u}\left(\frac{1}{\alpha}, u\right) - i\alpha P\left(\frac{1}{\alpha}, u\right) \text{ for all } u$$

so that

$$\frac{\partial}{\partial u}\left(e^{-i\alpha u} P\left(\frac{1}{\alpha}, u\right)\right) = e^{-i\alpha u}(1-u^2)^{\frac{d-3}{2}}.$$

Note that this is possible by the assumption d is odd. If we set

$$(1-u^2)^{\frac{d-3}{2}} = \sum_{l=0}^{d-3} a_l u^l \text{ and } P\left(\frac{1}{\alpha}, u\right) = \sum_{l=0}^{d-3} b_l(\alpha) u^l,$$

then the coefficients b_l can be computed (backwards) recursively by the formula

$$\begin{cases} (l+1)b_{l+1}(\alpha) - i\alpha b_l(\alpha) = a_l, & \text{for } n = 0, \cdots, d-4. \\ b_{d-3}(\alpha) = \frac{i}{\alpha} a_{d-3}. \end{cases}$$

Now, a straightforward calculation shows that the polynomial P fulfils the desired conditions. $\square$

Using Lemma 4.3.2 we write

$$\begin{aligned} \hat{f}_t(Z) &= \frac{1}{(2\pi t)^d} \int_0^\infty \left(S_{d-2} \int_0^\pi e^{-\frac{ir}{t}\|Z\|\cos\theta} \sin(\theta)^{d-2} d\theta\right) \left(\frac{r}{\sinh r}\right)^N r^{d-1} dr \\ &= \frac{1}{(2\pi t)^d} \int_0^\infty S_{d-2} \left[e^{-i\frac{r\|Z\|}{t}u} P\left(\frac{t}{r\|Z\|}, u\right)\right]_{-1}^1 \left(\frac{r}{\sinh r}\right)^N r^{d-1} dr \\ &= \frac{-S_{d-2}}{(2\pi t)^d} \int_{-\infty}^\infty e^{i\frac{r\|Z\|}{t}} \left(\frac{r}{\sinh r}\right)^N r^{d-1} P\left(\frac{t}{r\|Z\|}, -1\right) dr. \end{aligned}$$

The proof of the main result of this section is an adaptation of the proof given in [96] for the Riemannian case:

Theorem 4.3.3. *Let $\mathbb{G}$ be an H-type group with lattice Γ. We assume that d is odd. Then for $t > 0$, it holds:*

$$tr(e^{-t\Delta_{sub}^{\Gamma\backslash\mathbb{G}}}) = \frac{C_{N,d}}{t^{N+d}} + \frac{vol(\mathbb{T}_1)}{(4\pi t)^N} \sum_{X\in\Gamma_1\backslash\{0\}} e^{-\frac{\|X\|^2}{4t}} + \sum_{Z\in\Gamma_2\backslash\{0\}} \sum_{k=1}^\infty \|Z\|^{-(N+d)} \varphi_k\left(\frac{\|Z\|}{t}\right) e^{-\frac{k\pi\|Z\|}{t}},$$

where φ_k, for $k = 1, 2, \cdots$, is a polynomial of degree $\leq 2N + d - 1$ and $C_{N,d}$ is a constant given by

$$C_{N,d} = \frac{2^d vol(\mathbb{T}_1) vol(\mathbb{T}_2)}{(4\pi)^{N+d}} \int_{\mathbb{R}^d} \left(\frac{\|\xi\|}{\sinh\|\xi\|}\right)^N d\xi.$$

Proof. By formula (4.5), the theorem is proven if we show the existence of polynomials φ_k such that

$$\frac{\mathrm{vol}(\mathbb{T}_1)\mathrm{vol}(\mathbb{T}_2)}{(4\pi t)^N}\hat{f}_t(Z) = \|Z\|^{-(N+d)}\sum_{k=1}^{\infty}\varphi_k\left(\frac{\|Z\|}{t}\right)e^{-\frac{k\pi\|Z\|}{t}}.$$

For fixed $t > 0$ and $Z \neq 0$, consider the following function of u on $\mathbb{R}$:

$$F_{Z,t} : u \longmapsto -S_{d-2}e^{i\frac{\|Z\|}{t}u}P\left(\frac{t}{u\|Z\|}, -1\right)\left(\frac{u}{\sinh u}\right)^N u^{d-1}.$$

Then $F_{Z,t}$ has a meromorphic extension to $\mathbb{C}$ with poles $ik\pi$ for $k \in \mathbb{Z}\backslash\{0\}$. We apply the residue theorem to obtain the desired result. For this, let us consider the following contours in the complex plane which were axis-aligned squares with sides passing through $\pm(k\pi + \frac{\pi}{2})$ and $i(k\pi + \frac{\pi}{2})$:

$$C_k := C_k^1 \cup C_k^2 \cup C_k^{\pm}$$

with

- $C_k^1 = \{|x| \leq k\pi + \frac{\pi}{2} \text{ and } x \in \mathbb{R}\}$.
- $C_k^2 = \{x + i(k\pi + \frac{\pi}{2}) : |x| \leq k\pi + \frac{\pi}{2} \text{ and } x \in \mathbb{R}\}$.
- $C_k^{\pm} = \{\pm(k\pi + \frac{\pi}{2}) + iy : 0 \leq y \leq k\pi + \frac{\pi}{2} \text{ and } y \in \mathbb{R}\}$.

A straightforward calculation shows that

$$\lim_{k\to\infty}\int_{\mathcal{C}_k^{\pm}} F_{Z,t}(u)du = 0.$$

For $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, it holds:

$$\sinh\left(x + i(k\pi + \frac{\pi}{2})\right) = (-1)^k\cosh x.$$

Hence it follows that

$$\lim_{k\to\infty}\int_{\mathcal{C}_k^{2}} F_{Z,t}(u)du = 0.$$

Applying the Residue theorem we obtain:

$$\hat{f}_t(Z) = \frac{1}{(2\pi t)^d}\int_{\mathbb{R}} F_{Z,t}(r)dr = \frac{2\pi i}{(2\pi t)^d}\sum_{k=1}^{\infty}\mathrm{Res}(F_{Z,t}, ik\pi).$$

We use Laurent serie expansion to compute the residue of F at a pole $u_k := ik\pi$ as follows:

- $$\begin{aligned} e^{i\frac{\|Z\|}{t}(u+u_k)} &= e^{-\frac{k\pi\|Z\|}{t}}e^{i\frac{\|Z\|}{t}u} \\ &= e^{-\frac{k\pi\|Z\|}{t}}\sum_{l=0}^{\infty}\frac{(i\|Z\|u)^l}{t^l l!}. \end{aligned}$$

- We set
$$Q(u,t) := -S_{d-2}P\left(\frac{t}{u\|Z\|}, -1\right)u^{d-1},$$
which is a polynomial in u (resp. t) of degree $d-1$ (resp. $d-2$). Therefore we can write
$$Q(u+u_k,t) = \sum_{l=0}^{d-1} a_l\left(\frac{t}{\|Z\|}\right)u^l,$$
where a_l are polynomials of degree $\leq d-2$.

- $$\left(\frac{(u+u_k)}{\sinh(u+u_k)}\right)^N = \sum_{l=-N}^{\infty} b_l u^l.$$

Now, a straightforward calculation shows that there is a polynomial $\widetilde{\varphi}_k$ of degree less than or equal $N+d-1$ such that
$$\operatorname{Res}(F_{Z,t}, ik\pi) = \left(\frac{t}{\|Z\|}\right)^d \widetilde{\varphi}_k\left(\frac{\|Z\|}{t}\right)e^{-\frac{k\pi\|Z\|}{4t}}.$$
The result follows by setting
$$\varphi_k\left(\frac{\|Z\|}{t}\right) := C\left(\frac{\|Z\|}{t}\right)^{N+d}\widetilde{\varphi}_k\left(\frac{\|Z\|}{t}\right),$$
with
$$C := 2\pi i\cdot\frac{\mathrm{vol}(\mathbb{T}_1)\mathrm{vol}(\mathbb{T}_2)}{(4\pi)^N(2\pi)^d}.$$

□

Remark 4.3.4. 1. Since $\sinh(u+ik\pi) = (-1)^k\sinh u$, it follows from the calculation of the residue at $ik\pi$ that the polynomial functions φ_k fulfil
$$\varphi_k\left(\frac{\|Z\|}{t}\right) \leq Ck^\alpha\left(\frac{\|Z\|}{t}\right)^{2N+d-1} \quad \text{for all } k \text{ and Z}$$
with some positive constants C and α. Therefore, it follows that the serie
$$\sum_{Z\in\Gamma_2\setminus\{0\}}\sum_{k=1}^{\infty}\|Z\|^{-(N+d)}\varphi_k\left(\frac{\|Z\|}{t}\right)e^{-\frac{k\pi\|Z\|}{t}}$$
converges absolutely.

2. By Theorem 4.3.3, we also obtain short-time asymptotics of the heat trace:
$$\operatorname{tr}(e^{-t\Delta_{sub}^{\Gamma\setminus\mathbb{G}}}) = \frac{C_{N,d}}{t^{N+d}} + O(t^\infty), \text{ as } t\to 0^+.$$
Note that $N+d$ is half the Hausdorff dimension of the metric space $(\mathbb{G}, d)$ where d denotes the Carnot-Carathéodory distance.

3. From the proof of Theorem 4.3.3, we see that for $k = 1, 2, \cdots$, φ_k cannot by identically zero, because this will imply that the function $F_{Z,t}$ does not have a pole at $ik\pi$, which is obviously not true. Furthermore, a detailed analysis of the polynomial P from Lemma 4.3.2 shows that in fact, φ_k has degree $2N + (d-1)/2$. Therefore, we conclude that knowing the spectrum of the sublaplacian, we can completely extract the length spectrum of the subriemannian manifold $\Gamma\backslash\mathbb{G}$ by following the same idea in [96].

4. If d is even, then the function $I(\alpha)$ from Lemma 4.3.2 can be expressed with the help of modified Bessel functions. However, the method above does not work.

5. It is interesting to compare the Poisson summation formula from Theorem 4.3.3 with the one proved in [96]. Therein it was proved, that the heat trace of the Laplacian on $\Gamma\backslash\mathbb{G}$ ($\mathbb{G}$ is the Heisenberg group) with respect to the standard Riemannian structure can be expressed in the form

$$\sum_l \varphi_l \left(\frac{1}{t}\right) e^{-\frac{l^2}{4t}},$$

where the summation runs over lengths of closed Riemannian geodesics in $\Gamma\backslash\mathbb{G}$ and φ_l are functions, but not necessary polynomials like in the subriemannian case.

Chapter 5

Trivializable and quaternionic subriemannian structure on $\mathbb{S}^7$

This Chapter is taken from the article *A3* and is organized as follows. In Sections 5.1 and 5.2 we recall the construction of two different subriemannian structures on $\mathbb{S}^7$ and we list some of their properties. Then we compute the Popp volume induced by these structures in Section 5.3. In Section 5.4 we show that the tangent groups of $\mathbb{S}^7$ endowed with the trivializable subriemannian structure may change from point to point and that this structure is not locally isometric to the quaternionic contact structure. The type of the trivializable structure and a family of subriemannian isometries are determined in Section 5.5. In Section 5.6 we compute the first heat invariants in the small-time asymptotics of the heat trace by using an approximation method in [39, 110]. Comparing both we show that the above subriemannian structures on $\mathbb{S}^7$ are not isospectral with respect to the sublaplacians. In Chapter 5.7 we consider the (non-intrinsic) sublaplacian $\widetilde{\Delta}_{\text{sub}}^{T}$ on $\mathbb{S}^7_T$ induced by the standard measure on $\mathbb{S}^7$. In Theorem 5.7.2 we prove the inclusion $\sigma(\Delta_{\text{sub}}^{\text{Q}}) \subset \sigma(\widetilde{\Delta}_{\text{sub}}^{\text{T}})$ of spectra where $\Delta_{\text{sub}}^{\text{Q}}$ denotes the sublaplacian corresponding to the quarternionic contact structure. However, we mention that both operators are not isospectral. Chapter 5.7 extends former results in [21].

5.1 Quaternionic Hopf structure:

Let $\mathbb{H} \simeq \mathbb{R}^4$ denote the quaternionic space

$$\mathbb{H} := \{x + y\mathbf{i} + z\mathbf{j} + \omega\mathbf{k} : x, y, z, \omega \in \mathbb{R}\},$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$.
For $n \geq 0$, we consider the $(n+1)$-dimensional quaternionic space $\mathbb{H}^{n+1}$ as a right $\mathbb{H}$-module with the hermitian form:

$$\langle p, q \rangle_{\mathbb{H}} := \sum_{l=0}^{n} \overline{p_l} \cdot q_l$$

for $p = (p_0, \cdots, p_n), q = (q_1, \cdots, q_n) \in \mathbb{H}^{n+1}$. The real part of this hermitian form which we denote by $\langle \cdot, \cdot \rangle$, is the usual real inner product on $\mathbb{H}^{n+1}$, corresponding to the identification $\mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)}$.

Let us consider the sphere $\mathbb{S}^7$ embedded into $\mathbb{H}^2$ as the set of elements of norm 1:

$$\mathbb{S}^7 = \{q = (q_0, q_1) \in \mathbb{H}^2 : \|q_0\|^2_{\mathbb{H}} + \|q_1\|^2_{\mathbb{H}} = 1\}.$$

There is a natural diagonal right action of $\mathbb{S}^3$ on $\mathbb{S}^7$ which induces the quaternionic Hopf fibration:

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4.$$

The *quaternionic Hopf distribution* $\mathcal{H}_Q$ is the corank 3 connection of this $\mathbb{S}^3$-principal bundle. It is given by the orthogonal complement to the following orthonormal vector fields induced by the right multiplication with the curves $e^{t\mathbf{i}}, e^{t\mathbf{j}}, e^{t\mathbf{k}}$:

$$V_{\mathbf{i}}(q) = -y_0\partial_{x_0} + x_0\partial_{y_0} + \omega_0\partial_{z_0} - z_0\partial_{\omega_0} - y_1\partial_{x_1} + x_1\partial_{y_1} + \omega_1\partial_{z_1} - z_1\partial_{\omega_1}$$

$$V_{\mathbf{j}}(q) = -z_0\partial_{x_0} - \omega_0\partial_{y_0} + x_0\partial_{z_0} + y_0\partial_{\omega_0} - z_1\partial_{x_1} - \omega_1\partial_{y_1} + x_1\partial_{z_1} + y_1\partial_{\omega_1}$$

$$V_{\mathbf{k}}(q) = -\omega_0\partial_{x_0} + z_0\partial_{y_0} - y_0\partial_{z_0} + x_0\partial_{\omega_0} - \omega_1\partial_{x_1} + z_1\partial_{y_1} - y_1\partial_{z_1} + x_1\partial_{\omega_1}$$

at each $q = (x_0, y_0, z_0, \omega_0, x_1, y_1, z_1, \omega_1) \in \mathbb{S}^7$ and with respect to the standard Riemannian metric of $\mathbb{S}^7$.

As is well-known the quaternionic Hopf distribution $\mathcal{H}_Q$ is bracket generating, [20, 84, 85]. Moreover, if we endow $\mathcal{H}_Q$ with the pointwise inner product obtained by restriction from the standard Riemannian metric we obtain a subriemannian structure on $\mathbb{S}^7$ which we call *quaternionic structure*. In the following, we write $\mathbb{S}^7_Q$ for the sphere $\mathbb{S}^7$ endowed with this subriemannian structure.

Note that $\mathbb{S}^7_Q$ can also be considered as a quaternionic contact manifold as follows. Let $\eta_{\mathbf{i}}, \eta_{\mathbf{j}}, \eta_{\mathbf{k}}$ denote the dual frame of the frame $V_{\mathbf{i}}, V_{\mathbf{j}}, V_{\mathbf{k}}$. Then the quaternionic Hopf distribution $\mathcal{H}_Q$ is locally given by

$$\mathcal{H}_Q = \bigcap_{\mathbf{l} \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}} \mathrm{Ker}(\eta_{\mathbf{l}}).$$

Furthermore, if we denote by $I_{\mathbf{i}}, I_{\mathbf{j}}, I_{\mathbf{k}}$ the right multiplications by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then it is known that $\{I_{\mathbf{l}} : \mathbf{l} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}\}$ are almost complex structures satisfying the quaternionic relations compatible with the metric on $\mathcal{H}_Q$, i.e.

$$2\langle I_{\mathbf{l}} X, Y \rangle = d\eta_{\mathbf{l}}(X, Y)$$

for $X, Y \in \mathcal{H}_Q$ and $\mathbf{l} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Recall that the symplectic group $\mathbf{Sp}(2)$ is the subgroup of $\mathbb{H}$-linear elements of the orthogonal group $\mathbf{O}(8)$ which preserve the quaternionic inner product. Note that this is a subgroup of the group of all subriemannian isometries $\mathcal{I}(\mathbb{S}^7_Q)$ of $\mathbb{S}^7_Q$. Hence, by representing elements of $\mathbf{Sp}(2)$ as 2×2 matrices whose rows build an $\mathbb{H}$-orthonormal basis of $\mathbb{H}^2$, we see that $\mathbf{Sp}(2)$ (and hence $\mathcal{I}(\mathbb{S}^7_Q)$) acts transitively on $\mathbb{S}^7$.

The tangent bundle of the sphere $\mathbb{S}^7$ and the orthogonal complement of the quaternionic Hopf distribution $\mathcal{H}_Q$ in $T\mathbb{S}^7$ are both trivial as vector bundles. Hence it is natural to ask wether $\mathcal{H}_Q$ is trivial itself or whether $\mathcal{H}_Q$ admits at least one globally defined and nowhere vanishing smooth vector field. In fact, this question was posed as an open problem in [85, p. 1018] and will be answered below.

Given a globally defined smooth vector field X on $\mathbb{S}^7$, we consider it as a smooth function $X:\mathbb{S}^7\longrightarrow\mathbb{H}^2$ such that

$$\langle q, X(q)\rangle = 0 \qquad \textit{for all} \qquad q\in\mathbb{S}^7.$$

Definition 5.1.1. Let X be a globally defined vector field on $\mathbb{S}^7$. We call X a *quaternionic vector field* on $\mathbb{S}^7$ if $\langle q, X(q)\rangle_{\mathbb{H}} = 0$ for all $q\in\mathbb{S}^7$.

The next lemma states that the quaternionic Hopf distribution is precisely the quaternionic tangent space of the sphere:

Lemma 5.1.2. *Horizontal vector fields on $\mathbb{S}^7$ are the quaternionic vector fields.*

Proof. By definition, a vector field X on $\mathbb{S}^7$ is horizontal if and only if for all $q\in\mathbb{S}^7$:

$$\langle q, X(q)\rangle = \langle V_{\mathbf{i}}(q), X(q)\rangle = \langle V_{\mathbf{j}}(q), X(q)\rangle = \langle V_{\mathbf{k}}(q), X(q)\rangle = 0.$$

Note that the components of the vector fields $V_{\mathbf{i}}, V_{\mathbf{j}}$ and $V_{\mathbf{k}}$ at a point q coincide with the components of $q\mathbf{i}, q\mathbf{j}$ and $q\mathbf{k}$. Furthermore, a straightforward calculation shows that for $p,q\in\mathbb{H}^2$:

$$\langle p,q\rangle_{\mathbb{H}} = \langle p,q\rangle + \mathbf{i}\langle p\mathbf{i},q\rangle + \mathbf{j}\langle p\mathbf{j},q\rangle + \mathbf{k}\langle p\mathbf{k},q\rangle.$$

This implies that X is horizontal if and only if $\langle q, X(q)\rangle_{\mathbb{H}} = 0$ for all $q\in\mathbb{S}^7$, i.e. X is horizontal if and only if X is a quaternionic vector field. $\square$

Now we recall the following quaternionic version of Adam's theorem in [1] on the maximal dimension of a trivial subbundle of the tangent bundle of a sphere. Theorem 5.1.3 below was proven in [81] from methods in topological K-theory.

Theorem 5.1.3 ([81])**.** *For $n\geq 1$, the sphere $\mathbb{S}^{4n+3}$ admits a nowhere vanishing and globally defined quaternionic vector field if and only if $n\equiv -1 \bmod 24$.*

By combining this result with Lemma 5.1.2 we obtain:

Corollary 5.1.4. *The quaternionic Hopf distribution $\mathcal{H}_Q$ on $\mathbb{S}^7$ does not admit a nowhere vanishing and globally defined vector field (section of the bundle). In particular, the distribution $\mathcal{H}_Q$ is not trivial.*

5.2 Trivializable subriemannian structure

In the following we recall the definition of a second remarkable subriemannian structure on $\mathbb{S}^7$, called trivializable subriemannian structure [21, 25]. According to [21, Theorem 4.4] such structures only exist on the spheres $\mathbb{S}^3, \mathbb{S}^7$ and $\mathbb{S}^{15}$.

By $\mathbb{K}(n)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ we denote the space of all $n \times n$-matrices with entries in $\mathbb{K}$. Let $A_1, \cdots, A_m \in \mathbb{R}(8)$ be a family of skew-symmetric real matrices that fulfill the anti-commutation relation:

$$A_i A_j + A_j A_i = -2\delta_{ij} \quad \textit{for} \quad i, j = 1, \ldots, m. \tag{5.1}$$

Then a collection of m linear vector fields $X(A_1), \ldots, X(A_m)$ on $\mathbb{S}^7$ orthonormal at each point (*canonical vector fields*) can be defined in global coordinates of $\mathbb{R}^8$ by:

$$X(A_k) := \sum_{i,j=1}^{8} (A_k)_{ij} x_j \frac{\partial}{\partial x_i} \quad \textit{for} \quad k = 1, \ldots, m.$$

Due to the representation theory for Clifford algebras, the maximal number m of matrices in $\mathbb{R}(8)$ such that the relations (5.1) hold is $m = 7$. We recall the following properties of the above linear vector fields on spheres.

Lemma 5.2.1 ([21]). *Let $A_1, \ldots, A_7 \in \mathbb{R}(8)$ be a collection of matrices with (5.1). For $j = 1, \cdots, 7$ we set*

$$X_j := X(A_j).$$

Then it holds:

1. *For $i, j = 1, \cdots, 7$ with $i \neq j$:*

$$[X_i, X_j] = -X([A_i, A_j]) = -2X(A_i A_j).$$

2. *All higher Lie brackets $[X_{i_1}[X_{i_2}, [X_{i_3}, \ldots]]]$ are contained in*

$$\mathrm{Span}\Big\{X_i, [X_j, X_k] : i, j, k = 1, \cdots, 7\Big\}.$$

3. *Let $i_1, i_2, i_3, i_4 \in \{1, \cdots, 7\}$. The rank-4 distribution $\mathcal{H}$ on $\mathbb{S}^7$ generated by the vector fields $\{X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}\}$ is bracket generating of step two.*

Remark 5.2.2. Let $\{A_1^{(1)}, \ldots, A_4^{(1)}\}$ and $\{A_1^{(2)}, \ldots, A_4^{(2)}\}$ be two families of skew-symmetric and anti-commuting matrices in $\mathbb{R}(8)$. Then it was shown in [21] that there exists $C \in \mathbf{O}(8)$ such that

$$A_i^{(1)} = C^{-1} A_i^{(2)} C \quad \textit{for} \quad i = 1, \ldots, 4.$$

Therefore, if we define the following bracket generating distributions:

$$\mathcal{H}^{(k)} := \mathrm{Span}\{X(A_i^{(k)}) : i = 1, \dots, 4\} \quad \textit{for} \quad k = 1, 2$$

then the subriemannian structures $(\mathbb{S}^7, \mathcal{H}^{(k)}, \langle \cdot, \cdot \rangle)$ for $k = 1, 2$ are isometric, i.e. the above defined trivializable subriemannian structure on $\mathbb{S}^7$ is, up to subriemannian isometries, independent of the choice of linear vector fields induced by the Clifford module structure of $\mathbb{R}^8$ and spanning the distribution.

In the following we give an explicit family of skew-symmetric and anti-commuting matrices which will serve as a model for the study of a trivializable subriemannian structure on $\mathbb{S}^7$ induced by matrices which fulfill the relations (5.1). Consider $A_4, A_5, A_6, A_7 \in \mathbb{H}(2)$ defined by:

$$A_4 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; A_5 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \; A_6 := \begin{pmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{pmatrix} \tag{5.2}$$
$$A_7 := \begin{pmatrix} \mathbf{k} & 0 \\ 0 & -\mathbf{k} \end{pmatrix}.$$

One easily verifies that $\{A_4, A_5, A_6, A_7\} \subset \mathbb{H}(2)$ are anti-commuting and skew-symmetric with respect to the standard inner product on $\mathbb{H}^2$.
Via the standard basis of $\mathbb{R}^8$ we may regard A_j as skew-symmetric element in $\mathbb{R}(8)$.

Lemma 5.2.3. *There are three skew-symmetric matrices $A_1, A_2, A_3 \in \mathbb{R}(8)$ such that $\{A_j \,:\, j = 1, \dots, 7\} \subset \mathbb{R}(8)$ are anti-commuting and skew-symmetric.*

Proof. Consider the following skew-symmetric real matrices:

$$B_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; B_2 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; B_3 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that B_1 (resp. B_2 and B_3) corresponds to the right quaternionic multiplication by $\mathbf{k}$ (resp. $\mathbf{j}$ and $-\mathbf{i}$). Now we define for $i = 1, 2, 3$:

$$A_i := \begin{pmatrix} 0 & B_i \\ B_i & 0 \end{pmatrix}.$$

A straightforward calculation shows that the following relations hold:

$$[B_i, \mathbf{l}] = 0 \quad \textit{for} \quad \mathbf{l} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \quad \textit{and} \quad i = 1, 2, 3$$

and

$$B_i B_j + B_j B_i = -2\delta_{ij} \quad \textit{for} \quad i, j \in \{1, 2, 3\}.$$

By a direct calculation based on these relations it follows that $A_1, \dots, A_7$ have the desired properties. $\square$

We consider the following trivializable distribution on $\mathbb{S}^7$:

$$\mathcal{H}_T := \operatorname{Span}\{X(A_i) : i = 1, 2, 3, 4\},$$

and we denote by $\mathbb{S}^7_T$ the trivializable subriemannian manifold $(\mathbb{S}^7, \mathcal{H}_T, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ denotes the restriction of the standard Riemannian metric on $\mathbb{S}^7$ to the trivial bundle $\mathcal{H}_T$.

Remark 5.2.4. According to Corollary 5.1.4, the quaternionic Hopf structure $\mathbb{S}^7_Q$ does not admit globally defined and nowhere vanishing horizontal vector fields and hence it cannot be isometric (as a subriemannian manifold) to the trivializable structure $\mathbb{S}^7_T$. We will see that both structures not even are locally isometric.

5.3 The Popp measures

Recall that the *Popp measure* on $\mathbb{S}^7$ is a smooth measure which intrinsically can be assigned to a given regular subriemannian structure (see [4, 11, 19, 93]). In the present section we determine the Popp measures $\mathcal{P}_Q$ and $\mathcal{P}_T$ on $\mathbb{S}^7$ corresponding to the quaternionic and the trivializable subriemannian structure, respectively.

Let $X_1, \cdots, X_4$ be a local orthonormal frame for the distribution $\mathcal{H}_Q$. Then an adapted frame for $\mathbb{S}^7_Q$ is given by $\mathcal{F} = [X_1, \cdots, X_4, V_{\mathbf{i}}, V_{\mathbf{j}}, V_{\mathbf{k}}]$. According to [11, Theorem 1] the Popp measure $\mathcal{P}_Q$ for the quaternionic subriemannian structure can be expressed locally in the form:

$$\mathcal{P}_Q(z) = \frac{1}{\sqrt{\det B_Q(z)}} \eta_1 \wedge \cdots \wedge \eta_7, \qquad z \in \mathbb{S}^7. \tag{5.3}$$

Here $B_Q(z)$ is a certain matrix which is obtained from the adapted structure constants of the geometric structure and $\eta_1, \cdots, \eta_7$ denotes the dual basis to the frame $\mathcal{F}$ (see [11] for more details).

Since the vector fields $X_1, \cdots, X_4, V_{\mathbf{i}}, V_{\mathbf{j}}, V_{\mathbf{k}}$ are orthonormal with respect to the standard Riemannian metric on $\mathbb{S}^7$, the volume form

$$d\sigma := \eta_1 \wedge \cdots \wedge \eta_7$$

is the standard volume form on $\mathbb{S}^7$.

Lemma 5.3.1. *The Popp volume $\mathcal{P}_Q$ for the quaternionic structure equals the standard volume form $d\sigma$ up to a constant factor.*

Proof. We can write

$$\mathcal{P}_Q(z) = f(z) d\sigma(z)$$

with a nowhere vanishing function $f \in C(\mathbb{S}^7)$. We know that the symplectic group $\mathbf{Sp}(2)$ is a subgroup of the isometry group $\mathcal{I}(\mathbb{S}^7_Q)$. But $\mathbf{Sp}(2)$ is also a subgroup of $\mathbf{O}(8)$ which is the isometry group of $\mathbb{S}^7$ with respect to the standard Riemannian metric. It follows that the Popp volume $\mathcal{P}_Q$ [11, Proposition 7] and the standard volume $d\sigma$ are invariant under $\mathbf{Sp}(2)$, and therefore f must be also invariant under the action of $\mathbf{Sp}(2)$. Now, the assumption follows from the fact that $\mathbf{Sp}(2)$ acts transitively on $\mathbb{S}^7$. $\square$

Contrary to the quaternionic Hopf structure, we do not have enough information about the isometry group of the trivializable structure $\mathbb{S}^7_T$ to conclude in a similar way. Therefore, we compute the Popp volume $\mathcal{P}_T$ directly using the adapted structure constants. An adapted frame for the trivializable structure is given globally by the orthonormal vector fields $X_1, \cdots, X_7$ defined from the matrices $A_1, \cdots, A_7$ in Lemma 5.2.3. According to [11, Theorem 1] the Popp volume can be written as

$$\mathcal{P}_T(z) = \frac{1}{\sqrt{\det B_T(z)}} d\sigma(z),$$

where $B_T(z) = (B_T^{kl}(z))_{k,l=5}^7$ is the 3×3 matrix function on $\mathbb{S}^7$ with coefficients

$$B_T^{kl}(z) = \sum_{i,j=1}^4 b_{ij}^k(z) b_{ij}^l(z), \qquad z \in \mathbb{S}^7.$$

For $i, j = 1, \ldots, 4$ and $k = 5, 6, 7$ the functions $b_{ij}^k(z)$ are defined by:

$$b_{ij}^k(z) = \langle [X_i, X_j](z), X_k(z) \rangle = -2\langle A_i A_j z, A_k z \rangle \quad \textit{for} \quad z \in \mathbb{S}^7. \tag{5.4}$$

In (5.4) we have used the notation $\langle \cdot, \cdot \rangle$ for the Euclidean inner product on $\mathbb{R}^8$ and its restriction to the sphere, respectively. In the following, we write $\|A\|_{\mathrm{HS}}$ for the Hilbert-Schmidt norm of $A \in \mathbb{R}(8)$.

Lemma 5.3.2. *The Popp measure $\mathcal{P}_T$ with respect to the trivializable subriemannian structure $\mathbb{S}^7_T$ is given by*

$$\mathcal{P}_T(z) = g(z) d\sigma,$$

where

$$g(z) := \left[16(1 - 2\|x\|^2\|y\|^2)\right]^{-3/2} \quad \textit{for} \quad z = (x, y) \in \mathbb{S}^7 \subset \mathbb{R}^8.$$

Proof. We introduce the following notations:

$$A_{\mathbf{i}} := A_5,\ A_{\mathbf{j}} := A_6,\ A_{\mathbf{k}} := A_7 \quad \textit{and} \quad A_8 := Id.$$

Let $l \in \{5, 6, 7\}$ and $z = (x, y) \in \mathbb{S}^7$. Using the fact that the skew-symmetric and anti-commuting matrices $A_1, \cdots, A_7$ lie in $\mathbf{O}(8)$ and that $\{A_1 z, \cdots, A_8 z\}$ forms an orthonormal basis of $\mathbb{R}^8$, we can write:

$$B_T^{ll}(z) = 4 \cdot \sum_{i,j=1}^4 \left\langle A_l A_i z, A_j z \right\rangle^2$$

$$= 4 \cdot \left(\|A_l\|_{\mathrm{HS}}^2 - \sum_{i=5}^{8}\sum_{j=1}^{8} \left\langle A_l A_i z, A_j z\right\rangle^2 - \sum_{i=1}^{4}\sum_{j=5}^{8} \left\langle A_l A_i z, A_j z\right\rangle^2 \right)$$

$$= 4 \cdot \left(\|A_l\|_{\mathrm{HS}}^2 - \sum_{i=5}^{8} \underbrace{\left\|A_l A_i z\right\|^2}_{=1} - \sum_{i=1}^{4}\sum_{j=5}^{7} \left\langle A_l A_i z, A_j z\right\rangle^2 \right).$$

Furthermore, a straightforward calculation shows that for $\mathbf{l} \neq \mathbf{m} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$:

$$|\langle A_{\mathbf{l}} A_i z, A_{\mathbf{m}} z\rangle| = 2|\langle B_i x, (\mathbf{l}\cdot\mathbf{m})y\rangle| \quad \textit{for} \quad i = 1, \cdots, 4.$$

Here B_1, B_2 and B_3 are the matrices defined in Lemma 5.2.3 and $B_4 := Id$.

We assume that $x \neq 0$. Since $\{\|x\|^{-1} B_i x : i = 1, \cdots, 4\}$ is an orthonormal basis of $\mathbb{H} \cong \mathbb{R}^4$ it follows that for $\mathbf{l}, \mathbf{m} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$:

$$\sum_{i=1}^{4} \langle A_{\mathbf{l}} A_i z, A_{\mathbf{m}} z\rangle^2 = 4\|x\|^2 \|(\mathbf{l}\cdot\mathbf{m})y\|^2 = 4\|x\|^2\|y\|^2.$$

Equality also holds in the case $x = 0$. Therefore, we find for $l = 5, 6, 7$:

$$B_T^{ll}(z) = 4(4 - 8\|x\|^2\|y\|^2) = 16(1 - 2\|x\|^2\|y\|^2).$$

For $l \neq m \in \{5, 6, 7\}$ it holds:

$$\begin{aligned}
\frac{1}{4} B_T^{lm}(z) &= \sum_{i_1, i_2 = 1}^{4} \left\langle A_l A_{i_1} z, A_{i_2} z\right\rangle \left\langle A_m A_{i_1} z, A_{i_2} z\right\rangle \\
&= \Big(\sum_{i_1, i_2 = 1}^{8} - \sum_{i_1=5}^{8}\sum_{i_2=1}^{8} - \sum_{i_1=1}^{4}\sum_{i_2=5}^{8} \Big) \left\langle A_l A_{i_1} z, A_{i_2} z\right\rangle \left\langle A_m A_{i_1} z, A_{i_2} z\right\rangle \\
&= \sum_{i_1=1}^{8} \underbrace{\left\langle A_m A_{i_1} z, A_l A_{i_1} z\right\rangle}_{=0} - \sum_{i_1=5}^{8} \underbrace{\left\langle A_m A_{i_1} z, A_l A_{i_1} z\right\rangle}_{=0} \\
&\quad - \sum_{i_1=1}^{4}\sum_{i_2=5}^{8} \left\langle A_l A_{i_1} z, A_{i_2} z\right\rangle \left\langle A_m A_{i_1} z, A_{i_2} z\right\rangle.
\end{aligned}$$

Since the matrices $A_1, \cdots, A_7$ are anti-commuting, it follows that

$$\left\langle A_l A_{i_1} z, A_{i_2} z\right\rangle \left\langle A_m A_{i_1} z, A_{i_2} z\right\rangle = 0 \quad \textit{for} \quad i_2 \in \{l, m\}.$$

Hence we can write with $i_2 \in \{5, 6, 7\} \backslash \{l, m\}$ and $\mathbf{i_2} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ defined by $A_{i_2} = A_{\mathbf{i_2}}$:

$$\frac{1}{4} B_T^{lm}(z) = - \sum_{i_1=1}^{4} \left\langle A_l A_{i_1} z, A_{i_2} z\right\rangle \left\langle A_m A_{i_1} z, A_{i_2} z\right\rangle$$

$$
\begin{aligned}
&= -4 \sum_{Q \in \{I, B_1, B_2, B_3\}} \big\langle Qx, (\mathbf{l} \cdot \mathbf{i_2})y \big\rangle \big\langle Qx, (\mathbf{m} \cdot \mathbf{i_2})y \big\rangle \\
&= -4 \big\langle (\mathbf{l} \cdot \mathbf{i_2})y, (\mathbf{m} \cdot \mathbf{i_2})y \big\rangle \\
&= -4 \big\langle \mathbf{l}y, \mathbf{m}y \big\rangle = 0.
\end{aligned}
$$

We obtain:

$$B_T(z) = 16(1 - 2\|x\|^2\|y\|^2) \cdot \mathrm{Id} \in \mathbb{R}(3) \tag{5.5}$$

and therefore, the Popp measure $\mathcal{P}_T$ has the form:

$$\mathcal{P}_T(z) = \big[16(1 - 2\|x\|^2\|y\|^2)\big]^{-\frac{3}{2}} d\sigma.$$

$\square$

5.4 The nilpotent approximation

Let $z = (x, y) \in \mathbb{S}^7 \subset \mathbb{R}^8$. Since $X_1, \cdots, X_7$ is an adapted orthonormal frame for $\mathbb{S}^7_T$, the tangent algebra at z for $\mathbb{S}^7_T$ is (isomorphic to) the Lie algebra of step 2 given by

$$\mathfrak{g}_z = \mathcal{H}_z \oplus \mathcal{V}_z \simeq \mathbb{R}^7, \tag{5.6}$$

where

$$
\begin{aligned}
\mathcal{H}_z &:= \mathrm{Span}\{X_i(z) : i = 1, \cdots, 4\}, \\
\mathcal{V}_z &:= \mathrm{Span}\{X_k(z) : k = 5, 6, 7\}.
\end{aligned}
$$

For $i, j = 1, \cdots, 4$ the (possibly non-trivial) Lie brackets are given by:

$$[X_i(z), X_j(z)] := \sum_{k=5}^{7} \langle [X_i, X_j], X_k \rangle_z X_k(z).$$

Note that the inner product $\langle \cdot, \cdot \rangle_z$ on $\mathcal{H}_z$ induces an inner product on the first layer of the graded Lie algebra $\mathfrak{g}_z$, i.e. $\mathfrak{g}_z$ is a Carnot Lie algebra.

In the following, we need a technical lemma on the local comparison of two subriemannian manifolds. First, we recall the definition of a *nonsingular Carnot algebra*, see [49, 59] for more details.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Carnot algebra of step 2, i.e.

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \quad \textit{and} \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \text{ for } i + j > 2.$$

We assume that an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$ is given. Then every element $\mathbf{n} \in \mathfrak{g}_2^*$ induces a representation map $J_{\mathbf{n}} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$ defined by

$$\langle J_{\mathbf{n}} X, Y \rangle := \mathbf{n}([X, Y]) \quad \textit{for} \quad X, Y \in \mathfrak{g}_1.$$

Definition 5.4.1. We say that the Carnot algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is *nonsingular*, if for all $\mathbf{n} \in \mathfrak{g}_2^* \backslash \{0\}$, the induced map $J_{\mathbf{n}}$ is invertible. Otherwise, $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called *singular*.

Note that if $\varphi : (\mathfrak{g}, \langle\cdot,\cdot\rangle) \to (\mathfrak{g}', \langle\cdot,\cdot\rangle')$ is a Lie algebra isomorphism which preserves the inner products (i.e. an isometry), then $(\mathfrak{g}', \langle\cdot,\cdot\rangle')$ will be nonsingular (resp. singular) if and only if $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is. Hence we obtain:

Lemma 5.4.2. *Let $(M, \mathcal{H}, g)$ and $(M', \mathcal{H}', g')$ be step two subriemannian manifolds which near a point $x \in M$ are locally isometric by ϕ. Then the Lie algebras $\mathfrak{g}M(x)$ and $\mathfrak{g}M'(\phi(x))$ are isometric. In particular, if the tangent algebra of M at $x \in M$ is nonsingular, then so is the tangent algebra of M' at $\phi(x)$.*

By considering $\mathbb{S}^7_Q$ as a quaternionic contact manifold, it is easy to see that its tangent algebra can be identified at every point with the quaternionic Heisenberg Lie algebra, which, in particular, is non-singular. For the trivializable subriemannian structure on $\mathbb{S}^7$, the situation is completely different. As we will see, its tangent algebra can be different from point to point.

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and consider the vertical vector

$$Z := \alpha X_5(z) + \beta X_6(z) + \gamma X_7(z) \in \mathcal{V}_z.$$

By declaring the vectors $X_5(z), X_6(z), X_7(z)$ to be orthonormal, we obtain an inner product on $\mathcal{V}_z$ which again is denoted by $\langle\cdot,\cdot\rangle$. This induces an identification of $\mathcal{V}_z^*$ with $\mathcal{V}_z$ so that we can write for $J_Z : \mathcal{H}_z \longrightarrow \mathcal{H}_z$:

$$\langle J_Z X, Y\rangle_z = \langle Z, [X, Y]\rangle_z \quad \textit{for} \quad X, Y \in \mathcal{H}_z.$$

Let $A(\alpha, \beta, \gamma)$ denote the following element of $\mathbb{H}$:

$$A(\alpha, \beta, \gamma) := \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}.$$

Then a straightforward calculation shows that:

$$\begin{aligned}
\langle Z, [X_1(z), X_2(z)]\rangle_z &= 2\langle A(\alpha,\beta,\gamma)x, B_3x\rangle - 2\langle A(\alpha,\beta,\gamma)y, B_3y\rangle = a - d\\
\langle Z, [X_1(z), X_3(z)]\rangle_z &= -2\langle A(\alpha,\beta,\gamma)x, B_2x\rangle + 2\langle A(\alpha,\beta,\gamma)y, B_2y\rangle = -b + e\\
\langle Z, [X_1(z), X_4(z)]\rangle_z &= 2\langle A(\alpha,\beta,\gamma)x, B_1x\rangle + 2\langle A(\alpha,\beta,\gamma)y, B_1y\rangle = c + f\\
\langle Z, [X_2(z), X_3(z)]\rangle_z &= 2\langle A(\alpha,\beta,\gamma)x, B_1x\rangle - 2\langle A(\alpha,\beta,\gamma)y, B_1y\rangle = c - f\\
\langle Z, [X_2(z), X_4(z)]\rangle_z &= 2\langle A(\alpha,\beta,\gamma)x, B_2x\rangle + 2\langle A(\alpha,\beta,\gamma)y, B_2y\rangle = b + e\\
\langle Z, [X_3(z), X_4(z)]\rangle_z &= 2\langle A(\alpha,\beta,\gamma)x, B_3x\rangle + 2\langle A(\alpha,\beta,\gamma)y, B_3y\rangle = a + d.
\end{aligned}$$

Hence, with respect to the basis $\{X_i(z) : i = 1, \cdots, 4\}$, the map J_Z can be represented by a skew-symmetric matrix of the form:

$$\begin{pmatrix} 0 & d-a & b-e & -c-f\\ a-d & 0 & -c+f & -b-e\\ -b+e & c-f & 0 & -a-d\\ c+f & b+e & a+d & 0 \end{pmatrix} \tag{5.7}$$

with $a, b, c, d, e, f \in \mathbb{R}$ as above.

Note that the matrix in (5.7) has the determinant:

$$(a^2 + b^2 + c^2 - d^2 - e^2 - f^2)^2.$$

By using the following identity for $\omega \in \mathbb{R}^4$:

$$\langle A(\alpha,\beta,\gamma)\omega, C\omega\rangle^2 + \langle A(\alpha,\beta,\gamma)\omega, D\omega\rangle^2 + \langle A(\alpha,\beta,\gamma)\omega, E\omega\rangle^2 = (\alpha^2+\beta^2+\gamma^2)\|\omega\|^4,$$

we calculate the determinant of J_Z:

$$\det J_Z = 16(\|x\|^2 - \|y\|^2)^2(\alpha^2+\beta^2+\gamma^2)^2.$$

Hence, if $\|x\| \neq \|y\|$ then the operator J_Z is invertible for all $Z \in \mathcal{V}_z \backslash \{0\}$.

Lemma 5.4.3. *Let $z = (x, y) \in \mathbb{S}^7_T$. Then the tangent algebra of $\mathbb{S}^7_T$ at z is non-singular if and only if $\|x\| \neq \|y\|$.*

Using Lemma 5.4.2 and Lemma 5.4.3 we conclude that:

Theorem 5.4.4. *The subriemannian manifolds $\mathbb{S}^7_Q$ and $\mathbb{S}^7_T$ are not locally isometric. Furthermore, the isometry group $\mathcal{I}(\mathbb{S}^7_T)$ of the trivializable subriemannian structure does not act transitively on $\mathbb{S}^7$.*

Remark 5.4.5. The above calculations show also that the trivializable distribution $\mathcal{H}_T$ has the strong bracket generating property at a point $z = (x, y) \in \mathbb{S}^7$ if and only if $\|x\| \neq \|y\|$. In fact, the strong bracket generating condition holds if and only if for non-zero horizontal vector field X, the map

$$F_X : \mathcal{H}_T \longrightarrow \mathcal{V} \simeq \mathcal{H}^2_T/\mathcal{H}_T$$

defined by $F_X(Y) := [X, Y] \mod \mathcal{H}_T$ is surjective. Note that this map is tensorial, i.e. for all $z \in \mathbb{S}^7$ the element $[X, Y](z) \mod (\mathcal{H}_T)_z$ depends only on $X(z)$ and $Y(z)$. Now, the surjectivity of the map F_X is equivalent to the injectivity of the dual map $F^*_X : \mathcal{V}^* \simeq \mathcal{V} \longrightarrow \mathcal{H}^*_T \simeq \mathcal{H}_T$ given by $Z \longmapsto J_Z(X)$. By the above calculations, the following equation in Z: $J_Z(X) = 0$ has a non-trivial solution if and only if $z = (x, y)$ with $\|x\| = \|y\|$.

5.5 On the type of distributions

On a 7-dimensional manifold there is a particular class of distributions called *elliptic*. Such distributions are interesting from a geometric point of view. In fact, using the so-called *Cartan's approach* it was shown in [93] that the symmetry group of the induced geometry is always a finite dimensional Lie group. Therefore, it is interesting to know the type of the distributions $\mathcal{H}_Q$ and $\mathcal{H}_T$ on $\mathbb{S}^7$. In the following we briefly recall how they are defined (see [93] for more details). Let $\mathcal{H}$ be a co-rank 3, bracket generating distribution of step two on a 7-dimensional manifold M and let us consider the so-called *curvature* (linear) bundle map of $\mathcal{H}$

$$F : \Lambda^2\mathcal{H} \longrightarrow TM/\mathcal{H} \tag{5.8}$$

defined by $F(X,Y) = -[X,Y] \bmod \mathcal{H}$ for $X, Y \in \mathcal{H}$. Write $\mathcal{H}^\perp \subset T^*M$ for the bundle of covectors that annihilate $\mathcal{H}$. We consider now the dual curvature map ω:

$$\omega := F^* : \mathcal{H}^\perp \longrightarrow \Lambda^2\mathcal{H}^*. \tag{5.9}$$

Since $\mathcal{H}$ is bracket generating, the curvature map is onto. Furthermore, the real vector space $\Lambda^4\mathcal{H}^*$ is 1-dimensional, hence the squared dual curvature map

$$\begin{aligned} \omega^2 : \mathcal{H}^\perp &\longrightarrow \Lambda^4\mathcal{H}^* \\ \lambda &\longmapsto \omega(\lambda) \wedge \omega(\lambda) \end{aligned}$$

is a quadratic form on the 3-dimensional space $\mathcal{H}^\perp$ with values in the 1-dimensional vector space $\Lambda^4\mathcal{H}^*$. We say that $\mathcal{H}$ is *elliptic* if this quadratic form has signature $(3,0)$ or $(0,3)$. Note that we do not have a canonical choice of an element in $\Lambda^4\mathcal{H}^*$ and hence, the signature is only defined up to a sign $\pm$. In general, we say that $\mathcal{H}$ is of type (r,s) if this quadratic form has signature (r,s) or (s,r).

Note that the type of a distribution $\mathcal{H}$ is independent of the chosen metric on $\mathcal{H}$. Note also that the symmetry group considered in [93] is defined as the group of diffeomorphisms of M which preserve the distribution $\mathcal{H}$. In particular, if we endow the distribution $\mathcal{H}$ with a metric, the associated subriemannian isometry group will be a subgroup of the symmetry group.

We have seen that the tangent algebras of $\mathbb{S}^7_T$ are nonsingular outside the set

$$\mathcal{S} := \{z = (x,y) \in \mathbb{S}^7 : \|x\| = \|y\|\}.$$

In the following we show that the trivializable distribution $\mathcal{H}_T$ fails to be elliptic on this singular set $\mathcal{S}$.

Recall that the curvature map (5.8) of the distribution $\mathcal{H}_T$ is defined by

$$\begin{aligned} F : \Lambda^2\mathcal{H}_T &\longrightarrow T\mathbb{S}^7/\mathcal{H}_T \\ (X,Y) &\longmapsto F(X,Y) := -[X,Y] \bmod \mathcal{H}_T. \end{aligned}$$

The dual curvature map ω in (5.9) is then given as the dual map, i.e.

$$\begin{aligned} \omega : \mathcal{H}_T^\perp &\longrightarrow \Lambda^2\mathcal{H}_T^* \\ \lambda &\longmapsto \omega(\lambda), \end{aligned}$$

with

$$\omega(\lambda)(X \wedge Y) := -\lambda([X,Y]) \quad \textit{for all} \quad X, Y \in \mathcal{H}_T.$$

Using the standard Riemannian metric on $\mathbb{S}^7$, we identify $\mathcal{H}_T^{\perp}$ with

$$\mathcal{V} := \mathrm{Span}\{\langle X_j, \cdot\rangle : j = 5, 6, 7\}.$$

The distribution $\mathcal{H}_T$ is generated by globally defined vector fields $X_1, \cdots, X_4$ and this induces a specific horizontal form, namely

$$\eta_{\mathcal{H}_T} := \eta_1 \wedge \cdots \wedge \eta_4 \in \Lambda^4 \mathcal{H}_T^*,$$

where $\eta_1, \cdots, \eta_4$ denotes the frame dual to $X_1, \cdots, X_4$. Now the dual curvature map ω induces a family parametrized over M of real quadratic forms $Q := \omega^2/\eta_{\mathcal{H}_T}$ on $\mathcal{H}_T^{\perp} \simeq \mathcal{V}$ defined by:

$$\begin{aligned} \omega^2 : \mathcal{H}_T^{\perp} &\longrightarrow \Lambda^4 \mathcal{H}_T^* \\ \lambda &\longmapsto \omega(\lambda) \wedge \omega(\lambda) = Q(\lambda)\eta_{\mathcal{H}_T}. \end{aligned}$$

In the following lemma we compute the quadratic form Q for the trivializable subriemannian structure on $\mathbb{S}^7$.

Lemma 5.5.1. *Let $\lambda = \sum_{l=5}^{7} \lambda^l X_l \in \mathcal{V} \simeq \mathcal{H}_T^{\perp}$. Then the quadratic form Q is given by*

$$Q(\lambda) = 2 \sum_{k,l=5}^{7} \left(b_{12}^l b_{34}^k + b_{14}^l b_{23}^k - b_{13}^l b_{24}^k \right) \lambda^l \lambda^k,$$

where for $i, j = 1, \ldots, 4$ the coefficients b_{ij}^k have been defined in (5.4).

Proof. For $X = \sum_{i=1}^{4} \alpha_i X_i$ and $Y = \sum_{j=1}^{4} \beta_j X_j \in \mathcal{H}_T$ it holds:

$$\begin{aligned} \omega(\lambda)(X \wedge Y) &= -\langle \lambda, [X,Y]\rangle \\ &= -\sum_{i,j=1}^{4} \alpha_i \beta_j \langle \lambda, [X_i, X_j]\rangle \\ &= -\sum_{1 \le i < j \le 4} (\alpha_i \beta_j - \alpha_j \beta_i)\langle \lambda, [X_i, X_j]\rangle \\ &= -\sum_{1 \le i < j \le 4} \langle \lambda, [X_i, X_j]\rangle \eta_i \wedge \eta_j(X, Y). \end{aligned}$$

Hence the dual curvature map ω is given by :

$$\omega(\lambda) = -\sum_{1 \le i < j \le 4} \langle \lambda, [X_i, X_j]\rangle \eta_i \wedge \eta_j,$$

with

$$\langle \lambda, [X_i, X_j] \rangle = \sum_{l=5}^{7} b_{ij}^l \lambda^l.$$

A straightforward calculation shows now that

$$\omega(\lambda)^2 = \left(2 \sum_{k,l=5}^{7} \left(b_{12}^l b_{34}^k + b_{14}^l b_{23}^k - b_{13}^l b_{24}^k \right) \lambda^l \lambda^k \right) \eta_{\mathcal{H}_T}.$$

□

We set for $k, l \in \{5, 6, 7\}$:

$$T^{lk} := b_{12}^l b_{34}^k + b_{12}^k b_{34}^l + b_{14}^l b_{23}^k + b_{14}^k b_{23}^l - b_{13}^l b_{24}^k - b_{13}^k b_{24}^l.$$

Using similar arguments as for the computation of the Popp volume for $\mathbb{S}^7_T$, we find that the off-diagonal symbols T^{lk} vanish and that

$$T^{11} = T^{22} = T^{33} = 2(\|x\|^2 - \|y\|^2).$$

Hence, it follows that the quadratic form Q for the trivializable structure $\mathbb{S}^7_T$ is given explicitly by

$$Q(\lambda) = 2 \sum_{l=5}^{7} (\|x\|^2 - \|y\|^2)(\lambda^l)^2.$$

Corollary 5.5.2. *The trivializable distribution $\mathcal{H}_T$ on $\mathbb{S}^7$ is of elliptic type on the open dense subset $\{(x, y) \in \mathbb{S}^7 : \|x\| \neq \|y\|\}$. Otherwise, it is of type* $(0, 0)$.

It was shown in [93] that the quaternionic Hopf distribution $\mathcal{H}_Q$ is of elliptic type everywhere on $\mathbb{S}^7$. Furthermore, the sphere $\mathbb{S}^7_Q$ equipped with $\mathcal{H}_Q$ has a symmetry group of maximal dimension ($= 21$). In comparison, the trivializable structure on $\mathbb{S}^7$ is everywhere elliptic on $\mathbb{S}^7$ except on $\mathcal{S}$ which is a closed submanifold of $\mathbb{S}^7$ of dimension 6. As it was mentioned in [93], the presence of such degeneracies may add a new level of complexity to the analysis of such structures. Therefore, the symmetry group of $\mathbb{S}^7_T$ seems to be interesting and the study of this group is postponed to a future work.

In the following, by giving a 3-dimensional family of subriemannian isometries of $\mathbb{S}^7_T$, we show that the isometry group of $\mathbb{S}^7_T$ is at least three dimensional. Let $x = (x_0, x_1, x_2, x_3) \in \mathbb{S}^3$ and consider the following matrix

$$C := \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & -x_2 & x_1 & -x_0 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_1 & x_0 & x_3 & -x_2 \end{pmatrix} \in \mathbf{O}(4).$$

Then the following relations hold:

$$B_3C = CB_1, \quad CB_3 = -B_1C \quad \textit{and} \quad CB_2 = B_2C. \tag{5.10}$$

Let us define the following block matrix in $\mathbf{O}(8)$:

$$U := \begin{pmatrix} 0 & C \\ CB_1 & 0 \end{pmatrix}.$$

Then based on the relations (5.10) and the commutation relations of the matrices B_j (s. Lemma 5.2.3) we have:

$$UA_1 = A_4U \quad \textit{and} \quad UA_j = A_{j-1}U \quad \textit{for} \quad j = 2, 3, 4.$$

In particular, this implies that U defines a subriemannian isometry of $\mathbb{S}^7_T$.

5.6 Small time asymptotics of the heat kernel

An analysis of the intrinsic sublaplacian induced by the quaternionic Hopf structure on $\mathbb{S}^{4n+3}$ was done in [15]. In particular, the first heat invariants c_0 and c_1, i.e. the first two coefficients in the small time asymptotic expansion of the heat trace, have been explicitly calculated. In this section, we use the nilpotent approximation to compute the first heat invariant for the trivializable subriemannian structure $\mathbb{S}^7_T$.

For the trivializable subriemannian structure on $\mathbb{S}^7$ we have two choices of a natural smooth measure. The first one is the measure induced by the standard Riemannian metric on $\mathbb{S}^7$ which we denote by $d\sigma$ and the second one is the Popp measure $\mathcal{P}_T$. The sublaplacian with respect to the Popp measure can be expressed as (see [4, 11]):

$$\Delta^T_{\mathrm{sub}} = -\sum_{i=1}^{4} \left(X_i^2 + \mathrm{div}_{\mathcal{P}_T}(X_i)X_i \right).$$

Here $\{X_1, \cdots, X_4\}$ denotes the globally defined orthonormal frame of $\mathcal{H}_T$. We recall that by Lemma 5.3.2 the Popp measure is given by

$$\mathcal{P}_T(z) = g(z)d\sigma(z) \quad \textit{where} \quad g(z) = \left(16(1 - 2\|x\|^2\|y\|^2)\right)^{-\frac{3}{2}}, \qquad z = (x, y).$$

Therefore, using the fact that $X_1, \cdots, X_4$ are Killing vector fields and hence $d\sigma$-divergence free and by using the formula

$$\mathrm{div}_{\mathcal{P}_T}(X) = \mathrm{div}_{d\sigma}(X) + X(\log g)$$

for a smooth vector field X on M, we see that

$$\mathrm{div}_{\mathcal{P}_T}(X_i) = X_i(h) \quad \textit{for} \quad i = 1, \cdots, 4$$

with

$$h(z) := -\frac{3}{2}\log\left(1 - 2\|x\|^2\|y\|^2\right) \quad \textit{for} \quad z = (x,y) \in \mathbb{S}^7.$$

Hence we have the following formula:

Lemma 5.6.1. *The intrinsic sublaplacian Δ_{sub}^T on the trivializable SR manifold $\mathbb{S}^7_T$ acting on $C^\infty(\mathbb{S}^7)$ is given by:*

$$\Delta_{\mathrm{sub}}^T = -\sum_{i=1}^{4}\left(X_i^2 + X_i(h)X_i\right).$$

Remark 5.6.2. A different choice of the anti-commuting skew-symmetric matrices A_j leads to a subriemannian structure on $\mathbb{S}^7$ (with intrinsic sublaplacian $\Delta_{\mathrm{sub}}^{T'}$) equivalent to $\mathbb{S}^7_T$ (s. Remark 5.2.2). Furthermore, a subriemannian isometry preserves the Popp measure (s. [11]) and hence, the intrinsic sublaplacians $\Delta_{\mathrm{sub}}^{T'}$ and Δ_{sub}^T are unitary equivalent. The last fact can be also directly seen from the representation of the intrinsic sublaplacian in [11, Corollary 2] and the representation of B_T in (5.5). In particular, both sublaplacians have the same spectrum, i.e. the spectrum of the trivializable subriemannian structure on $\mathbb{S}^7$ does not depend on the specific choice of the anti-commuting skew-symmetric matrices A_j.

In the following we use the nilpotent approximation to compute the first heat invariant for the trivializable subriemannian structure endowed with the Popp measure. For this, let $z \in \mathbb{S}^7$ be fixed. Since $X_1, \cdots, X_7$ is an adapted frame for $\mathbb{S}^7_T$ at z, the inverse of the local diffeomorphism

$$\phi^{-1} : (u_1, \cdots, u_7) \longmapsto \exp\left(u_1X_1 + \cdots + u_7X_7\right)(z)$$

defines a system of local adapted coordinates at z. Because the adapted frame is a frame of linear vector fields, i.e.

$$X_i(z) = A_iz \quad \textit{for} \quad i = 1, \cdots, 7 \quad \textit{and} \quad z \in \mathbb{S}^7,$$

the integral curve $\gamma(t)$ of the vector field $u_1X_1 + \cdots + u_7X_7$ with $u = (u_1, \cdots, u_7) \in \mathbb{R}^7$ and starting at z can be explicitly calculated as:

$$\gamma(t) = \cos\left(\|u\|t\right)z + \frac{\sin\left(\|u\|t\right)}{\|u\|}A_uz,$$

where

$$A_u := \sum_{i=1}^{7} u_iA_i \quad \textit{and} \quad \|u\| = \sqrt{u_1^2 + \cdots + u_7^2}.$$

Hence ϕ^{-1} is given by:

$$\phi^{-1}(u) = \cos\left(\|u\|\right)z + \frac{\sin\|u\|}{\|u\|}A_uz \quad \textit{for} \quad u \in \mathbb{R}^7.$$

We recall that by the anti-commutation relations (5.1) of the matrices $A_1, \cdots, A_7$, the matrix A_u fulfills the identity:

$$A_u^2 = -\|u\|^2 \mathrm{Id} \quad \textit{for all} \quad u \in \mathbb{R}^7.$$

Now, let $w \in \mathbb{S}^7 \backslash \{-z\}$ and consider the following equation in $u \in B(0,\pi) := \{u \in \mathbb{R}^7 : \|u\| < \pi\}$:

$$w = \cos(\|u\|)z + \frac{\sin\|u\|}{\|u\|} A_u z. \tag{5.11}$$

Again by using the relations (5.1) we can write:

$$\langle w, z\rangle = \cos\|u\| \quad \textit{and} \quad \langle w, A_i z\rangle = \frac{\sin\|u\|}{\|u\|} u_i,$$

for $i = 1, \cdots, 7$. Hence the equation (5.11) has the unique solution $u \in B(0,\pi)$ given by:

$$u_i = \frac{\arccos\langle w, z\rangle}{\sqrt{1-\langle w,z\rangle^2}} \langle w, A_i z\rangle \quad \textit{for} \quad i = 1, \cdots, 7. \tag{5.12}$$

We summarize the above calculations in:

Lemma 5.6.3. *Canonical coordinates of the first kind at $z \in \mathbb{S}^7$ are given by*

$$\begin{aligned} \phi : \mathbb{S}^7\backslash\{-z\} &\longrightarrow B(0,\pi) \\ w &\longmapsto \phi(w) = u, \end{aligned}$$

and $\phi(z) = 0$, where $u = (u_1, \cdots, u_7)$ is given by (5.12).

Next, we compute the expansion of the horizontal vector fields $X_1, \cdots, X_4$ near 0 in the chart ϕ. Let us define the following smooth functions on $[0,\pi[$:

$$F(u) := \frac{1}{\|u\|^2} - \frac{\cot\|u\|}{\|u\|} \quad \textit{and} \quad G(u) := \|u\|\cot\|u\|,$$

with $F(0) := \frac{1}{3}$ and $G(0) := 1$.

Then a straightforward computation shows that the pushforwards of the horizontal vector fields $X_1, \cdots, X_4$ by ϕ are given on $B(0,\pi)$ by:

$$(X_i)_* = \sum_{j=1}^{7} a_{ij} \frac{\partial}{\partial u_j},$$

where the functions a_{ij} with b_{ik}^k in (5.4) are defined by:

$$a_{ij}(u) := G(u)\delta_{ij} + F(u)u_i u_j + \frac{1}{2}\sum_{k=1}^{7} b_{ij}^k(z) u_k. \tag{5.13}$$

For $\epsilon > 0$ small, consider the expansion of X_i^ϵ around 0:

$$X_i^\epsilon := \epsilon\delta_\epsilon^*(X_i)_* \simeq X^{(-1)} + \epsilon X_i^{(0)} + \epsilon^2 X_i^{(1)} + \cdots,$$

where $X_i^{(l)}$ is the homogeneous part of X_i of order l.

Lemma 5.6.4. *For $i = 1, \cdots, 4$, it holds:*

$$X_i^{(-1)} = \frac{\partial}{\partial u_i} + \frac{1}{2}\sum_{j=5}^{7}\sum_{k=1}^{4} b_{ij}^k u_k \frac{\partial}{\partial u_j},$$

$$X_i^{(0)} = \frac{1}{2}\sum_{j=1}^{4}\sum_{k=1}^{4} b_{ij}^k u_k \frac{\partial}{\partial u_j} + \frac{1}{2}\sum_{j=5}^{7}\sum_{k=5}^{7} b_{ij}^k u_k \frac{\partial}{\partial u_j},$$

$$X_i^{(1)} = \frac{1}{2}\sum_{j=1}^{4}\sum_{k=5}^{7} b_{ij}^k u_k \frac{\partial}{\partial u_j} + \frac{1}{3}\sum_{j=1}^{7} u_i u_j \frac{\partial}{\partial u_j} - \frac{1}{3}\sum_{k=1}^{4} u_k^2 \frac{\partial}{\partial u_i}.$$

Furthermore, for $l \geq 2$:

$$X_i^{(l)} = G^{(l+1)}(u)\frac{\partial}{\partial u_i} + \sum_{j=1}^{7} F^{(l-1)}(u) u_i u_j \frac{\partial}{\partial u_j},$$

where $F^{(l)}(u)$ (resp. $G^{(l)}(u)$) denotes the homogeneous part of order l in the anisotropic expansion of F (resp. G).

Proof. According to (5.13) we need to compute the expansion of a_{ij} near 0. We recall that the function $u \longmapsto u_i$ for $i = 1, \cdots, 4$ (resp. $i = 5, \cdots, 7$) has order 1 (resp. 2). Also the vector field $\frac{\partial}{\partial u_i}$ for $i = 1, \cdots, 4$ (resp. $i = 5, \cdots, 7$) has order -1 (resp. -2). The third term of (5.13):

$$\frac{1}{2}\sum_{k=1}^{7} b_{ij}^k(z)u_k = \underbrace{\frac{1}{2}\sum_{k=1}^{4} b_{ij}^k(z)u_k}_{\text{order 1}} + \underbrace{\frac{1}{2}\sum_{k=5}^{7} b_{ij}^k(z)u_k}_{\text{order 2}}$$

give us only homogeneous terms of order less than 2. Furthermore, a straightforward calculation shows that for $\epsilon \to 0$:

$$F(\delta_\epsilon(u)) \simeq \frac{1}{3} + \sum_{l\geq 1} F^{(l)}(u)\epsilon^l$$

$$G(\delta_\epsilon(u)) \simeq 1 - \frac{1}{3}\sum_{j=1}^{4} u_j^2\epsilon^2 + \sum_{l\geq 3} G^{(l)}(u)\epsilon^l.$$

Here $F^{(l)}(u)$ and $G^{(l)}(u)$ are homogeneous polynomials in u of order l. By arranging homogeneous terms in the expression (5.13) and writing

$$X_i^{(l)} = \sum_{j=1}^{4} a_{ij}^{(l+1)}(u)\frac{\partial}{\partial u_j} + \sum_{j=5}^{7} a_{ij}^{(l+2)}(u)\frac{\partial}{\partial u_j},$$

where $a_{ij}^{(l)}$ denotes the homogeneous term of a_{ij} of order l, we obtain the desired result. $\square$

Note that Lemma 5.6.4 not only holds for the trivializable subriemannian structure defined by the specific matrices $A_1, \cdots, A_7$ from Lemma 5.2.3, but also for arbitrary skew-symmetric matrices with relations (5.1).

Remark that only the first three homogeneous terms in the anisotropic expansion of X_i encode the geometric data (b_{ij}^k) of our subriemannian manifold $\mathbb{S}_T^7$. The remaining homogeneous terms are completely given by the functions F and G, which are independent of the chosen matrices $A_1, \cdots, A_7$.

The tangent group of $\mathbb{S}_T^7$ at z is isomorphic to the unique simply connected nilpotent Lie group $\mathbb{G}(z)$ corresponding to the Lie algebra generated by the vector fields:

$$X_1^{(-1)}, \cdots, X_4^{(-1)}.$$

By definition, the nilpotentization of the Popp measure at z is the Haar measure $\hat{\mathcal{P}}_T^z$ on $\mathbb{G}(z) \simeq \mathbb{R}^7$ given in global exponential coordinates $u_1, \cdots, u_7$ by:

$$\hat{\mathcal{P}}_T^z = g(z) du_1 \wedge \cdots \wedge du_7.$$

Here $g(z)$ denotes the density appearing in Lemma 5.3.2. In order to compute the first heat invariant c_0, according to (2.6), we need to derive the heat kernel $K_t^{\mathbb{G}(z)}$ of the sublaplacian

$$\Delta_{\text{sub}}^{\mathbb{G}(z)} := \sum_{i=1}^{4} \left(X_i^{(-1)} \right)^2$$

on $\mathbb{G}(z) \simeq \mathbb{R}^7$ with respect to the Haar measure $\hat{\mathcal{P}}_T^z$. This explicitly is obtained by the *Beals-Gaveau-Greiner formula* for the sublaplacian on general step two nilpotent Lie groups in [26, 36], which we recall next. For $\alpha, \beta \in \mathbb{G}(z)$ it holds:

$$K_t^{\mathbb{G}(z)}(\alpha, \beta) = \frac{1}{(4\pi t)^5} \int_{\mathbb{R}^3} e^{-\frac{\varphi(\tau, \alpha^{-1} * \beta)}{2t}} W(\tau) \frac{d\tau}{g(z)}, \tag{5.14}$$

where the *action function* $\varphi = \varphi(\tau, \alpha) \in C^\infty(\mathbb{R}^3 \times \widetilde{\mathbb{G}}(z))$ and the *volume element* $W(\tau) \in C^\infty(\mathbb{R}^3)$ are given as follows: put $\alpha = (a, b) \in \mathbb{R}^4 \times \mathbb{R}^3$, then

$$\begin{aligned} \varphi(\tau, g) &= \varphi(\tau, a, b) = \sqrt{-1}\langle \tau, b \rangle + \frac{1}{2}\Big\langle \sqrt{-1} J_{\tau/2} \coth\big(\sqrt{-1} J_{\tau/2}\big) \cdot a, a \Big\rangle, \\ W(\tau) &= \left\{ \det \frac{\sqrt{-1} J_{\tau/2}}{\sinh \sqrt{-1} J_{\tau/2}} \right\}^{1/2}, \end{aligned}$$

where $\langle b, b' \rangle = \sum_{k=1}^{3} b_k b_k'$ denotes the Euclidean inner product on $\mathbb{R}^3$ and J_τ denotes the map defined in Section 5.4.

Next, we compute the eigenvalues of the representation maps J_Z, for $Z \in \mathcal{V}_z \simeq \mathbb{R}^3$.

Lemma 5.6.5. *Let $z = (x, y) \in \mathbb{S}^7$ and $Z \in \mathcal{V}_z$. Then the eigenvalues of J_Z are*

$$\pm 2i(\|x\|^2 \pm \|y\|^2)\|Z\|.$$

Proof. According to (5.7) the characteristic polynomial $P(\lambda)$ of J_Z is given by:

$$P(\lambda) = \lambda^4 + 8(1 - 2\|x\|^2\|y\|^2)\|Z\|^2\lambda^2 + 16(1 - 4\|x\|^2\|y\|^2)\|Z\|^4.$$

Hence, a straightforward calculation shows that the roots of $P(\lambda)$ are exactly

$$\pm 2i(\|x\|^2 \pm \|y\|^2)\|Z\|.$$

□

Theorem 5.6.6. *The first heat invariant c_0^T of the trivializable subriemannian structure on $\mathbb{S}^7$ is given by*

$$c_0^T(z) = \frac{1}{(4\pi)^5 g(z)} \int_{\mathbb{R}^3} \frac{\|\tau\|}{\sinh\|\tau\|} \cdot \frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh(\|x\|^2 - \|y\|^2)\|\tau\|} d\tau$$

for $z = (x, y) \in \mathbb{S}^7$.

Proof. let $z = (x, y) \in \mathbb{S}^7$ and $Z \in \mathcal{V}_z$. By Lemma 5.6.5, the eigenvalues of the skew-symmetric operator J_Z are $\pm 2i(\|x\|^2 \pm \|y\|^2)\|Z\|$. We assume that z fulfills:

$$\|x\| \neq \|y\| \quad \textit{and} \quad x \neq 0.$$

Such points form an open dense subset in $\mathbb{S}^7$ and therefore, due to the smoothness of the local assignment $z \longmapsto c_0^T(z)$ (see [110]) we only need to compute $c_0^T(z)$ at such points. The advantage of considering such points is that the eigenvalues of the map J_Z for all $Z \in \mathcal{V}_z$, are simple. Hence the expression of the function $W(\tau)$ takes the form:

$$\det\left(\frac{iJ_\tau/2}{\sinh(iJ_\tau/2)}\right) = \left(\frac{\|\tau\|}{\sinh\|\tau\|}\right)^2 \left(\frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh((\|x\|^2 - \|y\|^2)\|\tau\|)}\right)^2.$$

Hence, by (2.6) and (6.8), we can write:

$$\begin{aligned} c_0^T(z) &= \frac{1}{(4\pi)^5} \int_{\mathbb{R}^3} \sqrt{\det\left(\frac{iJ_\tau/2}{\sinh iJ_\tau/2}\right)} \frac{d\tau}{g(z)} \\ &= \frac{1}{(4\pi)^5 g(z)} \int_{\mathbb{R}^3} \frac{\|\tau\|}{\sinh\|\tau\|} \cdot \frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh(\|x\|^2 - \|y\|^2)\|\tau\|} d\tau. \end{aligned}$$

□

Remark 5.6.7. At points $z = (x, y) \in \mathbb{S}^7$ with $x = 0$ or $y = 0$, a straightforward computation using the representation (5.7) shows that the maps J_{X_5}, J_{X_6} and J_{X_7} fulfill the quaternionic relations and hence the tangent groups of the subriemannian manifolds $\mathbb{S}^7_T$ and $\mathbb{S}^7_Q$ are isometric. Hence it follows that at these points the first heat invariants coincide:

$$c_0^Q(z) = c_0^T(z).$$

Also it is not hard to see that the infimum of $c_0^T(z)$ over $\mathbb{S}^7$ is attained at these points and therefore we can write

$$\inf\{c_0^T(z) : z \in \mathbb{S}^7\} = \hat{c}_0^Q. \tag{5.15}$$

Here $\hat{c}_0^Q$ denotes the value of the constant function $z \longmapsto c_0^Q(z)$ which will be calculated explicitly below.

We remark that the remaining heat invariants $c_1, c_2, \cdots$ might not be equal at these special points. In fact, in order to compute these numbers we have to take into account the local behavior of the corresponding subriemannian structures at such points.

As a corollary we prove now that the subriemannian manifolds $\mathbb{S}^7_T$ and $\mathbb{S}^7_Q$ are not *isospectral* with respect to the intrinsic sublaplacians:

Corollary 5.6.8. *Let $\mathbb{S}^7_T$ and $\mathbb{S}^7_Q$ be considered with the induced Popp measures. Then the intrinsic sublaplacians Δ^T_{sub} and Δ^Q_{sub} are not isospectral.*

Proof. By considering the subriemannian manifold $\mathbb{S}^7_Q$ as a quaternionic contact manifold and using the quaternionic relations of the almost complex structures $I_{\mathbf{l}}$ for $\mathbf{l} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, we see that the Popp measure is given by, (see Lemma 5.3.1):

$$\mathcal{P}_Q(z) = \frac{1}{(16)^{3/2}} d\sigma(z).$$

According to the results in [15], where the first heat invariant has been computed with respect to the standard measure on $\mathbb{S}^7_Q$, it follows that the first heat invariant $c_0^Q(z)$ (with respect to the Popp measure) is given by

$$c_0^Q(z) = \frac{16^{3/2}}{(4\pi)^5} \int_{\mathbb{R}^3} \left(\frac{\|\tau\|}{\sinh \|\tau\|} \right)^2 d\tau \text{ for } z \in \mathbb{S}^7.$$

We set

$$c_0^T - c_0^Q := \frac{1}{(4\pi)^5} \int_{\mathbb{S}^7} \int_{\mathbb{R}^3} \frac{\|\tau\|}{\sinh \|\tau\|} \left(\frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh\left((\|x\|^2 - \|y\|^2)\|\tau\|\right)} - \frac{\|\tau\|}{\sinh \|\tau\|} \right) d\tau d\sigma(z). \tag{5.16}$$

Note that the function $u \longmapsto u/\sinh(u)$ is even, smooth and monotone decreasing on the interval $[0, \infty[$. This shows that the integrand in (5.16) is a non-negative function on $\mathbb{S}^7 \times \mathbb{R}^3$ and non-vanishing on an open dense subset. Therefore $c_0^T > c_0^Q$ and the subriemannian manifolds $\mathbb{S}^7_Q$ and $\mathbb{S}^7_T$ cannot be isospectral. $\square$

5.7 Sublaplacian induced by the standard measure

If we consider the subriemannian manifold $\mathbb{S}^7_T$ endowed with the standard volume $d\sigma$, then the corresponding sublaplacian $\widetilde{\Delta}^T_{\text{sub}}$ will be a sum of squares:

$$\widetilde{\Delta}^T_{\text{sub}} = -\sum_{i=1}^{4} X_i^2. \tag{5.17}$$

Here $X_i = X(A_i)$ for $i = 1, \dots, 4$ with A_j defined in (5.2) and Lemma 5.2.3 denotes the system of linear vector fields generating the distribution $\mathcal{H}_T$ of $\mathbb{S}^7_T$. According to [61] the operator (5.17) is subelliptic, positive and with discrete spectrum consisting of eigenvalues. We recall that a part of this spectrum has been determined in [21]. Moreover, Corollary 5.4 of [21] implies that a different choice of the generating anti-commuting skew-symmetric matrices A_j leads to a sublaplacian which is unitary equivalent to (5.17) and therefore has the same spectrum. Hence, when studying the spectrum of the trivializable subriemannian structure, we can restrict ourselves to a specific choice the generators of a Clifford algebra (s. Remarks 5.2.2 and 5.6.2).

In this section a relation between the spectrum of $\widetilde{\Delta}^T_{\text{sub}}$ and the spectrum of the sublaplacian

$$\Delta^Q_{\text{sub}} = \Delta_{\mathbb{S}^7} + X(A_6)^2 + X(A_7)^2 + X(A_6A_7)^2$$

induced by the quaternionic Hopf fibration (s. [15]) will be shown. Here

$$\Delta_{\mathbb{S}^7} = -\sum_{j=1}^{7} X(A_j)^2$$

denotes the Laplace-Beltrami operator on $\mathbb{S}^7$ with respect to the standard metric. Via the inclusion $\mathbb{S}^3 \subset (\mathbb{H}, *)$ and for $\ell \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ consider the vector fields:

$$W_\ell f(z) = \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} f\big(z * e^{t\ell}\big) \quad \textit{where} \quad f \in C^\infty(\mathbb{S}^3).$$

By the same formula W_ℓ can be interpreted as a (linear) vector field on $\mathbb{R}^4 \cong \mathbb{H}$. A direct calculation using the decomposition $(x, y) \in \mathbb{S}^7 \subset \mathbb{R}^4 \times \mathbb{R}^4 \subset \mathbb{H}^2$ and the form of the matrices in (5.2) shows:

$$\widetilde{\Delta}^T_{\text{sub}} = \Delta_{\mathbb{S}^7} - \Delta_{\mathbb{S}^3} \otimes I - I \otimes \Delta_{\mathbb{S}^3} - 2B \tag{5.18}$$

$$\Delta^Q_{\text{sub}} = \Delta_{\mathbb{S}^7} - \Delta_{\mathbb{S}^3} \otimes I - I \otimes \Delta_{\mathbb{S}^3} + 2B. \tag{5.19}$$

Here $\Delta_{\mathbb{S}^3} = -\sum_{\ell \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}} W_\ell^2$ denotes the Laplace-Beltrami operator on $\mathbb{S}^3$ with respect to the standard metric and

$$B := \sum_{\ell \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}} W_\ell \otimes W_\ell. \tag{5.20}$$

The tensor product notation $A \otimes C$ means that an operator A acts with respect to the variable x and C with respect to y. Note that B in (5.20) vanishes on smooth functions $f(x,y) = \tilde{f}(x)$ and $g(x,y) = \tilde{g}(y)$ which only depend on x and y of $\mathbb{R}^4$, respectively. Therefore, $\widetilde{\Delta}_{\mathrm{sub}}^T$ and Δ_{sub}^Q act in the same way on functions g and f of the above type.

With the notation $\omega = (\omega_1, \omega_2) \in \mathbb{R}^4 \times \mathbb{R}^4$ we write $K_t^{\mathrm{Q}}(\omega, \mathrm{np}) = \widetilde{k}_t^{\mathrm{Q}}(\omega_1)$ for the heat kernel of $\Delta_{\mathrm{sub}}^{\mathrm{Q}}$ at the north pole $\mathrm{np} = (1, 0, \dots, 0) \in \mathbb{S}^7 \subset \mathbb{R}^8$. The function $\widetilde{k}_t^{\mathrm{Q}}$ has been calculated in [15] and only depends on ω_1. It follows from the previous remark that:

$$\widetilde{\Delta}_{\mathrm{sub}}^T K_t^{\mathrm{Q}}(\cdot, \mathrm{np}) = \Delta_{\mathrm{sub}}^{\mathrm{Q}} K_t^{\mathrm{Q}}(\cdot, \mathrm{np}) = -\frac{\mathrm{d}}{\mathrm{dt}} K_t^{\mathrm{Q}}(\cdot, \mathrm{np}). \tag{5.21}$$

Choose an orthonormal system $[\phi_\ell : \ell \in \mathbb{N}]$ of $L^2(\mathbb{S}^7) = L^2(\mathbb{S}^7, \sigma)$ consisting of smooth eigenfunctions of $\widetilde{\Delta}_{\mathrm{sub}}^T$ with corresponding eigenvalues $\lambda_\ell \geq 0$. We obtain an expansion of the heat kernel:

$$K_t^{\mathrm{Q}}(\omega, \mathrm{np}) = \sum_{\ell=1}^{\infty} c_\ell(t) \phi_\ell(\omega) = \sum_{\ell=1}^{\infty} c_\ell(t) \phi_\ell(\omega_1, 0),$$

which converges in $C^\infty(\mathbb{S}^7)$. From (5.21) one concludes that $c_\ell'(t) + \lambda_\ell \cdot c_\ell(t) = 0$ for each $\ell \in \mathbb{N}$. Hence there are constants γ_ℓ such that:

$$c_\ell(t) = \gamma_\ell e^{-\lambda_\ell t} \qquad \textit{where} \qquad t > 0.$$

Moreover, for all $\ell \in \mathbb{N}$:

$$\phi_\ell(\mathrm{np}) = \lim_{t \downarrow 0} \int_{\mathbb{S}^7} \phi_\ell(\omega) K_t^{\mathrm{Q}}(\omega, \mathrm{np}) d\sigma(\omega) = \lim_{t \downarrow 0} c_\ell(t) = \gamma_\ell.$$

Let $K_t^T(\omega, \mathrm{np})$ denote the heat kernel of $\widetilde{\Delta}_{\mathrm{sub}}^T$. From our calculation we conclude:

$$K_t^{\mathrm{Q}}(\omega, \mathrm{np}) = \sum_{\ell=1}^{\infty} e^{-\lambda_\ell t} \phi_\ell(\omega) \phi_\ell(\mathrm{np}). \tag{5.22}$$

On the other hand, we can choose an orthonormal basis $[\psi_\ell : \ell \in \mathbb{N}]$ of $L^2(\mathbb{S}^7)$ consisting of eigenfunctions of $\Delta_{\mathrm{sub}}^{\mathrm{Q}}$ with corresponding eigenvalue sequence $(\mu_\ell)_{\ell \in \mathbb{N}}$. We write:

$$K_t^{\mathrm{Q}}(\omega, \mathrm{np}) = \sum_{\ell=1}^{\infty} e^{-\mu_\ell t} \psi_\ell(\omega) \psi_\ell(\mathrm{np}) = \sum_{\ell=1}^{\infty} e^{-\widetilde{\mu}_\ell t} \Psi_\ell(\omega). \tag{5.23}$$

On the right hand side we have used the definition:

$$\Psi_\ell(\omega) := \sum_{\substack{j \text{ s.t.} \\ \mu_j = \widetilde{\mu}_\ell}} \psi_j(\omega) \psi_j(\mathrm{np}),$$

where $0 \leq \widetilde{\mu}_1 < \widetilde{\mu}_2 < \widetilde{\mu}_3 \dots$ denotes the sequence of distinct eigenvalues of $\Delta_{\mathrm{sub}}^{\mathrm{Q}}$ in increasing order. We write $m(\mu)$ for the multiplicity of an eigenvalue μ of $\Delta_{\mathrm{sub}}^{\mathrm{Q}}$.

Lemma 5.7.1. *For $\ell \in \mathbb{N}$ the sum $\sum_{\mu_j=\widetilde{\mu}_\ell} |\psi_j(x)|^2 \equiv \|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)}$ is constant on $\mathbb{S}^7$ and*

$$m(\mu_\ell) = \mathrm{vol}(\mathbb{S}^7)\|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)} \neq 0.$$

Proof. Since $\{\psi_\ell\}_\ell$ is an orthonormal basis of $L^2(\mathbb{S}^7)$ we have:

$$\|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)} = \sum_{\substack{j \text{ s.t.} \\ \mu_j=\widetilde{\mu}_\ell}} |\psi_j(\mathrm{np})|^2.$$

Consider the subriemannian isometry group $\mathcal{I}(\mathbb{S}^7_Q)$. Recall that $\mathbf{Sp}(2) \subset \mathcal{I}(\mathbb{S}^7_Q)$ and $\mathbf{Sp}(2)$ acts transitively on $\mathbb{S}^7$ (see the proof of Lemma 5.3.1). For all $g \in \mathbf{Sp}(2)$ we define the unitary operator V_g on $L^2(\mathbb{S}^7)$ by composition, i.e. $V_g f := f \circ g$ for all $f \in L^2(\mathbb{S}^7)$. Note that

$$\left[\Delta^{\mathrm{Q}}_{\mathrm{sub}}, V_g\right] = 0$$

and put $\psi^g_\ell := V_g\psi_\ell = \psi_\ell \circ g$. Then $\{\psi^g_\ell\}_\ell$ defines an orthonormal basis of $L^2(\mathbb{S}^7)$ consisting of eigenfunctions of $\Delta^{\mathrm{Q}}_{\mathrm{sub}}$ corresponding to the sequence $(\mu_\ell)_\ell$ of eigenvalues, as well, and the heat kernel expansion of $\Delta^{\mathrm{Q}}_{\mathrm{sub}}$ can be rewritten as:

$$K^{\mathrm{Q}}_t(\omega, \mathrm{np}) = \sum_{\ell=1}^{\infty} e^{-\mu_\ell t}\psi_\ell \circ g(\omega) \cdot \psi_\ell \circ g(\mathrm{np}), \quad \textit{where} \quad \omega \in \mathbb{S}^7.$$

It follows for all $g \in H$:

$$\|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)} = \sum_{\substack{j \text{ s.t.} \\ \mu_j=\widetilde{\mu}_\ell}} |\psi_j \circ g(\mathrm{np})|^2.$$

Since $\mathbf{Sp}(2)$ acts transitively on $\mathbb{S}^7$ we conclude that the finite sum below is constant on $\mathbb{S}^7$ with value:

$$\sum_{\substack{j \text{ s.t.} \\ \mu_j=\widetilde{\mu}_\ell}} |\psi_j(x)|^2 \equiv \|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)}, \qquad (x \in \mathbb{S}^7). \tag{5.24}$$

Hence:

$$m(\widetilde{\mu}_\ell) = \#\{j \ : \mu_j = \widetilde{\mu}_\ell\} = \int_{\mathbb{S}^7} \sum_{\substack{j \text{ s.t.} \\ \mu_j=\widetilde{\mu}_\ell}} |\psi_j(x)|^2 d\sigma(x) = \mathrm{vol}(\mathbb{S}^7)\|\Psi_\ell\|^2_{L^2(\mathbb{S}^7)}.$$

This proves the assertion. □

Lemma 5.7.1 implies that in each eigenspace of $\Delta^{\mathrm{Q}}_{\mathrm{sub}}$ there is an element ψ such that $\psi(\mathrm{np}) \neq 0$. As usual let $\sigma(A)$ denote the spectrum of an operator A and put:

$$\Lambda := \Big\{\lambda \in \sigma\big(\widetilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}}\big) \ : \ \exists\, \phi \in \ker\big(\widetilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}} - \lambda\big) \textit{ such that } \ \phi(\mathrm{np}) \neq 0\Big\}.$$

Consider the following subset of distinct eigenvalues:

$$\Lambda := \big\{\widetilde{\lambda}_\ell \in \Lambda \ : \ \widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \dots\big\} \subset \sigma\big(\widetilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}}\big).$$

From (5.22) and (5.23) we have for all $t > 0$:

$$\sum_{\ell=1}^{\infty} e^{-\widetilde{\lambda}_\ell t}\Phi_\ell(\omega) = \sum_{\ell=1}^{\infty} e^{-\widetilde{\mu}_\ell t}\Psi_\ell(\omega), \tag{5.25}$$

where for each $\widetilde{\lambda}_\ell \in \Lambda$:

$$\Phi_\ell(\omega) := \sum_{\substack{j \text{ s.t.} \\ \lambda_j = \widetilde{\lambda}_\ell}} \phi_j(\omega)\phi_j(\mathrm{np}).$$

Note that $\Phi_\ell(\mathrm{np}) \neq 0$ by definition of Λ.

Theorem 5.7.2. *We have the inclusion of spectra* $\Lambda = \sigma(\Delta_{\mathrm{sub}}^{\mathrm{Q}}) \subset \sigma(\widetilde{\Delta}_{\mathrm{sub}}^{\mathrm{T}})$.

Proof. Assume that $\widetilde{\mu}_1 \neq \widetilde{\lambda}_1$. Without loss of generality assume that $\widetilde{\lambda}_1 < \widetilde{\mu}_1$. Then

$$0 \neq \Phi_1(\mathrm{np}) = \sum_{\ell=1}^{\infty} e^{-(\widetilde{\mu}_\ell - \widetilde{\lambda}_1)t}\Psi_\ell(\mathrm{np}) - \sum_{\ell=2}^{\infty} e^{-(\widetilde{\lambda}_\ell - \widetilde{\lambda}_1)t}\Phi_\ell(\mathrm{np}).$$

Since the right hand side tends to zero as $t \to \infty$ we obtain a contradiction. Hence $\widetilde{\lambda}_1 = \widetilde{\mu}_1$ and

$$\Phi_1(\mathrm{np}) = \Psi_1(\mathrm{np}) = m(\mu_1).$$

Therefore

$$\sum_{\ell=2}^{\infty} e^{-\widetilde{\lambda}_\ell t}\Phi_\ell(\mathrm{np}) = \sum_{\ell=2}^{\infty} e^{-\widetilde{\mu}_\ell t}\Psi_\ell(\mathrm{np}),$$

and proceeding inductively in this way we obtain the result. $\square$

Remark 5.7.3. The spectrum $\sigma\big(\Delta_{\mathrm{sub}}^{\mathrm{Q}}\big)$ is known explicitly, see [15]. Moreover, the multiplicities of eigenvalues $\lambda \in \Lambda$ with respect to the operators $\Delta_{\mathrm{sub}}^{\mathrm{Q}}$ and $\widetilde{\Delta}_{\mathrm{sub}}^{\mathrm{T}}$ may not coincide. The statement in Theorem 5.7.2 generalizes results in [21] where a (smaller) part of the spectrum $\sigma(\widetilde{\Delta}_{\mathrm{sub}}^{T})$ has been calculated.

Using the nilpotent approximation as in Section 5.6 we can show that the first heat invariant $\tilde{c}_0^T(z)$ of $\widetilde{\Delta}_{\mathrm{sub}}^{T}$ is exactly:

$$\tilde{c}_0^T(z) = \frac{1}{16^{3/2}(4\pi)^5}\int_{\mathbb{R}^3} \frac{\|\tau\|}{\sinh\|\tau\|} \cdot \frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh\left((\|x\|^2 - \|y\|^2)\|\tau\|\right)} d\tau$$

for $z = (x, y) \in \mathbb{S}^7$. Hence using the same arguments as in the proof of Corollary 5.6.8, it follows that the operators $\Delta_{\mathrm{sub}}^{Q}$ and $\widetilde{\Delta}_{\mathrm{sub}}^{T}$ are not isospectral, as well, i.e. the inclusion of spectra in Theorem 5.7.2 is strict or they have the same spectrum but the eigenvalues have different multiplicities.

5.8 Open problems

Finally, we mention some open problems which have been left in the analysis of the trivializable subriemannian manifold $\mathbb{S}^7_T$.

1. What is the significance of the second heat invariant c_1^T for the trivializable subriemannian structure on $\mathbb{S}^7$? We recall that in the framework of Riemannian geometry, the second heat invariant can be interpreted as the scalar curvature. Furthermore, for contact subriemannian structures on 3-dimensional manifolds an interpretation of the second heat invariant in terms of certain curvature terms has given by D. Barilari in [7].

2. Derive an explicit formula for the heat kernel of the sublaplacian Δ_{sub}^T on $\mathbb{S}^7_T$ and on $\mathbb{S}^{15}_T$ equipped with the rank eight trivializable subriemannian structure of step two in [21]. In case of the quaternionic contact structure such a formula is known and can be found in [15].

3. Is the symmetry group of $\mathbb{S}^7_T$ a finite dimensional Lie group?

4. As is known, the Carnot-Carathéodory distance d on $\mathbb{S}^7_T$ appears in the exponent of the off-diagonal small time asymptotics of the subelliptic heat kernel of Δ_{sub}^T. Can one (at least locally) obtain formulas or estimates on d via a heat kernel analysis?

Chapter 6

Heat kernel asymptotics for quaternionic contact manifolds

This Chapter is taken from the article $A4$ and is organized as follows. In Section 6.1 we recall the definition of qc manifolds and we list some of their properties. Then we compute the Popp volume and the intrinsic sublaplacian induced by a qc structure in Section 6.2. In Section 6.3 we compute the first coefficients in the small time asymptotics of the heat kernel by using an approximation method [39, 110] and the qc normal coordinates [73].

6.1 Quaternionic contact manifolds

In the following we recall briefly the definition of qc manifolds and some of their properties. We follow the presentations in [33, 66]. For a more detailed exposition of the properties of qc structures we refer to [10, 33, 65, 66].

Definition 6.1.1. A quaternionic contact manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a $(4n+3)$-dimensional connected manifold M (with $n \geq 2$) together with a corank 3 distribution $\mathcal{H}$ and a fiberwise inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ such that

1. $\mathcal{H}$ is given locally as the kernel of an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$:
$$\mathcal{H} = \bigcap_{i=1}^{3} \mathrm{Ker}(\eta_i). \tag{6.1}$$

2. There are three almost complex structures I_1, I_2 and I_3 on $\mathcal{H}$ that satisfy the quaternionic commutation relations:
$$(I_i)^2 = I_1 I_2 I_3 = -\mathrm{Id} \text{ for } i = 1, 2, 3. \tag{6.2}$$
Furthermore, the following compatibility conditions hold:
$$2\langle I_i X, Y\rangle = d\eta_i(X, Y) \tag{6.3}$$
for all horizontal vector fields $X, Y \in \mathcal{H}$ and $i = 1, 2, 3$.

Note that the choice of 1-forms η_1, η_2, η_3 and almost complex structures I_1, I_2, I_3 with the above properties (6.1)-(6.3) is not unique. If $\psi \in SO(3)$ then $\psi(\eta)$ and $\psi(I)$ with $I := (I_1, I_2, I_3)$ satisfy the above relations as well [65]. Hence we have a 2-sphere bundle of almost complex structures over M (locally) given by

$$\mathbb{I} := \{aI_1 + bI_2 + cI_3 : a^2 + b^2 + c^2 = 1\}.$$

An important fact is that the 1-forms η_1, η_2, η_3 and the almost complex structures I_1, I_2, I_3 uniquely determine the metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. Note also that a qc manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is an equiregular fat SR manifold of step two [10].

Due to the presence of the three almost complex structures and their relation to the metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$, it is useful to work with the so-called $Sp(n)Sp(1)$-frames. By an $Sp(n)Sp(1)$-frame $\{X_1, \cdots, X_{4n}\}$ we mean an orthonormal frame of the distribution $\mathcal{H}$ such that

$$I_i X_{4k+1} = X_{4k+i+1} \text{ for } k = 0, \cdots, n-1 \text{ and } i = 1, 2, 3.$$

In the following we denote by $X_{\mathcal{H}}$ (resp. $X_{\mathcal{V}}$) the orthogonal projection of a vector field X onto $\mathcal{H}$ (resp. $\mathcal{V}$).
On a qc manifold with $n \geq 2$ there exists a canonical connection defined by Biquard. In [33] it is shown that there is a unique connection ∇ with torsion T and a unique complementary subspace $\mathcal{V}$ to $\mathcal{H}$ in TM such that

1. ∇ preserves the decomposition $\mathcal{H} \oplus \mathcal{V}$ and the $Sp(n)Sp(1)$-structure on $\mathcal{H}$:
$$\nabla \langle \cdot, \cdot \rangle = 0 \text{ and } \nabla \sigma \in \Gamma(\mathbb{I}) \text{ for } \sigma \in \Gamma(\mathbb{I}).$$
2. The torsion T on $\mathcal{H}$ fulfils
$$T(X, Y) = -[X, Y]_{\mathcal{V}} \text{ for } X, Y \in \mathcal{H}.$$
3. For $V \in \mathcal{V}$, the endomorphism
$$\begin{aligned} T(V, \cdot) : \mathcal{H} &\longrightarrow \mathcal{H} \\ X &\longmapsto T(V, X)_{\mathcal{H}} \end{aligned}$$
lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$.
4. There is a natural identification $\varphi : \mathcal{V} \longrightarrow Sp(1)$ with $\nabla \varphi = 0$.

This connection is known as the *Biquard connection*. Furthermore, the vertical distribution $\mathcal{V}$ is locally generated by the (quaternionic contact) Reeb vector fields V_1, V_2, V_3 defined by

$$\eta_i(V_j) = \delta_{ij} \text{ and } (V_i \lrcorner d\eta_j)_{\mathcal{H}} = -(V_j \lrcorner d\eta_i)_{\mathcal{H}} \text{ for } i, j = 1, 2, 3. \tag{6.4}$$

Using the Reeb vector fields V_1, V_2, V_3 we extend the metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ to a Riemannian metric g by requiring $\mathcal{H} \perp \mathcal{V}$ and

$$g := \langle \cdot, \cdot \rangle \oplus \left(\eta_1^2 + \eta_2^2 + \eta_3^2 \right).$$

Note that neither the extended Riemannian metric nor the Biquard connection depends on the action of $SO(3)$ on $\mathcal{V}$.

Remark 6.1.2. In case of $n = 1$, the conditions (6.4) are not satisfied in general [33]. Furthermore, it was shown in [47], that if we assume in addition the existence of Reeb vector fields as in (6.4), then the existence of a linear connection with similar properties as above is assured.

Let us denote by T (resp. R) the torsion (resp. curvature) tensor of the Biquard connection defined by

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for smooth vector fields X, Y and Z. In the following we denote by $\{\theta_1, \cdots \theta_{4n}\}$ the dual frame of a horizontal frame $\{X_1, \cdots, X_{4n}\}$. It is occasionally useful to have a notation for the entire frame $\{X_1, \cdots, X_{4n}, V_1, V_2, V_3\}$ and coframe $\{\theta_1, \cdots, \theta_{4n}, \eta_1, \eta_2, \eta_3\}$. Therefore, we may refer to V_i as X_{4n+i} and η_i as θ_{4n+i}. Furthermore, in order to have a consistent index notation we will use different letters for different ranges of indices like in [73] as follows:

$$a, b, c \in \{1, \cdots, 4n+3\}, \ \ \alpha, \beta, \gamma, \delta \in \{1, \cdots, 4n\}, \ \ i, j, k \in \{1, 2, 3\},$$

and $\bar{i} := 4n + i$, for $i = 1, 2, 3$. With this convention we set

$$I^i_{\alpha\beta} := g(I_i X_\alpha, X_\beta), \ \ T^c_{ab} := \theta_c \left(T(X_a, X_b) \right) \text{ and } R^d_{abc} := \theta_d \left(R(X_a, X_b) X_c \right).$$

Definition 6.1.3. The quaternionic contact scalar curvature κ is defined by

$$\kappa := \sum_{\alpha,\beta=1}^{4n} R^\alpha_{\alpha\beta\beta}.$$

We recall the following identities proved in [33, 65] which we shall need later:

Proposition 6.1.4. *It holds:*

1. *For* $\alpha, \beta = 1, \cdots, 4n$ *and* $i = 1, 2, 3$*:*

$$T^{\bar{i}}_{\alpha\beta} = -2 I^i_{\alpha\beta};$$

2. *For* $i, j = 1, 2, 3$:
$$T^{\bar{i}}_{\bar{i}\bar{j}} = 0;$$

3. *For* $V \in \mathcal{V}$ *and* $I \in \mathbb{I}$*:*
$$\sum_{\alpha=1}^{4n} g(T(V, X_\alpha), IX_\alpha) = 0;$$

4. *For* $i = 1, 2, 3$*:*
$$\sum_{\alpha,\beta=1}^{4n} g(R(X_\alpha, I_i X_\alpha) I_i X_\beta, X_\beta) = \frac{2n\kappa}{n+2};$$

5. *For* $i = 1, 2, 3$*:*
$$\sum_{\alpha,\beta=1}^{4n} g(R(X_\beta, I_i X_\alpha) X_\alpha, I_i X_\beta) = -\frac{n\kappa}{n+2}.$$

6.2 Popp measure and intrinsic sublaplacian

Let $\{X_1, \cdots, X_{4n}, V_1, V_2, V_3\}$ be an orthonormal frame near $q \in M$ such that $\{X_1, \cdots, X_{4n}\}$ is an $Sp(n)Sp(1)$-frame. Then this frame is also an adapted frame for the SR structure $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$. According to [11], the associated Popp measure $\mathcal{P}$ can be expressed locally in the form
$$\mathcal{P} = \frac{1}{\sqrt{\det B}} \theta_1 \wedge \cdots \wedge \theta_{4n} \wedge \eta_1 \wedge \eta_2 \wedge \eta_3.$$
Here $B = (B_{ij})_{ij}$ is the 3×3-matrix function locally defined near q with coefficients given by
$$B_{ij} := \sum_{\alpha,\beta=1}^{4n} b^i_{\alpha\beta} b^j_{\alpha\beta},$$
where $b^i_{\alpha\beta}$ are defined for $\alpha, \beta = 1, \cdots, 4n$ and $i = 1, 2, 3$ by
$$b^i_{\alpha\beta} := g([X_\alpha, X_\beta], V_i).$$

Now by (6.3) we can write:
$$\begin{aligned} b^i_{\alpha\beta} &= g([X_\alpha, X_\beta], V_i) \\ &= \eta_i([X_\alpha, X_\beta]) \\ &= -d\eta_i(X_\alpha, X_\beta) \\ &= -2g(I_i X_\alpha, X_\beta). \end{aligned}$$

Hence it follows that
$$B_{ij} = 4 \cdot \sum_{\alpha,\beta=1}^{4n} g(I_i X_\alpha, X_\beta) g(I_j X_\alpha, X_\beta)$$

$$
\begin{aligned}
&= 4 \cdot \sum_{\alpha=1}^{4n} g(I_i X_\alpha, I_j X_\alpha) \\
&= 16n\delta_{ij}.
\end{aligned}
$$

In the last equality we used the skew-symmetry of the almost complex structures and the commutation relations (6.2). This shows that B is a diagonal matrix:

$$B = 16n \cdot \mathrm{Id} \in \mathbb{R}^{3\times 3}$$

and hence we obtain the following formula for the Popp measure:

Lemma 6.2.1. *The Popp measure $\mathcal{P}$ for the quaternionic contact manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ has the form*

$$\mathcal{P} = \frac{1}{(16n)^{3/2}} d\sigma,$$

where $d\sigma := \theta_1 \wedge \cdots \wedge \theta_{4n} \wedge \eta_1 \wedge \eta_2 \wedge \eta_3$.

In terms of the orthonormal frame $\{X_1, \cdots, X_{4n}, V_1, V_2, V_3\}$, the intrinsic sublaplacian Δ_{sub} with respect to the Popp measure $\mathcal{P}$ can be expressed in the form (see [11])

$$\Delta_{sub} = -\left(\sum_{\alpha=1}^{4n} X_\alpha^2 + \mathrm{div}_{\mathcal{P}}(X_\alpha)X_\alpha\right). \tag{6.5}$$

Here the divergence operator is defined in terms of the Lie derivative $\mathcal{L}_{X_\alpha}$ by

$$\mathcal{L}_{X_\alpha}(\mathcal{P}) = \mathrm{div}_{\mathcal{P}}(X_\alpha)\mathcal{P}.$$

To find out an expression of the first-order term in the formula (6.5) involving the structure constants, we proceed like in the 3D contact case [7]. For this we use the formula

$$\mathcal{L}_X(\mu \wedge \nu) = \mathcal{L}_X(\mu) \wedge \nu + \mu \wedge \mathcal{L}_X(\nu)$$

for any vector field X and differential forms μ and ν on M. With this in mind, we obtain

$$\mathcal{L}_{X_\alpha}(\mathcal{P}) = \frac{1}{(16n)^{3/2}} \sum_{a=1}^{4n+3} \theta_1 \wedge \cdots \wedge \theta_{a-1} \wedge \mathcal{L}_{X_\alpha}(\theta_a) \wedge \theta_{a+1} \wedge \cdots \wedge \theta_{4n+3}.$$

Since $\{\theta_1, \cdots, \theta_{4n+3}\}$ is a coframe, we write for $\alpha = 1, \cdots, 4n$ and $a = 1, \cdots, 4n+3$:

$$\mathcal{L}_{X_\alpha}(\theta_a) = \sum_{b=1}^{4n+3} f_{\alpha ab}\theta_b.$$

The functions $f_{\alpha ab}$ can be calculated as follows

$$f_{\alpha ab} = \mathcal{L}_{X_\alpha}(\theta_a)(X_b) = \theta_a([X_b, X_\alpha]) =: c_{b\alpha}^a.$$

Hence we obtain

$$\mathrm{div}_{\mathcal{P}}(X_\alpha) = \sum_{\beta=1}^{4n} \theta_\beta([X_\beta, X_\alpha]) + \sum_{i=1}^{3} \eta_i([V_i, X_\alpha]) = \sum_{a=1}^{4n+3} c_{a\alpha}^a.$$

Lemma 6.2.2. *The intrinsic sublaplacian Δ_{sub} has the expression*

$$\Delta_{sub} = -\left(\sum_{\alpha=1}^{4n} X_\alpha^2 + \left(\sum_{a=1}^{4n+3} c_{a\alpha}^a\right) X_\alpha\right),$$

where $c_{a\alpha}^a = \theta_a([X_a, X_\alpha])$.

6.3 First and second heat invariant

In case of a general subriemannian manifold (even step two), a classical problem is to identify the second heat invariant c_1 (associated to the intrinsic sublaplacian) with geometric invariants of the underlying structure, specially if this subriemannian structure is induced from a specific structure (e.g. H-type subriemannian manifolds). In the case of 3D contact SR manifolds, such a result was established by D. Barilari in [7]. Moreover, on a strictly pseudoconvex CR manifold with a Levi-metric a similar result holds [28]. In both cases, a crucial tool for obtaining a geometric interpretation of c_1 was the construction of special privileged coordinates [5, 69].

In the following, we consider special coordinates on qc manifolds constructed in [73], the so-called *qc normal coordinates*. In these coordinates, by considering a special adapted frame at a given point q, we are able to express $\mathcal{P}_2$ from (2.5) as a second-order differential operator with polynomial coefficients in the torsion and curvature tensors of the Biquard connection.

6.3.1 QC normal coordinates

We recall the construction of the qc normal coordinates from [73].

Theorem 6.3.1 ([73])**.** *Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a quaternionic contact manifold with Biquard connection ∇ and the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Let $q \in M$ and $X_q + V_q \in \mathcal{H}_q \oplus \mathcal{V}_q$. Consider the (geodesic) curve $\gamma_{X_q+V_q}$ starting at q and satisfying*

$$D_t^2 \dot{\gamma}_{X_q+V_q} = 0, \;\; \dot{\gamma}_{X_q+V_q}(0) = X_q \text{ and } D_t \dot{\gamma}_{X_q+V_q}(0) = V_q.$$

Then there are neighborhoods $0 \in U \subset T_qM$ and $q \in U_M \subset M$ such that the function

$$\begin{aligned} \psi : U &\longrightarrow U_M \\ X_q + V_q &\longmapsto \gamma_{X_q+V_q}(1) \end{aligned}$$

is a diffeomorphism. Furthermore, the following scaling by t holds:

$$\psi(tX_q + t^2 V_q) = \gamma_{X_q+V_q}(t)$$

whenever either side is defined. Here D_t denotes the covariant derivative along $\gamma_{X_q+V_q}$.

Now, let $\{V_1, V_2, V_3\}$ be an oriented basis of $\mathcal{V}_q$ and I_1, I_2, I_3 be the associated almost complex structures at q. Choose an $Sp(n)Sp(1)$-frame $\{X_1, \cdots, X_{4n}\}$ of $\mathcal{H}_q$. Extending these vectors to be parallel along the geodesics (in the sense of the above theorem) starting at q, one obtains a smooth local frame of $TM = \mathcal{H} \oplus \mathcal{V}$. Let us consider the dual frame $\{\theta_1, \cdots, \theta_{4n}, \eta_1, \eta_2, \eta_3\}$ of the frame $\{X_1, \cdots, X_{4n}, V_1, V_2, V_3\}$. Furthermore, extend the almost complex structures by defining

$$I_i X_{4k+1} = X_{4k+i+1} \text{ for } k = 0, \cdots, n-1 \text{ and } i = 1, 2, 3.$$

Then one obtains the so-called special frame and co-frame at q (s. [73]). Now, the qc normal coordinates (x_α, z_i) at q are defined by composing the inverse of the map ψ with the map

$$\begin{aligned} \lambda : T_qM &\longrightarrow \mathbb{R}^{4n+3} \\ X &\longmapsto (\theta_1(X), \cdots, \theta_{4n}(X), \eta_1(X), \eta_2(X), \eta_3(X))^t. \end{aligned}$$

From now on, the torsion and curvature tensors T, R are considered with respect to this special frame, co-frame and 1-forms η_1, η_2, η_3. As we will see below, the qc normal coordinates at q are privileged coordinates at q, when the qc manifold M is considered as a SR manifold.
The most important fact is that the infinitesimal generator

$$P = \sum_{\alpha=1}^{4n} x_\alpha \frac{\partial}{\partial x_\alpha} + 2\sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i}$$

of the action

$$\begin{aligned} \delta_t : \mathbb{R}^{4n+3} &\longrightarrow \mathbb{R}^{4n+3} \\ (x, z) &\longmapsto (tx, t^2 z) \end{aligned}$$

can be expressed (in qc normal coordinates) in terms of the special frame $\{X_\alpha, V_i\}$ in the form

$$P = \sum_{\alpha=1}^{4n} x_\alpha X_\alpha + \sum_{i=1}^{3} z_i V_i.$$

Using this fact, the following expression of the homogeneous terms of the special co-frame in qc normal coordinates was obtained in [73] (we omit the summation signs for repeated indices) :

Proposition 6.3.2. *In the qc normal coordinates centered at $q \in M$, the low order homogeneous terms of the special co-frame and connection 1-forms ω_{ab} are:*

1. $\eta_i^{(2)} = \frac{1}{2} dz_i - I^i_{\alpha\beta} x_\alpha dx_\beta$, $\eta_i^{(3)} = 0$ *and*

$$\eta_i^{(4)} = \frac{1}{4}\left(z_j \omega^{(2)}_{\bar{j}\,\bar{i}} + T^{\bar{i}}_{\bar{j}\,\bar{k}}(q) z_j \eta_k^{(2)} - 2 I^i_{\alpha\beta} x_\alpha \theta^{(3)}_\beta \right).$$

2. $\theta_\alpha^{(1)} = dx_\alpha, \;\; \theta_\alpha^{(2)} = 0$ *and*

$$\theta_\alpha^{(3)} = \frac{1}{3}\left(x_\beta \omega_{\beta\alpha}^{(2)} - T_{\bar{i}\gamma}^{\alpha}(q) x_\gamma \eta_i^{(2)} + T_{\bar{i}\beta}^{\alpha}(q) z_i \theta_\beta^{(1)} \right).$$

3. $\omega_{ab}^{(1)} = 0$ *and* $\omega_{ab}^{(2)} = \frac{1}{2} R_{\alpha\beta a}^{b}(q) x_\alpha \theta_\beta^{(1)}$.

Here the connection 1-forms ω_{ab} are defined by

$$\nabla X_\alpha = \omega_{\alpha\beta} \otimes X_\beta, \; \nabla V_i = \omega_{\bar{i}\bar{j}} \otimes V_j \text{ and } \omega_{\alpha\bar{i}} = \omega_{\bar{i}\alpha} = 0.$$

Let us consider the anisotropic expansion of the special frame $\{X_1, \cdots, X_{4n}, V_1, V_2, V_3\}$ in the qc normal coordinates (x_α, z_i) near 0:

$$\begin{aligned} X_\alpha &= X_\alpha^{(-1)} + X_\alpha^{(0)} + X_\alpha^{(1)} + \cdots \\ V_i &= V_i^{(-2)} + V_i^{(-1)} + V_i^{(0)} + \cdots . \end{aligned}$$

It was shown in [73] that the homogeneous terms of the lowest order are given by:

$$X_\alpha^{(-1)} = \frac{\partial}{\partial x_\alpha} + 2 \sum_{\beta,i} I_{\beta\alpha}^{i} x_\beta \frac{\partial}{\partial z_i} \quad \text{for } \alpha = 1, \cdots, 4n$$

and

$$V_i^{(-2)} = 2\frac{\partial}{\partial z_i} \quad \text{for } i = 1, 2, 3.$$

Using Proposition 6.3.2, we compute now the homogeneous terms in the expansion of the special frame that we need in the computation of the second heat invariant.
Note that the left-invariant vector fields

$$\{X_1^{(-1)}, \cdots, X_{4n}^{(-1)}, V_1^{(-2)}, V_2^{(-2)}, V_3^{(-2)}\}$$

on $\mathbb{R}^{4n+3}$ are linearly independent and therefore, every vector field on $\mathbb{R}^{4n+3}$ can be expressed as linear combination of these vector fields.

In the following we set:

$$\widetilde{X}_\alpha := X_\alpha^{(-1)} \text{ and } \widetilde{V}_i := V_i^{(-2)}$$

for $\alpha = 1, \cdots, 4n$ and $i = 1, 2, 3$.
The following properties about the order of tensor fields can be proved using (2.1) (s. [6]).

Lemma 6.3.3. *The following hold (whenever it makes sense):*

1. *Let X and Y be homogeneous vector fields. Then $[X, Y]$ is homogeneous of order $ord(X) + ord(Y)$.*

2. *let X resp. ω be a homogeneous vector field resp. 1-form. Then $\omega(X)$ is homogeneous of order $ord(\omega) + ord(X)$.*

3. *Let f resp. X be a homogeneous function resp. vector field. Then fX is homogeneous of order $ord(f) + ord(X)$.*

Using Lemma 6.3.3 we obtain:

Lemma 6.3.4. *In qc normal coordinates centered at q we have:*

1. *For $\alpha = 1, \cdots, 4n$, it holds:*
$$X_\alpha^{(0)} = 0 \text{ and } X_\alpha^{(1)} = \sum_\beta s_\alpha^{\beta(2)} \widetilde{X}_\beta + \sum_j r_\alpha^{\bar{j}(3)} \widetilde{V}_j$$
with
$$s_\alpha^{\beta(2)} = -\frac{1}{6} \sum_{\gamma,\delta} R^\beta_{\gamma\alpha\delta}(q) x_\gamma x_\delta - \frac{1}{3} \sum_i T^\beta_{\bar{i}\alpha}(q) z_i$$
and
$$r_\alpha^{\bar{j}(3)} = -\frac{1}{8} \sum_{\gamma,i} R^{\bar{j}}_{\gamma\alpha\bar{i}}(q) x_\gamma z_i + \frac{1}{2} \sum_{\gamma',\delta'} I^j_{\gamma'\delta'} x_{\gamma'} \left(\frac{1}{6} \sum_{\gamma,\delta} R^{\delta'}_{\delta\alpha\gamma}(q) x_\gamma x_\delta + \frac{1}{3} \sum_k T^{\delta'}_{\bar{k}\alpha}(q) z_k \right).$$

2. *For $i = 1, 2, 3$, it holds:*
$$V_i^{(-1)} = 0 \text{ and } V_i^{(0)} = \sum_\beta s_{\bar{i}}^{\beta(1)} \widetilde{X}_\beta + \sum_j r_{\bar{i}}^{\bar{j}(2)} \widetilde{V}_j$$
with
$$s_{\bar{i}}^{\beta(1)} = \frac{1}{3} \sum_\gamma T^\beta_{\bar{i}\gamma}(q) x_\gamma$$
and
$$r_{\bar{i}}^{\bar{j}(2)} = -\frac{1}{4} \sum_k T^{\bar{j}}_{\bar{k}\bar{i}}(q) z_k - \frac{1}{6} \sum_{\gamma,\delta,\delta'} I^j_{\gamma\delta} T^\delta_{\bar{i}\delta'}(q) x_\gamma x_{\delta'}.$$

Proof. 1. We can write locally near 0:
$$X_\alpha = \sum_\beta s_\alpha^\beta \widetilde{X}_\beta + \sum_j r_\alpha^{\bar{j}} \widetilde{V}_j \tag{6.6}$$
for some smooth functions s_α^β and $r_\alpha^{\bar{j}}$. Let us consider the anisotropic expansion of these functions at 0:
$$s_\alpha^\beta = s_\alpha^{\beta(0)} + s_\alpha^{\beta(1)} + s_\alpha^{\beta(2)} + \cdots$$
$$r_\alpha^{\bar{j}} = r_\alpha^{\bar{j}(0)} + r_\alpha^{\bar{j}(1)} + r_\alpha^{\bar{j}(2)} + \cdots.$$

Here $s_\alpha^{\beta(l)}$ (resp. $r_\alpha^{\bar{j}(l)}$) (for $l \geq 0$) denotes the homogeneous term of order l in the expansion of s_α^β (resp. $r_\alpha^{\bar{j}}$).

Now, applying θ_γ (resp. η_i) to both sides of (6.6) and taking the homogeneous parts of order l (see Lemma 6.3.3), we obtain the following recursive formulas for $s_\alpha^{\gamma(l)}$ and $r_\alpha^{\bar{i}(l)}$ (for $l \geq 1$):

$$\begin{aligned} s_\alpha^{\gamma(l)} &= -\sum_{m=0}^{l-1}\sum_\beta s_\alpha^{\beta(m)}\theta_\gamma^{(l-m+1)}(\widetilde{X}_\beta) - \sum_{m=0}^{l}\sum_j r_\alpha^{\bar{j}(m)}\theta_\gamma^{(l-m+2)}(\widetilde{V}_j) \\ r_\alpha^{\bar{i}(l)} &= -\sum_{m=0}^{l-1}\sum_\beta s_\alpha^{\beta(m)}\eta_i^{(l-m+1)}(\widetilde{X}_\beta) - \sum_{m=0}^{l-1}\sum_j r_\alpha^{\bar{j}(m)}\eta_i^{(l-m+2)}(\widetilde{V}_j) \end{aligned}$$

with initial terms $s_\alpha^{\gamma(0)} = \delta_{\alpha\gamma}$ and $r_\alpha^{\bar{i}(0)} = 0$.

Combining these recursive formulas with Proposition 6.3.2 we find

$$s_\alpha^{\beta(1)} = 0 \text{ and } r_\alpha^{\bar{j}(0)} = r_\alpha^{\bar{j}(1)} = r_\alpha^{\bar{j}(2)} = 0.$$

Hence, it follows that

$$s_\alpha^{\beta(2)} = -\theta_\beta^{(3)}(\widetilde{X}_\alpha) = -\frac{1}{6}\sum_{\gamma,\delta} R^\beta_{\gamma\alpha\delta}(q)x_\gamma x_\delta - \frac{1}{3}\sum_i T^\beta_{\bar{i}\alpha}(q)z_i,$$

and

$$\begin{aligned} r_\alpha^{\bar{j}(3)} &= -\eta_j^{(4)}(X_\alpha) \\ &= -\frac{1}{8}\sum_{\gamma,i} R^{\bar{j}}_{\gamma\alpha\bar{i}}(q)x_\gamma z_i + \frac{1}{2}\sum_{\gamma',\delta'} I^j_{\gamma'\delta'}x_{\gamma'}\left(\frac{1}{6}\sum_{\gamma,\delta} R^{\delta'}_{\delta\alpha\gamma}(q)x_\gamma x_\delta + \frac{1}{3}\sum_k T^{\delta'}_{\bar{k}\alpha}(q)z_k\right). \end{aligned}$$

2. The proof is similar to (1) and is left to the reader.

□

Remark 6.3.5. In qc normal coordinates at q, the homogeneous terms of order 0 of the special frame vanishes, i.e.

$$X_\alpha^{(0)} = 0 \text{ for all } \alpha = 1, \cdots, 4n. \tag{6.7}$$

The same holds for the privileged coordinates considered in [7] for $3D$ contact manifolds and in [69] for CR manifolds. In comparison, in [112] a system of privileged coordinates for Riemannian contact manifolds was constructed using similar techniques as in [69]. But therein, the obstruction for the considered special frame to have the property (6.7) is the obstruction for the Riemannian manifold to be a CR manifold. It is natural to ask in general, if there is some relation between the construction of privileged coordinates for which the property (6.7) holds (if it is possible) and the geometry of the manifold.

6.3.2 First heat invariant

In the following, we assume that the qc manifold M is complete. For every $q \in M$, the tangent group $\mathbb{G}(q)$ of the subriemannian manifold M at q is isomorphic to the unique connected, simply connected, step two nilpotent Lie group associated to the Lie algebra generated by the vector fields

$$\{\widetilde{X}_1, \cdots, \widetilde{X}_{4n}\}.$$

Using global exponential coordinates, the group law on $\mathbb{G}(q) \simeq \mathbb{R}^{4n+3}$ is given by

$$(x,z) * (x',z') = (x'', z''),$$

for $(x,z), (x',z') \in \mathbb{R}^{4n+3}$ with

$$x''_\alpha = x_\alpha + x'_\alpha \text{ and } z''_i = z_i + z'_i + 2 \sum_{\alpha,\beta=1}^{4n} I^i_{\alpha\beta} x_\alpha x'_\beta.$$

According to (2.6), in order to compute the first heat invariant we need the nilpotentization of the Popp measure and the heat kernel of the sublaplacian on the tangent group with respect to this measure.

By definition and according to Proposition 6.3.2, the nilpotentization of the Popp measure $\mathcal{P}$ at q is the Haar measure $\widetilde{\mathcal{P}}_q$ on $\mathbb{G}(q)$ given by

$$\begin{aligned}\widetilde{\mathcal{P}}_q &= \frac{1}{(16n)^{3/2}} \theta_1^{(1)} \wedge \cdots \wedge \theta_{4n}^{(1)} \wedge \eta_1^{(2)} \wedge \eta_2^{(2)} \wedge \eta_3^{(2)} \\ &= \frac{1}{8(16n)^{3/2}} dx_1 \wedge \cdots \wedge dx_{4n} \wedge dz_1 \wedge dz_2 \wedge dz_3.\end{aligned}$$

Furthermore, the heat kernel $K_t^{\mathbb{G}(q)}$ of the sublaplacian

$$\widetilde{\Delta}_{sub} := -\sum_{\alpha=1}^{4n} \widetilde{X}_\alpha^2$$

on $\mathbb{G}(q)$ with respect to the Haar measure $\widetilde{\mathcal{P}}_q$ is explicitly obtained by the *Beals-Gaveau-Greiner formula* for the sublaplacian on general step two nilpotent Lie groups, which we recall next. For $h, h' \in \mathbb{G}(q)$ it holds:

$$K_t^{\mathbb{G}(q)}(h,h') = \frac{8(16n)^{3/2}}{(4\pi t)^{2n+3}} \int_{\mathbb{R}^3} e^{-\frac{\varphi(\tau,\, h^{-1} * h')}{2t}} W(\tau)\, d\tau, \tag{6.8}$$

where the *action function* $\varphi = \varphi(\tau, h) \in C^\infty(\mathbb{R}^3 \times \mathbb{G}(q))$ and the *volume element* $W(\tau) \in C^\infty(\mathbb{R}^3)$ are given as follows: Put $h = (x,z) \in \mathbb{R}^{4n} \times \mathbb{R}^3$, then

$$\varphi(\tau, h) = \varphi(\tau, x, z) = \sqrt{-1}\langle \tau,\, z\rangle + \frac{1}{2}\Big\langle \sqrt{-1}\Omega_\tau \coth\big(\sqrt{-1}\Omega_\tau\big) \cdot x, x\Big\rangle,$$

$$W(\tau) = \left\{ \det \frac{\sqrt{-1}\Omega_\tau}{\sinh \sqrt{-1}\Omega_\tau} \right\}^{1/2},$$

where $\langle \tau, \tau' \rangle = \sum_{i=1}^{3} \tau_i \tau_i'$ denotes the Euclidean inner product on $\mathbb{R}^3$ and the matrix Ω_τ encodes the structure constants of the Lie algebra and is given by

$$\Omega_\tau = 2 \sum_{i=1}^{3} \tau_i (I^i_{\alpha\beta})_{\alpha,\beta} \in \mathbb{R}^{4n \times 4n}.$$

Using the relations (6.2), we see that the eigenvalues of $\sqrt{-1}\Omega_\tau$ are $\pm 2\|\tau\|$. Hence the functions $\varphi(\tau, h)$ and $W(\tau)$ take the forms

$$\begin{aligned} \varphi(\tau, h) &= \varphi(\tau, x, z) = \sqrt{-1}\langle \tau,\ z \rangle + \frac{1}{2} \left(\|2\tau\| \coth \|2\tau\| \right) \|x\|^2, \\ W(\tau) &= \left(\frac{\|2\tau\|}{\sinh \|2\tau\|} \right)^{2n}. \end{aligned}$$

Recalling that the first heat invariant is given by (see (2.6))

$$c_0(q) = K_t^{\mathbb{G}(q)}(0, 0),$$

we have the following formula:

Theorem 6.3.6. *The first heat invariant $c_0(q)$ of the complete qc manifold M is independent of the point $q \in M$ and is given by*

$$c_0(q) = \frac{(16n)^{3/2}}{(4\pi)^{2n+3}} \int_{\mathbb{R}^3} \left(\frac{\|\tau\|}{\sinh \|\tau\|} \right)^{2n} d\tau.$$

Remark 6.3.7. The expression for c_0 obtained in [15] for the case where $M = \mathbb{S}^{4n+3}$ with the standard qc structure differs from our formula by the factor $(16n)^{3/2}$. This is due to the fact that our sublaplacian is defined with respect to the Popp measure, which differs from the Riemannian measure on $\mathbb{S}^{4n+3}$ (with respect to the standard metric) by the factor $\frac{1}{(16n)^{3/2}}$ (see Lemma 6.2.1).

6.3.3 Second heat invariant

In general, computing the remaining heat invariants $c_1, c_2, \cdots$ with the help of the nilpotentization is rather complicated. However, using the qc normal coordinates on a qc manifold, it is possible to identify the second heat invariant $c_1(q)$ with a geometric invariant associated to the considered qc structure and this is the main goal of the present section.

For this, by (2.7), Lemma 6.2.2 and Lemma 6.3.4, additionally we need the homogeneous term $Y^{(0)}$ in the expansion of the horizontal vector field

$$Y := \sum_{\alpha=1}^{4n} \left(\sum_{a=1}^{4n+3} c_{a\alpha}^{a} \right) X_\alpha.$$

For this, we consider the asymptotic expansion of the vector field $\epsilon Y^\epsilon = \epsilon^2 \delta_\epsilon^*(Y)$ as $\epsilon \to 0$:

$$\epsilon Y^\epsilon = \epsilon Y^{(-1)} + \epsilon^2 Y^{(0)} + \cdots$$

A straightforward calculation shows that ϵY^ϵ can be expressed in the form

$$\epsilon Y^\epsilon = \sum_{\alpha=1}^{4n} \left(\sum_{a=1}^{4n+3} c_{a\alpha}^{a}(\epsilon) \right) X_\alpha^\epsilon,$$

where $c_{a\alpha}^{a}(\epsilon)$ is defined by

$$c_{a\alpha}^{a}(\epsilon) := \theta_a^\epsilon([X_a^\epsilon, X_\alpha^\epsilon])$$

with $\theta_a^\epsilon := \frac{1}{\epsilon}\delta_\epsilon^*(\theta_a)$ for $a = 1, \cdots, 4n$ and $\theta_a^\epsilon := \frac{1}{\epsilon^2}\delta_\epsilon^*(\theta_a)$ for $a = 4n+1, 4n+2, 4n+3$. Therefore, using Lemma 6.3.3 it is sufficient to consider the asymptotic expansions of X_α^ϵ and $\sum_{a=1}^{4n+3} c_{a\alpha}^{a}(\epsilon)$.

Lemma 6.3.8. *Let $\alpha = 1, \cdots, 4n$. As $\epsilon \to 0$, it holds:*

$$\sum_{a=1}^{4n+3} c_{a\alpha}^{a}(\epsilon) = \epsilon^2 \left(\sum_{\beta=1}^{4n} \widetilde{X}_\beta(s_\alpha^{\beta(2)}) - \widetilde{X}_\alpha(s_\beta^{\beta(2)}) + \sum_{i=1}^{3} \widetilde{V}_i(r_\alpha^{\bar{i}(3)}) - \widetilde{X}_\alpha(r_{\bar{i}}^{\bar{i}(2)}) \right) + O(\epsilon^3).$$

Hence, the homogeneous term $Y^{(0)}$ can be expressed as

$$Y^{(0)} = \sum_{\alpha=1}^{4n} \left(\sum_{\beta=1}^{4n} \widetilde{X}_\beta(s_\alpha^{\beta(2)}) - \widetilde{X}_\alpha(s_\beta^{\beta(2)}) + \sum_{i=1}^{3} \widetilde{V}_i(r_\alpha^{\bar{i}(3)}) - \widetilde{X}_\alpha(r_{\bar{i}}^{\bar{i}(2)}) \right) \widetilde{X}_\alpha.$$

Proof. By Proposition 6.3.2 and Lemma 6.3.4, as $\epsilon \to 0$ it holds:

$$X_\gamma^\epsilon = \widetilde{X}_\gamma + \epsilon^2 X_\gamma^{(1)} + O(\epsilon^3)$$

and

$$\theta_\gamma^\epsilon = \theta_\gamma^{(1)} + \epsilon^2 \theta_\gamma^{(3)} + O(\epsilon^3)$$

for $\gamma = 1, \cdots, 4n$. Hence, we can write

$$\begin{aligned} c_{\beta\alpha}^{\beta}(\epsilon) &= \theta_\beta^\epsilon([X_\beta^\epsilon, X_\alpha^\epsilon]) \\ &= \left(\theta_\beta^{(1)} + \epsilon^2\theta_\beta^{(3)} + O(\epsilon^3)\right) \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] + \epsilon^2[\widetilde{X}_\beta, X_\alpha^{(1)}] + \epsilon^2[X_\beta^{(1)}, \widetilde{X}_\alpha] + O(\epsilon^4)\right) \\ &= \theta_\beta^{(1)}([\widetilde{X}_\beta, \widetilde{X}_\alpha]) + \epsilon^2 \left(\theta_\beta^{(1)} \left([\widetilde{X}_\beta, X_\alpha^{(1)}] + [X_\beta^{(1)}, \widetilde{X}_\alpha]\right) + \theta_\beta^{(3)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha]\right)\right) + O(\epsilon^3). \end{aligned}$$

Using the fact that $\theta_\beta^{(1)} = dx_\beta$ and that the Lie brackets of two horizontal left-invariant vector fields is a vertical vector field, i.e.

$$\theta_\beta^{(1)}([\widetilde{X}_\beta, \widetilde{X}_\alpha]) = dx_\beta \left(\sum_{i=1}^{3} 4I_{\beta\alpha}^i \frac{\partial}{\partial z_i} \right) = 0,$$

it follows that

$$c_{\beta\alpha}^\beta(\epsilon) = \epsilon^2 \left(\theta_\beta^{(1)} \left([\widetilde{X}_\beta, X_\alpha^{(1)}] + [X_\beta^{(1)}, \widetilde{X}_\alpha] \right) + \theta_\beta^{(3)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] \right) \right) + O(\epsilon^3). \tag{6.9}$$

We recall that by Lemma 6.3.4, it holds for $\delta = 1, \cdots, 4n$:

$$X_\delta^{(1)} = \sum_\gamma s_\delta^{\gamma(2)} \widetilde{X}_\gamma + \sum_j r_\delta^{\bar{j}(3)} \widetilde{V}_j.$$

Inserting the last identity into Equation (6.9) and using the fact that $\theta_\beta^{(1)}(\widetilde{V}_i) = 0$ for all i and β, we obtain

$$c_{\beta\alpha}^\beta(\epsilon) = \epsilon^2 \left(\widetilde{X}_\beta(s_\alpha^{\beta(2)}) - \widetilde{X}_\alpha(s_\beta^{\beta(2)}) + \theta_\beta^{(3)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] \right) \right) + O(\epsilon^3). \tag{6.10}$$

In the same way, we can prove that for $i = 1, 2, 3$:

$$c_{\bar{i}\alpha}^{\bar{i}}(\epsilon) = \epsilon^2 \left(\widetilde{V}_i(r_\alpha^{\bar{i}(3)}) - \widetilde{X}_\alpha(r_{\bar{i}}^{\bar{i}(2)}) + \sum_{\beta=1}^{4n} s_{\bar{i}}^{\beta(1)} \eta_i^{(2)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] \right) \right) + O(\epsilon^3). \tag{6.11}$$

Again by using Proposition 6.3.2 and Lemma 6.3.4, a straightforward calculation shows that

$$\sum_{\beta=1}^{4n} \theta_\beta^{(3)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] \right) = -\sum_{\beta=1}^{4n} \sum_{i=1}^{3} s_{\bar{i}}^{\beta(1)} \eta_i^{(2)} \left([\widetilde{X}_\beta, \widetilde{X}_\alpha] \right).$$

Now the assertion follows by adding Equations (6.10) and (6.11). □

Using Lemma 6.3.8 and Lemma 6.2.2 we can write

$$\Delta_{sub}^\epsilon = \widetilde{\Delta}_{sub} + \epsilon^2 \mathcal{P}_2 + \epsilon^3 \mathcal{R}(\epsilon),$$

where $\mathcal{P}_2$ and $\mathcal{R}(\epsilon)$ are second-order differential operators with (we omit the summation signs for repeated indices)

$$\mathcal{P}_2 := \widetilde{X}_\alpha X_\alpha^{(1)} + X_\alpha^{(1)} \widetilde{X}_\alpha + \left(\widetilde{X}_\beta(s_\alpha^{\beta(2)}) - \widetilde{X}_\alpha(s_\beta^{\beta(2)}) + \widetilde{V}_i(r_\alpha^{\bar{i}(3)}) - \widetilde{X}_\alpha(r_{\bar{i}}^{\bar{i}(2)}) \right) \widetilde{X}_\alpha. \tag{6.12}$$

We recall that the second heat invariant $c_1(q)$ can be expressed in the from (see (2.7))

$$c_1(q) = \int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0, (x, z)) \mathcal{P}_2 \left(K_s^{\mathbb{G}(q)}((x, z), 0) \right) dxdzds. \tag{6.13}$$

Note also that the function

$$K_t^{\mathbb{G}(q)}(0,(x,z)) = \frac{(16n)^{3/2}}{(4\pi t)^{2n+3}} \int_{\mathbb{R}^3} e^{-(\sqrt{-1}\langle \tau, z\rangle + \frac{1}{2}\|\tau\| \coth(\|\tau\|)\|x\|^2)/2t} \left(\frac{\|\tau\|}{\sinh \|\tau\|} \right)^{2n} d\tau$$

is invariant under the action of the orthogonal group $\mathbf{O}(4n)$ (resp. $\mathbf{O}(3)$) on the variable x (resp. z).
According to (6.12), the second-order differential operator $\mathcal{P}_2$ can be completely expressed through the vector fields

$$\widetilde{X}_\alpha,\ X_\alpha^{(1)},\ \widetilde{V}_i$$

and the functions

$$s_\alpha^{\beta(2)}, r_\alpha^{\bar{i}(3)}, r_{\bar{i}}^{\bar{i}(2)}.$$

Furthermore, by Lemma 6.3.4 these data can be written in terms of the torsion and curvature tensors of the Biquard connection at q and the left-invariant vector fields $\widetilde{X}_\alpha, \widetilde{V}_i$. Hence, it follows from (6.13) that $c_1(q)$ can be expressed as a polynomial in the torsion and curvature tensors at q. In fact, using the $\mathbf{O}(4n) \times \mathbf{O}(3)$-invariance of $K_t^{\mathbb{G}(q)}$ and the identities from Proposition 6.1.4 , more can be proved:

Theorem 6.3.9. *Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complete quaternionic contact manifold and let $q \in M$. Then it holds:*

$$K_t(q,q) = \frac{1}{t^{2n+3}} \left(c_0 + C_n \kappa(q) t + o(t) \right) \ \textit{as } t \to 0.$$

Here $\kappa(q)$ denotes the qc scalar curvature of the Biquard connection (see Definition 6.1.3) and C_n is a universal constant depending only on n and independent of the qc manifold M.

Proof. We use the symmetries of the heat kernel $K_t^{\mathbb{G}(q)}$ on the quaternionic Heisenberg group and standard identities of torsion and curvature tensors like the ones in Proposition 6.1.4 to obtain our result. In the following we omit the summation signs for repeated indices to simplify the notations. We set

$$\mathcal{P}_2 = \mathcal{P}_{21} + \mathcal{P}_{22},$$

where

$$\begin{aligned} \mathcal{P}_{21} :&= \widetilde{X}_\alpha X_\alpha^{(1)} + X_\alpha^{(1)} \widetilde{X}_\alpha \\ &= \widetilde{X}_\alpha(s_\alpha^{\beta(2)})\widetilde{X}_\beta + \widetilde{X}_\alpha(r_\alpha^{\bar{j}(3)})\widetilde{V}_j + 2r_\alpha^{\bar{j}(3)}\widetilde{X}_\alpha\widetilde{V}_j + s_\alpha^{\beta(2)}\left(\widetilde{X}_\beta\widetilde{X}_\alpha + \widetilde{X}_\alpha\widetilde{X}_\beta\right). \end{aligned}$$

and

$$\mathcal{P}_{22} := \left(\widetilde{X}_\beta(s_\alpha^{\beta(2)}) - \widetilde{X}_\alpha(s_\beta^{\beta(2)}) + \widetilde{V}_i(r_\alpha^{\bar{i}(3)}) - \widetilde{X}_\alpha(r_{\bar{i}}^{\bar{i}(2)}) \right) \widetilde{X}_\alpha.$$

Using the $\mathbf{O}(4n) \times \mathbf{O}(3)$-invariance and the exponential decay of the heat kernel $K_t^{\mathbb{G}(q)}$ and parity arguments, we obtain the following formulas:

1. For $\alpha, \beta = 1, \cdots, 4n$ and $i = 1, 2, 3$:
$$\int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) x_\alpha x_\beta \frac{\partial}{\partial z_i} \left(K_s^{\mathbb{G}(q)}(s,(x,z),0) \right) dxdz = 0,$$

2. For $\alpha, \beta, \gamma, \delta = 1, \cdots, 4n$:
$$\int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) x_\alpha x_\beta \frac{\partial}{\partial x_\gamma} \frac{\partial}{\partial x_\delta} \left(K_s^{\mathbb{G}(q)}((x,z),0) \right) dxdz = 0,$$
unless one of the following cases holds:
$$\alpha = \beta \text{ and } \gamma = \delta, \;\; \alpha = \gamma \text{ and } \beta = \delta \text{ or } \alpha = \delta \text{ and } \beta = \gamma.$$

3. For $i = 1, 2, 3$:
$$\int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) \frac{\partial}{\partial z_i} \left(K_s^{\mathbb{G}(q)}((x,z),0) \right) dxdz = 0.$$

4. For $\alpha, \beta, \gamma, \delta = 1, \cdots, 4n$ and $i, j = 1, 2, 3$:
$$\int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) x_\alpha x_\beta x_\gamma x_\delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \left(K_s^{\mathbb{G}(q)}((x,z),0) \right) dxdz = 0,$$
unless $i = j$ and one of the following cases hold:
$$\alpha = \beta \text{ and } \gamma = \delta, \;\; \alpha = \gamma \text{ and } \beta = \delta \text{ or } \alpha = \delta \text{ and } \beta = \gamma.$$
In each case, the above integral does not depend on $i = j$ and $\alpha, \beta, \gamma, \delta$.

To prove the theorem, we need to treat every term in the expression of $\mathcal{P}_2$. Since the calculations are similar, we only consider the last summation in the expression of $\mathcal{P}_{21}$:
$$s_\alpha^{\beta(2)} \widetilde{X}_\alpha \widetilde{X}_\beta.$$
Recalling that
$$s_\alpha^{\beta(2)} = -\frac{1}{6} \sum_{\gamma,\delta} R_{\gamma\alpha\delta}^{\beta}(q) x_\gamma x_\delta - \frac{1}{3} \sum_i T_{\bar{i}\alpha}^{\beta}(q) z_i,$$
and using the formulas cited above, we find that
$$\int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) R_{\gamma\alpha\delta}^{\beta}(q) x_\gamma x_\delta \widetilde{X}_\alpha \widetilde{X}_\beta \left(K_s^{\mathbb{G}(q)}((x,z),0) \right) dxdzds$$
is a linear combination of
$$R_{\alpha\beta\beta}^{\alpha}(q), I_{\alpha\beta}^{i} I_{\gamma\delta}^{i} R_{\alpha\beta\gamma}^{\delta}(q) \text{ and } I_{\alpha\beta}^{i} I_{\gamma\delta}^{i} R_{\gamma\alpha\beta}^{\delta}(q), \tag{6.14}$$

Furthermore, the second part of $s_\alpha^{\beta(2)}$ give us terms involving the torsion at q. More precisely, there is some constant C depending only on n such that

$$\int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) T_{\bar{i}\alpha}^{\beta}(q) z_i \widetilde{X}_\alpha \widetilde{X}_\beta \left(K_s^{\mathbb{G}(q)}((x,z),0)\right) dxdzds = C \cdot I_{\alpha\beta}^i T_{\bar{i}\alpha}^{\beta}(q).$$

Therefore, the expression

$$\int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) s_\alpha^{\beta(2)} \widetilde{X}_\alpha \widetilde{X}_\beta \left(K_s^{\mathbb{G}(q)}((x,z),0)\right) dxdzds \tag{6.15}$$

is reduced to a linear combination of

$$R_{\alpha\beta\beta}^{\alpha}(q), I_{\alpha\beta}^i T_{\bar{j}\alpha}^{\beta}(q), I_{\alpha\beta}^i I_{\gamma\delta}^i R_{\alpha\beta\gamma}^{\delta}(q) \text{ and } I_{\alpha\beta}^i I_{\gamma\delta}^i R_{\gamma\alpha\beta}^{\delta}(q). \tag{6.16}$$

The first term of (6.16) is the qc scalar curvature at q: $R_{\alpha\beta\beta}^{\alpha}(q) = \kappa(q)$. Furthermore, using Proposition 6.1.4 and the fact that $I_i X_\alpha = I_{\beta\alpha}^i X_\beta$, the remaining terms of (6.16) can be interpreted as follows:

$$\begin{aligned}
I_{\alpha\beta}^i T_{\bar{j}\alpha}^{\beta}(q) &= -g(T(V_j, X_\alpha), I_i X_\alpha) = 0,\\
I_{\alpha\beta}^i I_{\gamma\delta}^i R_{\gamma\alpha\beta}^{\delta}(q) &= -g(R(X_\beta, I_i X_\alpha) X_\alpha, I_i X_\beta) = \frac{n\kappa(q)}{n+2}, \text{ for all } i,\\
I_{\alpha\beta}^i I_{\gamma\delta}^i R_{\alpha\beta\gamma}^{\delta}(q) &= -g(R(X_\alpha, I_i X_\alpha) I_i X_\beta, X_\beta) = -\frac{2n\kappa(q)}{n+2}, \text{ for all } i.
\end{aligned}$$

It follows that each non-trivial term of (6.16) equals the qc scalar curvature at q up to a constant multiple. Hence (6.15) equals the qc scalar curvature at q up to a constant multiple.

Similar arguments as above show that

$$\begin{aligned}
\mathcal{I}_1(q) : &= \int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) \mathcal{P}_{21} \left(K_s^{\mathbb{G}(q)}((x,z),0)\right) dxdzds\\
&= C_1(n)\kappa(q),
\end{aligned}$$

where $C_1(n)$ is a constant depending only on n.

Again, by using the formulas cited above, we can show that

$$\mathcal{I}_2(q) := \int_0^1 \int_{\mathbb{R}^{4n+3}} K_{1-s}^{\mathbb{G}(q)}(0,(x,z)) \mathcal{P}_{22} \left(K_s^{\mathbb{G}(q)}((x,z),0)\right) dxdzds$$

is a linear combination of the following terms:

$$R_{\alpha\beta\beta}^{\alpha}(q), I_{\alpha\beta}^i T_{\bar{j}\alpha}^{\beta}(q) \text{ and } T_{\bar{j}\,\bar{i}}^{\bar{i}}(q).$$

Using Proposition 6.1.4 shows that the second term $I_{\alpha\beta}^i T_{\bar{j}\alpha}^{\beta}(q)$ and the last term $T_{\bar{j}\,\bar{i}}^{\bar{i}}(q)$ vanish. Therefore, we conclude that the second heat invariant $c_1(q)$ equals the curvature at q up to a constant multiple depending only on n. $\square$

Remark 6.3.10. In the case of a SR structure associated to a CR structure with a Levi metric (s. [28]), the problem of computing the second heat invariant is reduced to classical $\mathbf{U}(n)$-invariant theory for linear functionals on the space of mixed tensors. On the other side, our structure has two symmetries, the first one being the $Sp(n)Sp(1)$-action on the $Sp(n)Sp(1)$-frame $\{X_\alpha\}$ and the second one the $SO(3)$-action on the vertical frame $\{V_i\}$. That is, since $c_1(q)$ depends only on the point q and not on the choice of the special frame $\{X_\alpha, V_i\}$, the expression of $c_1(q)$ which involves the torsion and curvature tensors at q must be invariant under the action of these groups. Therefore, we could use these symmetries to find the explicit dependence of $c_1(q)$ on the torsion and curvature tensors at q like in the CR case or the Riemannian case. However, we do not know if similar results of $\mathbf{U}(n)$-invariant theory holds for the group $Sp(n)Sp(1)$.

For $M = \mathbb{S}^{4n+3}$ with the standard qc structure, the second coefficient c_1 was computed in [15]:

$$c_1(q) = \frac{1}{(4\pi)^{2n+2}} \int_0^\infty \frac{y^{2n+2}}{(\sinh y)^{2n}} \left((2n+1)^2 - \frac{2n(2n+1)(\sinh y - y\cosh y)}{y^2 \sinh y} \right) dy,$$

for $q \in \mathbb{S}^{4n+3}$. Furthermore, the qc curvature $\kappa_{\mathbb{S}^{4n+3}}$ of the sphere $\mathbb{S}^{4n+3}$ is constant with value $16n(n+2)$ (see [65]). Hence we deduce the value of the universal constant C_n in Theorem 6.3.9:

$$C_n = \frac{1}{16(n^2+2n)(4\pi)^{2n+2}} \int_0^\infty \frac{y^{2n+2}}{(\sinh y)^{2n}} \left((2n+1)^2 - \frac{2n(2n+1)(\sinh y - y\cosh y)}{y^2 \sinh y} \right) dy.$$

In the following we give an application of Theorem 6.3.9 to the case of qc-Einstein compact manifolds. We recall that a qc-Einstein manifold M is a qc manifold such that the torsion T_V vanishes identically on $\mathcal{H}$. In this case, it was proven in [65] that the qc scalar curvature κ of such a manifold is constant. We recall that if the qc manifold M is compact, then the sublaplacian Δ_{sub} (which is subelliptic due to the bracket generating property of the distribution $\mathcal{H}$) has compact resolvent and thus has discrete spectrum $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \cdots \to \infty$ only consisting of eigenvalues with finite multiplicities. Furthermore, $e^{-t\Delta_{sub}}$ is of trace class for every $t > 0$.
Now, let M and M' be two qc-Einstein compact manifolds with qc scalar curvatures κ and κ', respectively. Assume that M and M' are isospectral with respect to the intrinsic sublaplacians, i.e. the associated intrinsic sublaplacians Δ_{sub} and Δ'_{sub} have the same spectrum with the same multiplicities of eigenvalues. Hence we can write

$$\mathrm{tr}(e^{-t\Delta_{sub}}) = \mathrm{tr}(e^{-t\Delta'_{sub}}).$$

By Theorems 6.3.6 and 6.3.9 we obtain

$$\dim(M) = \dim(M'),\ \mathcal{P}(M) = \mathcal{P}'(M') \text{ and } \kappa = \kappa',$$

i.e. the dimension, the Popp volume and the qc scalar curvature of a qc-Einstein compact manifold are spectral invariants.

Corollary 6.3.11. *Let M and M' be two qc-Einstein compact manifolds, which are isospectral with respect to the intrinsic sublaplacians. Then they have the same dimension, Popp's volume and qc scalar curvature.*

6.4 Open problems and future works

Finally, we mention some open problems to be treated in the future.

1. Is there a relation between the Popp volume of small subriemannian balls and the qc scalar curvature on qc manifolds? We recall that in the Riemannian case, the asymptotic expansion of the Riemannian volume of $B(q, \epsilon)$ has an expression in terms of the volume v_n of the unit ball in $\mathbb{R}^n$ and the scalar cuvature $s(q)$ at q:

$$\operatorname{vol}(B(q,\epsilon)) = v_n \epsilon^n \left(1 - \frac{s(q)}{6(n+2)}\epsilon^2 + O(\epsilon^3)\right) \text{ as } \epsilon \to 0.$$

   Note also that a similar formula holds for 3-dimensional subriemannian contact manifolds (s. [8]).

2. There are another interesting geometric balls which are natural to consider. More precisely, we consider the ball $B(q, \epsilon)$ of radius ϵ centered at q defined as the image under the parabolic exponential map $\psi : T_qM \longrightarrow M$ (see Theorem 6.3.2) of the so-called *Koranyi ball*: $\{(x, z) \in \mathbb{R}^{4n+3} : (\|x\|^4 + \|z\|^2)^{\frac{1}{4}} < \epsilon\}$ on $\mathbb{R}^{4n+3}$. Here we conjecture that the second coefficient in the asymptotic expansion of the Popp volume of small balls equals the qc scalar curvature up to a (univeral) constant multiple. Note that a similar result holds for CR structures (s. [69]).

3. For contact Riemannian manifolds, a system of privileged coordinates was constructed in [112]. We should use these coordinates to express the second heat invariant of contact Riemannaian manifolds in terms of geometric data of the underlying structure.

4. Another interesting class of subriemannian manifolds are the so-called *H-type foliations* [14]. These are subriemannian structures where the tangent groups are H-type groups. Furthermore, there is a metric connection adapted to the structure called *Bott* connection. It would be interesting to know if we can give a geometric meaning to the second heat invariant in this setting.

Bibliography

[1] J.F. Adams, *Vector fields on spheres*, Ann. of Math. 75 (1962), 603-632.

[2] M.F. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology **3** (1964), Suppl. I, 3-38.

[3] A. Agrachev, D. Barilari, U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, Cambridge University press, 2019.

[4] A. Agrachev, U. Boscain, J.-P. Gauthier, F. Rossi, *The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups*. J. Funct. Anal. 256 (2009), no. 8, 2621 - 2655.

[5] A. Agrachev, EL-H. Chakir, J.-P. Gauthier, *SubRiemannian metrics on* $\mathbb{R}^3$, CMS Conf. Proc., 25, Amer. Math. Soc., Providence (1998), 29-78.

[6] A. Bellaïche, *The tangent space in sub-Riemannian geometry*, In Sub-Riemannian geometry, volume 144 of Progr. Math., pages 1-78. Birkhäuser, Basel, 1996.

[7] D. Barilari, *Trace heat kernel asymptotics in 3D contact sub-Riemannian geometry*, J. Math. Sci. (2013), 195(3), 391-411.

[8] D. Barilari, I. Beschastnyi, A. Lerario, *Volume of small balls and sub-Riemannian curvature in 3D contact manifolds*, arXiv:1802.10155, 2018.

[9] D. Barilari, U. Boscain, R. W. Neel, *Heat kernel asymptotics on sub-Riemannian manifolds with symmetries and applications to the bi-Heisenberg group*, Ann. Fac. Sci. Toulouse Math. Vol. 28, No. 4, 707-732 (2019).

[10] D. Barilari, S. Ivanov, *A Bonnet–Myers type theorem for quaternionic contact structures*, Calc. Var. Partial Differential Equations (2019), 58(1).

[11] D. Barilari, L. Rizzi, *A formula for Popp's volume in sub-Riemannian geometry*, Anal. Geom. Metr. Spaces (2013), 42-57.

[12] D. Barilari, L. Rizzi, *Sharp measure contraction property for generalized H-type Carnot groups*, Commun. Contemp. Math. 20.06, 2018.

[13] F. Baudoin, M. Bonnefont, *The subelliptic heat kernel on* SU(2)*: representations, asymptotics and gradient bounds*, Math. Z. 263 (2009), 647-672.

[14] F. Baudoin, E. Grong, G. Molino, L. Rizzi, *H-type foliations*, arXiv:1812.02563v1, 2018.

[15] F. Baudoin, J. Wang, *The subelliptic heat kernels of the quarternionic Hopf fibration*, Potential Anal. (2014), 41, 959-982.

[16] W. Bauer, K. Furutani, C. Iwasaki, *Spectral zeta function of the sublaplacian on two step nilmanifolds*, J. Math. Pures Appl. (9) 97 no. 3 (2012), 242-261.

[17] W. Bauer, K. Furutani, C. Iwasaki, *Spectral zeta function on pseudo H-type nilmanifolds*, Indian J. Pure Appl. Math. 46 no. 4 (2015), 539-582.

[18] W. Bauer, K. Furutani, C. Iwasaki, *Fundamental solution for a higher step Grushin type operator*, Adv. Math. 271 (2015), 188-234.

[19] W. Bauer, K. Furutani, C. Iwasaki, *Sub-Riemannian structures in a principal bundle and their Popp measures*, Appl. Anal. 96 (2017), no. 14, 2390-2407.

[20] W. Bauer, K. Furutani, C. Iwasaki, *A codimension 3 sub-Riemannian structure on the Gromoll-Meyer exotic sphere*, Differential Geom. Appl. 53 (2017), 114 - 136.

[21] W. Bauer, K. Furutani, C. Iwasaki, *Trivializable sub-Riemannian structures on spheres*, Bull. Sci. Math. 137 (2013), no. 3, 361-385.

[22] W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi, *Spectral theory of a class of nilmanifolds attached to Clifford modules*, Math. Z. (2021), 297(1), 557-583.

[23] W. Bauer, A. Laaroussi, *Trivializable and quaternionic subriemannian structure on $\mathbb{S}^7$ and subelliptic heat kernel*, arXiv:2102.04784, 2021.

[24] W. Bauer, A. Laaroussi, D. Tarama, *Rank* 5 *trivializable subriemannian structure on $\mathbb{S}^7$ and subelliptic heat kernel*, preprint.

[25] W. Bauer, D. Tarama, *Subriemannian geodesic flow on $\mathbb{S}^7$*, RIMS Proceedings *Kôkyûroku* 2137 of the workshop Symmetry and singularity of geometric structures and differential equations, ISSN 1880-2818, (2018) 42-59.

[26] R. Beals, B. Gaveau, P. Greiner, *The Green function of model step two hypoelliptic operators and the analysis of certain tangential Cauchy Riemann complexes*, Adv. Math. (1996), 121(2), 288-345.

[27] R. Beals, B. Gaveau, P. Greiner, *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups*, J. Math. Pures Appl. **79**(7) (2000), 633-689.

[28] R. Beals, P. C. Greiner, N. K. Stanton, *The heat equation on a CR manifold*, J. Differential Geom. (1984), 20(2), 343-387.

[29] G. Ben Arous, *Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale*, Ann. Inst. Fourier (Grenoble) (1989), 73-99.

[30] G. Ben Arous, R. Léandre, *Décraissance exponentielle du noyau de la chaleur sur la diagonale. I + II (French)*, Probab. Theory Related fields 90 (1991), no. 2, 175 - 202 and no. 3, 377-402.

[31] M. Berger, P. Gauduchon, E. Mazet, *Le spectre d'une variété Riemannienne*, Springer-Verlag, Berlin (1971).

[32] J. Berndt, L. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture notes in Math. Vol. 1598, Springer-Verlag (1995).

[33] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque (2000).

[34] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Science and Business Media 2007.

[35] B. Bonnard, M. Chyba, J. Rouot, D. Takagi, *Sub-Riemannian geometry, Hamiltonian dynamics, micro-swimmers, copepod nauplii and copepod robot*, Pac. J. Math. Ind. (2018), 10(1), 1-27.

[36] O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki, *Heat kernels for elliptic and sub-elliptic operators. Methods and techniques*, Birkhäuser Boston (2010).

[37] O. Calin, D.-C. Chang, and I. Markina, *Subriemannian geometry on the sphere S^3*, Canad. J. Math. 61 (2009), no. 4, 721 - 739.

[38] L. Capogna, E. Le Donne, *Smoothness of subRiemannian isometries*, Amer. J. Math. 138(5), 1439-1454 (2016).

[39] W. Choi, R. Ponge, *Privileged coordinates and nilpotent approximation of Carnot manifolds, I. General results.* J. Dyn. Control Syst. (2019), 25. Jg., Nr. 1, S. 109-157.

[40] W.L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. 117(1) (1939), 98 - 105.

[41] P. Ciatti, *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Ann. Mat. Pura Appl. (4) 178 (2000), 1-31.

[42] G. Citti, B. Franceschiello, G. Sanguinetti, A. Sarti, *Sub-Riemannian mean curvature flow for image processing*, SIAM J. Imaging Sci. 9(1), 212-237 (2016).

[43] G. Citti, A. Sarti, *A cortical based model of perceptual completion in the roto-translation space*, J. Math. Imaging Vis. 24(3), 307-326 (2006).

[44] L. Corwin, F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications: Volume 1, Part 1, Basic theory and examples*, Cambridge university press, 2004.

[45] G. Crandall, J. Dodziuk, *Integral structures on H-type Lie algebras*, J. Lie Theory 12 no. 1 (2002), 69-79.

[46] S. Dave, S. Haller, *The heat asymptotics on filtered manifolds*, J. Geom. Anal. 30 (2020), no. 1, 337-389.

[47] D. Duchemin, *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) (2006), 851-885.

[48] J. J. Duistermaat, V. W. Guillemin, *On the spectrum of positive elliptic operators and periodic bicharacteristics*, In Fourier Integral Operators and Partial Differential Equations (pp. 15-22), Springer Berlin, Heidelberg (1975).

[49] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. Éc. Norm. Supér. Vol. 27. No. 5. 1994, pp. 611-660.

[50] G. B. Folland, *Compact Heisenberg manifolds as CR manifolds*, J. Geom. Anal. (2004), 14(3), 521-532.

[51] G. B. Folland, E. M. Stein, *Estimates for the $\overline{\partial}_b$ complex and analysis on the heisenberg group*, Communications on Pure and applied Mathematics (1974), 27(4), 429-522.

[52] K. Furutani, *Heat kernels of the sublaplacian and Laplacian on nilpotent Lie groups*, Analysis, Geometry and Topology of Elliptic Operators, papers in honor of Krzysztof P. Wojciechowsky, (World Scientific, London-Singapore) (2006), 185-226.

[53] K. Furutani, I. Markina, *Complete classification of pseudo H-type algebras: II*, Geom. Dedicata 202 (2019), 233-264.

[54] K. Furutani, I. Markina, *Complete classification of pseudo H-type algebras: I*, Geom. Dedicata 190 (2017), 23-51.

[55] K. Furutani, I. Markina, *Existence of lattices on general H-type groups*, J. Lie Theory 24 (2014), 979-1011.

[56] P. B. Gilkey, *Invariance theory: the heat equation and the Atiyah-Singer index theorem*, CRC press Vol. 16.

[57] C. Gordon, D. Webb, S. Welpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. 110 no. 1 (1992), 1-22.

[58] R. Gorneth, *A new construction of isospectral Riemannian nilmanifolds with examples*, Michigan Math. J. 43 (1996), 159-188.

[59] R. Gorneth, M.B. Mast, *The length spectrum of Riemannian two-step nilmanifolds*, Ann. Sci. de l'Ecole Norm. Superieure vol. 33, No. 2 (2000), 181-209.

[60] P. C. Greiner, D. Holcman, Y. Kannai, *Wave kernels related to second-order operators*, Duke Math. J. 114(2), 329-386 (2000).

[61] L. Hörmander, *Hypo-elliptic second order differential equations*, Acta Math. 119 (1967), 147 - 171.

[62] A. Ikeda, *On lens spaces which are isospectral but not isometric*, Ann. Sci. École Norm Sup. (4) 13 no. 3 (1980), 303-315.

[63] A. Ikeda, *On spherical space forms which are isospectral but not isometric*, J. Math. Soc. of Japan, 35 (1983), 437-444.

[64] Y. Inahama, S. Taniguchi, *Heat trace asymptotics on equiregular sub-Riemannian manifolds*, J. Math. Soc. Japan 72(4), 1049-1096 (2020).

[65] S. Ivanov, I. Minchev, D. Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. Amer. Math. Soc. (2014), 231(1086).

[66] S. Ivanov, D. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Scientific Publishing (2011).

[67] C. Iwasaki, N. Iwasaki, *Parametrix for a degenerate parabolic equation and its application to the asymptotic behavior of spectral functions for stationary problems*, Publ. Res. Inst. Math. Sci.17 no.2 (1981), 577-655.

[68] I. M. James, *The topology of Stiefel manifolds*, London Math. Soc. Lecture Note Ser., vol. 24, Cambridge Univ. Press, 1976.

[69] D. Jerison, J.M. Lee, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Differential Geom. (1989), 29(2), 303-343.

[70] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly 73 no. 4, part II (1966), 1-23.

[71] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. 258 (1980), 147-153.

[72] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. Lond. Math. Soc. 15(1) (1983), 35-42.

[73] C. S. Kunkel, *Quaternionic contact pseudohermitian normal coordinates*, Thesis (Ph. D.), Mathematics Research Library, University of Washington (arXiv:0807.0465), 2008.

[74] A. A. Kirillov, *Representation theory and noncommutative harmonic analysis I*, Springer Berlin Heidelberg.

[75] A. Laaroussi, *Heat kernel asymptotics for quaternionic contact manifolds*, arXiv:2103.00892 (2021).

[76] A. Laaroussi, *A Poisson formula for subriemannian H-type nilmanifolds*, in preparation.

[77] H.B. Lawson, M.-L. Michelson, *Spin geometry*, Princeton University Press, 1989.

[78] C. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. 63(3), 723-743, 1990.

[79] A. Lerario, A. Mondino, *Homotopy properties of horizontal loop spaces and applications to closed sub-Riemannian geodesics*, Trans. Amer. Math. Soc. (2019), Series B, 6(6), 187-214.

[80] W. Liu, H. J. Sussmann, *Shortest paths for sub-Riemannian metrics on rank-two distributions*, Mem. Amer. Math. Soc. 118 (1995), no. 564.

[81] N. Mahammed, R. Piccinini, U. Suter, *Some applications of topological K-theory*, Notas de Mathemática 74, North-Holland Publishing Co. Amsterdam-New York, 1980.

[82] A. I. Malćev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation 39 (1951); Izv. Akad. Nauk USSR, Ser. Mat. 13 (1949), 9-32.

[83] I. Markina, *Geodesics in geometry with constraints and applications*, in: Quantization, PDEs, and geometry, 153 - 314, Oper. Theory Adv. Appl., 251, Adv. Partial Differ. Equ. (Basel), Birkhäuser/Springer, Cham, 2016.

[84] I. Markina, M. Godoy Molina, *Sub-Riemannian geodesics and heat operator on odd dimensional spheres*, Anal. Math. Phys. 2, (2012), 123 - 147.

[85] I. Markina, M. Godoy Molina, *Sub-Riemannian geometry of parallelizable spheres*, Rev. Mat. Iberoam. 27 (2011), no. 3, 997 - 1022.

[86] H. P. Mckean, *Selberg's trace formula as applied to a compact surface*, Comm. Pure Appl. Math. 25 (1972), 225-246.

[87] G. Métivier, *Fonction spectrale et valeurs propres d'une classe d'opérateurs non elliptiques*, Journées équations aux dérivées partielles, 1-54 (1976).

[88] R. B. Melrose, *Hypoelliptic operators with characteristic variety of codimension two and the wave equation*, Séminaire Équations aux dérivées partielles (Polytechnique) (1979), 1-12.

[89] R. B. Melrose, *The wave equation for a hypoelliptic operator with symplectic characteristics of codimension two*, J. Anal. Math. (1984) 44, 134-182.

[90] A. Menikoff, J. Sjöstrand, *The eigenvalues of hypoelliptic operators. III. The nonsemibounded case*, J. Analyse. Math. 35 (1979), 123-150.

[91] A. Menikoff, J. Sjöstrand, *On the eigenvalues of a class of hypoelliptic operators*, Math. Ann. (1978) 235, 55-85.

[92] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 542.

[93] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Math. Surveys Monogr. 2002.

[94] C.C. Moore, *Representations of solvable and nilpotent groups and harmonic analysis on nil and solvmanifolds Harmonic Analysis on Homogeneous Spaces (Williamstown, Mass., 1972)*, Proc. Sympo. Pure Math. (Vol. 26, pp. 3-44). Amer. Math. Soc..

[95] H. Pesce, *Calcul du spectre d'une nilvariété de rang deux et applications*, Trans. Amer. Math. Soc. 339.1 (1993): 433-461.

[96] H. Pesce, *Une formule de Poisson pour les variétés de Heisenberg*, Duke Math. J. 73.1 (1994): 79-95.

[97] L. Pukánszky, *Lecons sur les représentations des groupes*, Monographies de la Société Mathématique de France, No. 2, Dunod, Paris, 1967.

[98] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, New York, Heidelberg, Berlin, 1972.

[99] P.K. Rashevskii, *About connecting two points of complete nonholonomic space by admissible curve*, Uch. Zapiski ped. inst. Libknexta 2, (1938), 83 - 94 (in Russian).

[100] L. Rizzi, T. Rossi, *Heat content asymptotics for sub-Riemannian manifolds*, J. Math. Pures Appl. 148, 267-307 (2021).

[101] S. Rosenberg, *The Laplacian on a Riemannian manifold*, Cambridge Univ. Press, Cambridge (1997).

[102] L. P. Rothschild, E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Mathematica (1976), 137(1), 247-320.

[103] A. Sarti, G. Citti, J. Petitot, *The symplectic structure of the primary visual cortex* , Biological Cybernetics, 98(1), 33-48. (2008).

[104] N. Savale, *Spectrum and abnormals in sub-Riemannian geometry: the 4D quasi-contact case*, arXiv:1909.00409 (2019).

[105] R. S. Strichartz, *Spectral asymptotics on compact Heisenberg manifolds*, J. Geom. Anal. 26 no. 3 (2016), 2450-2458.

[106] R. Strichartz, *Sub-Riemannian Geometry*, J. Differential Geom. 24 (1986), no. 2, 221-263.

[107] R. Strichartz, *Corrections to "Sub-Riemannian Geometry"*, J. Differential Geom. 30 (1989), no. 2, 595-596.

[108] T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. 121(2) no. 1 (1985), 169-186.

[109] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Kinokuniya Co. Ltd., Tokyo (1975).

[110] Y. C. de Verdiére, L. Hillairet, E. Trélat, *Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results*, arXiv:2004.06461 (2020).

[111] Y. C. de Verdière, *Spectre du Laplacien et longueurs des géodésiques périodiques. II*, Compos. Math. 27.2 (1973): 159-184.

[112] W. Wang, F. Wu, *On the Yamabe problem on contact Riemannian manifolds*, Ann. Global Anal. Geom. (2019), 56(3), 465-506.

[113] S. M. Webster, *Pseudo-hermitian structures on a real hypersurface*, J. Differential Geom. 13 (1979), 25-41.

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Liste of Publications:

(1) W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi, *Spectral theory of a class of nilmanifolds attached to Clifford modules*, Mathematische Zeitschrift (2021), 297(1), 557-583.

(2) W. Bauer, A. Laaroussi, *Trivializable and quaternionic subriemannian structure on* $\mathbb{S}^7$ *and subelliptic heat kernel*, submitted (arXiv:2102.04784).

(3) A. Laaroussi, *Heat kernel asymptotics for quaternionic contact manifolds*, arXiv:2103.00892 (2021).