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Abstract

Let $W \subset \mathrm{GL}(V)$ be a complex reflection group and $\mathscr{A}(W)$ the set of the mirrors of the complex reflections in W. It is known that the complement $X(\mathscr{A}(W))$ of the reflection arrangement $\mathscr{A}(W)$ is a $K(\pi,1)$ space. For Y an intersection of hyperplanes in $\mathscr{A}(W)$, let $X(\mathscr{A}(W)^Y)$ be the complement in Y of the hyperplanes in $\mathscr{A}(W)$ not containing Y. We hope that $X(\mathscr{A}(W)^Y)$ is always a $K(\pi,1)$. We prove it in case of the monomial groups $W = G(r,p,\ell)$. Using known results, we then show that there remain only three irreducible complex reflection groups, leading to just eight such induced arrangements for which this $K(\pi,1)$ property remains to be proved.

1. Introduction

An *arrangement* in a vector space V is a finite set of homogeneous hyperplanes in V. For integers $\ell \geqslant 2$, $0 \leqslant k \leqslant \ell$ and $r \geqslant 1$, we define $\mathscr{A}_\ell^k(r)$ to be the arrangement in the complex vector space $\mathbb{C}^\ell$ (coordinates $y_1,\dots,y_\ell$) consisting of the first k coordinate hyperplanes $y_a = 0$ $(1 \leqslant a \leqslant k)$ and of the hyperplanes $y_i = \zeta y_j$ for $i \neq j$ and ζ an rth root of unity.

For $\mathscr{A}$ an arrangement in a complex vector space V, we define $X(\mathscr{A})$ to be the complement in V of the union of the hyperplanes in $\mathscr{A}$. We say that the arrangement $\mathscr{A}$ is of $K(\pi,1)$ type, or a $K(\pi,1)$-arrangement, if $X(\mathscr{A})$ is a $K(\pi,1)$, that is, if the homotopy groups π_i of $X(\mathscr{A})$ are trivial for $i \geqslant 2$ or equivalently if the universal covering of $X(\mathscr{A})$ is contractible. Our main result is the following.

Theorem 1. *The arrangements $\mathscr{A}_\ell^k(r)$ are of $K(\pi,1)$ type.*

For V a finite-dimensional complex vector space, an element s of $\mathrm{GL}(V)$ is a *complex reflection* if its fixed point set is a hyperplane. This hyperplane is called the *mirror* of s. If $W \subset \mathrm{GL}(V)$ is a complex reflection group, that is, a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections, the *arrangement* $\mathscr{A}(W)$ is the set of mirrors of the complex reflections in W. The arrangements so obtained are the *reflection arrangements*. Note that for $\ell \geqslant 3$, $r \geqslant 2$ and $0 < k < \ell$, the $\mathscr{A}_\ell^k(r)$ are not reflection arrangements.

If $X \subset V$ is the intersection of some hyperplanes belonging to an arrangement $\mathscr{A}$ in V, the arrangement $\mathscr{A}^X$ *induced* by $\mathscr{A}$ on X is the set of the traces on X of the hyperplanes in $\mathscr{A}$ not containing X. The arrangements *induced* by $\mathscr{A}$ are the arrangements so obtained.

The $K(\pi,1)$-property is not generic among all arrangements. A *generic* complex ℓ-arrangement $\mathscr{A}$ for $\ell \geqslant 3$ is an ℓ-arrangement with at least $\ell+1$ hyperplanes and the property

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that the hyperplanes of every subarrangement $\mathscr{B} \subseteq \mathscr{A}$ with $|\mathscr{B}| = \ell$ are linearly independent. It follows from work of Hattori [Hat75] that generic arrangements are never $K(\pi, 1)$.

By Deligne [Del72], complexified simplicial arrangements are $K(\pi, 1)$. Likewise for complex fiber-type arrangements; cf. [FR85] and [Ter86]. As restrictions of simplicial (respectively fiber-type) arrangements are again simplicial (respectively fiber-type), the $K(\pi, 1)$-property of these kinds of arrangements is inherited by their restrictions. However, we emphasize that, in general, a restriction of a $K(\pi, 1)$-arrangement need not be $K(\pi, 1)$ again; see [AMR18] for examples of this kind.

Along with the previously known instances of $K(\pi, 1)$ restrictions of reflection arrangements, it follows from Theorem 1 that only a small number of restrictions of rank 3 or 4 of arrangements associated to some non-real exceptional groups remain unresolved. This provides strong evidence towards the following statement.

Hope. *Any arrangement induced from a reflection arrangement $\mathscr{A}(W)$ is a $K(\pi, 1)$.*

This Hope reduces to the case of arrangements induced from reflection arrangements $\mathscr{A}(W)$, for $W \subset \mathrm{GL}(V)$, such that V is an irreducible representation of W. In what follows we, sometimes tacitly, only consider this case.

The Hope is true for W the complexification of a real reflection group. Indeed, after reduction to the case where the intersection of all mirrors is reduced to $\{0\}$, such an $\mathscr{A}(W)$ is the complexification of a simplicial arrangement [Bou68, V 3.9]. The property of being the complexification of a simplicial arrangement is stable by induction and one applies Deligne [Del72].

All reflection arrangements $\mathscr{A}(W)$ are of $K(\pi, 1)$ type. This theorem is due to Fadell and Neuwirth [FN62], Brieskorn [Bri73], Nakamura [Nak83] and Orlik and Solomon [OS88] in special cases, and to Bessis [Bes15] in the general case.

It follows from Theorem 1 that the arrangements induced from the reflection arrangements $\mathscr{A}_\ell^0(r)$ and $\mathscr{A}_\ell^\ell(r)$ are $K(\pi, 1)$-arrangements.

Using those results and the trivial fact that for $\dim V \leqslant 2$ any arrangement is of $K(\pi, 1)$ type, one gets our Hope in all but finitely many cases. More precisely, by Theorem 1, our discussion above and from the classification of the irreducible complex reflection groups, our Hope reduces to 13 instances when the underlying reflection group is of exceptional type. Following [OS82], we label the W-orbit of $Y \in L(\mathscr{A}(W))$ by the pair (G_n, T), where G_n is the relevant reflection group, in the Shephard–Todd numbering [ST54], and T is the type of the reflection subgroup of G_n fixing pointwise the intersection of mirrors Y we are considering. In our cases, the type T determines Y up to G_n-conjugacy. The 13 instances are (G_{29}, A_1), (G_{31}, A_1), $(G_{32}, C(3))$, (G_{33}, A_1), (G_{33}, A_1^2), (G_{33}, A_2), (G_{34}, A_1), (G_{34}, A_1^2), (G_{34}, A_2), (G_{34}, A_1^3), (G_{34}, A_1A_2), (G_{34}, A_3) and $(G_{34}, G(3, 3, 3))$; see [OS82, § 3, Appendix]. The cases $(G_{32}, C(3))$ and $(G_{34}, G(3, 3, 3))$ can be handled as follows. The lattices of intersections of $(G_{32}, C(3))$ and $(G_{34}, G(3, 3, 3))$ are both isomorphic to the lattice of $\mathscr{A}(G_{26})$; cf. [OT92, Appendix D]. Viewed projectively, the arrangement $\mathscr{A}(G_{26})$ is the extended Hessian configuration of 21 lines in $\mathbb{P}^2(\mathbb{C})$; cf. [OT92, Example 6.30]. It is classical that this configuration, as a set of 21 lines, is determined by its combinatorics, i.e. by the isomorphism class of the corresponding lattice: the arrangements $(G_{32}, C(3))$ and $(G_{34}, G(3, 3, 3))$ are linearly isomorphic to the reflection arrangement $\mathscr{A}(G_{26})$. Therefore, since $\mathscr{A}(G_{26})$ is a $K(\pi, 1)$-arrangement, so are the restrictions $(G_{32}, C(3))$ and $(G_{34}, G(3, 3, 3))$. Moreover, since the localization of a $K(\pi, 1)$-arrangement is again $K(\pi, 1)$ (cf. [Par93, Lemma 1.1]) and since G_{33} is a parabolic subgroup of G_{34} [OS82, Table 11], the three instances stemming from G_{33} are localizations of the corresponding restrictions from G_{34}. So, verifying our Hope reduces to the remaining eight restrictions in the list above.

2. Method of proof

A reasonable topological space X, for instance a manifold or a CW complex, is a $K(\pi,1)$ if and only if it is connected (hence, by definition, not empty) and if for some (equivalently, any) base point $o \in X$, the homotopy groups $\pi_i(X,o)$ are trivial for $i \geqslant 2$. The long exact sequence of homotopy groups implies the following result.

Lemma 2.1. *Suppose that X is connected and that $f : X \to Y$ is a fibration. Then, if Y is a $K(\pi,1)$ and if some connected component of some fiber $f^{-1}(y)$ is a $K(\pi,1)$, so is X.*

We fix k,ℓ,r as in Theorem 1. We simply write $\mathscr{A}$ for the arrangement $\mathscr{A}_\ell^k(r)$ in $\mathbb{C}^\ell$. The coordinates of $\mathbb{C}^\ell$ are denoted $y_1,\ldots,y_\ell$.

Let us consider another copy of $\mathbb{C}^\ell$, with coordinates $x_1,\ldots,x_\ell$. Let V be the quotient of this vector space $\mathbb{C}^\ell$ by its diagonal subspace $\mathbb{C}$. The action of the symmetric group S_ℓ on $\mathbb{C}^\ell$ passes to the quotient. So do the linear forms $x_i - x_j$. The *arrangement* $\mathrm{A}_{\ell-1}$ on V is the set of the hyperplanes $x_i - x_j = 0$ of V. It is the reflection arrangement $\mathscr{A}(S_\ell)$ defined by the action of S_ℓ on V. It is of $K(\pi,1)$ type and the fundamental group of $X(\mathrm{A}_{\ell-1})$ is the pure braid group on ℓ strands.

The $z_i := x_i - x_\ell$ $(1 \leqslant i \leqslant \ell-1)$ form a system of coordinates on V. In this system of coordinates, the arrangement $\mathrm{A}_{\ell-1}$ is the arrangement $\mathscr{A}_{\ell-1}^{\ell-1}(1)$ consisting of the coordinate hyperplanes $z_i = 0$ and of the hyperplanes $z_i = z_j$ for $i \neq j$.

Our *deus ex machina* is the composite map

$$f : \mathbb{C}^\ell \text{ (coordinates } y_i) \longrightarrow \mathbb{C}^\ell \text{ (coordinates } x_i) \longrightarrow V, \tag{2.2}$$

where the first map in (2.2), or rather its graph, is given by

$$x_i = y_1 \cdots y_k \; y_i^r. \tag{2.3}$$

It is equivariant for the subgroup $S_k \times S_{\ell-k}$ of S_ℓ, acting on $\mathbb{C}^\ell$ and on V: the coordinates $y_1,\ldots,y_k$, as well as $y_{k+1},\ldots,y_\ell$, play symmetric roles.

The inverse image by f of the union of the hyperplanes in $\mathrm{A}_{\ell-1}$ is the union of the hyperplanes in $\mathscr{A}$. Indeed, the inverse image of the hyperplane $x_i - x_j = 0$ is the union of the coordinate hyperplanes $y_a = 0$ for $1 \leqslant a \leqslant k$, and of the hyperplanes $y_i = \zeta y_j$ for ζ an rth root of unity.

In the coordinate system (z_i) of V, the map (2.2) is given by

$$z_i = y_1 \cdots y_k \; (y_i^r - y_\ell^r). \tag{2.4}$$

It induces a map, still denoted by f,

$$f : X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1}). \tag{2.5}$$

Theorem 2. *The map $f : X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1})$ realizes $X(\mathscr{A})$ as a smooth fiber space over $X(\mathrm{A}_{\ell-1})$.*

A consequence of Theorem 2 is that the fibers of $f : X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1})$ are non-empty smooth affine curves. The connected components of such a curve are again smooth and affine and hence are $K(\pi,1)$. Indeed, the only Riemann surface which is not a $K(\pi,1)$ is the sphere. As $X(\mathscr{A})$ is connected and $X(\mathrm{A}_{\ell-1})$ is a $K(\pi,1)$, Theorem 1 is a consequence of Theorem 2 and Lemma 2.1.

The proof of Theorem 2 is given in the next section, where we use the following lemma.

LEMMA 2.6. *Let M and B be C^∞-manifolds, N a closed submanifold of M and $f : M \longrightarrow B$ a morphism. If f is proper, submersive, and with a restriction to N submersive, then, locally on $B, f : (M, N) \longrightarrow B$ is isomorphic to a projection $(M_0 \times B, N_0 \times B) \longrightarrow B$. A fortiori, $f : M - N \longrightarrow B$ is a smooth fiber bundle.*

For N empty, the lemma first appeared without proof in [Ehr47, Proposition 1]. For the sake of completeness, we now explain the folklore proof of Lemma 2.6 in the case when N is empty and then explain how to extend it to the general case.

The question being local on B, we may assume that B is of the form $]-1,1[^\ell$ (coordinates $t_i, \dots, t_\ell$) and we proceed by induction on ℓ, the case $\ell = 0$ being trivial. The vector field ∂_{t_ℓ} on B can be lifted to a vector field X on M. Indeed, such a lifting exists locally on M and one uses a partition of unity to get a global lifting from local liftings. As $df(X) = \partial_{t_\ell}$, by integrating X, we obtain isomorphisms between the fibers of f at $(t_1, \dots, t_{\ell-1}, 0)$ and $(t_1, \dots, t_{\ell-1}, t_\ell)$. These isomorphisms identify $M \longrightarrow B$ with the pull-back by $]-1,1[^\ell \longrightarrow\,]-1,1[^{\ell-1}$ of the restriction of $M \longrightarrow B$ to $]-1,1[^{\ell-1} \times \{0\} \subset B$. One concludes using the induction hypothesis.

The proof of Lemma 2.6 is identical: one just needs to choose the lifting X of ∂_{t_ℓ} to be tangent to N.

3. Proof of Theorem 2

The fiber F_z at $z \in X(\mathrm{A}_{\ell-1})$ of $f : X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1})$ is given, in $\mathbb{C}^\ell$, by the equations

$$y_1 \cdots y_k (y_i^r - y_\ell^r) = z_i \quad (i = 1, \dots, \ell - 1). \tag{3.1}$$

Any of these equations implies that $y_1, \dots, y_k \neq 0$. Their system is equivalent to the first equation

$$y_1 \cdots y_k (y_1^r - y_\ell^r) = z_1, \tag{3.2}$$

supplemented by the equations

$$\frac{1}{z_1}(y_1^r - y_\ell^r) = \frac{1}{z_i}(y_i^r - y_\ell^r) \quad (2 \leqslant i \leqslant \ell - 1), \tag{3.3}$$

which are homogeneous of degree r in the y_i.

Let us compactify $\mathbb{C}^\ell$ into $\mathbb{P}^\ell(\mathbb{C})$. In $\mathbb{P}^\ell(\mathbb{C})$, we will use the homogeneous coordinates $y_0, y_1, \dots, y_\ell$, $y_0 = 0$ being the equation of the hyperplane at infinity added to $\mathbb{C}^\ell$. To compactify the fiber F_z, it suffices to take the projective variety $\overline{F}_z$ defined by the homogeneous equations (3.3), and by (3.2) made homogeneous, that is,

$$y_1 \cdots y_k (y_1^r - y_\ell^r) = z_1\ y_0^{k+r}, \tag{3.2$'$}$$

an equation homogeneous of degree $k + r$ in the y_i.

It will be convenient to define $z_\ell := 0$. With this notation, (3.3) tells that the $y_i^r - y_\ell^r$ are proportional to the $z_i - z_\ell$ and it follows that all $y_i^r - y_j^r$ are proportional to the $z_i - z_j$: for some u, $y_i^r - y_j^r = u(z_i - z_j)$.

To compute the intersection of this compactification $\overline{F}_z$ with the hyperplane at infinity H_∞, it suffices to put y_0 equal to 0 and to view $y_1, \dots, y_\ell$ as projective coordinates for the hyperplane at infinity. We obtain $kr^{\ell-2} + r^{\ell-1}$ distinct points, as follows. One of the factors at the left-hand side of (3.2$'$) must vanish. If $y_i = 0$ $(1 \leqslant i \leqslant k)$, the $y_j^r = y_j^r - y_i^r$ are proportional to the $z_j - z_i$ and we get the $r^{\ell-2}$ points with coordinates

$$(y_i = 0, y_j = (z_j - z_i)^{1/r})$$

(to be taken up to multiplying by a common rth root of unity).

If $y_1^r - y_\ell^r = 0$, all $y_i^r - y_j^r$ must vanish. We get the $r^{\ell-1}$ points 'all y_i are an rth root of unity', again taken up to multiplication by a common rth root of unity.

Lemma 3.4. *The compactification $\overline{F}_z$ of the fiber F_z of f at z defined by the $\ell-1$ equations (3.2′) and (3.3) is a complete intersection curve, smooth at infinity, and meeting transversally the hyperplane at infinity H_∞.*

Proof. If $\overline{F}_z$ had an irreducible component of dimension >1, the intersection of this component with H_∞ would be of dimension >0, contradicting the finiteness of $\overline{F}_z \cap H_\infty$. It follows that $\overline{F}_z$, being defined by $\ell - 1$ equations, is a complete intersection curve. By Bezout, the number of points in $\overline{F}_z \cap H_\infty$, each counted with its intersection multiplicity, is $(k+r)r^{\ell-2}$. It follows that each intersection multiplicity is one. As $\overline{F}_z$ and H_∞ are local complete intersections, this implies that $\overline{F}_z$ is smooth at each point of $\overline{F}_z \cap H_\infty$ and that the intersection is transversal. □

It follows from Lemma 3.4 that $\overline{F}_z$ is simply the closure of F_z in $\mathbb{P}^\ell(\mathbb{C})$ and that the curve $\overline{F}_z$ is generically reduced, that is, generically smooth, as a non-reduced component would intersect H_∞.

The same argument shows the following result.

Lemma 3.5. *For $k+1 \leqslant i \leqslant \ell$, the curve $\overline{F}_z$ is smooth at each of its intersection points with the hyperplane $y_i = 0$.*

Proof. It suffices to show that the number of intersection points is $(k+r)r^{\ell-2}$. As the hyperplanes $y_i = 0$ $(k+1 \leqslant i \leqslant \ell)$ play symmetric roles, it suffices to consider the case of the hyperplane $y_\ell = 0$. Equations (3.3) tell us that $(y_1^r, \ldots, y_{\ell-1}^r)$ is proportional to $(z_1, \ldots, z_{\ell-1})$, while by (3.2′) they cannot be all zero, as otherwise y_0 would be zero too. If we fix the indeterminacy 'multiplication by a common constant' by requiring $y_{\ell-1}$ to be a specified root of $z_{\ell-1}$, (3.3) tells that each y_i $(1 \leqslant i \leqslant \ell-2)$ is an rth root of z_i. This gives $r^{\ell-2}$ possibilities, while (3.2′) leaves $k+r$ possibilities for y_0. □

Lemma 3.6. *The curve $\overline{F}_z$ is smooth.*

Proof. By Lemma 3.4, it suffices to show that F_z is smooth. By (3.2), F_z does not intersect the hyperplanes $y_i = 0$ for $1 \leqslant i \leqslant k$. By Lemma 3.5, it hence suffices to check that in the open set where none of the y_i vanishes, F_z is smooth. Locally on $(\mathbb{C}^*)^\ell$, we can take as local coordinates the $Y_i = y_i^r$. In these local coordinates, (3.3) tells us that F_z is on the surface $Y_i = az_i + b$ (coordinates a, b). In the coordinates a, b, the equation (3.2) becomes

$$\prod_{i=1}^{k} (az_i + b)^{1/r} \cdot az_1 = z_1$$

for some branches of the rth roots. It follows that $a \neq 0$ and that F_z is contained in the curve of the plane (a, b),

$$a^r \prod_{i=1}^{k} (az_i + b) = 1.$$

One concludes by invoking the following well-known result.

Lemma 3.7. *If $F(a_1, \ldots, a_n)$ is a homogeneous polynomial of degree $d \geqslant 1$, the hypersurface $F(a_1, \ldots, a_n) = 1$ is non-singular.*

By homogeneity, the hypersurfaces $F = c$ are, for $c \neq 0$, all isomorphic. By Sard's theorem, almost all are non-singular. One could rather use the Jacobian criterion: at a point where $F = 1$, Euler's identity

$$\Sigma a_i \partial_i F = dF = d$$

shows that not all $\partial_i F$ can vanish.

If we now let z vary in $X(\mathrm{A}_{\ell-1})$, we obtain a family of smooth complete intersection curves in the projective space $\mathbb{P}^\ell(\mathbb{C})$ completing $\mathbb{C}^\ell$, transversal to the hyperplane at infinity. The total space is contained in $\mathbb{P}^\ell(\mathbb{C}) \times X(\mathrm{A}_{\ell-1})$ and one applies Lemma 2.6 to it. $\square$

4. Complements

The projective completion $\overline{F}_z$ of the fiber F_z of the fiber bundle $f : X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1})$ is a smooth complete intersection in $\mathbb{P}^\ell(\mathbb{C})$, of multidegree $(k+r, r, \dots, r)$. As $\overline{F}_z$ is defined by $\ell - 2$ homogeneous equations, its complement U is the union of $\ell - 2$ smooth open affine varieties of dimension ℓ, so that $H^1_c(U) = 0$ and $H^0(\mathbb{P}^\ell(\mathbb{C})) \xrightarrow{\sim} H^0(\overline{F}_z)$: so $\overline{F}_z$ is connected. To see this, one could rather make an iterated application to the Lefschetz hyperplane theorem, again for H^0.

The canonical line bundle of a complete intersection Y of degrees $(d_1, \dots, d_{\ell-1})$ in $\mathbb{P}^\ell(\mathbb{C})$ is isomorphic to the restriction to Y of $\mathcal{O}(\Sigma d_i - \ell - 1)$. In our case, it follows that the degree $2g - 2$ of the canonical line bundle of the curve $\overline{F}_z$ is given by

$$2g - 2 = (k + (r-1)(\ell-1) - 2)(k+r)r^{\ell-2}.$$

The fiber F_z is the complement in $\overline{F}_z$ of $(k+r)r^{\ell-2}$ points. Its fundamental group is hence a free group with N generators, where

$$\begin{aligned} N &= 2g + \text{number of removed points } - 1 \\ &= (k + (r-1)(\ell-1) - 1)\ (k+r)r^{\ell-2} + 1. \end{aligned}$$

Each curve $\overline{F}_z$ contains at infinity the point $(0, 1, \dots, 1)$. If $M \longrightarrow X(\mathrm{A}_{\ell-1})$ is the total space of the family of the $\overline{F}_z$ (contained in $\mathbb{P}^\ell(\mathbb{C}) \times X(\mathrm{A}_{\ell-1})$), this common point gives us a section s of the fiber bundle $M \longrightarrow X(\mathrm{A}_{\ell-1})$. The vertical tangent bundle, restricted to this section, is a trivial line bundle, because any line bundle on $X(\mathrm{A}_{\ell-1})$ is trivial. Let v be, along s, a nowhere-vanishing section of the vertical tangent bundle. Pushing s in the direction of v, one obtains a C^∞ section of $X(\mathscr{A}) \longrightarrow X(\mathrm{A}_{\ell-1})$. The fundamental group of $X(\mathscr{A})$ is hence a semi-direct product of the fundamental group of the basis by the fundamental group of the fiber: a semi-direct product of the braid group on ℓ strands by the free group on N generators.

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