# Elliptic K3 surfaces and their moduli: dynamics, geometry and arithmetic

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*A babbo e mamma*

# Abstract

This thesis deals with K3 surfaces and their moduli spaces. In the first part we identify a class of complex K3 surfaces, called of zero entropy, with a particularly simple (but infinite) automorphism group, naturally arising from complex dynamics. We provide a lattice-theoretical classification of their Néron-Severi lattices. In the second part we move to the study of the Kodaira dimension of the moduli spaces of elliptic K3 surfaces of Picard rank 3. We show that almost all of them are of general type, by using the low-weight cusp form trick developed by Gritsenko, Hulek and Sankaran. Moreover, we prove that many of the remaining moduli spaces are unirational, by providing explicit projective models of the corresponding K3 surfaces. In the final part, we investigate the set of rational points on K3 and Enriques surfaces over number fields. We show that all Enriques surfaces over number fields satisfy (a weak version of) the potential Hilbert property, thus proving that, after a field extension, the rational points on their K3 cover are dense and do not come from finite covers.

**Key words:** K3 surfaces, elliptic fibrations, lattices, dynamical systems, automorphisms, moduli spaces, Kodaira dimension, Enriques surfaces, Hilbert property, rational points.

# Contents

# Preface

K3 surfaces are ubiquitous in modern mathematics and physics. They arise naturally in many different contexts, from string theory to representation theory and Lie groups, from particle scattering in particle physics to differential geometry. From the point of view of algebraic geometry K3 surfaces are very special, since they are the only surfaces (together with abelian surfaces) that have a trivial canonical bundle. This peculiarity makes K3 surfaces objects of great interest, as they are the simplest algebraic varieties that are not trivial to study. Since the end of the 19th century, when the italian school led by Castelnuovo and Enriques started studying the theory of algebraic surfaces, many breakthroughs have been accomplished in the understanding of K3 surfaces. Nevertheless, K3 surfaces continue to offer such a variety of deep and interesting questions, that we are still naturally brought back to studying their properties.

As natural 2-dimensional generalizations of elliptic curves, K3 surfaces can be studied from different perspectives, from an algebraic, a geometric and an arithmetic point of view. The goal of this thesis is to offer insights in all three directions, and to show some underlying connections between them. The first problem that we tackle is trying to understand the group of automorphisms of K3 surfaces. Despite a huge theoretical machinery that allows us to view their automorphisms as isometries of certain lattices, our grasp of such groups of automorphisms is very limited. Thanks to works by Nikulin, Vinberg and Kondō, we have a complete classification of complex K3 surfaces with a finite automorphism group depending on their Néron-Severi lattice, but practically nothing is known beyond this. Therefore we try to identify K3 surfaces with an infinite, but simple, automorphism group. The idea comes from complex dynamics: if $f$ is an automorphism of a complex surface, we can attach to it a number $h(f) \geq 0$, called the *entropy* of $f$, that measures the complexity of its dynamics. The most “regular” automorphisms are those of zero entropy. Hence we say that a complex K3 surface has *zero entropy* if all its automorphisms have zero entropy. Otherwise the K3 surface has *positive entropy*. Unsurprisingly, the automorphism group of K3 surfaces with zero entropy is very special: Cantat and Oguiso show that it is almost-abelian, that is, abelian up to a finite group (cf. Theorem 2.1.1). As Nikulin showed that there are only finitely many Néron-Severi lattices of K3 surfaces with zero entropy, we attempt to classify them. Our main result is the following:

**Theorem 1.** *Let $X$ be a complex K3 surface admitting an elliptic fibration with only irreducible fibers. $X$ has zero entropy if and only if its Néron-Severi lattice belongs to an explicit* *list of* 32 *lattices.*

The list can be found in Theorem 2.5.9. We are also able to remove the extra, technical assumption if the Picard rank of $X$ is big enough:

**Theorem 2.** *Let $X$ be a complex K3 surface with Picard rank $\rho(X) \geq 19$. Then either $X$ has a finite automorphism group, or it has positive entropy.*

Theorem 1 singles out the Néron-Severi lattices of K3 surfaces with a very simple automorphism group. More specifically, such K3 surfaces are shown to admit a *unique* elliptic fibration $|F|$, and their automorphism group is isomorphic, up to a finite group, to the $\mathbb{Z}$-module $\mathrm{MW}(F)$ of rational sections of $|F|$. This gives an explicit characterization of the automorphism groups of K3 surfaces of zero entropy.

Then we move to the study of the geometry of K3 surfaces and their moduli spaces. The reason for this is to investigate whether K3 surfaces with zero entropy have some additional special geometric properties. For instance, thanks to recent works by Roulleau, we know that the moduli spaces of many K3 surfaces with finite automorphism group are unirational. This shows that K3 surfaces with finite automorphism group are very special from a geometric point of view as well: indeed, they can be realized as very explicit projective models. This led us to wonder whether K3 surfaces with zero entropy are special from this point of view as well. The "simplest" K3 surfaces with zero entropy are those of Picard rank 3: more specifically, they are the K3 surfaces with Néron-Severi lattice isometric to $U \oplus \langle -2k \rangle$, with $k \in \{2, 3, 4, 5, 7, 9, 13, 25\}$ (if $k = 1$ the K3 surfaces have a finite automorphism group). Thus we try to understand whether the moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces are unirational. Our main results are the following. We refer to Theorems 3.1.1 and 4.1.1 for the complete and precise statements.

**Theorem 3.** *The moduli space $\mathcal{M}_{2k}$ is of general type for $k \geq 220$ and for some smaller values of $k$, until $k = 170$. Moreover, the Kodaira dimension of $\mathcal{M}_{2k}$ is non-negative for $k \geq 176$ and for some smaller values of $k$, until $k = 140$.*

**Theorem 4.** *The moduli space $\mathcal{M}_{2k}$ is unirational for $k \leq 34$ and for some larger values of $k$, up to $k = 97$. Therefore the moduli spaces of K3 surfaces with zero entropy and Picard rank* 3 *are all unirational.*

Theorems 3 and 4 provide an almost exhaustive classification of the Kodaira dimensions of the moduli spaces $\mathcal{M}_{2k}$. Moreover Theorem 4 answers positively the question whether the moduli spaces of K3 surfaces with zero entropy are unirational. This confirms our expectation that K3 surfaces with zero entropy are very special, even from a purely geometric standpoint.

In the last chapter of the thesis we move to the study of a fundamental, arithmetic property of algebraic varieties, the Hilbert property. Roughly speaking, an algebraic variety over a number field $K$ satisfies the Hilbert property if its $K$-rational points are dense, and they do not come from a finite number of finite covers of $X$. The study of the Hilbert property has several ramifications into many number-theoretic problems, such as the inverse Galois problem and weak approximation. The reason why the Hilbert property arises naturally in the context of entropy on K3 surfaces is very concrete, and it depends on a recent theorem by Demeio (cf. Theorem 5.2.4). In a nutshell, it states that the K3 surfaces admitting many elliptic fibrations sharing few reducible components satisfy the Hilbert property. Such a result builds a bridge between the Hilbert property and the entropy on K3 surfaces: indeed, K3 surfaces with positive entropy have at least 2 elliptic fibrations with infinitely many sections, and consequently we expect many of them to satisfy the Hilbert property.

Nevertheless, deciding whether a K3 surface over a number field satisfies the Hilbert property remains a hard problem for the majority of K3 surfaces. For instance, we do not even know in general whether their $K$-rational points are dense. However, we know that rational points are *potentially* dense, that is, they are dense after a finite field extension. Hence we try to implement a similar approach, and we ask the question of which K3 surfaces satisfy the *potential* Hilbert property, that is, which K3 surfaces satisfy the Hilbert property after a finite field extension. Our two main results in this direction are:

**Theorem 5.** *All K3 surfaces over a number field K that cover an Enriques surface satisfy the potential Hilbert property. In particular, all Kummer surfaces satisfy the potential Hilbert property.*

**Theorem 6.** *Let $X$ be a K3 surface over a number field $K$. If $\rho(X_{\mathbb{C}}) < 10$, and $X$ admits two distinct genus 1 fibrations, then $X$ satisfies the potential Hilbert property.*

Notice that Theorem 5 deals with K3 surfaces with "large" Picard rank, while Theorem 6 with K3 surfaces with "small" Picard rank. Together, they show that a huge class of K3 surfaces satisfies the Hilbert property after a finite field extension. Our future goal is to show that all K3 surfaces with positive entropy satisfy the potential Hilbert property. This would reveal a strong connection between the dynamical side of the entropy and the arithmetic side of the Hilbert property, thus achieving a big breakthrough in our understanding of rational points on K3 surfaces over number fields.

**Contents.** Let us briefly explain how the thesis is organized. In Chapter 1 we recall the basics of lattices and K3 surfaces that we will use throughout the thesis. We do not include the proof of the majority of the statements, but we provide detailed references. In Chapter 2 we study K3 surfaces of zero entropy, and we prove Theorems 1 and 2. The contents of the chapter follow the paper [Mez21]. Chapters 3 and 4 are devoted to the study of the moduli spaces $\mathcal{M}_{2k}$. More precisely, in Chapter 3 we prove Theorem 3, while in Chapter 4 we prove Theorem 4. Chapter 3 follows the paper [FM21], joint with Mauro Fortuna (Leibniz Universität Hannover), while Chapter 4 follows the paper [FHM20], joint with Mauro Fortuna and Michael Hoff (Universität des Saarlandes). The case $k = 11$ in Section 4.6 is new, and it is part of a joint work in progress with Michael Hoff. Finally in Chapter 5 we study the Hilbert property on K3 and Enriques surfaces, and we prove Theorems 5 and 6. This is part of a joint work in progress with Damián Gvirtz (University College London).

**Computational data.** Throughout the thesis, and especially in Chapter 2, we need the help of the computer to carry out some computations. The programming languages that we use are `Magma` and `Macaulay2`. We provide the detailed pseudocode of the most important algorithms that we implement. The interested reader can ask for the actual implemented algorithms to the author.

# 1 | Preliminaries

## 1.1 Lattices

In this section we recall some basics about lattices that we will use throughout the thesis. The main references are [Nik79b], [CS99], [Kne02], [Ebe13], [Kon20, Chapter 1] and [Huy16, Chapter 14].

**Definition 1.1.1.** A *lattice* is a finitely generated free $\mathbb{Z}$-module $L$ endowed with a non-degenerate symmetric bilinear form $(,)$ with values in $\mathbb{Z}$. The *rank* $\mathrm{rk}(L)$ is the rank of $L$ as a $\mathbb{Z}$-module. $L$ is called *even* if $(x,x) \in 2\mathbb{Z}$ for all $x \in L$. The *signature* of $L$ is the signature of the natural extension of the bilinear form to the real vector space $L \otimes \mathbb{R}$. Hence $L$ is called *positive* (resp. *negative*) *definite* if the signature of $L$ is $(\mathrm{rk}(L), 0)$ (resp. $(0, \mathrm{rk}(L))$). The *determinant* $\det(L)$ of $L$ is the determinant of any matrix representing the bilinear product $(,)$ on $L$. If there is no chance of ambiguity, we will simply write the bilinear product $(v, w)$ as $v \cdot w$.

*Example* 1.1.2. The lattice $U$ of rank 2 with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is called the *hyperbolic plane*. It has signature $(1,1)$ and it is *unimodular*, that is, $|\det(U)| = 1$.

We will be primarily interested in even, negative definite lattices for applications to the theory of K3 surfaces. Therefore in the following we assume $L$ to be even and negative definite.

To the lattice $L$ we can attach the *dual lattice*

$$L^\vee = \{x \in L \otimes \mathbb{Q} \mid (x, L) \in \mathbb{Z}\}.$$

$L^\vee$ is a lattice with the natural extension of the bilinear product on $L$. It has the same rank of $L$, so the quotient $A_L = L^\vee/L$ is a finite group. We can endow $A_L$ with the quadratic form $q_L$ with values in $\mathbb{Q}/2\mathbb{Z}$ such that

$$q_L(\overline{x}, \overline{x}) = (x, x) \pmod{2\mathbb{Z}},$$

where $\overline{x}$ denotes the class of $x \in L^\vee$ in the quotient $A_L$. The finite group $A_L$, endowed with this quadratic form $q_L$, is called the *discriminant group* of $L$. The cardinality of $A_L$ coincides with the absolute value of the determinant of $L$. The *length* of the group $A_L$, denoted by $\ell(A_L)$, is the minimum number of generators of the abelian group $A_L$. The *p-length* $\ell_p(A_L)$ of $A_L$ is the length of its $p$-part.

If $L$ is any lattice and $n \geq 1$, we denote by $L(n)$ the lattice whose bilinear product coincides with the bilinear product of $L$, multiplied by $n$. Therefore $\det(L(n)) = n^{\mathrm{rk}(L)} \det(L)$. We will say that $L(n)$ is a *multiple* of $L$.

**Definition 1.1.3.** An embedding of lattices $i : L \hookrightarrow M$ is called *primitive* if the cokernel $M/i(L)$ is torsion-free. The *saturation* of $i(L)$ in $M$, denoted by $i(L)_{sat}$, is the smallest primitive sublattice of $M$ containing $i(L)$.

### 1.1.1 Root lattices

**Definition 1.1.4.** A *root lattice* is an even, negative definite lattice $L$ that admits a basis given by vectors of norm $-2$.

Vectors of norm $-2$ are usually called *roots*. The building blocks of root lattices are ADE lattices. These are the lattices $A_n$ (for $n \geq 1$), $D_n$ (for $n \geq 4$) and $E_n$ (for $6 \leq n \leq 8$). They can be interpreted as the root lattices associated to the Dynkin diagrams of the corresponding type. We refer to [Kon20, Section 1.1] for a precise definition of ADE lattices, but we list in the following table their basic properties. The subscript $n$ denotes the rank of the lattice.

| $L$ | $\lvert\det(L)\rvert$ | $A_L$ |
|---|---|---|
| $A_n$ | $n+1$ | $\mathbb{Z}/(n+1)\mathbb{Z}$ |
| $D_{2n}$ | 4 | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ |
| $D_{2n+1}$ | 4 | $\mathbb{Z}/4\mathbb{Z}$ |
| $E_6$ | 3 | $\mathbb{Z}/3\mathbb{Z}$ |
| $E_7$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $E_8$ | 1 | $\{0\}$ |

Table 1.1: Discriminant groups of ADE lattices

**Proposition 1.1.5** ([Kon20], Proposition 1.12)**.** *Any root lattice is the direct sum of ADE lattices.*

If $L$ is an even, negative definite lattice, we denote by $L_{root}$ the *root part* of $L$, that is, the sublattice of $L$ generated by the vectors of $L$ of norm $-2$. Clearly, a lattice coincides with its root part if and only if it is a root lattice. On the other hand, the root part of a lattice $L$ is empty if and only if $L$ has no roots.

### 1.1.2 Overlattices

Let $L$ be an even, negative definite lattice.

**Definition 1.1.6.** An *overlattice* of $L$ is an even, negative definite lattice $L'$ containing $L$ such that the quotient $L'/L$ is finite. If $L'$ is an overlattice of $L$, we denote by $[L' : L]$ the *index* of the overlattice, i.e. the index of $L$ as a subgroup of $L'$.

**Lemma 1.1.7** ([Nik79b], Section 1.4)**.** *Let $L'$ be an overlattice of $L$. Then the following equality holds:*

$$\frac{|\det(L)|}{|\det(L')|} = [L' : L]^2.$$

The following fundamental proposition gives a concrete way to characterize the overlattices of a given lattice $L$.

**Proposition 1.1.8** ([Nik79b], Proposition 1.4.1)**.** *There exists a 1:1 correspondence between overlattices of $L$ and isotropic subgroups of $A_L$, i.e. subgroups $H < A_L$ such that $q_L|_H \equiv 0$.*

*Proof.* We will give the main ideas of the proof, as they will be useful in the thesis. If $L'$ is an overlattice of $L$, we can associate to it the subgroup $H = L'/L < A_L$. Since the bilinear product on $L'$ is even and has values in $\mathbb{Z}$, the quadratic form $q_L$ is equivalently zero on the subgroup $H$.
Conversely, given a $q_L$-isotropic subgroup $H < A_L$, we can consider the preimages $y_1, \ldots, y_n \in L^\vee \subseteq L \otimes \mathbb{Q}$ of a set of generators of $H$ under the projection $L^\vee \to A_L$. Then we associate to $H$ the overlattice $L'$ of $L$ obtained by adjoining to $L$ the vectors $y_1, \ldots, y_n$. $L'$ is an even lattice, since $H$ was $q_L$-isotropic. □

If $L'$ is an overlattice of $L$ given by an isotropic subgroup $H < A_L$, the discriminant group of $L'$ is simply $A_{L'} = H^\perp/H$ with the induced quadratic form in the quotient. This quadratic form is well-defined, as $H < A_L$ is isotropic.

**Definition 1.1.9.** A lattice $L$ is called a *root-overlattice* if one of the following equivalent conditions holds:

1. $L$ is an overlattice of a root lattice;
2. $L$ admits a $\mathbb{Q}$-basis given by vectors of norm $-2$;
3. The root part $L_{root}$ has the same rank as $L$.

Otherwise, we say that $L$ is a *non-root-overlattice*.

### 1.1.3 Genus

**Definition 1.1.10** ([Nik79b], Corollary 1.13.4)**.** Two even lattices $L$, $M$ with the same signature are said to be in the same *genus* if one of the following equivalent conditions holds:

1. The discriminant groups $A_L$, $A_M$ are isometric;

2. The lattices $U \oplus L$, $U \oplus M$ are isometric.

If $L$ is a lattice, we say that the *genus* of $L$ is the set of all lattices $M$ in the same genus of $L$, up to isometry. The genus of a lattice is always a finite set.

When the genus of a lattice $L$ is trivial, we will say that $L$ is *unique in its genus*. Only few negative definite lattices are unique in its genus, and they are completely classified, thanks to works by Watson, Lorch and Kirschmer (cf. [Wat63], [LK13]). More precisely, the following holds:

**Theorem 1.1.11** ([LK13])**.** *A negative definite lattice is unique in its genus if and only if it is a multiple of a lattice in the (finite) list avalaible at [LK13]. In particular, it has rank at most* 10*.*

We will need the following easy result.

**Lemma 1.1.12.**

1. *Let $L$ be an even, negative definite lattice, and consider two isotropic subgroups $H, H' < A_L$ such that there exists an isometry $\varphi \in \mathrm{O}(L)$ with $\overline{\varphi}(H) = H'$. Then $H, H'$ give rise to isometric overlattices of $L$.*

2. *Let $L, L'$ be two even, negative definite lattices in the same genus. Then, for every overlattice $P$ of $L$, there exists an overlattice $P'$ of $L'$ such that $P$ and $P'$ are in the same genus.*

*Proof.*

1. The two overlattices are obtained by adjoining the generators of $H$ (more precisely, their preimages in $L^\vee$ under the projection $L^\vee \to A_L$) to $L$, so the isometry $\varphi$ of $L$ extends to an isometry of the two overlattices.

2. $P$ corresponds to an isotropic subgroup $H < A_L$, which in turn can be seen as an isotropic subgroup $H'$ of $A_{L'}$ by using the isometry $A_L \cong A_{L'}$. The overlattice $P'$ of $L'$ corresponding to $H'$ is then in the genus of $P$, since they have isometric discriminant groups $A_P \cong H^\perp/H \cong (H')^\perp/H' \cong A_{P'}$.

□

### 1.1.4 Dense sphere packings

We start with an easy remark. If $R$ is a root-overlattice of rank $r$, then the (absolute value of the) determinant of $R$ is at most $2^r$. This is because by Lemma 1.1.7 this maximum is attained at a root lattice, and it is rather straightforward to see that the (only) root lattice attaining the maximum is $A_1^r$, with determinant $\pm 2^r$.

One of the main questions in the theory of dense sphere packings is to "reverse" this line of reasoning. More precisely, given a definite lattice $L$ with minimum $m$, can we bound the (absolute value of the) determinant of $L$ from below? The *minimum* $\min(L)$ of a lattice $L$ is simply the minimum absolute value of the norms of the vectors of $L$. The following is one of the main theorems in the theory of dense sphere packings.

**Theorem 1.1.13** ([CS99], Table 1.2)**.** *Let $L$ be an even, negative definite lattice of rank $r$, with $m = \min(L)$. Then there exists a number $\delta_r > 0$ such that*

$$\frac{(m/4)^{(r/2)}}{\sqrt{|\det(L)|}} \leq \delta_r.$$

*In other words, there exists a constant $\Delta_{m,r}$, depending on $r$ and $m = \min(L)$, such that $|\det(L)| \geq \Delta_{m,r}$.*

If $L$ has no roots, then clearly $\min(L) \geq 4$. Therefore [CS99, Table 1.2] provides some lower bounds for $|\det(L)|$:

| $r = \mathrm{rk}(L)$ | $\Delta_r$ | $r = \mathrm{rk}(L)$ | $\Delta_r$ | $r = \mathrm{rk}(L)$ | $\Delta_r$ |
|---|---|---|---|---|---|
| 1 | 4 | 7 | 256 | 13 | 192 |
| 2 | 12 | 8 | 256 | 14 | 146 |
| 3 | 32 | 9 | 278 | 15 | 106 |
| 4 | 64 | 10 | 283 | 16 | 73 |
| 5 | 128 | 11 | 266 | 17 | 48 |
| 6 | 192 | 12 | 233 | 18 | 29 |

Table 1.2: Lower bounds for the determinant of definite even lattices without roots.

*Remark* 1.1.14. The bounds in Table 1.2 are not known to be sharp if $n > 8$. It is in general very hard to find sharp bounds, and the interested reader can consult [CS99, Chapters 1–2] for a comprehensive survey about dense sphere packings.

Theorem 1.1.13 has many important consequences. For our purposes, we combine it with the initial remark in order to obtain the following result:

**Proposition 1.1.15.** *If $r \leq 8$, then any lattice $L$ of rank $r$ in the genus of a root-overlattice has $\min(L) = 2$. Equivalently, if $r \leq 8$, there are no root-overlattices in the genus of a lattice with no roots.*

*Proof.* Notice first that two lattices in the same genus have the same determinant. Therefore, if $r < 8$, the claim follows from the fact that $|\det(L)| \leq 2^r$ if $L$ is a root-overlattice, while $|\det(L)| \geq \Delta_r > 2^r$ if $L$ has no roots by Table 1.2.
If instead $r = 8$, we have that $\Delta_8$ is exactly $2^8 = 256$. This means that the same argument with the two inequalities works analogously if $L$ is not $A_1^8$, the only root-overlattice of rank

8 with determinant 256. However, we can check with the Siegel mass formula [CS88] that $A_1^8$ is unique in its genus: indeed, if $q$ is the quadratic form on the discriminant group of $A_1^8$, we have

$$m(q) = \frac{1}{10321920} = \frac{1}{|\operatorname{O}(A_1^8)|}.$$

□

*Remark* 1.1.16.
- A more geometric proof that $A_1^8$ is unique in its genus uses the theory of elliptic fibrations on K3 surfaces (cf. Section 1.2). Indeed by [Nik79a] any K3 surface $X$ with Néron-Severi lattice isometric to $U \oplus A_1^8$ has finite automorphism group, so every elliptic fibration on $X$ has finitely many sections. Therefore by the Shioda-Tate formula 1.2.6 any lattice $L$ such that $U \oplus L \cong U \oplus A_1^8$ is a root-overlattice. As $A_1^8$ is the unique root-overlattice with determinant 256, we conclude that $L \cong A_1^8$, i.e. $A_1^8$ is unique in its genus.
- The previous result is in general not true if $r > 8$. For instance, there exists a lattice with minimum 4 in the genus of $A_1^{12}$. We will see other similar examples in Chapter 2, arising from more geometric constructions.

## 1.2 K3 surfaces

In this section we recall some basic fundamental properties of K3 surfaces, with particular interest towards elliptic fibrations. The main references are [Mir89], [SS19], [Huy16, Chapter 11] and [SS10]. We fix an algebraically closed field $k = \overline{k}$ of characteristic $\neq 2,\ 3$; we will then point out which results are only valid when $k$ is the field $\mathbb{C}$ of complex numbers.

**Definition 1.2.1.** A smooth, projective surface $X$ over $k$ is a *K3 surface* if $H^1(X, \mathcal{O}_X) = 0$ and its canonical bundle $K_X$ is trivial.

*Remark* 1.2.2. In Chapter 5 we will be interested in K3 surfaces defined over number fields. A geometrically smooth, projective surface $X$ over a field $K$ is a K3 surface if the base change $X_{\overline{K}}$ to an algebraic closure $\overline{K}$ of $K$ is a K3 surface.

$H^2(X, \mathbb{Z})$ is naturally endowed with a unimodular intersection pairing, making it isometric to the *K3 lattice*

$$\Lambda_{K3} = U^3 \oplus E_8^2,$$

where $U$ is the hyperbolic plane and $E_8$ is the unique (up to isometry) even unimodular negative definite lattice of rank 8. In particular the signature of $H^2(X, \mathbb{Z})$ is $(3, 19)$. Since the canonical bundle of $X$ is trivial, there exists a unique (up to scalars) nowhere-vanishing $(2, 0)$-form $\omega_X$ on $X$.

The Néron-Severi group $\operatorname{NS}(X) = \operatorname{Pic}(X)$ is a hyperbolic sublattice of $H^2(X, \mathbb{Z})$, i.e. it has signature $(1, \rho(X) - 1)$, where $\rho(X) = \operatorname{rk}(\operatorname{NS}(X))$ is the *Picard rank* of $X$. The *transcendental lattice* $\operatorname{T}(X) = \operatorname{NS}(X)^\perp \subseteq H^2(X, \mathbb{Z})$ is the orthogonal complement of $\operatorname{NS}(X)$ in $H^2(X, \mathbb{Z})$, and its complexification $\operatorname{T}(X)_\mathbb{C} = \operatorname{T}(X) \otimes \mathbb{C}$ contains the $(2, 0)$-form $\omega_X$.

### 1.2.1 Elliptic fibrations

Let $X$ be a K3 surface.

**Definition 1.2.3.** A *genus* 1 *fibration* on $X$ is a proper, flat morphism $\pi : X \to \mathbb{P}^1$ such that the generic fiber of $\pi$ is a smooth elliptic curve. If $\pi$ admits a section, then it is called an *elliptic fibration*. A K3 surface admitting an elliptic fibration is called *elliptic*. If there is no chance of ambiguity, we will call *section* of $\pi$ the image $S = \sigma(\mathbb{P}^1)$ of a section $\sigma : \mathbb{P}^1 \to X$.

*Remark* 1.2.4. Definition 1.2.3 above is valid for any base field $K$, even if the characteristic of $K$ is 2 or 3 or if $K$ is not algebraically closed. The advantage, when the characteristic is $\neq 2, 3$, is that the condition that the generic fiber is smooth is equivalent to the existence of a smooth closed fiber. Instead, in the case when the characteristic is 2 or 3, *quasi-elliptic* fibrations can appear; we refer the interested reader to [SS19, Section 7.5] for more details.

On K3 surfaces, the genus formula for a smooth curve $C$ reads $C^2 = 2g(C) - 2$. Therefore smooth elliptic curves on K3 surfaces have self-intersection 0. Moreover by Riemann-Roch we have that $\dim |C| = g(C)$, so any smooth elliptic curve $E$ on a K3 surface $X$ induces a genus 1 fibration $|E|$. There exists a 1:1 correspondence between isotropic (that is, of norm 0), non-zero nef elements in $\mathrm{NS}(X)$ and genus 1 fibrations on $X$. Moreover, if $E \in \mathrm{NS}(X)$ is isotropic and nef, it induces an elliptic fibration if and only if there exists an element $S \in \mathrm{NS}(X)$ of square $-2$ such that $E \cdot S = 1$. Indeed, either $S$ or $-S$ is effective by Riemann-Roch, but $E \cdot S = 1$ and $E$ is nef, so $S$ must be effective. $S$ decomposes as the sum of some irreducible curves. As $E$ is nef, it has non-negative intersection with all irreducible curves on $X$, hence from the equation $E \cdot S = 1$ we deduce that there exists a component $S_0$ of $S$ such that $E \cdot S_0 = 1$. $S_0$ is a section of the elliptic fibration induced by $E$, since it is a smooth, rational curve that meets $E$ at precisely one point.

Now assume $k = \mathbb{C}$. If $|E|$ is a genus 1 fibration, we can define its *degree* as the lowest positive intersection of $E$ with a smooth curve on $X$. The degree of $|E|$ is 1 if and only if $E$ is an elliptic fibration. If instead the degree $d$ of $|E|$ is strictly bigger than 1, then we can consider the associated Jacobian fibration $J(X)$ (cf. [Huy16, Section 11.4]), which is again a K3 surface. The Néron-Severi lattice of $J(X)$ is closely related to the Néron-Severi lattice of $X$. For instance, we have that

$$\rho(J(X)) = \rho(X), \qquad \det(\mathrm{NS}(J(X))) = \det(\mathrm{NS}(X))/d^2$$

(cf. [Keu00, Lemma 2.1]).

### 1.2.2 The Shioda-Tate formula

Let $X$ be a K3 surface, and $|E|$ an elliptic fibration on $X$ with section $S_0$. The sublattice $\langle E, S_0 \rangle$ of $\mathrm{NS}(X)$ generated by the elliptic fiber and its section is isometric to the hyperbolic plane $U$. Since $U$ is unimodular, we have an orthogonal decomposition $\mathrm{NS}(X) = U \oplus L$, where $L$ is a negative definite lattice of rank $\rho(X) - 2$. $L$ encodes all the geometric information about the elliptic fibration $|E|$, including the structure of its reducible fibers. More precisely, the root part $L_{root}$ of $L$ is a root lattice, so it is sum of some ADE lattices. Each summand is generated by the irreducible components of a reducible fiber not intersecting the section $S_0$, and the Kodaira type of the reducible fiber determines the type of the ADE lattice. For instance, a fiber of type $I_n$ produces the lattice $A_{n-1}$, a fiber of type $I_n^*$ produces the lattice $D_{n+4}$, and so on (cf. [Mir89, Table II.3.1]).

**Definition 1.2.5.** The *Mordell-Weil group* of $|E|$ is the set $\mathrm{MW}(E)$ of sections of the elliptic fibration $|E|$. It has the structure of an abelian group, induced by the group structure on the generic fiber. The chosen section $S_0$ is the neutral element of this addition, and it is called the *zero section.*

We have an isomorphism of groups $L/L_{root} \cong \mathrm{MW}(E)$ (cf. [Mir89, Theorem VII.2.1]). From this it follows immediately (cf. also the original statement [Shi72, Corollary 1.5]):

**Theorem 1.2.6** (Shioda-Tate formula)**.** *Let $X$ be a K3 surface and $|E|$ an elliptic fibration on $X$. Then*

$$\rho(X) = 2 + \sum_{t \in \mathbb{P}^1} (n_t - 1) + \mathrm{rk}(\mathrm{MW}(E)),$$

*where $n_t$ is the number of irreducible components of the fiber over $t \in \mathbb{P}^1$.*

The sum in the right-hand side is clearly finite, since there are only finitely many reducible fibers (we are assuming $\mathrm{char}(k) \neq 2, 3$). We deduce that $\mathrm{rk}(\mathrm{MW}(E)) \leq \rho(X) - 2$, and equality holds if and only if $|E|$ has only irreducible fibers (i.e., all the fibers are either smooth elliptic curves, or nodal or cuspidal rational curves).

There is another important equality concerning the number of singular fibers and their Euler characteristic, when $k = \mathbb{C}$.

**Proposition 1.2.7** ([Mir89], Lemma IV.3.3)**.** *Assume $k = \mathbb{C}$. Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration induced by $E$. Denote by $S \subseteq \mathbb{P}^1$ the subset of points of $\mathbb{P}^1$ whose fiber is singular. Then*

$$24 = e(X) = \sum_{t \in S} (e(\pi^{-1}(t)),$$

*where $e$ denotes the Euler characteristic. Now define*

$$e(L) := \begin{cases} n+1 & \text{if } L = A_n \\ n+2 & \text{if } L = D_n \\ n+2 & \text{if } L = E_n. \end{cases}$$

*Then, if the root part $L_{root}$ of the orthogonal complement $L$ decomposes as $L_{root} = \bigoplus_{i \in I} R_i$, where the lattices $R_i$ are ADE lattices, we have*

$$\sum_{i \in I} e(R_i) \leq 24.$$

*Proof.* The first part is precisely [Mir89, Lemma IV.3.3]. For the second part, we can easily compute the Euler characteristic of each Kodaira fiber. The values can be found in [Mir89, Table IV.3.1]. Then the inequality follows immediately from this. Notice that we only have an inequality because the sum only runs over the reducible fibers. □

We conclude the paragraph with a fundamental geometric interpretation of the genus of a lattice. Let $X$ be a K3 surface admitting an elliptic fibration $|E|$, inducing a decomposition $\mathrm{NS}(X) = U \oplus L$. As explained above, $L$ encodes a lot of geometric information about the fibration $|E|$. Now by Definition 1.1.10 the genus $L$ consists of the negative lattices $M$ such that $\mathrm{NS}(X) = U \oplus L = U \oplus M$. Thus the genus of $L$ parametrizes the possible "structures" of elliptic fibrations on the K3 surface $X$. For instance, we have the following:

**Lemma 1.2.8.** *Let $X$ be a K3 surface admitting an elliptic fibration $|E|$, inducing a decomposition $\mathrm{NS}(X) = U \oplus L$. Then $|E|$ has finitely many sections if and only if $L$ is a root-overlattice.*

*Proof.* We have an isomorphism of groups $L/L_{root} \cong \mathrm{MW}(E)$, so $|E|$ has finitely many sections if and only if $L_{root}$ has finite index in $L$. Since $L_{root}$ is a root lattice, this is equivalent to $L$ being a root-overlattice. □

Elliptic fibrations with finitely many sections are very special. We deal with them in the next section.

### 1.2.3 Elliptic fibrations with finitely many sections

In this section we need $k = \mathbb{C}$. Let $X$ be a K3 surface and $|E|$ an elliptic fibration on $X$ admitting a finite number of sections. The choice of a zero section $S_0$ of $|E|$ induces an orthogonal decomposition $\mathrm{NS}(X) = U \oplus L$. By Lemma 1.2.8, we have that $L$ is a root-overlattice.

Since we are restricting ourselves to K3 surfaces, the finite group $\mathrm{MW}(E) \cong L/L_{root}$ can only be one of the following 12 groups (cf. [MP89, Table 4.5]):

$$\begin{gathered}\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \\ \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}.\end{gathered} \tag{1.1}$$

Moreover, the existence of torsion sections in $\mathrm{MW}(E)$ depends on the root lattice $L_{root}$, as the next proposition shows.

**Proposition 1.2.9.** *Denote by $\ell = |\mathrm{MW}(E)| = [L : L_{root}]$ the index of the overlattice $L$. Then:*

1. $\ell^2 \mid \det(L_{root})$;

2. *If $L_{root}$ contains a $D_n$ (resp. an $E_n$ lattice) as a summand, then $\ell$ divides* $\det(D_n)$ *(resp.* $\det(E_n)$*). In particular $\ell \leq 4$.*

*Proof.* 1. It follows directly from Lemma 1.1.7.

2. The restriction map

$$\mathrm{MW}(E) \to \mathrm{Tors}(E_t),$$

sending any torsion section of $|E|$ to its intersection point with the fiber $E_t$ over $t$, is injective for any $t \in \mathbb{P}^1$ (cf. [Mir89, Corollary VII.3.3]). Therefore the cardinality $\ell$ of $\mathrm{MW}(E)$ divides the cardinality of $\mathrm{Tors}(E_t)$ for any $t \in \mathbb{P}^1$. If $E_t$ is a singular fiber of additive type (which is assured if its dual graph of $(-2)$-curves not meeting the zero section is $D_n$ or $E_n$), then the number of its torsion points coincides with the determinant of the corresponding lattice (cf. [Mir89, Lemma VII.3.5]).

□

### 1.2.4 Positive and nef cones and automorphisms

We assume that the base field is $k = \mathbb{C}$. Let $X$ be a K3 surface. The *positive cone* $\mathcal{C}_X$ of $X$ is the cone

$$\mathcal{C}_X = \{v \in \mathrm{NS}(X) \otimes \mathbb{R} \mid v^2 > 0\}^+,$$

where the superscript + denotes the component containing an ample class. For any $\delta \in \mathrm{NS}(X)$ of square $-2$, we define the *reflection* associated with $\delta$ as the isometry $s_\delta$ of $\mathrm{NS}(X)$ such that

$$v \mapsto v + (v \cdot \delta)\delta$$

for any $v \in \mathrm{NS}(X)$. The subgroup of the group $\mathrm{O}(\mathrm{NS}(X))$ of isometries of the Néron-Severi lattice generated by all these reflections is called the *Weyl subgroup* of $\mathrm{NS}(X)$, and it is denoted $W(\mathrm{NS}(X))$. We denote by $\mathrm{O}^+(\mathrm{NS}(X))$ the subgroup of index 2 of $\mathrm{O}(\mathrm{NS}(X))$ preserving the positive cone. Notice that $W(\mathrm{NS}(X))$ is contained in $\mathrm{O}^+(\mathrm{NS}(X))$.

There exists a chamber decomposition of the positive cone $\mathcal{C}_X$, and the Weyl subgroup $W(\mathrm{NS}(X))$ acts transitively on the set of chambers (cf. [Huy16, Proposition 8.2.6]). The transitivity of the action of $W(\mathrm{NS}(X))$ can be rephrased in geometric terms: if $v \in \mathcal{C}_X$ is an element in the positive cone, either it is nef, or there exists a smooth rational curve $\delta$ such that $v \cdot \delta < 0$ (cf. [Huy16], Corollary 8.1.7). Then $v' := s_\delta(v)$ has positive intersection with $\delta$, and we can repeat the process with $v'$. After a *finite* number of reflections, the element $v$ becomes nef. This is because the intersection of $v$ with a fixed ample class gets smaller at each step. We conclude that there exists a unique nef element in the orbit $W(\mathrm{NS}(X)) \cdot v$. A fundamental domain for the action of $W(\mathrm{NS}(X))$ on the positive cone is given by the nef cone (cf. [Huy16, Corollary 8.2.11]). By the discussion above, the quotient $\mathrm{O}^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$ can be viewed as the subgroup of $\mathrm{O}^+(\mathrm{NS}(X))$ of isometries preserving the nef cone.

Let $f \in \mathrm{Aut}(X)$ be an automorphism of the K3 surface $X$. $f$ acts as an isometry $f^*$ on the Néron-Severi lattice, hence there exists a map

$$\mathrm{Aut}(X) \to \mathrm{O}^+(\mathrm{NS}(X)),$$

since automorphisms preserve the positive cone. The next fundamental proposition characterizes kernel and cokernel of this map. First, we need some notation. We denote by $\mathrm{O}_{\Delta^+}(\mathrm{NS}(X))$ the subgroup of $\mathrm{O}^+(\mathrm{NS}(X))$ of isometries preserving the set $\Delta^+ \subseteq \mathrm{NS}(X)$ of smooth $(-2)$-curves (i.e. those isometries $\varphi \in \mathrm{O}(\mathrm{NS}(X))$ such that $\varphi(C) = C$ for every class $C \in \mathrm{NS}(X)$ corresponding to a smooth $(-2)$-curve on $X$). Not all isometries preserve the set of smooth $(-2)$-curves. For instance reflections do not: if $\delta \in \mathrm{NS}(X)$ represents a smooth $(-2)$-curve on $X$, then $s_\delta(\delta) = -\delta$.
Moreover, we denote by $\mathrm{O}_{Hdg}(\mathrm{T}(X)) < \mathrm{O}(\mathrm{T}(X))$ the group of *Hodge isometries* of the transcendental part $T(X) \subseteq H^2(X, \mathbb{Z})$, i.e. the isometries of the transcendental lattice that preserve the integral Hodge structure on $\mathrm{T}(X)$. We refer to [Huy16, Section 3.2] for more details.

**Proposition 1.2.10** ([PŠ71], Section 7 - [Huy16], Chapter 15)**.** *Let $X$ be a smooth complex projective K3 surface. Then:*

1. *The homomorphism*
$$\mathrm{Aut}(X) \to \mathrm{O}^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$$
*has finite kernel and cokernel.*

2. *The group* $\mathrm{Aut}(X)$ *is isomorphic to the group*
$$\{(\alpha, \beta) \in \mathrm{O}_{\Delta^+}(\mathrm{NS}(X)) \times \mathrm{O}_{Hdg}(\mathrm{T}(X)) \mid \overline{\alpha} = \overline{\beta} \in \mathrm{O}(A_{\mathrm{NS}(X)}) = \mathrm{O}(A_{\mathrm{T}(X)})\},$$
*where $\overline{\alpha}$ and $\overline{\beta}$ are the induced isometries of the isometric discriminant groups $A_{\mathrm{NS}(X)} = A_{\mathrm{T}(X)}$.*

Now assume that the K3 surface $X$ admits an elliptic fibration $|E|$, and let $S \in \mathrm{MW}(E)$ be any section of $|E|$. The section $S$ induces the *translation* $\tau_S \in \mathrm{Aut}(X)$ via the group structure on the generic fiber. More precisely, $S$ induces an automorphism of the generic fiber $E_\eta$ by translation by $S \cap E_\eta$. This automorphism extends to a well-defined automorphism of the surface $X$. Since different sections produce different automorphisms, we obtain an injective map

$$\mathrm{MW}(E) \hookrightarrow \mathrm{Aut}(X). \tag{1.2}$$

The translations $\tau_S$ preserve the elliptic fibration $|E|$. Conversely, if $f$ is any automorphism of $X$ preserving the elliptic fibration $|E|$, then one of its powers is a translation. This is because by Theorem 1.2.6 and Proposition 1.2.7 any elliptic fibration on a K3 surface has at least 3 singular fibers, hence the action of any automorphism of $X$ preserving $|E|$ acts on the base $\mathbb{P}^1$ with finite order.

We conclude the section with a result that will come in handy later.

**Proposition 1.2.11.** *Let $X$ be an elliptic K3 surface, $|F|$ an elliptic fibration on $X$, and $f \in \mathrm{O}^+(\mathrm{NS}(X))$.*

1. *If $E = f(F)$ is nef, then there exists $s \in W(\mathrm{NS}(X)) < \mathrm{O}^+(\mathrm{NS}(X))$ such that $g = f \circ s$ preserves the nef cone and $g(F) = f(F) = E$.*

2. *If $|F|$ has only irreducible fibers and $E = f(F)$ is nef, then $f$ preserves the nef cone.*

3. *If $f$ preserves the nef cone and the set of Hodge isometries is trivial, i.e. $\mathrm{O}_{Hdg}(\mathrm{T}(X)) = \{\pm \mathrm{id}\}$, then $f$ corresponds to an automorphism of $X$ if and only if $\pm \mathrm{id} = \overline{f} \in \mathrm{O}(A_{\mathrm{NS}(X)})$.*

*Proof.* 1. If $f$ preserves the nef cone there is nothing to prove. So assume that there exists a nef divisor $H \in \mathrm{NS}(X)$ such that $f(H)$ is not nef, that is, there exists a $(-2)$-curve $C$ such that $f(H) \cdot C < 0$. Then $f^{-1}(C)$ has square $-2$, so by Riemann-Roch either $f^{-1}(C)$ or $-f^{-1}(C)$ is effective; since $H \cdot f^{-1}(C) = f(H) \cdot C < 0$, necessarily $-f^{-1}(C)$ is effective by the nefness of $H$. Now notice that $E \cdot C = f(F) \cdot C \geq 0$, as $E$ is nef and $C$ is effective, and also $-f(F) \cdot C = F \cdot (-f^{-1}(C)) \geq 0$, as $F$ is nef and $-f^{-1}(C)$ is effective; these two inequalities imply that $E \cdot C = 0$, so $C$ is a component of a reducible fiber of the elliptic fibration $|E|$. Now we can compose $f$ with the reflection $s_C$ associated to the $(-2)$-curve $C$ and obtain an isometry $f' = s_C \circ f$ such that $f'(F) = s_C(E) = E$, as $C$ is orthogonal to $E$, and

$$f'(H) \cdot C = f(H) \cdot s_C(C) = f(H) \cdot (-C) > 0.$$

Since there are only finitely many components in the reducible fibers of the elliptic fibration $|E|$, we can repeat the argument and obtain an isometry $g = s_{C_1} \circ \ldots \circ s_{C_r} \circ f$ such that $g(F) = E$ and $g$ preserves the nef cone.

2. We can repeat the argument of the previous point. As there are no $(-2)$-curves orthogonal to $F$ and $f$ is an isometry, then there are no $(-2)$-curves orthogonal to $E$, and we conclude that $f$ preserves the nef cone.

3. By Proposition 1.2.10 a power of $f$ corresponds to an automorphism of $X$, say $f^n$. We claim that $f \in \mathrm{O}_{\Delta^+}(\mathrm{NS}(X))$, i.e. $f$ preserves the set of $(-2)$-curves. Assume by contradiction that there exists a $(-2)$-curve $C$ such that $D = f^{-1}(C)$ is not irreducible. Surely $D$ is effective: for, let $H \in \mathrm{NS}(X)$ be a nef divisor not orthogonal to $D$. Then $H \cdot D = f(H) \cdot C > 0$ by nefness of $f(H)$, hence $D$ is effective. Therefore $D = C_1 + \ldots + C_r$ splits as the sum of $r \geq 2$ irreducible curves. Without loss of generality $C_1$ is a $(-2)$-curve, so we can repeat the previous argument to $f^{-1}(C_1)$, which is again effective of norm $-2$. After $n$ steps, we have that

$$f^{-n}(C) = C_1' + \ldots + C_s' \tag{1.3}$$

for some irreducible curves $C_1', \ldots, C_s'$, $s \geq 2$. Since $f^{-n}$ is an automorphism of $X$ by assumption, then $f^{-n}(C)$ is a $(-2)$-curve, and thus it is rigid by Riemann-Roch. Since

$s \geq 2$, the equality (1.3) is contradictory, thus showing that indeed $f$ preserves the set of $(-2)$-curves. Therefore by Proposition 1.2.10 $f$ corresponds to an automorphism of $X$ if and only if $\overline{f} \in \mathrm{O}(A_{\mathrm{NS}(X)})$ coincides with the restriction of a $g \in \mathrm{O}_{Hdg}(\mathrm{T}(X)) = \{\pm \mathrm{id}\}$. □

*Remark* 1.2.12. From [Ogu02], Lemma 4.1, we know that the assumption $\mathrm{O}_{Hdg}(\mathrm{T}(X)) = \{\pm \mathrm{id}\}$ is always satisfied if $X$ has odd Picard rank. Moreover, if $X$ has even Picard rank (and $\rho(X) < 20$) and the period $\omega_X \in \mathrm{T}(X)_{\mathbb{C}}$ is very general, then again $\mathrm{O}_{Hdg}(\mathrm{T}(X)) = \{\pm \mathrm{id}\}$. Indeed, any Hodge isometry of the transcendental lattice $\mathrm{T}(X)$ has $\omega_X$ as an eigenvector, so it suffices to choose $\omega_X$ outside the countable union of lines in $\mathrm{T}(X)_{\mathbb{C}}$ corresponding to the eigenvectors of isometries in $\mathrm{O}(\mathrm{T}(X))$.

## 1.3 Moduli spaces of lattice polarized K3 surfaces

In this section we review the construction of the moduli spaces of lattice polarized K3 surfaces. Our main reference is [Dol96].

Let $L$ be a lattice of signature $(2, n)$. Let $\Omega_L$ be one of the two connected components of

$$\{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, w) = 0,\ (w, \bar{w}) > 0\}.$$

It is a hermitian symmetric domain of type IV and dimension $n$. $\mathrm{O}^+(L)$ is the index two subgroup of the orthogonal group $\mathrm{O}(L)$ preserving $\Omega_L$. If $\Gamma < \mathrm{O}^+(L)$ is of finite index we denote by $\mathcal{F}_L(\Gamma)$ the quotient $\Gamma \backslash \Omega_L$. By a result of Baily and Borel [BB66], $\mathcal{F}_L(\Gamma)$ is a quasi-projective variety of dimension $n$. Such varieties are called *modular orthogonal varieties*. Recall that the boundary of the Baily-Borel compactification of $\mathcal{F}_L(\Gamma)$ is 1-dimensional: its 0-dimensional components are called 0-*cusps*, while its 1-dimensional components are called 1-*cusps*. We define:

$$\widetilde{\mathrm{O}}(L) := \ker(\mathrm{O}(L) \to \mathrm{O}(A_L))$$

and

$$\widetilde{\mathrm{O}}^+(L) := \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L).$$

We fix an integral even lattice $M$ of signature $(1, t)$ with $t \geq 0$.

**Definition 1.3.1.** An *$M$-polarized K3 surface* is a pair $(X, j)$ where $X$ is a K3 surface and $j : M \hookrightarrow \mathrm{NS}(X)$ is a primitive embedding.

Let

$$N := j(M)^{\perp}_{\Lambda_{K3}}$$

be the orthogonal complement of $M$ in $\Lambda_{K3}$. It is an integral even lattice of signature $(2, 19 - t)$. By the Torelli theorem (cf. [PŠ71], or [Dol96, Corollary 3.2]), the moduli spaces of $M$-polarized K3 surfaces can be identified with the quotient of a classical hermitian symmetric domain of type IV and dimension $19 - t$ by an arithmetic group. More precisely, the 2-form $\omega_X$ of a $M$-polarized K3 surface $X$ determines a point in the symmetric space $\Omega_N$ (the *period domain*), unique up to the action of the group (cf. [Dol96, Proposition 3.3])

$$\widetilde{\mathrm{O}}^+(N) = \{g \in \mathrm{O}^+(\Lambda_{K3}) \mid g|_M = \mathrm{id}\}.$$

**Theorem 1.3.2** ([Dol96], Section 3)**.** *The variety $\mathcal{F}_N(\widetilde{\mathrm{O}}^+(N))$ is isomorphic to the coarse moduli space of $M$-polarized K3 surfaces. Its dimension is $20 - \mathrm{rk}(M)$.*

## 1.4 Projective models of K3 surfaces

In this section we recall three well-known geometric constructions of K3 surfaces. Namely, double covers of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ (see Section 1.4.1) and of the Hirzebruch surface $\mathbb{F}_4$ (see Section 1.4.2) branched over suitable curves define lattice polarized K3 surfaces with respect to the lattices $U(2)$ and $U$ respectively. Furthermore, every elliptic K3 surface can be reconstructed from its Weierstrass fibration (see Section 1.4.3).

### 1.4.1 Double covers of $\mathbb{P}^1 \times \mathbb{P}^1$

Let $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$ be the smooth quadric surface in $\mathbb{P}^3$. Its Picard group is generated by the classes of the two pencils $\ell_1, \ell_2$ of lines, hence $\mathrm{Pic}(\mathbb{F}_0)$ endowed with the intersection form on $\mathbb{F}_0$ is isometric to the hyperbolic plane $U$. The canonical bundle is $K_{\mathbb{F}_0} = \mathcal{O}_{\mathbb{F}_0}(-2,-2)$.

Now let $\pi : X \to \mathbb{F}_0$ be the double cover branched over a smooth curve $B \in |-2K_{\mathbb{F}_0}| = |\mathcal{O}_{\mathbb{F}_0}(4,4)|$. Then $X$ is a smooth K3 surface. The pullbacks $E_i = \pi^*\ell_i$ for $i = 1, 2$ are smooth elliptic curves, and $E_1 \cdot E_2 = 2\ell_1 \cdot \ell_2 = 2$, so that

$$\langle E_1, E_2 \rangle = U(2) \hookrightarrow \mathrm{NS}(X).$$

This embedding is primitive, and $\mathrm{NS}(X) = U(2)$ for a very general branch divisor $B$.

Assume now that there exists a smooth rational curve $C \in |\mathcal{O}_{\mathbb{F}_0}(1,d)|$ for $d \geq 0$ intersecting $B$ with even multiplicities. For instance, $C$ can be simply tangent to $B$ in exactly $2d+2$ points. Then we have the following (cf. [Fes18, Proposition 5.1]):

**Lemma 1.4.1.** *Let $\nu : X \to Y$ be a double cover of smooth projective surfaces branched over a smooth curve $B$, and assume that there exists a smooth rational curve $C \subseteq Y$ intersecting $B$ with even multiplicities. Then the pullback $\nu^*C$ splits into two irreducible components, both isomorphic to $C$.*

*Proof.* Let $D := \nu^{-1}(C) \subseteq X$. The double cover $\nu$ induces a double cover $\overline{\nu} : \overline{D} \to C$, where $\overline{D}$ denotes the normalization of $D$. $\overline{\nu}$ is isomorphic to an unbranched double cover, because the branch locus of $\overline{\nu}$ coincides with the set $b(C) := \{x \in C \mid \mathrm{mult}_x(C,B) \equiv 1 \pmod 2\} = \varnothing$. The unique unbranched double cover of $C \cong \mathbb{P}^1$ is given by a disjoint union of two smooth rational curves isomorphic to $C$. This implies that $D$ splits as the union of two irreducible components (not necessarily disjoint). □

In the case $Y = \mathbb{F}_0$ as above, the pullback $D = \pi^*C = D_1 + D_2$ splits into the union of two irreducible components $D_1, D_2 \cong \mathbb{P}^1$. Since $D_1$ is smooth and rational, we have $D_1^2 = -2$, and moreover $D_1 \cdot E_1 = 1$, $D_1 \cdot E_2 = d$. This implies that there exists an embedding (not necessarily primitive)

$$\langle E_1, E_2, D_1 \rangle = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & d \\ 1 & d & -2 \end{pmatrix} \cong U \oplus \langle -2(2d+4) \rangle \hookrightarrow \mathrm{NS}(X).$$

If instead the branch divisor $B$ is not smooth, but has simple singularities, the double cover $\pi : X \to \mathbb{F}_0$ is a K3 surface with isolated simple singularities. Therefore the minimal desingularization $\widetilde{X} \to X$ is a smooth K3 surface, since simple singularities do not change adjunction.

The following result is well known, but we include its proof for the sake of completeness.

**Proposition 1.4.2.** *Let $X$ be an elliptic K3 surface with $\mathrm{NS}(X) \cong U \oplus \langle -2k \rangle$ for some $k \geq 1$. Then $X$ can be realized as a double cover of $\mathbb{F}_0$ if and only if $k$ is even and $k \geq 4$.*

*Proof.* If $X$ is a double cover of $\mathbb{F}_0$, the pullback map induces a primitive embedding

$$U(2) \hookrightarrow \mathrm{NS}(X) = U \oplus \langle -2k \rangle.$$

Any even lattice of rank 3 containing primitively $U(2)$ has determinant divisible by 4, so we conclude that $k = \frac{1}{2} \det(\mathrm{NS}(X))$ is even.

Conversely assume that $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 4$ even. Then as above we have an isomorphism

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & d \\ 1 & d & -2 \end{pmatrix} \cong U \oplus \langle -2k \rangle$$

for $d = \frac{1}{2}(k-4) \geq 0$, so there are two genus one fibrations $|E_1|, |E_2| : X \to \mathbb{P}^1$ induced by the two elements $E_1, E_2$ of the basis of square zero. We can now consider the surjective map

$$\pi = (|E_1|, |E_2|) : X \to \mathbb{F}_0.$$

It is a morphism of degree 2, since the preimage of any point of $\mathbb{F}_0$ consists of the two points of intersection of two elliptic curves in $|E_1|$ and $|E_2|$, as $E_1 \cdot E_2 = 2$. Consider the branch divisor $B$; if $B$ is smooth, then $\pi$ is a double cover, as claimed. Assume by contradiction that $B$ is singular. $B$ must have simple singularities, since otherwise the canonical divisor of $X$ would be strictly negative. Thus $X$ is the desingularization of the double cover $\widetilde{\pi} : \widetilde{X} \to \mathbb{F}_0$ branched over $B$, and therefore $\mathrm{NS}(X)$ contains the class of a smooth rational curve orthogonal to $U$. This is however absurd, since $\mathrm{rk}(\mathrm{NS}(X)) = 3$ and $\mathrm{NS}(X) \ncong U \oplus A_1$.

It only remains to deal with the case $k = 2$, so consider a K3 surface $X$ with $\mathrm{NS}(X) = U \oplus \langle -4 \rangle$. If by contradiction $X$ is a double cover of $\mathbb{F}_0$, then $\mathrm{NS}(X)$ contains primitively $U(2)$, so that

$$U \oplus \langle -4 \rangle \cong \begin{pmatrix} 0 & 2 & a \\ 2 & 0 & b \\ a & b & -2c \end{pmatrix}$$

for $a, b, c \in \mathbb{Z}$, $c \geq 1$. Say that this isomorphism is given by the choice of a basis $\{E_1, E_2, D\}$. The determinant of $\mathrm{NS}(X)$ is 4, and this forces $ab + 2c = 1$. Thus $a, b$ are odd, and without loss of generality $a < 0$, $b > 0$. Now choose $n \geq 0$ such that $a + 2n = 1$ and consider the divisor $D + nE_2$. It is effective by Riemann-Roch, since

$$(D + nE_2)^2 = -2c + 2nb = -2c + b(1 - a) = -2c - ab + b = b - 1 \geq 0$$

and $D + nE_2$ has intersection $1 \geq 0$ with the nef divisor $E_1$. Moreover $(D + nE_2) \cdot E_1 = 1$ means that $D + nE_2$ coincides with $kE_1 + S$ for a certain $k \geq 0$ and a section $S$ of the elliptic pencil $|E_1|$. In other words, $\mathrm{NS}(X)$ is generated by the three elements $E_1, E_2, S$. However the intersection form of $X$ with respect to this basis is

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & \alpha \\ 1 & \alpha & -2 \end{pmatrix}$$

and this matrix has determinant 4 only if $\alpha = -1$, which is a contradiction, as $E_2$ is nef and $S$ is effective. □

*Remark* 1.4.3. Let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 4$ even. Then an argument as above shows that a basis of $\mathrm{NS}(X)$ is given by $\{E_1, E_2, D\}$ with intersection matrix

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & d \\ 1 & d & -2 \end{pmatrix}$$

where $d = \frac{1}{2}(k-4)$, $\pi = (|E_1|, |E_2|) : X \to \mathbb{F}_0$ is the double cover branched over a $(4,4)$-curve $B$, and $C = \pi(D)$ is a smooth $(1, d)$-curve meeting $B$ with even multiplicities.

### 1.4.2 Double covers of $\mathbb{F}_4$

Consider the *Hirzebruch surface* $\mathbb{F}_4 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4))$. We denote by $p : \mathbb{F}_4 \to \mathbb{P}^1$ the $\mathbb{P}^1$-bundle structure. We have that $\mathrm{Pic}(\mathbb{F}_4) = \mathbb{Z}\langle f, s \rangle$, where $f$ is the class of a fiber $F$ of the projection $p$, while $s$ is the class of the unique curve $S \subseteq \mathbb{F}_4$ with negative self-intersection. The intersection form on $\mathrm{Pic}(\mathbb{F}_4)$ with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \cong U.$$

The canonical bundle of $\mathbb{F}_4$ is given by $K_{\mathbb{F}_4} = -2s - 6f$. Notice that $\varphi = \varphi_{|s+4f|} : \mathbb{F}_4 \to C_4$ is the desingularization of the quartic cone $C_4 \subseteq \mathbb{P}^5$ over the normal rational curve $C = \mathrm{Im}(|\mathcal{O}_{\mathbb{P}^1}(4)|) \subseteq \mathbb{P}^4$.

Now consider the double cover $\pi : X \to \mathbb{F}_4$ branched over a curve $B \in |-2K_{F_4}| = |4s + 12f|$. The linear system $|4s + 12f|$ has a fixed part, given by the curve $S$, and a moving part $|3s + 12f|$. Assume that $B$ splits as the sum $S + B_0$, where $B_0 \in |3s + 12f|$ is a smooth irreducible curve disjoint from $S$, as $s \cdot (3s + 12f) = 0$. Then the surface $X$ is a smooth K3 surface. The pullback $E = \pi^* F$ is a smooth elliptic curve, since the restricted double cover $E \to F$ is branched over $(4s + 12f) \cdot f = 4$ points. Moreover $\pi$ is totally ramified over $S \subseteq B$, so $\pi^* S = 2C$, where $C = \pi^{-1}(S) \cong \mathbb{P}^1$ is a smooth rational curve. Since $E \cdot C = \frac{1}{2}(\pi^* F) \cdot (\pi^* S) = F \cdot S = 1$, we have a primitive embedding

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \cong U \hookrightarrow \mathrm{NS}(X).$$

For a very general branch divisor $B$, we simply have $\mathrm{NS}(X) \cong U$.

Consider the linear system $|s + 2kf|$ for $k \geq 2$. Its general member $D$ is a smooth rational curve meeting $F$ in 1 point, $S$ in $2k - 4$ points and $B$ in $(s + 2kf) \cdot (4s + 12f) = 8k - 4$ points. Assume further that the curve $D$ intersects the branch divisor $B$ with even multiplicities. Then Lemma 1.4.1 ensures that the pullback $\pi^* D = D_1 + D_2$ splits into two disjoint components $D_1, D_2 \cong \mathbb{P}^1$. This implies that there exists an embedding (not necessarily primitive)

$$\langle E, C, D_1 \rangle = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & k-2 \\ 1 & k-2 & -2 \end{pmatrix} \cong U \oplus \langle -2k \rangle \hookrightarrow \mathrm{NS}(X),$$

since $D_1 \cdot E = \frac{1}{2}(\pi^* D) \cdot (\pi^* F) = D \cdot F = 1$ and $D_1 \cdot C = \frac{1}{4}(\pi^* D) \cdot (\pi^* S) = \frac{1}{2} D \cdot S = k - 2$.

**Proposition 1.4.4.** *Every elliptic K3 surface $X$ is the desingularization of a double cover of the Hirzebruch surface $\mathbb{F}_4$.*

*Proof.* Assume that $U \hookrightarrow \mathrm{NS}(X)$, and denote by $E, C$ the smooth curves in $X$ generating $U$ such that $E^2 = 0$, $C^2 = -2$. Consider the linear system $|4E + 2C|$. By [Huy16, Corollary 8.1.6] the divisor $4E + 2C$ is nef, as it has non-negative intersection with every smooth rational curve. Moreover $4E + 2C$ has intersection 0 with the curve $C$. Since $(4E + 2C)^2 = 8$ and $\dim |4E + 2C| = 5$, $\psi = \varphi_{|4E+2C|} : X \to \mathbb{P}^5$ is a morphism onto a surface $Y \subseteq \mathbb{P}^5$ contracting $C$. $C$ is a smooth $(-2)$-curve, so $Y$ is singular. Now the elliptic curve $E$ has intersection 2 with $4E + 2C$, so $\psi$ has degree 2 by [Sai74a, Theorem 5.2]. This implies that $\deg(Y) = 4$, so $Y \subseteq \mathbb{P}^5$ is a singular surface of minimal degree, hence $Y$ is the quartic cone $C_4$ (see [del87]). Therefore $\psi$ factors through the minimal resolution of $C_4$, which is $\mathbb{F}_4$, giving a morphism $\pi : X \to \mathbb{F}_4$ of degree 2. Now we can repeat the argument in the proof of Proposition 1.4.2, obtaining that $X$ is the desingularization of a double cover of $\mathbb{F}_4$. $\square$

*Remark* 1.4.5. Every K3 surface $X$ with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 2$ can be obtained as a double cover $\pi : X \to \mathbb{F}_4$ branched over a smooth curve $B \in |4s + 12f|$ admitting a rational curve $D \in |s + 2kf|$ intersecting $B$ with even multiplicities. If instead $X$ is a K3 surface with $\mathrm{NS}(X) = U \oplus \langle -2 \rangle$, then it is the desingularization of the double cover of $\mathbb{F}_4$ branched over a curve $B$ with a unique singularity of type $A_1$.

### 1.4.3 Weierstrass fibrations

Let $X$ be a smooth elliptic K3 surface. By [Mir89, Section II.3] $X$ is the desingularization of a *Weierstrass fibration* $\pi' : Y \to \mathbb{P}^1$, where $Y$ is defined by an equation

$$Y^2Z = X^3 + AXZ^2 + BZ^3 \tag{1.4}$$

in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ with $A \in H^0(\mathcal{O}_{\mathbb{P}^1}(8))$ and $B \in H^0(\mathcal{O}_{\mathbb{P}^1}(12))$ minimal and with $\Delta = 4A^3 + 27B^2$ not identically zero. Conversely, every such Weierstrass fibration desingularizes to a smooth elliptic K3 surface. We will usually restrict to the chart $\{Z \neq 0\}$ over the affine base $\mathbb{A}^1_t \subseteq \mathbb{P}^1$, where equation (1.4) becomes

$$y^2 = x^3 + A(t)x + B(t), \tag{1.5}$$

with $A$ and $B$ polynomials in $t$ of degree at most 8 and 12 respectively. Notice that this is the equation of the generic fiber of the Weierstrass fibration, which is an elliptic curve over $\mathbb{C}(t)$. Under this identification, sections of the fibration $\pi$ (or $\pi'$) correspond to $\mathbb{C}(t)$-rational points of equation (1.5). In particular the distinguished zero section is located at the point at infinity $S_0 = (0 : 1 : 0)$. Moreover we will write $S = (u(t), v(t))$ to denote the section $S$ of $\pi$ corresponding to the $\mathbb{C}(t)$-rational point $(u(t), v(t))$ of equation (1.5). By the above description, $u, v \in \mathbb{C}(t)$ are rational functions of degree at most $4, 6$ respectively.

*Remark* 1.4.6. Let $X$ be a $U \oplus \langle -2k \rangle$-polarized K3 surface. If $k \geq 2$, the given elliptic fibration on $X$ admits an extra section $S$ such that $S \cdot S_0 = k - 2$. This follows from the isomorphism of lattices

$$U \oplus \langle -2k \rangle \cong \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & k-2 \\ 1 & k-2 & -2 \end{pmatrix}.$$

Conversely, if $X$ is an elliptic K3 surface and $S$ is an extra section with $S \cdot S_0 = k - 2$, then there exists an embedding

$$U \oplus \langle -2k \rangle \hookrightarrow \mathrm{NS}(X).$$

This embedding is not necessarily primitive. However, it is primitive if the lattice $U \oplus \langle -2k \rangle$ has no non-trivial overlattices (for instance if $k$ is square-free, cf. [Nik79b, Proposition 1.4.1]).

## 1.5 Enriques surfaces

We conclude this preliminary chapter by recalling some basic facts about Enriques surfaces, that we will use in the last part of the thesis. The main references are [CD89] and [Cos85]. We will work over a field $K$ of characteristic different from 2 and 3.

**Definition 1.5.1.** An Enriques surface is a projective, geometrically smooth surface $S$ with $H^1(S, \mathcal{O}_S) = 0$ and such that the canonical bundle $K_S$ is 2-torsion.

The universal cover of an Enriques surface is a K3 surface, that we will call its *K3 cover*. Vice versa, the quotient of a K3 surface by an involution without fixed points is an Enriques surface. Much of the geometry of an Enriques surface can be understood from its K3 cover. The group $\mathrm{Num}(S)$, obtained by taking the quotient of the Néron-Severi group $\mathrm{NS}(S)$ of an Enriques surface $S$ by the 2-torsion element $K_S$, is a lattice, and it is isometric to the unimodular hyperbolic lattice $U \oplus E_8$ of rank 10.

Let $S$ be an Enriques surface, $X$ its K3 cover. The definition of a genus 1 fibration on $S$ is the same as the one for K3 surfaces (see Definition 1.2.3). Despite the many similarities between Enriques and K3 surfaces, genus 1 fibrations behave quite differently. We are going to recall in the following the basic properties of genus 1 fibrations on Enriques surfaces.

First of all, every Enriques surface admits a genus 1 fibration (see [CD89, Theorem 5.7.1]). However, no genus 1 fibration on $S$ has a section. This follows from the fact that any elliptic pencil on $S$ has precisely two double fibers. In order to see this, let $|F|$ be an elliptic pencil on $S$, $\pi : X \to S$ the K3 cover and $|E|$ the elliptic pencil on $X$ such that $\pi^*F = 2E$. Then we have a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & S \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow[\pi']{} & \mathbb{P}^1. \end{array}$$

If $t_1, t_2 \in \mathbb{P}^1$ are the branch points of the double cover $\pi' : \mathbb{P}^1 \to \mathbb{P}^1$, their fibers $F_1, F_2 \subseteq S$ are indeed double. Therefore, we will usually write elliptic pencils on $S$ as $|2F|$, where $F$ is called a *half-pencil*.

The Riemann-Roch formula on Enriques surfaces reads $C^2 = 2g(C) - 2$ whenever $C$ is an irreducible curve, as in the case of K3 surfaces. Therefore smooth rational curves are precisely the $(-2)$-curves. However, not all isotropic vectors induce elliptic pencils:

**Proposition 1.5.2** ([CD89], Corollary 5.7.2)**.** *Let $F \in \mathrm{NS}(S)$ be a non-zero isotropic vector, i.e. $F^2 = 0$. Then either $\dim |F| = 1$ or $\dim |2F| = 1$.*

Next, we move to the study of the (bi)sections of an elliptic pencil on $S$. The following is the main result in this direction:

**Theorem 1.5.3** ([Cos85], Proposition 3.4 and Theorem 4.1)**.** *Let $|2F|$ be an elliptic pencil on $S$. $|2F|$ admits a bisection, i.e. an irreducible curve $C \subseteq S$ such that $C \cdot F = 1$. Moreover, we can choose such a $C$ so that either $C^2 = -2$, so $C$ is smooth and rational (and $|2F|$ is called a special elliptic pencil), or $C^2 = 0$, and $C$ has (arithmetic) genus 1.*

Rational bisections are indeed very special. For instance, the generic Enriques surface contains no $(-2)$-curve. We say that an Enriques surface is *nodal* if it admits a $(-2)$-curve, otherwise it is said *unnodal*. If $S$ is nodal, then it admits a special elliptic pencil by [Cos85, Theorem 4.1].

*Remark* 1.5.4. Special elliptic pencils are also important because they pull back to elliptic fibrations on the K3 cover $X$ (cf. [Kon86a, Lemma 2.6]). Indeed, if $C$ is a rational bisection of $|2F|$, and $E = \pi^*F$, then the pullback $\pi^*C$ splits as the disjoint union of two $(-2)$-curves $S_1, S_2$, that are sections of $|E|$. This follows from the straightforward computation

$$2E \cdot S_1 = E \cdot (S_1 + S_2) = \pi^*F \cdot \pi^*C = 2F \cdot C = 2.$$

Let $|2F|$ be an elliptic pencil on an Enriques surface $S$, denote by $\pi : X \to S$ its K3 cover and by $E = \pi^*F$ the pullback of the pencil. We would like to understand the configuration of reducible fibers of $|E|$ from that of $|2F|$. This is completely answered by the following proposition. We adopt Kodaira's names for the possible reducible fibers on the elliptic pencils.

**Proposition 1.5.5** ([HS11], Section 3.2)**.** *Let $F_0 \in |2F|$ be a singular fiber.*

- *If $F_0$ is multiple, then $F_0 = 2F_0'$, where $F_0'$ is a fiber of type $I_n$. Moreover $\pi^*F_0$ is a fiber of type $I_{2n}$.*
- *If $F_0$ is not multiple, then $\pi^*F_0$ consists of two fibers isomorphic to $F_0$.*

It follows that, if $E_0 \in |E|$ is a reducible fiber not of type $I_{2n}$ on the K3 surface $X$, then there exists a second fiber $E_1 \in |E|$ isomorphic to $E_0$.

**Definition 1.5.6.** An *exceptional sequence* on $S$ is a list of 10 effective isotropic divisors $D_1, \ldots, D_{10} \in \mathrm{Num}(S)$ such $D_i \cdot D_j = 1$ for any $i \neq j$.

Exceptional sequences play a fundamental role towards the understanding of the geometry of an Enriques surface. Every Enriques surface admits an exceptional sequence. Notice that the 10 divisors $D_i$ span a lattice isometric to $U \oplus A_8$, which is an index 3 sublattice of $\mathrm{Num}(S)$. The following is the structure theorem for exceptional sequences:

**Theorem 1.5.7** ([Cos85], Theorem 3.3)**.** *1. Any exceptional sequence on $S$ is of the form*

$$F_1, F_1 + R_1, F_1 + R_1 + C_1^{(1)}, \ldots, F_2, F_2 + R_2, F_2 + R_2 + C_1^{(2)}, \ldots,$$

*where the $F_i$ are elliptic half-pencils, $R_i$ is a rational bisection of $|2F_i|$, and the $C_j^{(i)}$ are $(-2)$-curves orthogonal to $F_i$.*

*2. Every Enriques surface admits an exceptional sequence with at least 3 half-pencils.*

*3. There exists a unique exceptional sequence up to isometries of $\mathrm{NS}(S)$.*

The maximum number of elliptic half-pencils in exceptional sequences on $S$ is an important invariant of the Enriques surface $S$, and it usually denoted by $\mathrm{nd}(S)$. Clearly, if $S$ is unnodal, $\mathrm{nd}(S) = 10$. Moreover the previous theorem shows that $\mathrm{nd}(S) \geq 3$ for any Enriques surface $S$. Examples of Enriques surfaces $S$ with $\mathrm{nd}(S) = 4$ are Kondo's surfaces of type II (cf. [Kon86a]) with finite automorphism group.

# 2 | Classification of K3 surfaces of zero entropy

## 2.1 Introduction

Let $X$ be a smooth projective K3 surface over an algebraically closed field. The study of the group $\mathrm{Aut}(X)$ of automorphisms of $X$ is a central topic at the intersection of algebraic, arithmetic and differential geometry. Since the early works by Nikulin [Nik79a], Kondō [Kon86b] and Vinberg [Vin83], many have tried to understand explicitly the structure of the group $\mathrm{Aut}(X)$ using very different approaches. A very successful approach in the last 20 years has been via complex dynamics and entropy, pioneered by Cantat [Can01] and McMullen [McM02]. Our aim is to combine this with the huge lattice-theoretical machinery classically used to study K3 surfaces.

The first step towards the understanding of the group $\mathrm{Aut}(X)$ was made by Nikulin [Nik79a], Vinberg [Vin07] and Kondō [Kon86b], who completely classified the Néron-Severi lattices of complex K3 surfaces with a finite automorphism group (more precisely, Nikulin solved the case of Picard rank $\rho(X) \neq 4$, Vinberg the case $\rho(X) = 4$, and Kondō described their automorphism group). Their work relies on the theory of lattices developed by Nikulin in the 70's. However, when the automorphism group becomes infinite, very little is known. For example, we can describe the full automorphism group only of some K3 surfaces (see Vinberg's examples [Vin83] or Shimada's recent algorithm [Shi15]).

Our goal is to identify a class of complex K3 surfaces with an infinite but simple automorphism group. The *entropy* of an automorphism of a surface is a measure of the complexity of its dynamics (cf. Definition 2.2.3). The automorphisms with the most regular dynamics are those of zero entropy. A K3 surface $X$ is said to have *zero entropy* if all its automorphisms have zero entropy. The automorphism group of K3 surfaces of zero entropy is particularly simple:

**Theorem 2.1.1** ([Can99; Ogu07]). *Let $X$ be a smooth complex projective K3 surface with infinite automorphism group. The following are equivalent:*

1. *$X$ has zero entropy;*

2. *$X$ admits a unique genus* 1 *fibration with infinite automorphism group;*

3. $\mathrm{Aut}(X)$ *is almost-abelian, i.e. there exists a normal, finite index subgroup* $G < \mathrm{Aut}(X)$*, a finite group* $K$ *and a short exact sequence*

$$0 \to K \to G \to \mathbb{Z}^r \to 0$$

*for some* $r \geq 1$*.*

The advantage of this characterization is that it is purely lattice-theoretical, as the uniqueness of the genus 1 fibration with infinite automorphism group can be read off from the Néron-Severi lattice of $X$. A classification of complex K3 surfaces admitting a unique genus 1 fibration with infinite automorphism group was asked for by Nikulin in [Nik14].

We address this classification problem in the case when $X$ satisfies a technical condition. More precisely, we want $X$ to admit an elliptic fibration with only irreducible fibers, i.e. with only nodal or cuspidal singular fibers. Our main result is the following:

**Theorem 2.1.2.** *Let $X$ be a smooth complex projective K3 surface with infinite automorphism group. Suppose that $X$ admits an elliptic fibration with only irreducible fibers. Then $X$ has zero entropy if and only if the Néron-Severi lattice* $\mathrm{NS}(X)$ *belongs to an explicit list of* 32 *lattices.*

The reader can find this list in Theorem 2.5.9. Incidentally, we also classify which of these 32 classes of K3 surfaces admit other genus 1 fibrations during the proof of Theorem 2.1.2 (see Theorem 2.5.11).

The classification in Theorem 2.1.2 is obtained in three steps; we are going to outline the main ideas of each of them. If $X$ is a K3 surface admitting an elliptic fibration, then the sublattice of $\mathrm{NS}(X)$ generated by the elliptic curve $F$ and its zero section $S_0$ induces an orthogonal decomposition $\mathrm{NS}(X) = U \oplus L$. When $\rho(X) = 3$, the rank of $L$ is 1, hence the intersection form on $\mathrm{NS}(X)$ is completely governed by a unique number, which coincides with the determinant of $\mathrm{NS}(X)$. Since automorphisms preserve the nef cone (cf. Section 1.2.4), we can rephrase our problem in terms of the nef cone of such surfaces. We then show that the structure of the nef cone can be understood by solving some congruences involving the determinant of $\mathrm{NS}(X)$. This allows us to show that $X$ has zero entropy if and only if $\det(\mathrm{NS}(X))$ satisfies a certain arithmetic property (cf. Theorem 2.4.7).

When $\rho(X) \geq 4$, the intersection form on $\mathrm{NS}(X)$ depends on a lattice $L$ of rank $\rho(X) - 2 \geq 2$, hence it is impractical to generalize the previous approach. Nevertheless, we switch our attention to the study of the genus of $L$ (cf. Section 1.1.3). The second step of the proof of Theorem 2.1.2 consists in proving that if $X$ satisfies the assumptions of Theorem 2.1.2 and has zero entropy, then its Néron-Severi lattice must decompose as $\mathrm{NS}(X) = U \oplus L$, with $L$ having trivial genus. A priori it could happen that a K3 surface has many elliptic fibrations, but a unique one with infinite automorphism group. We rule this out by proving that if a K3 surface admits an elliptic fibration with only irreducible fibers and another elliptic fibration with finite automorphism group, then it admits a third "intermediate" elliptic fibration with infinite automorphism group (cf. Claim 2.6.3). We obtain this result by studying the genera of root lattices.

The third and final step of the proof of Theorem 2.1.2 amounts to studying the negative definite lattices $L$ unique in their genus. These are completely classified by Theorem 1.1.11. Despite the existence of infinitely many negative definite lattices unique in their genus, we are able to reduce to a finite number of cases by using a recursive argument based on Theorem 2.3.5 and the classification in Picard rank 3 obtained previously. The classification is then completed by checking individually these remaining lattices.

It is natural to ask what happens if we remove the technical condition in Theorem 2.1.2. If $\rho(X) = 20$ is maximal, the K3 surface is called *singular*, and in this case Oguiso has proved that $X$ always has positive entropy (cf. [Ogu07, Theorem 1.6]). Using the techniques introduced above, we are able to generalize his result to Picard rank 19:

**Theorem 2.1.3.** *All smooth complex projective K3 surfaces with Picard rank* $\geq 19$ *and infinite automorphism group have positive entropy.*

The outline of the chapter follows closely the previous discussion. In Section 2.2 we recall the definition of the entropy, the classification of automorphisms of K3 surfaces due to [Can99], and Theorem 2.1.1. In Section 2.3 we lay the groundwork to prove the main theorem 2.1.2. More precisely, we use Nikulin's theory of lattices to find sufficient conditions for a K3 surface to have positive entropy. In Sections 2.4, 2.5 and 2.6 we explain the three steps discussed above, in order to obtain the classification in Theorem 2.1.2. Finally, in Section 2.7 we prove Theorem 2.1.3.

**Convention 2.1.4.** Throughout the chapter we will always work over $\mathbb{C}$. We have used the software `Magma` to implement all the algorithms.

## 2.2 Entropy on K3 surfaces

Let $X$ be a K3 surface. The cohomology group $H^{1,1}(X,\mathbb{R})$ is a vector space of dimension 20, endowed with a hyperbolic nondegenerate metric $q_X$. Hence the sheet

$$\mathbb{H}_X = \{c \in H^{1,1}(X,\mathbb{R}) \mid q_X(c) = 1\}^+$$

intersecting the Kähler cone of $X$ is a model for the hyperbolic space $\mathbb{H}^{19}$. Since the automorphism group $\mathrm{Aut}(X)$ of the surface acts as isometries on $H^2(X,\mathbb{R})$ and preserves the Kähler cone, we have a natural map

$$\mathrm{Aut}(X) \to \mathrm{O}(\mathbb{H}_X).$$

Moreover $\mathrm{Aut}(X)$ can be seen as a discrete subgroup of isometries of $H^2(X,\mathbb{R})$, since it embeds into the group $\mathrm{O}(H^2(X,\mathbb{Z}))$ of isometries of the lattice $H^2(X,\mathbb{Z}) \subseteq H^2(X,\mathbb{R})$ (cf. [Huy16, Proposition 15.2.1]).

**Definition 2.2.1.** Let $\phi \in \mathrm{O}(\mathbb{H}_X)$ be an isometry of the hyperbolic space $\mathbb{H}_X$. $\phi$ is called

- *elliptic*, if $\phi$ fixes an inner point $x \in \mathbb{H}_X \backslash \partial\mathbb{H}_X$;
- *parabolic*, if $\phi$ is not elliptic and fixes a unique point in the boundary $\partial\mathbb{H}_X$;
- *hyperbolic*, if $\phi$ fixes two points in the boundary $\partial\mathbb{H}_X$.

The next classical theorem by Cantat classifies the automorphisms of a K3 surface $X$ into these three types.

**Theorem 2.2.2** ([Can99], Corollaire 2.2)**.** *Let $f \in \mathrm{Aut}(X)$, and denote by $f^* \in \mathrm{O}(\mathbb{H}_X)$ the induced isometry on the hyperbolic space $\mathbb{H}_X$.*

- *$f^*$ is elliptic if and only if $f$ has finite order.*
- *$f^*$ is parabolic if and only if $f$ is not periodic and it respects a genus 1 fibration on $X$ (i.e. there exists a primitive, nef element $F \in \mathrm{NS}(X)$ with $F^2 = 0$ such that $f^*F = F$). In this case, all eigenvalues of $\phi^*$ have norm 1.*
- *$f^*$ is hyperbolic otherwise. There exists an eigenvalue of $\phi^*$ of norm $> 1$.*

The concept of *entropy* of automorphisms is closely related to this classification. The entropy can be defined in much more generality, but we restrict ourselves to the case of a complex projective variety $Y$.

**Definition 2.2.3.** Let $Y$ be a complex projective variety and $g$ an automorphism of $Y$. The *entropy* of $g$ is defined as the quantity $h(g) = \log \lambda(g^*)$, where $\lambda(g^*)$ is the spectral radius of the pullback map $g^* : H^*(Y,\mathbb{C}) \to H^*(Y,\mathbb{C})$ on singular cohomology, i.e. the maximum norm of its eigenvalues.

*Remark* 2.2.4. Over $\mathbb{C}$ there exists an equivalent, more topological, definition of the entropy, that measures how fast the iterates of $g$ create distinct orbits. See [Can14] for a nice introduction, and [Gro03], [Yom87] for the equivalence of the two definitions. If instead the variety is defined on a field of positive characteristic, there exists a similar definition of the entropy that uses étale cohomology; the interested reader can consult [ES13].

If $X$ is a K3 surface and $f$ an automorphism of $X$, then the pullback $f^*$ acts as the identity on $H^0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C})$. Moreover $f^*$ acts with finite order on the complexification $\mathrm{T}(X)_\mathbb{C}$ of the transcendental lattice by [Huy16, Corollary 3.3.4], so the entropy of $f$ coincides with $\log \lambda(f^*|_{\mathrm{NS}(X)_\mathbb{C}})$, where $f^*|_{\mathrm{NS}(X)_\mathbb{C}}$ is the restriction of the pullback to $\mathrm{NS}(X)_\mathbb{C} \subseteq H^2(X,\mathbb{C})$. Hence Theorem 2.2.2 can be rephrased as follows:

**Corollary 2.2.5.** *Let $f \in \mathrm{Aut}(X)$ be an automorphism of the K3 surface $X$. Then $f$ has zero entropy if and only if $f^*$ is either elliptic or parabolic. In other words, $f$ has zero entropy if and only if $f$ has either finite order or it respects a genus 1 fibration on $X$.*

**Definition 2.2.6.** A K3 surface $X$ is said to have *zero entropy* if all of its automorphisms have zero entropy. Otherwise $X$ is said to have *positive entropy.*

If $X$ has a finite automorphism group, then every $f \in \mathrm{Aut}(X)$ is elliptic, hence $X$ has zero entropy. K3 surfaces with a finite automorphism group have been widely studied by Nikulin and Vinberg (see for instance [Nik80], [Nik81a], [Nik81b], [Nik84], [Nik87], [Nik96], [Nik99], [Vin07]); we have in fact a complete classification of Néron-Severi lattices of complex K3 surfaces with a finite automorphism group (see also [Kon86b] for a description of their automorphism groups). Therefore we are interested in studying K3 surfaces of zero entropy with an *infinite* automorphism group.

*Remark* 2.2.7. Having zero entropy only depends on the Néron-Severi lattice $\mathrm{NS}(X)$, and not on the K3 surface $X$ itself. For, let $f$ be an automorphism of $X$ of positive entropy. Then $f$ can be seen as an element of positive entropy in the group $\mathrm{O}^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$ by Proposition 1.2.10, which depends only on the Néron-Severi lattice $\mathrm{NS}(X)$. Conversely, if $f \in \mathrm{O}^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$ has positive entropy, then by Proposition 1.2.10 there exists a power $f^n$ that comes from an automorphism of $X$, and $f^n$ still has positive entropy.
In the following, we will say that a Néron-Severi lattice $N$ has *zero entropy* (resp. *positive entropy*) if any K3 surface $X$ with $\mathrm{NS}(X) = N$ has zero entropy (resp. positive entropy).

The following result characterizes the automorphism group of a K3 surface with zero entropy. If $|E|$ is a genus 1 fibration on $X$, the *automorphism group of* $|E|$ is the subgroup of automorphisms of $X$ that preserve $E \in \mathrm{NS}(X)$.

**Theorem 2.2.8** ([Ogu07], Theorem 1.4)**.** *Let $X$ be K3 surface with an infinite automorphism group. Then $X$ has zero entropy if and only if there exists a unique genus 1 fibration $|E|$ on $X$ with infinite automorphism group.*

We stress that the uniqueness above is *not* up to automorphism, but it is an absolute uniqueness. Since we will be mainly interested in *elliptic* K3 surfaces, we can state Theorem 2.2.8 in a more precise way:

**Corollary 2.2.9.** *Let $X$ be a K3 surface admitting an elliptic fibration $|F|$ with infinitely many sections. Then $X$ has zero entropy if and only if $|F|$ is the unique elliptic fibration on $X$ with infinitely many sections.*

*Proof.* Notice first that, if $|F|$ is an elliptic fibration on $X$ with infinitely many sections, then $|F|$ has an infinite automorphism group. This follows from the injectivity in equation (1.2). Thus, if $X$ admits two distinct elliptic fibrations with infinitely many sections, then $X$ has positive entropy by Theorem 2.2.8.
Conversely, assume that $|F|$ is the unique elliptic fibration on $X$ with infinitely many sections. If by contradiction $X$ has positive entropy, then by Theorem 2.2.8 there exists a genus 1

fibration $|E|$ on $X$ with infinite automorphism group. Choose any automorphism $f$ of $X$ of infinite order preserving $|E|$. Then $f$ also preserves the elliptic fibration $|F|$: indeed, the element $f^*F$ induce an elliptic fibration with infinitely many section, and $|F|$ is the unique such fibration. Hence $f$ preserves the sublattice $\langle F, E\rangle \subseteq \mathrm{NS}(X)$, and therefore it can be seen as an isometry of the orthogonal complement $L$ of $\langle F, E\rangle \subseteq \mathrm{NS}(X)$. $E$ and $F$ are isotropic, so the sublattice $\langle F, E\rangle$ has signature $(1, 1)$, and consequently $L$ is negative definite. $f$ is an isometry of infinite order of a definite lattice, a contradiction by [SS19, Theorem 2.14]. □

*Remark* 2.2.10.

1. K3 surfaces with a unique elliptic fibration with infinitely many sections were studied by Nikulin [Nik14]. He shows that as soon as a K3 surface admits two distinct elliptic fibrations with infinitely many sections, then it admits an infinite number of such fibrations (cf. [Nik14, Theorem 10]).
2. If a K3 surface admits an elliptic fibration with infinitely many sections, then its Picard rank $\rho(X)$ is at least 3. This follows from the Shioda-Tate formula 1.2.6, as the rank of the Mordell-Weil group of the fibration is non-zero.

## 2.3 Elliptic K3 surfaces of zero entropy

Let $X$ be a smooth complex projective elliptic K3 surface. Assume that $X$ admits an elliptic fibration $|F|$ with only irreducible fibers (i.e. all the singular fibers of the fibration are either nodal or cuspidal rational curves). Most of the results of this section heavily depend on this assumption; we will say explicitly when this assumption can be dropped.

If $S_0$ is the zero section of the fibration $|F|$, the unimodularity of the trivial lattice $\langle F, S_0\rangle \cong U$ induces an orthogonal decomposition

$$\mathrm{NS}(X) = \langle F, S_0\rangle \oplus L,$$

where $L$ is an even negative definite lattice of rank $r = \rho(X) - 2$. Notice that $L$ has no roots, because the elliptic fibration $|F|$ has no reducible fibers.

We will denote in the following by $[x, y, z] \in \mathrm{NS}(X)$ the divisor written with respect to the basis $\{F, S_0, \mathcal{B}\}$ of $\mathrm{NS}(X)$, where $\mathcal{B}$ is a basis of $L$, fixed once and for all. We will denote by $\|z\|_L$ the norm of the vector $z \in L$.

Let $0 \neq D = [x, y, z] \in \mathrm{NS}(X)$ be a divisor (not necessarily irreducible nor reduced) such that $D^2 \geq -2$. By Riemann-Roch one of $D$ and $-D$ is effective, and since $F \cdot D = y$, we have that $D$ is effective if $y > 0$, while $-D$ is effective if $y < 0$. This leads to the following useful characterization:

**Lemma 2.3.1.** *Let $X$ be a K3 surface admitting an elliptic fibration $|F|$ with only irreducible fibers. Let $A \in \mathrm{NS}(X)$ be a divisor with $A^2 \geq 0$ and $A \cdot F \geq 0$. Then $A$ is nef if and only if $A \cdot D \geq 0$ for all divisors $D = [x, y, z] \in \mathrm{NS}(X)$ with $D^2 = -2$ and $y > 0$.*

*Proof.* The divisor $A$ is nef if and only if it has non-negative intersection with all irreducible $(-2)$-curves on $X$ (cf. [Huy16, Corollary 8.1.4, 8.1.7]). Suppose that there exists $D = [x, y, z] \in \mathrm{NS}(X)$ with $D^2 = -2$ and $y > 0$ such that $A \cdot D < 0$. Then by the above discussion $D$ is effective, and, since $D^2 = -2$, it splits into the sum of some irreducible $(-2)$-curves. The inequality $A \cdot D < 0$ implies that there exists an irreducible $(-2)$-curve $C$ (which is a summand of $D$) such that $A \cdot C < 0$, contradicting the nefness of $A$.
Conversely, if $A$ is not nef, there exists an effective $(-2)$-curve $C = [x, y, z]$ such that $A \cdot C < 0$. As above $y \geq 0$, but $y = 0$ only if $C$ is contained in a fiber of the elliptic fibration $|F|$, hence $y > 0$ since by assumption there are no reducible fibers. □

**Lemma 2.3.2.** *Let $X$ be a K3 surface admitting an elliptic fibration $|F|$ with only irreducible fibers. Let $F \neq E = [\alpha, \beta, \gamma], C = [x, y, z] \in \mathrm{NS}(X)$ be effective, primitive divisors such that $E^2 = 0$ and $C^2 = -2$. Then the equation $E \cdot C = m$ can be equivalently written as*

$$-\frac{1}{2}\|v\|_L = \beta(\beta + my),$$

*where $v = y\gamma - \beta z$. In particular $E$ is nef if and only if*

$$-\frac{1}{2}\|v\|_L - \beta^2 \geq 0$$

*for any such $C$, and $E$ induces a genus 1 fibration with only irreducible fibers if and only if*

$$-\frac{1}{2}\|v\|_L - \beta^2 > 0$$

*for any such $C$.*

*Proof.* This is a straightforward computation. The self-intersections of $E, C$ force

$$\begin{cases} \alpha = \beta + \frac{-\frac{1}{2}\|\gamma\|_L}{\beta} \\ x = y + \frac{-\frac{1}{2}\|z\|_L - 1}{y} \end{cases} . \tag{2.1}$$

Notice that $\beta \neq 0$ since $E \neq F$, and $y \neq 0$ since we are assuming that $L$ has no roots. Substituting these expressions into the equation

$$m = E \cdot C = \alpha y + \beta x - 2\beta y + (\gamma, z)_L,$$

we easily obtain the desired equation. □

The next proposition highlights a certain "periodicity" of elliptic curves on $X$; notice that this highly depends on the assumption that $X$ admits an elliptic fibration with only irreducible fibers.

**Proposition 2.3.3.** *Let $X$ be a K3 surface admitting an elliptic fibration with only irreducible fibers. Let $E = [\alpha, \beta, \gamma], E' = [\alpha', \beta, \gamma'] \in \mathrm{NS}(X)$ be primitive isotropic elements with the same intersection number $F \cdot E = F \cdot E' = \beta$ and $\gamma' \equiv \gamma \pmod{\beta}$ (that is, all the entries are congruent modulo $\beta$). Then $E$ induces a genus 1 fibration (resp. an elliptic fibration) on $X$ if and only if $E'$ does so.*

*Proof.* Say $\gamma = [\gamma_1, \dots, \gamma_r]$ and assume $\gamma' = [\gamma_1 + \beta, \gamma_2, \dots, \gamma_r]$; $\alpha'$ is an integer by equation (2.1), since $\gamma', \gamma$ are congruent modulo $\beta$. Then $E$ is nef if and only if $E'$ is nef: indeed, if there exists an effective $C = [x, y, z]$ such that $C^2 = -2$ and $E \cdot C = m < 0$, by Lemma 2.3.2 we have

$$-\frac{1}{2}\|v\|_L = \beta(\beta + my),$$

where $v = y\gamma - \beta z$. If we put $z' = [z_1 + y, z_2, \dots, z_r]$, clearly $v = y\gamma' - \beta z'$ doesn't change, so $E' \cdot C' = m < 0$, where $C' = [x', y, z']$. Analogously, $E$ has a section if and only if $E'$ has one. We conclude by repeating the same argument for every coordinate of $\gamma$. □

*Remark* 2.3.4. This periodicity in Proposition 2.3.3 can be interpreted via translations with respect to the given elliptic fibration $|F|$. For, let $S = [x, 1, z]$, with $z \in L$. Notice that $S$ is irreducible, since if it split into the sum of irreducible $(-2)$-curves, one of these would be orthogonal to $F$, and we assumed that are no vertical $(-2)$-curves in $|F|$. Thus $S$ is a section of $|F|$ and it induces a translation $\tau_S$, that is an automorphism of the K3 surface preserving the class $F$. It is not hard to show that

$$\tau_S([\alpha, \beta, \gamma]) = [\alpha', \beta, \gamma + \beta z]$$

for a suitable $\alpha'$. Therefore, elements $E$ and $E'$ as in Proposition 2.3.3 are always conjugated under the action of $\mathrm{Aut}(X)$.

The following theorem will be one of our main tools to prove that many K3 surfaces have positive entropy. Let again $X$ be an elliptic K3 surface with $\mathrm{NS}(X) = U \oplus L$. For any primitive sublattice $L'$ of $L$ there exists an elliptic K3 surface $X'$ with Néron-Severi lattice $\mathrm{NS}(X') = U \oplus L'$. This follows from the surjectivity of the period map for K3 surfaces (cf. [Tod80], Theorem 1), since $U \oplus L' \hookrightarrow U \oplus L \hookrightarrow \Lambda_{K3}$ embeds primitively into the K3 lattice. The goal of the following theorem is to relate the entropy of $X$ to the entropy of $X'$. Recall that we say that $\mathrm{NS}(X)$ has positive entropy if $X$ has positive entropy (since having positive entropy only depends on the Néron-Severi lattice by Remark 2.2.7).

**Theorem 2.3.5.** *Let $X$ be a K3 surface admitting an elliptic fibration with only irreducible fibers, inducing a decomposition $\mathrm{NS}(X) = U \oplus L$. Assume that there exists a primitive sublattice $L'$ of $L$ of corank 1 such that $U \oplus L'$ has positive entropy. Then $X$ has positive entropy if one of the following conditions holds:*

- $|\det(L)| > 2|\det(L')|$
- $|\det(L)| = 2|\det(L')|$ *and* $\rho(X) \leq 10$.

*Proof.* By the primitivity of $L'$ in $L$, we can fix a basis for $L$ that completes a basis of $L'$. Then the intersection matrix of $L$ is

$$L = \left(\begin{array}{c|c} L' & M \\ \hline M^T & -2k \end{array}\right).$$

We denote by $[x, y, z, w] \in \mathrm{NS}(X)$ the divisor with coordinates $x, y$ wrt $U$, $z$ wrt $L'$ and $w$ wrt to $\langle -2k \rangle$. By assumption $U \oplus L'$ has positive entropy, so by Corollary 2.2.9 there exists a primitive effective divisor $E' = [\alpha, \beta, \gamma] \in U \oplus L'$, $E' \neq [1, 0, \dots, 0]$, that induces a second elliptic fibration with infinitely many sections on every K3 surface $X'$ with $\mathrm{NS}(X') = U \oplus L'$. We extend $E'$ to a divisor $E = [\alpha, \beta, \gamma, 0] \in \mathrm{NS}(X)$ by fixing the last coordinate to 0. Clearly $E$ is still primitive, effective and isotropic. We claim that $E$ is also nef. By construction we have that $E$ has non-negative intersection with all the $(-2)$-curves coming from $U \oplus L'$, i.e. $E \cdot C \geq 0$ for any effective $(-2)$-curve $C = [x, y, z, 0] \in \mathrm{NS}(X)$.
Therefore, let $C = [x, y, z, w] \in \mathrm{NS}(X)$ be an effective $(-2)$-curve with $w \neq 0$. We need to show that $E \cdot C \geq 0$. By Lemma 2.3.2 we have that the intersection $E \cdot C$ equals (up to a positive constant)

$$-\frac{1}{2} v^T L' v + M^T v \cdot (\beta w) + k \beta^2 w^2 - \beta^2,$$

where $v = y\gamma - \beta z$. We know that the minimum of the previous expression is attained at the vector $v$ where the gradient vanishes, i.e. at the vector $v$ such that $L'v = \beta w M$, and by substituting this in the previous expression we find that the minimum is in fact

$$\frac{1}{2} \beta^2 w^2 M^T L'^{-1} M + k \beta^2 w^2 - \beta^2.$$

After dividing by $\beta^2 w^2$, it is sufficient to show that

$$\frac{1}{2} M^T L'^{-1} M + k - \frac{1}{w^2} \geq 0.$$

Since $w \in \mathbb{Z}$ is an integer, it then suffices to prove that

$$M^T L'^{-1} M \geq -2k + 2. \tag{2.2}$$

Now consider the matrix

$$P = \left( \begin{array}{c|c} L' & M \\ \hline M^T & -2k+2 \end{array} \right),$$

obtained from $L$ by changing the $-2k$ in the last entry to $-2k + 2$. By using Laplace's formula for the determinant, we compute that

$$\det(P) = \det(L) + 2 \det(L').$$

$L'$ and $L$ are negative definite, so their determinants are opposite in sign. Since by assumption $|\det(L)| \geq 2|\det(L')|$, the deteminant of $P$ is also opposite in sign to $\det(L')$ (or 0 if $|\det(L)| = 2|\det(L')|$), hence $P$ is negative semidefinite. By the theory of Schur complement (see for instance [Gal19, Proposition 2.2]) this implies that

$$(-2k + 2) - M^T L'^{-1} M \leq 0,$$

which is the desired inequality (2.2). This proves that $E \in \mathrm{NS}(X)$ is nef.
Moreover $E$ induces an elliptic fibration on $X$, since it already had a section on $U \oplus L'$. It only remains to show that $E$ has infinitely many sections. We will do it separately for the two distinct assumptions.

- Assume that $|\det(L)| > 2|\det(L')|$. Then by the discussion above $\det(P) \neq 0$ is opposite in sign to $\det(L')$, so $P$ is negative definite. The theory of Schur complement implies that
$$(-2k+2) - M^T L'^{-1} M < 0,$$
i.e. the inequality (2.2) holds and is never an equality. In other words, $E$ is not only nef, but it has strictly positive intersection with all the effective $(-2)$-curves $C = [x, y, z, w]$ with $w \neq 0$. This means that the $(-2)$-curves orthogonal to $E$ are actually elements of the sublattice $U \oplus L'$, and therefore the elliptic fibration $|E|$ has at most $\mathrm{rk}(L') < \mathrm{rk}(L)$ vertical components not meeting the zero section. By the Shioda-Tate formula 1.2.6 we conclude that $|\operatorname{MW}(E)| = \infty$.

- Assume instead that $\rho(X) \leq 10$. The previous discussion implies that $E$ induces an elliptic fibration $|E|$ on $X$ with zero section $S_E$. Consider the primitive embedding $i : \langle E, S_E \rangle = U \hookrightarrow U \oplus L$. The orthogonal complement $i(U)^\perp$ is in the genus of $L$, and $L$ is a lattice without roots by assumption, thus $i(U)^\perp$ is not a root-overlattice by Proposition 1.1.15. We conclude again that $|\operatorname{MW}(E)| = \infty$

□

*Remark* 2.3.6. This criterion is not sharp, as we will see later, but it is quite powerful. Indeed, it will allow us to work only with a finite number of lattices (cf. Algorithm 2.5.7).

We now present a second method to prove that a certain K3 surface $X$ has positive entropy. Recall that this amounts to showing that $X$ admits (at least) two elliptic fibrations with infinitely many sections. Therefore it is natural to study the number of elliptic fibrations on $X$, and then investigate how many of them have infinitely many sections. However the number of elliptic fibrations is often infinite (cf. [Nik14, Theorem 10]), hence it is more significant to compute the number of elliptic fibrations *up to the action of* $\operatorname{Aut}(X)$. This number is finite by [Ste85, Proposition 2.6].
We denote
$$N(X) = \#\{\text{elliptic fibrations on } X\}/\operatorname{Aut}(X),$$
$$N^{\text{pos}}(X) = \#\{\text{elliptic fibrations on } X \text{ with infinitely many sections}\}/\operatorname{Aut}(X).$$
Clearly $N^{pos}(X) \leq N(X)$, and if $N^{pos}(X) > 1$, then $X$ has positive entropy by Corollary 2.2.9. Notice that in the next result we do not assume any extra assumption on the given elliptic fibration $|F|$. A generalization of the next theorem has recently been proved by Festi and Veniani [FV21b, Theorem 2.8].

**Theorem 2.3.7.** *Let $X$ be an elliptic K3 surface, $\operatorname{NS}(X) = U \oplus L$ and denote by $|F|$ the given elliptic fibration. If the Picard rank $\rho(X)$ is even (and $\rho(X) < 20$), assume that the period $\omega_X \in \mathrm{T}(X)_{\mathbb{C}}$ is very general. Then $N(X) = 1$ if and only if $L$ is unique in its genus and the restriction map $\mathrm{O}(L) \to \mathrm{O}(A_L)$ is surjective.*

*Proof.* Assume first that the restriction map $\mathrm{O}(L) \to \mathrm{O}(A_L)$ is not surjective, and let $\varphi \in \mathrm{O}(A_L)$ be an isometry of $A_L$ not in the image of the restriction map. By [Huy16, Theorem

14.2.4 ] we have an $f \in \mathrm{O}(\mathrm{NS}(X))$ such that $\overline{f} = \varphi \in \mathrm{O}(A_L) = \mathrm{O}(A_{\mathrm{NS}(X)})$. Up to composing $f$ with a finite number of elements in the Weyl group $W(\mathrm{NS}(X))$, we can assume that $E = f(F)$ is nef, and hence that it induces an elliptic fibration. Notice that the Weyl group $W(\mathrm{NS}(X))$ acts trivially on $A_{\mathrm{NS}(X)}$, so we still have that $\overline{f} = \varphi \in \mathrm{O}(A_L) = \mathrm{O}(A_{\mathrm{NS}(X)})$. We want to prove that $E$ and $F$ induce distinct elliptic fibrations under the action of $\mathrm{Aut}(X)$. Assume by contradiction that there exists $g \in \mathrm{Aut}(X)$ such that $g^*(F) = E$. Then $h = (g^*)^{-1} \circ f$ preserves the elliptic fibration $|F|$. Up to composing with a translation in $\mathrm{MW}(F)$, we can assume that $h$ preserves the lattice $U$ generated by $F$ and its zero section; hence $h \in \mathrm{O}(L)$ is an isometry of the orthogonal complement $L$ of $U$. By the generality assumption on $\omega_X$, Remark 1.2.12 and Lemma 1.2.11 imply that $\overline{(g^*)^{-1}} = \pm\,\mathrm{id} \in \mathrm{O}(A_{\mathrm{NS}(X)}) = \mathrm{O}(A_L)$. Hence $\overline{h} = \pm\overline{f} = \pm\varphi$ does not lift to an isometry of $L$, a contradiction.
Assume instead that $L$ is not unique in its genus, and let $M$ be a lattice in the genus of $L$ not isometric to $L$. By [Nik79b, Proposition 1.5.1] we have an embedding $j : M \hookrightarrow \mathrm{NS}(X)$ such that $j(M)^\perp = U$. Assume that $j(M)^\perp = \langle E, C\rangle$, where $E^2 = 0$, $C^2 = -2$ and $E \cdot C = 1$. Up to applying a certain (finite) number of isometries $s_i \in W(\mathrm{NS}(X))$, we can assume that $E$ is nef, and induces an elliptic fibration on $X$ with respect to which $\mathrm{NS}(X) = \langle E, C\rangle \oplus M$. $C$ is effective, since $E \cdot C = 1 > 0$, so $C$ splits as the union of some smooth $(-2)$-curves. Since $E$ is nef and $E \cdot C = 1$, necessarily $C$ is the union of a section $S_E$ of the elliptic fibration $|E|$ and some vertical components, say

$$C = S_E + \sum_{i,j} C_i^{(j)},$$

with $C_i^{(j)}$ a vertical $(-2)$-curve for every $i, j$, and $j$ indexing the reducible fibers of the fibration induced by $E$. Since

$$-2 = C^2 = S_E^2 + 2S_E\left(\sum_{i,j} C_i^{(j)}\right) + \sum_j \left(\sum_i C_i^{(j)}\right)^2,$$

and the intersection form restricted to the $(-2)$-curves of a reducible fiber not intersecting $S_E$ is negative definite, we have that for all $j$ there exists one and only one $i$ such that $S_E \cdot C_i^{(j)} = 1$. Hence applying the reflections $s_{C_i^{(j)}} \in W(\mathrm{NS}(X))$ we keep $E$ fixed and we map $S_E$ into $S'_E = S_E + \sum_j C_i^{(j)}$. By repeating the same argument to $S'_E$, we conclude that $C$ and $S_E$ are conjugated under the action of the Weyl group $W(\mathrm{NS}(X))$, and therefore

$$\langle E, S_E\rangle^\perp \cong \langle E, C\rangle^\perp \cong M.$$

Certainly $E, F$ are distinct up to automorphism, since the two orthogonal complements $L = \langle F, S_0\rangle^\perp$ and $M = \langle E, S_E\rangle$ are not isometric by assumption.
Finally we have to prove the converse. So assume that $L$ is unique in its genus and the restriction map $\mathrm{O}(L) \to \mathrm{O}(A_L)$ is surjective. Let $|E|$ be another elliptic fibration on $X$. The orthogonal complement $\langle E, S_E\rangle^\perp \subseteq \mathrm{NS}(X)$ is in the genus of $L$, hence it is isometric to $L$ by assumption. Therefore there exists an isometry $f \in \mathrm{O}^+(\mathrm{NS}(X))$ such that $f(F) = E$. The

restricted isometry $\overline{f} \in \mathrm{O}(A_L)$ comes by assumption from a $\varphi \in \mathrm{O}(L)$, so we can construct $g \in \mathrm{O}^+(\mathrm{NS}(X))$ such that $g|_{\langle E, S_E \rangle} = \mathrm{id}$ and $g|_L = \varphi$. Now $h = g^{-1} \circ f \in \mathrm{O}^+(\mathrm{NS}(X))$ satisfies by construction

$$h(F) = E \quad \text{and} \quad \overline{h} = \mathrm{id} \in \mathrm{O}(A_L),$$

so by Lemma 1.2.11.(1) there exists $h' = h \circ s \in \mathrm{O}^+(\mathrm{NS}(X))$ such that $h'(F) = E$, $h'$ preserves the nef cone and $\overline{h'} = \mathrm{id} \in \mathrm{O}(A_L)$. Thus $h'$ is an automorphism of $X$ by Lemma 1.2.11.(3), and consequently $F$ and $E$ are conjugated under the action of $\mathrm{Aut}(X)$. □

In a completely analogous manner we can prove:

**Theorem 2.3.8.** *Let $X$ be an elliptic K3 surface with $\mathrm{NS}(X) = U \oplus L$, where $L$ is not a root-overlattice. If the Picard rank $\rho(X)$ is even (and $\rho(X) < 20$), assume that the period $\omega_X \in \mathrm{T}(X)_{\mathbb{C}}$ is very general. Then $N^{pos}(X) = 1$ if and only if $L$ is the unique non-root-overlattice in its genus and the restriction map $\mathrm{O}(L) \to \mathrm{O}(A_L)$ is surjective.*

It is important to stress the fact that the numbers $N(X)$, $N^{pos}(X)$ really depend on the period of $X$, not only on the Néron-Severi lattice $\mathrm{NS}(X)$ (see for instance [FV21b, Theorem 2.8]). However in our case we can bypass the generality assumption, since we are only interested in whether the entropy of $X$ is zero or positive. Let us make this precise with the following corollary.

**Corollary 2.3.9.** *Let $X$ be an elliptic K3 surface with $\mathrm{NS}(X) = U \oplus L$. Assume that either:*

- *$L$ has at least two distinct non-root-overlattices in its genus, or*
- *The restriction map $\mathrm{O}(L) \to \mathrm{O}(A_L)$ is not surjective.*

*Then $X$ has positive entropy.*

*Proof.* Let $X'$ be a *very general* K3 surface with $\mathrm{NS}(X') = U \oplus L$. Then Theorem 2.3.8 implies that $X'$ has positive entropy. Since having positive entropy only depends on the Néron-Severi lattice and $\mathrm{NS}(X) = \mathrm{NS}(X')$, we conclude that $X$ also has positive entropy. □

Theorem 2.3.5 and Corollary 2.3.9 give two concrete methods to understand whether a K3 surface has zero or positive entropy. In the next sections we will perform an exhaustive inspection of the Néron-Severi lattices of elliptic K3 surfaces admitting an elliptic fibration with only irreducible fibers, in order to obtain the classification of Theorem 2.1.2.

## 2.4 K3 surfaces of Picard rank 3

By the Shioda-Tate formula 1.2.6, a K3 surface admitting an elliptic fibration with infinitely many sections must have Picard rank at least 3. The aim of this section is to study the number of elliptic fibrations on K3 surfaces of Picard number 3.

Let $X$ be an elliptic K3 surface of Picard rank 3, and denote by $|F|$ an elliptic fibration on $X$. The Néron-Severi lattice has the form

$$\mathrm{NS}(X) = U \oplus \langle -2k \rangle$$

for a certain $k \geq 1$. If $k = 1$, then the pencil $|F|$ admits a reducible fiber and only one section by Proposition 1.2.9. Thus we can assume $k \geq 2$. Shimada [Shi15] presents an algorithm to compute the automorphism group of these K3 surfaces; however the full automorphism group can only be computed for a finite number of Picard lattices (more precisely, Shimada computes it for $k \leq 16$). Notice that, if $k \geq 2$, the elliptic fibration induced by $F$ has no reducible fibers, thus infinitely many sections by Shioda-Tate's formula 1.2.6. In particular, any K3 surface $X$ with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$, $k \geq 2$, has infinite automorphism group.

*Remark* 2.4.1. Most results proved in this section are contained in Nikulin's paper [Nik99]; however, since some of the ideas used will be useful later on, we have decided to include the proofs. Moreover, our approach is rather different from Nikulin's. Of course the classification of K3 surfaces of Picard rank 3 admitting a unique elliptic (resp. genus 1) fibration we independently obtain coincides with Nikulin's (cf. Theorem 3 and the subsequent discussion in [Nik99]).

Let us fix a basis $\{F, S, D\}$ of the lattice $U \oplus \langle -2k \rangle$ such that the intersection matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2k \end{pmatrix}.$$

We will denote by $[\alpha, \beta, \gamma]$ the divisor $\alpha F + \beta S + \gamma D$ in $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$. For the sake of readability, we rewrite Lemma 2.3.2 and equation (2.1) in this setting.

*Remark* 2.4.2. Let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$. Since the rank 1 lattice $\langle -2k \rangle$ is unique in its genus, all elliptic fibrations on $X$ are isomorphic. Therefore, if $k \geq 2$, all elliptic fibrations on $X$ have no reducible fibers, and infinitely many sections.

**Lemma 2.4.3.** *Keep the notations as above. Let $F \neq E = [\alpha, \beta, \gamma], C = [x, y, z] \in \mathrm{NS}(X)$ be effective, primitive divisors such that $E^2 = 0$ and $C^2 = -2$. Then we have*

$$\beta \mid k\gamma^2, \qquad y \mid kz^2 - 1. \tag{2.3}$$

*Moreover the equation $E \cdot C = m$ can be equivalently written*

$$k(y\gamma - \beta z)^2 = \beta(\beta + my). \tag{2.4}$$

*In particular $E$ is nef if and only if*

$$k(y\gamma - \beta z)^2 - \beta^2 \geq 0$$

*for any such $C$, and $E$ induces a genus 1 fibration with only irreducible fibers if and only if*

$$k(y\gamma - \beta z)^2 - \beta^2 > 0$$

*for any such $C$.*

**Proposition 2.4.4.** *Let $X$ be an elliptic K3 surface with* $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$*, for* $k \geq 2$*. The number* $N(X)$ *of elliptic fibrations on* $X$ *up to automorphism is* $2^{m-1}$*, where* $m$ *is the number of distinct prime divisors of* $k$*.*

*Proof.* Let $|E|$ be an elliptic fibration on $X$, $S_E$ its zero section. The lattice $\langle E, S_E \rangle \cong U$ embeds primitively in the lattice $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$, and since the rank 1 lattice $\langle -2k \rangle$ is unique in its genus, we have that $\langle E, S_E \rangle^\perp \cong \langle -2k \rangle$. Therefore $E$ induces an isometry $f \in \mathrm{O}^+(\mathrm{NS}(X))$ such that $f(F) = E$. This gives a function

$$\{|E| \text{ elliptic fibration}\} \to \{f \in \mathrm{O}^+(\mathrm{NS}(X))\}.$$

Composing with the restriction map $\mathrm{O}(\mathrm{NS}(X)) \to \mathrm{O}(A_{\mathrm{NS}(X)}) = \mathrm{O}(A_L)$, we obtain another function

$$\{|E| \text{ elliptic fibration}\} \to \{\overline{f} \in \mathrm{O}(A_L)\}.$$

By the proof of Theorem 2.3.7 this map is surjective. Two elliptic fibrations $|E_1|, |E_2|$ are conjugated under the action of $\mathrm{Aut}(X)$ if and only if there exists $g \in \mathrm{Aut}(X)$ such that $g(E_1) = E_2$ thus, by Lemma 1.2.11, if and only if the induced $\overline{f_1}, \overline{f_2} \in \mathrm{O}(A_L)$ satisfy $\overline{f_1} = \pm \overline{f_2}$. Consequently we obtain a bijection

$$\{\text{elliptic fibrations}\} / \mathrm{Aut}(X) \longleftrightarrow \mathrm{O}(A_L)/\{\pm \mathrm{id}\}.$$

The discriminant group $A_L$ is cyclic, generated by the element $\frac{D}{2k}$, where $\{D\}$ is a basis for $L = \langle -2k \rangle$. Its norm in $A_L$ is $-\frac{1}{2k} \pmod{2\mathbb{Z}}$, hence we can identify $\mathrm{O}(A_L)$ with the group

$$G_k = \{x \in \mathbb{Z}/2k\mathbb{Z} \mid x^2 \equiv 1 \pmod{4k}\}.$$

An immediate application of the Chinese remainder theorem shows that $G_k$ has $2^m$ elements, concluding the proof. □

Proposition 2.4.4 implies that the K3 surfaces $X$ with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$, $k$ not a power of a prime, have at least two different elliptic fibrations with infinitely many sections, and thus positive entropy. In order to deal with the remaining cases, we need a more in-depth analysis.

**Proposition 2.4.5.** *Let $X$ be a K3 surface with* $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$*, and assume that there exists an integer* $q$ *such that* $q^2 < k$ *and* $q \nmid k - 1$*. Then* $X$ *has infinitely many elliptic fibrations, or equivalently it has positive entropy.*

*Proof.* If $k$ is not a power of a prime, then $X$ has positive entropy by Proposition 2.4.4. Therefore assume that $k = p^n$ is the power of a prime. We distinguish two cases depending on the exponent $n$.

$n \geq 3$: Consider the divisor $E = [p + p^{n-1}, p, 1]$ on $X$. We claim that $E$ induces a genus 1 fibration on $X$ with infinite automorphism group. First, $E$ is nef: indeed by Lemma 2.4.3 it amounts to showing that

$$p^n (y - pz)^2 \geq p^2$$

for any effective $(-2)$-curve $C = [x, y, z]$. Since $y \mid p^n z^2 - 1$ by equation (2.3), $y$ must be coprime with $p$. Therefore $y - pz \neq 0$ and

$$p^n(y - pz)^2 \geq p^n \geq p^3 \geq p^2,$$

as claimed. Notice that $E$ does not admit a section, as the intersection $E{\cdot}C$ is a multiple of $p$ for any curve $C$ in $X$. However the degree of the genus 1 fibration $|E|$ (cf. Section 1.2.1) is $p$, since $E$ admits the $p$-section $S_p = [p^{2n-4} + 2p^{n-2}, p^{n-2} - 1, p^{n-3}]$. Therefore we can consider the corresponding Jacobian fibration $J(X)$, which has Néron-Severi lattice isometric to $U \oplus \langle -2p^{n-2} \rangle$ by Section 1.2.1. Since $n \geq 3$ by assumption, $E$ becomes an elliptic fibration $J(E)$ with only irreducible fibers on $J(X)$ by Remark 2.4.2. Consequently the group $\mathrm{Aut}(J(E))$ is infinite, and it acts with infinite order on $E$: indeed, under the identifications $J(X) \cong \mathrm{Pic}^0(X/\mathbb{P}^1)$ and $X \cong \mathrm{Pic}^1(X/\mathbb{P}^1)$ (see [Huy16, Section 11.4.1]), the generic fiber of $J(X)$ acts by translation on $X$ and it preserves the fibration $|E|$. This shows that $\mathrm{Aut}(E)$ is infinite. We conclude that $X$ has positive entropy by Theorem 2.2.8.

$n \leq 2$: By assumption there exists a $q$ such that $q^2 < k = p^n$ and $q \nmid p^n - 1$. Since $n \leq 2$, $q$ must be coprime with $p$. Consider the primitive isotropic divisor $E = [q^2 + p^n, q^2, q]$. We claim that $E$ induces an elliptic fibration on $X$. In order to show that it is nef, by Lemma 2.4.3 it amounts to showing that

$$p^n(yq - q^2 z)^2 \geq q^4$$

for any effective $(-2)$-curve $C = [x, y, z]$. Since $p^n > q^2$, it suffices to show that $y - qz \neq 0$. Therefore assume by contradiction that $y = qz$. By equation (2.3) we have that $y = qz \mid p^n z - 1$, so necessarily $z = 1$. But then $y = q \mid p^n - 1$, contradicting the assumption on $q$. Then we have to show that $|E|$ has a section, or equivalently that its degree is 1. Since $\det(\mathrm{NS}(X)) = 2k = 2p^n$ with $n \leq 2$, it is sufficient to show that the degree of $|E|$ is not $p$. In fact $E \cdot C$ is never a multiple of $p$ if $C$ has square $-2$: indeed, if $E \cdot C = m$, by Lemma 2.4.3 we have that

$$p^n(y - qz)^2 = q^2 + my,$$

so if $p \mid m$ we would have that $p \mid q$, a contradiction.

□

The next lemma lists the integers $k$ that are not covered by Proposition 2.4.5:

**Lemma 2.4.6.** *The only natural numbers $k$ satisfying the condition*

*(C): For all $r \in \mathbb{N}$ with $r^2 < k$, $r \mid k - 1$*

*are*

$$\mathcal{L}_1 = \{2, 3, 4, 5, 7, 9, 13, 25\}.$$

*Proof.* Let $N$ be the number of distinct prime divisors of $k-1$, and assume $N \geq 5$. Put

$$e = \frac{1}{2}\log_2 k.$$

We have that $N \leq e$, since otherwise, denoted by $\{p_i\}$ the increasing sequence of prime numbers, we would have

$$\prod_{i=1}^{N} p_i > 4^N = 2^{2N} > 2^{2e} = k > k-1,$$

(remember that we have $N \geq 5$, and $2\cdot 3\cdot 5\cdot 7\cdot 11 > 4^5$), contradicting the fact that $k-1$ has $N$ distinct prime divisors. Now let $q$ be the smallest prime number not dividing $k-1$. If we can show that $q^2 < k$, we are done. $k-1$ has $N$ distict prime divisors, so $q$ is smaller or equal than the $(N+1)$-th prime number, which in turn is strictly smaller than $2^N$ (since there is always a prime number between $\alpha$ and $2\alpha$ for every $\alpha > 1$). Hence $q^2 < (2^N)^2 \leq 2^{2e} = k$. If instead $N \leq 4$, then as above we can choose $q$ as one of the first 5 prime numbers, hence $q \leq 11$. Therefore all natural numbers strictly greater than $11^2 = 121$ cannot satisfy (C). A quick inspection of the first 121 natural numbers yields the list $\mathcal{L}_1$ above. □

Combining Proposition 2.4.5 and Lemma 2.4.6 we are left with a finite number of Picard lattices, corresponding to the values of $k$ in the set $\mathcal{L}_1$ above. The next is the main result of the section, completing the classification of K3 surfaces of zero entropy and Picard number 3.

**Theorem 2.4.7.** *Let $X$ be an elliptic K3 surface of Picard rank 3, $\mathrm{NS}(X) = U \oplus \langle -2k\rangle$, $k \geq 2$. Then $X$ has zero entropy (or equivalently, a unique elliptic fibration) if and only if $k \in \mathcal{L}_1 = \{2, 3, 4, 5, 7, 9, 13, 25\}$.*

The 'only if' part is proved by Proposition 2.4.5 and Lemma 2.4.6. For the 'if' part, we are going to perform an exhaustive study of the possible elliptic fibers on these remaining surfaces. We start with a technical lemma.

**Lemma 2.4.8.** *Let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus \langle -2k\rangle$, and assume that $k = p^n$ is a power of a prime. Let $E \in \mathrm{NS}(X)$ be effective and primitive with $E^2 = 0$ and $C \in \mathrm{NS}(X)$ with $C^2 = -2$ and $E \cdot C = 1$. Then $E$ is either the given elliptic fiber $F$ on $X$, or is of one of the following two types:*

$$F'_{q,\gamma'} = [q^2 + k\gamma'^2, q^2, q\gamma'], \qquad F''_{q,\gamma'} = [q^2 k + \gamma'^2, q^2 k, q\gamma'],$$

*with $q > 0$ and $(q, \gamma') = 1$.*

*Proof.* Let $E = [\alpha, \beta, \gamma]$, and put $q = (\beta, \gamma) > 0$. By primitivity of $E$ we have that $q \nmid \alpha$, and the equation $E^2 = 0$ can be rewritten as $\alpha = \beta + \frac{k\gamma^2}{\beta}$. Therefore $q^2$ must divide $\beta$, say $\beta = q^2\beta'$, $\gamma = q\gamma'$, with $(\beta', \gamma') = 1$. Let us distinguish two cases.

- If $p \mid q$, then $p$ divides $\beta$ and $\gamma$, so, by primitivity of $E$, $p$ must not divide

$$\alpha = q^2\beta' + \frac{p^n\gamma'^2}{\beta'}.$$

  Since $\beta'$ is coprime with $\gamma'$, the only possibility for this to hold is that $\beta' = p^n = k$. This case yields an elliptic fiber of type $F''_{q,\gamma'}$.

- If instead $(p,q) = 1$, then as above $\beta' \mid p^n$, say $\beta' = p^m$. Let $C = [x,y,z]$. If $0 < m < n$, then equation (2.4)

$$p^n(\gamma' y - q\beta' z)^2 = \beta'(q^2\beta' + y)$$

  implies that $p \mid q^2\beta' + y$, thus $p \mid y$, and this is a contradiction since $p \mid y \mid p^n z^2 - 1$. Therefore $\beta'$ can only be either 1 (yielding a fiber of type $F'_{q,\gamma'}$) or $k = p^n$ (yielding a fiber of type $F''_{q,\gamma'}$).

The condition $(q,\gamma') = 1$ is necessary in order for $E$ to be primitive. □

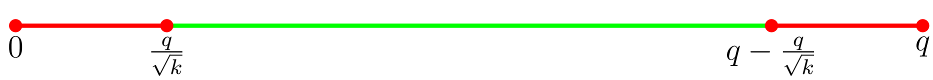
*Proof of Theorem 2.4.7.* We only have to show the 'if' direction by Proposition 2.4.5 and Lemma 2.4.6. Let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$, and assume that $k$ is one of the values in $\mathcal{L}_1$. We want to prove that the given elliptic fibration $|F|$ on $X$ is the unique elliptic fibration. By Lemma 2.4.8 we know that the possible other elliptic fibers on $X$ are either $F'_{q,\gamma'}$ or $F''_{q,\gamma'}$ for some coprime integers $q > 0$ and $\gamma'$. It is therefore sufficient to show that none of these divisors can be nef. We start with $F'_{q,\gamma'}$.
Let $E = F'_{q,\gamma'} = [q^2 + k\gamma'^2, q^2, q\gamma']$. By Proposition 2.3.3 we can assume that $0 < q\gamma' < q^2$, or equivalently that $\gamma' \in (0,q)$.
If $E$ induces an elliptic fibration, then we have that $E \cdot C > 0$ for every effective $(-2)$-curve $C$, since $E$ does not admit any reducible fiber. By Lemma 2.4.3, we can rewrite the inequality $E \cdot C > 0$ as

$$k(y\gamma' - qz)^2 - q^2 > 0$$

where $C = [x,y,z]$. Consider first the case when $C$ is a section of the given elliptic fibration $F$, i.e. $F \cdot C = 1$. Then $C = [kn^2, 1, n]$ for some $n \in \mathbb{Z}$, and the inequality $E \cdot C > 0$ becomes

$$k(\gamma' - qn)^2 > q^2, \quad \text{or equivalently} \quad |\gamma' - qn| > \frac{q}{\sqrt{k}}$$

for any $n \in \mathbb{Z}$. This inequality can hold for all $n \in \mathbb{Z}$ only if $\gamma' \notin (-q/\sqrt{k}, q/\sqrt{k})$ modulo $q$. The next picture summarizes the situation: $\gamma'$ can only be in the green part if $E$ is nef.

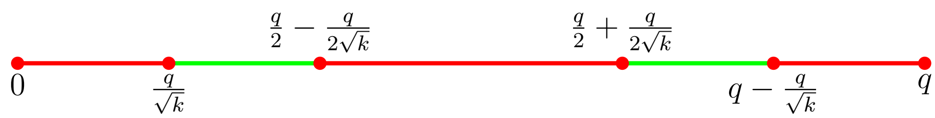



Notice that this is already sufficient to conclude if $k \leq 4$, since the green part is already empty.
If instead $k \geq 5$, consider next the case when $C = [x,y,z]$ is a bisection of $F$, i.e. $y = F \cdot C =$

2. By Lemma 2.4.3 we have that $x = 2 + \frac{kz^2-1}{2}$, and since $k \equiv 1 \pmod 2$, then necessarily $z = 2m + 1$ must be odd. Thus $C$ is of the form $C = [*, 2, 2m + 1]$ for some $m \in \mathbb{Z}$, and hence we can rewrite the inequality $E \cdot C > 0$ as

$$k(2\gamma' - q(2m+1))^2 > q^2, \quad \text{or equivalently} \quad |(2\gamma' - q) - 2mq| > \frac{q}{\sqrt{k}}$$

for any $m \in \mathbb{Z}$. This inequality can hold for all $m \in \mathbb{Z}$ only if $2\gamma' - q \notin (-q/\sqrt{k}, +q/\sqrt{k})$ modulo $2q$, that is if $\gamma' \notin (q/2 - q/2\sqrt{k}, q/2 + q/2\sqrt{k})$ modulo $q$. Again the picture summarizes the situation.




Notice that this is already sufficient to conclude if $k \leq 9$, since the green part is already empty.
If instead $k \geq 10$, we continue the process by considering 3-sections and 4-sections of the given elliptic fibration $F$, that are of the form $[*, 3, 3r \pm 1]$, $[*, 4, 4s \pm 1]$ respectively, for $r, s \in \mathbb{Z}$. It is straightforward to check that with these two further conditions the green part becomes empty for each value of $k$, thus concluding the proof. The reasoning for $F''_{q,\gamma'}$ is completely identical.

□

*Remark* 2.4.9. A natural, follow-up question is to study the moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces, and to investigate whether those $\mathcal{M}_{2k}$ corresponding to the zero entropy cases have some special geometric property. The same question has been asked for in the case of K3 surfaces with finite automorphism group: in many cases the corresponding moduli spaces are unirational (cf. [Rou19; Rou20]). In Chapters 3 and 4 we tackle the problem of studying the geometry of the moduli spaces $\mathcal{M}_{2k}$. It turns out that for the 8 cases $k \in \{2, 3, 5, 7, 9, 13, 25\}$ of zero entropy, the corresponding moduli space $\mathcal{M}_{2k}$ is unirational (cf. Theorem 4.1.1).

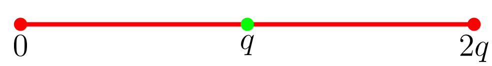
Another natural question is whether the elliptic K3 surfaces with $\mathrm{NS}(X)$ of one of these 8 types admit other genus 1 fibrations. As a corollary of the previous theorem, any other genus 1 fibration must have no sections.

**Theorem 2.4.10.** *Let $X$ be an elliptic K3 surface of Picard rank 3, $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$ and $k \in \{2, 3, 4, 5, 7, 9, 13, 25\}$. Denote by $F$ the fiber of the given elliptic fibration.*

1. *If $k \in \{2, 3, 5, 7, 13\}$ is prime, then $X$ admits a unique genus 1 fibration, induced by $F$.*
2. *If $k = p^2 \in \{4, 9, 25\}$ is a square, then $X$ admits infinitely many genus 1 fibrations, all meeting the given elliptic fiber $F$ at $p$ points (counted with multiplicity). Moreover, the number of genus 1 fibrations on $X$ up to the action of $\mathrm{Aut}(X)$ is 2 if $k \in \{4, 9\}$ and 3 if $k = 25$.*

*Proof.* 1. By Theorem 2.4.7, any genus 1 fibration $|E|$ on $X$ different from $|F|$ has no sections, so its degree $d$ is greater than 1. If $J(X)$ is the Jacobian fibration associated to $|E|$, we have that $\det(\mathrm{NS}(J(X))) = \det(\mathrm{NS}(X))/d^2 = 2k/d^2$ by Section 1.2.1. This is however a contradiction if $k > 2$ is a prime number. If instead $k = 2$, we only have the possibility $d = 2$; this would imply $\det(\mathrm{NS}(J(X))) = 1$, which is a contradiction, as there are no even unimodular lattices of rank 3.

2. Assume that $E = [\alpha, \beta, \gamma]$ induces a genus 1 fibration on $X$. Going through the proof of Lemma 2.4.8, we have that $\beta = q^2\beta'$, $\gamma = q\gamma'$, with $(q, \gamma') = 1$ and $\beta'$ a divisor of $k = p^2$. If $\beta'$ is either 1 or $k = p^2$, then $E$ is either of type $F'_{q,\gamma'}$ or of type $F''_{q,\gamma'}$, and we have proved in Theorem 2.4.7 that all such elements cannot be nef. So $E$ is of the form
$$E = [q^2p + p\gamma'^2, q^2p, q\gamma']$$
for some $q > 0$ and $\gamma' \neq 0$ coprime with $q'$. We are going to repeat the argument in the proof of Theorem 2.4.7.
Consider first the case $k = 4$, i.e. $p = 2$. Since $E$ is nef, it has non-negative intersection with all the sections of the elliptic fibrations $|F|$, that are of the form $S_n := [4n^2, 1, n]$ for $n \in \mathbb{Z}$. The inequality $E \cdot S_n \geq 0$ can be rewritten by Lemma 2.4.3 as
$$4(\gamma' - 2qn)^2 - 4q^2 \geq 0,$$
or equivalently
$$|\gamma' - 2qn| \geq q$$
for all $n \in \mathbb{Z}$. This inequality can hold for all $n \in \mathbb{Z}$ only if $\gamma' \notin (-q, q)$ modulo $2q$, as the picture shows.

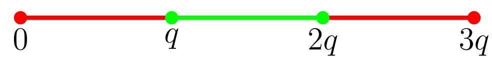



Therefore $\gamma' \equiv q \pmod{2q}$, hence $\gamma'$ is a multiple of $q$. But since $q$ and $\gamma'$ are coprime by assumption, we conclude that $q = 1$, i.e.
$$E = E_{\gamma'} := [2\gamma'^2 + 2, 2, \gamma']$$
for some odd $\gamma'$ (as if $\gamma'$ is even, $E$ is not primitive). By Proposition 2.3.3 we have that $E_{\gamma'}$ is nef if and only if $E_1 = [4, 2, 1]$ is nef, and it is straightforward to check that $E_1$ is indeed nef.
We move to the case $k = 9$, i.e. $p = 3$. We will use a similar approach as before. Since $E$ is nef, it has non-negative intersection with all the sections and bisections of the elliptic fibration $|F|$, that are of the form $S_n := [9n^2, 1, n]$, $B_n := [18m^2 + 18m + 6, 2, 2m + 1]$ respectively, for $n, m \in \mathbb{Z}$. The inequalities $E \cdot S_n \geq 0$ and $E \cdot B_m \geq 0$ can be rewritten by Lemma 2.4.3 as
$$\begin{cases} 9(\gamma' - 3qn)^2 - 9q^2 \geq 0 \\ 9(2\gamma' - 3q(2m+1))^2 - 9q^2 \geq 0 \end{cases}$$

or equivalently

$$\begin{cases} |\gamma' - 3qn| \geq q \\ |(2\gamma' - 3q) - 6mq| \geq q \end{cases} \tag{2.5}$$

for all $n, m \in \mathbb{Z}$. We look at $\gamma'$ modulo $3q$. The first inequality can hold for all $n \in \mathbb{Z}$ only if $\gamma' \notin (-q, q)$ modulo $3q$:




Moreover the second inequality can hold for all $m \in \mathbb{Z}$ only if $2\gamma' - 3q \notin (-q, q)$ modulo $6q$, that is if $\gamma' \notin (q, 2q)$ modulo $3q$:


We conclude that $\gamma' \equiv \pm q$ modulo $3q$, i.e. $\gamma'$ is a multiple of $q$, and thus $q = 1$ from the assumption that $q$ and $\gamma'$ are coprime. We have shown that

$$E = E_{\gamma'} := [3\gamma'^2 + 3, 3, \gamma']$$

for some $\gamma'$ coprime with 3 (otherwise $E$ is not primitive). In order to check that all these elements are nef, it is sufficient by Proposition 2.3.3 to show that $E_{\pm 1} = [6, 3, \pm 1]$ are nef. This is again a straightforward computation.
We conclude with the case $k = 25$, i.e. $p = 5$. The nefness of $E$ implies that $E$ has non-negative intersection with all the sections, bisections, trisections and 4-sections of the elliptic fibration $|F|$. Analogously to the above cases, these inequalities can be rewritten as

$$\begin{cases} |\gamma' - 5qn| \geq q \\ |(2\gamma' - 5q) - 10qm| \geq q \\ |(3\gamma' \pm 5q) - 15qr| \geq q \\ |(4\gamma' \pm 5q) - 20qs| \geq q \end{cases} \tag{2.6}$$

for all $n, m, r, s \in \mathbb{Z}$. In similar fashion to the cases $k = 4$, 9, these inequalities force $\gamma'$ to be a multiple of $q$, which in turn implies that $q = 1$ by the assumption that $q$ and $\gamma'$ are coprime. Thus

$$E = E_{\gamma'} := [5\gamma'^2 + 5, 5, \gamma']$$

for some $\gamma'$ coprime with 5 (otherwise $E$ is not primitive). In order to check that all these elements are nef, it is sufficient by Proposition 2.3.3 to show that $E_{\pm 1} = [10, 5, \pm 1]$ and $E_{\pm 2} = [25, 5, \pm 2]$ are nef. This is again a straightforward computation.
Summing up, we have shown that if $k = p^2 \in \{4, 9, 25\}$ is a square, then all the genus 1 fibrations on $X$ are either the elliptic fibration $|F|$ or they are of the form

$$E = E_{\gamma'} := [p\gamma'^2 + p, p, \gamma']$$

for some $\gamma'$ coprime with $p$. This proves the first part of the statement. In order to count the number of genus 1 fibrations up to $\mathrm{Aut}(X)$, we use the fact that the action of $\mathrm{Aut}(X)$ on $\mathrm{NS}(X)$ is generated by a translation $\tau \in \mathrm{MW}(F)$ and an involution $\sigma$:

$$\tau = \begin{pmatrix} 1 & k & 2k \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(see [Shi15, Section 9]). This can be proved by using the fact that any automorphism $f$ of $X$ preserves the elliptic fiber $F$, so up to composing with a power of $\tau$ we can assume that $f$ preserves the lattice $\langle F, S_0 \rangle \cong U$. But then $f$ can be seen as an automorphism of the orthogonal complement $U^\perp = \langle -2k \rangle$, hence necessarily $f \in \{\mathrm{id}, \sigma\}$.
Now notice that the automorphisms $\tau, \sigma$ act on the genus 1 fibrations $E_{\gamma'}$ as

$$\tau(E_{\gamma'}) = E_{\gamma'+p}, \qquad \sigma(E_{\gamma'}) = E_{-\gamma'}.$$

Thus all genus 1 fibrations $E_{\gamma'}$ are conjugated under the action of $\mathrm{Aut}(X)$ if $p$ is 2 or 3, while if $p = 5$ we have two distinct classes, represented for instance by $E_1$ and $E_2$.
□

## 2.5 K3 surfaces of Picard rank $4 \leq \rho(X) \leq 10$

Let $X$ be an elliptic K3 surface of Picard rank $4 \leq \rho(X) \leq 10$. From now on, we will assume:

**Assumption 2.5.1.** *$X$ admits an elliptic fibration $|F|$ with only irreducible fibers.*

Then $\mathrm{NS}(X) = U \oplus L$, where $L$ has no roots. In order to single out the Néron-Severi lattices of K3 surfaces of zero entropy, we want to apply Theorem 2.3.5 and Corollary 2.3.9. We will proceed inductively: we already have a complete list of lattices of rank 3 of zero entropy, and Theorem 2.3.5 allows us to obtain informations on the entropy of Néron-Severi lattices of higher rank. Recall that any even hyperbolic lattice of rank at most 10 embeds in the K3 lattice (cf. [Nik79b], Theorem 1.14.4), hence the orthogonal complement $L$ of $U$ in $\mathrm{NS}(X)$ can be any even negative definite lattice of rank $\mathrm{rk}(L) \leq 8$.

*Remark* 2.5.2. The reason for which we restrict ourselves first to the case $4 \leq \rho(X) \leq 10$ is basically Proposition 1.1.15. Indeed, let $X$ and $\mathrm{NS}(X) = U \oplus L$ be as above. Since $L$ has no roots, we know by Corollary 2.3.9 that, if $L$ admits a non-isometric non-root-overlattice in its genus, then $X$ has positive entropy. However, given the fact that $\mathrm{rk}(L) \leq 8$, Proposition 1.1.15 implies that no lattice in the genus of $L$ is a root-overlattice. In other words, if $L$ is not unique in its genus, then $X$ has positive entropy.

One could hope that the checks involving Theorem 2.3.5 and Corollary 2.3.9 are enough to single out the Néron-Severi lattices of the K3 surfaces of zero entropy. However this is not the case, as we are going to see in the following. We present an algorithm that searches for elliptic fibrations on a given Néron-Severi lattice. The algorithm is based on the following lemma.

**Lemma 2.5.3.** *Let $X$ be an elliptic K3 surface with $\mathrm{NS}(X) = U \oplus L$, $L$ without roots. Let $E = [\alpha, \beta, \gamma] \in \mathrm{NS}(X)$ be a primitive element with $E^2 = 0$. To any $v \in L$ we associate the finite set*

$$I(v) = \left\{ y \in \mathbb{N} : y \mid \left(-\frac{1}{2}\|v\|_L - \beta^2\right),\ z = \frac{1}{\beta}(y\gamma - v) \in L \text{ and } y \mid \left(-\frac{1}{2}\|z\|_L - 1\right) \right\},$$

*where $z \in L$ means that $z$ has integer entries. Then $E$ is nef if and only if*

$$I(v) = \varnothing \text{ for all } v \in L \text{ with } -\frac{1}{2}\|v\|_L < \beta^2.$$

*Proof.* $E$ is not nef if and only if there exists $C = [x_0, y_0, z_0]$ with $C^2 = -2$, $y_0 > 0$ such that $E \cdot C < 0$. Let $v = y_0\gamma - \beta z_0$. Then Lemma 2.3.2 shows that $y_0 \mid (-\frac{1}{2}\|z_0\|_L - 1)$ and $-\frac{1}{2}\|v\|_L < \beta^2$. Moreover

$$-\frac{1}{2}\|v\|_L - \beta^2 = -\frac{1}{2}\|y_0\gamma - \beta z_0\|_L - \beta^2 \equiv -\frac{1}{2}\|-\beta z_0\|_L - \beta^2 = \beta^2\left(-\frac{1}{2}\|z_0\|_L - 1\right) \equiv 0 \pmod{y_0},$$

hence $y_0 \in I(v)$ and thus $I(v) \neq \varnothing$. Conversely, assume that $y_0 \in I(v)$ for a $v \in L$ with $-\frac{1}{2}\|v\|_L < \beta^2$. Put $z_0 = \frac{1}{\beta}(y_0\gamma - v) \in L$ and choose $x_0 \in \mathbb{Z}$ such that $C = [x_0, y_0, z_0]$ has $C^2 = -2$ (such $x_0$ is an integer since $y_0 \mid (-\frac{1}{2}\|z_0\|_L - 1)$ by assumption). Then Lemma 2.3.2 shows that $E \cdot C < 0$, hence $E$ is not nef. □

*Remark* 2.5.4.

- Lemma 2.5.3 gives a practical way to decide whether a primitive isotropic divisor is nef. Indeed, the set of $v \in L$ satisfying $\frac{1}{2}\|v\|_L < \beta^2$ is finite, since $L$ is negative definite, so we only have to perform a finite number of checks.

- Lemma 2.5.3 can be generalized to any lattice $L$. Let $L$ be any even negative definite lattice, and $E \in U \oplus L$ primitive of square zero. Consider the root part $R = L_{root} \subseteq L$, and say that $R$ is generated by effective $(-2)$-roots $r_1, \dots, r_m$. Then the effective $(-2)$-roots in $U \oplus L$ can be orthogonal or not to the given elliptic fiber $F = [1, 0, 0] \in \mathrm{NS}(X)$. If $r$ is an effective root with $r \cdot F = 0$, then $r$ is a linear combination of $r_1, \dots, r_m$ with nonnegative coefficients. If instead $r \cdot F > 0$, then $r = [x, y, z] \in \mathrm{NS}(X)$ has $y > 0$, and hence we can apply the previous lemma. Summing up, we obtain that $E$ is nef if and only if the sets $I(v)$ as in the lemma are empty, and $Er_i \geq 0$ for all $i = 1, \dots, m$.

- Lemma 2.5.3 is a result analogous to Proposition 4.1 in [Shi14]. Shimada's algorithm checks the nefness of a divisor of positive square, while ours checks it for elements of square 0. Both algorithms boil down to listing some short vectors in $L = U^\perp \subseteq \mathrm{NS}(X)$.

**Algorithm 2.5.5** (`Search for elliptic fibrations`)**.** We are given a Néron-Severi lattice $U \oplus L$, with $n := \mathrm{rk}(L)$, and we want to search for elliptic fibers $E = [\alpha, \beta, \gamma] \in U \oplus L$ for a fixed $\beta \geq 2$.

– Choose a vector $\gamma = [\gamma_1, \ldots, \gamma_n]$ with $0 \le \gamma_i < \beta$ for all $1 \le i \le n$ such that

$$\alpha := \beta + \frac{-\frac{1}{2}\|\gamma\|_L}{\beta} \in \mathbb{Z} \quad \text{and} \quad E := [\alpha, \beta, \gamma] \text{ is primitive.}$$

– List all the vectors $v_1, \ldots, v_N \in L$ with $\frac{1}{2}\|v\|_L < \beta^2$.

– For each $v_i$ compute the set $I(v_i)$ described in Lemma 2.5.3. If the sets $I(v_i)$ are empty for $1 \le i \le N$, then $E$ is nef and we only have to search for a section of $E$.

– We compute the intersection $E \cdot C$ for all effective $(-2)$-curves $C = [x, y, z]$ such that $1 \le y \le 10$ and $z = [z_1, \ldots, z_n]$ with $-\beta < z_i < \beta$. As soon as one of these intersection numbers is 1, we stop the algorithm and we return $E$ together with its section.

Recall that, since $\rho(X) \le 10$, by Remark 2.5.2 we only have to consider the Néron-Severi lattices $U \oplus L$, with $L$ unique in its genus. By Theorem 1.1.11, we know that $L$ is unique in its genus if and only if it is a multiple of one of the lattices of an explicit list (that we can find at [LK13]). The first step is to reduce to a finite number of lattices, by using Theorem 2.3.5. We will use the following key idea.

*Remark* 2.5.6. Let $L$ be a lattice in the list [LK13] of rank $n$, and fix any primitive sublattice $L'$ of corank 1. Assume by inductive hypothesis that the list of Néron-Severi lattices of Picard rank $n+1 = 2+(n-1)$ is finite (it is indeed finite for $\rho(X) = 3$ by Theorem 2.4.7). Then there exists a positive integer $m_0$ such that the Néron-Severi lattices $U \oplus L'(m)$ for $m > m_0$ have positive entropy. Moreover notice that

$$\det(L'(m)) = m^{n-1}\det(L'), \qquad \det(L(m)) = m^n \det(L),$$

for any $m \ge 1$, hence there exists $m_1$ such that $|\det(L(m_1))| \ge 2|\det(L'(m_1))|$. Therefore by Theorem 2.3.5 we have that all the Néron-Severi lattices $U \oplus L(m)$ with $m \ge \max\{m_0, m_1\}$ have positive entropy.

**Algorithm 2.5.7** (`Candidate lattices`)**.** We create a finite list of lattices that are candidate to have zero entropy. For the moment we assume $n \le 8$.

– Put $n := 2$. Let $\mathcal{L}_{n-1}$ be the finite list of candidate lattices of rank $n-1$. Consider the list $\mathcal{L}_n^{prim}$ of lattices in [LK13] of rank $n$ (if the lattice is not even, multiply it by 2). We create an empty list $\mathcal{L}_{n+1}^{temp}$.

– Pick $L \in \mathcal{L}_n^{prim}$, and choose its first principal minor $L'$ of rank $n-1$. We set

$$m_0 := \max\{m \ge 1 \mid L'(m) \in \mathcal{L}_{n-1}\}, \qquad m_1 := \left\lceil 2\frac{|\det(L')|}{|\det(L)|} \right\rceil - 1$$

and $m := \max\{m_0, m_1\}$. We add to $\mathcal{L}_{n+1}^{temp}$ the lattices $L(u)$ for $1 \le u \le m$.

– Pick $L \in \mathcal{L}_{n+1}^{temp}$. If $L$ has a $(-2)$-root, we discard it from $\mathcal{L}_{n+1}^{temp}$. If not, then for any principal minor $L'$ of $L$ of rank $n-1$ we check whether $L' \notin \mathcal{L}_{n-1}$ and $|\det(L)| \geq 2|\det(L')|$. If both conditions hold, we discard $L$ from $\mathcal{L}_{n+1}^{temp}$.

– We repeat the previous check for many primitive sublattices of $L$ of corank 1. More precisely, choose a random upper-triangular matrix $T$ with ones in the diagonal. Then consider the primitive sublattice $L'$ of $L$ spanned by any $n-1$ of the $n$ columns of $T$. We check whether $L' \notin \mathcal{L}_{n-1}$ and $|\det(L)| \geq 2|\det(L')|$. If both conditions hold, we discard $L$ from $\mathcal{L}_{n+1}^{temp}$.

– For any $2 \leq \beta \leq 15$ we search for elliptic fibers on $U \oplus L$ with given $\beta$ by using Algorithm 2.5.5. If we find an elliptic fiber, we stop and we discard $L$ from $\mathcal{L}_{n+1}^{temp}$. This is sufficient to ensure that $U \oplus L$ has positive entropy, since $L$ is unique in its genus, hence any elliptic fibration on $U \oplus L$ has infinitely many sections.

– We return $\mathcal{L}_{n+1} := \mathcal{L}_{n+1}^{temp}$.

*Remark* 2.5.8. Algorithm 2.5.7 works analogously for $n > 8$, by changing the condition $|\det(L)| \geq 2|\det(L')|$ with the more restrictive $|\det(L)| > 2|\det(L')|$ (and by defining $m_1$ as the smallest integer *strictly* greater than $2\frac{|\det(L')|}{|\det(L)|}$). This is because the assumption in Theorem 2.3.5 becomes more restrictive when $\rho(X) > 10$.

After running Algorithm 2.5.7 for $2 \leq n \leq 8$, we obtain the following:

**Theorem 2.5.9.** *The only Néron-Severi lattices of K3 surfaces $X$ of Picard rank $\rho(X) \leq 10$ that satisfy Assumption 2.5.1 and that can have zero entropy are of the form $U \oplus L$, where $L$ is isometric to one of the following* 32 *lattices sorted by rank:*

$$1 : (-4), (-6), (-8), (-10), (-14), (-18), (-26), (-50)$$

$$2 : \begin{pmatrix} -14 & 3 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} -10 & 2 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} -10 & 0 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} -6 & 1 \\ 1 & -6 \end{pmatrix},$$

$$\begin{pmatrix} -6 & 2 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} -6 & 0 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$3 : \begin{pmatrix} -4 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -6 \end{pmatrix}, \begin{pmatrix} -4 & -1 & -1 \\ -1 & -4 & 1 \\ -1 & 1 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 2 & 2 \\ 2 & -6 & -1 \\ 2 & -1 & -6 \end{pmatrix}, \begin{pmatrix} -4 & 1 & 2 \\ 1 & -4 & 1 \\ 2 & 1 & -4 \end{pmatrix},$$

$$\begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & -1 \\ 1 & -1 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -6 \end{pmatrix}, \begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

$$4 : \begin{pmatrix} -4 & 0 & 0 & -2 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & -4 & -2 \\ -2 & -2 & -2 & -4 \end{pmatrix}, \begin{pmatrix} -4 & -2 & -1 & 1 \\ -2 & -4 & 1 & -1 \\ -1 & 1 & -4 & 1 \\ 1 & -1 & 1 & -4 \end{pmatrix}, \begin{pmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \end{pmatrix}, \begin{pmatrix} -4 & -1 & -2 & 2 \\ -1 & -4 & 1 & -1 \\ -2 & 1 & -4 & 1 \\ 2 & -1 & 1 & -4 \end{pmatrix}$$

$$5 : \begin{pmatrix} -4 & -1 & -1 & -1 & -2 \\ -1 & -4 & -1 & -1 & -2 \\ -1 & -1 & -4 & -1 & -2 \\ -1 & -1 & -1 & -4 & 1 \\ -2 & -2 & -2 & 1 & -4 \end{pmatrix}$$

$$6 : \begin{pmatrix} -4 & 1 & -1 & -1 & 0 & 0 \\ 1 & -4 & -2 & 1 & 0 & 0 \\ -1 & -2 & -4 & 2 & -3 & 0 \\ -1 & 1 & 2 & -4 & 3 & 0 \\ 0 & 0 & -3 & 3 & -6 & -3 \\ 0 & 0 & 0 & 0 & -3 & -6 \end{pmatrix}$$

$$8 : E_8(2).$$

*Remark* 2.5.10. The K3 surfaces with Picard lattice isometric to $U \oplus E_8(2)$ were already studied in [Nik81b] and proven to have zero entropy (cf. [Nik81b, Theorem 4.2.2 and 4.2.4]).

**Theorem 2.5.11.** *The elliptic K3 surfaces $X$ such that* $\mathrm{NS}(X) = U \oplus L$, *with $L$ one of the previous* 32 *lattices, have a unique elliptic fibration, hence zero entropy. Moreover the following table specifies whether such surfaces admit other genus* 1 *fibrations:*

| $\rho(X)$ | Nr° | Other genus 1 fibr.? | $E \cdot F$ |
|---|---|---|---|
| 4 | 1 | Yes | 5 |
| | 2 | Yes | 3 |
| | 3 | No | – |
| | 4 | Yes | 3 |
| | 5 | No | – |
| | 6 | No | – |
| | 7 | No | – |
| | 8 | No | – |
| | 9 | No | – |
| | 10 | Yes | 2 |

| $\rho(X)$ | Nr° | Other genus 1 fibr.? | $E \cdot F$ |
|---|---|---|---|
| 5 | 1 | No | – |
| | 2 | No | – |
| | 3 | Yes | 3 |
| | 4 | No | – |
| | 5 | Yes | 3 |
| | 6 | Yes | 3 |
| | 7 | Yes | 2 |
| 6 | 1 | Yes | 2 |
| | 2 | Yes | 3 |
| | 3 | Yes | 5 |
| | 4 | Yes | 3 |
| 7 | 1 | Yes | 3 |
| 8 | 1 | Yes | 3 |
| 10 | 1 | Yes | 2 |

Table 2.1: Genus 1 fibrations on K3 surfaces of zero entropy. The last column indicates the smallest intersection number of other genus 1 fibrations $|E|$ on $X$ with the fiber of the unique elliptic fibration $|F|$.

*Proof.* Let $L$ be one of the lattices above, $n = \mathrm{rk}(L)$. We want to prove that there exists a unique elliptic fibration on $X$, so let $E = [\alpha, \beta, \gamma] \in U \oplus L$ be primitive of square 0 with $\beta > 0$; the goal is to show that $E$ does not induce an elliptic fibration. In order to do so, we

generalize the approach of Theorem 2.4.7.
First, we can assume that the entries of $\gamma$ are in $[0, \beta-1]$ by Proposition 2.3.3. If $E$ induces an elliptic fibration, then $|E|$ has no reducible fibers, since $L$ is unique in its genus and it has no roots. This means that $E \cdot C > 0$ for any effective $(-2)$-divisor $C = [x, y, z] \in U \oplus L$. By Lemma 2.3.2, the inequality can be rewritten as

$$-\frac{1}{2}\|y\gamma - \beta z\|_L - \beta^2 > 0, \quad \text{or equivalently} \quad -\frac{1}{2}\left\|y\frac{\gamma}{\beta} - z\right\|_L > 1.$$

The vector $c = \frac{\gamma}{\beta}$ has rational entries between 0 and 1. If we are able to find finitely many effective $(-2)$-divisors $C_i := [x_i, y_i, z_i]$ such that the $n$-dimensional balls

$$B_i = \left\{-\frac{1}{2}\|y_i c - z_i\|_L \leq 1\right\}$$

cover the hypercube $[0,1]^n$, then we are done. Indeed this shows that, for any primitive isotropic $E$ different from the given elliptic fiber $F$, $E \cdot C_i \leq 0$ for at least one of our $(-2)$-divisors $C_i$. This implies that $E$ cannot induce an elliptic fibration.
We will explain one example in detail; the others are checked similarly with the help of a computer. Let

$$L = \begin{pmatrix} -6 & 2 \\ 2 & -4 \end{pmatrix}.$$

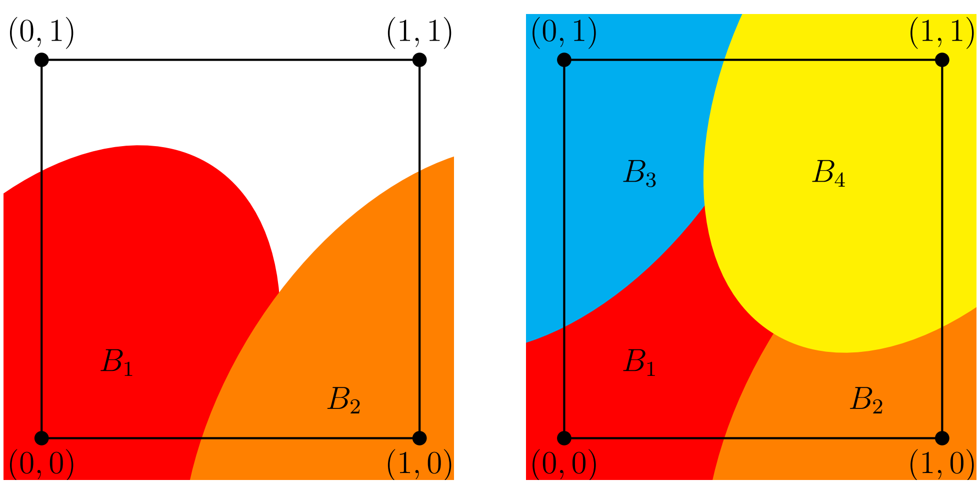
We consider the effective $(-2)$-divisors

$$C_1 = [0,1,0,0], \quad C_2 = [3,1,1,0], \quad C_3 = [2,1,0,1], \quad C_4 = [3,1,1,1].$$

These yield the balls

$$B_1 = \{3c_1^2 - 2c_1c_2 + 2c_2^2 \leq 1\}, \quad B_2 = \{3(c_1-1)^2 - 2(c_1-1)c_2 + 2c_2^2 \leq 1\},$$

$$B_3 = \{3c_1^2 - 2c_1(c_2-1) + 2(c_2-1)^2 \leq 1\}, \quad B_4 = \{3(c_1-1)^2 - 2(c_1-1)(c_2-1) + 2(c_2-1)^2 \leq 1\}.$$



As we can see from the picture, the balls indeed cover the square $[0,1]^2$, as wanted. For the second part of the statement we repeat a similar process. First, we run Algorithm 2.5.5 (more precisely, a modified version of it that only checks the nefness of an isotropic divisor) for "small" values of $\beta$, and we find other genus 1 fibrations on some of our lattices, as specified by Table 2.1. For the remaining lattices, we implement the exact same strategy as in the first part of the proof. More precisely, assume that the open balls

$$\mathring{B}_i = \left\{ -\frac{1}{2}\|y_i c - z_i\|_L < 1 \right\}$$

still cover the hypercube $[0,1]^n$. Then the corresponding Néron-Severi lattice does not admit any other genus 1 fibration. Indeed, if $E = [\alpha, \beta, \gamma]$ different from $F$ is nef, primitive and isotropic, then $E \cdot C_i \geq 0$ for all our effective $(-2)$-divisors $C_i$, implying that the vector $c = \frac{\gamma}{\beta} \notin \mathring{B}_i$ for all $i$, a contradiction.

□

## 2.6 K3 surfaces of Picard rank $\rho(X) > 10$

In order to complete our classification, we have to deal with K3 surfaces with Picard rank $\rho(X) > 10$. Let $X$ be an elliptic K3 surface of Picard rank $\rho(X) > 10$ satisfying Assumption 2.5.1, Then the Néron-Severi lattice of $X$ decomposes as $\mathrm{NS}(X) = U \oplus L$, $L$ without roots. The goal of the section is to prove the following theorem, that concludes our classification.

**Theorem 2.6.1.** *Let $X$ be a K3 surface with Picard rank $\rho(X) > 10$ satisfying Assumption 2.5.1. Then $X$ has positive entropy.*

The proof of this result relies on the help of the software `Magma`. Let us explain the strategy.

We consider first the case when the lattice $L$ is unique in its genus. This only happens when $L$ is a multiple of one of 6 lattices (cf. [LK13]) of rank 9 or 10. These cases are easily dealt with by using Algorithm 2.5.7 (with slight modifications, according to Remark 2.5.8). It turns out that all these Néron-Severi lattices have positive entropy.
Therefore, from now on, we assume that $L$ is not unique in its genus. This implies that any K3 surface $X$ with $\mathrm{NS}(X) = U \oplus L$ admits a second elliptic fibration, not isomorphic to the given $|F|$. If this second fibration has infinitely many sections, then it is already enough to conclude that $X$ has positive entropy by Theorem 2.2.8. However, it can happen that this second elliptic fibration has only finitely many sections, as the next example shows.

*Example* 2.6.2 ([Shi07]). Consider the singular K3 surface $X$ with transcendental lattice

$$\mathrm{T}(X) = \begin{pmatrix} 20 & 10 \\ 10 & 20 \end{pmatrix}$$

Shioda [Shi07] gives an explicit Weierstrass equation for $X$, namely

$$y^2 = x^3 + t^5 - \frac{1}{t^5} - 11,$$

and he proves that the Mordell-Weil group of $X$ over $\mathbb{P}^1_t$ has maximal rank 18. However, $X$ appears in Shimada-Zhang's list (cf. [SZ01, Table 2, Number 58]) of singular extremal K3 surfaces. These two results imply that $X$ admits both an elliptic fibration with only irreducible fibers, and an elliptic fibration with finitely many sections.

Our goal is to prove that any K3 surface admitting both an elliptic fibration with only irreducible fibers and an elliptic fibration with finitely many sections, must admit a third "intermediate" elliptic fibration. This can be rephrased more precisely in terms of genera of lattices:

**Claim 2.6.3.** *Let $R$ be a root-overlattice such that $U \oplus R$ embeds primitively in the K3 lattice. If the genus of $R$ contains a lattice $L$ without roots, then the genus of $R$ also contains a third lattice $M$ with $0 < \mathrm{rk}(M_{root}) < \mathrm{rk}(M)$.*

Theorem 2.6.1 will immediately follow from Claim 2.6.3. For, let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus L$, $L$ without roots and not unique in its genus (we have already dealt with the case when $L$ is unique in its genus). If the new lattice in the genus of $L$ is not a root-overlattice, then it induces an elliptic fibration with infinitely many sections by the Shioda-Tate formula 1.2.6. If instead it is a root-overlattice, then by Claim 2.6.3 we can choose a non-root-overlattice in the genus of $L$ and repeat the same argument.

*Remark* 2.6.4. The fact that Claim 2.6.3 holds is somewhat surprising. From the point of view of elliptic fibrations, it shows that the Mordell-Weil rank of elliptic fibrations on a given K3 surface $X$ cannot jump from 0 to the maximum $\rho(X) - 2$ without attaining some intermediate value. Moreover, Claim 2.6.3 becomes false as soon as we try to relax the assumption on $L$. For instance, the root lattice $A_1^9$ admits a unique non-isometric lattice in the genus, namely $A_1 \oplus E_8(2)$, that has only one root. In other words, if $X$ is a K3 surface with $\mathrm{NS}(X) = U \oplus A_1^9$, then the Mordell-Weil rank of elliptic fibrations on $X$ jumps from 0 to $8 = \rho(X) - 3$ without attaining any intermediate value.

It only remains to prove Claim 2.6.3. Notice that by Proposition 1.1.15 we can assume that $\mathrm{rk}(R) \geq 9$. We first consider the case when $R$ is a root lattice.

**Algorithm 2.6.5** (`RootLattices`)**.** We create a database $\mathcal{D}$ of root lattices not unique in their genus, and we show that any root lattice $R$ as in Claim 2.6.3 contains an element of $\mathcal{D}$ as a direct summand.

– Initialize an empty list $\mathcal{D}$.

– Create the list $\mathcal{R}$ of root lattices $R$ of rank $r$ satisfying the following properties:

1. $9 \leq r \leq 18$;

2. $\det(R) \geq \Delta_r$, where $\Delta_r$ can be found in Table 1.2;
3. If $R = \bigoplus_{i \in I} R_i$, where each $R_i$ is an ADE lattice, then

$$\sum_{i \in I} e(R_i) \leq 24,$$

where $e(R_i)$ equals $n+1$ (resp. $n+2$) if $R = A_n$ (resp. $R = D_n$ or $R = E_n$).

All root lattices $R$ as in Claim 2.6.3 must satisfy these three conditions. More specifically, condition (2) takes care of the fact that the genus of $R$ contains a lattice with no roots (cf. Theorem 1.1.13), while conditions (1) and (3) depend on the fact that $U \oplus R$ embeds primitively in the K3 lattice (cf. Proposition 1.2.7).

– For any root lattice $R = \bigoplus_{i \in I} R_i \in \mathcal{R}$, we check whether there is a proper sublattice $S = \bigoplus_{j \in J \subsetneq I}$ of $R$ that is in $\mathcal{D}$. If it exists, then we discard $R$ from the list $\mathcal{R}$.

– If no proper sublattice of $R$ is in $\mathcal{D}$, then we run through all these proper sublattices, until we find $S = \bigoplus_{j \in J \subsetneq I}$ that has a non-root-overlattice in its genus. We use the function `GenusRepresentatives` in order to compute the genus. If we find such an $S$, we add $S$ to $\mathcal{D}$ and we discard $R$ from $\mathcal{R}$.

– We return $\mathcal{R}$ and $\mathcal{D}$.

The output of Algorithm 2.6.5 is $\mathcal{R} = \varnothing$ and $\mathcal{D}$ consisting of 131 root lattices. This can be interpreted as follows. $\mathcal{D}$ is a list of root lattices admitting a non-root-overlattice in their genus. Moreover, for any root lattice $R$ as in Claim 2.6.3, there exists $S \in \mathcal{D}$ such that $R = S \oplus T$ for a certain root lattice $T$. This implies that the genus of $R$ contains a lattice $M$ with $0 < \mathrm{rk}(M_{root}) < \mathrm{rk}(M)$, as claimed in Claim 2.6.3. Indeed, if $S'$ is a non-root-overlattice in the genus of $S$, then $M = S' \oplus T$ is in the genus of $R$ and it satisfies

$$0 \leq \mathrm{rk}(S_{root}) < \mathrm{rk}(M_{root}) < \mathrm{rk}(M)$$

by construction.

In order to conclude the proof of Claim 2.6.3, we need to analyze in an analogous way the root-overlattices. For, we implement a function `AllOverLattices` that computes all the overlattices of a given lattice $L$ and with given quotient group $S < A_L$. More precisely, the algorithm computes all the overlattices $L'$ of $L$ such that $L'/L \cong S$. Then it only remains to run the following algorithm:

**Algorithm 2.6.6** (`RootOverlattices`).

– Create the list $\mathcal{R}$ of root lattices $R$ of rank $r$ satisfying the following properties (cf. Algorithm 2.6.5):

1. $9 \leq r \leq 18$;

2. $\det(R) \geq \Delta_r$, where $\Delta_r$ can be found in Table 1.2;

3. If $R = \bigoplus_{i \in I} R_i$, where each $R_i$ is an ADE lattice, then

$$\sum_{i \in I} e(R_i) \leq 24,$$

where $e(R_i)$ equals $n+1$ (resp. $n+2$) if $R = A_n$ (resp. $R = D_n$ or $R = E_n$).

– Pick $R \in \mathcal{R}$. For any group $S$ listed in equation (1.1), check whether the following conditions on $k = \#S$ hold:

1. $k^2 \mid \det(R)$;

2. $\det(R)/k^2 \geq \Delta_r$;

3. The condition of Proposition 1.2.9, part 2. holds.

4. If $\mathrm{rk}(R) = 18$, the pair $(R, S)$ belongs to the Shimada-Zhang list [SZ01, Table 2].

If $S$ satisfies these conditions, then compute the overlattices of $R$ with quotient group $S$ by using the algorithm `AllOverLattices`. Add to a list $\mathcal{O}$ the overlattices $R'$ of $R$ such that $R'_{root} = R$.

– Choose an overlattice $R' \in \mathcal{O}$ of $R$. We know from Algorithm 2.6.5 that there exist non-root-overlattices $M_1, \ldots, M_s$ in the genus of $R$ (and we have stored such lattices during Algorithm 2.6.5). Then by Lemma 1.1.12 there are overlattices of $M_1, \ldots, M_s$ in the genus of $R'$. If one of these lattices in the genus of $R'$ is a non-root-overlattice and has minimum 2, we discard $R'$ from $\mathcal{O}$.

– For any remaining $R' \in \mathcal{O}$, we search for lattices in the genus of $R'$ that are non-root-overlattices and have minimum 2, by using `GenusRepresentatives`. If we find one, we discard $R'$ from $\mathcal{O}$.

– If $\mathcal{O} = \varnothing$, we discard $R$ from $\mathcal{R}$.

– We return $\mathcal{R}$.

*Remark* 2.6.7. The reason why we use the list $\mathcal{D}$ constructed during Algorithm 2.6.5 in order to find lattices in the genus of $R'$ is purely computational. In fact, the algorithm `GenusRepresentatives` is computationally very expensive, and the preliminary search that we implemented allows us to run Algorithm 2.6.6 in a reasonable amount of time.

The output of Algorithm 2.6.6 is $\mathcal{R} = \varnothing$, thus concluding the proof of Claim 2.6.3 and of Theorem 2.6.1.

Summing up the results of the last three sections, we have:

**Theorem 2.6.8.** *Let $X$ be a K3 surface satisfying Assumption 2.5.1. Then:*

1. *X has zero entropy, or equivalently X admits a unique elliptic fibration with infinitely many sections, if and only if* NS($X$) *belongs to an explicit list of* 32 *lattices. In particular* $\rho(X) \leq 10$.

2. *X admits a unique genus* 1 *fibration if and only if* NS($X$) *belongs to an explicit list of* 14 *lattices. In particular* $\rho(X) \leq 5$.

## 2.7 K3 surfaces of Picard rank $\geq 19$

The last section of this chapter is devoted to the proof of the following theorem:

**Theorem 2.7.1.** *All K3 surfaces with Picard rank* $\geq 19$ *and infinite automorphism group have positive entropy.*

When the K3 surface $X$ is *singular*, i.e. it has $\rho(X) = 20$, it has been proved by Oguiso (cf. [Ogu07, Theorem 1.6]) that $X$ has positive entropy. Using the methods introduced earlier in the chapter, we are able to extend his result to $\rho(X) = 19$.

Let $X$ be a K3 surface with Picard rank 19. $X$ is elliptic by [Huy16, Corollary 14.3.8], so its Néron-Severi lattice is $\mathrm{NS}(X) = U \oplus L'$, for a certain negative definite lattice $L'$ of rank 17. The transcendental lattice $\mathrm{T}(X) = \mathrm{NS}(X)^{\perp}$ has rank 3 and signature $(2, 1)$, so it embeds into the unimodular lattice $U^2 \oplus E_8$ by [Nik79b, Corollary 1.12.3]. This implies that $\mathrm{NS}(X)$ contains at least a copy of $E_8$, hence $\mathrm{NS}(X) = U \oplus E_8 \oplus L$ for a certain negative definite lattice $L$ of rank 9.

*Remark* 2.7.2. By [Nik79a] we know that the automorphism group of $X$ is finite if and only if $\mathrm{NS}(X) \cong U \oplus E_8 \oplus E_8 \oplus A_1$. In all the other cases of Picard rank 19, $X$ admits an elliptic fibration with infinitely many sections.

**Theorem 2.7.3.** *Let $X$ be a K3 surface with $\rho(X) = 19$ and an infinite automorphism group. Then $X$ admits at least two distinct elliptic fibrations with infinitely many sections. Equivalently, $X$ has positive entropy.*

*Proof.* Let $\mathrm{NS}(X) = U \oplus E_8 \oplus L$. We first consider the genus of $L$. Indeed, if the genus of $L$ contains at least two non-isometric non-root-overlattices, then any K3 surface $Y$ with $\mathrm{NS}(Y) = U \oplus L$ has positive entropy, and it admits two distinct elliptic fibrations with infinitely many sections. If these two elliptic fibrations are induced by $E_1, E_2 \in U \oplus L$, it is clear that the extensions $[E_1, 0], [E_2, 0] \in U \oplus L \oplus E_8$ induce distinct elliptic fibrations with infinitely many sections on $X$, thus $X$ has positive entropy.

Assume first that $L$ is unique in its genus. Then [LK13] shows that $L$ is a multiple of one of 4 lattices: $L_1 = E_8 \oplus A_1$, $L_2 = E_8(4) \oplus A_1$, $L_3, L_4$, where $L_3$ has no roots and $L_4$ has $\mathrm{rk}((L_4)_{root}) = 8$. Theorem 2.6.8 shows that $U \oplus L$ has positive entropy whenever $L$ is $L_2, L_3$, or any multiple $L_1(m), L_2(m), L_3(m), L_4(m)$ with $m > 1$. As above, if $U \oplus L$ has positive entropy, then also $U \oplus L \oplus E_8$ has positive entropy. Moreover, by Remark 2.7.2 we can discard $L_1$, as $U \oplus E_8 \oplus L_1$ has a finite automorphism group. We thus only have to consider $L = L_4$; we will deal with it at the end of the proof.

Assume instead that $L$ is not unique in its genus. If the genus of $L$ contains no root-overlattices, then $U \oplus L$ has positive entropy (and therefore $X$ has positive entropy) by Theorem 2.3.8. Hence we can assume that $L$ is a root-overlattice. We can easily list all root-overlattices of rank 9, obtaining 53 distinct genera of root-overlattices. Studying these genera with `Magma`, we find out that 41 of these 53 contain at least two non-isometric non-root-overlattices, hence by the remark at the beginning of the proof these give rise to K3 surfaces of positive entropy. After also discarding the lattice $E_8 \oplus A_1$ by Remark 2.7.2, we remain with the genera of the following 12 lattices:

$$A_1^9, D_4 \oplus A_1^5, D_4^2 \oplus A_1, D_4 \oplus D_5, E_6 \oplus A_2 \oplus A_1, D_6 \oplus A_3, E_7 \oplus A_1^2, D_7 \oplus A_2, E_7 \oplus A_2, D_9, O, L_4,$$

where $L_4$ is the lattice unique in its genus discussed above, and $O$ is an overlattice of $A_3 \oplus A_1^6$ of index 2.
It is easy to check that the genera of $D_4 \oplus D_5, D_6 \oplus A_3, E_7 \oplus A_2, D_9, O$ contain respectively the lattices $D_8 \oplus \langle -4 \rangle, E_7 \oplus A_1 \oplus \langle -4 \rangle, E_8 \oplus \langle -6 \rangle, E_8 \oplus \langle -4 \rangle, D_4^2 \oplus \langle -4 \rangle$. We then claim that the K3 surfaces $X$ with $\mathrm{NS}(X) = U \oplus E_8 \oplus L$, with $L$ one of these 5 lattices have positive entropy. Consider $\mathrm{NS}(X) = U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$, as the other 4 are analogous. The lattice $E_8 \oplus E_8$ is not unique in its genus (its genus contains the overlattice $D_{16}^+$ of $D_{16}$ of index 2). This implies the existence of two distinct elliptic fibrations on $U \oplus E_8 \oplus E_8$, say $E_1, E_2$. Then these two fibrations extend to elliptic fibrations $F_1 = [E_1, 0], F_2 = [E_2, 0]$ on $U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$ with infinitely many sections, as the orthogonal complements $F_1^\perp, F_2^\perp$ are not generated by roots (since both the orthogonal complements contain $\langle -4 \rangle$ as a direct summand).

In order to conclude the proof, we need to study the remaining 7 lattices. We switch back to the lattices $E_8 \oplus L$, namely

$$E_8 \oplus A_1^9, E_8 \oplus D_4 \oplus A_1^5, E_8 \oplus D_4^2 \oplus A_1, E_8 \oplus E_6 \oplus A_2 \oplus A_1, E_8 \oplus E_7 \oplus A_1^2, E_8 \oplus D_7 \oplus A_2, E_8 \oplus L_4.$$

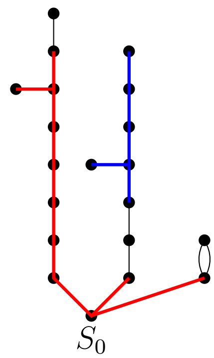
They all contain at least two non-isometric non-root-overlattices in the genus, thus concluding the proof. □

*Remark* 2.7.4. The same approach could be used to study K3 surfaces of smaller Picard rank. Indeed [Nik79b, Corollary 1.12.3] shows that any transcendental lattice $\mathrm{T}(X)$ of rank $\leq 6$ embeds into the unimodular lattice $U^2 \oplus E_8$. Therefore, if $X$ is a K3 surface with $\rho(X) \geq 16$, its Néron-Severi lattice is isometric to $U \oplus E_8 \oplus L$, for a certain negative definite lattice $L$. However, already in Picard rank 18, we find lattices $L$ such that $E_8 \oplus L$ admits a unique non-root-overlattice in the genus. Two examples are given by

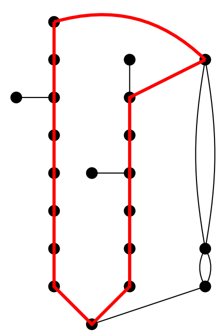
$$L = D_8, E_7 \oplus A_1.$$

This corresponds to the fact that all the elliptic fibrations with infinitely many sections on the K3 surfaces with $\mathrm{NS}(X) \cong U \oplus E_8 \oplus D_8$ or $\mathrm{NS}(X) \cong U \oplus E_8 \oplus E_7 \oplus A_1$ have the same *frame* (i.e., isometric orthogonal complements $U^\perp$ in $\mathrm{NS}(X)$).

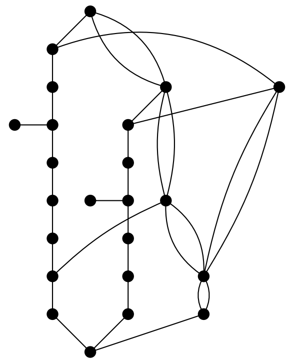
This approach based on the study of the genus is thus not sufficient to decide whether these K3 surfaces have positive entropy. One possible approach to deal with these two cases is to use the *neighboring method* (see [EK14, Section 5]). Let us explain it in the case of a K3 surface $X$ with $\mathrm{NS}(X) = U \oplus E_8 \oplus E_7 \oplus A_1$. The dual graph of these reducible fibers is:



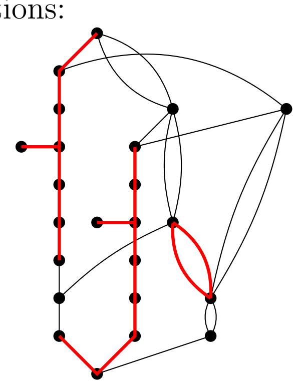

The red diagram highlights a new elliptic fiber $F'$ of type $I_6^*$. Since the blue diagram must be a reducible fiber in the pencil $|F'|$, it must be a fiber of type $I_2^*$:

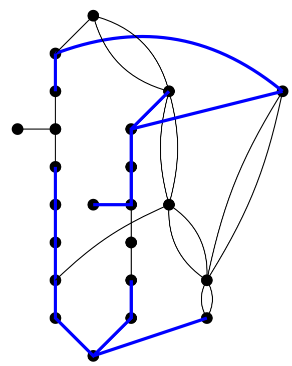


It is now possible to recognize a new elliptic fiber $F''$ of type $I_{16}$. The elliptic pencil $|F''|$ has infinitely many sections: if by contradiction $\mathrm{MW}(F'')$ was finite, then $F''$ would have a second reducible fiber of type $I_2$, so the root lattice $A_{15} \oplus A_1$ would embed in $E_8 \oplus E_7 \oplus A_1$. This is impossible, as they have the same rank and their determinants do not differ by a square (cf. Lemma 1.1.7). Now the computation of the genus explained at the beginning of the remark ensures that *all* elliptic fibrations on $X$ with infinitely many sections have a reducible fiber of type $I_{16}$. Moreover there exists a unique such elliptic fibration *up to automorphisms*, as the isometries of the discriminant group of the transcendental lattice are trivial (cf. Theorem 2.3.7). Thus the only way to decide whether $X$ has zero entropy is to investigate whether there is a second cycle of type $I_{16}$ on $X$. Nevertheless, this does not seem like an easy task. For instance, after doing some successive neighboring steps, we find the following graph of $(-2)$-curves on $X$:



A possible approach is to find isomorphic elliptic fibrations (for instance, other elliptic fibrations with reducible fibers $E_8, E_7$ and $A_1$), construct the corresponding isometry $\varphi$ of $\mathrm{NS}(X)$, and look at the image of the $I_{16}$ cycle under $\varphi$. We tried this for the following two elliptic fibrations:





Unfortunately, it turns out that both isometries preserve the $I_{16}$ cycle. A much more detailed study of the elliptic fibrations on such surfaces is probably needed to determine their entropy.

# 3 | The Kodaira dimension of some moduli spaces of elliptic K3 surfaces

## 3.1 Introduction

Moduli spaces of complex K3 surfaces are a fundamental topic of interest in algebraic geometry. One of the first geometric properties one wants to understand is their Kodaira dimension. Towards this direction, the seminal work [GHS07b] of Gritsenko, Hulek and Sankaran proved that the moduli space $\mathcal{F}_{2d}$ of polarized K3 surfaces of degree $2d$ is of general type for $d > 61$ and for other smaller values of $d$. It is then natural to address the general question about the Kodaira dimension of moduli spaces of lattice polarized K3 surfaces. We are interested in studying a particular class of such surfaces, namely elliptic K3 surfaces of Picard number at least 3.

When $X$ is elliptic, the classes of the fiber and the zero section in the Néron-Severi group of $X$ generate a lattice isometric to the hyperbolic plane $U$, and they span the whole Néron-Severi group if the elliptic K3 surface is very general. The geometry of elliptic surfaces can be studied via their realization as Weierstrass fibrations (cf. Section 1.4.3). By using this description, Miranda [Mir81] constructed the moduli space of elliptic K3 surfaces and showed its unirationality as a by-product. Later, Lejarraga [Lej93] proved that this space is actually rational. We want to study the divisors of the moduli space of elliptic K3 surfaces which parametrize the surfaces whose Néron-Severi groups contain primitively $U \oplus \langle -2k \rangle$, for $k \geq 1$. These are the moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. Geometrically we are considering elliptic K3 surfaces admitting an extra class in the Néron-Severi group: if $k = 1$, it comes from a reducible fiber of the elliptic fibration, while if $k \geq 2$ it is represented by an extra section, intersecting the zero section in $k-2$ points with multiplicity (cf. Remark 1.4.6).

In this chapter we aim at computing the Kodaira dimension of the moduli spaces $\mathcal{M}_{2k}$.

**Theorem 3.1.1.** *The moduli space $\mathcal{M}_{2k}$ is of general type for $k \geq 220$, or*

$$k \geq 208, \;\; k \neq 211, 219, \;\; \text{or } k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}.$$

*Moreover, the Kodaira dimension of $\mathcal{M}_{2k}$ is non-negative for $k \geq 176$, or*

$$k \geq 164, \;\; k \neq 169, 171, 175 \;\; \text{or } k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}.$$

The Torelli theorem for K3 surfaces (see [PŠ71]) allows the moduli spaces $\mathcal{M}_{2k}$ to be realized as quotients of bounded hermitian symmetric domains $\Omega_{L_{2k}}$ of type IV and dimension 17 by the stable orthogonal groups $\widetilde{\mathrm{O}}^+(L_{2k})$, where the lattice $L_{2k}$ is the orthogonal complement of $U \oplus \langle -2k \rangle$ in the K3 lattice $\Lambda_{K3}$. Via this description, one can apply the low-weight cusp form trick (Theorem 3.2.1) developed in [GHS07b]. This tool provides a sufficient condition for an orthogonal modular variety to be of general type. Namely, one has to find a non-zero cusp form on $\Omega^\bullet_{L_{2k}}$ of weight strictly less than the dimension of $\Omega_{L_{2k}}$ vanishing along the ramification divisor of the projection $\Omega_{L_{2k}} \to \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}$. In our case, to construct a suitable cusp form, we use the quasi-pullback method (Theorem 3.2.5) to pull back the Borcherds form $\Phi_{12}$ along the inclusion $\Omega^\bullet_{L_{2k}} \hookrightarrow \Omega^\bullet_{\mathrm{II}_{2,26}}$ induced by a lattice embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$. Here, the lattice $\mathrm{II}_{2,26}$ denotes the unique (up to isometry) even unimodular lattice of signature $(2, 26)$. The lattice embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$ determines the number $N(L_{2k})$ of effective roots in $L^\perp_{2k}$. If $N(L_{2k})$ is positive, the embedding determines the weight $12+N(L_{2k})$ of the cusp form. Therefore the whole proof of Theorem 3.1.1 boils down to finding the values of $k$ for which there exists a suitable primitive embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$, whose orthogonal complement contains at least 2 and at most 8 roots (cf. Problem 3.4.1).

The chapter is organized as follows. In Section 3.2 we describe the method used in proving Theorem 3.1.1, namely the low-weight cusp form trick (Theorem 3.2.1). The desired form is cooked up as a quasi-pullback of the Borcherds form $\Phi_{12}$ (Theorem 3.2.5). Section 3.3 is devoted to the proof of Proposition 3.3.1. Indeed, we study some special reflections in the stable orthogonal group $\widetilde{\mathrm{O}}^+(L_{2k})$. This is then used to impose the vanishing of the quasi-pullback $\Phi|_{L_{2k}}$ of the Borcherds form along the ramification divisor of the quotient map $\Omega_{L_{2k}} \to \mathcal{M}_{2k}$. In Section 3.4 we tackle Problem 3.4.1 of finding primitive embeddings $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$ with at least 2 and at most 8 orthogonal roots. First, we prove that for any $k \geq 4900$ such an embedding exists. Then, we perform an exhaustive computer analysis to find explicit embeddings for the remaining values of $k$. It relies on the geometry of K3 surfaces with Néron-Severi group isometric to $U \oplus E_8$.

**Convention 3.1.2.** Throughout the chapter we will always work over $\mathbb{C}$. We have used the software `Magma` to implement the algorithm described at the end of Section 3.4.

## 3.2 Low-weight cusp form trick

The computation of the Kodaira dimension of modular orthogonal varieties relies on the *low-weight cusp form trick* developed by Gritsenko, Hulek and Sankaran [GHS07b]. In order to describe it, we need to review some theory of modular forms on orthogonal groups.

Let $L$ be an even lattice of signature $(2, n)$. A *modular form* of weight $k$ and character $\chi : \Gamma \to \mathbb{C}^*$ for a finite index subgroup $\Gamma < \mathrm{O}^+(L)$ is a holomorphic function $F : \Omega^\bullet_L \to \mathbb{C}$ on the affine cone $\Omega^\bullet_L$ over $\Omega_L$ such that

$$F(tZ) = t^{-k}F(Z)\ \forall t \in \mathbb{C}^*, \quad \text{and} \quad F(gZ) = \chi(g)F(Z)\ \forall g \in \Gamma.$$

A modular form is a *cusp form* if it vanishes at every cusp (see Section 1.3). We denote the vector spaces of modular forms and cusp forms of weight $k$ and character $\chi$ for $\Gamma$ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively.

Recall that for any finite index subgroup $\Gamma < \mathrm{O}^+(L)$ we have denoted by $\mathcal{F}_L(\Gamma)$ the quotient $\Gamma \backslash \Omega_L$ (cf. Section 1.3). If $M$ is the Néron-Severi lattice of a K3 surface and $L = M^{\perp}_{\Lambda_{K3}}$ is the orthogonal complement of $L$ in the K3 lattice, then $\Omega_L$ is the period domain for $M$-polarized K3 surfaces, and $\mathcal{F}_L(\widetilde{\mathrm{O}}^+(L))$ is the coarse moduli space of $M$-polarized K3 surfaces (cf. Theorem 1.3.2)

**Theorem 3.2.1** ([GHS07b], Theorem 1.1)**.** *Let $L$ be a lattice of signature $(2, n)$ with $n \geq 9$, and let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index. The modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a non-zero cusp form $F \in S_k(\Gamma, \chi)$ of weight $k < n$ and character $\chi$ that vanishes along the ramification divisor of the projection $\pi : \Omega_L \to \mathcal{F}_L(\Gamma)$ and vanishes with order at least 1 at infinity.*

*If $S_n(\Gamma, \det) \neq 0$ then the Kodaira dimension of $\mathcal{F}_L(\Gamma)$ is non-negative.*

*Remark* 3.2.2. By [Ma18, Theorem 1.3] there are only finitely many lattices $L$ of signature $(2, n)$ with $n \geq 9$ such that $\mathcal{F}_L(\Gamma)$ is not of general type. Therefore, our moduli spaces $\mathcal{M}_{2k}$ are known to be of general type for $k$ large enough.

*Remark* 3.2.3. In the recent paper [Ma21] the author shows the necessity for an additional hypothesis in Theorem 3.2.1 concerning the so-called irregular cusps (cf. [Ma21, Theorem 1.2]). However, this does not affect our case as explained in [Ma21, Example 4.10].

### 3.2.1 Ramification divisor

First, we need to describe the ramification divisor of the orthogonal projection, which turns out to be the union of rational quadratic divisors associated to reflective vectors.

For any $v \in L \otimes \mathbb{Q}$ such that $v^2 < 0$ we define the *rational quadratic divisor*

$$\Omega_v(L) := \{[Z] \in \Omega_L \mid Z \cdot v = 0\} \cong \Omega_{v_L^{\perp}}$$

where $v_L^{\perp}$ is an even integral lattice of signature $(2, n-1)$.

The reflection with respect to the hyperplane defined by a non-isotropic vector $r \in L$ is given by

$$\sigma_r : l \mapsto l - 2\frac{(l, r)}{r^2} r.$$

If $r$ is primitive and $\sigma_r \in \mathrm{O}(L)$, then we say that $r$ is a *reflective vector*. We notice that $r$ is always reflective if $r^2 = \pm 2$, and we call it *root* in this case.

If $v \in L^{\vee}$ and $v^2 < 0$, the divisor $\Omega_v(L)$ is called a *reflective divisor* if $\sigma_v \in \mathrm{O}(L)$.

**Theorem 3.2.4** ([GHS07b], Corollary 2.13)**.** *For $n \geq 6$, the ramification divisor of the projection $\pi_\Gamma : \Omega_L \to \mathcal{F}_L(\Gamma)$ is the union of the reflective divisors with respect to $\Gamma < \mathrm{O}^+(L)$:*

$$\mathrm{Rdiv}(\pi_\Gamma) = \bigcup_{\substack{\mathbb{Z}r \subset L \\ \sigma_r \in \Gamma \cup -\Gamma}} \Omega_r(L)$$

### 3.2.2 Quasi pullback

To apply Theorem 3.2.1, we need a supply of modular forms for $\Gamma$. These are provided by quasi-pullbacks of modular forms with respect to some higher rank orthogonal group. In our case, let $\mathrm{II}_{2,26}$ denote the unique (up to isometry) even unimodular lattice of signature $(2,26)$:

$$\mathrm{II}_{2,26} = U^2 \oplus E_8^3.$$

Borcherds proved [Bor95] that $M_{12}(\mathrm{O}^+(\mathrm{II}_{2,26}), \det)$ is a 1-dimensional complex vector space spanned by a modular form $\Phi_{12}$, called the *Borcherds form*. The zeroes of $\Phi_{12}$ lie on rational quadratic divisors defined by $(-2)$-vectors in $\mathrm{II}_{2,26}$, i.e. $\Phi_{12}(Z) = 0$ if and only if there exists $r \in \mathrm{II}_{2,26}$ with $r^2 = -2$ such that $Z \cdot r = 0$. Moreover the multiplicity of the rational quadratic divisor of zeroes of $\Phi_{12}$ is one.

Given a primitive embedding of lattices $L \hookrightarrow \mathrm{II}_{2,26}$, with $L$ of signature $(2,n)$, we define

$$R_{-2}(L^\perp) := \{r \in \mathrm{II}_{2,26} \mid r^2 = -2,\ r \cdot L = 0\}.$$

To construct a modular form for some subgroup of $\mathrm{O}^+(L)$, one might try to pull back $\Phi_{12}$ along the closed immersion $\Omega_L^\bullet \hookrightarrow \Omega_{\mathrm{II}_{2,26}}^\bullet$. However, for any $r \in R_{-2}(L^\perp)$ one has $\Omega_L^\bullet \subset \Omega_{r^\perp}^\bullet$ and hence $\Phi_{12}$ vanishes identically on $\Omega_L^\bullet$. The method of the *quasi-pullback*, developed by Gritsenko, Hulek, and Sankaran [GHS07b], deals with this issue by dividing out by appropriate linear factors:

**Theorem 3.2.5** ([GHS15], Theorem 8.3). *Let $L \hookrightarrow \mathrm{II}_{2,26}$ be a primitive non-degenerate sublattice of signature $(2,n)$, $n \geq 3$, and let $\Omega_L \hookrightarrow \Omega_{\mathrm{II}_{2,26}}$ be the corresponding embedding of the homogeneous domains. The set of $(-2)$-roots $R_{-2}(L^\perp)$ in the orthogonal complement of $L$ is finite. We put $N(L) := |R_{-2}(L^\perp)|/2$. Then the function*

$$\Phi|_L(Z) := \left.\frac{\Phi_{12}(Z)}{\Pi_{r\in R_{-2}(L^\perp)/\pm 1}(Z,r)}\right|_{\Omega_L} \in M_{12+N(L)}(\widetilde{\mathrm{O}}^+(L), \det) \tag{3.1}$$

*is non-zero, where in the product over $r$ we fix a system of representatives in $R_{-2}(L^\perp)/\pm 1$. The modular form $\Phi|_L$ vanishes only on rational quadratic divisors of type $\Omega_v(L)$ where $v \in L^\vee$ is the orthogonal projection with respect to $L^\perp$ of a $(-2)$-root $r \in \mathrm{II}_{2,26}$ on $L^\vee$.*

*Moreover, if $N(L) > 0$, then $\Phi|_L$ is a cusp form.*

We want to apply the low-weight cusp form trick and Theorem 3.2.5 to the orthogonal variety isomorphic to the moduli space of $U \oplus \langle -2k\rangle$-polarized K3 surfaces.

First, we need to compute the orthogonal complement $L_{2k}$ of a primitive embedding $U \oplus \langle -2k\rangle \hookrightarrow \Lambda_{K3}$. Since there exists a unique primitive embedding $U \oplus \langle -2k\rangle \hookrightarrow \Lambda_{K3}$ up to isometry by [Nik79b, Theorem 1.14.4], we get the isomorphism

$$L_{2k} = U \oplus E_8^2 \oplus \langle 2k\rangle \cong (U \oplus \langle -2k\rangle)^\perp_{\Lambda_{K3}}.$$

We will denote by $\mathcal{M}_{2k} \cong \mathcal{F}_{L_{2k}}(\widetilde{\mathrm{O}}^+(L_{2k}))$ the moduli space of $U \oplus \langle -2k\rangle$-polarized K3 surfaces.

Second, we need to find a suitable primitive embedding of $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$, such that the quasi-pullback $\Phi|_{L_{2k}}$ is a cusp form of weight (strictly) less than 17 which vanishes along the ramification divisor of the projection

$$\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k} = \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}.$$

*Remark* 3.2.6. By [GHS09, Theorem 1.7] the abelianization of $\widetilde{\mathrm{O}}^+(L_{2k})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This is because $L_{2k}$ is isometric to $U^2 \oplus E_8 \oplus \langle -2k\rangle^{\perp}_{E_8}$, since the embedding $U \oplus \langle -2k\rangle \hookrightarrow \Lambda_{K3}$ is unique up to isometry (cf. [Nik79b, Theorem 1.14.4]). As a consequence, the Albanese varieties of the moduli spaces $\mathcal{M}_{2k}$ are all trivial (cf. [Kon88, Theorem 2.5]). Moreover, [GHS09, Corollary 1.8] implies that the unique non-trivial character of $\widetilde{\mathrm{O}}^+(L_{2k})$ is det.

## 3.3 Special reflections

Let $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$ be a primitive embedding. Since the embedding $U \oplus E_8^2 \hookrightarrow \mathrm{II}_{2,26}$ is unique up to isometry by [Nik79b, Theorem 1.14.4], we can assume that every summand of $U \oplus E_8^2$ is mapped identically onto the corresponding summand of $\mathrm{II}_{2,26}$. Therefore, any choice of a primitive vector $l \in U \oplus E_8$ of norm $l^2 = 2k$ gives a primitive embedding

$$L_{2k} = U \oplus E_8^2 \oplus \langle 2k\rangle \hookrightarrow \mathrm{II}_{2,26}.$$

In this section we prove the following:

**Proposition 3.3.1.** *The quasi pullback $\Phi|_{L_{2k}}$ defined in Thereom 3.2.5 vanishes along the ramification divisor of*

$$\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k} = \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}$$

*for any primitive embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$ such that $(L_{2k})^{\perp}_{\mathrm{II}_{2,26}}$ does not contain a copy of $E_8$.*

For any $l \in L$ we define its *divisibility* $\mathrm{div}(l)$ to be the unique $m > 0$ such that $l \cdot L = m\mathbb{Z}$ or, equivalently, the unique $m > 0$ such that $l/m \in L^\vee$ is primitive. Since $\mathrm{div}(r) > 0$ is the smallest intersection number of $r$ with any other vector, $\mathrm{div}(r)$ divides $r^2$. Moreover, if $r$ is reflective, the number $2\frac{l \cdot r}{r^2}$ must be an integer, so $r^2$ divides $2(l \cdot r)$ for all $l \in L$, i.e. $r^2 \mid 2\,\mathrm{div}(r)$. Summing up

$$\mathrm{div}(r) \mid r^2 \mid 2\,\mathrm{div}(r).$$

**Proposition 3.3.2.** *Let $r \in L_{2k}$ be a reflective vector. Then $\sigma_r$ induces $\pm\mathrm{id}$ in $A_{L_{2k}}$, i.e. $\pm\sigma_r \in \widetilde{\mathrm{O}}(L)$, if and only if $r^2 = \pm 2$ or $r^2 = \pm 2k$ and $\mathrm{div}(r) \in \{k, 2k\}$.*

*Proof.* Similar to [GHS07b, Proposition 3.2, Corollary 3.4] . □

Now $\sigma_r \in \mathrm{O}^+(L \otimes \mathbb{R})$ if and only if $r^2 < 0$ (see [GHS07a]). Recall that a lattice $T$ is called *2-elementary* if $A_T$ is an abelian 2-elementary group.

**Proposition 3.3.3.** *Let $r \in L_{2k}$ be primitive with $r^2 = -2k$ and $\mathrm{div}(r) \in \{k, 2k\}$. Then $L_r := r^{\perp}_{L_{2k}}$ is a 2-elementary lattice of signature $(2,16)$ and determinant 4.*

*Proof.* We have the following well-known formula for $\det(L_r)$ (see for instance [GHS07b, equation 20]):

$$\det(L_r) = \frac{\det(L_{2k}) \cdot r^2}{\mathrm{div}(r)^2} \in \{1, 4\}.$$

Since $L_{2k}$ has signature $(2,17)$ and $r^2 < 0$, we have that $L_r$ has signature $(2,16)$. Therefore $\det(L_r)$ cannot be 1, because there are no even unimodular lattices with signature $(2,16)$ (see [Nik79b, Theorem 0.2.1]). This shows that $\mathrm{div}(r) = k$. Therefore the reflection $\sigma_r$ acts as $-\mathrm{id}$ on the discriminant group $A_{L_{2k}}$ (see [GHS07b, Corollary 3.4]). Now we can extend $-\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$ to an element $\bar{\sigma}_r \in \mathrm{O}(\Lambda_{K3})$ by defining $\bar{\sigma}_r|_{U\oplus\langle -2k\rangle} = \mathrm{id}$ on the orthogonal complement of $L_{2k} \hookrightarrow \Lambda_{K3}$. Put $S_r := (L_r)^{\perp}_{\Lambda_{K3}}$. It is easy to see that

$$\bar{\sigma}_r|_{L_r} = -\mathrm{id} \quad \text{and} \quad \bar{\sigma}_r|_{S_r} = \mathrm{id}\,.$$

Then $L_r$ is 2-elementary by [Nik79b, Corollary I.5.2]. □

**Proposition 3.3.4.** *Given any embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$, let $r \in L_{2k}$ be a primitive reflective vector with $r^2 = -2k$, and consider $L_r = r^{\perp}_{L_{2k}}$ as above. Under the chosen embedding, the orthogonal complement $(L_r)^{\perp}_{\mathrm{II}_{2,26}}$ is isometric to either*

$$D_{10} \text{ or } E_8 \oplus A_1^2.$$

*Proof.* Since $\mathrm{II}_{2,26}$ is unimodular, the discriminant groups of $L_r$ and $(L_r)^{\perp}_{\mathrm{II}_{2,26}}$ are isometric up to a sign. Proposition 3.3.3 thus implies that $(L_r)^{\perp}_{\mathrm{II}_{2,26}}$ is a 2-elementary, negative definite lattice of rank 10 and determinant 4. By [Nik79b, Proposition 1.8.1], any 2-elementary discriminant form is isometric to a direct sum of finite quadratic forms, each of which is isometric to one of four finite quadratic forms, namely the discriminant forms of the 2-elementary lattices $A_1$, $A_1(-1)$, $U(2)$, $D_4$. Since $(L_r)^{\perp}_{\mathrm{II}_{2,26}}$ has signature $-2 \pmod 8$ and determinant 4, it is immediate to see that its discriminant form must be isometric to the discriminant form of $A_1^2$. Now we notice that the lattice $E_8 \oplus A_1^2$ is a 2-elementary, negative definite lattice of rank 10 with the desired discriminant form. Finally it is enough to compute the genus of $E_8 \oplus A_1^2$. A quick check with `Magma` yields that the whole genus consists of $E_8 \oplus A_1^2$ and $D_{10}$. Alternatively, one can use the Siegel mass formula [CS88] and check that the mass of the quadratic form $f$ associated to the lattice $E_8 \oplus A_1^2$ is

$$m(f) = \frac{5}{2^8 \cdot 4! \cdot 1814400} = \frac{1}{2229534720}.$$

Since a straightforward check shows that $D_{10}$ is in the genus of $E_8 \oplus A_1^2$, and the equality

$$\frac{1}{|\,\mathrm{O}(D_{10})|} + \frac{1}{|\,\mathrm{O}(E_8 \oplus A_1^2)|} = \frac{1}{3715891200} + \frac{1}{5573836800} = \frac{1}{2229534720} = m(f)$$

holds, we deduce that $\{D_{10}, E_8 \oplus A_1^2\}$ is the whole genus of $E_8 \oplus A_1^2$. □

Now we are ready to prove Proposition 3.3.1.

*Proof of Proposition 3.3.1.* In order to prove that $\Phi|_{L_{2k}}$ vanishes along the ramification divisor of the projection $\pi$, we have to show that $\Phi|_{L_{2k}}$ vanishes on the $(-2k)$-divisors $\Omega_r(L_{2k})$ given by reflective vectors $r \in L_{2k}$ of norm $-2k$ (see Theorem 3.2.4). Hence let $r$ be a $(-2k)$-reflective vector. By Proposition 3.3.4, $(L_r)^{\perp}_{\mathrm{II}_{2,26}}$ is a root lattice with at least 180 roots ($E_8 \oplus A_1^2$ has 244 and $D_{10}$ has 180). Since by assumption the orthogonal complement of $L_{2k}$ in $\mathrm{II}_{2,26}$ does not contain a copy of $E_8$, the root lattice generated by $R_{-2}(L_{2k}^{\perp})$ has rank at most 9 and does not contain a copy of $E_8$. By checking all such root lattices, we obtain $|R_{-2}(L_{2k}^{\perp})| \leq |\{\text{roots of } D_9\}| = 144$ (just recall that $A_n$ has $n(n+1)$ roots, $D_n$ has $2n(n-1)$ roots, $E_6, E_7$ have 72 and 126 roots respectively). Consequently $\Phi|_{L_{2k}}$ vanishes along the $(-2k)$-divisor $\Omega_r(L_{2k})$ given by $r$ with order $\geq (180-144)/2 > 0$, as claimed. $\square$

## 3.4 Lattice engineering

By the previous discussion, we have transformed our original question of determining the Kodaira dimension of $\mathcal{M}_{2k}$ to the following

**Problem 3.4.1.** *For which $2k > 0$ does there exist a primitive vector $l \in U \oplus E_8$ with norm $l^2 = 2k$ such that $l$ is orthogonal to at least 2 and at most 8 roots?*

We want to find a lower bound for the values $2k$ answering Problem 3.4.1 positively (see Proposition 3.4.5). Since $U \oplus E_8$ contains infinitely many roots, we want to start by reducing to the more manageable case of $E_8$, whose number of roots is finite.

For simplicity we define

$$R(l) := \{r \in U \oplus E_8 \mid r^2 = -2, r \cdot l = 0\} = R_{-2}(L_{2k}^{\perp}).$$

The following is a slight generalization of [TV19, Lemma 4.1, 4.3], proved in [Pet19, Lemma 3.3 and 3.4].

**Lemma 3.4.2.** *Let $l = \alpha e + \beta f + v$, where $U = \langle e, f\rangle$ such that $e^2 = f^2 = 0$ and $e \cdot f = 1$, $v \in E_8$ and $\alpha, \beta \in \mathbb{Z}$, with norm $l^2 = 2k > 0$. Let $r = \alpha' e + \beta' f + v'$ be a vector of $R(l)$, where $v' \in E_8$ and $\alpha', \beta' \in \mathbb{Z}$. If $\alpha \neq \beta$, $\alpha, \beta > \sqrt{k}$ and $\alpha\beta < \frac{5}{4}k$, then $\alpha' = \beta' = 0$.*

In other words, if $l = \alpha e + \beta f + v \in U \oplus E_8$ is a vector of norm $2k$ satisfying the assumptions of Lemma 3.4.2, then the roots of $U \oplus E_8$ orthogonal to $l$ are roots of $E_8$. Therefore the set $R(l)$ coincides with the set of roots in $v^{\perp}_{E_8}$. The following lemma, inspired by [GHS07b, Theorem 7.1], controls the number of roots of $E_8$ orthogonal to $v$.

**Lemma 3.4.3.** *There exists $v \in E_8$ with $v^2 = 2n$ and such that $v^{\perp}_{E_8}$ contains at least 2 and at most 8 roots if the inequality*

$$2N_{E_7}(2n) > 28N_{E_6}(2n) + 63N_{D_6}(2n), \tag{3.2}$$

*holds, where $N_L(2n)$ denotes the number of representations of $2n$ by the positive definite lattice $L$.*

*Proof.* We follow closely [GHS07b, Theorem 7.1]. Let $a \in E_8$ be a root. Its orthogonal complement $E_7^{(a)} := a^{\perp}_{E_8}$ is isometric to $E_7$. The set of 240 roots in $E_8$ consists of the 126 roots in $E_7^{(a)}$ and 114 other roots, forming the subset $X_{114}$. Assume that every $v \in E_7^{(a)}$ with $v^2 = 2n$ is orthogonal to at least 10 roots in $E_8$, including $\pm a$. By [GHS07b, Lemma 7.2] we know that every such $v$ is contained in the union

$$\bigcup_{i=1}^{28} (A_2^{(i)})^{\perp}_{E_8} \sqcup \bigcup_{j=1}^{63} (A_1^{(j)})^{\perp}_{E_7^{(a)}}, \tag{3.3}$$

where $A_2^{(i)}$ (resp. $A_1^{(j)}$) are root systems of type $A_2$ (resp. $A_1$) contained in $X_{114}$ (resp. $E_7^{(a)}$). Denote by $n(v)$ the number of components in the union (3.3) containing $v$. Since $(A_2^{(i)})^{\perp}_{E_8} \cong E_6$ and $(A_1^{(j)})^{\perp}_{E_7^{(a)}} \cong D_6$, we have counted the vector $v$ exactly $n(v)$ times in the sum

$$28 N_{E_6}(2n) + 63 N_{D_6}(2n).$$

We distinguish three cases.

1. If $v \cdot c \neq 0$ for every $c \in X_{114} \setminus \{\pm a\}$, then $v$ is orthogonal to at least 4 copies of $A_1$ in $E_7^{(a)}$, so $n(v) \geq 4$.
2. If $v$ is orthogonal to only one $A_2^{(i)}$ (6 roots), then $v$ is orthogonal to at least 2 copies of $A_1$ in $E_7^{(a)}$, so $n(v) \geq 3$.
3. If $v$ is orthogonal to at least two $A_2^{(i)}$, then $n(v) \geq 2$.

In conclusion $n(v) \geq 2$ for every $v \in E_7^{(a)}$. Therefore, under our assumption that every $v \in E_7^{(a)}$ with $v^2 = 2n$ is orthogonal to at least 10 roots, we have shown that any such $v$ is contained in at least 2 sets of the union (3.3), i.e.

$$2 N_{E_7}(2n) \leq 28 N_{E_6}(2n) + 63 N_{D_6}(2n).$$

□

**Proposition 3.4.4.** *Let $n \geq 952$. Then there exists $v \in E_8$ with $v^2 = -2n$ such that $v^{\perp}_{E_8}$ contains at least 2 and at most 8 roots.*

*Proof.* [GHS07b, equations (31), (33) and (34)] give the following estimates:

$$N_{E_7}(2n) > 123.8\, n^{5/2}, \qquad N_{E_6}(2n) < 103.69\, n^2, \qquad N_{D_6}(2n) < 75.13\, n^2.$$

By Lemma 3.4.3, we immediately obtain the claim. □

We are now ready to answer Problem 3.4.1:

**Proposition 3.4.5.** *Let $k \geq 4900$. Then there exists a primitive $l \in U \oplus E_8$ with $l^2 = 2k$ and $2 \leq |R(l)| \leq 8$.*

*Proof.* Pick $k > 0$ and consider $l = \alpha e + \beta f + v$, where $l^2 = 2k$, $v^2 = -2n$, so that $\alpha\beta = n + k$. Suppose that there exist $\alpha$ and $\beta$ satisfying the hypotheses of Lemma 3.4.2 such that $n = \alpha\beta - k \geq 952$. Then Proposition 3.4.4 implies that we can find a $v \in E_8$ with $v^2 = -2n$ such that $v^{\perp}_{E_8}$ contains at least 2 and at most 8 roots. Moreover Lemma 3.4.2 ensures that the roots of $U \oplus E_8$ orthogonal to $l = \alpha e + \beta f + v$ are contained in $E_8$, so that $l^{\perp}_{U\oplus E_8}$ also contains at least 2 and at most 8 roots. Therefore the existence of such $\alpha, \beta$ is sufficient for the existence of $l \in U \oplus E_8$ with $2 \leq |R(l)| \leq 8$.

Now let $k \geq 4900 = 70^2$, and consider

$$\alpha = \lceil \sqrt{k} + 6 \rceil, \qquad \beta = \alpha + 1.$$

Clearly $\alpha \neq \beta$, $\gcd(\alpha, \beta) = 1$ and $\alpha, \beta > \sqrt{k}$. Moreover

$$\frac{5}{4}k - \alpha\beta \geq \frac{5}{4}k - (\sqrt{k} + 7)(\sqrt{k} + 8) = \frac{1}{4}k - 15\sqrt{k} - 56 > 0,$$

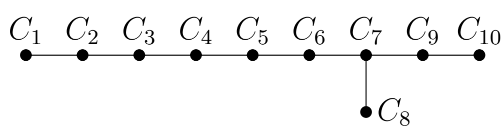
and

$$n = \alpha\beta - k \geq (\sqrt{k} + 6)(\sqrt{k} + 7) - k = 13\sqrt{k} + 42 \geq 952,$$

completing the proof. □

In order to deal with the remaining values of $k$, we use of the geometry of the K3 surfaces with Néron-Severi lattice isometric to $U \oplus E_8$. We recall in the following the main properties of such surfaces.

Let $X$ be a K3 surface with $\mathrm{NS}(X) = U \oplus E_8$. Then $X$ has finite automorphism group and a finite number of irreducible $(-2)$-curves (see e.g. [Nik79c] or [Kon89]). More precisely, if $|E|$ denotes the unique elliptic fibration on $X$, then the irreducible $(-2)$-curves on $X$ are the 9 curves $C_2, \ldots, C_{10}$ contained in the reducible fiber of $|E|$, plus the unique section of $E$, which we will denote by $C_1$. The dual graph of such $(-2)$-curves is




(3.4)

Now let $D \in \mathrm{NS}(X) = U \oplus E_8$ be a primitive divisor of norm $2k > 0$ with $2 \leq |R(D)| \leq 8$. In other words, $D^{\perp}$ contains at least 1 and at most 4 effective $(-2)$-divisors. Up to the action of the Weyl group $W < \mathrm{O}(U \oplus E_8)$ we can assume that $D$ is nef, since isometries of $U \oplus E_8$ do not change the number of orthogonal roots. The nef cone $\mathrm{Nef}(X)$ is rational polyhedral, and a basis can be computed in `Magma`. It turns out that one such basis $\{D_1, \ldots, D_{10}\}$ is the dual basis of $\{C_1, \ldots, C_{10}\}$, i.e. $D_i \cdot C_j = \delta_{ij}$ for all $1 \leq i, j \leq 10$. For instance, $D_1 = E$ defines the only elliptic fibration on $X$, and $D_1^2 = 0$. Moreover $D_i^2 > 0$ for $i \geq 2$. This means that the nef divisor $D$ is a linear combination of $D_1, \ldots, D_{10}$ with non-negative coefficients

$$D = \sum_{i=1}^{10} d_i D_i.$$

By construction, the $(-2)$-curve $C_j$ is orthogonal to $D$ if and only if $d_j = 0$. This implies that the root part of $D^\perp$ is a root lattice $R$ generated by the $(-2)$-curves $\{C_j \mid d_j = 0\}$. Since $R$ contains at most 4 effective roots, it is one of the following root lattices:

$$A_1,\ A_1^2,\ A_1^3,\ A_1^4,\ A_2,\ A_2 \oplus A_1.$$

Choose one of these root lattices $R$, and fix one of the finitely many sub-diagrams $J \subseteq \{1, \dots, 10\}$ of the dual graph (3.4) giving rise to a root lattice $\langle C_j \mid j \in J\rangle$ isometric to $R$. Thus the nef divisors $D$ orthogonal precisely to $\{C_j \mid j \in J\}$ are all those of the form

$$D = \sum_{i \notin J} d_i D_i$$

for some $d_i > 0$. Since we are only interested in divisors of norm $2k < 2 \cdot 4900$, we can use the inequality

$$D^2 \geq \sum_{i \notin J} d_i^2 D_i^2 + 2 \sum_{1 \neq i \notin J} d_1 d_i (D_1 D_i),$$

to bound the $d_i$ for $i \notin J$. More precisely, we have that

$$d_i^2 \leq \frac{2 \cdot 4900}{D_i^2}\ \forall i \geq 2, \quad \text{and} \quad d_1 \leq \frac{4900}{\sum_{1 \neq i \notin J} D_1 D_i}.$$

By varying the coefficients $d_i$'s in these ranges, we obtain all primitive vectors $D \in U \oplus E_8$ with $D^2 \leq 2 \cdot 4900$ and $2 \leq |R(D)| \leq 8$ up to the action of $\mathrm{O}(U \oplus E_8)$. Therefore this search is completely exhaustive.

A similar list can be obtained if we allow $D$ to have up to 10 orthogonal roots. All the previous discussion works analogously, with the only difference that the root part of $D^\perp$ can also be isometric to $A_1^5$ or $A_2 \oplus A_1^2$.

We get the following: a primitive vector $l \in U \oplus E_8$ with $l^2 = 2k < 2 \cdot 4900$ and $2 \leq |R(l)| \leq 8$ exists if and only if

$$k \geq 208,\ k \neq 211, 219 \text{ or } k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}. \qquad (3.5)$$

Moreover a similar vector $l$ with $2 \leq |R(l)| \leq 10$ exists if and only if

$$k \geq 164,\ k \neq 169, 171, 175 \text{ or } k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}. \qquad (3.6)$$

We have implemented the algorithm described above in `Magma`. We are now ready to prove Theorem 3.1.1.

*Proof of Theorem 3.1.1.* Proposition 3.4.5 combined with the subsequent search ensures that there exists a primitive $l \in U \oplus E_8$ with norm $l^2 = 2k$ and $2 \leq |R(l)| \leq 8$ if $k \geq 4900$ or $k$ belongs to the list (3.5), in particular for any $k \geq 220$. Such an $l \in U \oplus E_8$ determines an embedding $L_{2k} \hookrightarrow \mathrm{II}_{2,26}$ with the property

$$1 \leq N(L_{2k}) \leq 4,$$

where $N(L_{2k})$ is the number of effective roots in the orthogonal complement $(L_{2k})^{\perp}_{\mathrm{II}_{2,26}}$. Hence Theorem 3.2.5 provides a non-zero cusp form $\Phi|_{L_{2k}}$ of weight $12 + N(L_{2k}) \leq 12 + 4 < 17 = \dim(\mathcal{M}_{2k})$, which vanishes along the ramification divisor of $\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k}$ in view of Proposition 3.3.1, since $l^{\perp}$ does not contain $E_8$, otherwise $l$ would be orthogonal to at least 240 roots. Then the low-weight cusp form trick (Theorem 3.2.1) ensures that $\mathcal{M}_{2k}$ is of general type.

An analogous argument shows that $\mathcal{M}_{2k}$ has non-negative Kodaira dimension if $k$ belongs to the list (3.6), in particular for any $k \geq 176$. □

# 4 | Unirationality of some moduli spaces of elliptic K3 surfaces

## 4.1 Introduction

In the previous chapter we were able to compute the Kodaira dimension of almost all moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. In particular, we showed that these spaces are of general type if $k$ is "big enough" (see Theorem 3.1.1). It is natural to ask what happens for smaller values of $k$. For instance, for which values of $k$ is $\mathcal{M}_{2k}$ (uni)rational? Unirational varieties have Kodaira dimension $-\infty$, so $\mathcal{M}_{2k}$ is surely not unirational for all the values of $k$ listed in Theorem 3.1.1. Unirational moduli spaces are in fact very rare, but at the same time they are of fundamental importance: indeed, unirational moduli spaces are the easiest to describe, as their objects can be explicitly parametrized by a finite number of parameters.

The problem of studying whether moduli spaces of K3 surfaces are unirational is very classical, and it dates back to works of Mukai [Muk88; Muk96; Muk06; Muk12; Muk92]. His contribution was showing that the moduli spaces $\mathcal{F}_{2d}$ of K3 surfaces of degree $2d$ are unirational for $d \leq 11$ and $d = 12, 15, 17, 19$. This was later improved by Farkas and Verra [FV18; FV21a] extending the unirationality result to K3 surfaces of degree $d = 13, 21$ by using the connection to special cubic fourfolds. Recently, the moduli spaces of $n$-pointed K3 surfaces of degree $d \leq 21$ were studied systematically in [Ma19]. It is then natural to ask the more general question about the unirationality of moduli spaces of lattice polarized K3 surfaces. Farkas and Verra [FV12; FV16; Ver16] worked out the case of polarized Nikulin surfaces. The case of 2-elementary K3 surfaces was studied in [Ma15], and further results in this direction were obtained in [BHK16] by using orbits of representation of algebraic groups. Furthermore, Roulleau investigated the unirationality of moduli spaces of K3 surfaces with finite automorphism group (cf. [Rou19; Rou20]).

The main result of the chapter is the following:

**Theorem 4.1.1.** *The moduli space $\mathcal{M}_{2k}$ is unirational for $k \leq 34$, for $36 \leq k \leq 50$ and $k \notin \{42, 48\}$ and for the following values of $k$:*

$$\{52, 53, 54, 59, 60, 61, 62, 64, 68, 69, 73, 79, 81, 94, 97\}.$$

The strategy for the proof of Theorem 4.1.1 involves a systematic study of projective models of K3 surfaces in $\mathcal{M}_{2k}$. More precisely, we realize such surfaces as:

1. Double covers of $\mathbb{P}^2$;
2. Double covers of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (cf. Section 1.4.1);
3. Double covers of $\mathbb{F}_4$ (cf. Section 1.4.2);
4. Weierstrass fibrations (cf. Section 1.4.3);
5. Complete intersections in $\mathbb{P}^3$, $\mathbb{P}^4$ and $\mathbb{P}^5$ containing either two smooth rational curves or an elliptic curve and a rational curve meeting at one point (cf. Section 4.3.1).
6. Complete intersections in the Grassmannian $\mathrm{G}(2,5) \subseteq \mathbb{P}^9$ (cf. Section 4.6).

We stress the fact that the cases not covered by Theorem 4.1.1 are still open, and we have no information about their Kodaira dimension at the moment. We are currently working on trying to prove unirationality of some of them, by realizing their objects as complete intersections in homogeneous spaces. At the present time, the only case that we were able to solve in this way is $k = 11$ (see Section 4.6).

It would be of great interest to find connections between the spaces $\mathcal{M}_{2k}$ and other known geometric objects. One such connection was found in [BH17], where the authors showed that the moduli space $\mathcal{M}_{56}$ is birational to a $\mathbb{P}^1$-bundle over the universal Brill–Noether variety $\mathcal{W}^1_{9,6}$ parametrizing curves of genus 9 together with a pencil of degree 6. Their strategy was to use the relative canonical resolution of these curves on rational normal quartic scrolls. We hope that our methods can be used to reveal new reincarnations of the moduli spaces $\mathcal{M}_{2k}$ since we provide explicit methods to study such elliptic K3 surfaces and the geometry of their moduli spaces.

The chapter is divided as follows. In Section 4.2 we prove the unirationality of $\mathcal{M}_{2k}$ for $k \leq 6$ and $k = 8$ by realizing the K3 surfaces in $\mathcal{M}_{2k}$ as either Weierstrass fibrations or as double covers of $\mathbb{P}^2$, $\mathbb{F}_0$ and $\mathbb{F}_4$. In Section 4.3 we then complete the proof of Theorem 4.1.1 by showing the unirationality of $\mathcal{M}_{2k}$ for the remaining values of $k$. More precisely, we first find an exhaustive list of possible projective models for $U \oplus \langle -2k \rangle$-polarized K3 surfaces as complete intersections in $\mathbb{P}^3$, $\mathbb{P}^4$ and $\mathbb{P}^5$. This is then used to construct a dominant rational map $I \dashrightarrow \mathcal{M}_{2k}$ (as in diagram 4.4), where $I$ is a unirational parameter space (cf. Section 4.3.2), concluding our proof. Finally, in Section 4.6 we solve the last remaining case $k = 11$.

**Convention 4.1.2.** We have used the software `Macaulay2` to perform the computations.

## 4.2 Unirationality of $\mathcal{M}_{2k}$ for small $k$

The aim of this section is to show the unirationality of $\mathcal{M}_{2k}$ for $k \leq 6$ and $k = 8$. Our strategy will involve the geometric constructions of Section 1.4. Since the arguments in each case are somewhat different, we present a case-by-case analysis.

### 4.2.1 $k = 1$

The variety $\mathcal{M}_2$ is the moduli space of $U \oplus \langle -2 \rangle$-polarized K3 surfaces. If $X$ is a general K3 surface in $\mathcal{M}_2$, then $X$ is the desingularization of a Weierstrass fibration $Y$ with an $A_1$ singularity. Hence $X$ admits an elliptic fibration with a reducible fiber, consisting of two irreducible smooth rational curves. A quick inspection of the Kodaira fibers [Mir89, Table I.4.1] yields that this reducible fiber can be either of type $I_2$ (two smooth rational curves meeting transversely at two distinct points) or $III$ (two smooth rational curves simply tangent at one point). This depends on whether the $A_1$ singularity on $Y$ belongs to a nodal or cuspidal rational curve respectively. After moving the singular fiber to $t = 0$, $Y$ can be written as a Weierstrass fibration

$$y^2 = x^3 + a(t)x^2 + b(t)x + c(t)$$

satisfying $t \mid b(t)$ and $t^2 \mid c(t)$. Up to a change of coordinates in $x$, this equation is equivalent to the one in (1.5). Conversely, a general such Weierstrass equation desingularizes to an elliptic K3 surface with an $I_2$ or a $III$ fiber at $t = 0$. From this description we can define a dominant rational map

$$\mathcal{P}_2 := \{(a, b, c) \in H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(12)) : t \mid b(t),\ t^2 \mid c(t)\} \dashrightarrow \mathcal{M}_2$$

sending the polynomials $(a, b, c)$ into the isomorphism class of the desingularization of the corresponding Weierstrass equation. Since $\mathcal{P}_2$ is an affine space, $\mathcal{M}_2$ is unirational.

### 4.2.2 $k = 2$

An $U \oplus \langle -4 \rangle$-polarized K3 surface $X$ is an elliptic K3 surface admitting a section $S$ disjoint from the zero section $S_0$ of the given elliptic fibration by Remark 1.4.6. Let

$$y^2 = x^3 + a(t)x^2 + b(t)x + c(t)$$

be a Weierstrass equation for $X$, where the point at infinity $S_0 = (0 : 1 : 0)$ is the zero section. Let $S = (u(t), v(t))$ be the extra section. Notice that the points of intersection of $S$ and $S_0$ coincide with the poles of $v$ (or equivalently of $u$), as $(u(t_0) : v(t_0) : 1) = (0 : 1 : 0)$ if and only if $t_0$ is a pole for $v$. But $S$ and $S_0$ are disjoint by assumption, so $u, v$ are simply polynomials of degree at most $4, 6$ respectively. After the change of variables $x \mapsto x - u$, $y \mapsto y - v$, the Weierstrass equation becomes

$$y^2 + 2v(t)y = x^3 + d(t)x^2 + e(t)x, \tag{4.1}$$

for polynomials $d, e, v$ of degree at most $4, 8, 6$ respectively. Conversely, a general Weierstrass equation as in (4.1) defines an elliptic K3 surface containing the disjoint sections $S_0 = (0 : 1 : 0)$, $S = (0, 0)$, and therefore an $U \oplus \langle -4 \rangle$-polarized K3 surface. This implies that there exists a dominant rational map

$$\mathcal{P}_4 := \{(d, e, v) \in H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(6))\} \dashrightarrow \mathcal{M}_4.$$

$\mathcal{P}_4$ is an affine space, so $\mathcal{M}_4$ is unirational.

### 4.2.3 $k = 3$

Let $X$ be the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic $B$ with an $A_2$ singularity. Then $X$ is a K3 surface with

$$\langle 2 \rangle \oplus A_2 \cong U \oplus \langle -6 \rangle \hookrightarrow \mathrm{NS}(X).$$

Since 6 is square-free, this embedding is primitive. Conversely, if $X$ is a K3 surface with $\mathrm{NS}(X) \cong \langle 2 \rangle \oplus A_2$, the linear system associated to the first element of the basis induces a morphism $X \to \mathbb{P}^2$ of degree 2 contracting the two $(-2)$-curves, so $X$ is the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic with an $A_2$ singularity. Up to a projective transformation, we can assume that the sextic $B \subseteq \mathbb{P}^2$ has an $A_2$ singularity at $P = (0 : 0 : 1) \in \mathbb{P}^2$, and that the unique line of $\mathbb{P}^2$ meeting $B$ in $P$ with multiplicity 3 is $V(x_0)$. This forces $B$ to be given by an equation $f(x_0, x_1, x_2) \in H^0(\mathcal{O}_{\mathbb{P}^2}(6))$ with the coefficients of $x_2^6$, of $x_0 x_2^5$, of $x_1 x_2^5$, of $x_0 x_1 x_2^4$ and of $x_1^2 x_2^4$ all being zero. We denote by $\mathcal{P}_6$ the linear subspace of $H^0(\mathcal{O}_{\mathbb{P}^2}(6))$ consisting of all such polynomials. Therefore there exists a dominant rational map

$$\mathcal{P}_6 \dashrightarrow \mathcal{M}_6.$$

$\mathcal{P}_6$ is an affine space, hence $\mathcal{M}_6$ is unirational.

### 4.2.4 $k = 4$

By Proposition 1.4.2 and Remark 1.4.3 a general $U \oplus \langle -8 \rangle$-polarized K3 surface $X$ is the double cover of $\mathbb{F}_0$ branched over a smooth $(4, 4)$-curve $B$ admitting a line $L$ simply tangent to $B$ at 2 points. Choose coordinates $((x_0 : x_1), (y_0 : y_1))$ on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ such that $L = V(x_0)$. If $B$ is given by a bihomogeneous polynomial $f(x_0, x_1, y_0, y_1)$ of bidegree $(4, 4)$, then $B$ is tangent to $L$ at 2 points if and only if

$$f(x_0, x_1, y_0, y_1) = x_0 g(x_0, x_1, y_0, y_1) + x_1^4 h_1(y_0, y_1)^2 h_2(y_0, y_1)^2$$

for $g \in H^0(\mathcal{O}_{\mathbb{F}_0}(3, 4))$, $h_1, h_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Therefore we get a dominant rational map

$$\mathcal{P}_8 := \{(g, h_1, h_2) \in H^0(\mathcal{O}_{\mathbb{F}_0}(3, 4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(1))\} \dashrightarrow \mathcal{M}_8$$

sending $(g, h_1, h_2)$ to the isomorphism class of the double cover of $\mathbb{F}_0$ branched along the divisor $f = 0$ defined above. It follows that $\mathcal{M}_8$ is unirational.

### 4.2.5 $k = 5$

Let $X$ be the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic $B$ with a simple node and admitting a tritangent line. Then $X$ is a K3 surface with

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -10 \rangle \hookrightarrow \mathrm{NS}(X)$$

(see Lemma 1.4.1). Since 10 is square-free, this embedding is primitive. Conversely, let $X$ be a K3 surface with $\mathrm{NS}(X) \cong U \oplus \langle -10 \rangle$ and take a basis $\{H, L, C\}$ with intersection matrix as above. The linear system $|H|$ induces a morphism $X \to \mathbb{P}^2$ of degree 2 contracting $C$. Let $Y \to \mathbb{P}^2$ denote the double cover obtained contracting $C$. Then $Y$ has a singular point of type $A_1$, so the branch locus $B \subseteq \mathbb{P}^2$ has a node. Moreover $L$ is mapped onto a line of $\mathbb{P}^2$ meeting $B$ with even multiplicities, so generically it will be a tritangent of $B$. Now, up to a projective transformation, we can assume that the tritangent line is given by $V(x_0)$, so that $B$ is given by an equation of the form

$$f = x_0 g(x_0, x_1, x_2) + h_1(x_1, x_2)^2 h_2(x_1, x_2)^2 h_3(x_1, x_2)^2. \tag{4.2}$$

We can also assume that the node of $B$ is located at $P = (1 : 0 : 0)$. This forces the coefficients of $g$ of the terms $x_0^5, x_0^4 x_1$ and $x_0^4 x_2$ to be zero. We denote by $\mathcal{Q}_{10}$ the linear subspace of $H^0(\mathcal{O}_{\mathbb{P}^2}(5))$ consisting of all such polynomials. Then there exists a dominant rational map

$$\mathcal{P}_{10} = \mathcal{Q}_{10} \times H^0(\mathcal{O}_{\mathbb{P}^1}(1))^3 \dashrightarrow \mathcal{M}_{10}.$$

sending $(g, h_1, h_2, h_3)$ to the isomorphism class of the double cover of $\mathbb{P}^2$ branched over $f$ defined as in equation (4.2). As $\mathcal{P}_{10}$ is an affine space, $\mathcal{M}_{10}$ is unirational.

### 4.2.6 $k = 6$

By Proposition 1.4.2 and Remark 1.4.3, a general such K3 surface is the double cover of $\mathbb{F}_0$ branched over a $(4,4)$-curve $B$ admitting a smooth $(1,1)$-curve $C$ intersecting $B$ in 4 points with multiplicity 2. We can choose coordinates on $\mathbb{F}_0$ so that $C = V(x_0 y_1 - x_1 y_0)$ and $B$ does not pass through the point $((0:1),(0:1)) \in C$, so that the intersection $B \cap C$ is contained in the chart $W = \{x_0 \neq 0, y_0 \neq 0\}$, with coordinates $(1 : u), (1 : v)$. Say that $B$ is given by the equation

$$f(x_0, x_1, y_0, y_1) = \sum_{i+j=k+l=4} \alpha_{ijkl} x_0^i x_1^j y_0^k y_1^l$$

with $\alpha_{0404} = 1$. Since $C|_W = V(u - v)$, the intersection $B \cap C \subseteq W$ is given by the vanishing of

$$g(u) = f(1, u, 1, u) = \sum_{i+j=k+l=4} \alpha_{ijkl} u^{j+l} = \sum_{\eta=0}^{8} \beta_\eta u^\eta,$$

where $\beta_\eta = \sum_{j+l=\eta} \alpha_{ijkl}$ and $\beta_8 = \alpha_{0404} = 1$. Now $g(u)$ has 4 double roots at $u = \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ if and only if

$$g(u) = (u - \varepsilon_1)^2 (u - \varepsilon_2)^2 (u - \varepsilon_3)^2 (u - \varepsilon_4)^2.$$

The choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ uniquely determines the coefficients $\beta_\eta$ for $\eta \leq 7$, which in turn uniquely determine 8 of the $\alpha_{ijkl}$. The 17 other coefficients $\alpha_{ijkl}$ are free parameters so, if we denote them by $\alpha'_1, \ldots, \alpha'_{17}$, we have a rational dominant map

$$\mathcal{P}_{12} := \{(\varepsilon_i, \alpha'_j) \in (\mathbb{A}^1)^4 \times (\mathbb{A}^1)^{17}\} \dashrightarrow \mathcal{M}_{12}.$$

$\mathcal{P}_{12}$ is an affine space, so $\mathcal{M}_{12}$ is unirational.

### 4.2.7 $k = 8$

By Proposition 1.4.2 and Remark 1.4.3, a general such K3 surface is the double cover of $\mathbb{F}_0$ branched over a $(4,4)$-curve $B$ admitting a smooth $(1,2)$-curve $C$ intersecting $B$ in 6 points with multiplicity 2. We can choose coordinates on $\mathbb{P}^3$ such that $\mathbb{F}_0 = V(x_0x_3 - x_1x_2)$ and $C$ is the twisted cubic curve $V(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$. We may also assume that $B$ does not pass through the point $(0:0:0:1) \in C$, so that the intersection $B \cap C$ is contained in the chart $W = \{x_0 \neq 0\} \subseteq \mathbb{P}^3$ with coordinates $(1 : u : v : w)$. Then $C|_W = V(v - u^2, w - u^3)$, so if $B$ is given by a quartic $f(x_0, x_1, x_2, x_3) \in H^0(\mathcal{O}_{\mathbb{P}^3}(4))$, the intersection $B \cap C \subseteq W$ is given by the vanishing of

$$g(u) = f(1, u, u^2, u^3).$$

Now an argument as in the case $k = 6$ shows that $\mathcal{M}_{16}$ is unirational.

## 4.3 Unirationality of $\mathcal{M}_{2k}$ via projective models in $\mathbb{P}^n$

This aim of this section is to continue the proof of Theorem 4.1.1. Our strategy is to realize $U \oplus \langle -2k \rangle$-polarized K3 surfaces as complete intersections in $\mathbb{P}^n$ for $3 \leq n \leq 5$ containing suitable pairs of curves. More precisely, if $X \in \mathcal{M}_{2k}$ is very general, with $\mathrm{NS}(X) = U \oplus \langle -2k \rangle$, we want to find a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ given by a very ample polarization and two other smooth curves, where we distinguish two cases:

**Case 1**: Two rational curves meeting transversely at several points in general position;

**Case 2**: A rational curve and an elliptic curve meeting transversely at one point.

In Case 2 the resulting K3 surfaces are automatically elliptic. In Case 1 we check this case by case by computing the genus of the resulting lattice (cf. Section 4.3.3).

### 4.3.1 The construction

We describe the construction in detail for Case 2. Let $n, d, \gamma$ be integers such that $3 \leq n \leq 5$, $3 \leq d \leq 8$ and $\gamma \geq 1$.

**Step 1**: We construct a smooth elliptic curve $E$ of degree $d$ in $\mathbb{P}^n$ with a distinguished point $p$.

**Step 2**: We construct a smooth rational curve $\Gamma$ of degree $\gamma$ intersecting $E$ transversely only at the point $p$.

**Step 3**: We choose (if it exists) a smooth K3 surface $X$ in $\mathbb{P}^n$ of degree $2n-2$ containing $E \cup \Gamma$.

Then, we get a K3 surface $X$ containing an elliptic curve $E$ and a rational curve $\Gamma$ with the following lattice embedding

$$\begin{pmatrix} 2n-2 & \gamma & d \\ \gamma & -2 & 1 \\ d & 1 & 0 \end{pmatrix} \cong U \oplus \langle 2n-2-2d\gamma-2d^2 \rangle \hookrightarrow \mathrm{NS}(X). \tag{4.3}$$

In particular, we set $k = d^2 + d\gamma - n + 1$. We check in each case that the resulting surface $X$ is a $U \oplus \langle -2k \rangle$-polarized K3 surface by showing that such a lattice embedding is generically primitive (see Section 4.3.4). By abuse of notation, we denote by $E$ and $\Gamma$ the classes in $\mathrm{NS}(X)$ of the corresponding curves under this lattice embedding. Recall that the moduli space $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces is an irreducible, quasi-projective variety of dimension 17.

We can easily adapt the above strategy to Case 1. First, we construct a smooth rational curve $\Gamma_1 \subseteq \mathbb{P}^n$ of degree $\gamma_1$, together with $m$ points $p_1, \ldots, p_m \in \Gamma_1$. Then, we construct a second smooth rational curve $\Gamma_2$ intersecting $\Gamma_1$ transversely, precisely at $p_1, \ldots, p_m$. Finally, Step 3 remains unchanged: we just choose (if it exists) a smooth K3 surface $X$ in $\mathbb{P}^n$ containing $\Gamma_1 \cup \Gamma_2$.

We compute in `Macaulay2` that the constructed curves are smooth points of a component of the right dimension in the corresponding Hilbert schemes. By standard semicontinuity arguments (see e.g. [Sch13]), we will perform our computations over a finite field (the main reason for doing this is that the computation is much faster over a finite field, but our constructions also work over the rationals). Finally, a dimension count shows that the construction dominates the corresponding moduli space $\mathcal{M}_{2k}$. We will present more details in the rest of the section.

### 4.3.2 Unirationality

In this section we show that the constructions described above in Step 1, 2 and 3 can be realized as incidence varieties, which are then shown to be unirational.

*Remark* 4.3.1. We denote by $H_{d,g,n}$ the open subscheme of the Hilbert scheme parametrizing smooth irreducible curves of degree $d$ and genus $g$ in $\mathbb{P}^n$. We notice that $H_{d,g,n}$ is irreducible if $g \in \{0,1\}$ and $n \in \{3,4,5\}$ by [Ein86]. Moreover, we can easily compute the dimension of $H_{d,g,n}$ for $g \in \{0,1\}$ by using the fact that every smooth curve $C \subseteq \mathbb{P}^n$ of genus $g \leq 1$ and degree $d > 0$ is *non-special*, that is, $H^1(\mathcal{O}_C(1)) = 0$. Indeed, this implies that $H^1(\mathcal{N}_{C/\mathbb{P}^n}) = 0$, and thus

$$\dim H_{d,g,n} = h^0(\mathcal{N}_{C/\mathbb{P}^n}) = \begin{cases} (n+1)d + (n-3) & \text{if } g = 0 \\ (n+1)d & \text{if } g = 1. \end{cases}$$

**Step 1:** We include the proofs of the following classical results for the sake of completeness.

**Lemma 4.3.2.** *Let $n$ and $d$ be integers with $3 \leq n \leq 5$ and $3 \leq d \leq 8$. The incidence variety*

$$H^1_{d,1,n} := \{(E,p) \mid p \in E\} \subseteq H_{d,1,n} \times \mathbb{P}^n$$

*of elliptic curves of degree $d$ in $\mathbb{P}^n$ with a marked point is unirational. Its dimension is*

$$\dim H^1_{d,1,n} = \dim H_{d,1,n} + 1.$$

*Proof.* The moduli space $\mathcal{M}_{1,2}$ of elliptic curves marked with 2 points is rational by [Bel98]. In order to construct an elliptic curve $E$ of degree $d \geq 3$ in $\mathbb{P}^n$ together with a marked point $p$, we start with a plane cubic curve $E'$ with two distinguished points $p$ and $q$. The choice of a basis of the vector space $V = H^0(E', \mathcal{O}_{E'}(dq))$ of dimension $d$ yields a birational map

$$E' \to E \subset \mathbb{P}^{d-1}$$

where the image $E$ is a smooth elliptic curve of degree $d$ (recall that all line bundles on $E'$ of degree $d$ are of the form $\mathcal{O}_{E'}(dq)$, see e.g. [Eis05, Theorem 6.16]). The choice of the basis is unirational since it is parametrized by an open subset of $V^d$. If $d-1 > n$, then we project the curve $E$ birationally to a smooth elliptic curve of degree $d$ in $\mathbb{P}^n$. If $d-1 < n$, then we embed the ambient space $\mathbb{P}^{d-1}$ into $\mathbb{P}^n$ in order to get again an elliptic curve of degree $d$ in $\mathbb{P}^n$. In both cases, we have to choose an appropriate linear subspace of $\mathbb{P}^n$ or $\mathbb{P}^{d-1}$, and this choice is clearly unirational. □

*Remark* 4.3.3. Let $p_1, \ldots, p_m \in \mathbb{P}^n$ be a set of points spanning a linear subspace $\mathbb{P}^l \subseteq \mathbb{P}^n$. We say that $p_1, \ldots, p_m$ are in general position if they are in general position inside $\mathbb{P}^l$. In particular, we have $m \leq l + 1 \leq n + 1$.

**Lemma 4.3.4.** *Let $n$, $\gamma$ and $m$ be integers with $3 \leq n \leq 5$, $1 \leq \gamma \leq 8$ and $m \leq \min\{n+1, \gamma+1\}$. Fix points $p_1, \ldots, p_m \in \mathbb{P}^n$ in general position. The variety*

$$H_{\gamma,0,n}(p_1, \ldots, p_m) := \{\Gamma \ni p_1, \ldots, p_m\} \subseteq H_{\gamma,0,n}$$

*of rational curves of degree $\gamma$ in $\mathbb{P}^n$ passing through $p_1, \ldots, p_m$ is irreducible and unirational. Moreover, it is non-empty, of dimension*

$$\dim(H_{\gamma,0,n}(p_1, \ldots, p_m)) = \dim(H_{\gamma,0,n}) - m(n-1).$$

*Proof.* Since the variety of embeddings $\mathrm{Emb}_\gamma(\mathbb{P}^1,\mathbb{P}^n)$ of degree $\gamma$ is rational and the morphism $\mathrm{Emb}_\gamma(\mathbb{P}^1,\mathbb{P}^n) \to H_{\gamma,0,n}$ sending a morphism to its image is dominant, we have that $H_{\gamma,0,n}$ is irreducible and unirational. If $p_1,\dots,p_m$ are points in $\mathbb{P}^n$ in general position, we consider the subvariety $T$ of $\mathrm{Emb}_\gamma(\mathbb{P}^1,\mathbb{P}^n) \times (\mathbb{P}^1)^m$ consisting of embeddings $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ and $m$ points $x_1,\dots,x_m \in \mathbb{P}^1$ such that $f(x_i) = p_i$ for all $1 \le i \le m$. Since the conditions $f(x_i) = p_i$ are linear and $T$ dominates $H_{\gamma,0,n}(p_1,\dots,p_m)$, we deduce that $H_{\gamma,0,n}(p_1,\dots,p_m)$ is irreducible and unirational.

In order to show that $H_{\gamma,0,n}(p_1,\dots,p_m)$ is non-empty, fix a general linear map $h : \mathbb{P}^\gamma \to \mathbb{P}^n$ and points $q_1,\dots,q_m \in \mathbb{P}^\gamma$ in general position such that $h(q_i) = p_i$ for all $1 \le i \le m$. Since there is always a smooth rational normal curve $C \subseteq \mathbb{P}^\gamma$ of degree $\gamma$ passing through $m \le \gamma+1$ points in general position, then $\Gamma := h(C) \subseteq \mathbb{P}^n$ is the desired smooth rational curve (by the generality assumption on $h$).

Finally, we have to compute $\dim(H_{\gamma,0,n}(p_1,\dots,p_m))$. Let $\Gamma \in H_{\gamma,0,n}(p_1,\dots,p_m)$ be a rational curve as constructed above, and consider the divisor $D := p_1 + \ldots + p_m$ over $\Gamma$. Let $H$ be the restriction of the hyperplane class on $\mathbb{P}^n$ to $\Gamma$. Then the exact sequences

$$0 \to \mathcal{T}_\Gamma(-D) \to \mathcal{T}_{\mathbb{P}^n}|_\Gamma(-D) \to \mathcal{N}_{\Gamma/\mathbb{P}^n}(-D) \to 0$$

and

$$0 \to \mathcal{O}_\Gamma(-D) \to \mathcal{O}_\Gamma(H-D)^{n+1} \to \mathcal{T}_{\mathbb{P}^n}|_\Gamma(-D) \to 0,$$

combined with the fact that $H^1(\mathcal{O}_\Gamma(H-D)) = 0$ (by Serre duality, as $\deg(\mathcal{O}_\Gamma(H-D)) > -2$), imply that $H^1(\mathcal{N}_{\Gamma/\mathbb{P}^n}(-D)) = 0$. Thus, the dimension of $H_{\gamma,0,n}(p_1,\dots,p_m)$ coincides with $\chi(\mathcal{N}_{\Gamma/\mathbb{P}^n}(-D))$, and a straightforward computation using the two previous exact sequences yields

$$\chi(\mathcal{N}_{\Gamma/\mathbb{P}^n}(-D)) = (\gamma+1)(n+1) - 4 - m(n-1) = \dim(H_{\gamma,0,n}) - m(n-1).$$

□

**Step 2:** As a consequence of the previous discussion and the fact that $m$ general points on a rational curve of degree $\gamma$ lie in general position for $m \le \gamma+1$, we obtain:

**Lemma 4.3.5.** *Let $\gamma_1, \gamma_2$ and $m$ be integers with $1 \le \gamma_2 \le \gamma_1 \le 8$ and $m \le \min\{n+1, \gamma_2+1\}$. The incidence variety*

$$I^m_{\gamma_1,\gamma_2,n} = \{(\Gamma_1,\Gamma_2) \mid \Gamma_1 \cap \Gamma_2 = \{p_1,\dots,p_m\}\} \subseteq H_{\gamma_1,0,n} \times H_{\gamma_2,0,n}$$

*of two rational curves intersecting transversely at $m$ points in general position is irreducible and unirational, of dimension*

$$\dim I^m_{\gamma_1,\gamma_2,n} = \dim H_{\gamma_1,0,n} + m + \dim H_{\gamma_2,0,n} - m(n-1).$$

Notice that the irreducibility and the unirationality of $I^m_{\gamma_1,\gamma_2,n}$ follow from the fact that by Lemma 4.3.4, $I^m_{\gamma_1,\gamma_2,n}$ is dominated by a projective bundle over $H_{\gamma_1,0,n}$. The transversality of the intersection does not affect the result since it is an open condition. Similarly, we have:

**Lemma 4.3.6.** *Let $n, d$ and $\gamma$ be integers with $3 \leq n \leq 5$, $3 \leq d \leq 8$ and $\gamma \geq 1$. The incidence variety*

$$I_{d,\gamma,n} = \{(E, \Gamma) \mid E \cap \Gamma = \{pt\}\} \subseteq H_{d,1,n} \times H_{\gamma,0,n}$$

*of an elliptic and a rational curve intersecting transversely at one point is irreducible and unirational, of dimension*

$$\dim I_{d,\gamma,n} = \dim H_{d,1,n} + 1 + \dim H_{\gamma,0,n} - (n-1).$$

**Step 3:** The choice of a K3 surface $X$ containing $E \cup \Gamma$ (or $\Gamma_1 \cup \Gamma_2$) is parametrized by an iterated Grassmannian $G$. In the case $n = 3$, $X$ is a quartic surface, so $G = |\mathcal{I}_{E\cup\Gamma}(4)|$. In the case $n = 4$, $X$ is a complete intersection of a quadric and a cubic, hence $G = \mathbb{P}\mathcal{E}$ is a projective bundle over $|\mathcal{I}_{E\cup\Gamma}(2)|$, whose fiber over $q \in |\mathcal{I}_{E\cup\Gamma}(2)|$ is $\mathbb{P}\mathcal{E}_q$ defined in the exact sequence

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \xrightarrow{\cdot q} H^0(\mathcal{I}_{E\cup\Gamma}(3)) \to \mathcal{E}_q \to 0.$$

Finally, in the case $n = 5$, $X$ is a complete intersection of 3 quadrics, thus our parameter space is $G = \mathrm{Gr}(3, H^0(E \cup \Gamma, \mathcal{I}_{E\cup\Gamma}(2)))$. All these parameter spaces $G$ are rational. We are going to discuss in detail Case 2, as Case 1 can be handled analogously.

Let $k = d\gamma + d^2 - n + 1$. We want to construct the surfaces in $\mathcal{M}_{2k}$ by using the embedding of lattices as in equation (4.3). We consider the incidence variety

$$I := \{(X, E, \Gamma) \mid E \cup \Gamma \subseteq X,\ E \cap \Gamma = \{pt\}\} \subseteq G \times I_{d,\gamma,n}$$

of triples $(X, E, \Gamma)$, where $X$ is a complete intersection in $G$ containing the two curves $E, \Gamma$. We denote by $I^{sm} \subseteq I$ the open subvariety of $I$ containing triples $(X, E, \Gamma)$ with $X$ a smooth K3 surface.

**Lemma 4.3.7.** *The image of the morphism $\pi_1 : I^{sm} \to \mathcal{M}_{2k}$, sending a triple $(X, E, \Gamma)$ to the isomorphism class of the smooth K3 surface $X$, is open in $\mathcal{M}_{2k}$. In particular, as $\mathcal{M}_{2k}$ is irreducible, either $I^{sm} = \emptyset$ or $\pi_1$ is dominant.*

*Proof.* Choose a basis $\{H, E, \Gamma\}$ of $U \oplus \langle -2k \rangle$ such that the intersection matrix of $\{H, E, \Gamma\}$ is as in equation (4.3), and let $X$ be a $U \oplus \langle -2k \rangle$-polarized K3 surface. Up to the action of the Weyl group, we can assume that $H \in \mathrm{NS}(X)$ is big and nef. By Saint-Donat's result [Sai74b, Theorem 5.2], $H$ is very ample if and only if there is no element $C \in \mathrm{NS}(X)$ with $C^2 = -2$ and $H \cdot C = 0$, and no element $D \in \mathrm{NS}(X)$ with $D^2 = 0$ and $H \cdot D = 2$. Both conditions are closed in $\mathcal{M}_{2k}$, and if $H$ is indeed very ample, it embeds $X$ in $\mathbb{P}^n$ as a surface containing the desired pair of curves. □

In order to show the unirationality of $\mathcal{M}_{2k}$, we will use the following proposition. We denote by $\pi_2 : I \to I_{d,\gamma,n}$ the forgetful map that sends the triple $(X, E, \Gamma)$ to the pair $(E, \Gamma)$.

**Proposition 4.3.8.** *Assume that there exists a pair* $(E, \Gamma) \in I_{d,\gamma,n}$ *such that the fiber* $F := \pi_2^{-1}(E, \Gamma)$ *contains an element in* $I^{sm}$*. If the number* $\dim I_{d,\gamma,n} + \dim F$ *coincides with the expected dimension of* $I$

$$\dim \mathcal{M}_{2k} + \dim \mathrm{PGL}(n+1) + \dim |E| + \dim |\Gamma| = 18 + \dim \mathrm{PGL}(n+1),$$

*then* $\pi_2$ *is dominant. As a consequence* $I$ *is unirational,* $\pi_1 : I \dashrightarrow \mathcal{M}_{2k}$ *is a dominant rational map, and thus* $\mathcal{M}_{2k}$ *is unirational.*

*Proof.* Since by assumption $I^{sm} \neq \emptyset$, then by Lemma 4.3.7 the morphism $\pi_1 : I^{sm} \to \mathcal{M}_{2k}$ is dominant and induces a dominant rational map $\pi_1 : I \dashrightarrow \mathcal{M}_{2k}$. The map $\pi_1$ can be decomposed as the composition $q_2 \circ q_1$, where $q_1$ sends a triple $(X, E, \Gamma)$ to the K3 surface $X$, and $q_2$ sends a (smooth) K3 surface to its isomorphism class. The set of automorphisms of $\mathbb{P}^n$ fixing a K3 surface $X \subseteq \mathbb{P}^n$ is finite so that the dimension of the fiber of $q_2$ is equal to $\dim \mathrm{PGL}(n+1)$. Moreover, the fiber of $q_1$ is 1-dimensional since the elliptic curve $E$ moves in a 1-dimensional pencil and $\Gamma$ is rigid on $X$. Since $\pi_1$ is dominant, necessarily $\dim I$ is at least the expected dimension

$$\dim \mathcal{M}_{2k} + \dim \mathrm{PGL}(n+1) + \dim |E| + \dim |\Gamma| = 18 + \dim \mathrm{PGL}(n+1).$$

Now let $d \leq \dim F$ be the minimal dimension of the fibers of $\pi_2$. By semicontinuity of the fiber dimension, there exists on open dense subset $U$ of $I_{d,\gamma,n}$ such that $\dim \pi_2^{-1}(E', \Gamma') = d$ for all $(E', \Gamma') \in U$. If $d < \dim F$, then

$$\dim I = \dim I_{d,\gamma,n} + d < \dim I_{d,\gamma,n} + \dim F,$$

which is a contradiction, since by assumption the number $\dim I_{d,\gamma,n} + \dim F$ coincides with the expected dimension of $I$. This implies that $\pi_2$ is dominant, and thus gives to $I$ (generically) the structure of a projective bundle over $I_{d,\gamma,n}$ (with fiber isomorphic to $G$). In particular, $I$ is unirational, and therefore $\mathcal{M}_{2k}$ is unirational, too. $\square$

The previous proposition proves the existence of the diagram

$$\begin{array}{ccc} & I & \\ {\scriptstyle \pi_1} \swarrow\!\!\!\cdots & & \searrow {\scriptstyle \pi_2} \\ \mathcal{M}_{2k} & & I_{d,\gamma,n} \end{array} \tag{4.4}$$

whenever $I^{sm} \neq \emptyset$ and a certain equality of dimensions holds. As we already remarked, the general fiber of $\pi_2$ is isomorphic to the iterated Grassmannian $G$ defined above, of dimension

$$\dim \pi_2^{-1}(E, \Gamma) = \begin{cases} h^0(E \cup \Gamma, \mathcal{I}_{E \cup \Gamma}(4)) - 1 & \text{for } n = 3, \\ h^0(E \cup \Gamma, \mathcal{I}_{E \cup \Gamma}(2)) - 1 + h^0(E \cup \Gamma, \mathcal{I}_{E \cup \Gamma}(3)) - 6 & \text{for } n = 4, \\ 3 \cdot (h^0(E \cup \Gamma, \mathcal{I}_{E \cup \Gamma}(2)) - 3) & \text{for } n = 5. \end{cases}$$

In the case $n = 4$, we choose a quadric hypersurface through $E \cup \Gamma$ and then a cubic hypersurface through $E \cup \Gamma$ which is not a multiple of the chosen quadric.

We check with `Macaulay2` that there exists a pair $(E, \Gamma) \in I_{d,\gamma,n}$ such that the fiber $F = \pi_2^{-1}(E, \Gamma)$ contains one element in $I^{sm}$ and that

$$\dim I_{d,n,\gamma} + \dim F - \dim \mathrm{PGL}(n+1) = 18.$$

In order to compute the dimension of $I_{d,n,\gamma}$, we check with `Macaulay2` that $(E, \Gamma)$ is a smooth point of $I_{d,n,\gamma}$. Notice that the same strategy works analogously for Case 1, with the only difference that the previous number has to be 17 instead of 18. This follows from the fact that the two smooth rational curves are rigid on the K3 surface, and thus the fiber of $q_1$ in the proof of Proposition 4.3.8 is 0-dimensional. All the experimental data can be found in Tables 4.1, 4.2, 4.3.

*Remark* 4.3.9. We are also adapting the above strategy to prove unirationality for some quasi-polarized K3 surfaces. We construct nodal K3 surfaces having a node (i.e., having a unique $A_1$-singularity) and containing either a smooth rational curve or a smooth elliptic curve (see Section 4.5 for more details).

### 4.3.3 Search for the lattices

In this section we explain how we obtain an exhaustive list of projective models of elliptic K3 surfaces of Picard number 3 in Case 1 and 2.

In Case 2, the K3 surface is automatically elliptic. In Case 1, we actually have to check that the lattice contains a copy of the hyperbolic plane. By [Nik79b, Corollary 1.13.3], the lattices $U \oplus \langle -2k \rangle$ are unique in their genus, so an even lattice $L$ of rank 3, signature $(1, 2)$ and determinant $2k$ contains a copy of $U$ if and only if $L$ is in the genus of $U \oplus \langle -2k \rangle$ (see Section 1.1.3). This amounts to computing the discriminant groups of $L$; we do not report these straightforward computations here.

In order to obtain an exhaustive, but finite, list of possible projective models for such K3 surfaces in $\mathbb{P}^3$, $\mathbb{P}^4$ and $\mathbb{P}^5$, we want to bound the degrees of the elliptic and rational curves. This bound arises from the fact that curves of fixed genus and "high" degree do not lie on hypersurfaces of "small" degree. We explain this in detail for the case of two rational curves in $\mathbb{P}^3$ meeting transversely at $m$ points in general position; the strategy in the other cases is completely analogous.

Let $\Gamma_1, \Gamma_2$ be smooth rational curves in $\mathbb{P}^3$ of degree $\gamma_1$ and $\gamma_2$, respectively, meeting at $m$ points. The short exact sequence

$$0 \to \mathcal{I}_{\Gamma_1 \cup \Gamma_2}(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_{\Gamma_1 \cup \Gamma_2}(4) \to 0$$

combined with the maximal rank conjecture (see e.g., [Lar17]) implies that $h^0(\mathcal{I}_{\Gamma_1 \cup \Gamma_2}(4)) > 0$ if and only if

$$35 = h^0(\mathcal{O}_{\mathbb{P}^3}(4)) > h^0(\mathcal{O}_{\Gamma_1 \cup \Gamma_2}(4)) = 4(\gamma_1 + \gamma_2) + 2 - m. \tag{4.5}$$

Notice that the number $m$ of intersection points is bounded by the degrees $\gamma_1, \gamma_2$, since there are no smooth rational curves of degree $\gamma$ passing through more than $2\gamma$ points of $\mathbb{P}^n$ in general position (cf. the formula in Lemma 4.3.4). Therefore, the inequality (4.5) provides a bound $d_{max}$ for the degrees $\gamma_1, \gamma_2$ of the two rational curves.

Now, we can produce the list of all possible projective models of elliptic K3 surfaces of Picard number 3 in Case 1 and 2: for every $\gamma_1 \leq \gamma_2 \leq d_{max}$ and every $m \leq 2\gamma_1$ satisfying the inequality (4.5), we check whether the corresponding lattice contains a copy of the hyperbolic plane by looking at its genus.

*Remark* 4.3.10. The same search works analogously in the case of nodal K3 surfaces since we only have to bound the degree of the smooth rational curve or of the smooth elliptic curve.

### 4.3.4 Primitivity

The K3 surfaces that we construct in Tables 4.1, 4.2, 4.3 are in fact $U \oplus \langle -2k \rangle$-polarized K3 surfaces for suitable values of $k$. In order to show this, we have to prove that the embeddings as in equation (4.3) are primitive. We will perform a case-by-case inspection, depending on the divisibility of $k$. Recall that, if $D \in U \oplus \langle -2k \rangle$ is primitive and divisible by $r$ in $\mathrm{NS}(X)$, then $r^2 \mid k$ (see Lemma 1.1.7).

**Table 4.1:** We consider the lattice embedding $\Lambda^1_{n,\gamma_1,\gamma_2} \hookrightarrow \mathrm{NS}(X)$ in equation (4.6). If $k \neq 50$, it is easy to see that the embedding is not primitive if and only if the divisor $\Gamma_1 + \Gamma_2$ is divisible in $\mathrm{NS}(X)$. However, $\Gamma_1 + \Gamma_2$ has square 0, it is reduced and connected, hence it is primitive in $\mathrm{NS}(X)$. If instead $k = 50$, the embedding is primitive if and only if the divisor $H - \Gamma_2$ is primitive in $\mathrm{NS}(X)$. If by contradiction $H - \Gamma_2$ would be divisible by 5, then $\frac{1}{5}(H - \Gamma_2)$ would have square 0 and intersection 1 with $H$; this is a contradiction since there are no curves of degree 1 and arithmetic genus 1.

**Table 4.2:** We consider the lattice embedding $\Lambda^2_{n,d,\gamma} \hookrightarrow \mathrm{NS}(X)$ in equation (4.7). We are going to apply the following strategy for all the cases.
Let $D \in \Lambda^2_{n,d,\gamma}$ be a generator of $\langle E, \Gamma \rangle^\perp$. Since $\langle E, \Gamma \rangle$ is a copy of the hyperbolic plane, we have that $D^2 = -2k$ and the basis $\{E, \Gamma, D\}$ gives an explicit isomorphism $\Lambda^2_{n,d,\gamma} \cong U \oplus \langle -2k \rangle$. Therefore, the embedding $\Lambda^2_{n,d,\gamma} \hookrightarrow \mathrm{NS}(X)$ is primitive if and only if $D$ is primitive in $\mathrm{NS}(X)$. Hence assume that $\frac{1}{2}D \in \mathrm{NS}(X)$ (thus $4 \mid k$); if $D$ is divisible by some other number $r$ the argument is analogous. A straightforward computation yields

$$D = H - (\gamma + 2d)E - d\Gamma.$$

We distinguish some cases depending on the parity of $d, \gamma$.
• $d \equiv \gamma \equiv 0 \pmod 2$: In this case $D$ is divisible by 2 if and only if $H$ is divisible by 2, but the hyperplane section is primitive in $\mathrm{NS}(X)$.
• $d \equiv 1, \gamma \equiv 0 \pmod 2$: $D$ is divisible by 2 if and only if $H - \Gamma$ is divisible by 2. But $\frac{1}{2}(H - \Gamma)$ would have square 0 and intersection 2 with $H$, and there are no curves of degree 2 and arithmetic genus 1.

• $d \equiv 0, \gamma \equiv 1 \pmod 2$: $D$ is divisible by 2 if and only if $H - E$ is divisible by 2. If $k = 24$, $\frac{1}{2}(H-E)$ would be a curve of degree 2 and arithmetic genus 1. If $k = 40$, $\frac{1}{2}(H-E)$ would have square $-2$, so it would be either effective or anti-effective. But $(H-E)H < 0$, $(H-E)E > 0$, and this is a contradiction since $H$ and $E$ are nef. Finally, if $k = 68$, $\frac{1}{2}(H-E)$ would have square $-2$ and intersection 0 with $H$; this is a contradiction since the K3 surfaces we are considering are generically smooth.
• $d \equiv \gamma \equiv 1 \pmod 2$: $D$ is divisible by 2 if and only if $H - E - \Gamma$ is divisible by 2. If $k \in \{16, 28\}$, $\frac{1}{2}(H-E-\Gamma)$ would have square $-2$, but $(H-E-\Gamma)H < 0$ and $(H-E-\Gamma)E > 0$ leads to a contradiction as above. If instead $k = 52$, $\frac{1}{2}(H-E-\Gamma)$ would be a $(-2)$-curve orthogonal to $H$ which is again a contradiction.

**Table 4.3:** The reasoning is completely analogous to the previous case.

## 4.4 Experimental data

### 4.4.1 Case 1: Two rational curves

Let $\gamma_1, \gamma_2$ and $m$ be integers. Let $\Gamma_1, \Gamma_2 \subset \mathbb{P}^n$ be two rational curves of degree $\gamma_1$ and $\gamma_2$, respectively, intersecting transversely at $m$ points. If $X \subseteq \mathbb{P}^n$ is a K3 surface containing the union $\Gamma_1 \cup \Gamma_2$, then there exists a lattice embedding

$$\Lambda^1_{n,\gamma_1,\gamma_2} := \begin{pmatrix} 2n-2 & \gamma_1 & \gamma_2 \\ \gamma_1 & -2 & m \\ \gamma_2 & m & -2 \end{pmatrix} \hookrightarrow \mathrm{NS}(X). \tag{4.6}$$

For suitable choices, this lattice is isometric to $U \oplus \langle -2k \rangle$ for some integer $k$. In Table 4.1 we specify this integer $k$ in every case and list the data obtained with our `Macaulay2` program. Note that there is more than one configuration of two rational curves yielding a K3 surface in $\mathcal{M}_{2k}$. We only list one possiblity for each $k$ in Table 4.1.

*Example* 4.4.1. $k = 10$: Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface containing two disjoint lines $\Gamma_1, \Gamma_2$. Then $X$ is a K3 surface with the following primitive lattice embedding

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -20 \rangle \hookrightarrow \mathrm{NS}(X).$$

We recall the dimension count in this example:

$$\begin{aligned} &2 \cdot \dim(\mathrm{Hilb}_{t+1}(\mathbb{P}^3)) + (h^0(\Gamma_1 \cup \Gamma_2, \mathcal{I}_{\Gamma_1 \cup \Gamma_2}(4)) - 1) - \dim \mathrm{PGL}(4) = \\ &= 2 \cdot 4 + 24 - (4^2 - 1) = 17 = \dim \mathcal{M}_{2k} + \dim |\Gamma_1| + \dim |\Gamma_2|. \end{aligned}$$

| $k$ | $n$ | $\gamma_1$ | $\gamma_2$ | $m$ | $\dim(\mathrm{Hilb}_{\gamma_1\cdot t+1}(\mathbb{P}^n))$ | $\dim(\mathrm{Hilb}_{\gamma_2\cdot t+1}(\mathbb{P}^n))$ | $\dim \pi_2^{-1}(\Gamma_1,\Gamma_2)$ |
|---|---|---|---|---|---|---|---|
| 10 | 3 | 1 | 1 | 0 | 4 | 4 | 24 |
| 13 | 3 | 2 | 1 | 0 | 8 | 4 | 20 |
| 17 | 4 | 2 | 1 | 0 | 11 | 6 | 24 |
| 19 | 3 | 3 | 1 | 1 | 12 | 4 | 17 |
| 21 | 4 | 2 | 2 | 1 | 11 | 11 | 21 |
| 25 | 3 | 4 | 1 | 0 | 16 | 4 | 12 |
| 26 | 3 | 3 | 3 | 0 | 12 | 12 | 8 |
| 29 | 4 | 4 | 1 | 0 | 21 | 6 | 14 |
| 31 | 5 | 3 | 2 | 1 | 20 | 14 | 21 |
| 34 | 3 | 5 | 1 | 0 | 20 | 4 | 8 |
| 36 | 5 | 3 | 3 | 2 | 20 | 20 | 18 |
| 37 | 3 | 5 | 1 | 1 | 20 | 4 | 9 |
| 39 | 5 | 3 | 3 | 1 | 20 | 20 | 15 |
| 41 | 5 | 4 | 3 | 0 | 26 | 20 | 6 |
| 43 | 3 | 4 | 3 | 1 | 16 | 12 | 5 |
| 46 | 5 | 5 | 3 | 4 | 32 | 20 | 12 |
| 49 | 3 | 6 | 1 | 1 | 24 | 4 | 5 |
| 50 | 5 | 5 | 3 | 0 | 32 | 20 | 0 |
| 53 | 5 | 6 | 1 | 0 | 38 | 8 | 6 |
| 59 | 5 | 5 | 3 | 3 | 32 | 20 | 9 |
| 61 | 5 | 5 | 3 | 1 | 32 | 20 | 3 |
| 64 | 3 | 5 | 3 | 2 | 20 | 12 | 2 |
| 69 | 5 | 6 | 3 | 4 | 38 | 20 | 6 |
| 73 | 5 | 5 | 4 | 4 | 32 | 26 | 6 |
| 79 | 5 | 6 | 3 | 3 | 38 | 20 | 3 |
| 81 | 5 | 6 | 3 | 2 | 38 | 20 | 0 |
| 94 | 5 | 7 | 3 | 4 | 44 | 20 | 0 |
| 97 | 3 | 5 | 4 | 4 | 20 | 16 | 0 |

Table 4.1: List of lattices in Case 1

### 4.4.2 Case 2: An elliptic and a rational curve

Let $d, n, \gamma$ be integers such that $3 \leq d \leq 8$, $3 \leq n \leq 5$ and $1 \leq \gamma \leq 7$. Let $E \subset \mathbb{P}^n$ be an elliptic curve of degree $d$, and let $\Gamma \subset \mathbb{P}^n$ be a rational curve of degree $\gamma$ intersecting $E$ transversely at one point. We denote by $X \subseteq \mathbb{P}^n$ a K3 surface with the following lattice embedding

$$\Lambda^2_{n,d,\gamma} := \begin{pmatrix} 2n-2 & d & \gamma \\ d & 0 & 1 \\ \gamma & 1 & -2 \end{pmatrix} \cong U \oplus \langle -2(d^2 + d\gamma + -n + 1)\rangle \hookrightarrow \mathrm{NS}(X). \tag{4.7}$$

We set $k = d^2 + d\gamma - n + 1$. In Table 4.2 we specify this integer $k$ in every case and list the data obtained with our `Macaulay2` program.

| $k$ | $n$ | $d$ | $\gamma$ | $\dim(\mathrm{Hilb}_{d\cdot t}(\mathbb{P}^n))$ | $\dim(\mathrm{Hilb}_{\gamma\cdot t+1}(\mathbb{P}^n))$ | $\dim \pi_2^{-1}(E,\Gamma)$ |
|---|---|---|---|---|---|---|
| 9 | 4 | 3 | 1 | 15 | 6 | 23 |
| 10 | 3 | 3 | 1 | 12 | 4 | 19 |
| 12 | 4 | 3 | 2 | 15 | 11 | 18 |
| 13 | 3 | 3 | 2 | 12 | 8 | 15 |
| 15 | 4 | 3 | 3 | 15 | 16 | 13 |
| 16 | 3 | 3 | 3 | 12 | 12 | 11 |
| 16 | 5 | 4 | 1 | 24 | 8 | 24 |
| 17 | 4 | 4 | 1 | 20 | 6 | 18 |
| 18 | 3 | 4 | 1 | 16 | 4 | 15 |
| 20 | 5 | 4 | 2 | 24 | 14 | 18 |
| 21 | 4 | 4 | 2 | 20 | 11 | 13 |
| 22 | 3 | 4 | 2 | 16 | 8 | 11 |
| 24 | 5 | 4 | 3 | 24 | 20 | 12 |
| 26 | 3 | 4 | 3 | 16 | 12 | 7 |
| 26 | 5 | 5 | 1 | 30 | 8 | 18 |
| 27 | 4 | 5 | 1 | 25 | 6 | 13 |
| 28 | 3 | 5 | 1 | 20 | 4 | 11 |
| 28 | 5 | 4 | 4 | 24 | 26 | 6 |
| 30 | 3 | 4 | 4 | 16 | 16 | 3 |
| 31 | 5 | 5 | 2 | 30 | 14 | 12 |
| 32 | 4 | 5 | 2 | 25 | 11 | 8 |
| 33 | 3 | 5 | 2 | 20 | 8 | 7 |
| 38 | 3 | 5 | 3 | 20 | 12 | 3 |
| 38 | 5 | 6 | 1 | 36 | 8 | 12 |
| 39 | 4 | 6 | 1 | 30 | 6 | 8 |
| 40 | 3 | 6 | 1 | 24 | 4 | 7 |
| 44 | 5 | 6 | 2 | 36 | 14 | 6 |
| 46 | 3 | 6 | 2 | 24 | 8 | 3 |
| 52 | 5 | 7 | 1 | 42 | 8 | 6 |
| 54 | 3 | 7 | 1 | 28 | 4 | 3 |
| 59 | 5 | 7 | 2 | 42 | 14 | 0 |
| 68 | 5 | 8 | 1 | 48 | 8 | 0 |

Table 4.2: List of lattices in Case 2

*Example* 4.4.2. $k = 10$: Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface containing a line $\Gamma$ and an elliptic curve $E$ of degree 3 intersecting transversely at one point. Then $X$ is a K3 surface with the following primitive lattice embedding

$$\begin{pmatrix} 4 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cong U \oplus \langle -20 \rangle \hookrightarrow \mathrm{NS}(X).$$

We recall the dimension count in this example:

$$\begin{aligned} &\dim(\mathrm{Hilb}_{3\cdot t}(\mathbb{P}^3)) + 1 + \dim(\mathrm{Hilb}_{t+1}(\mathbb{P}^3)) - 2 + (h^0(E \cup \Gamma, \mathcal{I}_{E\cup\Gamma}(4)) - 1) - \dim \mathrm{PGL}(4) = \\ &= 12 + 1 + (4-2) + (19-1) - (4^2 - 1) = 18 = \dim \mathcal{M}_{2k} + \dim |E| + \dim |\Gamma|. \end{aligned}$$

## 4.5 Construction of nodal elliptic K3 surfaces

We adapt our above construction in order to deal with nodal K3 surfaces having one $A_1$-singularity. As in Step 1 of the construction in Section 4.3.1, we construct a curve of degree $d$ (that can either be a rational or an elliptic curve). In the second step we choose a point $p \in \mathbb{P}^n$, that can either belong or not to the chosen curve, depending on the situation. The final step is to construct a K3 surface $X$ containing the given curve and having an $A_1$-singularity at $p$.

We restrict our considerations to the case of an elliptic curve $E$ with a point $p$ on it, as the other cases are treated similarly. The desingularization of $X$ contains the exceptional divisor $C_p \cong \mathbb{P}^1$ over the $A_1$-singularity. Then we have the following lattice embedding (which will also be primitive by Section 4.3.4)

$$\begin{pmatrix} 2n-2 & 0 & d \\ 0 & -2 & 1 \\ d & 1 & 0 \end{pmatrix} \cong U \oplus \langle -2 \cdot (d^2 - n + 1) \rangle \hookrightarrow \mathrm{NS}(X).$$

given by the intersection matrix with respect to the basis $\langle \mathcal{O}_{\mathbb{P}^n}(1)|_X, C_p, E \rangle$. We set $k' = d^2 - n + 1$. Hence, we have an incidence variety

$$I' := \{(X, E) \mid (E, p) \in H^1_{d,1,n} \text{ and } E \subset X \in \mathcal{M}_{2k'}\} \subset \mathcal{M}_{2k'} \times H^1_{d,1,n}$$

and the natural projections, denoted by $\pi_1'$ and $\pi_2'$. The fiber $\pi_2'^{-1}(E, p)$ is unirational. Indeed, to obtain a K3 surface that is nodal in a point, one has to solve linear equations in the coefficients of the equations generating the K3 surface and of their derivatives. But the choice is unirational, and therefore the incidence variety $I'$ is unirational. The unirationality of $\mathcal{M}_{2k'}$ follows by the construction of a smooth K3 surface with the desired properties and a dimension count, as in Case 1 and 2 (Proposition 4.3.8 holds almost verbatim in the nodal case as well). The following table lists our experimental data for such nodal K3 surfaces. We denote by $\dim \pi_2'^{-1}(E, p)$ the dimension of the space of nodal K3 surfaces containing $E$ and having a node at $p \in E$.

| $k'$ | $n$ | $d$ | $\dim(\mathrm{Hilb}_{d\cdot t}(\mathbb{P}^n))$ | $\dim \pi_2'^{-1}(E,p)$ |
|---|---|---|---|---|
| 6 | 4 | 3 | 15 | 24 |
| 7 | 3 | 3 | 12 | 20 |
| 12 | 5 | 4 | 24 | 28 |
| 13 | 4 | 4 | 20 | 19 |
| 14 | 3 | 4 | 16 | 18 |
| 21 | 5 | 5 | 30 | 22 |
| 22 | 4 | 5 | 25 | 14 |
| 23 | 3 | 5 | 20 | 12 |
| 32 | 5 | 6 | 36 | 16 |
| 33 | 4 | 6 | 30 | 9 |
| 34 | 3 | 6 | 24 | 8 |
| 45 | 5 | 7 | 42 | 10 |
| 47 | 3 | 7 | 28 | 4 |
| 60 | 5 | 8 | 48 | 4 |
| 62 | 3 | 8 | 32 | 0 |

Table 4.3: List of values for nodal K3 surfaces

*Remark* 4.5.1. One can repeat the same process as above by considering a rational curve instead of an elliptic curve. This leads to the unirationality of $\mathcal{M}_{2k}$ for the following values of $k$:

$$k \in \{13, 17, 21, 25, 31, 37, 41, 61\}.$$

However we have already shown the unirationality of $\mathcal{M}_{2k}$ for these values of $k$ before (see Table 4.1). Therefore, we do not treat these cases in detail.

We end this final section with three examples demonstrating how to choose a nodal K3 surface in $\mathbb{P}^n$ for $n = 3, 4$ and 5 containing a given curve and having an $A_1$-singularity at a point. Furthermore, we recall the dimension count in these examples showing that the projection $\pi_1' : I' \dashrightarrow \mathcal{M}_{2k'}$ is dominant.

*Example* 4.5.2 (Nodal quartic surfaces: $k' = 7$.). Given an elliptic curve $E$ together with a point $p$ in $\mathbb{P}^3$, a nodal K3 surface containing $E$ and with a node at $p$ is a quartic generator of the ideal $\mathcal{I}_{Ep^2} := \mathcal{I}_E \cap (\mathcal{I}_p)^2$.

Let $X' \subset \mathbb{P}^3$ be a nodal quartic surface containing an elliptic curve $E$ of degree 3 with an $A_1$-singularity at a point of $E$. Then the desingularization $X$ of $X'$ is a smooth K3 surface with the following primitive lattice embedding

$$\begin{pmatrix} 4 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \cong U \oplus \langle -14 \rangle \hookrightarrow \mathrm{NS}(X).$$

We recall the dimension count in this case:

$$\begin{aligned} &\dim(\mathrm{Hilb}_{3\cdot t}(\mathbb{P}^3)) + 1 + (h^0(\mathcal{I}_{Ep^2}(4)) - 1) - \dim \mathrm{PGL}(4) \\ &= 12 + 1 + (21 - 1) - (4^2 - 1) = 18 = \dim \mathcal{M}_{2k'} + \dim |E|. \end{aligned}$$

*Example* 4.5.3 (Nodal complete intersections of a quadric and a cubic: $k' = 6.$). Given an elliptic curve $E$ together with a point $p$, we get a nodal K3 surface by choosing two generators of degree 2 and 3 in the ideal $\mathcal{I}_E$ with the same tangent space at the point $p$. Therefore, we compute all quadric and cubic hypersurfaces containing $E$ and being tangent at $p$ to a $\mathbb{P}^3$ that contains the tangent line of $E$ at $p$. The ideal of such hypersurfaces, denoted $\mathcal{I}_{E,\mathbb{P}^3}$, is the intersection of $\mathcal{I}_E$ and $\mathcal{I}_p^2 \cap \mathcal{I}_{\mathbb{P}^3}$.

Let $X' \subset \mathbb{P}^4$ be a nodal K3 surface containing an elliptic curve $E$ of degree 3 with an $A_1$-singularity at a point of $E$. Then the desingularization $X$ of $X$' is a smooth K3 surface with the following primitive lattice embedding

$$\begin{pmatrix} 6 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \cong U \oplus \langle -12 \rangle \hookrightarrow \mathrm{NS}(X).$$

We recall the dimension count in this case:

$$\begin{aligned} &\dim(\mathrm{Hilb}_{3\cdot t}(\mathbb{P}^3)) + 1 + \dim(\mathbb{P}^3 \text{ containing the tangent line } T_p(E)) \\ &+ (h^0(\mathcal{I}_{E,\mathbb{P}^3}(2)) + h^0(\mathcal{I}_{E,\mathbb{P}^3}(3)) - 7) - \dim \mathrm{PGL}(5) \\ &= 15 + 1 + 2 + (7 + 24 - 7) - (5^2 - 1) = 18 = \dim \mathcal{M}_{2k'} + 1. \end{aligned}$$

*Example* 4.5.4. Nodal complete intersections of three quadrics: $k' = 12$. Given an elliptic curve $E$ together with a point $p$, we obtain a nodal K3 surface by choosing a nodal quadric in the ideal $\mathcal{I}_{Ep^2} := \mathcal{I}_E \cap (\mathcal{I}_p)^2$ and two further quadrics in the ideal of $E$.

Let $X' \subset \mathbb{P}^5$ be a nodal K3 surface containing an elliptic curve $E$ of degree 4 with an $A_1$-singularity at a point of $E$. Then the desingularization $X$ of $X'$ is a smooth K3 surface with the following primitive lattice embedding

$$\begin{pmatrix} 8 & 0 & 4 \\ 0 & -2 & 1 \\ 4 & 1 & 0 \end{pmatrix} \cong U \oplus \langle -24 \rangle \hookrightarrow \mathrm{NS}(X).$$

We recall the dimension count in this case:

$$\begin{aligned} &\dim(\mathrm{Hilb}_{4\cdot t}(\mathbb{P}^5)) + 1 + (h^0(\mathcal{I}_{Ep^2}(2)) - 1) + (2 \cdot (h^0(\mathcal{I}_E(2)) - 1 - 2)) - \dim \mathrm{PGL}(6) \\ &= 24 + 1 + (9 - 1) + (2 \cdot (13 - 1 - 2)) - (6^2 - 1) = 18 = \dim \mathcal{M}_{2k'} + 1. \end{aligned}$$

## 4.6 The case $k = 11$

We end the chapter by dealing with the last remaining case, namely $k = 11$. We have decided to dedicate its own section to this case, as the strategy involved is different from the ones of the previous sections. More precisely, we realize the generic K3 surface in $\mathcal{M}_{22}$ as a complete intersection in the Grassmannian $\mathrm{G}(2,5) \subseteq \mathbb{P}^9$. Then a dimension count following Proposition 4.3.8 (together with an explicit smooth example) allows us to show the unirationality of $\mathcal{M}_{22}$. Let us first recall some well-known facts about K3 surfaces of degree 10, following [Muk88].

We denote by $\mathrm{G}(2,5) \subseteq \mathbb{P}^9$ the Grassmannian of lines in $\mathbb{P}^4$ embedded in $\mathbb{P}^9$ via the usual Plücker embedding. We denote by $x_0, \dots, x_9$ a set of coordinates for $\mathbb{P}^9$. $\mathrm{G}(2,5) \subseteq \mathbb{P}^9$ is given by the vanishing of 4 quadrics, it has degree 5 and dimension 6.

The intersection of $\mathrm{G}(2,5)$ with a general linear subspace $\mathbb{P}^6 \subseteq \mathbb{P}^9$ of dimension 6 is a quintic del Pezzo 3-fold $Y$ (see [Muk88, Theorem 0.9]). There are many equivalent definitions of such del Pezzo 3-folds: for instance they are the zero set in $\mathbb{P}^6$ of the maximal pfaffians of a $5 \times 5$ skew-symmetric matrix with general linear polynomials as entries by the structure theorem by Buchsbaum and Eisenbud [BE77]. Moreover, they are precisely those Fano 3-folds in $\mathbb{P}^6$ whose linear sections are del Pezzo surfaces of degree 5. Recall that a del Pezzo surface of degree 5 is the blow-up of the plane $\mathbb{P}^2$ at 4 general points.

Let $X$ be a general K3 surface of degree 10. The polarization of $X$ given by the very ample line bundle $L$ on $X$ with $L^2 = 10$ embeds $X$ in $\mathbb{P}^9$ as the complete intersection of a quintic del Pezzo 3-fold and a quadric hypersurface (see [Muk88, Corollary 0.3]). In other words, a general K3 surface of degree 10 is a complete intersection in $\mathrm{G}(2,5)$ of a quadric and a codimension 3 linear space.

We are interested in the projective model of K3 surfaces in $\mathcal{M}_{22}$ given by the intersection matrix

$$\begin{pmatrix} 10 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \cong U \oplus \langle -22 \rangle.$$

*Remark* 4.6.1. The moduli space $\mathcal{M}_{22}$ is a codimension 2 subvariety of the moduli space $\mathcal{F}_{10}$ of K3 surfaces of degree 10, so we have two possibilities: either no K3 surface in $\mathcal{M}_{22}$ can be realized as a complete intersection in $\mathrm{G}(2,5)$, or the general one can. We will exclude the first possibility during the course of the construction, when we will find an explicit example.

By looking at the intersection matrix above, we want to realize the K3 surfaces in $\mathcal{M}_{22}$ as complete intersections in $\mathrm{G}(2,5)$ with a singular point $V$, and with an elliptic pencil of degree 4 having $V$ as a base point.

We start with the Segre embedding $s : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Denote by $[t_0 : t_1]$, $[y_0 : y_1 : y_2]$ and $[x_0 : \dots : x_5]$ the coordinates on $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^5$ respectively. The image of $s$ is a 3-fold $T$. We embed $\mathbb{P}^5 \hookrightarrow \mathbb{P}^6$ as the hyperplane $\{x_6 = 0\}$, and we construct the cone $\mathcal{S}$ with vertex $V = [0 : \dots : 0 : 1]$ over $T$. We can see $\mathcal{S}$ as the image of the morphism

$$\mathrm{Spec}(R) := \mathrm{Spec}(\mathbb{C}[t_0, t_1, y_0, y_1, y_2, x_6]) \hookrightarrow \mathbb{P}^6$$

extending the Segre embedding $s$ so that $x_6 \mapsto x_6$. We consider $t_0, t_1$ (resp. $y_0, y_1, y_2$) to have bidegree $(1,0)$ (resp. $(0,1)$) in $R$; by homogeneity, $x_6$ must have bidegree $(1,1)$ in $R$.

$\mathcal{S} \subseteq \mathbb{P}^6$ is a cubic scroll of dimension 4, given by the equations

$$\mathcal{S} = \{x_0x_4 - x_1x_3 = x_0x_5 - x_2x_3 = x_1x_5 - x_2x_4 = 0\}.$$

Moreover $\mathcal{S}$ admits a ruling

$$\begin{array}{rccc} \rho: & \mathcal{S} & \to & \mathbb{P}^1 \\ & [x_0 : \ldots : x_6] & \mapsto & [x_0 : x_1] = [x_3 : x_4] = [x_5 : x_6] \end{array}$$

with fiber $\mathbb{P}^3 = \{x_0 = x_1 = x_2 = 0\}$.

Now let $Y$ be the subvariety of $\mathcal{S}$ given by an equation of bidegree $(1,2)$ in $R$. First, $Y$ contains the vertex $V$, so $V$ is singular at $V$. This is because the only monomials in $R$ of bidegree $(1,2)$ containing $x_6$ are $y_i x_6$ for $i \in \{0,1,2\}$. Moreover $Y$ is a (singular) quintic del Pezzo 3-fold. This is well-known, but let us sketch the argument. Clearly $Y$ is a Fano 3-fold (for instance, the projection onto the second factor $\mathbb{P}^2$ gives a morphism $Y \to \mathbb{P}^2$ with fiber $\mathbb{P}^1$). Moreover the linear section $\{x_6 = 0\}$ of $Y$ is a surface $Z$ in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(1,2)$, which is a del Pezzo surface of degree 5 (indeed, the projection $Z \to \mathbb{P}^2$ onto the second factor is the blow-up of 4 points).

The ruling $\rho$ restricts to a morphism $\sigma : Y \to \mathbb{P}^1$ with fiber $Q_1 = \{x_0 = x_1 = x_2 = 0\} \cap Y$, which is a quadric surface. In other words, $Y$ is covered by a pencil of quadric surfaces passing through the vertex $V$.

*Remark* 4.6.2. Not all quintic del Pezzo 3-folds in $\mathbb{P}^6$ lie inside a cubic scroll. Let $\mathcal{H}_{dP,5}$ be the space of quintic del Pezzo 3-folds, and $\mathcal{H}_{dP,5}^{scroll} \subseteq \mathcal{H}_{dP,5}$ the subspace of those lying inside a cubic scroll. One can check that $\mathcal{H}_{dP,5}^{scroll}$ is of codimension 1 in $\mathcal{H}_{dP,5}$. However, both spaces are unirational. $\mathcal{H}_{dP,5}$ is unirational by the structure theorem by Buchsbaum and Eisenbud [BE77]. For the other one, notice first that the space $\mathcal{H}_{scroll}$ of cubic scrolls in $\mathbb{P}^6$ is unirational (they are Cohen-Macaulay subschemes of $\mathbb{P}^6$ of codimension 2, so we can apply Ellingsrud's result [Ell75]). Then $\mathcal{H}_{dP,5}^{scroll}$ is (generically) a projective bundle over $\mathcal{H}_{scroll}$ with fiber isomorphic to

$$\mathbb{P}H^0(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}(1,2)) \cong \mathbb{P}^{14}.$$

Finally, we choose a quadric $Q_2 \subseteq \mathbb{P}^6$ passing through the vertex $V$. The intersection $X = Y \cap Q_2$ is a K3 surface of degree 10 singular at $V$, and the morphism $\sigma$ restricts to a genus 1 fibration $\pi : X \to \mathbb{P}^1$ of degree 4, since the fibers are intersections of two generic quadrics in $\mathbb{P}^3$. All the elliptic curves in this pencil pass through the vertex $V$. After blowing up $V$, we obtain a smooth K3 surface with the desired Néron-Severi lattice (the exceptional divisor becomes a section of the elliptic pencil).

We obtain with `Macaulay2` an explicit smooth example. Therefore it only remains to show that the parameter space of such complete intersections is unirational, and that the dimension count works as in Proposition 4.3.8. We have shown in Remark 4.6.2 that the space $\mathcal{H}_{dP,5}^{scroll}$ is unirational, and certainly the space of quadrics of $\mathbb{P}^6$ passing through $V$ is rational. So the parameter space is indeed unirational. For the dimension count, we have

$$h^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^6}) + (h^0(\mathcal{O}_{\mathrm{Spec}(R)}(1,2)) - 1) + h^0(\mathcal{I}_V(2)) - \dim \mathrm{PGL}(7) = 30 + 14 + 21 - 48 = 17,$$

thus concluding the proof.

# 5 | Enriques surfaces satisfy the potential Hilbert Property

## 5.1 Introduction

Given an algebraic variety $X$ defined over a field $K$, one would like to understand "how many" $K$-rational points the variety $X$ has. This problem has been at the heart of fundamental discoveries in algebraic and arithmetic geometry in the past century. Depending on the base field $K$, the question of determining the set of $K$-rational points of $X$ can have slightly different flavours. For instance, if $K = \mathbb{F}_p$ is a finite field, the problem is equivalent to understanding how many solutions there are to the defining equations of $X$ modulo $p$. If instead $K$ is an infinite field, then one needs a way to establish whether $X$ has a "small" or a "large" number of $K$-rational points. The coarsest possibility is to ask whether the $K$-rational points are (Zariski) dense in $X$. However, this remains a very hard question to answer for many classes of varieties. Indeed, the only completely settled case is the one of curves: for rational and elliptic curves, the problem was classically solved, while Faltings' theorem [Fal83] shows that $C(K)$ is not dense in $C$ for any non-singular algebraic curve $C$ of genus $g \geq 2$ over a number field $K$. In the case of varieties of higher dimension, Vojta's conjectures [Voj87] predict when rational points should be dense, but in very few cases we actually have a definitive answer.

Considering instead varieties that admit a dense set of $K$-rational points, one might find finer ways to understand how "large" the set $X(K)$ of $K$-rational points is. If $K$ is a number field there are multiple, classical possibilities. The most important for us is the so-called *Hilbert property*, that was first studied by Hilbert. Roughly speaking, a variety $X$ over a number field $K$ satisfies the Hilbert property if the set $X(K)$ of $K$-rational points is dense in $X$, and it does not come from a finite number of finite covers $Y_i \to X$ (see Definition 5.2.1). The Hilbert property is closely related to weak approximation (see [Ser92]).

The importance of the Hilbert property derives from its several connections to other fundamental problems in Arithmetic Geometry and Number Theory. For instance the following conjecture, attributed to Colliot-Thélène and Sansuc [CS87], would settle, if proved, the inverse Galois problem:

**Conjecture 5.1.1** (Colliot-Thélène, Sansuc)**.** *Every unirational variety over a number field* *satisfies HP.*

While on one side varieties with Kodaira dimension $-\infty$ are believed to have "many" $K$-rational points, on the other hand by Vojta's conjectures one expects varieties with positive Kodaira dimension to have fewer $K$-rational points. Therefore, as for curves, the case where we expect a more varied behaviour is the one of Kodaira dimension zero. In the present chapter we will focus on K3 and Enriques surfaces. In recent years much work has been devoted to the study of the Hilbert property on such classes of surfaces. For instance, Corvaja and Zannier [CZ17, Appendix 1] proved that, if the Vojta's conjectures hold, then every Kummer surface satisfies the Hilbert property. Moreover Demeio provided a link between the Hilbert property and the presence of many genus 1 fibrations on K3 surfaces (cf. [Dem21, Theorem 1.2]).

For Enriques surfaces, one needs to adjust the definition of the Hilbert property. In fact, as already noted by Corvaja and Zannier [CZ17, Section 2.2], the definition of the Hilbert property is not suitable for varieties with a torsion fundamental group. Therefore they introduce a modified version of the Hilbert property, called the *weak* Hilbert property. An Enriques surface satisfies the weak Hilbert property if and only if its K3 cover satisfies the Hilbert property (cf. Definition 5.2.3).

Despite these important results, it is still very hard to understand whether many K3 surfaces satisfy the Hilbert property. In fact, the density of $K$-rational points is still not known for the majority of K3 surfaces. Therefore we slightly change perspective, and we focus on studying whether K3 and Enriques surfaces satisfy the Hilbert property *after a finite field extension*. We call this the *potential* Hilbert property. Our main result is the following:

**Theorem 5.1.2.** *Every Enriques surface over a number field satisfies the potential weak Hilbert property. Equivalently, every K3 surface over a number field covering an Enriques surface satisfies the potential Hilbert property.*

Theorem 5.1.2 shows that, if $X$ is a K3 surface over $K$ covering an Enriques surface, there exists a finite field extension $L/K$ such that the base change $X_L$ satisfies the Hilbert property. During the course of the proof of Theorem 5.1.2 we are able to effectively find such a field extension $L/K$: in fact we show that, if $X(L)$ is dense and the geometric Néron-Severi group of $X$ is defined over $L$, then $X_L$ satisfies the Hilbert property.

As an immediate consequence we have (see [Keu90]):

**Corollary 5.1.3.** *Every Kummer surface over a number field satisfies the potential Hilbert property.*

Notice that Theorem 5.1.2 only deals with K3 surfaces with large Picard rank, i.e. $\rho > 10$. However, we are also able to prove the following:

**Theorem 5.1.4.** *Let $X$ be a K3 surface over a number field $K$. If $X_{\mathbb{C}}$ admits at least two distinct genus 1 fibrations, and $\rho(X_{\mathbb{C}}) < 10$, then $X$ satisfies the potential Hilbert property.*

Our proof relies on a detailed study of the embeddings of root lattices into the Néron-Severi lattice of K3 surfaces, and on Demeio's result [Dem21, Theorem 1.2]. We stress the fact that our methods can be easily adapted to a much wider class of K3 surfaces, thus showing some evidence towards the following conjecture:

**Conjecture 5.1.5.** *Let $X$ be a K3 surface over a number field $K$. If $X_{\mathbb{C}}$ admits at least two genus 1 fibrations and $\mathrm{Aut}(X_{\mathbb{C}})$ is infinite, then $X$ satisfies the potential Hilbert property. In particular, if $X_{\mathbb{C}}$ has positive entropy, then $X$ satisfies the potential Hilbert property.*

Conjecture 5.1.5, if proved, would provide a surprising link between the geometric side of complex dynamics on K3 surfaces, and a purely arithmetic question as the Hilbert property.

The chapter is organized as follows: in Section 5.2 we review the basic facts about the Hilbert property, with special interest towards K3 surfaces. Section 5.3 contains the technical core of the chapter. We provide an in-depth study of root lattices on K3 surfaces, and we prove Propositions 5.3.4 and 5.3.7, the two main ingredients of the proof of Theorem 5.1.2. Finally, in Section 5.4 we prove Theorem 5.1.2 by using the results of Section 5.3.

## 5.2 The Hilbert property

Throughout the section $K$ is a number field and $X$ is a geometrically irreducible, projective variety over $K$.

**Definition 5.2.1.** $X$ satisfies the *Hilbert property* (HP for short) if the set of its $K$-rational points $X(K)$ is not *thin*, i.e. $X(K)$ is (Zariski) dense in $X$ and for any finite morphism $p : Y \to X$ of degree $\geq 2$ such that $X(K) \smallsetminus p(Y(K))$ is not dense in $X$, there exists a rational section of $p$.

Thinness is a measure of "how many" $K$-rational points the variety $X$ has. Clearly, the fact that a variety satisfies HP highly depends on the base field $K$. For instance, if the set of $K$-rational points $X(K)$ is not dense, then $X$ cannot satisfy HP. However, if a variety $X$ over $K$ satisfies HP, then the base change $X_L$ satisfies HP for any finite field extension $L/K$ [Ser92, Proposition 3.2.1]. Therefore it makes sense to introduce the following definition.

**Definition 5.2.2.** A variety $X$ over $K$ satisfies the *potential Hilbert property* (PHP for short) if the base change $X_L$ satisfies HP for a certain finite field extension $L/K$.

This definition allows much more flexibility: for instance, if $X$ is an elliptic K3 surface or an Enriques surface, we can always choose a finite field extension $L/K$ such that $X(L)$ is dense (see [BT98; BT00]). Therefore a much larger class of varieties can satisfy PHP.

We will be mainly interested in studying the Hilbert property for Enriques and K3 surfaces. We remark immediately that the "classical" definition of Hilbert property (i.e. Definition 5.2.1) is not very suitable in the case of Enriques surfaces, and more generally for all varieties with a finite, non-trivial fundamental group. Indeed Corvaja and Zannier prove in [CZ17, Theorem 1.6] that any smooth projective variety satisfying HP must be algebraically simply connected (and thus its fundamental group cannot have subgroups of finite index).

Consequently, we formulate the following definition:

**Definition 5.2.3.** An Enriques surface satisfies the *weak Hilbert property* (WHP for short) if its K3 cover satisfies HP. We will write PWHP for the *potential* weak Hilbert property.

This definition coincides with the one in [CZ17, Section 2.2] in the case of Enriques surfaces.

An important sufficient condition for a K3 surface to satisfy HP, due to Demeio, is the following:

**Theorem 5.2.4** ([Dem21], Theorem 1.2)**.** *Let $X$ be a K3 surface over $K$ admitting at least two distinct genus 1 fibrations. Denote by $D \subseteq X$ the divisor obtained as the union of all the irreducible curves on $X$ that are orthogonal to all genus 1 fibrations on $X$. If $X \setminus D$ is simply connected and $X(K)$ is dense in $X$, then $X$ satisfies HP.*

*Remark* 5.2.5. When we ask $X \setminus D$ to be simply connected, we mean that the base change $X_{\mathbb{C}} \setminus D$ over $\mathbb{C}$ has to be simply connected. In reality, for Theorem 5.2.4 we only need that there are no unramified covers of degree $> 1$ over $X \setminus D$, which is a weaker condition than $\pi_1(X_{\mathbb{C}} \setminus D) = 1$ (cf. [CZ17, Proposition 1.1]).

The power of Theorem 5.2.4 derives from the fact that it allows us to infer the Hilbert property for a K3 surface $X$ just by looking at its Néron-Severi lattice. The downside, however, is that sometimes the geometric Néron-Severi lattice $\mathrm{NS}(X_{\mathbb{C}})$ is not entirely defined over $K$, and consequently some genus 1 fibrations on $X_{\mathbb{C}}$ do not exist over $K$. Our solution to this problem is to extend the base field to a number field $L/K$ such that $\mathrm{NS}(X_L) = \mathrm{NS}(X_{\mathbb{C}})$. Then we can show that $X_L$ satisfies HP by lattice-theoretical methods.

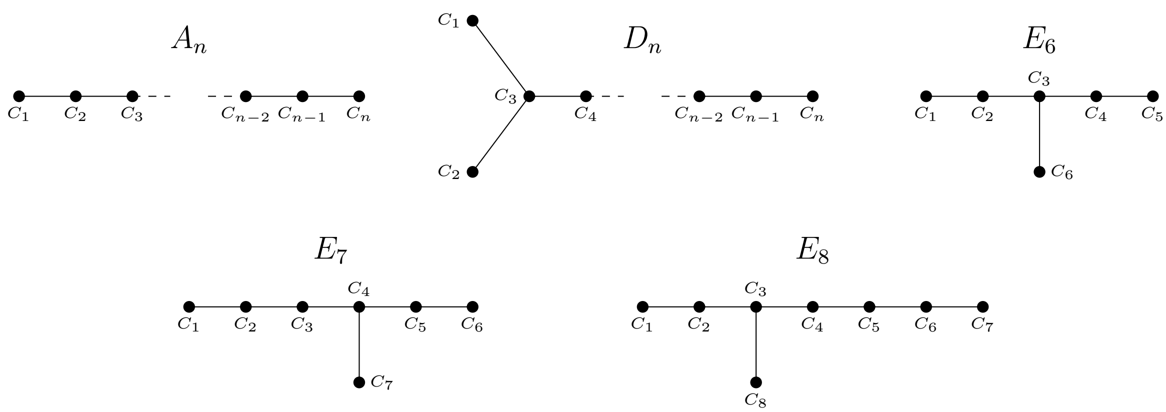
*Remark* 5.2.6. The divisor $D$ in Theorem 5.2.4 (to be more precise, a very closely related object) already appeared in the paper [Nik14]. More precisely, Nikulin defines the *exceptional sublattice* of a complex K3 surface to be the sublattice $E(\mathrm{NS}(X)) \subseteq \mathrm{NS}(X)$ spanned by all the irreducible curves on $X$ that are orthogonal to all genus 1 fibrations on $X$ *with infinite stabilizer*. Clearly, if $E(\mathrm{NS}(X)) = 0$, then $D = \emptyset$, and therefore $X$ satisfies HP by Theorem 5.2.4. Nikulin shows that, if $\rho(X) \geq 6$, there are only finitely many Néron-Severi lattices of K3 surfaces with a non-zero exceptional lattice; combining this with the above discussion, we obtain that there are only finitely many Néron-Severi lattices of rank $\geq 6$ of K3 surfaces not satisfying HP. However, a list of lattices $N$ for which $E(N) \neq 0$ is not known at the moment. For instance, all K3 surfaces with zero entropy have a non-zero exceptional lattice (cf. Chapter 2). Nevertheless, notice that many of the K3 surfaces with zero entropy found in Theorem 2.5.11 (or better, their models over some number field) actually satisfy PHP. Indeed, those who admit some extra genus 1 fibrations have an empty divisor $D$: the only elliptic fibration $|F|$ has irreducible fibers, and the fiber itself $F$ is not a part of the divisor $D$, as it is not orthogonal to the extra genus 1 fibrations.

*Remark* 5.2.7. Theorem 5.2.4 already shows that any unnodal Enriques surface satisfies PWHP. Indeed, any Enriques surface $S$ admits at least three genus 1 fibrations (possibly after base changing to a larger number field) by Theorem 1.5.7, induced by elliptic half-fibers $F_1, F_2, F_3$. If $S$ is unnodal, $|2F_i|$ has only irreducible fibers for each $i$, and therefore the pullbacks $E_i = \pi^* F_i$ induce genus 1 fibrations on the K3 cover $X$ with at most two reducible fibers (with two irreducible components each). Denote by $C_1$ and $C_2$ the two irreducible components of such an elliptic fiber of $|E_1|$. $C_1$ and $C_2$ are interchanged by the Enriques involution $\sigma$, and, up to reordering, $C_1 \cdot E_2 \neq 0$. Therefore $C_2 \cdot \sigma(E_2) = C_1 \cdot E_2 \neq 0$, and consequently no curve on $X$ is orthogonal to all genus 1 fibrations on $X$. Since rational points are potentially dense on $X$ by [BT00], we conclude that $X$ satisfies PHP.

## 5.3 Root lattices on K3 surfaces

We devote this section to an in-depth study of the possible embeddings of a root lattice $R$ into the Néron-Severi lattice of a K3 surface.

**Convention 5.3.1.** Let us fix some notations and conventions about root lattices that will be used throughout the chapter. We will keep the following numbering of the components:




If $R$ is a direct sum of some ADE lattice, we will number its components following the order of the direct summands, and we will use the above convention for the numbering of the basis vectors in each summand.

The discriminant groups of ADE lattices are discussed in Section 1.1.1. We will often write vectors in $A_R = R^\vee / R$ as vectors in the dual lattice $R^\vee$.

In order to show that many K3 surfaces satisfy PHP, by Theorem 5.2.4 we need to understand when the complement of some $(-2)$-curves on a K3 surface is simply connected. The problem of determining the fundamental group of an "open" K3 surface was initialized by works of Shimada, Zhang and Keum [SZ01; KZ02]. We recall the main result:

**Theorem 5.3.2** ([SZ01], Theorem 4.3)**.** *Let $X$ be a complex K3 surface, and consider a root lattice $R$ spanned by some $(-2)$-curves $C_1, \ldots, C_r$ on $X$. Assume that $R$ satisfies the following three conditions:*

1. $\mathrm{rk}(R) \leq 18$*;*
2. $\ell(A_R) \leq 20 - \mathrm{rk}(R)$*;*
3. *the embedding $R \hookrightarrow \mathrm{NS}(X)$ is primitive.*

*Then the complement $X \smallsetminus (C_1 \cup \ldots \cup C_r)$ is simply connected.*

*Remark* 5.3.3. If the embedding $R \hookrightarrow \mathrm{NS}(X)$ is primitive, then the inequality

$$\ell(A_R) \leq 22 - \mathrm{rk}(R)$$

holds. For, denote by $R^{\perp}$ the orthogonal complement of $R$ in the K3 lattice $\Lambda_{K3}$. The unimodularity of the K3 lattice implies that $A_R$ and $A_{R^{\perp}}$ are isomorphic as finite groups, and hence

$$\ell(A_R) = \ell(A_{R^{\perp}}) \leq \mathrm{rk}(R^{\perp}) = 22 - \mathrm{rk}(R).$$

Therefore we would like to understand what happens when the embedding $R \hookrightarrow \mathrm{NS}(X)$ is not primitive. The following result already appeared in the preprint [Sch18, Theorem 1.1]; nevertheless, our proof is somewhat different from the one there.

**Proposition 5.3.4.** *Let $X$ be a complex K3 surface, and consider a root lattice $R$ spanned by some $(-2)$-curves in $X$. If the embedding of lattices $i : R \hookrightarrow \mathrm{NS}(X)$ is not primitive, then the roots of the saturation $i(R)_{sat}$ are roots of $R$. In particular, $i(R)_{sat}$ is not a root lattice.*

*Proof.* Assume by contradiction that in the saturation $i(R)_{sat}$ of $i(R)$ there is a vector $D$ of norm $-2$ that is not in $i(R)$. If $R$ is spanned by $(-2)$-curves $C_1, \ldots, C_r$, we can write $D = \sum_i \alpha_i C_i$ for $\alpha_i \in \mathbb{Q}$. Since the norm of $D \in \mathrm{NS}(X)$ is $-2$, one of $D$ and $-D$ is effective, and without loss of generality we can assume that $D$ is effective. Hence at least one of the $\alpha_i$ is positive. Moreover we can write $D = C'_1 + \ldots + C'_s$ for some (not necessarily distinct) irreducible curves on $X$. After separating the $C_i$ with positive and negative coefficients $\alpha_i$, we get an equality

$$\sum_j C'_j + \sum_{\alpha_i < 0} (-\alpha_i) C_i = \sum_{\alpha_i \geq 0} \alpha_i C_i \tag{5.1}$$

inside $\mathrm{NS}(X)$. Up to multiplying both sides by a positive integer, we can assume that the coefficients $\alpha_i$ are integers. Since the right hand side belongs to the negative definite lattice $i(R)$, its associated linear system contains only one effective divisor, which is itself (both sides of the equation are effective, since they are positive linear combinations of effective curves). Therefore equality (5.1) is an equality of divisors, so the curves $C'_1, \ldots, C'_s$ coincide with some of the $C_i$, and thus they are contained in $i(R)$. We conclude that $D \in i(R)$, a contradiction. □

Combining Theorem 5.3.2 with Proposition 5.3.4, we obtain the following:

**Corollary 5.3.5.** *Let $X$ be a complex K3 surface, and consider a root lattice $R$ spanned by some $(-2)$-curves $C_1, \ldots, C_r$ in $X$. Assume that $R$ satisfies the following three conditions:*

1. $\mathrm{rk}(R) \leq 18$*;*
2. $\ell(A_R) \leq 20 - \mathrm{rk}(R)$*;*
3. *all the overlattices of $R$ (as an abstract lattice) are root lattices.*

*Then the complement $X \smallsetminus (C_1 \cup \ldots \cup C_r)$ is simply connected.*

*Remark* 5.3.6. Notice that the condition (3) above is equivalent to the condition

3'. all the overlattices of $R$ (as an abstract lattice) contain more roots than $R$.

Indeed, both conditions are equivalent to the following: all isotropic vectors in the discriminant group $A_R$ have a preimage in $R^\vee$ of norm $-2$. For instance, $A_1^m$ satisfies condition (3) above if and only if $m < 8$, while $A_2^m$ does so if and only if $m < 6$. Similarly, it is straightforward to check that any root lattice of rank $< 8$ satisfies condition (3).

Let $R$ be a root lattice, $R'$ an overlattice of $R$ of index $p$ prime. We say that $R'$ is *given* by $v \in R^\vee$ if $R' = R[v]$. Moreover, we consider $v$ to be *normalized*, i.e. $v = \frac{1}{p}(\alpha_1 C_1 + \ldots + \alpha_r C_r)$ for some $0 \leq \alpha_i \leq p-1$ (cf. Convention 5.3.1). We say that the overlattice $R'$ is *concentrated* at some curves $\{C_i\}_{i \in I}$ if $\alpha_i = 0$ for any $i \notin I$. The following result characterizes overlattices of root lattices.

**Proposition 5.3.7.** *Let $R$ be a root lattice, $R'$ an overlattice of $R$ of index $p$ prime. Then the curves over which $R'$ is concentrated span a sublattice of $R$ isometric to $A_{p-1}^m$, for some $m \geq 1$. Therefore, if $R'$ is not a root lattice, then $A_{p-1}^m$ admits an overlattice that is not a root lattice as well.*

*Proof.* Assume that $R = R_1 \oplus \ldots \oplus R_N$ splits as the direct sum of certain ADE lattices. The overlattice $R'$ of $R$ is given by a normalized vector $v \in R^\vee$, and its image $\overline{v} \in A_R = A_{R_1} \times \ldots \times A_{R_N}$ has order $p$, so $v = (v_1, \ldots, v_N)$ for certain $v_i \in A_{R_i}$ of order $p$ (or $v_i = 0$). Therefore we only need to classify the vectors of order $p$ in $A_R$, where $R$ is an ADE lattice. Assume $R = A_{pr-1}$, spanned by $C_1, \ldots, C_{pr-1}$ (cf. Convention 5.3.1). The discriminant group $A_R$ is cyclic of order $pr$, and its generator is the image of $e = \frac{1}{pr}(\sum jC_j) \in R^\vee$. All vectors of order $p$ in $A_R$ are the images of $mr \cdot e \in R^\vee$, for $m \in \{1, \ldots, p-1\}$. It is easy to notice that, after normalizing $mr \cdot e$, its coefficients are zero precisely in the positions $p, 2p, \ldots, (r-1)p$. Thus $mr \cdot e \in R^\vee$ is concentrated at the curves $\{C_i : p \nmid i\}$, that span a sublattice of $R$ isometric to $A_{p-1}^r$.
The reasoning for $D_n$ (and $p = 2$) and for $E_6, E_7$ (and $p = 3$, $p = 2$ respectively) is completely analogous. □

*Remark* 5.3.8. We can interpret Proposition 5.3.7 in a more geometric way, at least when $R$ is a root lattice generated by $(-2)$-curves on a K3 surface $X$. If the embedding $i : R \hookrightarrow \mathrm{NS}(X)$ is not primitive, then the saturation of $i(R)$ contains an overlattice $R'$ of $R$ of index $p$ prime. We denote by $X_0$ the K3 surface obtained from $X$ by contracting all the $(-2)$-curves in $R$ where the overlattice $R'$ is concentrated, and by $t_1, \ldots, t_m$ the resulting singular points of $X_0$. The fact that $R' \subseteq \mathrm{NS}(X)$ implies that there exists a cyclic $p : 1$ cover $X' \to X_0$ branched precisely over the singular points $t_1, \ldots, t_m \in X_0$, where $X'$ is smooth. Since the image of a smooth ramification point of a $p : 1$ cover is a singular point of type $A_{p-1}$ (see [Bar+04, Proposition III.5.3]), we conclude that the root type of the curves of $X$ where $R'$ is concentrated is $A_{p-1}^m$.

Proposition 5.3.7 will be a key ingredient for the proof of Theorem 5.1.2. In order to explain its potential, we prove the following result, which anticipates the strategy that we will implement to prove our main theorem.

**Theorem 5.3.9.** *Let $X$ be a K3 surface over a number field $K$. If $X_{\mathbb{C}}$ admits at least two distinct genus 1 fibrations, and $\rho(X_{\mathbb{C}}) < 10$, then $X$ satisfies PHP.*

*Proof.* We choose a number field $L/K$ over which the base change $X_L$ admits two genus 1 fibrations $|E_1|, |E_2|$, and $X(L)$ is dense in $X$. Let $H = \langle E_1, E_2 \rangle \subseteq \mathrm{NS}(X_{\mathbb{C}})$. If $C$ is an irreducible curve orthogonal to all genus 1 fibrations on $X_L$, then clearly $C \in H^{\perp}$. Since $H$ is isometric to $U(m)$ for some $m \geq 1$, $H^{\perp}$ is negative definite, and therefore $C$ is a $(-2)$-curve. Denote by $R$ the root lattice spanned by all $(-2)$-curves $C_1, \ldots, C_r$ on $X_L$ that are orthogonal to all genus 1 fibrations on $X$. $R$ embeds into $H^{\perp}$, so $\mathrm{rk}(R) < 8$. All the overlattices of $R$ are root lattices by Remark 5.3.6, hence the complement $X_L \smallsetminus (C_1 \cup \ldots \cup C_r)$ is simply connected by Corollary 5.3.5, and finally $X_L$ satisfies HP by Theorem 5.2.4. ☐

## 5.4 Proof of the main theorem

We have already seen in Remark 5.2.7 that every unnodal Enriques surface satisfies PWHP. Therefore let $S$ be a nodal Enriques surface over a number field $K$, and denote by $\pi : X \to S$ its K3 cover. Up to enlarging the number field $K$, we assume that $X(K)$ is dense in $X$ (see [BT00]), and that $\mathrm{NS}(X_K) = \mathrm{NS}(X_{\mathbb{C}})$.

$S$ admits a special elliptic pencil $|2F|$ by [Cos85, Theorem 4.1], with rational bisection $R_0$. Let $M \subseteq \mathrm{NS}(S_{\mathbb{C}})$ be the root lattice spanned by all $(-2)$-curves on $S$ orthogonal to all genus 1 fibrations on $S$.

The pullback of the special elliptic pencil $|2F|$ induces an elliptic fibration $|E|$ on the K3 cover $X$. More precisely, each of the two disjoint components $S_0, P$ of the pullback of $R_0$ is a section of $|E|$. Moreover the pullbacks of the $(-2)$-curves in $M$ span a root lattice $R \subseteq \mathrm{NS}(X)$, isometric to $M^2$.

Our goal is to show that the Enriques surface $S$ satisfies PWHP. By virtue of Theorem 5.2.4, it is sufficient to show that the complement of the curves in $X$, that are orthogonal to all genus 1 fibrations on the K3 surface $X$, is simply connected. Notice that all such curves are $(-2)$-curves by an argument similar to the one in the proof of Theorem 5.3.9, since there are at least 3 distinct genus 1 fibrations on $X$ (the pullbacks of the 3 genus 1 fibrations on $S$).

*Remark* 5.4.1. The $(-2)$-curves on $X$ that are orthogonal to all genus 1 fibrations on $X$ surely lie in the root lattice $R$, since $R$ is spanned by the pullbacks of the curves on $S$ orthogonal to all genus 1 fibrations on $S$. However, it can happen that a $(-2)$-curve $C \in R$ intersects non-trivially a genus 1 fibration $E'$ on $X$, not coming from $S$. If $\sigma$ denotes the Enriques involution of $X$, we have that $\sigma(C)$ intersects non-trivially the genus 1 fibration $\sigma(E')$ on $X$, so we can remove $C$ and $\sigma(C)$ from $R$. Equivalently, we can make $M$ smaller by removing the $(-2)$-curve $\pi(C)$. Therefore we can assume, just by making $M$ smaller, that the $(-2)$-curves in $R \cong M^2$ are precisely the curves on $X$ that are orthogonal to all genus 1 fibrations.

First, let us find some restrictions on the root lattice $M \subseteq \mathrm{NS}(S_{\mathbb{C}})$. Up to enlarging the number field $K$, the Enriques surface $S$ admits two further elliptic pencils $|2F'|$, $|2F''|$, with $F \cdot F' = F \cdot F'' = F' \cdot F'' = 1$. By looking at the intersection matrix, it is easy to realize that the lattice $H \subseteq \mathrm{NS}(S)$ spanned by $F$, $F'$ and $F''$ is isometric to $U \oplus A_1$. Since by assumption the root lattice $M \subseteq \mathrm{NS}(S)$ is spanned by $(-2)$-curves that are orthogonal to all genus 1 fibrations on $S$, we have that $M \hookrightarrow H^{\perp} \cong E_7$. The isometry $H^{\perp} \cong E_7$ follows from the fact that the embedding $H \cong U \oplus A_1 \hookrightarrow U \oplus E_8 \cong \mathrm{NS}(S_{\mathbb{C}})$ is unique up to isometry by [Nik79b, Theorem 1.14.4], and the orthogonal complement of $A_1$ in $E_8$ is clearly isometric to $E_7$. As a consequence $\mathrm{rk}(M) \leq 7$. Up to making $M$ smaller, we can assume that the $(-2)$-curves in $R \cong M^2$ are precisely the curves on $X$ that are orthogonal to all genus 1 fibrations (see Remark 5.4.1). We have three possibilities for the root lattice $M^2 \cong R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$:

(1) The embedding $R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$ satisfies the three conditions of Theorem 5.3.2.

(2) The embedding $R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$ is primitive, but $\ell(A_R) > 20 - \mathrm{rk}(R)$.

(3) The embedding $R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$ is not primitive.

Clearly $X$ satisfies HP in situation (1), by combining Theorems 5.3.2 and 5.2.4. In order to deal with situation (2), we show the following:

**Lemma 5.4.2.** *There are no root lattices $M \hookrightarrow E_7$ with $\ell(A_M) = 11 - \mathrm{rk}(M)$.*

*Proof.* Clearly no root lattice of rank $\leq 5$ can satisfy the property $\ell(A_M) = 11 - \mathrm{rk}(M)$, since $\ell(A_M) \leq \mathrm{rk}(M)$. Assume instead that $r := \mathrm{rk}(M) \in \{6, 7\}$. Since the length of $A_M$ coincides with the maximum of the $p$-lengths, we have $\ell_p(A_M) = 11 - r \in \{4, 5\}$ for some prime $p$. If $p > 2$, then each ADE summand of $M$ with determinant multiple of $p$ has rank at least 2 and $p$-length 1 (see Table 1.1), and hence $\mathrm{rk}(M) \geq 8$, a contradiction. Thus $\ell_2(A_M) = 11 - r$. By checking all ADE lattices with even determinant (namely $A_{2n-1}$, $D_n$ and $E_7$), we observe from Table 1.1 that

$$\mathrm{rk}(L) + \ell_2(A_L) \equiv 0 \pmod 2$$

for any ADE lattice $L$. This implies that $\mathrm{rk}(M) + \ell_2(A_M)$ is even as well, a contradiction since $\mathrm{rk}(M) + \ell_2(A_M) = 11$. □

Lemma 5.4.2 implies that situation (2) actually never occurs: indeed, if $M^2 \cong R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$ is primitive and $\ell(A_R) > 20 - \mathrm{rk}(R)$, then

$$20 - \mathrm{rk}(R) = 20 - 2\,\mathrm{rk}(M) < \ell(A_R) = 2\ell(A_M) \leq 22 - \mathrm{rk}(R)$$

by Remark 5.3.3, i.e. $\ell(A_M) = 11 - \mathrm{rk}(M)$. However there are no such root lattices $M$ that embed into $E_7$ by Lemma 5.4.2.

Finally, we need to deal with the situation when the embedding $R \hookrightarrow \mathrm{NS}(X_{\mathbb{C}})$ is not primitive. Proposition 5.3.4 implies that the saturation of $R$ has the same roots as $R$, and therefore $R$ admits an overlattice $R'$ of index $p$ prime that is not a root lattice. Applying Proposition 5.3.7 to $R \cong M^2$, we know that the overlattice $R'$ is concentrated at a sublattice $\widetilde{R} \subseteq R$ isometric to $A_{p-1}^r$ for some $r \geq 1$. The image $\pi(\widetilde{R}) \hookrightarrow E_7$ is isometric to $A_{p-1}^s$ for some $s \geq \frac{r}{2}$. Moreover $\widetilde{R}$ has an overlattice that is not a root lattice by Proposition 5.3.7, and by Remark 5.3.6 this leaves us with the possibilities

$$\widetilde{R} = A_1^r,\ 8 \leq r \leq 14, \qquad \widetilde{R} = A_2^6. \tag{5.2}$$

In order to conclude the proof of Theorem 5.1.2, we will assume the existence of many $(-2)$-curves on $X$ orthogonal to all genus 1 fibrations on $X$, and spanning a non-primitive sublattice $R \subseteq \mathrm{NS}(X_{\mathbb{C}})$ isometric to one of the lattices in (5.2); we will see that such an assumption leads to a contradiction, as we are always able to find an extra genus 1 fibration intersecting non-trivially one of these $(-2)$-curves. We split the argument in two sections, one for $A_1^r$ and one for $A_2^6$.

### 5.4.1 Case $R = A_1^r$

We keep the notations as in the previous section. We start by studying the overlattices of $R = A_1^r$ that are not root lattices. The discriminant group of $A_1$ is cyclic of order 2, generated by an element of square $-\frac{1}{2} \pmod{2\mathbb{Z}}$. If $8 \leq r \leq 11$, there exists a unique overlattice of index 2 of $R$ that is not a root lattice up to isometry, corresponding to the vector $\frac{1}{2}(C_1 + \ldots + C_8) \in R^\vee$. If instead $12 \leq r \leq 14$, we have a second overlattice of index 2 that is not a root lattice, given by $\frac{1}{2}(C_1 + \ldots + C_{12}) \in R^\vee$. We can deal with the two cases, corresponding to the two overlattices, in a completely analogous way. We will present a detailed argument for the first case, that works verbatim for the second case as well.

Consequently, throughout the section, we will assume the following:

**Assumption 5.4.3.** *On the K3 cover $X$ there are eight $(-2)$-curves $C_1, \ldots, C_8$ orthogonal to all genus 1 fibrations on $X$, and spanning a root lattice $R \cong A_1^8$ that is not primitive in $\mathrm{NS}(X)$.*

The goal of the section is to show that this is impossible. Notice that eight $(-2)$-curves span a lattice isometric to $A_1^8$ if and only if they are disjoint. By the discusssion at the beginning of the section, the embedding $R \hookrightarrow \mathrm{NS}(X)$ is not primitive if and only if the divisor $D = \frac{1}{2}(C_1 + \ldots + C_8)$ is integral, that is if $D \in \mathrm{NS}(X)$.

Recall that the special elliptic pencil $|2F|$ on the Enriques surface $S$ pulls back to an elliptic fibration $|E|$ on $X$, with disjoint sections $S_0$ and $P$. Moreover $C_1, \ldots, C_8$ are orthogonal to $E$ by Assumption 5.4.3. Therefore $C_1, \ldots, C_8$ are vertical components in reducible fibers of $|E|$. We start with a lemma concerning the possible configurations of the curves $C_1, \ldots, C_8$ in the reducible fibers of $|E|$. Notice that, since the divisor $2D = C_1 + \ldots + C_8$ is divisible by 2, the sum $C_1 + \ldots + C_8$ must have even intersection number with any other curve on $X$.

**Lemma 5.4.4.** *Let $C_1, \ldots, C_r$ be disjoint $(-2)$-curves in a reducible fiber $E_0$ of the elliptic fibration $|E|$, and assume that $Z = C_1 + \ldots + C_r$ has even intersection with all the $(-2)$-curves in $E_0$. Then we have the following possibilities for the fiber type of $E_0$ and $r \geq 1$, shown in the picture below:*

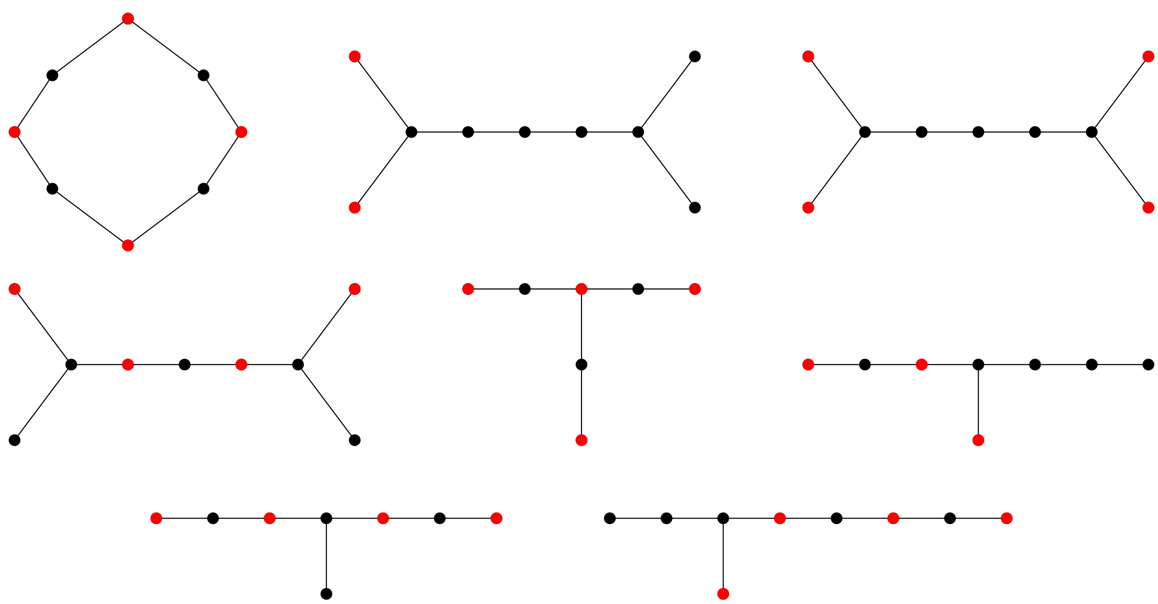
1. *$E_0 = I_{2n}$, $r = n$, or $E_0 = III$, $r = 1$;*

2. *$E_0 = I_n^*$, $r \in \{2, 4\}$;*

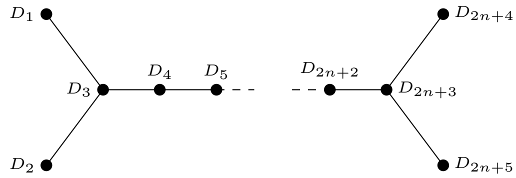
3. *$E_0 = I_{2n}^*$, $r = n + 2$;*

4. *$E_0 = III^*$, $r = 3$;*

5. *$E_0 \in \{IV^*, III^*, II^*\}$, $r = 4$.*



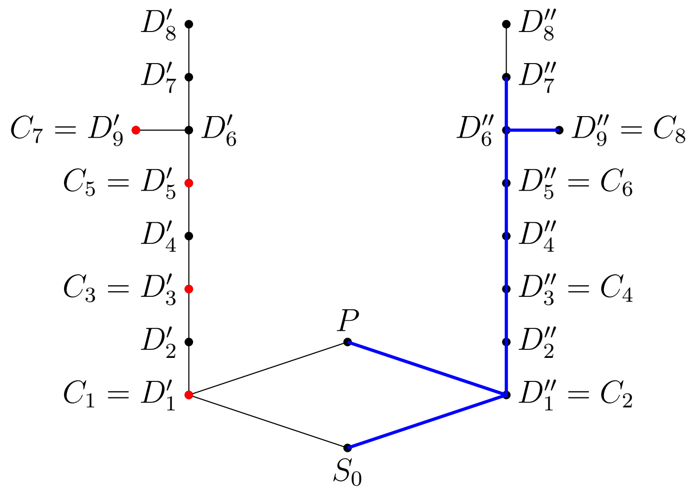
*Proof.* All the cases are handled similarly; we are going to show one of them in detail. Consider the case $E_0 = I_{2n}^*$:




Assume first that $D_1$ and $D_2$ belong to the divisor $Z$. Then $D_3 \notin Z$ because it meets $D_1$ and $D_2$, and $D_4 \notin Z$, otherwise $D_3 \cdot Z = 3$. Now $D_5 \notin Z$, since otherwise $D_4 \cdot Z = 1$, and so on, until $D_{2n+3} \notin Z$. Now we have two choices: either $D_{2n+4}$ and $D_{2n+5}$ both belong to $Z$, or none does.
Assume instead that $D_1 \in Z$, but $D_2 \notin Z$. Then $D_4 \in Z$ (otherwise $D_3 \cdot Z = 1$), and so on until $D_{2n+2} \in Z$. In this case exactly one of $D_{2n+4}$ and $D_{2n+5}$ belongs to $Z$.
Assume finally that $D_1, D_2, D_{2n+4}, D_{2n+5} \notin Z$. Then $D_3, D_{2n+3} \notin Z$ (otherwise $D_1 \cdot Z = 1$), and hence $D_4, D_{2n+2} \notin Z$, and so on, until no component belongs to $Z$. □

We need to show that Assumption 5.4.3 leads to a contradiction. We divide our argument in several subcases, depending on the reducible fibers of the elliptic fibration $|E|$ on $X$.

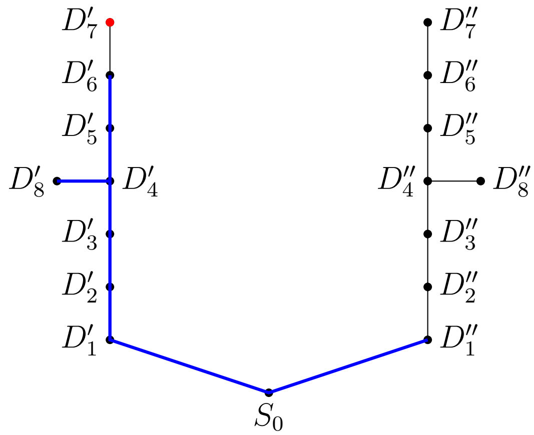
**Case 1: There is a $II^*$ fiber.** Since the elliptic fibration $|E|$ is the pullback of an elliptic fibration on the Enriques surface $S$, by Proposition 1.5.5 there must exist two fibers of type $II^*$. The dual graph of $(-2)$-curves is as follows, where the red vertices correspond to the curves $C_i$:




The configuration of the red vertices follows from Lemma 5.4.4. Moreover $S_0$ and $P$, being sections of $|E|$, must intersect the $II^*$ fibers at the only component of multiplicity 1.

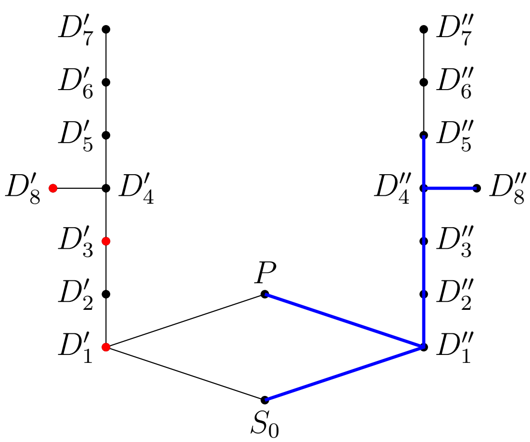
Now the blue diagram in the picture highlights a new elliptic fiber of type $I_5^*$, intersecting $C_1$ non-trivially. This contradicts Assumption 5.4.3.

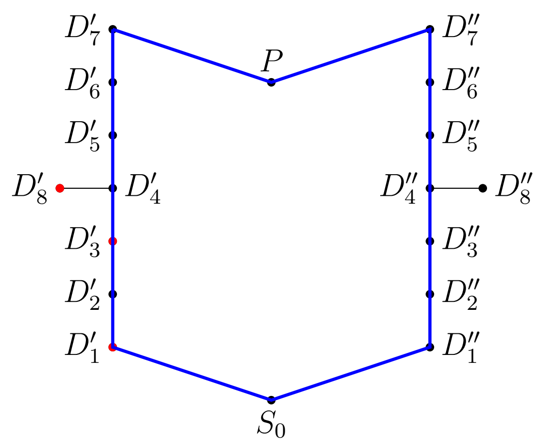
**Case 2: There is a $III^*$ fiber.** A singular fiber of type $III^*$ on the Enriques surface $S$ gets doubled on the K3 cover by Proposition 1.5.5.




We assume first that $D'_7$ is one of our curves $C_i$. Then the blue diagram in the picture above highlights a new elliptic fiber of type $II^*$, intersecting $D'_7$ non-trivially. This contradicts Assumption 5.4.3.

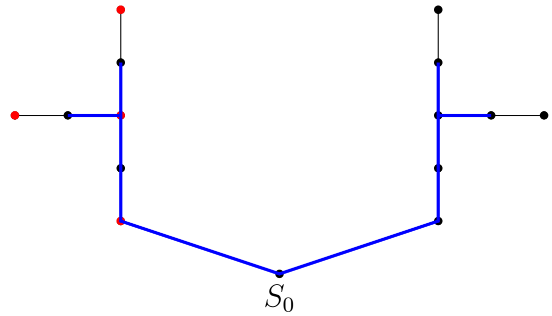
If instead $D'_7$ is not one of the $C_i$, then by Lemma 5.4.4 we have only one possible configuration for the red vertices in the left fiber, consisting of the three curves $D'_1$, $D'_3$ and $D'_8$. Depending on the position of the second section $P$ of $|E|$, we have one of the following two dual graphs:



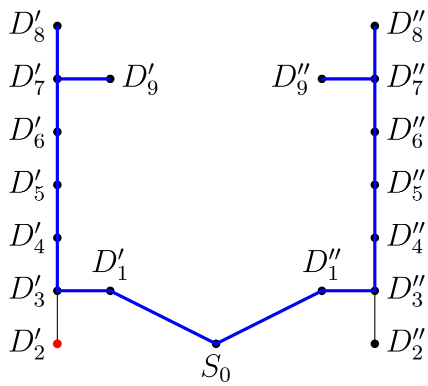

The blue diagrams in the pictures, of type $I_3^*$ and $I_{16}$ respectively, intersect $D'_1$ and $D'_8$ non-trivially, contradicting Assumption 5.4.3.

**Case 3: There is a $IV^*$ fiber.** The dual graph is as follows:

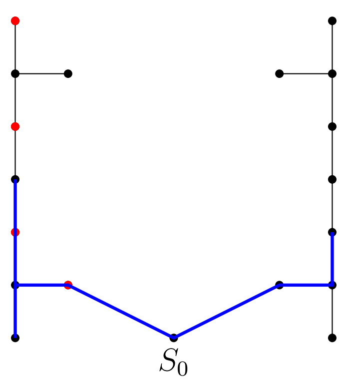



The new elliptic fiber of type $I_6^*$ highlighted in blue in the picture contradicts Assumption 5.4.3.

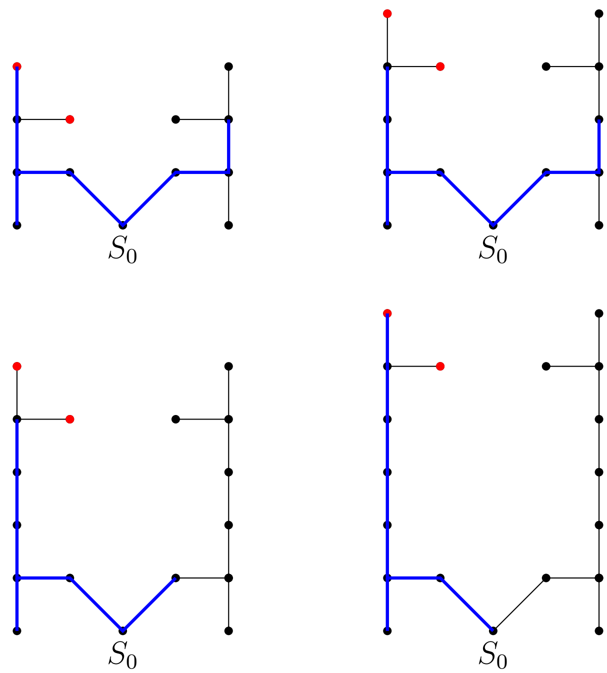
**Case 4: There is an $I_n^*$ fiber.** By Lemma 5.4.4 there are three cases to consider for the configuration of our curves in an $I_n^*$ fiber. Recall that a fiber of type $I_n^*$ on the Enriques surface $S$ gets doubled on the K3 cover by Proposition 1.5.5.



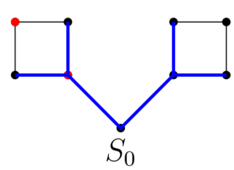

If the component $D_2'$ is one of the $C_i$, then the diagram in blue gives a contradiction. Notice that this already covers all possible configurations if the fiber is of type $I_0^*$. If instead $n \geq 2$ and the configuration of the $C_i$ is as follows, then a new elliptic fiber of type $II^*$ gives the desired contradiction.



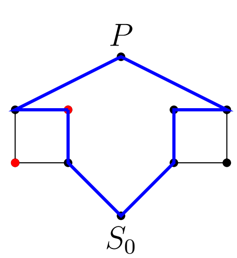
For the last configuration, namely the one containing only two red vertices far from $S_0$, we find the following new elliptic fibers, depending on $1 \leq n \leq 4$ (notice that $n \leq 4$, since the rank of the Néron-Severi lattice of the Enriques surface $S$ is 10):



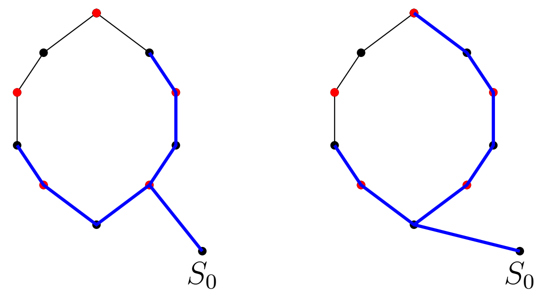

**Case 5: There are at least 2 fibers of type $I_{2n}$, with $n \geq 2$.** Here we only work on the K3 cover $X$. Recall that the elliptic fibration $|E|$ admits (at least) two sections, $S_0$ and $P$. If $S_0$ (or $P$) meets one of the $I_{2n}$ fibers at a component that is one of the $C_i$, then the next picture gives a contradiction (we draw only the case $n = 2$, as the other ones are completely analogous):



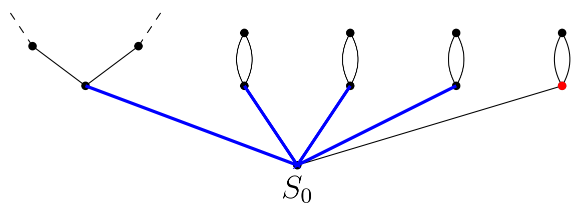

If instead both $S_0$ and $P$ meet the two fibers away from the components $C_i$, then the following new elliptic fiber gives the desired contradiction:



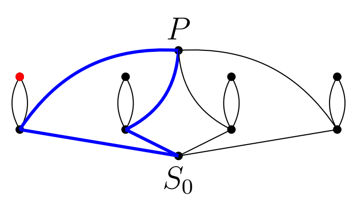
**Case 6: There is a fiber of type $I_{2n}$, $n \geq 5$.** Depending on the intersection of $S_0$ with the fiber, a new elliptic fiber of type $III^*$ or $II^*$ gives the desired contradiction:




**Case 7: There are at least 4 fibers of type $I_2$ (or $III$).** Notice that this is the last case: combining Cases 5 and 6 we know that there is at most one fiber $I_{2n}$ with $2 \leq n \leq 4$, so the remaining $\geq 4$ curves $C_i$ must lie on $I_2$ (or $III$) fibers. If $S_0$ (or $P$) meets at least one of the $I_2$ (or $III$) fibers at a component that is one of the $C_i$, then the new elliptic fiber




of type $I_0^*$ gives the desired contradiction. Otherwise both $S_0$ and $P$ meet the four $I_2$ (or $III$) fibers away from the $C_i$, and hence the new elliptic fiber




of type $I_4$ concludes the case.

### 5.4.2 Case $R = A_2^6$

Throughout the section, we will assume the following:

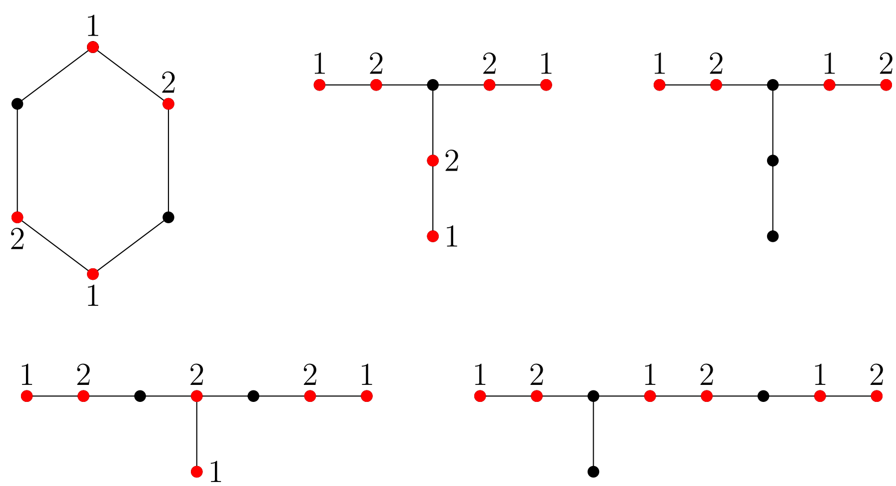
**Assumption 5.4.5.** *On the K3 cover $X$ there are twelve $(-2)$-curves $C_1, \ldots, C_{12}$ orthogonal to all genus 1 fibrations on $X$, and spanning a root lattice $R \cong A_2^6$ that is not primitive in* $\mathrm{NS}(X)$.

The goal of the section is to show that this is impossible. Our strategy will be completely analogous to the case $M = A_1^r$ solved in Section 5.4.1. First, we have to classify the overlattices of $R = A_2^6$ of index 3 that are not root lattices. The discriminant group of $A_2$ is cyclic of order 3, generated by a vector of norm $-\frac{2}{3} \pmod{2\mathbb{Z}}$. Therefore there exists a unique overlattice of index 3 of $R$ that is not a root lattice, namely the one corresponding to $\frac{1}{3}(C_1 + 2C_2 + \ldots + C_{11} + 2C_{12}) \in R^\vee$ (cf. Convention 5.3.1). Therefore, the embedding $R \hookrightarrow \mathrm{NS}(X)$ is not primitive if and only if the divisor $D = \frac{1}{3}(C_1 + 2C_2 + \ldots + C_{11} + 2C_{12})$ is integral, that is if $D \in \mathrm{NS}(X)$.

We keep the notations as in Section 5.4. In particular, we denote by $|E|$ the elliptic fibration on the K3 cover $X$, and by $S_0, P$ its sections. Notice that, since the divisor $3D = C_1 + 2C_2 + \ldots + C_{11} + 2C_{12}$ is divisible by 3, the sum $C_1 + 2C_2 + \ldots + C_{11} + 2C_{12}$ must have intersection $\equiv 0 \pmod 3$ with any other curve on $X$.

**Lemma 5.4.6.** *Let $C_1, \ldots, C_{2r}$ be $(-2)$-curves in a reducible fiber $E_0$ of the elliptic fibration $|E|$ generating a root lattice of type $A_2^r$, and assume that $Z = C_1 + 2C_2 + \ldots + C_{2r-1} + 2C_{2r}$ has intersection $\equiv 0 \pmod 3$ with all the $(-2)$-curves in $E_0$. Then we have the following possibilities for the fiber type of $E_0$ and $r \geq 1$, shown in the picture below:*

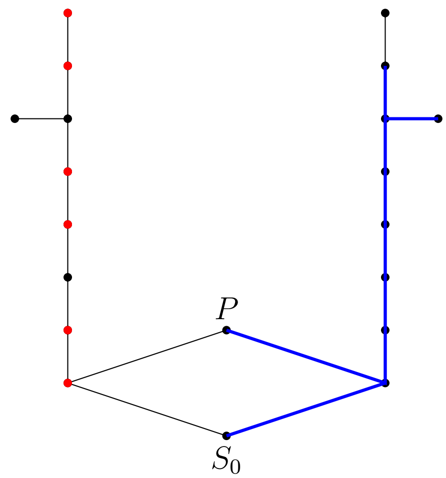
1. *$E_0 = I_{3n}$, $r = n$, or $E_0 = IV$, $r = 1$;*
2. *$E_0 = IV^*$, $r \in \{2, 3\}$;*
3. *$E_0 \in \{III^*, II^*\}$, $r = 3$.*




*Proof.* The proof is straightforward and completely analogous to the one of Lemma 5.4.4. $\square$

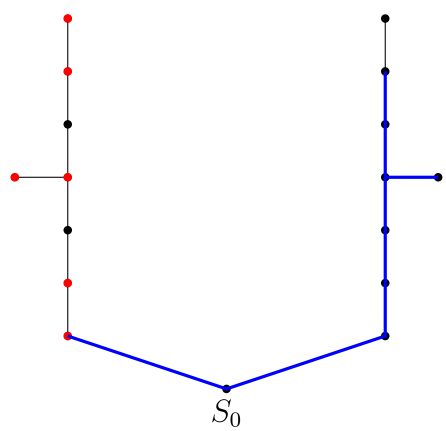
We perform a case-by-case analysis, as in the previous section.

**Case 1: There is a $II^*$ fiber.** The dual graph is as follows:



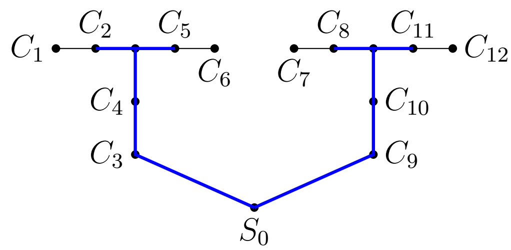

The blue diagram highlights a new elliptic fiber of type $I_5^*$, and it intersects a red component non-trivially, thus contradicting Assumption 5.4.5.

**Case 2: There is a $III^*$ fiber.** The dual graph is as follows:



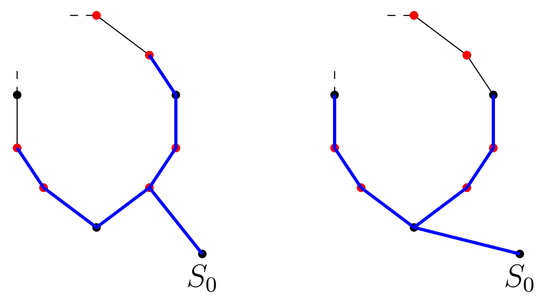

The blue diagram in the picture highlights a new elliptic fiber of type $II^*$, contradicting Assumption 5.4.5.

**Case 3: There is a $IV^*$ fiber.** By Lemma 5.4.6 we have two possibilities for the configuration of red components. The dual graph is as follows:



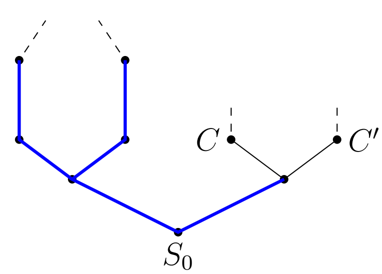

Surely at least one of $C_1$ or $C_6$ (and symmetrically, at least one of $C_7$ and $C_{12}$) is a red component by Lemma 5.4.6, so the new elliptic fiber of type $I_6^*$ highlighted in blue contradicts Assumption 5.4.5.

**Case 4: There is an $I_{18}$ fiber.** The dual graph is as follows, depending on which component the zero section $S_0$ meets:



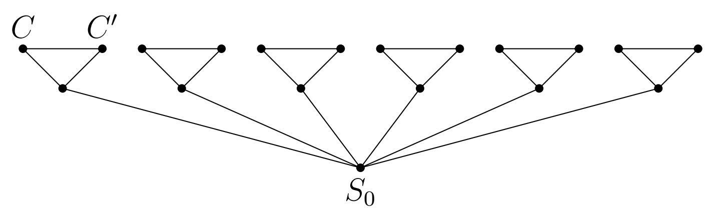

In both cases the new elliptic fiber of type $III^*$ contradicts Assumption 5.4.5.

**Case 5: There is an $I_{3n}$ fiber with $n > 1$.** After dealing with Cases 1 – 4, we can assume that all the reducible fibers of $|E|$ are of type $I_{3n}$ for some $n \geq 1$, and that our 12 components do not lie on a unique fiber. In this case the dual graph is as follows:

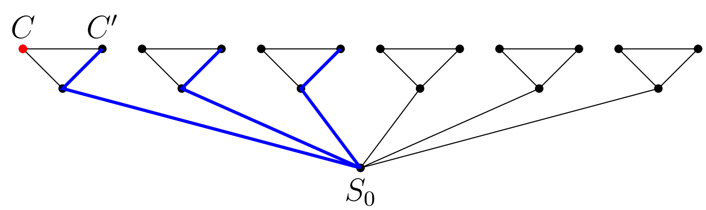



By Lemma 5.4.6 at least one of $C$ and $C'$ is a red component, so the new elliptic fiber of type $IV^*$ highlighted in blue contradicts Assumption 5.4.5.

**Case 6: There are 6 fibers of type $I_3$ (or $IV$).** We have the following dual graph:




At least one of $C$ and $C'$ is one of the $C_i$ by Lemma 5.4.6, say $C$ without loss of generality. Then the new elliptic fiber




of type $IV^*$ intersects $C$ non-trivially, contradicting Assumption 5.4.5.

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*E per ora la merenda regge!*

# Curriculum vitae

| | |
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